Numerical Treatment of the Black-Scholes Variational Inequality in Computational Finance

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Abstract

Among the central concerns in mathematical finance is the evaluation of American options. An American option gives the holder the right but not the obligation to buy or sell a certain financial asset within a certain time-frame, for a certain strike price. The valuation of American options is formulated as an optimal stopping problem. If the stock price is modelled by a geometric Brownian motion, the value of an American option is given by a deterministic parabolic free boundary value problem (FBVP) or equivalently a non-symmetric variational inequality on weighted Sobolev spaces on the entire real line $\mathbb{R}$.

To apply standard numerical methods, the unbounded domain is truncated to a bounded one. Applying the Fourier transform to the FBVP yields an integral representation of the solution including the free boundary explicitely. This integral representation allows to prove explicit truncation errors.

Since the variational inequality is formulated within the framework of weighted Sobolev spaces, we establish a weighted Poincaré inequality with explicit determined constants. The truncation error estimate and the weighted Poncaré inequality enable a reliable a posteriori error estimate between the exact solution of the variational inequality and the semi-discrete solution of the penalised problem on $\mathbb{R}$.

A sufficient regular solution provides the convergence of the solution of the penalised problem to the solution of the variational inequality. An a priori error estimate for the error between the exact solution of the variational inequality and the semi-discrete solution of the penalised problem concludes the numerical analysis.

The established a posteriori error estimates motivates an algorithm for adaptive mesh refinement. Numerical experiments show the improved convergence of the adaptive algorithm compared to uniform mesh refinement. The choice of different truncation points reveal the influence of the truncation error estimate on the total error estimator.

This thesis provides a semi-discrete reliable a posteriori error estimates for a variational inequality on an unbounded domain including explicit truncation errors. This allows to determine a truncation point such that the total error (discretisation and truncation error) is below a given error tolerance.

Keywords:
American options, variational inequality, Finite Element discretisation, error analysis

Zusammenfassung

In der Finanzmathematik hat der Besitzer einer amerikanische Option das Recht aber nicht die Pflicht, eine Aktie innerhalb eines bestimmten Zeitraums, für einen bestimmten Preis zu kaufen oder zu verkaufen. Die Bewertung einer amerikanische Option wird als so genanntes optimale stopping Problem formuliert. Erfolgt die Modellierung des Aktienkurses durch eine geometrische Brownsche Bewegung, wird der Wert einer amerikanischen Option durch ein deterministisches freies Randwertproblem (FRWP), oder eine äquivalente Variationsungleichung auf ganz $\mathbb{R}$ in gewichteten Sobolev Räumen gegeben.

Um Standardmethoden der Numerischen Mathematik anzuwenden, wird das unbeschränkte Gebiet zu einem beschränkten Gebiet abgeschnitten. Mit Hilfe der Fourier-Transformation wird eine Integraldarstellung der Lösung die den freien Rand explizit beinhaltet hergeleitet. Durch diese Integraldarstellung werden Abschneidefehlerschranken bewiesen.

Da die Variationsungleichungin gewichteten Sobolev Räume formuliert wird, werden gewichtete Poincare' expliziten Konstanten bewiesen. Der Abschneidefehler und die gewichtete Poincare' Ungleichung ermöglichen einen zuverlässigen a posteriori Fehlerschätzer zwischen der exakten Lösung der Variationsungleichung und der semidiskreten Lösung des penalisierten Problems auf $\mathbb{R}$ herzuleiten.

Eine hinreichend glatte Lösung der Variationsungleichung garantiert die Konvergenz der Lösung des penaltisierten Problems zur Lösung der Variationsungleichung. Ein a priori Fehlerschätzer für den Fehler zwischen der exakten Lösung der Variationsungleichung und der semidiskreten Lösung des penaltisierten Problems beendet die numerische Analysis.

Die eingeführten a posteriori Fehlerschätzer motivieren einen Algorithmus für adaptive Netzverfeinerung. Numerische Experimente zeigen die verbesserte Konvergenz des adaptiven Verfahrens gegenüber der uniformen Verfeinerung. Die Wahl von unterschiedlichen Abschneidepunkten illustrieren den Anteil des Abschneidefehlerschätzers an dem Gesamtfehlerschätzers.

Diese Arbeit präsentiert einen zuverlässigen semidiskreten a posteriori Fehlerschätzer für eine Variationsungleichung auf einem unbeschränkten Gebiet, der den Abschneidefehler berücksichtigt. Dieser Fehlerschätzer ermöglicht es, den Abschneidepunkt so zu wählen, daß der Gesamtfehler (Diskretisierungsfehler plus Abschneidefehler) kleiner als einer gegebenen Toleranz ist.

Schlagwörter:
Amerikanische Optionen, Variationsungleichung, Finite Elemente Diskretisierung, Fehleranalysis

To my parents and Christof

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Chapter 1

Introduction

In their celebrated paper Black and Scholes [1973] showed, that the pricing of options, which is a *stochastic* problem, can be formulated as a *deterministic* partial differential equation (PDE). Since then, the pricing of options by means of partial differential equations has become a standard device in quantitative finance.

An option gives the right *but not the obligation* to buy or sell a certain financial asset by a certain date T, for a certain strike price K. There are two main type of options, namely *call* options, which are options to *buy*, and *put* options which give the right to *sell*. Moreover, one distinguishes between *European* and *American* options. While an European option can only be exercised at the expiration date T, an American option may be exercised during the whole life time of the option. The theory of option pricing deals with the question of finding a fair price of an option.

On the exercise day, the value of an option can be easily calculated, since it only depends on the strike price K and the value S of the share on that day. Assume, for example, that the strike price of a call option is $K = 10$ and the share value equals $S = 15$. The holder of the call option exercises the option and has a gain of $S - K = 5$, because he buys the option for K and may sell it immediately for S. However, if $S = 8$, the (rational) holder would never exercise the option and so he gains 0. In other words, the pay-off function of a call option reads $h(S) = \max(0, S - K) =: (S - K)_+$, illustrated in Figure 1.1. In the case of put options it is the other way around: if the value of the share S is below the strike price K, the holder will exercise and gain $K - S$, otherwise the holder would not exercise and has a gain of 0, i.e. the pay-off function reads $h(S) = (K - S)_+$. Consequently, the value V of an option is known on the exercise day T and given by $V(S, T) = h(S)$ for American put options, illustrated in Figure 1.3 .

The question arises how to find a fair price for an option before the exercise day. Since the value of an option on the exercise day depends on the value of the share on the exercise day, which is a priori unknown, the stock price needs to be modelled by a stochastic process. Using geometric Brownian motion to modell the stock price, Black and Scholes [1973] prove that the value of an European option can be expressed by a deterministic parabolic PDE, the famous and Nobel-price awarded Black-Scholes equation.

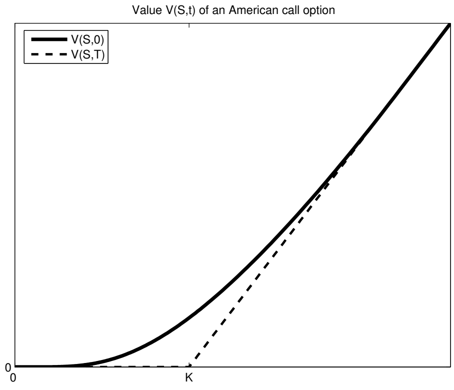

Figure 1.1: The value $V(S,t)$ of an American call option for $t = 0$ and $t = T$.

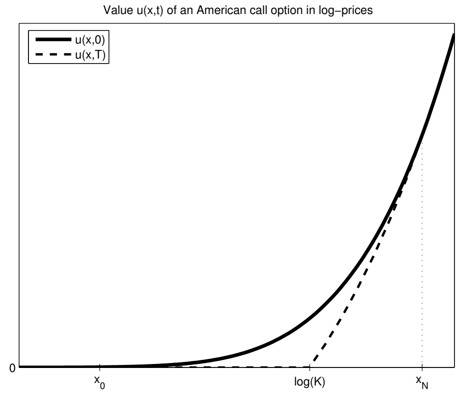

Figure 1.2: The value $V(S,t)$ of an American call option in log-prices, i.e. $S = e^x$ for $t = 0$ and $t = T$.

The valuation of American options is more involved, since the holder may exercise the option before the expire date T, which is called *early exercise* in the financial terminology. Due to this early exercise possibility and the so-called no-arbitrage assumption, the value of an American option can never be less than the pay-off function, i.e $V(S,t) \geq h(S)$.

The valuation of American options is formulated as an *optimal stopping problem* because of the possibility of early exercise. As in the case of European option, there exist deterministic formulations to describe the value of the option. The condition that the value must not be below the pay-off function already indicates that the deterministic formulation is an obstacle problem. Indeed, if the stock prices is modelled by geometric Brownian motion the value of an American option is described by a parabolic free boundary value problem (FBVP), cf. McKean [1965], Van Moerbeke [1976], Dewynne et al. [1993], Wilmott et al. [1995], Karatzas and Shreve [1998]. A formulation analogue to the Black-Scholes equation for European option is the linear complimentary formulation (LCF), cf. Wilmott et al. [1995], Lamberton and Lapeyre [1996]. The LCF can be considered as the strong form of a variational inequality, cf. Achdou and Pironneau [2005], Wilmott et al. [1995]. Figure 1.1 and Figure 1.3 show the value of an American call and put option for $t = 0$ and $t = T$, respectively. Recall that for $t = T$ the value is equal to the pay-off function h.

The spatial differential operator appearing in the Black-Scholes equation and in the three formulations for the evaluation of American options is a degenerated operator of Euler type. It is a standard procedure, cf. Wilmott et al. [1995], Seydel [2004] applying the transformation $S = e^x$ which yields to a non-degenerated operator. In financial terminology one speaks of transformation to log-prices. Figure 1.2 and Figure 1.4 show the value of American call and put options $u(x,t)$ for log-prices, i.e., $x = \log(S)$ at times $t = 0$ and $t = T$. The transformed pay-off functions in log-prices read $\psi(x) = (K - e^x)_+$ and $\psi(x) = (e^x - K)_+$ for a put and call option, respectively. Since the pay-off functions do not belong to $L^2(\mathbb{R})$, weighted Sobolev spaces are required to formulate a variational inequality for American options. In Jaillet et al. [1990] it is directly proved that the solution of the optimal stopping problem satisfies a non-symmetric variational inequality on weighted Sobolev spaces in an unbounded domain.

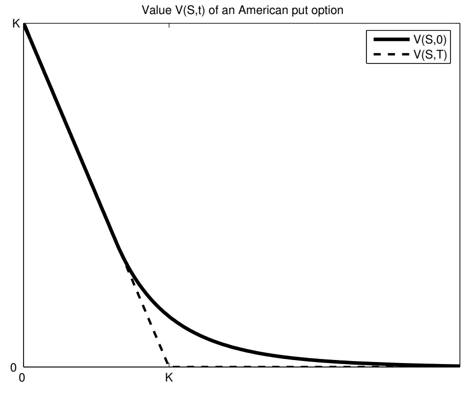

Figure 1.3: The value $V(S,t)$ of an American put option for $t=0$ and $t=T$.

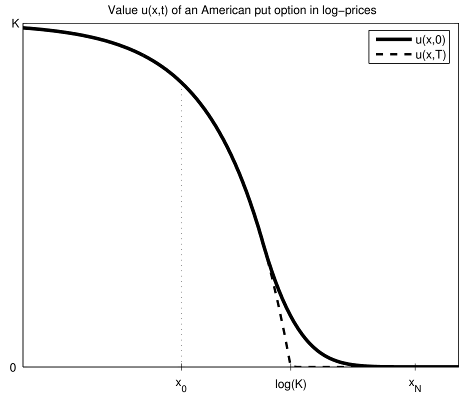

Figure 1.4: The value $V(S,t)$ of an American put option in log-prices, i.e. $S=e^x$ (right) for $t=0$ and $t=T$.

Since an analytical solution is only available in the case of perpetual options, cf. Kwok [1998], which means that the expiry date is at infinity, numerical methods are necessary to evaluate American options for finite T. One difficulty in the numerical solution of an option evaluation problem is the unbounded domain. In the majority of the publications and books on numerical method for option pricing, cf. Wilmott et al. [1995], Seydel [2004], Achdou and Pironneau [2005] the unbounded domain is truncated to a bounded one to apply standard tools of numerical mathematics. Suitable boundary conditions are set by heuristic financial arguments. Mathematical justification of this procedure by identifying truncation errors for European options is given in Kangro and Nicolaides [2000] and Matache et al. [2004]. In Kangro and Nicolaides [2000] the truncation error for European basket options, i.e., a multi-dimensional Black-Scholes equation, are proved. To be precise, the authors bound the truncation error in the supremum norm in the computational domain by the maximum error on the artifical boundary. In Matache et al. [2004] the authors prove truncation error estimates for European options on Lévy driven assets in the computational domain in the L^2-norm. For American options in the Black-Scholes setting, the convergence of the solution of the truncated problem to the solution to the untruncated problem is proven in Jaillet et al. [1990]. Allegretto et al. [2001], Han and Wu [2003], Ehrhardt and Mickens [2006], Achdou and Pironneau [2005] use transparent (also called artificial) boundary conditions for the truncated domain, which are mathematically exact. However, these boundary conditions are non-local and therefore their computation is costly.

Since the variational inequality is non-symmetric, energy techniques cannot be applied. Consequently, we cannot formulate an equivalent minimisation problem and apply standard tools for solving variational inequalities via minimisation techniques such as monotone multigrid, cf. Kornhuber [1997]. If the variational inequality with constant coefficients is formulated for a *bounded* domain with suitable boundary condition, it can be transformed to a variational inequality with a symmetric bilinear form, cf. Wilmott et al. [1995]. In Holtz [2004] monotone multigrid for American options is applied to such a transformed formulation. Such transformations are *not* possible with unbounded domains, because of non-smooth weighting functions. Since we are interested in analysing truncation errors and deriving error estimates on $\mathbb{R}$, there is no possibility to transform our problem to a symmetric variational inequality.

In the case of finite time horizon, i.e., $T < \infty$, reference solutions do not exists and the error between the exact solution and the numerical solution is unknown. Consequently, reliable a posteriori error estimates are important in order to determine the quality of the numerical solution.

At the heart of this thesis is the derivation of a priori and a posteriori error estimates on the whole domain $\mathbb{R}$ for American options. Since weighted Sobolev spaces are necessary to formulate the variational inequality, the error estimates are proved in corresponding weighted Sobolev norms. In the course of the analysis it becomes apparent that the interpolation error estimates in these weighted norms are a main difficulty in the derivation of the error bounds. The proof of a suitable weighted Poincare inequality is at the centre of the numerical analysis provided in this thesis. To apply efficient numerical methods, the unbounded domain is truncated to a bounded one. Consequently, truncation errors are part of the error bounds. Using the Fourier transform yields an integral representation of the solution of the FBVP incorporating the free boundary explicitely. This integral representation allows to prove explicit truncation error bounds, which complete the proof of the error bounds on the whole domain.

This thesis is organised as follows. Chapter 2 gives an introduction to the theory of option pricing. The presentation of the financial concept of options is followed by a stochastic formulation for the fair price of an option. For European options the fair price is given as an conditional expectation whereas for American options the fair price is defined by an optimal stopping problem. If the underlying stock is modelled by geometric Brownian motion, the value of an European options satisfies a deterministic PDE. For American options, however, the price is described by a deterministic FBVP or equivalently by an deterministic LCF. Finally, we give some properties of American options, which play an important role in the numerical analysis of this thesis.

Chapter 3 models European and American options in the framework of weighted Sobolev spaces, cf. Section 3.1 and 3.2. Starting from the strong formulation, we derive a variational formulation by means of weighted Sobolev spaces. The weights p are chosen such that the pay-off functions ψ satisfy $\psi\sqrt{p} \in L^2(\mathbb{R})$. Since the pay-off functions include the exponential function, it is straight forward to choose $p(x) = \exp(-2\eta|x|)$ for some $\eta > 0$. This weighted Sobolev spaces build a Gelfand triple and the bilinear form of the variational inequality is continuous and satisfies a Gårding inequality. These properties are essential to analyse the existence of a unique solution for time-dependent variational equalities and inequalities.

Section 3.3 is devoted to the derivation of an integral representation of the solution of American put and call options. The FBVP is transformed to a parabolic PDE on $\mathbb{R}$, with coefficients including the free boundary, which allows to apply the Fourier transform. Applying the Fourier transform to this PDE yields a first-order initial value problem, which can be solved by the variation of constants. Then, applying the inverse Fourier transform yields an integral representation for the solution of American options including the free boundary explicitely. Moreover, we provide a regularity result for the solution of parabolic PDEs on $\mathbb{R}$ by means of the Fourier transform. The first main result is established in Section 3.4. We provide an explicit pointwise truncation error estimate depending on given financial data. We show the exponential decay of the solution beyond a fully determined treshold using the integral representation derived by applying the Fourier transform to the FBVP. More precisely, there exists a treshold x_N^p for put options and x_0^c for call options (fully determined by given financial data) and some $\kappa > 0$, such that the solution u satisfies

$$|u(x,t)\exp(x^2/\kappa)| \leq C < \infty$$

for $x > x_N^p$ for put options and $x < x_0^c$ for call options, i.e., the solution decreases exponentially to zero. A refined analysis yields the constant C explicitly. This truncation error estimates are essential to prove reliable a posteriori error estimates on $\mathbb{R}$.

In Matache et al. [2004] the authors prove truncation errors for European options by transforming the Black-Scholes PDE which requires weighted Sobolev spaces, to a PDE which solution decreases exponentially. In Chapter 4 we analyse such transformations and show under which conditions on the coefficients such transformation are possible for general parabolic PDEs on unbounded domains. For American options, however, we prove, that such transformations are not possible, i.e., weighted Sobolev spaces are required in the (numerical) analysis. For constant coefficients, however, we proved the exponential decay of the exact solution. This and using that the solution equals the pay-off function ψ beyond a known treshold yields that non-weighted Sobolev spaces are admissible in the case for *constant* coefficients. We still consider the formulation with weighted Sobolev spaces, which allows to consider time- and space-dependent coefficients as in formulations with local volatility. Numerical experiments show that the a posteriori error estimate and the adaptive mesh-refinement are not sensitive on changes of η.

Chapter 5 concentrates on the numerical analysis for American options. Since the variational inequality is non-symmetric, we cannot formulate an equivalent minimisation problem and apply standard tools to solve variational inequalities via minimisation techniques. Instead of solving the variational inequality directly, penalisation techniques applied to the variational inequality yield a non-linear PDE. This is described in Section 5.1.

Since the spatial domain is unbounded, we split it into an inner domain (x_0, x_N) and two outer domain $T_0 = (-\infty, x_0)$ and $T_{N+1} = (x_N, \infty)$, cf. Figure 1.4 and 1.2. The truncation points x_0 and x_N, the ansatz functions and the discrete solution $u_{\varepsilon h}$ on the outer domains T_0 and T_{N+1} are determined in Section 5.2 by means of the truncation error estimates of Section 3.4, the FBVP, and properties of the free boundary. The inner domain (x_0, x_N) is approximated by P_1 finite elements. The time integration is done by the method of lines, i.e., we solve a system of ODEs. For the error analysis we assume that the time-integration is done sufficiently exact by proper chosen ODE solvers, i.e., we analyse the semi-discrete problem.

The variational inequality is formulated within the framework of weighted Sobolev spaces, hence we require interpolation error estimates in this weighted norms for the error analysis. The second main result is the proof of a weighted Poincare inequality for the non-smooth weights $p(x) = \exp(-2\eta|x|)$, $\eta \geq 0$ with explicit determined constants. More precisely, we extend a weighted Poincare inequality (for C^2-functions f with weighted integral mean zero) to our non-smooth weight. Then, by means of reflexion principles this estimates for functions f with weighted integral mean zero is extended to functions f with zero boundary values. Eventually, for $f \in H^1_\eta(a,b)$ and the nodal interpolation operator $\mathcal{I}$ there holds the next error estimate

$$\|f - \mathcal{I}f\|_{L^2_\eta(a,b)} \leq hC(\eta, h)\, \|f'\|_{L^2_\eta(a,b)}$$

with $h = b - a$ and the constant $C(\eta, h) \to \frac{2}{\pi}$ for $\eta h \to 0$ and the explicit representation

$$C(\eta, h) := \frac{2}{\pi}\sqrt{\frac{\cosh(2\eta h) - 1}{2\eta^2 h^2}}.$$

Then, we extend this estimate to functions $f \in H^1_\eta(a, b)$ which vanish at least at one point in (a, b). For such functions there holds

$$\|f\|_{L^2_\eta(a,b)} \leq hC(\eta, h)\, \|f'\|_{L^2_\eta(a,b)} \tag{1.1}$$

with constant

$$C(\eta, h) := \sqrt{\left(\frac{1}{\pi} + \Big(\frac{\cosh(2\eta h) - 1}{2\eta^2 h^2)}\Big)^{1/2}\right)^2 + \frac{4}{\pi^2}}. \tag{1.2}$$

The third main result is a residual-based a posteriori error estimate for the error $e = u - u_{\varepsilon h}$ on $\mathbb{R}$, i.e., the error between the exact solution u of the variational inequality and the semi-discrete solution $u_{\varepsilon h}$ of the penalised problem provided in Section 5.4. The special choice of the ansatz functions and the semi-discrete solution $u_{\varepsilon h}$ on the outer domains T_0 and T_{N+1} yield an estimate in which the right-hand side only depends on the semi-discrete solution $u_{\varepsilon h}$ and given data on inner domain (x_0, x_N) and terms including the truncation error for u at the truncation point. Since we determine an explicit truncation error in Section 3.4 we proved a fully computable reliable error bound consisting of a standard residual, penalisation error terms and the truncation error. By means of this a posteriori error estimator the error can be divided into the error involving through the truncation of the interval and discretisation error. The truncation point can be determined such that the order of magnitute of the truncation error is neglectable compared to the order of magnitute of the discretiasation error. Hence, artifical boundary conditions (which are non-local and costly in their computation) are not necessary to guarantee that the error is below a given tolerance.

In Section 5.5 we prove that for sufficient regular solution the penalisation error $e = u - u_\varepsilon$, i.e., the error between the exact solution of the variational inequality and the exact solution u_ε of the penalised problem, is of order $\sqrt{\varepsilon}$ as $\varepsilon \to 0$. An a priori error estimator for sufficiently regular solutions for the error $e = u - u_{\varepsilon h}$ on $\mathbb{R}$ constitutes the fourth main result. This estimate consits of three error types, namely the discretisation error, the penalisation error and the truncation error. An additional difficulty in the proof of the a priori and a posteriori error estimates is that the bilinear form is not elliptic, it only satisfies a Gårding inequality. We overcome this problem by using the L^∞-norm in time instead of the L^2-norm in time on the left hand side of the estimators. This leads to a priori and a posteriori error estimates for sufficiently small time T, depending on the parameters in the Gårding inequality.

Chapter 6 is devoted to the implementation of the finite element solution of the semi-discrete solution of the penalised problem. We derive a system of ODEs which yields the semi-discrete solution. Then we formulate an adaptive algorithm for the mesh refinement based on the a posteriori error estimate proved in Chapter 5. Although the convergence of the adaptive mesh refinement cannot be proved, numerical experiments show the convergence of the adaptive algorithm. We compare the rates of convergence of uniform and adaptive mesh refinement. Since the problem does not contain any singularity, the convergence rate of adaptive and uniform mesh refinement is asymptotically equal. However, during the first refinement levels the adaptive algorithm refines more efficiently, so that it is superior to uniform mesh refinement. The a posteriori estimate contains truncation error terms. Hence we systematically carry out numerical experiments with different truncation points and compare the influence of the truncation error estimator on the total error estimator. Finally we investigate on the effect of different choices of the parameter η in the weight function on the error estimator and the refined meshes.

The numerical experiments show, that the pointwise truncation error derived in Section 3.4 is sufficiently sharp to be useful in practice. The truncation point can be chosen such that the total error is below a given error tolerance. From the numerical experiments it becomes evident that a simple penalisation technique combined with a standard ODE solver for the semi-discrete system leads to satisfactory results. For more evolved real time scientific computing a more sophisticated time discretisation has to be developed.

Frequently used notation and abbreviations are explained in Appendix A. Appendix B concisely describes the data structures and programme code of the FE implementation. Appendix C lists the maple code used for calculating the pointwise truncation error.

In this thesis we considered a one-dimensional Black-Scholes model for the evaluation of American put and call options. Several extentions for the model are possible and give an outlook on open questions beyond the scope of this thesis.

Introducing local volatility make the volatility space- and time dependent. For sufficient regular and bounded volatility functions, the numerical analysis from this thesis is transferable, with exception the derivation of truncation errors. This relies on the Fourier analysis, which is not applicable for PDEs with space dependent coefficients.

Modelling the asset with Lévy processes yields an partial integro differential equation. Hence, instead applying the Fourier transform, the theory of pseudo-differential operators needs to applied and it is totally unclear, if truncation errors can be derived in such way. Moreover, the problem is non-local, which means that the numerical analysis is much more intricate and the numerical solution requires special tools such as wavelets, cf. Matache et al. [2004] or H-matrices to reduce the complexity resulting from the non-locality.

Basket options, which are options on more than one asset, are valued as a multi - dimensional Black-Scholes problem. It remains unclear, if in this generalised context the truncation error estimates can be still proved by applying the Fourier transform. However, a FE implementation with adaptive mesh refinement returns very encouraging for up to three space dimensions. For higher dimension, which are standard in the evaluation of basket options, other methods such as sparse grids for dimension reductions need to be applied, cf. Reisinger and Wittum [2004].

Chapter 2

Option Pricing

This chapter briefly introduces the theory of option pricing. For a more detailed treatment we refer to Lamberton and Lapeyre [1996], Karatzas and Shreve [1998] with emphasis on stochastic analysis and to Kwok [1998], Wilmott et al. [1995] more in tune with applied mathematics and PDEs. The first section treats European options. After explaining the concept of options, a stochastic model for option pricing is introduced, mainly following Lamberton and Lapeyre [1996]. In this often called Black-Scholes model the assets follow a geometric Brownian motion. Finally, we establish the famous results of Black and Scholes [1973] and Merton [1973], for which Scholes and Merton received the Nobel price in economics in 1997. The second section deals with American options. We start with explaining the concept of American options. Then we introduce the stochastic formulation as an optimal stopping problem. Finally we present deterministic models for pricing American options.

2.1 Pricing European Options

An option is a contract that gives the holder the right but not the obligation to buy or sell a certain financial asset by a certain date T, for a certain strike price K. There are two main type of options, namely a *call* option, which is a option to *buy*, and a *put* option, which gives the right to *sell*. In the contract the following features need to be specified:

(i) the type of the option, i.e. call or put,

(ii) the underlying asset, usually shares, bonds or currencies,

(iii) the amount of the underlying asset,

(iv) the expiration date or maturity T, and

(v) the exercise price, or strike price K.

The question arises how to obtain a fair price of an option. On the exercise day the holder must decide if he exercises the option. First we consider a call option, i.e. an option to buy,

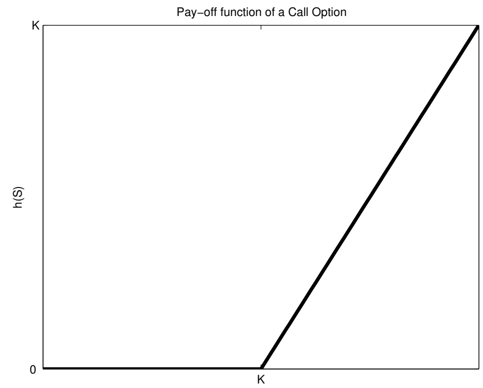

Figure 2.1: The pay-off function $h(S) = (K - S)_+$ for call options.

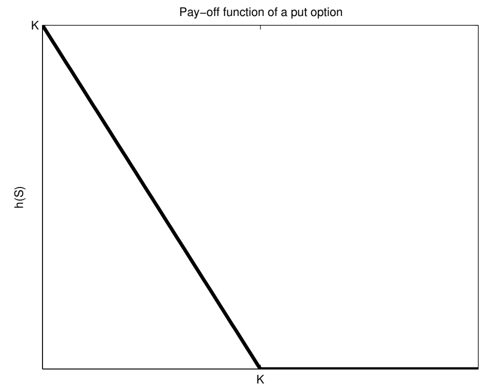

Figure 2.2: The pay-off function $h(S) = (K - S)_+$ for put options.

with strike price $K = 10$. Assume that the share value equals $S = 15$ on the exercise day. The holder will exercise the option and has a gain of $S - K = 5$, because he buys the share for K and may sell it immediately for S. If the value of the share is 8, the (rational) holder would not exercise and he does not gain anything. In other words, if the strike price K is greater than the value S of the share, the holder will exercise and gain $S - K$, if the strike price K is greater than the share value S, he would not exercise. Hence, the pay-off function reads $h = (S - K)_+ := \max(S - K, 0)$, illustrated in Figure 2.1. In the case of put options it is the other way around: if the value of the share S is below the strike price K, the holder will exercise and gain $K - S$, else he would not exercise and consequently gain nothing, i.e., the pay-off function reads $h(S) = (K - S)_+$, illustrated in Figure 2.2.

We aim to find a fair price for an option before the exercise day.

Since the value of the option depends on the share value at the exercise day, which is a priori unknown, the stock price needs to be modelled by a stochastic processes. If the stock price follows a geometric Brownian motion, Black and Scholes showed, that the value of an European option is described by a deterministic backward parabolic PDE.

2.1.1 The Black-Scholes Model

The classical option pricing theory of Black and Scholes (1973) is based on a continuous time model with one risky asset (with price S_t at time t) and a risk-less asset with price S_t^0 at time t satisfying the ordinary differential equation

$$dS_t^0 = rS_t^0 dt, \quad t \geq 0,$$

where $r > 0$ is the risk-less interest rate.

Remark 2.1. In the financial literature it is common to denote the time dependence of a stochastic process in the subscript, i.e., one writes S_t instead of $S(t)$, while in PDE literature the subscript t denotes the time derivative.

To model the stock price we fix a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the so-called usual conditions , i.e. it is right continuous and complete, cf. Karatzas and Shreve [1991]. A filtration $(\mathcal{F}_t)_{t\geq 0}$ is an increasing family of σ-algebras included in $\mathcal{F}$. The σ-algebra $\mathcal{F}_t$ represents the information available at time t. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion on that propability space. With the drift μ and the volatility σ the stock price is determined by the following stochastic differential equation (SDE)

$$dS_t = S_t(\mu dt + \sigma dB_t), \quad t \geq 0. \tag{2.1}$$

Applying Itô's formula yields the explicit representation of the unique solution of the SDE (2.1)

$$S_t = S_0 \exp\big(\mu t - \sigma^2/2\, t + \sigma B_t\big), \quad t \geq 0.$$

A formal definition of a Brownian motion is given in the subsequent definition, cf. Lamberton and Lapeyre [1996].

Definition 2.2 (Brownian motion). A stochastic process $(B_t)_{t\geq 0}$ is called a Brownian motion (or a Wiener process) with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ if the conditions (i)-(v) hold:

(i) $B_0 = 0$ almost surely;

(ii) $(B_t)_{t\geq 0}$ has independent increments, i.e. if $t_1 < t_2 \leq t_3 < t_4$ then $B_{t_4} - X_{t_3}$ and $B_{t_2} - B_{t_1}$ are independent stochastic variables.;

(iii) $B_t - B_s$ has the Gaussian distribution $N(0, t-s)$ for $s < t$;

(iv) B_t is $\mathcal{F}_t$ measurable;

(v) B_t has continuous trajectories almost surely.

The classical option pricing theory relies on the fact that the pay-off of every option can be duplicated by a portfolio consisting of an investment in the underlying stock and in the riskless asset. In this so-called *complete markets* there exists a unique probability measure $\mathcal{Q}$ equivalent to the measure $\mathcal{P}$ (the 'real world' measure) under which the discounted stock price $\tilde{S}_t := \exp(-rt)S_t$ is a martingale, cf. Lamberton and Lapeyre [1996]. A process $(X_t)_{t\geq 0}$ is called a martingale under $\mathcal{Q}$ if $E_{\mathcal{Q}}(X_t|\mathcal{F}_s) = X_s$ for all $s \leq t$. An European option which is defined by a non-negative, $\mathcal{F}_T$-measurable random variable $h(S_T)$, the pay-off function, is replicable and the value at time t is given by

$$V(S_t, t) = E_{\mathcal{Q}}\big(e^{-r(T-t)}h(S_T)|\mathcal{F}_t\big). \tag{2.2}$$

The model of Black and Scholes is based on the subsequent set of assumptions:

- trading takes place continuously in time;

- the riskless interest rate r is known and constant over time;

- the asset pays no dividends;

- there are no transaction costs in buying or selling the asset or the option and no taxes;
- the assets are perfectly divisable, i.e., it is possible to trade fractions of assets;
- there are no penalties to short selling and the full use of proceeds is permitted;
- there are no riskless arbitrage possibilities;
- the asset follows the geometric Brownian motion (2.1).

Then, the price $V = V(S,t)$ of an European option satisfies the backward parabolic PDE

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \quad \text{for } (S,t) \in (0,\infty) \times [0,T). \tag{2.3}$$

With the pay-off function $h(S)$ the terminal condition reads

$$V(S,T) = h(S). \tag{2.4}$$

The derivation can be found in Black and Scholes [1973] and many finance books, e.g. Karatzas and Shreve [1998], Lamberton and Lapeyre [1996], Wilmott et al. [1995], Kwok [1998].

2.1.2 Options on Dividend Paying Assets

Dividends are payments from the company, that issued the shares, to the share holders. Typically dividends are paid once or twice a year. Since dividend payments effect the stock price (note that on the day of the dividend payment the stock price decreases by the amount of the dividend), dividends need to be included in the evaluation for options. In this thesis we only consider deterministic dividends, i.e. that the amount is a priori known. This is a reasonable assumption since many companies try to maintain a similar payment from year to year. Although dividends are paid at discrete times we modell them by a continuous dividend yield, cf. Wilmott et al. [1995], Kwok [1998]. The dividend yield d is defined as the ratio of the dividends to the asset price. Merton [1973] extended the Black-Scholes equation to options on dividend paying shares, which reads

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r-d)S\frac{\partial V}{\partial S} - rV = 0 &\quad \text{for } (S,t) \in (0,\infty) \times [0,T), \\ V(S,T) = h(S) &\quad \text{for } S \in (0,\infty). \end{aligned} \tag{2.5}$$

2.2 Pricing American Options

In contrast to European options which can only be exercised at the expiration date T, American options can be exercised at any time until expiration. Due to this early exercise possibility the value of an American option is at least the same of the price of an European option with the same contract attributes. Moreover, due to the no-arbitrage assumption the value of an option must always be greater than the pay-off function, i.e. $V(S,t) \geq h(S)$. This is easily understood by the following argument. Assume that the value of an American put option is less than its pay-off function, i.e., $0 \leq V(S,t) < (K-S)_+$. Since we deal with an American option, it can be exercised immediately. If we buy now the option for V, exercise the option by selling the share for K and repurchasing the share at the market for S. Thus, we make a riskless profit of $-V+K-S > 0$, which contradicts the no-arbitrage assumption. Hence, the value of an American option satisfies $V(S,t) \geq h(S)$.

Since the evaluation of European options leads to a deterministic PDE, the condition $V(S,t) \geq h(S)$ indicates, that the valuation of American options may be written as an obstacle problem. Indeed, we will show later in this section, that American options can be formulated as free boundary value problems (FBVP) or linear complimentary formulations (LCP). Since the option holder has to decide if he exercises the option early, the evaluation is formulated as an optimal stopping problem in stochastics. From an economic point of view the holder has to decide if his gain by exercising the option immediately exceeds the current value of the option.

2.2.1 An Optimal Stopping Problem

In this thesis we concentrate on American option where the asset is modelled by a geometric Brownian motion, cf. Subsection 2.1.1 for European options. Again, the evaluation relies on finding an equivalent probability measure $\mathcal{Q}$ under which the discounted price process S_t is a martingale. With the definitions and notations from Subsection 2.1.1 the value $V(S_t,t)$ of an American option at time t is given by the following optimal stopping problem, cf. Lamberton and Lapeyre [1996] with stopping time τ

$$V(S_t,t) = \sup_{t\leq\tau\leq T} E_{\mathcal{Q}}\big(e^{-r(\tau-t)}h(S_\tau)|\mathcal{F}_t\big).$$

As in the case of European options in the Black-Scholes setting, there exists a deterministic formulation to evaluate American options. The next subsections deal with deterministic formulations for the evaluations of American options.

2.2.2 A Free Boundary Value Problem

Let S denote the value of the underlying share, and $V(S,t)$ the value of an American put option at time $t \in [0,T)$ and share value S. Further, let $\gamma(t) \geq 0$ be the early exercise curve (or free boundary). Given the positive constants T (exercise date), K (strike price), r (riskless interest rate), d (dividend yields), and σ (volatility), the value $V(S,t)$ of an American put option satisfies (cf. Karatzas and Shreve [1998], Wilmott et al. [1995])

$$\begin{aligned} V_t + \sigma^2/2\, S^2 V_{SS} + (r-d)SV_S - rV = 0 &\quad \text{for } \gamma(t) < S < \infty,\ t \in [0,T), \\ V(S,T) = (K-S)_+ &\quad \text{for } \gamma(T) \leq S < \infty, \\ V(\gamma(t),t) = K-\gamma(t) \ \text{ and } \lim_{S\to\infty} V(S,t) = 0 &\quad \text{for } t \in [0,T), \\ V_S(\gamma(t),t) = -1 &\quad \text{for } t \in [0,T). \end{aligned} \tag{2.6}$$

The condition $V_S(\gamma(t), t) = -1$ is called high contact condition. It arises from the no-arbitrage assumption, cf. Kwok [1998], Wilmott et al. [1995]. Note that if the holder exercised the option, the value equals its pay-off function, i.e., $V(S, t) = K - S$ if $S \leq \gamma(t)$.

The value $V(S, t)$ of an American call is given by (cf. Karatzas and Shreve [1998], Wilmott et al. [1995])

$$\begin{aligned} V_t + \sigma^2/2\, S^2 V_{SS} + (r-d) S V_S - rV = 0 \quad & \text{for } 0 \leq S \leq \gamma(t),\ t \in [0, T), \\ V(S, T) = (S - K)_+ \quad & \text{for } 0 \leq S \leq \gamma(T), \\ V(\gamma(t), t) = \gamma(t) - K \text{ and } V(0, t) = 0 \quad & \text{for } t \in [0, T), \\ V_S(\gamma(t), t) = 1 \quad & \text{for } t \in [0, T). \end{aligned} \tag{2.7}$$

The no-arbitrage assumption (cf. Kwok [1998], Wilmott et al. [1995]) yields the so-called high contact condition $V_S(\gamma(t), t) = 1$. Note that if the holder exercised the option, the value equals its pay-off function, i.e., $V(S, t) = S - K$ if $S \geq \gamma(t)$.

2.2.3 A Linear Complimentary Formulation

The FBVP can be written as a LCF by using that an American option satisfies the Black-Scholes inequality (cf. Wilmott et al. [1995]), namely,

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r-d) S \frac{\partial V}{\partial S} - rV \leq 0 \quad \text{for } (S, t) \in (0, \infty) \times [0, T). \tag{2.8}$$

Together with the fact, that either the Black-Scholes equation holds or the option value equals the pay-off function h, the value $V(S, t)$ of an American Option is given by Wilmott et al. [1995], Lamberton and Lapeyre [1996]

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r-d) S \frac{\partial V}{\partial S} - rV \leq 0 \quad & \text{in } [0, T) \times \mathbb{R}^+, \\ \big(V - h\big)\Big(\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r-d) S \frac{\partial V}{\partial S} - rV\Big) = 0 \quad & \text{in } [0, T) \times \mathbb{R}^+, \\ V(\cdot, t) \geq h(\cdot) \quad & \text{in } [0, T) \times \mathbb{R}^+, \\ V(\cdot, T) = h(\cdot) \quad & \text{in } \mathbb{R}^+. \end{aligned} \tag{2.9}$$

Multiplying (2.9) by appropriate test functions and using the complimentary conditions yields a variational inequality, cf. Wilmott et al. [1995], Friedman [1982], Baiocchi and Capelo [1984]. Since the initial condition h for call and put options does not belong to $L^2(\mathbb{R})$, weighted Sobolev spaces are required, cf. Section 3.1. Therefore we postpone the formulation as a variational inequality to Section 3.2 where we will introduce the necessary functional analytic background.

2.2.4 Some Properties of American options

In this subsection we state some properties of American options from Kwok [1998]. Let $\tau = T - t$ denote the time until expiry and $\gamma(\tau)$ the early exercise curve. Recall that r denote the risk-less interest rate and d the constant dividend yield.

American call options

The early exercise curve $\gamma(\tau)$ of an American call option is a continuous increasing function of τ for $\tau > 0$. In Kwok [1998], Karatzas and Shreve [1998] it is proved that

$$\lim_{\tau \to 0^+} \gamma(\tau) = \max\Big(\frac{r}{d}K, K\Big). \tag{2.10}$$

In particular, when $d = 0$, i.e., no dividends are paid, it follows that $\gamma(\tau) \to \infty$ as $\tau \to 0^+$. Since $\gamma(\tau)$ is monotonically increasing this implies that $\gamma(\tau) \to \infty$ for all τ. In order words, early exercise is never optimal if $d = 0$. Consequently, we will assume throughout this thesis that dividends are paid, i.e., the constant dividend yield is $d > 0$, when American call options are under consideration. By means of perpetual options (which are options with an infinite horizon), one proves that the early exercise curve of an American call option is bounded above. With

$$\mu_+ = \frac{-(r - d - \sigma^2/2) + \sqrt{(r - d - \sigma^2/2)^2 + 2r\sigma^2}}{\sigma^2} \geq 1$$

there holds for all $\tau > 0$

$$\gamma(\tau) \leq \frac{\mu_+}{\mu_+ - 1}K. \tag{2.11}$$

Note that $\mu_+ = 1$ if and only if $d = 0$. With (2.10) and (2.11) we see that the free boundary is bounded by explicit bounds. This property will be uses for the numerical analysis.

American put options

The early exercise curve $\gamma(\tau)$ of an American put option is a continuous monotone decreasing function of τ for $\tau > 0$. Moreover, there holds

$$\lim_{\tau \to 0^+} \gamma(\tau) = \min\Big(\frac{r}{d}K, K\Big). \tag{2.12}$$

In particular, when $r = 0$, $\gamma(\tau) \to 0$ as $\tau \to 0^+$. Since $\gamma(\tau)$ is monotone decreasing, we conclude that $\gamma(\tau) = 0$ for all $\tau > 0$ if the risk-less interest rate $r = 0$. In order words, it is never optimal to exercise an American put option early, if the risk-less interest rate $r = 0$. Hence, we always assume that $r > 0$ if we deal with American put options. As in the case of American call options it is possible to give a lower bound of the early exercise curve by means of perpetual put options. In Kwok [1998], Karatzas and Shreve [1998] it is proved that with

$$\mu_- = \frac{-(r - d - \sigma^2/2) - \sqrt{(r - d - \sigma^2/2)^2 + 2r\sigma^2}}{\sigma^2} \leq 0$$

there holds for all $\tau \geq 0$

$$\gamma(\tau) \geq \frac{\mu_-}{\mu_- - 1}K. \tag{2.13}$$

Note that $\mu_- = 0$ if and only if $r = 0$. The lower bound is important if log-prices are considered, i.e., we set $x = \log S$, which yields constant coefficient in the FBVP and LCF.

Remark 2.3. The free formulations of the evaluation of American options, i.e., the free boundary value problem, the linear complimentary formulation and the variational inequality are equivalent, cf. Wilmott et al. [1995], Friedman [1982].

Chapter 3

Mathematical Analysis for European and American Options

The proof of the existence of a unique weak solution of the Black-Scholes equation and inequality opens this chapter. First, certain transformations applied to the strong formulation, yield a forward parabolic PDE (European options) or VI (American options) with constant coefficients. The weak formulation with properly chosen weighted Sobolev spaces allows for a weak solution within this framework. The Fourier transform yields an integral representation of the solution including the free boundary. Starting from this integral representation we prove a truncation error for American options.

3.1 European Options

3.1.1 Black-Scholes Equation

The Black-Scholes equation (2.5) on page 11, is a backward PDE with space-dependent coefficients of Euler type. It is a well known fact that a transformation of the form $S = e^x$ removes the space-dependence of the coefficients, cf. Wilmott et al. [1995]. The financial interpretation is that the option value is considered in logarithmic prices, i.e. $w(x,t) := V(e^x, t)$.

Transformation 3.1 (leading to constant coefficients). Let V solve (2.5) and set $S = e^x$, $w(x,t) := V(e^x,t)$, and $h(e^x) =: \psi(x)$. Then the partial derivatives satisfy

$$\partial_t V = \partial_t w, \quad \partial_S V = S^{-1}\partial_x w, \quad \partial_{SS} V = S^{-2}(\partial_{xx} w - \partial_x w). \tag{3.1}$$

Inserting this in (2.5) yields

$$\begin{aligned} \frac{\partial w}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 w}{\partial x^2} + \Big(r - d - \frac{\sigma^2}{2}\Big)\frac{\partial w}{\partial S} - rw = 0 \quad &\text{for } (x,t) \in \mathbb{R} \times [0,T), \\ w(x,T) = \psi(x) \quad &\text{for } x \in \mathbb{R}. \end{aligned} \tag{3.2}$$

The next step is to transform (3.2), a PDE in backward-time, to a PDE in forward-time.

Transformation 3.2 (leading to forward time). In order to obtain a forward parabolic equation, set $\tau = T - t$ and $u(x, \tau) := w(x, T - t)$. Then, $\partial_t w = -\partial_\tau u$ and (3.2) reads

$$\begin{aligned} \frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} - \left(r - d - \frac{\sigma^2}{2}\right)\frac{\partial u}{\partial x} + ru = 0 \quad & \text{for } (x,t) \in \mathbb{R} \times (0, T], \\ u(x, 0) = \psi(x) \quad & \text{for } x \in \mathbb{R}. \end{aligned} \tag{3.3}$$

Remark 3.3. The transformations also apply for time-dependent coefficients $r(t)$ and $\sigma(t)$ in Transformation 3.1 and 3.2.

From now on, we will always use the formulation in forward time and log-prices. For brevity we write t instead of τ. Financially speaking, time t means now time till expiration. We denote the spatial derivative $\frac{\partial u}{\partial x}$ by u' and the time derivative $\frac{\partial u}{\partial t}$ by $\dot{u}$.

3.1.2 Existence and Uniqueness of the Solution

This subsection aims to provide existence and uniqueness for a weak solution for the evaluation of European options. Since the pay-off functions for call and put options do not belong to $L^2(\mathbb{R})$ we introduce weighted Sobolev spaces to formulate a variational problem.

Problem 3.4 (Strong formulation). To simplify notation we define the operator $\mathcal{A} : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$\mathcal{A}[u] := -\frac{\sigma^2}{2}u'' - \left(r - d - \frac{\sigma^2}{2}\right)u' + ru. \tag{3.4}$$

Then (3.3) reads

$$\begin{aligned} \dot{u} + \mathcal{A}[u] = 0 \quad & \text{for } (x,t) \in \mathbb{R} \times (0, T], \\ u(\cdot, 0) = \psi(\cdot) \quad & \text{for } x \in \mathbb{R}. \end{aligned} \tag{3.5}$$

Since typical pay-off functions (e.g., $\psi(x) = (e^x - K)_+$ for a call option, $\psi(x) = (K - e^x)_+$ for a put option) do not belong to $L^2(\mathbb{R})$, we introduce weighted Lebesgue and Sobolev spaces H^1_η and L^2_η, cf. Jaillet et al. [1990].

Definition 3.5 (Weighted Sobolev spaces). The weighted Sobolev spaces L^2_η and H^1_η are defined for $\eta \in \mathbb{R}$ by

$$L^2_\eta := L^2_\eta(\mathbb{R}) := \left\{ v \in L^1_{loc}(\mathbb{R}) | v e^{-\eta|x|} \in L^2(\mathbb{R}) \right\}, \tag{3.6}$$

$$H^1_\eta := H^1_\eta(\mathbb{R}) := \left\{ v \in L^1_{loc}(\mathbb{R}) | v e^{-\eta|x|}, v' e^{-\eta|x|} \in L^2(\mathbb{R}) \right\}. \tag{3.7}$$

The respective norms are defined by

$$\|u\|^2_{L^2_\eta} := \int_{\mathbb{R}} u^2 e^{-2\eta|x|}\, dx \quad \text{and} \quad \|u\|^2_{H^1_\eta} := \|u\|^2_{L^2_\eta} + \|u'\|^2_{L^2_\eta}. \tag{3.8}$$

The L^2_η-scalar product is denoted by

$$(u\ ,\ v)_{L^2_\eta} := \int_{\mathbb{R}} uv\, e^{-2\eta|x|}\, dx. \tag{3.9}$$

Sometimes we abbreviate the weight $p(x) := e^{-2\eta|x|}$ and the weighted scalar product $(u\ ,\ v)_{L^2_\eta}$ by $(u,v)_\eta$. Then there holds $\left\|f\sqrt{p}\right\|_{L^2} = \|f\|_{L^2_\eta}$.

Lemma 3.1. *The space $\mathcal{D}(\mathbb{R})$ is dense in H^1_η.*

Before proving this lemma, we show that a function $f \in L^2_\eta$ convolved with a mollifier φ_ε converges strongly to f in L^2_η as $\varepsilon \to 0$.

Lemma 3.2. *Let $f \in L^2_\eta$. For $\varphi \in C_0^\infty$, $\operatorname{supp}\varphi = (-1,1)$, $\int_{\mathbb{R}} \varphi\, dx = 1$, and $\varphi_\varepsilon := \frac{1}{\varepsilon}\varphi(\frac{x}{\varepsilon})$ we have*

$$\varphi_\varepsilon * f \longrightarrow f \quad \text{in } L^2_\eta \text{ as } \varepsilon \to 0.$$

Proof. For brevity set $q := \sqrt{p} = e^{-\eta|x|}$. We make the following decomposition

$$(\varphi_\varepsilon * f)\cdot q = \varphi_\varepsilon * (f\cdot q) + \big((\varphi_\varepsilon * f)\cdot q - \varphi_\varepsilon * (f\cdot q)\big).$$

Since $f\cdot q \in L^2$ and $\varphi_\varepsilon * (f\cdot q) \longrightarrow f\cdot q$ in L^2 it suffices to show that

$$\|g_\varepsilon(x)\|_{L^2} := \|(\varphi_\varepsilon * f)\cdot q - \varphi_\varepsilon * (f\cdot q)\|_{L^2} \longrightarrow 0 \quad \text{for } \varepsilon \to 0.$$

The Fundamental Theorem of Calculus for absolutely continuous q yields

$$g_\varepsilon(x) = \int_{\mathbb{R}} \varphi_\varepsilon(x-y) f(y)\big(q(x)-q(y)\big)\, dy = \int_{\mathbb{R}} \varphi_\varepsilon(x-y) f(y) \int_y^x q'(z)\, dz\, dy.$$

Since $\operatorname{supp}\varphi_\varepsilon = (-\varepsilon,\varepsilon)$,

$$\begin{aligned} |g_\varepsilon(x)| &\le \int_{\mathbb{R}} \varphi_\varepsilon(x-y)\,|f(y)|\,\big(2\varepsilon \sup_{z\in(x,y)} |q'(z)|\big)\, dy \\ &= 2\varepsilon \underbrace{\int_{\mathbb{R}} \varphi_\varepsilon(x-y)\,|f(y)|\,\big(\sup_{|s|\le\varepsilon} |q'(y+s)|\big)\, dy}_{=:\tilde g_\varepsilon(x)}. \end{aligned}$$

Since $|q'(x)| = |\eta|\exp(\eta|x|)$ there holds

$$\sup_{|s|\le\varepsilon} |q'(y+s)| = |\eta| \begin{cases} 1 & \text{if } |y| \le \varepsilon, \\ e^{\eta|y|-\eta\varepsilon} & \text{if } |y| > \varepsilon. \end{cases}$$

Then, for some constant $C > 0$

$$\sup_{|s|\le\varepsilon} |q'(y+s)| \le Cp(y) \quad \text{for all } \varepsilon < \varepsilon_0$$

and so

$$\sup_{|s|\le\varepsilon} |q'(y+s)|\,|f(y)| \in L^2(\mathbb{R}).$$

Since $\tilde g_\varepsilon \in L^2$ uniformly and $|g_\varepsilon(x)| \le 2\varepsilon|\tilde g_\varepsilon(x)|$ there holds $\|g_\varepsilon\|_{L^2} \le C\varepsilon \to 0$ as $\varepsilon \to 0$. $\square$

Proof (of Lemma 3.1). Let $f \in H^1_\eta$ and the truncation function $\Phi \in C^\infty$ such that

$$\Phi(x) := \begin{cases} 1 & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| \geq 2. \end{cases}$$

We show now that $f_\varepsilon := \big(f \cdot \Phi(\varepsilon(\cdot))\big) * \varphi_\varepsilon \in C_0^\infty$ with φ_ε from Lemma 3.2

$$f_\varepsilon \longrightarrow f \quad \text{in } H^1_\eta,$$

i.e., $f_\varepsilon \longrightarrow f$ and $\nabla f_\varepsilon \longrightarrow \nabla f$ in L^2_η, holds true. Since

$$\nabla f_\varepsilon = \big(\nabla f \cdot \Phi(\varepsilon(\cdot))\big) * \varphi_\varepsilon + \varepsilon\big(f \cdot (\nabla\Phi)(\varepsilon(\cdot))\big) * \varphi_\varepsilon$$

it suffices to show

$$\big(f \cdot \Phi(\varepsilon(\cdot))\big) * \varphi_\varepsilon \longrightarrow f \quad \text{in } L^2_\eta.$$

The Lebesgue Dominated Convergence Theorem yields

$$f \cdot \Phi(\varepsilon(\cdot))\sqrt{p} \longrightarrow f\sqrt{p} \quad \text{in } L^2$$

and consequently

$$f \cdot \Phi(\varepsilon(\cdot)) \longrightarrow f \quad \text{in } L^2_\eta.$$

Thus Lemma 3.2 concludes the proof. □

Definition 3.6. We define the dual of H^1_η with respect to the pivot space L^2_η, i.e., for $u \in (H^1_\eta)^* =: H^{-1}_\eta$ and $v \in H^1_\eta$ the dual pairing $\langle \cdot \, , \cdot \rangle_{H^{-1}_\eta \times H^1_\eta}$ is defined by

$$\langle u \, , \, v \rangle_\eta := \langle u \, , \, v \rangle_{H^{-1}_\eta \times H^1_\eta} := \int_{\mathbb{R}} uv e^{-2\eta|x|} \, dx.$$

Remark 3.7. The density of the test functions $\mathcal{D}(\mathbb{R})$ in H^1_η and the definition of $\langle \cdot \, , \cdot \rangle_\eta$ allows to interpret the operator $\mathcal{A}$ as a mapping from $H^1_\eta \to H^{-1}_\eta$.

Definition 3.8 ($\boldsymbol{a_\eta(\cdot,\cdot)}$). We define the bilinear form $a_\eta(\cdot,\cdot) : H^1_\eta \times H^1_\eta \to \mathbb{R}$ as

$$\begin{aligned} a_\eta(u,v) &:= \int_{\mathbb{R}} \mathcal{A}[u] v \exp(-2\eta|x|) \, dx \\ &= \frac{\sigma^2}{2} \int_{\mathbb{R}} u'v' \exp(-2\eta|x|) \, dx + r \int_{\mathbb{R}} uv \exp(-2\eta|x|) \, dx \\ &\quad - \int_{\mathbb{R}} \Big(\eta\sigma^2 \operatorname{sign}(x) + r - d - \frac{\sigma^2}{2}\Big) u'v \exp(-2\eta|x|) \, dx. \end{aligned} \tag{3.10}$$

Remark 3.9. Since $\mathcal{D}(\mathbb{R})$ is dense in $H^1_\eta(\mathbb{R})$, the boundary terms vanish in the integration by part in (3.10).

Before we formulate the weak problem we introduce the so-called Bochner spaces, cf. Zeidler [1990], Dautray and Lions [1992].

Definition 3.10. Let X be a Banach space and $a, b \in \mathbb{R}$ with $a < b$, $1 \leq p < \infty$. Then $L^2(a,b;X)$ and $L^\infty(a,b;X)$ denote the spaces of measurable functions u defined on (a,b) with values in V such that the function $t \to \|u(\cdot,t)\|_X$ is square integrable, respectively, essentially bounded. The respective norms are defined by

$$\begin{aligned} \|u\|_{L^2(a,b;X)} &= \left(\int_a^b \|u(\cdot,t)\|_X^2 \, dt \right)^{1/2}, \\ \|u\|_{L^\infty(a,b;X)} &= \operatorname{ess.sup}_{a \leq t \leq b} \|u(\cdot,t)\|_X \end{aligned}$$

For details on this function spaces we refer to Zeidler [1990], Dautray and Lions [1992]. By means of the introduced weighted norms and the bilinear form $a_\eta(\cdot,\cdot)$ we give a weak formulation of Problem 3.4.

Problem 3.11 (Weak formulation). The weak formulation of Problem 3.4 reads: Given $\psi \in H^1_\eta$ seek $u \in L^2(0,T;H^1_\eta)$ with $\dot{u} \in L^2(0,T;(H^1_\eta)^*)$ such that $u(\cdot,0) = \psi(\cdot)$ almost everywhere in $\mathbb{R}$ and for almost all times $t \in (0,T]$,

$$\frac{\partial}{\partial t}(u(\cdot,t),v)_{L^2_\eta} + a_\eta(u(\cdot,t),v) = 0 \quad \text{for all } v \in H^1_\eta(\mathbb{R}). \tag{3.11}$$

The main theorem on first-order linear evolution equations in Zeidler [1990] proves existence of a unique solution of problem (3.11). The next definition explains the concept of a Gelfand triple which is used in the main theorem.

Definition 3.12 (Gelfand Triple). $V \subseteq H \subseteq V^*$ is called Gelfand triple if V is a real separable and reflexive Banach space while H is a real separable Hilbert space and V is dense in H with continuous embedding $V \subseteq H$, i.e. for some $C < \infty$,

$$\|v\|_H \leq C \|v\|_V \text{ for all } v \in V.$$

The following theorem is the main theorem on first-order linear evolution equations.

Theorem 3.3 (Zeidler IIa, Chapt. 23). *Suppose $u_0 \in H$, $f \in L^2(0,T;V^*)$ and the conditions (H1)-(H3).*

(H1) $V \subseteq H \subseteq V^*$ is a Gelfand triple with $\dim V = \infty$, $0 < T < \infty$; H and V are real Hilbert spaces.

(H2) The mapping $a : V \times V \to \mathbb{R}$ is bilinear, bounded and strongly positive.

(H3) $(w_1, w_2, \dots)$ is a basis in V, and (u_{n0}) is a sequence in H with $u_{n0} \in \operatorname{span}\{w_1,\dots,w_n\}$ for all n and

$$(u_{n0}) \to u_0 \textit{ in } H \textit{ as } n \to \infty.$$

Then there exists a unique solution $u \in W^1_2(0,T;V,H) := \{u \in L^2(0,T;V) : u_t \in L^2(0,T;V^*)\}$ satisfying

$$\begin{aligned} \frac{\partial}{\partial t}\left(u(\cdot,t),v(\cdot)\right)_H + a(u(\cdot,t),v(\cdot)) &= \langle f(\cdot,t),v\rangle_{V^*\times V} \quad \textit{for all } v \in V, \\ u(\cdot,0) &= u_0(\cdot), \end{aligned} \tag{3.12}$$

Remark 3.13. The existence of a basis in condition (H3) follows directly from (H1) and $u_0 \in H$.

Corollary 3.4. *$H^1_\eta \subseteq L^2_\eta \subseteq H^{-1}_\eta$ form a Gelfand triple.*

Proof. The density of the test functions $\mathcal{D}(\mathbb{R})$ in H^1_η and L^2_η implies the separability of L^2_η and H^1_η and also the density of H^1_η in L^2_η. The continuity of the embedding $H^1_\eta \subseteq L^2_\eta$ is easily seen because the H^1_η-norm is by definition stronger then the L^2_η-norm. □

To apply Theorem 3.3 to the weak formulation (3.11), the bilinear form $a_\eta(\cdot,\cdot)$ defined in (3.10) needs to be bounded and elliptic. The next proposition proves boundedness and a Gårding inequality of a_η. This proposition can be found for $d = 0$ in Matache et al. [2004] where most of the proof is left to the reader. Therefore we give a more detailed proof including the explicit determination of the constants α and λ which play a crucial part in the error analysis in Chapter 5.

Proposition 3.5 (Gårding inequality and ellipticity of $\boldsymbol{a_\eta(\cdot,\cdot)}$). *The bilinear form $a_\eta(\cdot,\cdot) : H^1_\eta \times H^1_\eta \to \mathbb{R}$ is continuous and satisfies a Gårding inequality: with*

$$C := \max\left\{|r - d - \sigma^2/2 + \eta\sigma^2|, |r - d - \sigma^2/2 - \eta\sigma^2|\right\} \tag{3.13}$$

and

$$M = \max(\sigma^2/2, r) + C > 0, \qquad \alpha = \sigma^2/4 > 0, \qquad \lambda = C^2/\sigma^2 + \sigma^2/4 - r$$

there holds

$$|a_\eta(u,v)| \leq M \|u\|_{H^1_\eta(\mathbb{R})} \|v\|_{H^1_\eta(\mathbb{R})} \text{ for all } u, v \in H^1_\eta(\mathbb{R}); \tag{3.14}$$

$$a_\eta(u,u) \geq \alpha \|u\|^2_{H^1_\eta(\mathbb{R})} - \lambda \|u\|^2_{L^2_\eta(\mathbb{R})} \text{ for all } u \in H^1_\eta(\mathbb{R}). \tag{3.15}$$

Proof. Some straight-forward estimates lead to

$$\begin{aligned}
&|a_\eta(u,v)| \\
&= \left|\frac{\sigma^2}{2}\int_{\mathbb{R}} u'v'e^{-2\eta|x|}\,dx + r\int_{\mathbb{R}} uve^{-2\eta|x|}\,dx - \int_{\mathbb{R}} \big(\eta\sigma^2\operatorname{sign}(x) + r - d - \sigma^2/2\big)u've^{-2\eta|x|}\,dx\right| \\
&\leq \max(\sigma^2/2, r)\int_{\mathbb{R}} \left|(u'v' + uv)\,e^{-2\eta|x|}\right|\,dx + \left\|\eta\sigma^2\operatorname{sign}(x) + r - d - \frac{\sigma^2}{2}\right\|_{L^\infty}\int_{\mathbb{R}}\left|u've^{-2\eta|x|}\right|\,dx \\
&\leq M\|u\|_{H^1_\eta}\|v\|_{H^1_\eta}.
\end{aligned}$$

Conversely, with $C := \max(|r - d - \frac{\sigma^2}{2} + \eta\sigma^2|, |r - d - \frac{\sigma^2}{2} - \eta\sigma^2|)$ and $\varepsilon := \sigma^2/(4C)$ one estimates

$$\begin{aligned}
a_\eta(u,u) &\geq \frac{\sigma^2}{2}\|u'\|^2_{L^2_\eta} + r\|u\|^2_{L^2_\eta} - C\|u'u\exp(-2\eta|\cdot|)\|_{L^1} \\
&\geq \frac{\sigma^2}{2}\|u'\|^2_{L^2_\eta} + r\|u\|^2_{L^2_\eta} - C\left(\varepsilon\|u'\|^2_{L^2_\eta} + \frac{1}{4\varepsilon}\|u\|^2_{L^2_\eta}\right) \\
&= \frac{\sigma^2}{4}\|u'\|^2_{L^2_\eta} + (r - C^2/\sigma^2)\|u\|^2_{L^2_\eta} \\
&= \alpha\|u\|^2_{H^1_\eta} - \lambda\|u\|^2_{L^2_\eta}.
\end{aligned}$$

□

Remark 3.14 (Ellipticity vs. Gårding inequality for European Options). Although Theorem 3.3 demands the strict positivity of the bilinear form $a(\cdot,\cdot)$, it is sufficient that $a(\cdot,\cdot)$ satisfies the Gårding inequality; i.e. there exists $\alpha > 0$, $\lambda \in \mathbb{R}$ such that

$$a(u,u) \geq \alpha \|u\|_V^2 - \lambda \|u\|_H^2 \text{ for all } u \in V. \tag{3.16}$$

Proof. The transformation $u = e^{\lambda t} w$ in equation (3.12) and setting $a_1(w,v) := a(w,v) + \lambda(w,v)_H$, $f_1 := e^{-\lambda t} f$ leads to

$$\begin{aligned} \frac{\partial}{\partial t}(w(\cdot,t), v(\cdot))_H + a_1(w(\cdot,t), v(\cdot)) &= \langle f_1(\cdot,t), v\rangle_{V^* \times V} \quad \text{for all } v \in V, \\ w(0) &= u_0 \in H. \end{aligned}$$

Note that the bilinear form $a_1(\cdot,\cdot)$ is strictly positive. Since this transformation only effects the time-derivative, it can be applied as well to the weak form without changing the solution spaces. □

The next theorem provides the unique existence of a weak solution of Problem 3.11, the variational formulation of the transformed Black-Scholes equation.

Corollary 3.6. *Problem 3.11 has a unique solution*

$$u \in \left\{ u \in L^2(0,T; H^1_\eta) : u_t \in L^2(0,T; (H^1_\eta)^*) \right\}.$$

Proof. Since the weighted Sobolev spaces H^1_η and L^2_η form a Gelfand triple, cf. Corollary 3.4, and the bilinear form $a_\eta(\cdot,\cdot)$ is bounded and a Gårding inequality is applicable, cf. Proposition 3.5, (H1) and (H2) of Theorem 3.3 are satisfied; which implies Corollary 3.6. □

3.2 American Options

As already explained in Chapter 2 the optimal stopping problem for the evaluation of American options corresponds to a determinitstic system of partial differential (in)equalities, also called linear complimentary formulation (LCF), cf. Lamberton and Lapeyre [1996], Wilmott et al. [1995]. In this section we will derive the weak formulation, namely the variational inequality. This formulation can be found in Wilmott et al. [1995]. In Jaillet et al. [1990] it is directly shown that the solution of the optimal stopping problem can be written as a deterministic variational inequality in weighted Sobolev spaces. The formulation as a free boundary value problem (FBVP) can be found in McKean [1965], Van Moerbeke [1976], Karatzas and Shreve [1998], Dewynne et al. [1993], Wilmott et al. [1995]. We will use this formulation to derive an integral representation of the value of an American option and truncation error estimates.

3.2.1 The Black-Scholes inequality

In the original variables S and t the LCF for the valuation of American options, cf. (2.9) on page 13, is a backward time formulation with space-dependent coefficients. As in the case of European option we transform the problem to forward time and log-prices, cf. Transformation 3.1 on page 15 and 3.2 on page 16.

Problem 3.15 (Strong formulation / Linear complementary formulation (LCF)). Recall the definition of the operator $\mathcal{A} : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ from (3.4) on page 16,

$$\mathcal{A}[u] := -\frac{\sigma^2}{2}u'' - \Big(r - d - \frac{\sigma^2}{2}\big)u' + ru. \tag{3.17}$$

Set $S = e^x$, $\tau = T - t$, $u(x, \tau) := V(e^x, T - t)$, and $\psi(x) := h(e^x)$, cf. Transformation 3.1 and 3.2. Then, with t instead of τ, (2.9) reads

$$\begin{aligned} \dot{u} + \mathcal{A}[u] &\geq 0 \quad \text{in } (0, T] \times \mathbb{R}, \\ \big(u - \psi\big)\Big(\dot{u} + \mathcal{A}[u]\Big) &= 0 \quad \text{in } (0, T] \times \mathbb{R}, \\ u(\cdot, t) &\geq \psi(\cdot) \quad \text{in } (0, T] \times \mathbb{R}, \\ u(\cdot, 0) &= \psi(\cdot) \quad \text{in } \mathbb{R}. \end{aligned} \tag{3.18}$$

3.2.2 Variational Formulation

As for the European option case we introduce weighted Sobolev spaces L^2_η and H^1_η as in Definition 3.5 on page 16. The bilinear form $a_\eta(\cdot, \cdot)$ corresponding to the space operator $\mathcal{A}$ is defined in (3.10) on page 18. Define the set of admissible solutions

$$\mathcal{K}_\psi := \{v \in H^1_\eta(\mathbb{R}) | v \geq \psi \text{ a.e.}\}. \tag{3.19}$$

Note that $\mathcal{K}_\psi$ is a closed, convex, and non-empty subset of H^1_η.

Problem 3.16 (Weak formulation). The variational formulation of problem (3.18) reads: Find $u \in L^2(0, T; H^1_\eta)$, $\dot{u} \in L^2(0, T; L^2_\eta)$ such that $u \in K_\psi$ almost everywhere in $(0, T]$,

$$\begin{aligned} (\dot{u}, v - u)_{L^2_\eta} + a_\eta(u, v - u) \geq 0 \quad &\text{for all } v \in K_\psi \\ u(\cdot, 0) = \psi(\cdot). \end{aligned} \tag{3.20}$$

Remark 3.17 (Variational formulation vs. complimentary formulation). If a solution
$u \in L^2(0, T; H^2_\eta)$ solves the variational inequality (3.20) then it is also a solution of the LCF (3.18), cf. Bensoussan and Lions [1982].

3.2.3 Existence and Uniqueness

To prove the existence of a unique solution of the variational inequality (3.20) we cite the main theorem on evolution variational inequalities of first order from Zeidler [1985].

Theorem 3.7 (Zeidler III. Chapt 55). *We consider the following problem*

$$\begin{aligned} (\dot{u}, v-u)_H + a(u, v-u) \geq 0 \quad & \text{for all } v \in M \text{ and almost all } t \in [0,T], \\ u(0) = u_0 \in V. & \end{aligned} \tag{3.21}$$

This problem has exactly one solution $\{u \in L^2(0,T;V) \,:\, \dot{u} \in L^2(0,T;H)\}$ if the following hold true.

(i) *"$V \subseteq H \subseteq V^*$" is a Gelfand triple.*

(ii) *M is a closed convex nonempty set in V.*

(iii) *The bilinear form $a : V \times V \to \mathbb{R}$ is bounded, and there exist real numbers ω and $\beta > 0$ such that*
$$a(u,v) + \omega \|u\|_H^2 \geq \beta \|u\|_V^2 \;\text{ for all } v \in V.$$

(iv) *For a fixed $g \in H$ and for all $v \in M$ there holds*
$$a(u_0, v-u_0) \geq (g, v-u_0)_H.$$

Corollary 3.8. *With $M := \mathcal{K}_\psi$, $H := L^2_\eta(\mathbb{R})$, $V := H^1_\eta(\mathbb{R})$, $a(\cdot,\cdot) := a^\eta(\cdot,\cdot)$, and $u_0 := \psi$ problem (3.20) has exactly one solution.*

Proof. Since (i), (ii), and (iii) are satisfied (cf. Corollary 3.4 on page 20, the definition of $\mathcal{K}_\psi$ (3.19) on page 22, and Proposition 3.5 on page 20), it remains to show that (iv) holds true. In case of a put option, i.e. $u_0 = (K - e^x)_+$, define $g \in L^2_\eta$ by

$$g(x) := \begin{cases} -\sigma^2/2\,K - de^x + rK & \text{for } -\infty < x \leq \log K, \\ 0 & \text{for } x > \log K. \end{cases}$$

Set $\tilde{v} := v - u_0$. Since $v \in \mathcal{K}_\psi$ there holds $0 \leq \tilde{v} \in H^1_\eta$; then one obtains for all $\tilde{v} \geq 0$

$$\begin{aligned} a_\eta(u_0, \tilde{v}) &= -\frac{\sigma^2}{2} \int_{-\infty}^{\log K} e^x \tilde{v}_x \exp(-2\eta|x|)\,dx + r \int_{-\infty}^{\ln K} (K - e^x)\tilde{v} \exp(-2\eta|x|)\,dx \\ &\quad + \int_{-\infty}^{\log K} (r - d - \sigma^2/2 + \eta\sigma^2 \operatorname{sign}(x)) e^x \tilde{v} \exp(-2\eta|x|)\,dx \\ &= \frac{\sigma^2}{2} \int_{-\infty}^{\log K} (1 - 2\sigma^2 \operatorname{sign}(x)) e^x \tilde{v} \exp(-2\eta|x|)\,dx \\ &\quad + \int_{-\infty}^{\log K} (r - d - \sigma^2/2 + \eta\sigma^2 \operatorname{sign}(x)) e^x \tilde{v} \exp(-2\eta|x|)\,dx \\ &\quad + r \int_{-\infty}^{\log K} (K - e^x)\tilde{v} \exp(-2\eta|x|)\,dx - \frac{\sigma^2}{2} K \tilde{v}(\ln K) \exp(-2\eta|\ln K|) \\ &= \int_{-\infty}^{\log K} \big(-\sigma^2/2\,K\delta_{\log K} - de^x + rK\big)\tilde{v} \exp(-2\eta|x|)\,dx \\ &\geq \int_{-\infty}^{\log K} \big(-\sigma^2/2\,K - de^x + rK\big)\tilde{v} \exp(-2\eta|x|)\,dx \\ &= \int_{\mathbb{R}} g\tilde{v} \exp(-2\eta|x|)\,dx = (g, \tilde{v})_{L^2_\eta}. \end{aligned}$$

For a call option, i.e. $u_0 = (K - e^x)_+$, we choose $g \in L^2_\eta$

$$g(x) := \begin{cases} -\sigma^2/2\,K + de^x - rK & \text{for } \log K \le x < \infty, \\ 0 & \text{for } x < \log K. \end{cases}$$

Similar calculations show that there holds $a_\eta(u_0, \tilde{v}) \ge (g, \tilde{v})_{L^2_\eta}$ for all $0 \le \tilde{v} \in H^1_\eta$. □

3.3 Solving American Options via the Fourier Transform

The derivation of an integral representation formula for the value of an American option is the main task of this section. Using the Fourier transform to solve the free boundary value problem (FBVP) arising in the evaluation of American options, cf. Wilmott et al. [1995], yields an integral representation depending on the free boundary. This approach can be found for American call options in Underwood and Wang [2002]. Since we need this representation also for put options and in the transformed spaces for the error analysis in Chapter 5 we give a detailed derivation.

3.3.1 American Put

Recall that early exercise of an American put option is never optimal for the riskless interest rate $r = 0$, cf. Subsection 2.2.4. Therefore we assume that $r > 0$ to guarantee the existence of a contact region, i.e. $\gamma(\tau) > 0$. This assumption makes perfectly sense because if it is a priori known that there does not exist a free boundary, the evaluation problem can be considered as a PDE. Note that the value of an American put option is $V(S,\tau) = K - S$ for $S < \gamma(\tau)$ and $\tau \in [0,T)$.

A Problem formulation

Let S denote the value of the underlying share, $\tau \in [0,T)$ the 'backward' time, and $V(S,\tau)$ the value of an American put option at time τ and share value S. Further, let $\gamma(\tau) \ge 0$ be the early exercise curve (or free boundary). Given the positive constants T, K, r, d, and σ^2, the value $V(S,\tau)$ of an American put option satisfies (cf. Subsection 2.2.2)

$$\begin{aligned} V_\tau + \sigma^2/2\,S^2 V_{SS} + (r-d)SV_S - rV = 0 \quad & \text{for } \gamma(\tau) < S < \infty,\ \tau \in [0,T), \\ V(S,T) = (K-S)_+ \quad & \text{for } \gamma(T) \le S < \infty, \\ V(\gamma(\tau),\tau) = K - \gamma(\tau) \text{ and } \lim_{S\to\infty} V(S,\tau) = 0 \quad & \text{for } \tau \in [0,T), \\ V_S(\gamma(\tau),\tau) = -1 \quad & \text{for } \tau \in [0,T). \end{aligned} \tag{3.22}$$

B Transformations to a formulation which admits Fourier transformation

Note that the Fourier transform cannot be applied directly to (3.22) owing to the space dependent coefficients and the time-dependent domain. The forthcoming Transformations 3.18-3.20 lead to formulation (3.25) with constant coefficients and spatial domain $\mathbb{R}$.

Transformation 3.18 (leading to forward time and constant coefficients). A logarithmic price and forward time transformation in formulation (3.22), i.e. $S = e^x$, $t = T - \tau$, $u(x,t) := V(e^x, T-\tau)$, and $x_f(t) := \log(\gamma(t))$ yields

$$\begin{aligned} u_t - \sigma^2/2\, u_{xx} - (r - d - \sigma^2/2)u_x + ru = 0 &\quad \text{for } x_f(t) \leq x < \infty,\ t \in (0,T], \\ u(x,0) = (K - e^x)_+ &\quad \text{for } x_f(0) \leq x < \infty, \\ u(x_f(t),t) = K - e^{x_f(t)} \text{ and } \lim_{x\to\infty} u(x,t) = 0 &\quad \text{for } t \in (0,T], \\ u_x(x_f(t),t) = -e^{x_f(t)} &\quad \text{for } t \in (0,T]. \end{aligned} \tag{3.23}$$

Transformation 3.19 (leading to a time-invariant domain). A shift $y := x - x_f(t)$ leads to a time-invariant domain in (3.23). Moreover, the left boundary lies at the origin. Then, $w(y,t) := u(x - x_f(t), t)$ satisfies

$$\begin{aligned} w_t - \sigma^2/2\, w_{yy} - (r - d - \sigma^2/2 + \dot{x}_f(t))w_y + rw = 0 &\quad \text{for } 0 \leq y < \infty,\ t \in (0,T], \\ w(y,0) = (K - e^{y + x_f(0)})_+ &\quad \text{for } 0 \leq y < \infty, \\ w(0,t) = K - e^{x_f(t)} \text{ and } \lim_{y\to\infty} w(y,t) = 0 &\quad \text{for } t \in (0,T], \\ w_x(0,t) = -e^{x_f(t)} &\quad \text{for } t \in (0,T]. \end{aligned} \tag{3.24}$$

Transformation 3.20 (leading to homogeneous boundary conditions at zero). The extension from $(0,\infty)$ to $\mathbb{R}$ in (3.24) requires homogeneous boundary conditions at zero. Decompose $w(y,t) = v(y,t) + g(y,t)$ such that $v(0,t) = v_y(0,t) = 0$ for all t, i.e. $g(y,t) = (K - e^{y+x_f(t)})_+$. Then, v satisfies

$$\begin{aligned} v_t - \sigma^2/2\, v_{yy} - (r - d - \sigma^2/2 + \dot{x}_f(t))v_y + rv = f(y,t) &\quad \text{for } 0 \leq y < \infty,\ t \in (0,T], \\ v(y,0) = 0 &\quad \text{for } 0 \leq y < \infty, \\ v(0,t) = 0 \text{ and } \lim_{y\to\infty} v(y,t) = 0 &\quad \text{for } t \in (0,T], \\ v_y(0,t) = 0 &\quad \text{for } t \in (0,T], \end{aligned} \tag{3.25}$$

with

$$f(y,t) = -g_t + \sigma^2/2 g_{yy} + (r - d - \sigma^2/2 + \dot{x}_f(t))g_y - rg. \tag{3.26}$$

Remark 3.21. Note that an extension of $v(y,t)$ and $f(y,t)$ by zero for $y < 0$ and all $t \in [0,T]$, yields that the extended v satisfies (3.25) for all $y \in \mathbb{R}$ and all $0 \leq t \leq T$.

C Calculation of the right-hand side f

Since $g(y,t)$ is continuous, $g_t(y,t)$, $g_y(y,t)$, and $g_{yy}(y,t)$ are understood in the weak sense. Then,

$$g_t = \begin{cases} -\dot{x}_f(t)e^{y+x_f(t)} & \text{if } y \le \log K - x_f(t), \\ 0 & \text{otherwise;} \end{cases}$$

$$g_y = \begin{cases} -e^{y+x_f(t)} & \text{if } y \le \log K - x_f(t), \\ 0 & \text{otherwise;} \end{cases}$$

$$g_{yy} = \begin{cases} K\delta_{\log K - x_f(t)} - e^{y+x_f(t)} & \text{if } y \le \log K - x_f(t), \\ 0 & \text{otherwise;} \end{cases}$$

with the Dirac measure δ, i.e. $\langle \delta_a, \varphi \rangle := \varphi(a)$ for all $\varphi \in \mathcal{D}(\mathbb{R})$.

Inserting g_t, g_y, and g_{yy} in (3.26) yields for $y \le \log K - x_f(t)$

$$\begin{aligned} f(y,t) &= \dot{x}_f(t)e^{y+x_f(t)} + \sigma^2/2(K\delta_{\log K - x_f(t)} - e^{y+x_f(t)}) + (r - d - \sigma^2/2 + \dot{x}_f(t))(-e^{y+x_f(t)}) \\ &\quad - r(K - e^{y+x_f(t)}) \\ &= \sigma^2/2\, K\delta_{\log K - x_f(t)} - rK + de^{y+x_f(t)}. \end{aligned} \tag{3.27}$$

Since f is extended by zero for $y < 0$, the measure f reads

$$f(y,t) = \begin{cases} \sigma^2/2\, K\delta_{\log K - x_f(t)} - rK + de^{y+x_f(t)} & \text{if } 0 \le y \le \log K - x_f(t), \\ 0 & \text{otherwise.} \end{cases} \tag{3.28}$$

Remark 3.22. Note that f has compact support because $x_f(t) \ge C > -\infty$. Since, in addition, the Dirac measure $\delta \in L^\infty(0,T;H^{-s}(\mathbb{R}))$ for $s > 1/2$ there holds $f \in L^\infty(0,T;H^{-s}(\mathbb{R}))$ for all $s > 1/2$.

D Fourier Transformation

Throughout the remainder of this section, any reference to (3.25) is understood in the sense of Remark 3.21. Before solving (3.25) via the Fourier transform, we give a formal definition of the Fourier transform and state some of its properties. An introduction on the Fourier transform can be found for example in Yosida [1995].

Definition 3.23 (Fourier transform). Let $\mathcal{S}(\mathbb{R})$ denote the space of rapidly decreasing smooth functions, i.e., $f \in C^\infty(\mathbb{R})$ such that $\sup_{x\in\mathbb{R}} |x^k f^{(\alpha)}(x)| < \infty$ for all $k, \alpha \in \mathbb{N}$. Then, the Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is defined by

$$(\mathcal{F}f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\xi x}\, dx =: \hat{f}(\xi). \tag{3.29}$$

The inverse Fourier transform $\mathcal{F}^{-1} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ reads

$$(\mathcal{F}^{-1}f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) e^{i\xi x}\, dx =: \tilde{f}(x). \tag{3.30}$$

Theorem 3.9 (Properties of the Fourier transform). *The Fourier transform $\mathcal{F}$ maps $\mathcal{S}(\mathbb{R})$ linearly and continuously into $\mathcal{S}(\mathbb{R})$. The same holds for the inverse Fourier transform. Moreover,*

$$\hat{\tilde{f}} = f, \quad \tilde{\hat{f}} = f, \quad \widehat{f * g} = \sqrt{2\pi}\hat{f}\hat{g}, \quad \sqrt{2\pi}\widehat{fg} = \hat{f} * \hat{g}. \tag{3.31}$$

Proof. The proof can be found in Yosida [1995].

Remark 3.24. The Fourier transform can be extended to $\mathcal{S}'(\mathbb{R})$, the space of tempered distributions, and to $L^2(\mathbb{R})$. Theorem 3.9 holds literally.

E Fourier transform to solve parabolic PDEs

Consider the following abstract PDE with an elliptic operator $\mathcal{A}[u] := -au_{xx} + b(t)u_x + cu$ ($a > 0$, $b(t) \in L^1(0,T), c \in \mathbb{R}$) and right-hand-side $f \in L^\infty(0,T; H^{-s}(\mathbb{R}))$ for $s > 1/2$,

$$\begin{aligned} u_t + \mathcal{A}[u] &= f, \\ u(0) &= 0. \end{aligned} \tag{3.32}$$

Since $f \in L^\infty(0,T; H^{-s}(\mathbb{R}))$ for $s > 1/2$ there holds $\hat{f} \in L^\infty(0,T; L^1_{\ell oc}(\mathbb{R}))$ and

$$(1 + |\xi|^2)^{-s/2} \hat{f}(\xi, t) \in L^\infty(0,T; L^2(\mathbb{R})).$$

Then, applying the Fourier transform on (3.32) yields the ODE

$$\hat{u}_t + p(\xi,t)\hat{u} = \hat{f}. \tag{3.33}$$

Note that $\hat{u}(0) = 0$ and

$$p(\xi,t) = a|\xi|^2 + b(t)i\xi + c.$$

The solution $\hat{u}$ of (3.33) reads

$$\hat{u}(\xi,t) = e^{-\int_0^t p(\xi,s)\,ds} \int_0^t \hat{f}(\xi,s) e^{\int_0^s p(\xi,r)\,dr}\, ds. \tag{3.34}$$

Lemma 3.10 (Regularity of u). *Let u be a solution of (3.32). Given that $f(\xi,t) \in L^\infty(0,T; H^s(\mathbb{R}))$ there holds $u(x,t) \in L^2(0,T; H^{s+2}(\mathbb{R}))$.*

Proof. Recall that by definition of $v \in H^s(\mathbb{R})$, the Fourier transform $\hat{v}$ of v satisfies

$$(1 + |\xi|^2)^{s/2} \hat{v}(\xi) \in L^2(\mathbb{R}).$$

Then,

$$\begin{aligned}\|u\|^2_{L^2(0,T;H^{s+2}(\mathbb{R}))} &= \int_0^T \int_{\mathbb{R}} (1+|\xi|^2)^{s+2} \hat{u}^2(\xi)\, d\xi\, dt \\ &= \int_0^T \int_{\mathbb{R}} (1+|\xi|^2)^{s+2} \Big(\int_0^t \hat{f}(\xi,\cdot) e^{-\int_s^t p(\xi,r)\,dr}\, ds\Big)^2 d\xi\, dt.\end{aligned}$$

Set $\int_s^t b(r)\,dr = B(t)-B(s)$. Hölder's inequality in the inner time integral yields

$$\begin{aligned}&\|u\|^2_{L^2(0,T;H^{s+2}(\mathbb{R}))} \\ &\quad \le \int_0^T \int_{\mathbb{R}} (1+|\xi|^2)^{s+2} \left\|\hat{f}(\xi,\cdot)\right\|^2_{L^\infty(0,t)} \Big(\int_0^t |e^{-a|\xi|^2(t-s)-c(t-s)-i\xi(B(t)-B(s))}|\, ds\Big)^2 d\xi\, dt.\end{aligned}$$

Observe $|e^{-i\xi(B(t)-B(s))}| = 1$ and $|e^{-c(t-s)}| \le C(T)$ with $C(T) := e^{-cT}$ for some $c<0$ and $C(T)=1$ for $c \ge 0$. This yields

$$\begin{aligned}\|u\|^2_{L^2(0,T;H^{s+2}(\mathbb{R}))} &\le C(T) \int_0^T \int_{\mathbb{R}} (1+|\xi|^2)^{s+2} \left\|\hat{f}(\xi,\cdot)\right\|^2_{L^\infty(0,T)} \Big(\frac{1-e^{-at|\xi|^2}}{a|\xi|^2}\Big)^2 d\xi\, dt \\ &= C(T) \int_0^T \int_{\mathbb{R}} \frac{(1+|\xi|^2)^2}{a^2|\xi|^4} \big(1-e^{-at|\xi|^2}\big)^2 \big(1+|\xi|^2\big)^s \left\|\hat{f}(\xi,\cdot)\right\|^2_{L^\infty(0,t)} d\xi\, dt.\end{aligned}$$

Since $(1+|\xi|^2)^{s/2} \left\|\hat{f}(\xi,\cdot)\right\|_{L^\infty(0,t)} \in L^2(\mathbb{R})$ it suffices to show that $g(\xi,t) := \frac{(1+|\xi|^2)^2}{a^2|\xi|^4}\big(1-e^{-at|\xi|^2}\big)^2$ is bounded. Note that

$$\lim_{|\xi|\to\infty} g(\xi,t) = 0$$

for all $t \in [0,T]$. To analyse the limit $|\xi| \to 0$ write g in the form

$$g(\xi,t) = t^2(1+|\xi|^2)^2\Big(\frac{1-e^{-at|\xi|^2}}{at|\xi|^2}\Big)^2.$$

The rule of l'Hospital yields

$$\lim_{|\xi|\to 0} \frac{1-e^{-at|\xi|^2}}{at|\xi|^2} = e^{-at|\xi|^2}.$$

It follows that $g(\xi,t)$ is uniformly bounded which finishes the proof. □

F Solving (3.25) by Fourier transformation

Applying the Fourier transform on both sides of equation (3.25) and setting $\mathcal{F}(v(y,t)) =: \hat{v}(\xi,t)$ and $\mathcal{F}(f(y,t)) =: \hat{f}(\xi,t)$ yields

$$\hat{v}_t(\xi,t) + p(\xi,t)\hat{v}(\xi,t) = \hat{f}(\xi,t), \tag{3.35}$$

with initial condition

$$\hat{v}(\xi,0) = \mathcal{F}(v(x,0)) = 0,$$

and

$$p(\xi,t) := \sigma^2/2\,\xi^2 - \Big(r-d-\sigma^2/2+\dot{x}_f(t)\Big)i\xi + r.$$

Inserting

$$\int_0^t p(\xi,s)\,ds = \big(\sigma^2/2\,\xi^2 - \big(r-d-\sigma^2/2\big)i\xi + r\big)t - i\xi\big(x_f(t)-x_f(0)\big) \tag{3.36}$$

in (3.34) yields

$$\hat{v}(\xi,t) = \int_0^t \hat{f}(\xi,s)e^{-\left(\sigma^2/2\,\xi^2-\left(r-d-\sigma^2/2\right)i\xi+r\right)(t-s)+i\xi\left(x_f(t)-x_f(s)\right)}\,ds. \tag{3.37}$$

Applying the inverse Fourier transformation on $V(\xi,t)$ gives the solution $v(y,t)$ of equation (3.25). For brevity set

$$P(\xi,t,s) := e^{-\left(\sigma^2/2\,\xi^2-\left(r-d-\sigma^2/2\right)i\xi+r\right)(t-s)+i\xi\left(x_f(t)-x_f(s)\right)}.$$

Then $\hat{v}$ can be written as

$$\hat{v}(\xi,t) = \int_0^t \hat{f}(\xi,s)P(\xi,s)\,ds.$$

Hence, the inverse Fourier transform of $\hat{v}$ reads

$$\mathcal{F}^{-1}(\hat{v}(\xi,t)) = \int_0^t \mathcal{F}^{-1}\big(\hat{f}(\xi,s)P(\xi,s)\big)\,ds = \frac{1}{\sqrt{2\pi}}\int_0^t \mathcal{F}^{-1}(\hat{f}(\xi,s)) * \mathcal{F}^{-1}(P(\xi,t,s))\,ds. \tag{3.38}$$

The next step is to calculate the inverse Fourier transform of P. Note that P has the form $P(\xi,t) = e^{-a\xi^2+bi\xi+c}$ with $a=\sigma^2/2\,(t-s)$, $b=\big(r-d-\sigma^2/2\big)(t-s)+(x_f(t))-x_f(s))$, and $c=-r(t-s)$. Then,

$$\sqrt{2\pi}\mathcal{F}^{-1}(P(\xi,t)) = \int_{-\infty}^{\infty} e^{-a\xi^2+bi\xi+c}e^{i\xi y}\,d\xi = e^{c-\frac{(b+y)^2}{4a}}\int_{-\infty}^{\infty} e^{-\left(\sqrt{a}\xi+\frac{(b+y)i}{\sqrt{2a}}\right)^2}\,d\xi = \sqrt{\frac{\pi}{a}}e^{c-\frac{(b+y)^2}{4a}}.$$

Consequently the inverse Fourier transform of $P(\xi,t)$ reads

$$\mathcal{F}^{-1}(P(\xi,t,s)) = \frac{e^{-r(t-s)}}{\sigma\sqrt{t-s}}e^{-\frac{1}{2\sigma^2(t-s)}\left((r-d-\sigma^2/2)(t-s)+(x_f(t)-x_f(s))+y\right)^2}. \tag{3.39}$$

Inserting (3.39) in (3.38) yields the solution $v(y,t)=\mathcal{F}^{-1}(\hat{v}(\xi,t))$ of (3.25), namely

$$v(y,t) = \frac{1}{\sigma\sqrt{2\pi}}\int_0^t \frac{e^{-r(t-s)}}{\sqrt{t-s}}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2(t-s)}\left((r-d-\sigma^2/2)(t-s)+(x_f(t)-x_f(s))-w+y\right)^2} f(w,s)\,dw\,ds. \tag{3.40}$$

G Solution and backward transformation

The next step is to insert f from (3.28) in (3.40). Applying the inverse transformations of Transformation 3.18 and 3.19 yields $u(x,t)$, the solution of (3.23), i.e. the value of an

American put option in logarithmic prices and forward time. Using $u(x,t)-(K-e^x)_+ = v(x-x_f(t),t)$ (cf. Transformation 3.18 and 3.19) and f from (3.28) in (3.40) yields

$$\begin{aligned}
&u(x,t)-(K-e^x)_+ = v(x-x_f(t),t)\\
&= \frac{1}{\sigma\sqrt{2\pi}}\int_0^t \frac{e^{-r(t-s)}}{\sqrt{t-s}}\int_0^{\log K - x_f(s)} e^{-\frac{1}{2\sigma^2(t-s)}\left((r-d-\sigma^2/2)(t-s)-x_f(s)-w+x\right)^2} f(w,s)\,dw\,ds\\
&= \frac{\sigma K}{2\sqrt{2\pi}}\int_0^t \frac{e^{-r(t-s)}}{\sqrt{t-s}} e^{-\frac{1}{2\sigma^2(t-s)}\left((r-d-\sigma^2/2)(t-s)-\log K+x\right)^2}\,ds\\
&\quad - rK\frac{1}{\sigma\sqrt{2\pi}}\int_0^t \frac{e^{-r(t-s)}}{\sqrt{t-s}}\int_0^{\log K-x_f(s)} e^{-\frac{1}{2\sigma^2(t-s)}\left((r-d-\sigma^2/2)(t-s)-x_f(s)-w+x\right)^2}\,dw\,ds\\
&\quad + d\frac{1}{\sigma\sqrt{2\pi}}\int_0^t \frac{e^{-r(t-s)+x_f(s)}}{\sqrt{t-s}}\int_0^{\log K-x_f(s)} e^w e^{-\frac{1}{2\sigma^2(t-s)}\left((r-d-\sigma^2/2)(t-s)-x_f(s)-w+x\right)^2}\,dw\,ds.
\end{aligned} \tag{3.41}$$

To simplify (3.41) note that the two spatial integrals are of the form

$$I_1 := \int_0^c e^{-\frac{1}{b}(a-w)^2}\,dw, \quad I_2 := \int_0^c e^{-\frac{1}{b}(a-w)^2+w}\,dw.$$

Integral I_1 can easily be computed by using the transformation $z = {}^1\!/\!\sqrt{b}(a-w)$ and the error function $\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$,

$$I_1 = -\sqrt{b}\int_{a/\sqrt{b}}^{1/\sqrt{b}(a-c)} e^{-z^2}\,dz = \frac{\sqrt{b\pi}}{2}\Big(\operatorname{erf}\Big(\frac{a}{\sqrt{b}}\Big)-\operatorname{erf}\Big(\frac{1}{\sqrt{b}}(a-c)\Big)\Big). \tag{3.42}$$

To calculate I_2 substitute $z = \frac{y}{\sqrt{b}} - \frac{2a+b}{2\sqrt{b}}$. Then

$$I_2 = \sqrt{b}e^{a+b/4}\int_{b+2a/2\sqrt{b}}^{b+2a-2c/2\sqrt{b}} e^{-z^2}\,dz = \frac{\sqrt{b\pi}}{2}e^{a+b/4}\Big(\operatorname{erf}\Big(\frac{b+2a}{2\sqrt{b}}\Big)-\operatorname{erf}\Big(\frac{b+2a-2c}{2\sqrt{b}}\Big)\Big). \tag{3.43}$$

Using I_1 and I_2 with $a = (r-d-\sigma^2/2)(t-s)-x_f(s)+x$, $b = 2\sigma^2(t-s)$, and $c = \log K - x_f(s)$ in (3.41) yields the solution u of (3.23), namely

$$\begin{aligned}
&u(x,t)-(K-e^x)_+ =\\
&= \frac{1}{\sigma\sqrt{2\pi}}\int_0^t \frac{e^{-r(t-s)}}{\sqrt{t-s}} e^{-\frac{1}{2\sigma^2(t-s)}\left((r-d-\sigma^2/2)(t-s)-\log K+x\right)^2}\,ds\\
&- \frac{rK}{2}\int_0^t e^{-r(t-s)}\Big(\operatorname{erf}\Big(\frac{(r-d-\sigma^2/2)(t-s)-x_f(s)+x}{\sqrt{2\sigma^2(t-s)}}\Big)\\
&\qquad\qquad - \operatorname{erf}\Big(\frac{(r-d-\sigma^2/2)(t-s)+x-\log K}{\sqrt{2\sigma^2(t-s)}}\Big)\Big)\,ds\\
&+ \frac{d}{2}\int_0^t e^{-d(t-s)+x}\Big(\operatorname{erf}\Big(\frac{(r-d)(t-s)-x_f(s)+x}{\sqrt{2\sigma^2(t-s)}}\Big)-\operatorname{erf}\Big(\frac{(r-d)(t-s)-\log K+x}{\sqrt{2\sigma^2(t-s)}}\Big)\Big)\,ds.
\end{aligned} \tag{3.44}$$

Remark 3.25. Note that this integral representation for u only holds for $x > x_f(t)$. For $x \le x_f(t)$ the solution u satisfies $u(x,t) = K - e^x$.

Remark 3.26. The free boundary $x_f(t)$ is given as a solution of an integral equation. Its derivation can be found in Goodman and Ostrov [2002], Karatzas and Shreve [1998], Kwok [1998]. In Kwok [1998] the early exercise boundary for American put options is given as the following integral equation, where $N(x)$ denotes the distribution function of the standard normal distribution,

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt. \tag{3.45}$$

Then, $\gamma(t) = \exp(x_f(t))$ satisfies

$$K - \gamma(t) = Ke^{-rt}N(-d_2) - \gamma(t)e^{-dt}N(-d_1) + \int_0^t \big(rKe^{-r\xi}N(-d_{\xi,2}) - d\gamma(t)e^{-d\xi}N(d_{\xi,1})\big)\, d\xi.$$

The parameters d_1, d_2, $d_{\xi,1}$, and $d_{\xi,2}$ are given by

$$\begin{aligned} d_1 &= \frac{\log\big(\gamma(t)/K\big) + (r - d + \sigma^2/2)t}{\sigma\sqrt{t}}, \qquad && d_2 = d_1 - \sigma\sqrt{t},\\ d_{\xi,1} &= \frac{\log\big(\gamma(t)/\gamma(t-\xi)\big) + (r - d + \sigma^2/2)\xi}{\sigma\sqrt{\xi}}, \qquad && d_{\xi,2} = d_{\xi,1} - \sigma\sqrt{\xi}. \end{aligned} \tag{3.46}$$

3.3.2 American Call

This subsection aims at deriving an integral representation for American call options. The procedure is the same as for American put option. Note that the American call and put option differ in the domain, the initial and boundary conditions whereas the differential operator is the same. In contrast to American put options, where early exercise is only optimal in case of a strictly positive interest rate r, American calls are only exercised early, if dividends are paid, i.e. $d > 0$, cf. Subsection 2.2.4. Therefore, we assume that $d > 0$ and consequently $\gamma(\tau) \le C < \infty$ for all τ. The value of an American call for $S \ge \gamma(\tau)$ is known, namely $V(S, \tau) = S - K$.

A Problem formulation

Let S denote the value of the underlying share, $\tau \in [0, T)$ the time, and $V(S, \tau)$ the value of an American call option at time τ and share value S. Further, let $\gamma(\tau) \ge K$ be the early exercise curve (or free boundary). Given the positive constants T, K, r, d, and $\sigma^2/2$, the value $V(S, \tau)$ of an American call option satisfies

$$\begin{aligned} V_\tau + \sigma^2/2\, S^2 V_{SS} + (r-d)SV_S - rV = 0 \quad & \text{for } 0 \le S \le \gamma(\tau),\ \tau \in [0, T),\\ V(S, T) = (S - K)_+ \quad & \text{for } 0 \le S \le \gamma(T),\\ V(\gamma(\tau), \tau) = \gamma(\tau) - K \text{ and } V(0, \tau) = 0 \quad & \text{for } \tau \in [0, T),\\ V_S(\gamma(\tau), \tau) = 1 \quad & \text{for } \tau \in [0, T). \end{aligned} \tag{3.47}$$

B Transformations to a formulation which admits Fourier transformation

As in the case of the American put options the Fourier transform cannot be applied directly to (3.47) owing to the space dependent coefficients and the time-dependent domain. The forthcoming Transformations 3.27-3.29 lead to formulation (3.50) with constant coefficients and spatial domain $\mathbb{R}$.

Transformation 3.27 (leading to forward time and constant coefficients). A logarithmic price and forward time transformation in formulation (3.47), i.e. $S = e^x$, $t = T - \tau$, $u(x,t) := V(e^x, T-\tau)$, and $x_f(t) := \log(\gamma(t))$ yields

$$\begin{aligned} u_t - \sigma^2/2\, u_{xx} - (r-d-\sigma^2/2)u_x + ru = 0 \quad & \text{for } -\infty < x \le x_f(t),\ t \in (0,T], \\ u(x,0) = (e^x - K)_+ \quad & \text{for } -\infty < x < x_f(0), \\ u(x_f(t),t) = e^{x_f(t)} - K \text{ and } \lim_{x\to-\infty} u(x,t) = 0 \quad & \text{for } t \in (0,T], \\ u_x(x_f(t),t) = e^{x_f(t)} \quad & \text{for } t \in (0,T]. \end{aligned} \tag{3.48}$$

Transformation 3.28 (leading to a time-invariant domain). A shift $y := x - x_f(t)$ leads to a time-invariant domain in (3.48). Moreover, the left boundary lies at the origin. Then, $w(y,t) := u(x - x_f(t), t)$ satisfies

$$\begin{aligned} w_t - \sigma^2/2\, w_{yy} - \big(r-d-\sigma^2/2 + \dot{x}_f(t)\big)w_y + rw = 0 \quad & \text{for } -\infty < y \le 0,\ t \in (0,T], \\ w(y,0) = (e^{y+x_f(0)} - K)_+ \quad & \text{for } -\infty < y \le 0, \\ w(0,t) = e^{x_f(t)} - K \text{ and } \lim_{y\to-\infty} w(y,t) = 0 \quad & \text{for } t \in (0,T], \\ w_x(0,t) = e^{x_f(t)} \quad & \text{for } t \in (0,T]. \end{aligned} \tag{3.49}$$

Transformation 3.29 (leading to homogeneous boundary conditions at zero). The extension from $(0,\infty)$ to $\mathbb{R}$ in (3.49) requires homogeneous boundary conditions at zero. Decompose $w(y,t) = v(y,t) + g(y,t)$ such that $v(0,t) = v_y(0,t) = 0$ for all t, i.e. $g(y,t) = (e^{y+x_f(t)} - K)_+$. Then, v satisfies

$$\begin{aligned} v_t - \sigma^2/2\, v_{yy} - \big(r-d-\sigma^2/2 + \dot{x}_f(t)\big)v_y + rv = f(y,t) \quad & \text{for } -\infty < y \le 0,\ t \in (0,T], \\ v(y,0) = 0 \quad & \text{for } -\infty < y \le 0, \\ v(0,t) = 0 \text{ and } \lim_{y\to-\infty} v(y,t) = 0 \quad & \text{for } t \in (0,T], \\ v_y(0,t) = 0 \quad & \text{for } t \in (0,T], \end{aligned} \tag{3.50}$$

with

$$f(y,t) = -g_t + \sigma^2/2\, g_{yy} + (r-d-\sigma^2/2 + \dot{x}_f(t))g_y - rg. \tag{3.51}$$

Remark 3.30. Note that an extension of $v(y,t)$ and $f(y,t)$ by zero for $y > 0$ and all $t \in [0,T]$, yields that the extended v satisfies (3.50) for all $y \in \mathbb{R}$ and all $0 \le t \le T$.

C Calculation of the right-hand side f

Since $g(y,t)$ is continuous, $g_t(y,t)$, $g_y(y,t)$, and $g_{yy}(y,t)$ are understood in the weak sense,

$$g_t = \begin{cases} \dot{x}_f(t)e^{y+x_f(t)} & \text{if } y \geq \log K - x_f(t), \\ 0 & \text{otherwise;} \end{cases} \tag{3.52}$$

$$g_y = \begin{cases} e^{y+x_f(t)} & \text{if } y \geq \log K - x_f(t), \\ 0 & \text{otherwise;} \end{cases} \tag{3.53}$$

$$g_{yy} = \begin{cases} K\delta_{\log K - x_f(t)} + e^{y+x_f(t)} & \text{if } y \geq \log K - x_f(t), \\ 0 & \text{otherwise.} \end{cases} \tag{3.54}$$

Inserting g_t, g_y, and g_{yy} in (3.51) one obtains for $\log K - x_f(t) \leq y$ that

$$\begin{aligned} f(y,t) &= -\dot{x}_f(t)e^{y+x_f(t)} + \sigma^2/2(K\delta_{\log K - x_f(t)} + e^{y+x_f(t)}) + (r - d - \sigma^2/2 + \dot{x}_f(t))e^{y+x_f(t)} \\ &\quad - r(e^{y+x_f(t)} - K) \\ &= \sigma^2/2\, K\delta_{\log K - x_f(t)} + rK - de^{y+x_f(t)}. \end{aligned} \tag{3.55}$$

Note that f was extended by zero for $y > 0$, so the measure f reads

$$f(y,t) = \begin{cases} \sigma^2/2\, K\delta_{\log K - x_f(t)} + rK - de^{y+x_f(t)} & \text{if } \log K - x_f(t) \geq y \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.56}$$

Remark 3.31. Note that f has compact support because $x_f(t) \leq C < \infty$. Since, in addition, the Dirac measure $\delta \in L^\infty(0,T;H^{-s}(\mathbb{R})$ for $s > 1/2$ there holds $f \in L^\infty(0,T;H^{-s}(\mathbb{R}))$ for all $s > 1/2$ and t fixed.

D Solution and backward transformation

Note that after the extension to the spatial domain $\mathbb{R}$ problems (3.25) and (3.50) only differ in the right-hand side f. Hence, the representation of the solution v (3.40) holds literally. Using the backward transformation $u(x,t) - (e^x - K)_+ = v(x - x_f(t), t)$ cf. Transformation 3.28 and 3.29 , the representation of v from (3.40), and the right-hand side f from (3.56) yields

$$\begin{aligned}
u(x,t) - (e^x - K)_+ &= v(x - x_f(t), t) \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_0^t \frac{e^{-r(t-s)}}{\sqrt{t-s}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2(t-s)}\left((r-d-\sigma^2/2)(t-s)-x_f(s)-w+x\right)^2} f(w,s)\, dw\, ds \\
&= \frac{K\sigma}{2\sqrt{2\pi}} \int_0^t \frac{e^{-r(t-s)}}{\sqrt{t-s}} e^{-\frac{1}{2\sigma^2(t-s)}\left((r-d-\sigma^2/2)(t-s)-\log K+x\right)^2} ds \\
&\quad + \frac{rK}{\sigma\sqrt{2\pi}} \int_0^t \frac{e^{-r(t-s)}}{\sqrt{t-s}} \int_{\log K - x_f(s)}^{0} e^{-\frac{1}{2\sigma^2(t-s)}\left((r-d-\sigma^2/2)(t-s)-x_f(s)-w+x\right)^2} dw\, ds \\
&\quad - \frac{d}{\sigma\sqrt{2\pi}} \int_0^t \frac{e^{-r(t-s)+x_f(s)}}{\sqrt{t-s}} \int_{\log K - x_f(s)}^{0} e^{w} e^{-\frac{1}{2\sigma^2(t-s)}\left((r-d-\sigma^2/2)(t-s)-x_f(s)-w+x\right)^2} dw\, ds.
\end{aligned} \tag{3.57}$$

Note that the two spatial integrals of (3.57) are almost the same as in the integral representation of the American put (3.41), only the upper and the lower bounder are interchanged. Consequently using (3.42) and (3.43) in (3.57) yields the solution u of (3.48), namely

$$\begin{aligned}
&u(x,t) - (e^x - K)_+ = \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_0^t \frac{e^{-r(t-s)}}{\sqrt{t-s}} e^{-\frac{1}{2\sigma^2(t-s)}\left((r-d-\sigma^2/2)(t-s)-\log K+x\right)^2} ds \\
&- \frac{rK}{2} \int_0^t e^{-r(t-s)} \Bigg(\operatorname{erf}\left(\frac{(r-d-\sigma^2/2)(t-s) - x_f(s) + x}{\sqrt{2\sigma^2(t-s)}}\right) \\
&\qquad\qquad - \operatorname{erf}\left(\frac{(r-d-\sigma^2/2)(t-s) + x - \log K}{\sqrt{2\sigma^2(t-s)}}\right)\Bigg) ds \\
&+ \frac{d}{2} \int_0^t e^{-d(t-s)+x} \left(\operatorname{erf}\left(\frac{(r-d)(t-s) - x_f(s) + x}{\sqrt{2\sigma^2(t-s)}}\right) - \operatorname{erf}\left(\frac{(r-d)(t-s) - \log K + x}{\sqrt{2\sigma^2(t-s)}}\right)\right) ds.
\end{aligned} \tag{3.58}$$

Remark 3.32. Note that this integral representation for u only holds for $x < x_f(t)$. For $x \geq x_f(t)$ the solution u satisfies $u(x,t) = e^x - K$.

Remark 3.33. The free boundary $x_f(t)$ is given as a solution of an integral equation. Its derivation can be found in Goodman and Ostrov [2002], Karatzas and Shreve [1998], Kwok [1998]. In Kwok [1998] the early exercise boundary for an American call option is given as the following integral equation with $N(x)$ defined in (3.45) on page 31. Then, $\gamma(t) = \exp(x_f(t))$ satisfies

$$\gamma(t) - K = \gamma(t)e^{-dt}N(d_1) - Ke^{-rt}N(d_2) + \int_0^t \left(d\gamma(t)e^{-d\xi}N(d_{\xi,1}) - rKe^{-r\xi}N(d_{\xi,2})\right) d\xi.$$

The parameters d_1, d_2, $d_{\xi,1}$, and $d_{\xi,2}$ are given in (3.46) on page 31.

3.4 Truncation Error Estimates for American Options

In this section we prove truncation error estimates for American call and put options. Recall that the value of an American option is given on an unbounded spatial domain (cf. the variational formulation (3.20) on page 22 and the FBVP (3.23) on page 25 for American put options). For efficient numerical simulation it is necessary to truncate the unbounded domain to a bounded computational domain. Hence an estimation of the truncation error is essential for a relaible a posteriori error analysis in Chapter 5. We investigate the exact quantitative decay behaviour in terms of all involved parameters. Indeed, we explicitely determine a treshold x_N depending only on given financial data which guarantees the exponential decay for $x > x_N$. In other words, we prove the existence of $x_N = x_N(r, d, \sigma^2, K, T)$ such that for all $\kappa > 2\sigma^2 t$ and $x > x_0$

$$|u(x,t)e^{x^2/\kappa}| \leq C < \infty. \tag{3.59}$$

All details may be found in Theorem 3.20 below. To prove this result we proceed as follows. First we prove some preliminary estimates. Then we prove (3.59) for each integral terms in the solution u from (3.44) on page 30. Finally we combine these estimates to obtain (3.59).

3.4.1 Some Calculus

In this subsection we prove some preliminary estimates which we need throughout this section. The first lemma provides sufficient conditions on the coefficients α, β, and γ such that a function $f : \mathbb{R}_{>0} \times [0,T] \to \mathbb{R}$ with

$$f(x,t) := \alpha(t) + \frac{\beta(t)}{x} + \frac{\gamma(t)}{x^2} \tag{3.60}$$

is bounded from below by a positive constant C for x sufficiently large.

Lemma 3.11. *Let $\alpha(t) \geq \alpha_0 > 0$, $\beta(t) \geq \beta_0 \in \mathbb{R}$, $\gamma(t) \geq \gamma_0 \geq 0$, and f as in (3.60). Then there exists a constant C_p such that*

$$f(x,t) \geq C_p > 0 \tag{3.61}$$

for all $t \in [0,T]$ and $x \geq x_N > \max(0, x_p)$, with

$$x_p := \begin{cases} 0 & \text{if } \beta_0^2 - 4\alpha_0\gamma_0 < 0, \\ (2\alpha_0)^{-1}(-\beta_0 + \sqrt{\beta_0^2 - 4\alpha_0\gamma_0}) & \text{if } \beta_0^2 - 4\alpha_0\gamma_0 \geq 0. \end{cases} \tag{3.62}$$

Moreover, with $x^* = -2\frac{\gamma_0}{\beta_0}$ and $\tilde{f}(x) = \alpha_0 + \frac{\beta_0}{x} + \frac{\gamma_0}{x^2}$ the constant C_p in (3.61) reads

$$C_p := \begin{cases} \alpha_0 & \text{for } \beta_0 \geq 0, \\ \tilde{f}(x_N) & \text{for } \beta_0 < 0, \tilde{f}(x^*) \leq 0, \\ \tilde{f}(x^*) & \text{for } \beta_0 < 0, \tilde{f}(x^*) > 0, x_N \leq x^*, \\ \tilde{f}(x_N) & \text{for } \beta_0 < 0, \tilde{f}(x^*) > 0, x_N > x^*. \end{cases} \tag{3.63}$$

Remark 3.34. Note that the condition $\gamma_0 \geq 0$ is not necessary for the result. Since we only require this bound for non-negative functions $\gamma(t)$ we use this condition for simplicity.

Remark 3.35. The subscript p in x_p and C_p stands for put options, because these constants are required in the estimates for American put options.

Proof. Since the coefficients of f are bounded below there holds

$$f(x,t) \geq \alpha_0 + \frac{\beta_0}{x} + \frac{\gamma_0}{x^2} =: \tilde{f}(x).$$

Since $\tilde{f}$ is continuous for $x > 0$ and $\lim_{x\to\infty} \tilde{f}(x) = \alpha_0 > 0$, $\tilde{f}$ is positive for x sufficiently large. Note that $\tilde{f}$ vanishes at $x_{1,2} = \frac{-\beta_0 \pm \sqrt{\beta_0^2 - 4\alpha_0\gamma_0}}{2\alpha_0}$. Consequently, if $\beta_0^2 - 4\alpha_0\gamma_0 \geq 0$, $\tilde{f}$ is positive for $x > \max(x_1, x_2) = \frac{-\beta_0 + \sqrt{\beta_0^2 - 4\alpha_0\gamma_0}}{2\alpha_0} =: x_p$. Otherwise $\tilde{f}$ is positive for all $x > 0$. Consequently, $\tilde{f}(x) > 0$ for $x \geq x_N > \max(0, x_p)$ holds true. It remains to prove (3.63). Since $\tilde{f}'(x) = -\frac{\beta_0}{x^2} - \frac{2\gamma_0}{x^3}$, $\tilde{f}$ is monotone-decreasing for $\beta_0 \geq 0$ and therefore $\tilde{f} > \alpha_0 =: C_p$. If $\beta_0 < 0$, $\tilde{f}$ has a minimum at $x^* = -\frac{2\gamma_0}{\beta_0}$. We distinguish three cases:
(i) $\tilde{f}(x^*) \leq 0$. Then, $\tilde{f}$ is strictly positive for $x > x_N$ and there holds $\tilde{f} > \tilde{f}(x_N) =: C_p$ for all $x > x_N$.
(ii) $\tilde{f}(x^*) > 0$ and $x_N \leq x^*$, i.e. the minima $x^* \in [x_N, \infty)$. Then, $\tilde{f} \geq \tilde{f}(x^*) =: C_p$ for all $x > x_N$.
(iii) $\tilde{f}(x^*) > 0$ and $x_N > x^*$. Then $\tilde{f}(x) \geq f(x_N) =: C_p$ for all $x > x_N$. □

The next lemma gives a similar result for the function (3.60) for strictly negative x.

Lemma 3.12. *Let $f : \mathbb{R}_{<0} \times [0,T] \to \mathbb{R}$ be defined by*

$$f(x,t) := \alpha(t) + \frac{\beta(t)}{x} + \frac{\gamma(t)}{x^2}. \tag{3.64}$$

If $\alpha(t) \geq \alpha_0 > 0$, $\beta(t) \leq \beta_0 \in \mathbb{R}$, and $\gamma(t) \geq \gamma_0 \geq 0$ there exists a constant C_c such that

$$f(x,t) \geq C_c > 0 \tag{3.65}$$

for all $t \in [0,T]$ and $x \leq x_N < \min(0, x_c)$ with

$$x_c := \begin{cases} 0 & \text{if } \beta_0^2 - 4\alpha_0\gamma_0 < 0, \\ (2\alpha_0)^{-1}(-\beta_0 - \sqrt{\beta_0^2 - 4\alpha_0\gamma_0}) & \text{if } \beta_0^2 - 4\alpha_0\gamma_0 \geq 0. \end{cases} \tag{3.66}$$

Moreover, with $x^* = -2\frac{\gamma_0}{\beta_0}$ and $\tilde{f}(x) = \alpha_0 + \frac{\beta_0}{x} + \frac{\gamma_0}{x^2}$ the constant C_c is given by

$$C_c := \begin{cases} \alpha_0 & \text{for } \beta_0 \leq 0, \\ \tilde{f}(x_N) & \text{for } \beta_0 > 0, \tilde{f}(x^*) \leq 0, \\ \tilde{f}(x^*) & \text{for } \beta_0 > 0, \tilde{f}(x^*) > 0, x_N \geq x^*, \\ \tilde{f}(x_N) & \text{for } \beta_0 > 0, \tilde{f}(x^*) > 0, x_N < x^*. \end{cases} \tag{3.67}$$

Proof. Since $\beta(t)$ is bounded from above and $\alpha(t)$ and $\gamma(t)$ are bounded from below there holds

$$f(x,t) \geq \alpha_0 + \frac{\beta_0}{x} + \frac{\gamma_0}{x^2} =: \tilde{f}(x).$$

Using the same arguments as in the proof of Lemma 3.11 and taking into account that we consider now $x < 0$ leads to (3.66) and (3.67). □

Remark 3.36. The subscript c in x_c and C_c stands for call options, because these constants are required in the estimates for American call options.

The next two lemmas bound the difference of two error functions by the exponential function. The first one plays an important rule in the analysis for American put option.

Lemma 3.13. *Let* $0 < \alpha < \beta \in \mathbb{R}$ *and*

$$\operatorname{erf}(x) := \frac{2}{\pi} \int_0^x e^{-t^2}\, dt.$$

Then,

$$\operatorname{erf}(\beta) - \operatorname{erf}(\alpha) \le \frac{2e^{-\alpha^2}}{\pi\alpha}. \tag{3.68}$$

Proof. The assertion follows from

$$\operatorname{erf}(\beta) - \operatorname{erf}(\alpha) = \frac{2}{\pi} \int_\alpha^\beta e^{-t^2}\, dt \le \frac{2}{\pi} \int_\alpha^\infty e^{-\alpha t}\, dt = \frac{2e^{-\alpha^2}}{\pi\alpha}. \qquad \square$$

The subsequent lemma is required in the analysis of the decay behaviour of American call options.

Lemma 3.14. *Let* $\beta < \alpha < 0 \in \mathbb{R}$, $\operatorname{erf}(x) := 2/\pi \int_0^x e^{-t^2}\, dt$. *Then,*

$$\operatorname{erf}(\alpha) - \operatorname{erf}(\beta) \le -\frac{2e^{-\alpha^2}}{\pi\alpha}. \tag{3.69}$$

Proof. The assertion follows from

$$\operatorname{erf}(\alpha) - \operatorname{erf}(\beta) = \frac{2}{\pi} \int_\beta^\alpha e^{-t^2}\, dt \le \frac{2}{\pi} \int_{-\infty}^\alpha e^{-\alpha t}\, dt = -\frac{2e^{-\alpha^2}}{\pi\alpha}. \qquad \square$$

The next lemma states explicit formulae of some integrals arising below.

Lemma 3.15. *For* $t > 0$ *there holds*

$$\int_0^t \sqrt{\tau} e^{-\frac{x^2}{\tau}}\, d\tau = \frac{2}{3}\sqrt{t} e^{-\frac{x^2}{t}}(t - 2x^2) + \frac{4\sqrt{\pi}}{3} x^3 \Big(1 - \operatorname{erf}(\frac{x}{\sqrt{t}})\Big), \tag{3.70}$$

$$\int_0^t \frac{e^{-\frac{x^2}{\tau}}}{\sqrt{\tau}}\, d\tau = 2\sqrt{t} e^{-\frac{x^2}{t}} + 2x\sqrt{\pi}\Big(\operatorname{erf}\,(\frac{x}{\sqrt{t}}) - 1\Big), \tag{3.71}$$

$$\int_0^t \frac{e^{-\frac{x^2}{\tau}}}{\tau^{3/2}}\, du = \frac{\sqrt{\pi}}{x}\Big(1 - \operatorname{erf}(\frac{x}{\sqrt{t}})\Big), \tag{3.72}$$

$$\int_0^t \frac{e^{-\frac{x^2}{\tau}}}{\tau^{5/2}}\, d\tau = \frac{\sqrt{\pi}}{2x^3}\Big(1 - \operatorname{erf}(\frac{x}{\sqrt{t}})\Big) + \frac{1}{\sqrt{t}x^2} e^{-\frac{x^2}{t}}. \tag{3.73}$$

Proof. The above integrals can be evaluated by suitable substitutions or using maple. $\square$

The next lemma will be frequently used to show that some constants are uniformly bounded.

Lemma 3.16. *For any positive integer k holds*

$$\lim_{x\to\pm\infty} x^k\big(\operatorname{erf}(x)-1\big)=0$$

Proof. The rule of l'Hospital yields

$$\lim_{x\to\pm\infty} x^k\big(\operatorname{erf}(x)-1\big)=\lim_{x\to\pm\infty}\frac{\operatorname{erf}(x)-1}{x^{-k}}=\lim_{x\to\pm\infty}-\frac{2}{k\pi}\frac{e^{-x^2}}{x^{-k-1}}=0. \qquad \square$$

3.4.2 Decay Behaviour for American Put Options

This section aims at analysing the behaviour of $u(x,t)$ for $x\to\infty$ for American put options. Recall that the integral representation (3.44) on page 30 only holds for $x>x_f(t)$. For brevity we split the integral representation for the option value into

$$u(x,t)-(K-e^x)_+ =: \frac{1}{\sigma\sqrt{2\pi}}I_1(x,t)-\frac{rK}{2}I_2(x,t)+\frac{d}{2}I_3(x,t). \tag{3.74}$$

With $a:=2\sigma^2>0$, $b:=r-d-\sigma^2/2$, $c:=r-d$, and $e:=\log K$ there holds

$$I_1(x,t):=\int_0^t \frac{e^{-r(t-s)}}{\sqrt{t-s}}e^{-\frac{1}{a(t-s)}\left(b(t-s)-e+x\right)^2}\,ds, \tag{3.75}$$

$$I_2(x,t):=\int_0^t e^{-r(t-s)}\Big(\operatorname{erf}\Big(\frac{b(t-s)-x_f(s)+x}{\sqrt{a(t-s)}}\Big)-\operatorname{erf}\Big(\frac{b(t-s)+x-e}{\sqrt{a(t-s)}}\Big)\Big)\,ds, \tag{3.76}$$

$$I_3(x,t):=\int_0^t e^{-d(t-s)+x}\Big(\operatorname{erf}\Big(\frac{c(t-s)-x_f(s)+x}{\sqrt{a(t-s)}}\Big)-\operatorname{erf}\Big(\frac{c(t-s)-e+x}{\sqrt{a(t-s)}}\Big)\Big)\,ds. \tag{3.77}$$

Note that $r>0$ and $d\geq 0$.

The next three propositions prove for $j=1,2,3$ that

$$|I_j(x,t)\exp(x^2/\kappa)|\leq C. \tag{3.78}$$

Remark 3.37. The proofs are organised as follows.

(i) Bound the difference of two error functions appearing in I_2 and I_3 in terms of the exponential function by using Lemma 3.13 on page 37.

(ii) Factorise $-\frac{x^2}{u}$ in the exponent.

(iii) Bound the remainder by means of Lemma 3.11 on page 35 .

(iv) The resulting integral is such that Lemma 3.15 on page 37 applies.

For brevity we introduce the following notation. These constant will always appear in connection with I_1 and I_2 as in the two Propositions 3.17 and 3.18 and later in Theorem 3.20.

Notation 3.38. Let $\alpha_0^{(1)}$, $\beta_0^{(1)}$, and $\gamma_0^{(1)}$ be defined by

$$\alpha_0^{(1)} = \frac{\kappa - at}{a\kappa}, \quad \beta_0^{(1)} = \min\Big(-\frac{2e}{a}, \quad \frac{2(bt-e)}{a}\Big), \quad \gamma_0^{(1)} = \min\Big(\frac{e^2}{a}, \frac{(bt-e)^2 + art^2}{a}\Big). \tag{3.79}$$

Then, with $\alpha_0 = \alpha_0^{(1)}$, $\beta_0 = \beta_0^{(1)}$, and $\gamma_0 = \gamma_0^{(1)}$ we recall that

$$x_p^{(1)} := \begin{cases} 0 & \text{if } \beta_0^2 - 4\alpha_0\gamma_0 < 0, \\ (2\alpha_0)^{-1}(-\beta_0 + \sqrt{\beta_0^2 - 4\alpha_0\gamma_0}) & \text{if } \beta_0^2 - 4\alpha_0\gamma_0 \geq 0. \end{cases} \tag{3.80}$$

With some $x_N > 0$, $x^* = -2\frac{\gamma_0}{\beta_0}$, and $\tilde{f}(x) = \alpha_0 + \frac{\beta_0}{x} + \frac{\gamma_0}{x^2}$ the constant C_p from (3.63) reads

$$C_{p,1} := \begin{cases} \alpha_0 & \text{for } \beta_0 \geq 0, \\ \tilde{f}(x_N) & \text{for } \beta_0 < 0,\ \tilde{f}(x^*) \leq 0, \\ \tilde{f}(x^*) & \text{for } \beta_0 < 0,\ \tilde{f}(x^*) > 0,\ x_N \leq x^*, \\ \tilde{f}(x_N) & \text{for } \beta_0 < 0,\ \tilde{f}(x^*) > 0,\ x_N > x^*. \end{cases} \tag{3.81}$$

With Notation 3.38 and $I_1(x,t)$ from (3.75) we are able to formulate the decay behaviour for I_1.

Proposition 3.17 ($I_1(x,t)$). *Let*

$$x_N^{(1)} > \max\Big(0, x_p^{(1)}\Big). \tag{3.82}$$

Then, there is a universal constant $C > 0$ so that there holds for $x \geq x_N^{(1)}$, $t > 0$, and each $\kappa > at$

$$|I_1(x,t)\exp(x^2/\kappa)| \leq C. \tag{3.83}$$

.

Remark 3.39. More precisely, the proof shows that for $I_1 = I_1(x,t)$ with $x_N := x_N^{(1)}$ and $C := C_{p,1}$ as defined in (3.81)

$$|I_1(x,t)\exp(x^2/\kappa)| \leq 2\sqrt{t}e^{-\frac{Cx^2}{t}} + 2x\sqrt{C\pi}\Big(\operatorname{erf}\big(\frac{\sqrt{C}x}{\sqrt{t}}\big) - 1\Big), \tag{3.84}$$

which is uniformely bounded in x and t.

For the sake of completeness we give a detailed proof for the decay behaviour of I_1. Since all the proofs in this section follow the same scheme, cf. Remark 3.37, we will later omit details.

Proof. Use the definition of I_1 , substitute $\tau = t - s$ and factorise $-\frac{x^2}{\tau}$ in the exponent. This yields

$$\begin{aligned} |I_1(x,t)\exp(x^2/\kappa)| &= \int_0^t \frac{1}{\sqrt{\tau}} \exp\Big(- \frac{\big(b\tau - e + x\big)^2}{a\tau} + \frac{x^2}{\kappa} - r\tau\Big)\, d\tau \\ &= \int_0^t \frac{1}{\sqrt{\tau}} \exp\Big(- \frac{x^2}{\tau}\Big(\frac{\kappa - a\tau}{\kappa a} + \frac{2(b\tau - e)}{ax} + \frac{(b\tau - e)^2 + ar\tau^2}{ax^2}\Big)\Big)\, d\tau. \end{aligned} \tag{3.85}$$

Set $f(x,\tau) = \frac{\kappa - a\tau}{\kappa a} + \frac{2(b\tau-e)}{ax} + \frac{(b\tau-e)^2+ar\tau^2}{ax^2}$ and $\alpha(\tau) = \frac{\kappa-a\tau}{\kappa a}$, $\beta(\tau) = \frac{2(b\tau-e)}{a}$, and $\gamma(\tau) = \frac{(b\tau-e)^2+ar\tau^2}{a}$. Note that $\alpha(\tau)$, $\beta(\tau)$, and $\gamma(\tau)$ are bounded from below by $\alpha_0^{(1)}$, $\beta_0^{(1)}$, and $\gamma_0^{(1)}$ from Notation 3.38, respectively. By means of Lemma 3.11 $f(x,\tau) \geq C_{p,1}$ for $x \geq x_N^{(1)}$ holds true. Using this and (3.71) in (3.85) yields for $\kappa > at$ and $x \geq x_N^{(1)}$ with $C := C_{p,1}$

$$|I_1(x,t)\exp(x^2/\kappa)| \leq \int_0^t \frac{e^{-\frac{Cx^2}{\tau}}}{\sqrt{\tau}}\, d\tau = 2\sqrt{t}e^{-\frac{Cx^2}{t}} + 2x\sqrt{\pi C}\Big(\operatorname{erf}\big(\frac{\sqrt{C}x}{\sqrt{t}}\big) - 1\Big) \tag{3.86}$$

which is uniformly bounded since $C_{p,1} > 0$ and Lemma 3.16 applies. □

Next we show (3.78) for I_2. Since the exponential part in the integral in the second line in (3.90) is the same as the exponential part in the first line in (3.85) we may apply Lemma 3.11 with the same constants as in Notation 3.38.

Proposition 3.18 ($I_2(x,t)$). *Let $x_N^{(1)} > \max(0, x_p^{(1)})$. Then, there is a universal constant $C > 0$ so that there holds for all*

$$x \geq \max(x_N^{(1)}, e + |b|t + \varepsilon) \tag{3.87}$$

with $\varepsilon > 0$, $t > 0$, and each $\kappa > at$

$$|I_2(x,t)\exp(x^2/\kappa)| \leq C. \tag{3.88}$$

Remark 3.40. More precisely, the proof shows that for $I_2 = I_2(x,t)$ with $x_N := x_N^{(1)}$ and $C := C_{p,1}$ as defined in (3.81)

$$|I_2(x,t)\exp(x^2/\kappa)| \leq \frac{2\sqrt{a}}{\varepsilon\pi}\Big(\frac{2}{3}\sqrt{t}e^{-\frac{Cx^2}{t}}(t - 2Cx^2) + \frac{4\sqrt{\pi C^3}}{3}x^3\Big(1 - \operatorname{erf}(\frac{\sqrt{C}x}{\sqrt{t}})\Big)\Big), \tag{3.89}$$

which is uniformly bounded in x and t.

Proof. Use the definition of I_2, substitute $\tau = t - s$ and apply Lemma 3.13. Then factorising $-\frac{x^2}{\tau}$ in the exponent and applying Lemma 3.11 to the remainder yields for $\kappa > at$ and $x > e + |b|t + \varepsilon$, $\varepsilon > 0$ with $C := C_{p,1}$

$$\begin{aligned} &|I_2(x,t)\exp(x^2/\kappa)| \\ &\quad = \int_0^t \exp\big(- r(t-s) + x^2/\kappa\big)\Big(\operatorname{erf}\Big(\frac{b(t-s) - x_f(s) + x}{\sqrt{a(t-s)}}\Big) - \operatorname{erf}\Big(\frac{b(t-s) - e + x}{\sqrt{a(t-s)}}\Big)\Big)\, ds \\ &\quad \leq \frac{2\sqrt{a}}{\pi} \int_0^t \frac{\sqrt{\tau}}{b\tau + x - e} \exp\Big(- \frac{\big(b\tau - e + x\big)^2}{a\tau} + \frac{x^2}{\kappa} - r\tau\Big)\, d\tau \\ &\quad \leq \frac{2\sqrt{a}}{\varepsilon\pi} \int_0^t \sqrt{\tau}e^{-\frac{Cx^2}{\tau}}\, d\tau. \end{aligned} \tag{3.90}$$

With (3.70) from Lemma 3.15 one obtains (3.89), which is uniformly bounded since $C_{p,1} > 0$ and Lemma 3.16 applies. $\square$

Since the constants appearing in the error function in I_3 differ from those appearing in the error function in I_2 we need other constants in Lemma 3.11. Therefore, we introduce some notation which is used always in connection with I_3 as in the next Proposition 3.19 and later in Theorem 3.20.

Notation 3.41. Let $\alpha_0^{(3)}$, $\beta_0^{(3)}$, and $\gamma_0^{(3)}$ be defined by

$$\alpha_0^{(3)} = \frac{\kappa - at}{a\kappa}, \quad \beta_0^{(3)} = \min\Big(-\frac{2e}{a}, \quad \frac{2(ct-e)}{a}\Big), \quad \gamma_0^{(3)} = \min\Big(\frac{e^2}{a}, \frac{(ct-e)^2 + adt^2}{a}\Big). \tag{3.91}$$

Then, with $\alpha_0 = \alpha_0^{(3)}$, $\beta_0 = \beta_0^{(3)}$, and $\gamma_0 = \gamma_0^{(3)}$ we recall that

$$x_p^{(3)} := \begin{cases} 0 & \text{if } \beta_0^2 - 4\alpha_0\gamma_0 < 0, \\ (2\alpha_0)^{-1}(-\beta_0 + \sqrt{\beta_0^2 - 4\alpha_0\gamma_0}) & \text{if } \beta_0^2 - 4\alpha_0\gamma_0 \geq 0. \end{cases} \tag{3.92}$$

With some $x_N > 0$, $x^* = -2\frac{\gamma_0}{\beta_0}$, and $\tilde{f}(x) = \alpha_0 + \frac{\beta_0}{x} + \frac{\gamma_0}{x^2}$ the constant C_p from (3.63) by

$$C_{p,3} := \begin{cases} \alpha_0 & \text{for } \beta_0 \geq 0, \\ \tilde{f}(x_N) & \text{for } \beta_0 < 0,\ \tilde{f}(x^*) \leq 0, \\ \tilde{f}(x^*) & \text{for } \beta_0 < 0,\ \tilde{f}(x^*) > 0,\ x_N \leq x^*, \\ \tilde{f}(x_N) & \text{for } \beta_0 < 0,\ \tilde{f}(x^*) > 0,\ x_N > x^*. \end{cases} \tag{3.93}$$

With Notation 3.41 and $I_3(x,t)$ from (3.77) we formulate the decay behaviour for I_3.

Proposition 3.19 ($I_3(x,t)$). *Let*

$$x_N^{(3)} > \max\Big(0, x_p^{(3)}\Big). \tag{3.94}$$

Then, there is a universal constant $C > 0$ so that there holds for $x \geq \max(x_N^{(3)}, e + |c|t + \varepsilon)$ with $\varepsilon > 0$, $t > 0$, and each $\kappa > at$

$$|I_3(x,t)\exp(x^2/\kappa)| \leq C. \tag{3.95}$$

Remark 3.42. More precisely, the proof shows that for $I_3 = I_3(x,t)$ with $x_N := x_N^{(3)}$ and $C := C_{p,3}$ as defined in (3.93)

$$|I_3(x,t)\exp(x^2/\kappa)| \leq \frac{2\sqrt{a}}{\pi\varepsilon}\Big(\frac{2}{3}\sqrt{t}e^{-\frac{Cx^2}{u}}(t-2x^2) + \frac{4\sqrt{\pi C^3}}{3}x^3\Big(1 - \operatorname{erf}(\frac{\sqrt{C}x}{\sqrt{t}})\Big)\Big) \tag{3.96}$$

which is uniformly bounded in x and t.

Proof. Use the definition of I_3, substitute $\tau = t-s$ and apply Lemma 3.13. Then, factorising $-\frac{x^2}{\tau}$ in the exponent and applying Lemma 3.11 to the remainder yields for $\kappa > at$ and

$x > e + |c|t + \varepsilon$, $\varepsilon > 0$ with $C := C_{p,3}$

$$
\begin{aligned}
&|I_3(x,t)\exp(x^2/\kappa)| \\
&= \int_0^t \exp\big(-d(t-s) + x + x^2/\kappa\big)\Big(\operatorname{erf}\Big(\frac{c(t-s) - x_f(s) + x}{\sqrt{a(t-s)}}\Big) - \operatorname{erf}\Big(\frac{c(t-s) - e + x}{\sqrt{a(t-s)}}\Big)\Big)\, ds \\
&\leq \frac{2\sqrt{a}}{\pi}\int_0^t \exp\Big(-\frac{\big(c\tau - e + x\big)^2}{a\tau} + x + \frac{x^2}{\kappa} - d\tau\Big)\frac{\sqrt{\tau}}{c\tau + x - e}\, d\tau \\
&\leq \frac{2\sqrt{a}}{\pi\varepsilon}\int_0^t \sqrt{\tau}\exp\Big(-\frac{x^2}{\tau}\Big(\frac{\kappa - a\tau}{\kappa a} + \frac{2(c\tau - e) - au}{ax} + \frac{(c\tau - e)^2 + ad\tau^2}{ax^2}\Big)\Big)\, d\tau \\
&\leq \frac{2\sqrt{a}}{\pi\varepsilon}\int_0^t \sqrt{\tau} e^{-\frac{Cx^2}{\tau}}\, d\tau.
\end{aligned}
$$

With (3.70) one obtains (3.96), which is bounded since $C_{p,3} > 0$ and Lemma 3.16 applies.□

Combining the last three propositions which gave a decay behaviour for I_1, I_2, and I_3 yields an estimate for $u(x,t)$ of the form (3.59). With Notation 3.38 and 3.41, the definition of $u(x,t)$ from (3.74), and $x_N^{(1)}$ and $x_N^{(3)}$ from (3.82) and (3.94), respectively, we formulate the theorem describing the rate of decay for the solution $u(x,t)$ of an American put option.

Theorem 3.20. *There exists a universal constant $C > 0$ so that there holds for all*

$$
x > \max\Big(x_N^{(1)}, x_N^{(3)}, e + \max\big((|c| + |b|)t\big) + \varepsilon\Big) =: x_N^P, \tag{3.97}
$$

$\varepsilon > 0$, $t > 0$ and each $\kappa >$ at that

$$
|u(x,t)\exp(x^2/\kappa)| \leq C. \tag{3.98}
$$

Proof. Since $e = \log K$ there holds $x > \log K$ (cf. (3.97)) and therefore $(K - e^x)_+ = 0$ in the representation for u (3.74). Combining the estimates for I_1, I_2, and I_3 from Proposition 3.17, 3.18, and 3.19, respectively yields (3.98). □

Remark 3.43. The constant C in (3.98) can be explicitely determined on behalf of Remark 3.39, 3.40, and 3.42.

3.4.3 Decay Behaviour of the First and Second Spatial Derivative

In the last subsection we proved the decay behaviour for solutions $u(x,t)$ for American put options as $x \to \infty$. Theorem 3.20 provides an explicit threshold x_N depending only on given financial data which guarantees exponential decay for the solution for $x > x_N$. This suggests to set the numerical solution $u(x,t) = 0$ for $x > x_N$. Given an error tolerance $\varepsilon > 0$ we determine an explicit x_N such that some norm of truncation error is smaller than ε, i.e. $\|u\|_{(x_N,\infty)} \leq \varepsilon$. Since we need in Chapter 5 Sobolev norms of the solution u on (x_N,∞) this subsection aims to determine the decay behaviour for the first and second spatial derivative, i.e. we prove that for $j = 1,2$ there exists some x_N fully determined by given financial data so that there holds for $x > x_N$

$$
\Big|\frac{\partial^j u}{\partial x^j}\exp(x^2/\kappa)\Big| \leq C. \tag{3.99}
$$

Consequently the first and second spatial derivatives of I_1, I_2, and I_3 are required. First we verify that we are allowed to interchange the order of differentiation and integration.

Remark 3.44. The class of integrands we consider is covered by functions $\Phi_1(x,t)$ and $\Phi_2(x,t)$ defined below. For a smooth function $g(x,t)$ satisfying $g(x,t) \geq \lambda_0 > 0$ and $|g_x(x,t)| \leq \Lambda_0 < \infty$ for $0 \leq t \leq u_0$ and $r_0 \leq x \leq R_0$ and a smooth function $h(x,t)$ satisfying $|h(x,t)| \leq C_1$ and $|h_x(x,t)| \leq C_2$ for $r_0 \leq x \leq R_0$ define $\Phi^1(\cdot,t) \in C^1(r_0,R_0)$ by

$$\Phi_1(x,t) = h(x,t)\frac{\exp\big(-\frac{1}{t}(g(x,t))\big)}{t^\alpha}.$$

Then,

$$\begin{aligned}|\partial_x\Phi_1(x,t)| &= \Big|h_x(x,t)\frac{\exp\big(-\frac{1}{t}(g(x,t))\big)}{t^\alpha} + h(x,t)\frac{\exp\big(-\frac{1}{t}(g(x,t))\big)}{t^{\alpha+1}}g_x(x,t)\Big| \\ &\leq C_2\frac{\exp(-\frac{\lambda_0}{t})}{t^\alpha} + C_1\Lambda_0\frac{\exp(-\frac{\lambda_0}{t})}{t^{\alpha+1}} \in L^1(0,u_0).\end{aligned}$$

Take $g(x,t)$ and $h(x,t)$ as above. Define $\Phi_2(\cdot,t) \in C^1(r_0,R_0)$ by

$$\Phi_2(x,t) = h(x,t)\operatorname{erf}\Big(\frac{g(x,t)}{t^\alpha}\Big).$$

Then,

$$\begin{aligned}|\partial_x\Phi_2(x,t)| &= \Big|\frac{2}{\sqrt{\pi}}h(x,t)t^\alpha\exp\big(-\frac{g(x,t)^2}{t^{2\alpha}}\big) + h_x(x,t)\operatorname{erf}\Big(\frac{g(x,t)}{t^\alpha}\Big)\Big| \\ &\leq C\Bigg(t^{-\alpha}\exp\Big(-\frac{\lambda_0^2}{t^{2\alpha}}\Big) + \operatorname{erf}\Big(\frac{g(x,t)}{t^\alpha}\Big)\Bigg) \in L^1(0,u_0).\end{aligned}$$

Consequently, there holds for $j = 1,2$

$$\frac{\partial}{\partial x}\int_0^{u_0}\Phi_j(x,t)\,dt = \int_0^{u_0}\frac{\partial}{\partial x}\Phi_j(x,t)\,dt,$$

i.e., we are allowed to interchange integration and differentiation for $r_0 \leq x \leq R_0$.

The proofs follow the same scheme as in the last subsection, cf. Remark 3.37. Since the exponents in I_1 and $\partial_x I_1$ are equal (cf. Φ_1 and $\partial_x\Phi_1$ in Remark 3.44), we may apply Lemma 3.11 with the same constants $\alpha_0^{(1)}$, $\beta_0^{(1)}$, and $\gamma_0^{(1)}$ defined in Notation 3.38 on page 39. With I_1 from (3.75) we formulate the decay behaviour for $\partial_x I_1(x,t)$ as follows.

Proposition 3.21 ($\boldsymbol{\partial_x I_1(x,t)}$). *Let*

$$x_N^{(1)} > \max\Big(0, x_p^{(1)}\Big). \tag{3.100}$$

Then, there exists a universal constant $C > 0$ so that there holds for $x \geq \max(x_N^{(1)}, e + |b|t)$, $t > 0$, and each $\kappa > at$

$$\Big|\frac{\partial I_1}{\partial x}\exp(x^2/\kappa)\Big| \leq C. \tag{3.101}$$

Remark 3.45. More precisely, the proof shows that for $I_1 = I_1(x,t)$ with $x_N := x_N^{(1)}$ and $C := C_{p,1}$ defined in (3.81), the weighted derivative

$$\Big|\frac{\partial I_1}{\partial x}\exp(x^2/\kappa)\Big| \leq \frac{2}{a}\Big(2|b|\sqrt{t}e^{-\frac{Cx^2}{t}} + \big(2|b|\sqrt{C\pi}x - \frac{\pi|x-e|}{\sqrt{C}x}\big)\big(\operatorname{erf}(\frac{\sqrt{C}x}{\sqrt{t}}) - 1\big)\Big) \tag{3.102}$$

is uniformly bounded in x and t.

Proof. Since $g(x,\tau) = (b\tau - e + x)^2$ and $h(x,\tau) = \exp(-r\tau)$ satisfy all the conditions on g and h in Remark 3.44 for $r_0 > e - b\tau$ and arbitrary $R_0 > r_0$ we may interchange the order of integration and differentiation for $x > e + |b|t$. Taking the derivative of I_1 from (3.75), substituting $\tau = t - s$, factorising $-\frac{x^2}{\tau}$ in the exponent and applying Lemma 3.11 yields with $C := C_{p,1}$

$$\begin{aligned}\Big|\frac{\partial I_1}{\partial x}\exp(x^2/\kappa)\Big| &= \Big|\frac{2}{a}\int_0^t \frac{b\tau - e + x}{\tau^{3/2}}\exp\Big(-\frac{\big(b\tau - e + x\big)^2}{a\tau} + \frac{x^2}{\kappa} - r\tau\Big)\,d\tau\Big| \\ &\leq \frac{2}{a}\int_0^t \frac{|b\tau - e + x|}{\tau^{3/2}}e^{-\frac{Cx^2}{\tau}}\,d\tau.\end{aligned}$$

Using the triangle inequality and (3.71) and (3.72) yields (3.102), which is uniformly bounded since $C_{p,1} > 0$ and Lemma 3.16 on page 38 holds true. □

Since we consider now $\partial_x I_2$ we make use of Notation 3.38 on page 39 .

Proposition 3.22 ($\partial_x I_2(x,t)$). *Let*

$$x_N^{(1)} > \max\Big(0, x_p^{(1)}\Big). \tag{3.103}$$

Then, there is a universal constant $C > 0$ so that there holds for $x > \max(x_N^{(1)}, e + |b|t)$, $t > 0$, and each $\kappa > at$

$$\Big|\frac{\partial I_2}{\partial x}\exp(x^2/\kappa)\Big| \leq C. \tag{3.104}$$

Remark 3.46. More precisely, the proof shows that for $I_2 = I_2(x,t)$ with $x_N := x_N^{(1)}$ and $C := C_{p,1}$ defined in (3.81), the weighted derivative

$$\Big|\frac{\partial I_2}{\partial x}\exp(x^2/\kappa)\Big| \leq \frac{8}{\sqrt{a\pi}}\Big(\sqrt{t}e^{-\frac{Cx^2}{t}} + x\sqrt{\pi C}\Big(\operatorname{erf}\big(\frac{\sqrt{C}x}{\sqrt{t}}\big) - 1\Big)\Big) \tag{3.105}$$

is uniformly bounded in x and t.

Proof. According to Remark 3.44 we may interchange the order of differentiation and integration for $x > e + |b|t$ to calculate $\partial_x I_2(x,t)$,

$$\begin{aligned}&\frac{\partial I_2(x,t)}{\partial x} \\ &= \int_0^t \frac{2e^{-r(t-s)}}{\pi\sqrt{a(t-s)}}\left(\exp\Big(-\frac{(b(t-s) - x_f(s) + x)^2}{a(t-s)}\Big) - \exp\Big(-\frac{(b(t-s) + x - e)^2}{a(t-s)}\Big)\right)ds.\end{aligned}$$

Since $x_f(s) \leq \log K = e$ for all $s \in [0,t]$, we may bound the absolute value of the difference of the two exponential functions by twice the exponent containing e. Substituting $\tau = t - s$, factorising $-\frac{x^2}{\tau}$ in the exponent and applying Lemma 3.11 to the remainder yields with $C := C_{p,1}$

$$\begin{aligned}\Big|\frac{\partial I_2}{\partial x}\exp(x^2/\kappa)\Big| &\leq \frac{4}{\sqrt{a\pi}}\Big|\int_0^t \frac{1}{\sqrt{\tau}}\exp\Big(-\frac{(b\tau - e + x)^2}{a\tau} + \frac{x^2}{\kappa} - r\tau\Big)\,d\tau\Big| \\ &\leq \frac{4}{\sqrt{a\pi}}\int_0^t \frac{1}{\sqrt{\tau}}e^{-\frac{Cx^2}{\tau}}\,d\tau.\end{aligned}$$

Using (3.71) yields (3.105), which is uniformly bounded since $C_{p,1} > 0$ and Lemma 3.16 applies. □

Since I_3 is under consideration, Notation 3.41 on page 41 is used.

Proposition 3.23 ($\partial_x I_3(x,t)$). *Let*

$$x_N^{(3)} > \max\Big(0, x_p^{(3)}\Big). \tag{3.106}$$

Then, there is a universal constant $C > 0$ so that there holds for $x \geq \max(x_N^{(3)}, e + |c|t + \varepsilon)$ with $\varepsilon > 0$, $t > 0$, and each $\kappa > at$

$$\Big|\frac{\partial I_3}{\partial x}\exp(x^2/\kappa)\Big| \leq C. \tag{3.107}$$

Remark 3.47. More precisely, the proof shows that for $I_{3,2} = I_{3,2}(x,t)$ with $x_N := x_N^{(3)}$ and $C := C_{p,3}$ defined in (3.93),

$$|I_{3,2}\exp(x^2/\kappa)| \leq \frac{4}{\sqrt{a}}\Big(\sqrt{t}e^{-\frac{Cx^2}{t}} + x\sqrt{C\pi}\Big(\operatorname{erf}\big(\frac{\sqrt{C}x}{\sqrt{t}}\big) - 1\Big)\Big) \tag{3.108}$$

is uniformly bounded in x and t.

Proof. Remark 3.44 shows that we may interchange the order of differentiation and integration for $x > e + |c|t$. Thus, the first partial derivative $\partial_x I_3(x,t)$ for I_3 defined in (3.77) is given by

$$\begin{aligned}\frac{\partial I_3(x,t)}{\partial x} &= \int_0^t e^{-d(t-s)+x}\Big(\operatorname{erf}\Big(\frac{c(t-s) - x_f(s) + x}{\sqrt{a(t-s)}}\Big) - \operatorname{erf}\Big(\frac{c(t-s) + x - e}{\sqrt{a(t-s)}}\Big)\Big)\,ds \\ &+ \int_0^t \frac{e^{-d(t-s)+x}}{\sqrt{a(t-s)}}\Big(\exp\Big(-\frac{\big(c(t-s) - x_f(s) + x\big)^2}{a(t-s)}\Big) - \exp\Big(-\frac{\big(c(t-s) + x - e\big)^2}{a(t-s)}\Big)\Big)\,ds \\ &:= I_{3,1}(x,t) + I_{3,2}(x,t).\end{aligned}$$

Note that $I_{3,1} = I_3$ and apply Proposition 3.19. To estimate the difference between the two exponential functions in $I_{3,2}$ we use the same argument as in the proof of Proposition 3.22.

As before, substitute $\tau = t - s$, factorise $-\frac{x^2}{\tau}$ in the exponent and apply Lemma 3.11 to the remainder. This yields with $C := C_{p,3}$

$$|I_{3,2}(x,t)\exp(x^2/\kappa)| \leq \frac{2}{\sqrt{a}}\int_0^t \frac{1}{\sqrt{\tau}}\exp\Big(-\frac{\big(c\tau - e + x\big)^2}{a\tau} + \frac{x^2}{\kappa} + x - d\tau\Big)\,d\tau$$
$$\leq \frac{2}{\sqrt{a}}\int_0^t \frac{1}{\sqrt{\tau}}e^{-\frac{Cx^2}{\tau}}\,d\tau.$$

With (3.71) one obtains (3.108), which is uniformly bounded since $C_{p,3} > 0$ and Lemma 3.16 holds true. □

Combining the Propositions 3.21, 3.22, and 3.23 lead to the next theorem describing the decay behaviour for $u_x(x,t)$. Note that $x_p^{(1)}$ and $x_p^{(3)}$ are always the same in the propositions for I_j and $\partial_x I_j$, $j = 1, 2, 3$. Hence we use the same notation for Theorem 3.24 as in Theorem 3.20, i.e. Notation 3.38 on page 39 and 3.41 on page 41.

Theorem 3.24. *There exists a universal constant $C > 0$ so that there holds for all*

$$x > \max\Big(x_N^{(1)}, x_N^{(3)}, e + \max\big((|c| + |b|)t\big) + \varepsilon\Big), \tag{3.109}$$

$\varepsilon > 0$, $t > 0$ and each $\kappa > at$

$$|u_x(x,t)\exp(x^2/\kappa)| \leq C. \tag{3.110}$$

Proof. Since $e = \log K$ there holds $x > \log K$ (cf. (3.109)) and therefore $(K - e^x)_+ = 0$ in the representation for u (3.74). Combining the estimates for $\partial_x I_1$, $\partial_x I_2$, and $\partial_x I_3$ from Proposition 3.22, 3.22, and 3.22, respectively yields (3.110). □

Remark 3.48. The constant C in (3.110) can be explicitely determined on behalf of Remark 3.45, 3.46, 3.42, and 3.47.

After determining the decay behaviour of u and u_x we study now u_{xx}. With Notation 3.38 we state the decay behaviour for $\partial_{xx} I_1$.

Proposition 3.25 ($\boldsymbol{\partial_{xx} I_1(x,t)}$). *Let*

$$x_N^{(1)} > \max\Big(0, x_p^{(1)}\Big). \tag{3.111}$$

Then, there exists a universal constant C so that there holds for $x \geq x_N$, $t > 0$, and each $\kappa > at$

$$\Big|\frac{\partial^2 I_1}{\partial x^2}\exp(x^2/\kappa)\Big| \leq C. \tag{3.112}$$

Remark 3.49. More precisely, the proof shows that for $I_1 = I_1(x,t)$ with $x_N := x_N^{(1)}$ and $C := C_{p,1}$ defined in (3.81), the weighted second derivative

$$\begin{aligned}\Big|\frac{\partial^2 I_1}{\partial x^2}\exp(x^2/\kappa)\Big| \leq \frac{8b^2}{a^2}\Big(\sqrt{t}e^{-\frac{Cx^2}{t}} + x\sqrt{\pi C_{p,1}}\big(\operatorname{erf}(\frac{\sqrt{C}x}{\sqrt{t}}) - 1\big)\Big)\\ + \frac{2\sqrt{\pi}|a - 4b(x-e)|}{\sqrt{C}a^2 x}\big(1 - \operatorname{erf}(\frac{\sqrt{C}x}{\sqrt{t}}\big)\\ + \frac{4(x-e)^2}{a^2}\Big(\frac{\sqrt{\pi}}{2C^{3/2}x^3}\big(1 - \operatorname{erf}(\frac{\sqrt{C}x}{\sqrt{t}})\big) + \frac{1}{\sqrt{t}}e^{-\frac{Cx^2}{t}}\Big),\end{aligned} \tag{3.113}$$

is uniformly bounded in x and t.

Proof. According to Remark 3.44 we may interchange the order of differentiation and integration for $x > |b|t + e$. Then, the second partial derivative $\partial_{xx} I_1(x,t)$ for I_1 defined in (3.75) is given by

$$\frac{\partial^2 I_1(x,t)}{\partial x^2}$$
$$= -\frac{2}{a}\int_0^t \Big(\frac{1}{(t-s)^{3/2}} - \frac{2(b(t-s)-e+x)^2}{a(t-s)^{3/2}}\Big) \exp\Big(-\frac{\big(b(t-s)-e+x\big)^2}{a(t-s)} - r(t-s)\Big)\, ds.$$

Substituting $\tau = t - s$, factorising $-\frac{x^2}{\tau}$ in the exponent and applying Lemma 3.11 yields with $C := C_{p,1}$

$$\Big|\frac{\partial^2 I_1}{\partial x^2}\exp(x^2/\kappa)\Big|$$
$$\leq \frac{2}{a}\int_0^t \Big| -\frac{2b^2}{a\tau^{1/2}} + \frac{a-4b(x-e)}{a\tau^{3/2}} - \frac{2(x-e)^2}{a\tau^{5/2}}\Big|$$
$$\exp\Big(-\frac{x^2}{\tau}\Big(\frac{\kappa - a\tau}{\kappa a} + \frac{2(b\tau - e)}{ax} + \frac{(b\tau-e)^2 + ar\tau^2}{ax^2}\Big)\Big)\, d\tau$$
$$\leq \frac{2}{a}\int_0^t \Big| -\frac{2b^2}{a\tau^{1/2}} + \frac{a-4b(x-e)}{a\tau^{3/2}} - \frac{2(x-e)^2}{a\tau^{5/2}}\Big| e^{-\frac{Cx^2}{u}}\, d\tau.$$

Using the triangle inequality and (3.71)-(3.73) one obtains (3.113), which is uniformly bounded since $C_{p,1} > 0$ and Lemma 3.16 applies. □

To state the decay behaviour for $\partial_{xx} I_2$ we apply Notation 3.38 on page 39.

Proposition 3.26 ($\boldsymbol{\partial_{xx} I_2(x,t)}$). *Let*

$$x_N^{(1)} > \max\Big(0, x_p^{(1)}\Big). \tag{3.114}$$

Then, there is a universal constant $C > 0$ so that there holds for $x \geq \max(x_N, e + b|t| + \varepsilon)$ with $\varepsilon > 0$, $t > 0$, and each $\kappa > at$

$$\Big|\frac{\partial^2 I_2}{\partial x^2}\exp(x^2/\kappa)\Big| \leq C. \tag{3.115}$$

Remark 3.50. More precisely, the proof shows that for $I_2 = I_2(x,t)$ with $x_N := x_N^{(1)}$ and $C := C_{p,1}$ defined in (3.81), the weighted second derivative

$$\Big|\frac{\partial I_2}{\partial x}\exp(x^2/\kappa)\Big| \leq \frac{8}{\sqrt{a^3\pi}}\left(2|b|\sqrt{t}e^{-C\frac{x^2}{t}} + \Big(2|b|\sqrt{C\pi}x - \frac{\pi(x+|e|)}{\sqrt{C}x}\Big)\Big(\operatorname{erf}\big(\frac{\sqrt{C}x}{\sqrt{t}}\big) - 1\Big)\right) \tag{3.116}$$

is uniformly bounded in x and t.

Proof. By means of Remark 3.44 interchange the order of integration and differentiation for $x > |b|t+e$. Then, the second partial derivative $\partial_x I_2(x,t)$ for I_2 defined in (3.76) is given by

$$\frac{\partial^2 I_2(x,t)}{\partial x^2} = \frac{2}{\sqrt{a}\pi}\int_0^t \frac{e^{-r(t-s)}}{\sqrt{(t-s)}}\Bigg(-\frac{2(b(t-s)-x_f(s)+x)}{a(t-s)}\exp\Big(-\frac{(b(t-s)-x_f(s)+x)^2}{a(t-s)}\Big) + \frac{2(b(t-s)+x-e)}{a(t-s)}\exp\Big(-\frac{(b(t-s)+x-e)^2}{a(t-s)}\Big)\Bigg)\,ds. \tag{3.117}$$

Substituting $\tau = t-s$, using that $x > e+|b|t$ and $x_f(s) \leq e$ for $s \in [0,t]$ and then factorising $-\frac{x^2}{\tau}$ in the exponent yields together with bounding the remainder by means of Lemma 3.11 with $C := C_{p,1}$

$$\begin{aligned}\Big|\frac{\partial^2 I_2(x,t)}{\partial x^2}\exp(x^2/\kappa)\Big| &\leq \frac{8}{\sqrt{a^3}\pi}\int_0^t \frac{|b\tau+|e|+x|}{\tau^{3/2}}\exp\Big(-\frac{(b\tau-e+x)^2}{a\tau} - r\tau + \frac{x^2}{\kappa}\Big)\,d\tau \\ &\leq \frac{8}{\sqrt{a^3}\pi}\int_0^t \frac{|b\tau+|e|+x|}{\tau^{3/2}} e^{-C\frac{x^2}{\tau}}\,d\tau.\end{aligned} \tag{3.118}$$

With the triangle inequality and (3.71) and (3.72) one obtains (3.116), which is uniformly bounded since $C_{p,1} > 0$ and Lemma 3.16 applies. □

To state the decay behaviour for $\partial_{xx} I_3$ we apply Notation 3.41 on page 41.

Proposition 3.27 ($\boldsymbol{\partial_{xx} I_3(x,t)}$). *Let*

$$x_N^{(3)} > \max\Big(0, x_p^{(3)}\Big). \tag{3.119}$$

Then, there is a universal constant $C > 0$ so that there holds for $x \geq \max(x_N, e+|c|t+\varepsilon)$ with $\varepsilon > 0$, $t > 0$, and each $\kappa > at$

$$|\frac{\partial^2 I_3}{\partial x^2}\exp(x^2/\kappa)| \leq C. \tag{3.120}$$

Remark 3.51. More precisely, the proof shows that for $I_{3,3} = I_{3,3}(x,t)$, $x_N := x_N^{(3)}$ and $C := C_{p,3}$ defined in (3.93),

$$|I_{3,3}(x,t)\exp(x^2/\kappa)| \leq \frac{8|c|}{a^{3/2}}\Big(\sqrt{t}e^{-C\frac{x^2}{t}} + x\sqrt{C\pi}\Big(\operatorname{erf}\big(\frac{\sqrt{C}x}{\sqrt{t}}\big)-1\Big)\Big) + \frac{4\sqrt{\pi}(|e|+x)}{a^{3/2}C^{1/2}x}\Big(1-\operatorname{erf}(\frac{\sqrt{C}x}{\sqrt{t}})\Big) \tag{3.121}$$

is uniformly bounded in x and t.

Proof. By means of Remark 3.44 we may interchange the order of integration and differentiation for $x > |c|t+e$. Then the second partial derivative $\partial_{xx} I_3(x,t)$ for I_3 defined in (3.77) is given by

$$
\begin{aligned}
\frac{\partial^2 I_3(x,t)}{\partial x^2} &= \int_0^t e^{-d(t-s)+x}\Bigg(\operatorname{erf}\Big(\frac{c(t-s)-x_f(s)+x}{\sqrt{a(t-s)}}\Big) - \operatorname{erf}\Big(\frac{c(t-s)+x-e}{\sqrt{a(t-s)}}\Big)\Bigg)\,ds \\
&+2\int_0^t \frac{e^{-d(t-s)+x}}{\sqrt{a(t-s)}}\Bigg(\exp\Big(-\frac{\big(c(t-s)-x_f(s)+x\big)^2}{a(t-s)}\Big) - \exp\Big(-\frac{\big(c(t-s)+x-e\big)^2}{a(t-s)}\Big)\Bigg)\,ds \\
&+\int_0^t \frac{e^{-d(t-s)+x}}{\sqrt{a^3(t-s)^3}}\Bigg(-2\big(c(t-s)-x_f(s)+x\big)\exp\Big(-\frac{\big(c(t-s)-x_f(s)+x\big)^2}{a(t-s)}\Big) \\
&\qquad + 2\big(c(t-s)+x-e\big)\exp\Big(-\frac{2\big(c(t-s)+x-e\big)}{a(t-s)}\Big)\Bigg)\,ds \\
&= I_{3,1}(x,t) + 2I_{3,2}(x,t) + I_{3,3}(x,t).
\end{aligned} \tag{3.122}
$$

Note that we have already considered $I_{3,1}$ and $I_{3,2}$ in Proposition 3.23. Substitute $\tau = t - s$ in $I_{3,3}$, take into account $x_f(s) \geq e$, factorise $-\frac{x^2}{\tau}$ in the exponent and bound the remainder by means of Lemma 3.11. This yields with $C := C_{p,3}$

$$
\begin{aligned}
|I_{3,3}(x,t)\exp(x^2/\kappa)| &\leq \frac{4}{a^{3/2}}\int_0^t \frac{|cu+|e|+x|}{u^{3/2}}\exp\Big(-\frac{\big(cu-e+x\big)^2}{au} - du + \frac{x^2}{\kappa} + x\Big)\,d\tau \\
&\leq \frac{4}{a^{3/2}}\int_0^t \frac{|cu+|e|+x|}{u^{3/2}}e^{-\frac{Cx^2}{\tau}}\,d\tau.
\end{aligned} \tag{3.123}
$$

Using the triangle inequality, (3.71) and (3.72) yields (3.121) which is uniformly bounded since $C_{p,3} > 0$ and Lemma 3.16 applies. □

By combining the Propositions 3.25, 3.26, and 3.27 we state the decay rate for $u_{xx}(x,t)$ by making use of Notation 3.38 on page 39 and 3.41 on page 41.

Theorem 3.28. *There exists a universal constant $C > 0$ so that there holds for all*

$$
x > \max\Big(x_N^{(1)}, x_N^{(3)}, e + \max\big((|c|+|b|)t\big) + \varepsilon\Big), \tag{3.124}
$$

$\varepsilon > 0$, $t > 0$ *and each* $\kappa > at$

$$
|u_{xx}(x,t)\exp(x^2/\kappa)| \leq C. \tag{3.125}
$$

Proof. Since $e = \log K$ there holds $x > \log K$ (cf. (3.97)) and therefore $(K - e^x)_+ = 0$ in the representation for $u(x,t)$ (3.74). Combining the estimates for $\partial_{xx}I_1$, $\partial_{xx}I_2$, and $\partial_{xx}I_3$ from Proposition 3.25, 3.26, and 3.27, respectively, yields (3.125). □

Remark 3.52. The constant C in (3.125) can be explicitely determined on behalf of Remark 3.49, 3.50, 3.42, 3.47, and 3.51.

3.4.4 Decay Behaviour for American Call Options

In the previous subsection we proved the decay behaviour for American put options. We give a similar result for American call options. In contrast to put options the integral representation for the value of American call options (3.58) holds for $x < x_f(t)$. Consequently, we are interesteted in the behaviour of the solution for small values of x. For brevity set $a := 2\sigma^2 > 0$, $b := r - d - \sigma^2/2$, $c := r - d$, and $e := \log K$. Then the value $u(x,t)$ for an American call is given by

$$\begin{aligned} u(x,t) - (e^x - K)_+ = \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_0^t \frac{e^{-r(t-s)}}{\sqrt{t-s}} e^{-\frac{1}{a(t-s)}\left(b(t-s)-e+x\right)^2} \, ds \\ &- \frac{rK}{2} \int_0^t e^{-r(t-s)} \Big(\operatorname{erf}\Big(\frac{b(t-s) - x_f(s) + x}{\sqrt{a(t-s)}} \Big) - \operatorname{erf}\Big(\frac{b(t-s) + x - e}{\sqrt{a(t-s)}} \Big) \Big) \, ds \\ &+ \frac{d}{2} \int_0^t e^{-d(t-s)+x} \Big(\operatorname{erf}\Big(\frac{c(t-s) - x_f(s) + x}{\sqrt{a(t-s)}} \Big) - \operatorname{erf}\Big(\frac{c(t-s) - e + x}{\sqrt{a(t-s)}} \Big) \Big) \, ds \\ &= \frac{1}{\sigma\sqrt{2\pi}} I_1(x,t) - \frac{rK}{2} I_2(x,t) + \frac{d}{2} I_3(x,t). \end{aligned} \tag{3.126}$$

Similar to the previous subsection we introduce some notation, cf. Notation 3.38 on page 39 and 3.41 on page 41. Instead of referring to Lemma 3.11 on page 35 we refer to Lemma 3.12 on page 36. The next notation is the counterpart to Notation 3.38, i.e. the constants are connected with the integrals I_1 and I_2.

Notation 3.53. Let $\alpha_0^{(1)}$, $\beta_0^{(1)}$, and $\gamma_0^{(1)}$ be defined by

$$\alpha_0^{(1)} = \frac{\kappa - at}{a\kappa}, \quad \beta_0^{(1)} = \max\Big(-\frac{2e}{a}, \quad \frac{2(bt-e)}{a} \Big), \quad \gamma_0^{(1)} = \min\Big(\frac{e^2}{a}, \frac{(bt-e)^2 + art^2}{a} \Big). \tag{3.127}$$

Then, with $\alpha_0 = \alpha_0^{(1)}$, $\beta_0 = \beta_0^{(1)}$, and $\gamma_0 = \gamma_0^{(1)}$ we recall that

$$x_c^{(1)} := \begin{cases} 0 & \text{if } \beta_0^2 - 4\alpha_0\gamma_0 < 0, \\ (2\alpha_0)^{-1}(-\beta_0 - \sqrt{\beta_0^2 - 4\alpha_0\gamma_0}) & \text{if } \beta_0^2 - 4\alpha_0\gamma_0 \geq 0. \end{cases} \tag{3.128}$$

With some $x_N < 0$, $x^* = -2\frac{\gamma_0}{\beta_0}$, and $\tilde{f}(x) = \alpha_0 + \frac{\beta_0}{x} + \frac{\gamma_0}{x^2}$ the constant C_c from (3.67) reads

$$C_{c,1} := \begin{cases} \alpha_0 & \text{for } \beta_0 \leq 0, \\ \tilde{f}(x_N) & \text{for } \beta_0 > 0, \tilde{f}(x^*) \leq 0, \\ \tilde{f}(x^*) & \text{for } \beta_0 > 0, \tilde{f}(x^*) > 0, x_N \geq x^*, \\ \tilde{f}(x_N) & \text{for } \beta_0 > 0, \tilde{f}(x^*) > 0, x_N < x^*. \end{cases} \tag{3.129}$$

The subsequent notation is the counterpart to Notation 3.41 on page 41, i.e., the constants are connected with the integral I_3.

Notation 3.54. Let $\alpha_0^{(3)}$, $\beta_0^{(3)}$, and $\gamma_0^{(3)}$ be defined by

$$\alpha_0^{(3)} = \frac{\kappa - at}{a\kappa}, \quad \beta_0^{(3)} = \max\Big(-\frac{2e}{a}, \quad \frac{2(ct-e)}{a} \Big), \quad \gamma_0^{(3)} = \min\Big(\frac{e^2}{a}, \frac{(ct-e)^2 + adt^2}{a} \Big). \tag{3.130}$$

Then, with $\alpha_0 = \alpha_0^{(3)}$, $\beta_0^{(3)} = \beta_0^{(3)}$, and $\gamma_0^{(3)}$ we recall that

$$x_c^{(3)} := \begin{cases} 0 & \text{if } \beta_0^2 - 4\alpha_0\gamma_0 < 0, \\ (2\alpha_0)^{-1}(-\beta_0 + \sqrt{\beta_0^2 - 4\alpha_0\gamma_0}) & \text{if } \beta_0^2 - 4\alpha_0\gamma_0 \geq 0. \end{cases} \tag{3.131}$$

With some $x_N < 0$, $x^* = -2\frac{\gamma_0}{\beta_0}$, and $\tilde{f}(x) = \alpha_0 + \frac{\beta_0}{x} + \frac{\gamma_0}{x^2}$ the constant C_c from (3.67) reads

$$C_{c,3} := \begin{cases} \alpha_0 & \text{for } \beta_0 \leq 0, \\ \tilde{f}(x_N) & \text{for } \beta_0 > 0, \tilde{f}(x^*) \leq 0, \\ \tilde{f}(x^*) & \text{for } \beta_0 > 0, \tilde{f}(x^*) > 0, x_N \geq x^*, \\ \tilde{f}(x_N) & \text{for } \beta_0 > 0, \tilde{f}(x^*) > 0, x_N < x^*. \end{cases} \tag{3.132}$$

We state the decay behaviour for the value of an American call option $u = u(x,t)$ and its first and second spatial derivative by making use of Notation 3.53 and 3.54.

Theorem 3.29. *Suppose $x_N^{(j)} < \min(0, x_c^{(j)})$ for $j = 1, 3$ and $u = u(x,t)$ from (3.126) be the value of an American call option. Then there exists a universal constant $C > 0$ so that there holds for $t > 0$,*

$$x < \min\left(x_N^{(1)}, x_N^{(3)}, e - \max(|b|, |c|)t\right) =: x_0^C,$$

and each $\kappa >$ at

$$\left|\frac{\partial^k u}{x^k} \exp(x^2/\kappa)\right| \leq C \quad \text{for } k = 0, 1, 2. \tag{3.133}$$

Proof. The proof follows exactly the proofs for the propositions and theorems in Subsection 3.4.2 and 3.4.3 where we treated American put options. The only difference is that instead of applying Lemma 3.11 and 3.13, we apply Lemma 3.12 and 3.14, respectively. Moreover, we do not require some $\varepsilon > 0$ as for example in Proposition 3.18, since we can easily bound $\frac{1}{x+b\tau-e} < 0 < C$ for $x < e - |b|t$. $\square$

Chapter 4

Transformations

This chapter concerns the analysis of the Black-Scholes equation and the Black-Scholes inequality in a general parabolic framework. We show that suitable transformations applied to the Black-Scholes equation yield a formulation which admits a solution in non-weighted Sobolev spaces. For the Black-Scholes inequality, however, it turns out that the strong formulation cannot be transformed to a non-weighted setting. This underlines the necessity of weighted Sobolev spaces in the (numerical) analysis of American options.

4.1 The Black-Scholes Equation

Recall that the Black-Scholes Equation, cf. (3.3) on page 16, is a parabolic PDE with constant coefficients and inhomogeneous initial conditions. We proved that the spatial operator satisfies a Gårding inequality and that the solution belongs to a weighted Sobolev space. In this section we will prove that the Black-Scholes equation can be transformed to a parabolic problem with an elliptic spatial operator whose solution lies in a standard Sobolev space. This transformations are used in Matache et al. [2004] to identify truncation errors for the solution in the computational domain. However, we consider a general parabolic PDE on weighted Sobolev spaces H^1_η with weight $p(x) = \exp(2\xi|x|)$, $\xi \in \mathbb{R}$, i.e., $f \in H^1_\eta(\mathbb{R})$ if $f \exp(\xi|x|), f' \exp(\xi|x|) \in L^2(\mathbb{R})$. Note, that in this chapter we define the exponent in the weight function p without a negative sign. Moreover, we allow weights that grow exponentially which means that a solution that belongs to such spaces must decrease exponentially. We prove necessary and sufficient conditions on the coefficients of the PDE so that the solution belongs to H^1_η for some $\xi \in \mathbb{R}$ given.

Let a, b, $c \in \mathbb{R}$ and $a > 0$ be given and consider the following parabolic PDE in abstract form

$$\begin{aligned} \dot{u} + \mathcal{A}[u] &= 0 \quad \text{in } \mathbb{R} \times (0, T], \\ u(0, x) &= h(x) \quad \text{for all } x \in \mathbb{R}, \end{aligned} \tag{4.1}$$

where

$$\mathcal{A}[u] := -au'' + bu' + cu. \tag{4.2}$$

Recall that the initial condition for a call option reads $h_C(x) = (e^x - K)_+$ and for a put option $h_P(x) = (K - e^x)_+$.

The initial condition h for evaluating call or put options requires weighted Sobolev spaces to guarantee the existence of a weak solution as in Chapter 3. Recall that the solution space depends on the initial condition h and the right-hand side f. Consequently we need to transform (4.1) to a problem with initial condition in $H^1(\mathbb{R})$. A natural choice is to transform to homogenous initial conditions.

Transformation 4.1. Let u be a solution of (4.1) and set $v(x,t) = u(x,t) - h(x)$. Then v satisfies

$$\begin{aligned} \dot{v} + \mathcal{A}[v] &= -\mathcal{A}[h] =: f, \\ v(0,x) &= 0. \end{aligned} \tag{4.3}$$

It remains to show that the right-hand side f is contained in $H^{-1}(\mathbb{R}) = (H^1(\mathbb{R}))^*$. First we calculate f for call and put options, i.e., $f_C := -\mathcal{A}[h_C]$ and $f_P = -\mathcal{A}[h_P]$, respectively.

Lemma 4.1. *With $\mathcal{A}$ from (4.2), $h_C = (e^x - K)_+$, and $h_P(x) = (K - e^x)_+$ there hold*

$$f_C := -\mathcal{A}[h_C] = aK\delta_{(\log K)} + \big((a - b - c)e^x + cK)\big)\chi_{[\log K, \infty)}(x) \tag{4.4}$$

and

$$f_P := -\mathcal{A}[h_P] = aK\delta_{(\log K)} - \big((a - b - c)e^x + cK)\chi_{(-\infty, \log K]}(x). \tag{4.5}$$

Proof. Since the pay-off functions h_C and h_P are continuous, the first and second derivatives are understood in the distributional sense and read

$$h_C'(x) = \begin{cases} e^x & \text{for } x \geq \log K, \\ 0 & \text{otherwise} \end{cases}, \qquad h_P'(x) = \begin{cases} -e^x & \text{for } x \leq \log K, \\ 0 & \text{otherwise;} \end{cases}$$

and

$$h_C''(x) = \begin{cases} K\delta_{\log K} + e^x & \text{for } x \geq \log K, \\ 0 & \text{otherwise,} \end{cases} \qquad h_P''(x) = \begin{cases} K\delta_{\log K} - e^x & \text{for } x \leq \log(K), \\ 0 & \text{otherwise.} \end{cases}$$

Inserting the derivatives of h_C and h_P into $\mathcal{A}$ proves (4.4) and (4.5). □

In the next two propositions we give sufficient and necessary conditions on the coefficients a, b, and c in $\mathcal{A}[u]$ such that f_P and f_C belongs to $(H^1_\eta)^*$ for certain $\xi \in \mathbb{R}$ in the weight $p(x) = \exp(2\xi|x|)$.

Proposition 4.2. *With f_P from (4.5) it holds that*

$f_P \in (H^1_\eta)^*$ *for*	*if and only if*
$\xi < 0$	$a - b + c \neq 0,\ c \neq 0$
$\xi < 1$	$a \neq b,\ c = 0$
$\xi \in \mathbb{R}$	$a = b, c = 0$

(4.6)

Proof. By definition $f \in (H^1_\eta)^*$ if for all $v \in H^1_\eta$

$$\langle f\ ,\ v \rangle_{H^{-1}_\eta \times H^1_\eta} \le C \left\| v \right\|_{H^1_\eta} . \tag{4.7}$$

First we prove that the conditions (4.6) on a, b, and c are sufficient. Using the definition of f_P from (4.5) yields for all $v \in H^1_\eta$

$$\begin{aligned}\langle f_P\ ,\ v \rangle_{H^{-1}_\eta \times H^1_\eta} &= aKv(\log K)\exp(2\xi|\log K|) - \int_{-\infty}^{\log K} (a-b-c)e^x v e^{2\xi|x|}\,dx \\ &\quad - K\int_{-\infty}^{\log K} c v e^{2\xi|x|}\,dx \\ &\le \left\| aKve^{2\xi|x|} \right\|_{L^\infty(\log K-\varepsilon,\log K+\varepsilon)} \\ &\quad + |a-b-c| \left\| e^{x+\xi|x|} \right\|_{L^2(-\infty,\log K)} \left\| v \right\|_{L^2_\eta} \\ &\quad + K|c| \left\| e^{\xi|x|} \right\|_{L^2(-\infty,\log K)} \left\| v \right\|_{L^2_\eta} .\end{aligned} \tag{4.8}$$

Note that $\left\| aKve^{2\xi|x|} \right\|_{L^\infty(\log K-\varepsilon,\log K+\varepsilon)}$ is bounded for all $\xi \in \mathbb{R}$. The second term

$$|a-b-c| \left\| e^{x+\xi|x|} \right\|_{L^2(-\infty,\log K)} \left\| v \right\|_{L^2_\eta}$$

is bounded if $\left\| e^{x+\xi|x|} \right\|_{L^2(-\infty,\log K)} < \infty$ which holds for $\xi < 1$, otherwise $a-b-c = 0$ is necessary. The third term

$$K|c| \left\| e^{\xi|x|} \right\|_{L^2(-\infty,\log K)} \left\| v \right\|_{L^2_\eta}$$

is bounded if $\left\| e^{x+\xi|x|} \right\|_{L^2(-\infty,\log K)} < \infty$ which holds for $\xi < 0$, otherwise $c = 0$ is required. This proves that the conditions (4.6) are sufficient.

We turn to necessity next and aim to find a function $v \in H^1_\eta$ such that (4.7) is violated. Note that the term $aKv(\log K)e^{2\xi|\log K|}$ in (4.8) is bounded for all $\xi \in \mathbb{R}$. It remains to consider the last two terms. We split the integral from $(-\infty, \log K)$ into

$$I_1 + I_2 := -\int_{-\infty}^{R} \Big((a-b-c)e^x + Kc\Big) v e^{2\xi|x|}\,dx - \int_{R}^{\log K} \Big((a-b-c)e^x + Kc\Big) v e^{2\xi|x|}\,dx.$$

Note that the integral I_2 is bounded for all $\xi \in \mathbb{R}$. To analyse I_1 set $f(x) := \alpha e^x + \beta$ where $\alpha := -(a-b-c)$ and $\beta := -Kc$, i.e.,

$$I_1 = \int_{-\infty}^{R} \big(\alpha e^x + \beta\big) v e^{2\xi|x|}\,dx.$$

Given $R < \min(0, \log K)$ and $\nu \in \{-1, 1\}$ define a function $v \in H^1_\eta$

$$v := \begin{cases} \frac{\nu}{x} e^{-\xi|x|} & x < R, \\ \frac{\nu x}{R^2} e^{-\xi|R|} & R \le x < 0, \\ 0 & x \ge 0. \end{cases} \tag{4.9}$$

Then,

$$v' = \begin{cases} -\nu(\frac{1}{x^2} + \frac{\xi}{x}) e^{-\xi|x|} & x < R, \\ \frac{\nu}{R^2} e^{-\xi|R|} & R \le x < 0, \\ 0 & x \ge 0. \end{cases} \tag{4.10}$$

First we prove that $v \in H^1_\eta$, i.e.,

$$\left\| v e^{\xi|x|} \right\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{R} \frac{1}{x^2}\, dx + \frac{1}{R^2} \int_R^0 x^2 e^{2\xi(R-x)}\, dx < \infty$$

and

$$\left\| v' e^{\xi|x|} \right\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{R} \left(\frac{1}{x^2} + \frac{\xi}{x} \right)^2 dx + \frac{e^{2\xi R}}{R^2} \int_R^0 e^{-2\xi x}\, dx < \infty.$$

Hence $v \in H^1_\eta$.

For each of the cases in (4.6) we determine R and ν in (4.9) so that (4.7) does not hold, i.e., $f_P \notin (H^1_\eta)^*$. We start with the case where $\alpha = -a + b + c \neq 0$ and $\beta = -Kc \neq 0$ and $\xi \geq 0$. Note that $f(x) = \alpha e^x + \beta$ only contains a zero if $\alpha\beta < 0$. Then the sign changes at $x = \log(-\frac{\beta}{\alpha})$. Consequently, $\operatorname{sign}(f(x))$ is constant for $x < \log(-\frac{\beta}{\alpha})$. Moreover, $\operatorname{sign}(f(x)) = \operatorname{sign}(\beta)$ for $x < \log(-\frac{\beta}{\alpha})$. Hence, we choose $R < \min(0, \log K, \log(-\frac{\beta}{\alpha}))$ for $\alpha\beta < 0$ and $R < \min(0, \log K)$ for $\alpha\beta > 0$ in (4.9). Set $\nu = \operatorname{sign}(-\beta)$ in (4.9). Then, the integrand is positive and since $\xi \geq 0$ there holds

$$I_1 = \int_{-\infty}^{R} (\alpha e^x + \beta) \frac{\nu}{x} e^{\xi|x|}\, dx \geq C \int_{-\infty}^{R} -\frac{1}{x}\, dx$$

with the constant

$$C = \begin{cases} |\beta| & \text{if } \alpha\beta > 0, \\ |\alpha e^R + \beta| & \text{if } \alpha\beta < 0. \end{cases}$$

Hence I_1 is divergent and for $c \neq 0$ and $a - b - c \neq 0$ there holds $f_P \notin (H^1_\eta)^*$ for $\xi \geq 0$, i.e., the condition $\xi < 0$ is necessary.

To complete the proof, we consider the case where $c = 0$ but $a, b \in \mathbb{R}$. We show that (4.7) does not hold for $\xi \geq 1$. Inserting v from (4.9) in I_1 yields

$$I_1 = -(a - b - c) \int_{-\infty}^{R} \frac{\nu}{x} e^{-(\xi-1)x}\, dx.$$

Choose $R < \min(0, \log K)$ and $\nu = \operatorname{sign}(a - b - c)$. Then the integrand is positive and I_1 can be estimated from below by

$$I_1 > |a - b - c| \int_{-\infty}^{R} \frac{-1}{x}\, dx.$$

However, the right-hand side diverges. We constructed for each case in (4.6) a function $v \in H^1_\eta$ so that (4.7) is violated and consequently the conditions (4.6) are also necessary. □

The subsequent proposition provides a simular result for call options.

Proposition 4.3. *With f_C from (4.4) there holds*

$f_C \in (H^1_\eta)^*$ *for*	*if and only if*
$\xi < -1$	$a, b, c \in \mathbb{R}$
$\xi < 0$	$c \in \mathbb{R},\ a - b - c = 0$
$\xi \in \mathbb{R}$	$a = b, c = 0$

(4.11)

Proof. First we prove that the conditions (4.11) on a, b, and c are sufficient. Using the definition of f_C from (4.4) yields for all $v \in H^1_\eta$

$$\begin{aligned}\langle f_C\ ,\, v\rangle_{H^{-1}_\eta \times H^1_\eta} &= aKv(\log K)e^{2\xi|\log K|} + \int_{\log K}^{\infty} (a-b-c)e^x v e^{2\xi|x|}\, dx \\ &\quad + cK \int_{\log K}^{\infty} v e^{2\xi|x|}\, dx \\ &\le \left\| aKve^{2\xi|x|} \right\|_{L^\infty(\log K-\varepsilon, \log K+\varepsilon)} \\ &\quad + |a-b-c| \left\| e^{x+\xi|x|} \right\|_{L^2(\log K,\infty)} \|v\|_{L^2_\eta} \\ &\quad + K|c| \left\| e^{\xi|x|} \right\|_{L^2(\log K,\infty)} \|v\|_{L^2_\eta}\,. \end{aligned} \tag{4.12}$$

Note that the term $\left\| aKve^{2\xi|x|} \right\|_{L^\infty(\log K-\varepsilon, \log K+\varepsilon)}$ is bounded for all $\xi \in \mathbb{R}$. The second term

$$|a-b-c| \left\| e^{x+\xi|x|} \right\|_{L^2(\log K,\infty)} \|v\|_{L^2_\eta}$$

is bounded if $\xi < -1$ or vanishes if there holds $a - b - c = 0$. The third term

$$K|c| \left\| e^{\xi|x|} \right\|_{L^2(\log K,\infty)} \|v\|_{L^2_\eta}$$

is bounded if $\xi < 0$ or vanishes if $c = 0$. Consequently, the conditions (4.11) are sufficient.

It remains to show that the conditions on a, b, and c are also necessary. We proceed as in the proof of Proposition 4.2, finding functions $v \in H^1_\eta$ that violate (4.7). Since the first term in (4.12) is bounded we consider the last two terms. With $R > \max(0, \log K)$ there holds

$$\begin{aligned} &\int_{\log K}^{\infty} \big((a-b-c)e^x + cK\big) v e^{2\xi|x|}\, dx \\ &\quad = \int_{\log K}^{R} \big((a-b-c)e^x + cK\big) v e^{2\xi|x|}\, dx + \int_{R}^{\infty} \big((a-b-c)e^x + cK\big) v e^{2\xi|x|}\, dx = I_1 + I_2, \end{aligned}$$

I_1 is bounded so it remains to consider I_2. For brevity set $\alpha = a-b-c$ and $\beta = cK$. Then,

$$I_2 = \int_R^{\infty} (\alpha e^x + \beta) v e^{2\xi|x|}\, dx. \tag{4.13}$$

First we consider the case where $\alpha \neq 0$ and $\beta \neq 0$ and show that for $\xi \geq -1$, (4.7) does not hold. We start with $\xi = -1$ and take the test function v from (4.9). Then,

$$I_2 = \int_R^{\infty} \frac{\nu}{x} (\alpha + \beta e^{-x})\, dx.$$

The function $f(x) := \alpha + \beta e^{-x}$ only vanishes if $\alpha\beta < 0$. Then, the root is located at $x = \log(-\frac{\beta}{\alpha})$, i.e. $\operatorname{sign}(f(x)) = \operatorname{sign}(\alpha)$ for $x > \log(-\frac{\beta}{\alpha})$. Choosing $R > \max(0, \log K, \log(-\frac{\beta}{\alpha})$ for $\alpha\beta < 0$ and $R > \max(0, \log K)$ for $\alpha\beta > 0$, together with $\nu = \operatorname{sign}(\alpha)$ yields

$$(\alpha + \beta e^{-x})\frac{\nu}{x} > 0 \quad \text{for } x > R.$$

With

$$C := \min_{x\geq R}\left((\alpha+\beta e^{-x})\nu\right) = \begin{cases} |\alpha| & \alpha\beta > 0, \\ |\alpha+\beta e^{-R}| & \alpha\beta < 0, \end{cases}$$

I_2 is bounded below by

$$I_2 = \int_R^\infty \frac{\nu}{x}(\alpha+\beta e^{-x})\,dx \geq C\int_R^\infty \frac{1}{x}\,dx.$$

Again the right-hand side is divergent.

Let now $\xi > -1$ and take again v from (4.9). Then,

$$I_2 = \int_R^\infty \frac{\nu}{x}(\alpha e^x+\beta)e^{\xi x}\,dx.$$

Set $f(x) := (\alpha e^x + \beta)e^{\xi x}$, then $f'(x) = e^{\xi x}(\alpha(\xi+1)e^x + \beta)$. As $\xi + 1 > 0$ only α determines the sign and the monotony of f for $R > \max(0, \log K, \log(\frac{-\beta}{\alpha}), \log(\frac{-\beta}{\alpha(1+\xi)}))$ for $\alpha\beta < 0$ and $R > \max(0, \log K)$ for $\alpha\beta > 0$. To be precise, with $\nu = \mathrm{sign}(\alpha)$ there holds

$$\nu e^{\xi x}(\alpha e^x+\beta)\frac{1}{x} > 0 \quad \text{for } x > R,$$

and $\nu e^{\xi x}(\alpha e^x + \beta)\frac{1}{x}$ is monotonically increasing for $x > R$. Consequently,

$$I_2 = \int_R^\infty \frac{\nu}{x}(\alpha e^x+\beta)e^{\xi x}\,dx \geq |\alpha e^R+\beta|e^R\int_R^\infty \frac{1}{x}\,dx.$$

Again the right-hand side is divergent. Consequently, we have shown that for $a - b - c \neq 0$ and $c \neq 0$ there holds that $f_C \neq (H_\eta^1)^*$ for $\xi \geq -1$.

Finally we consider the case $a - b - c = 0$, i.e. $\alpha = 0$ and show that for $\xi \geq 0$ condition (4.7) does not hold. With $\alpha = 0$ and v from (4.9) with $\nu = \mathrm{sign}(\beta)$ I_2 reads

$$I_2 = \int_R^\infty \beta\frac{\nu}{x}ve^{2\xi|x|})\,dx = |\beta|\int_R^\infty \frac{1}{x}e^{\xi x}\,dx \geq |\beta|\int_R^\infty \frac{1}{x}\,dx,$$

which again is divergent. Consequently, we showed that the conditions (4.11) are necessary and sufficient. □

Proposition 4.2 and 4.3 demonstrated that f_P and f_C belong to H_η^{-1} for $\xi \geq 0$ if and only if $a = b$ and $c = 0$. Hence, before applying Transformation 4.1 we need to transform the spatial operator to an operator with $a = b$ and $c = 0$. The next two transformations show how this can be established.

Transformation 4.2 (leading to $\boldsymbol{b = \tilde{b}}$). For given $\tilde{b} \in \mathbb{R}$, let $u(x,t) = v(x+(\tilde{b}-b)t, t) = v(y,t)$. Then, we obtain with u-terms evaluated at (x,t) and v-terms evaluated at $(x + (\tilde{b} - b)t, t)$

$$\dot{u} = \dot{v} + (\tilde{b}-b)w_y, \quad u' = v_y, \quad u'' = v_{yy},$$

and thus v satisfies,

$$\begin{aligned} \dot{v} - av_{yy} + \tilde{b}v_y + cv &= 0, \\ v(0,y) &= h(y). \end{aligned} \tag{4.14}$$

Transformation 4.3 (leading to $c = \tilde{c}$). For given $\tilde{c} \in \mathbb{R}$, set $u(x,t) = e^{(\tilde{c}-c)t}v(x,t)$. This transformation yields

$$\dot{u} = e^{(\tilde{c}-c)t}(\dot{v} + (\tilde{c}-c)v), \quad u' = e^{(\tilde{c}-c)t}v', \quad u'' = e^{(\tilde{c}-c)t}v''.$$

Therefore v satisfies,

$$\begin{aligned} \dot{v} - av'' + bv' + \tilde{c}v = 0 \\ v(0,x) = h(x) \text{ for all } x \in \mathbb{R}. \end{aligned} \tag{4.15}$$

Finally, we are able to transform Problem (4.1) to a PDE with an elliptic spatial operator. Moreover, the solution u belongs to the space

$$\{u \in L^2(0,T;H^1(\mathbb{R})) \,:\, u_t \in L^2(0,T;H^{-1}(\mathbb{R}))\}.$$

The subsequent remark summerises the necessary transformations.

Remark 4.4. To transform the PDE (4.1) to a PDE which operator is elliptic in the spatial variable and a solution that belongs to standard Sobolev spaces, the following transformations can be applied. Note that the order of the transformations is essential.

(i) Apply Transformation 4.2 with $\tilde{b} = 0$ and Transformation 4.3 with $\tilde{c} = 0$. This yields a spatial operator $\mathcal{A}$ with $a = b$ and $c = 0$.

(ii) Then, applying Transformation 4.1 yields

$$\begin{aligned} \dot{u} - au'' + bu' &= f \quad \text{in } \mathbb{R} \times (0,T], \\ u(\cdot,0) &= 0 \quad \text{in } \mathbb{R}, \end{aligned} \tag{4.16}$$

with $f \in H^{-1}(\mathbb{R})$. The solution u belongs to

$$u \in \{u \in L^2(0,T;H^1(\mathbb{R})) \,:\, u_t \in L^2(0,T;H^{-1}(\mathbb{R}))\},$$

cf. Theorem 3.3 on page 19.

(iii) A transformation $u = e^{\lambda t}w$ where λ is the constant from the Gårding inequality, cf. Remark 3.14 on page 21, yields an elliptic spatial operator without changing the solution spaces.

4.2 The Black-Scholes Inequality

We consider the LCP in abstract form, cf. (3.18) on page 22,

$$\begin{aligned} \dot{u} + \mathcal{A}[u] &\geq 0, \\ (u-\psi)(\dot{u} + \mathcal{A}[u]) &= 0, \\ u(\cdot,t) &\geq \psi(x), \\ u(\cdot,0) &= \psi(x). \end{aligned} \tag{4.17}$$

with a general elliptic operator $\mathcal{A}$ as in the last section

$$\mathcal{A}[u] = -au'' + bu' + cu \tag{4.18}$$

where $a > 0$, $b, c \in \mathbb{R}$.

We want to transform system (4.17) such that the solution lies in unweighted Sobolev spaces, cf. Section 4.1. To guarantee the existence of a unique solution in unweighted Sobolev spaces, the following conditions must be satisfied, cf. Bensoussan and Lions [1982], Duvaut and Lions [1976], Zeidler [1985]:

(i) the right-hand side $f \in H^1(0, T; H^{-1}(\mathbb{R}))$,

(ii) the transformed obstacle $\psi(x, t) \in L^2(0, T; L^2(\mathbb{R}))$,

(iii) the initial condition $u_0 \in H^1(\mathbb{R})$.

4.2.1 First Approach – Homogenous Initial Condition

The first approach is the same as in the case of parabolic PDEs in the last subsection. First we transform the spatial operator such that $\mathcal{A}[\psi] \in H^{-1}(\mathbb{R})$. Then, we transform to homogenous initial conditions, which guarantees that condition (i) and (iii) are satisfied. It remains to show that the transformed obstacle satisfies $\psi_1(x, t) \in L^2(0, T; L^2(\mathbb{R}))$. Unfortunately this fails, due to the following argument.
Consider the pay-off function of an American call option, i.e., $\psi(x) = (e^x - K)_+$. We transform the spatial operator $\mathcal{A}$ such that $a = b$ and $c = 0$, i.e., $\tilde{b} = 0$ and $\tilde{c} = 0$ in Transformation 4.2 and 4.3, respectively. This means with $y = x - bt$ set

$$u(x, t) = e^{-ct} v(y, t).$$

Hence, $v(y, t) \geq e^{ct}\psi(y + bt) =: \psi_1(y)$ and the transformed obstacle reads

$$\psi_1(y) = e^{ct}(e^{y+bt} - K)_+.$$

Moreover, the initial condition reads $v(y, 0) = \psi(y)$. If we then transform to homogenous initial conditions, i.e., $w(y, t) = v(y, t) - \psi(y)$, the new obstacle reads

$$\psi_2(y) = \psi_1(y) - \psi(y) = e^{ct}(e^{y+bt} - K)_+ - (e^y - K)_+.$$

We show now, that $\psi_2 \notin L^2(\mathbb{R})$. with $C := \max(\log K, \log K - bt)$ there holds

$$\int_{\mathbb{R}} \psi_2(y)^2 \, dy \geq \int_C^\infty \left(K(1 - e^{ct}) + e^{y+bt} - e^y \right)^2 dy.$$

Since this integral only exists for $b = 0$ and $c = 0$ in (4.18), $\psi_2 \notin L^2(0, T; L^2(\mathbb{R}))$ for an elliptic operator with $b \neq 0$ and $c \neq 0$ and consequently weighted Sobolev spaces are necessary.
Similarly, for American put options with $\psi(x) = (K - e^x)_+$ the transformed obstacle ψ_2 reads

$$\psi_2(y) = \psi_1(y) - \psi(y) = e^{ct}(K - e^{y+bt})_+ - (K - e^y)_+.$$

Then, with $C := \min(\log K, \log K - bt)$ there holds

$$\int_{\mathbb{R}} \psi_2(y)^2 \, dy \geq \int_{-\infty}^{C} \big(K(e^{ct} - 1) - e^{y+bt} + e^y\big)^2 \, dy.$$

This integral only exists for $c = 0$ and consequently weighted Sobolev spaces are necessary.

4.2.2 Second Approach – Obstacle $\psi \equiv 0$

In the last subsection we showed, that a transformation to homogenous initial condition is not sufficient for the solution to be in a non-weighted Sobolev space. Here, we aim at problem formulation with homogenous initial conditions and obstacle $\psi(x) = 0$. Again it turns out that such transformations result in a problem that does not admit a solution in non-weighted Sobolev spaces.

Define the spatial operator $\tilde{\mathcal{A}}$ by

$$\tilde{\mathcal{A}}[v] := -av_{yy} + \tilde{b}v_y + \tilde{c}v. \tag{4.19}$$

Applying Transformation 4.2 and 4.3 to (4.17) yields with $u(x,t) = e^{(\tilde{c}-c)t}v(y,t)$, and $y = x + (\tilde{b} - b)t$, that v solves

$$\begin{aligned} \dot{v} + \tilde{\mathcal{A}}[v] &\geq 0, \\ (v - \psi_1)(\dot{v} + \tilde{\mathcal{A}}[v]) &= 0, \\ v(y,t) &\geq \psi_1(y,t), \\ v(y,0) &= \psi(y). \end{aligned} \tag{4.20}$$

The transformed obstacle ψ_1 is given by

$$\psi_1(y,t) = e^{-(\tilde{c}-c)t}\psi\big(y - (\tilde{b} - b)t\big).$$

A transformation $w(y,t) = v(y,t) - \psi_1(y,t)$ yields a homogenous initial condition and a zero-obstacle. It remains to calculate the resulting right-hand side $f := -(\dot{\psi}_1 + \tilde{\mathcal{A}}[\psi_1])$. We show this for a put option, namely $\psi(x) = (K - e^x)_+$. Note that the transformed obstacle ψ_1 reads, with $\beta := b - \tilde{b}$ and $\gamma = c - \tilde{c}$

$$\psi_1(y) = e^{\gamma t}(K - e^{y+\beta t})_+.$$

The first and second derivatives are understood in the distributional sense and read

$$\psi_1'(y,t) = \begin{cases} -e^{\gamma t}e^{y+\beta t} & \text{for } x \leq \log K - \beta t, \\ 0 & \text{otherwise;} \end{cases}$$

and

$$\psi_1''(y,t) = \begin{cases} e^{\gamma t}\big(K\delta_{\log K} - e^{y+\beta t}\big) & \text{for } y \geq \log K - \beta t, \\ 0 & \text{otherwise.} \end{cases}$$

The first time derivative reads

$$\dot{\psi}_1(y,t) = \begin{cases} e^{\gamma t}\Big(\gamma(K - e^{y+\beta t}) - \beta e^{y+\beta t}\Big) & \text{for } y \leq \log K - \beta t, \\ 0 & \text{otherwise.} \end{cases}$$

Hence $f = -(\dot{\psi}_1 + \tilde{\mathcal{A}}[\psi_1])$ reads

$$\begin{aligned} f(y,t) &= -e^{\gamma t}\Big((\gamma + \tilde{c})K + (-\gamma - \beta + a - \tilde{b} - \tilde{c})e^{y+\beta t} - aK\delta_{\log K}\Big)\chi_{(-\infty,\log K-\beta t]}(y) \\ &= -e^{\gamma t}\Big(cK + (a - b - c)e^{y+\beta t} - aK\delta_{\log K}\Big)\chi_{(-\infty,\log K-\beta t]}(y). \end{aligned} \tag{4.21}$$

Since a, b, and c are given data, $f(\cdot,t) \notin L^2(\mathbb{R})$ for arbitrary coefficients. Hence we cannot find the parameters $\tilde{a}$, $\tilde{b}$, and $\tilde{c}$ in Transformation 4.2 and 4.3 so that Problem (4.17) can be tranformed to a problem which admits a solution in unweighted Sobolev spaces.
For call options where $\psi(x) = (e^x - K)_+$ similar calculations show that the resulting right-hand side f does not belong to $L^2(\mathbb{R})$. Hence, also in the case of American call options, weighted Sobolev spaces are necessary to guarantee the existence of weak solutions on $\mathbb{R}$.

Chapter 5

Numerical Analysis for American Options

This chapter is devoted to the numerical analysis of American options. The unbounded domain in combination with the pay-off functions require weighted Sobolev spaces to guarantee existence of a unique solution as in Chapter 3. This chapter first recalls the formulation of the variational inequality in the weighted spaces from Chapter 3. Then we transform the variational inequality to a non-linear PDE (with solution u_ε) using penalisation techniques. We discretise the penalised problem with $P1$ finite elements in space and do the time integration with the method of lines to obtain a semi-discrete solution $u_{\varepsilon h}$. To prove a priori and a posteriori error estimates, interpolation estimates in the weighted norms are proved by a theorem from Payne and Weinberger [1960]. Then we prove an a posteriori error bound for the error $e = u - u_{\varepsilon h}$ and an a priori error estimates for the penalisation error $e = u - u_\varepsilon$ and the discretisation error $u - u_{\varepsilon h}$. For an introduction on a posteriori finite element analysis we refer to the books, cf. Ainsworth and Oden [2000], Bangerth and Rannacher [2003], Babuška and Strouboulis [2001], Johnson [1987].

5.1 Continuous Model

For the convenience of the reader we recall some formulations and results from Chapter 3.

Definition 5.1. Let H^1_η and L^2_η denote weighted Sobolev spaces defined as

$$\begin{aligned} H^1_\eta &:= H^1_\eta(\mathbb{R}) := \left\{ v \in L^1_{loc}(\mathbb{R}) | v \exp(-\eta|x|), v' \exp(-\eta|x|) \in L^2(\mathbb{R}) \right\}, \\ L^2_\eta &:= L^2_\eta(\mathbb{R}) := \left\{ v \in L^1_{loc}(\mathbb{R}) | v \exp(-\eta|x|) \in L^2(\mathbb{R}) \right\}. \end{aligned}$$

Definition 5.2. Given the volatility $\sigma > 0$, the risk less interest rate $r \geq 0$, and the constant dividend yield $d \geq 0$ we define the bilinear form $a_\eta : H^1_\eta \times H^1_\eta \to \mathbb{R}$ by

$$\begin{aligned} a_\eta(u, v) := \frac{\sigma^2}{2} \int_{\mathbb{R}} u'v' \exp(-2\eta|x|)\, dx + r \int_{\mathbb{R}} uv \exp(-2\eta|x|)\, dx \\ - \int_{\mathbb{R}} \big(\eta\sigma^2 \operatorname{sign}(x) + r - d - \frac{\sigma^2}{2}\big) u'v \exp(-2\eta|x|)\, dx. \end{aligned} \tag{5.1}$$

Proposition 5.1. *The bilinear form $a_\eta(\cdot,\cdot) : H^1_\eta \times H^1_\eta \to \mathbb{R}$ is continuous and satisfies a Gårding inequality, i.e. with*

$$C := \max\left\{|r-d-\sigma^2/2+\eta\sigma^2|, |r-d-\sigma^2/2-\eta\sigma^2|\right\}$$

there exist some

$$M = \max(\sigma^2/2, r) + C > 0, \qquad \alpha = \sigma^2/4 > 0, \qquad \lambda = C^2/\sigma^2 + \sigma^2/4 - r$$

such that

$$\begin{aligned} |a_\eta(u,v)| &\leq M \|u\|_{H^1_\eta(\mathbb{R})} \|v\|_{H^1_\eta(\mathbb{R})} \text{ for all } u, v \in H^1_\eta(\mathbb{R}), \qquad (5.2)\\ a_\eta(u,u) &\geq \alpha \|u\|^2_{H^1_\eta(\mathbb{R})} - \lambda \|u\|^2_{L^2_\eta(\mathbb{R})} \text{ for all } u \in H^1_\eta(\mathbb{R}). \qquad (5.3)\end{aligned}$$

Recall that the pay-off functions are $\psi(x) = (K-e^x)_+$ for a put option and $\psi(x) = (e^x-K)_+$ for a call option. With $\eta > 0$ and $\eta > 1$, respectively, it holds that $\psi \in H^1_\eta$. With the admissible set

$$\mathcal{K}_\psi := \{v \in H^1_\eta(\mathbb{R}) | v \geq \psi \text{ a.e.}\},$$

the valuation of American options leads to an unsymmetric variational inequality on an unbounded domain (cf. Jaillet et al. [1990]).

Problem 5.3 (*P*). Find $u \in L^2(0,T;H^1_\eta)$, $\dot{u} \in L^2(0,T;L^2_\eta)$ such that $u \in \mathcal{K}_\psi$ almost everywhere in $(0,T]$ and there holds

$$\begin{aligned} \left(\dot{u}, v-u\right)_{L^2_\eta} + a_\eta(u, v-u) \geq 0 \text{ for all } v \in \mathcal{K}_\psi,\\ u(\cdot,0) = \psi(\cdot) =: u_0(\cdot). \end{aligned} \qquad (5.4)$$

Theorem 5.2. *Given the constants r, σ, and d, the pay-off function $\psi(x)$ and η such that $\psi \in H^1_\eta$. Then there exists a unique solution $u \in L^2(0,T;H^1_\eta)$, $\dot{u} \in L^2(0,T;L^2_\eta)$ of problem (5.4).*

Instead of solving the variational inequality directly, we apply the Yosida approximation to obtain a nonlinear PDE as in Bensoussan and Lions [1982], Carstensen et al. [1999]. We define the penalisation term as follows.

Definition 5.4 (c^+_η). For $u, v \in H^1_\eta$ define $u^+ := (\psi - u)_+$ and set $c^+_\eta : H^1_\eta \times H^1_\eta \to \mathbb{R}$ by

$$c^+_\eta(u,v) := -\frac{1}{\varepsilon}\int_{\mathbb{R}} u^+ v \exp(-2\eta|x|)\, dx. \qquad (5.5)$$

Problem 5.5 (P_ε). The problem (P_ε) consists in finding $u_\varepsilon \in L^2(0,T;H^1_\eta)$ with $\frac{\partial u_\varepsilon}{\partial t} \in L^2(0,T;(H^1_\eta)^*)$, $u_\varepsilon(\cdot,0) = \psi(\cdot)$, and

$$(\dot{u}_\varepsilon, v)_{L^2_\eta} + a_\eta(u_\varepsilon, v) + c^+_\eta(u_\varepsilon, v) = 0 \quad \text{ for all } v \in V. \qquad (5.6)$$

Theorem 5.3. *Problem (5.6) has a unique solution $u_\varepsilon \in L^2(0,T;H^1_\eta)$ such that $\dot{u}_\varepsilon \in L^2(0,T;(H^1_\eta)^*)$.*

Proof. With the above properties of u_0, $\mathcal{K}_\psi$, ψ, and $a_\eta(\cdot,\cdot)$, Theorem 2.3 in Chapter 3, Section 2 of Bensoussan and Lions [1982] leads to the assertion. $\square$

Proposition 5.4. *The mapping $c_\eta^+ : V \times V \to \mathbb{R}$ is linear in the second component and monotone in the sense that there holds for all $u, v \in V$*

$$\frac{1}{\varepsilon}\int_{\mathbb{R}} (u^+ - v^+)^2 \exp(-2\eta|x|)\,dx \le c_\eta^+(u, u-v) - c_\eta^+(v, u-v). \tag{5.7}$$

Proof. The linearity of the second component follows directly from the definition. To prove (5.7) note that the right hand side of inequality (5.7) can be written as

$$-\frac{1}{\varepsilon}\int_{\mathbb{R}} (u^+ - v^+)(u-v)\exp(-2\eta|x|)\,dx.$$

Therefore it remains to show that $(u^+ - v^+)^2 \le -(u^+ - v^+)(u-v)$. Set $a := \psi - u$, $b := \psi - v$, $a^+ := \max(a,0)$, and $a^- = \min(a,0)$ and note that $a = a^+ + a^-$ and $a^+a^- = 0$. Using $b - a = u - v$ and $a^- \le 0 \le b^+$ one obtains

$$-(a^+ - b^+)(b-a) = (a^+ - b^+)^2 - (a^+b^- + b^+a^-) \ge (a^+ - b^+)^2. \qquad \square$$

5.2 Semi-discrete Model

To solve Problem 5.5 we discretise the space by P_1 finite elements corresponding to the basis functions φ_j, $j = 0, \dots, N$, defined below.

Definition 5.6 (FE-Basis). Set $-\infty < x_0 < x_1, \dots, < x_N < \infty$ where the interval (x_0, x_N) is chosen sufficiently large. The basis functions $\varphi_0, \dots, \varphi_N$ are defined by

$$\varphi_k(x) := \begin{cases} (x - x_{k-1})/h_k & \text{for } x \in (x_{k-1}, x_k], \\ (x_{k+1} - x)/h_{k+1} & \text{for } x \in (x_k, x_{k+1}], \\ 0 & \text{elsewhere,} \end{cases} \tag{5.8}$$

for $k = 1, \cdots N-1$ while, for $k = 0$ and $k = N$,

$$\varphi_0(x) := \begin{cases} f_{x_0}(x) & \text{for } x < x_0, \\ (x_1 - x)/h_1 & \text{for } x \in [x_0, x_1], \\ 0 & \text{elsewhere;} \end{cases} \tag{5.9}$$

$$\varphi_N(x) := \begin{cases} f_{x_N}(x) & \text{for } x > x_N, \\ (x - x_{N-1})/h_N & \text{for } x \in [x_{N-1}, x_N], \\ 0 & \text{elsewhere.} \end{cases} \tag{5.10}$$

with $f_{x_0} \in H^1_\eta(-\infty, x_0)$ and $f_{x_N} \in H^1_\eta(x_N, \infty)$ and satisfy $f_{x_0}(x_0) = 1 = f_{x_N}(x_N)$. Further let $h_j := x_j - x_{j-1}$ and $T_j := (x_{j-1}, x_j)$ for $j = 1, \dots, N$, $T_0 := (-\infty, x_0)$, and $T_{N+1} = (x_N, \infty)$. Finally, we define the discrete space $\bar{V}_h$ by

$$\bar{V}_h = \operatorname{span}\{\varphi_0, \dots, \varphi_N\}. \tag{5.11}$$

The determination of the truncation points x_0 and x_N is the main concern of Subsection 5.2.1 and 5.2.2. We fix the semi-discrete solution $u_{\varepsilon h}$ on the intervals T_0 and T_{N+1} according to information available from the free boundary value problem, cf. Subsection 5.2.1 and 5.2.2. Hence we seek the solution $u_{\varepsilon h}$ of Problem 5.5 on the inner nodes $x_1, \ldots, x_{N-1}$, i.e., we define the space of test functions by

$$V_h = \operatorname{span}\{\varphi_1, \ldots, \varphi_{N-1}\}.$$

Then, the solution $u_{\varepsilon h}$ in the inner domain (x_0, x_N) is determined by the following problem. Extending $u_{\varepsilon h}$ by $u_{\varepsilon h}(x) = \psi(x)$ for $x < x_0$ and $x > x_N$ (cf. Subsection 5.2.3) yields a semi-discrete solution $u_{\varepsilon h}$ on $\mathbb{R}$ given by the subsequent problem.

Problem 5.7 ($P_{\varepsilon h}$). The semi discrete problem $(P_{\varepsilon h})$ reads: Find $u_{\varepsilon h} \in H^1(0,T;\bar{V}_h)$ such that $u_{\varepsilon h}(\cdot,0) = \psi(\cdot)$, $u_{\varepsilon h}(x) = \psi(x)$ for $x < x_0$ and $x > x_N$, and

$$(\dot{u}_{\varepsilon h}, v_h)_{L^2_\eta} + a_\eta(u_{\varepsilon h}, v_h) + c^+_\eta(u_{\varepsilon h}, v_h) = 0 \quad \text{for all } v_h \in V_h. \tag{5.12}$$

Finally we define the semi-discrete solution $u_{\varepsilon h}$ in terms of the basis functions φ_j and time-dependent coefficients $c_j(t)$.

Definition 5.8. Given the time-dependent coefficients $c_0(t), \ldots, c_N(t) \in H^1(0,T)$ the semi-discrete solution $u_{\varepsilon h} \in \bar{V}_h$ is written by

$$u_{\varepsilon h}(x,t) = \sum_{j=0}^{N} c_j(t)\varphi_j(x). \tag{5.13}$$

Given the basis functions φ_j, $j = 1, \ldots, N-1$, from Definition 5.6 we define a nodal interpolation operator $\mathcal{I} : H^1_\eta(\mathbb{R}) \to V_h$.

Definition 5.9 (Nodal Interpolation). Given a partition $-\infty < x_0 < \cdots < x_N < \infty$ and subordinated nodal basis functions φ_j from Definition 5.6. Then, for a function $f \in H^1_\eta$, the nodal interpolant $\mathcal{I}f : H^1_\eta(\mathbb{R}) \to V_h$ is defined by

$$\mathcal{I}f(x) := \sum_{j=1}^{N-1} f(x_j)\varphi_j(x). \tag{5.14}$$

Remark 5.10. Note that $\mathcal{I}f(x) = 0$ if $x < x_0$ or $x > x_N$.

Finally, we define a nodal interpolation operator $\bar{\mathcal{I}} : H^1_\eta \to \bar{V}_h$. Given the basis functions φ_j, $j = 0, \ldots, N$ from Definition 5.6 we define a nodal interpolation operator $\bar{\mathcal{I}} : H^1_\eta(\mathbb{R}) \to \bar{V}_h$ by

$$\bar{\mathcal{I}}f(x) := \sum_{j=0}^{N} f(x_j)\varphi_j(x). \tag{5.15}$$

The next two subsections discuss appropriate truncation points x_0 and x_N to define the computational domain for American call and put options. Moreover, we define suitable functions f_{x_0} and f_{x_N} and the coefficients $c_0(t)$ and $c_N(t)$.

5.2.1 American Put Options

This subsection aims to find the truncation points x_0 and x_N as well as f_{x_0} and f_{x_N} for American put options. Recall that for American put options we always assume that $r > 0$ since early exercise is never optimal if $r = 0$, cf. Subsection 2.2.4. In Section 3.4 we proved that for $x > x_N^P = x_N^P(r, \sigma^2, d, K, T)$ (with x_N^P defined in Theorem 3.20 on page 42) and all $\kappa > \sigma^2 t$ there holds for the value $u(x,t)$ of an American put option

$$|u(x,t)\exp(-x^2/\kappa)| \leq C.$$

This indicates to choose $x_N > x_N^P$ and set the discrete solution $u_{\varepsilon h} = 0$ for $x \in T_{N+1}$.
Finding the truncation point x_0 is based on properties of the free boundary $x_f(t)$. In Subsection 2.2.4 we cited some properties of the free boundary from Kwok [1998]. Here, we repeat the main result in log-prices. The free boundary $x_f(t)$ is bounded for all $t \in [0,T]$ by

$$\log\Big(\frac{\mu_-}{\mu_- - 1}K\Big) \leq x_f(t) \leq \log\Big(\min\Big(K, \frac{r}{d}K\Big)\Big),$$

where

$$\mu_- = -\frac{(r-d-\sigma^2/2) + \sqrt{(r-d-\sigma^2/2)^2 + 2r\sigma^2}}{\sigma^2} < 0.$$

Since the solution u equals the pay-off function $\psi(x) = (K - e^x)_+$ for $x < x_f(t)$, cf. (3.23), we choose the truncation point $x_0 < \log\Big(\frac{\mu_-}{\mu_- - 1}K\Big)$ and set the semi-discrete solution $u_{\varepsilon h} = K - e^x$ on T_0.
It remains to find f_{x_0} and f_{x_N}. Since the semi-discrete solution $u_{\varepsilon h} = 0$ on T_{N+1} we set $f_{x_N} = 1$ and $c_N(t) = 0$. Note that $f_{x_N} \in H_\eta^1(T_{N+1})$. On T_0 the semi-discrete solution $u_{\varepsilon h} = K - e^x$. Thus, $f_{x_0}(x) = \frac{K - e^x}{K - e^{x_0}}$ and $c_0(t) = K - e^{x_0}$. Note that $f_{x_0} \in H_\eta^1(T_0)$ and $f_{x_0}(x_0) = 1$.

5.2.2 American Call Options

This subsection aims to determine x_0, x_N, f_{x_0}, and f_{x_N} for American call options. Recall that we assume $d > 0$, since early exercise is never optimal if $d = 0$ for American call options. In Section 3.4 we proved that there exists a treshold $x_0^C = x_0^C(r, d, \sigma^2, K, T)$ (with x_0^C defined in Theorem 3.29 on page 51) such that there holds for $x < x_0^C$, $t > 0$, and $\kappa > 2\sigma^2 t$ that

$$|u(x,t)\exp(x^2/\kappa)| \leq C.$$

This indicates to choose $x_0 < x_0^C$ and set the semi-discrete solution $u_{\varepsilon h} = 0$ in T_0. To find the truncation point x_N we state a main property of the free boundary for American call options, cf. Subsection 2.2.4. Here, we repeat the main result in log-prices. The free boundary $x_f(t)$ is bounded for all $t \in [0,T]$ by

$$\log\Big(\max\Big(K, \frac{r}{d}K\Big)\Big) \leq x_f(t) \leq \log\Big(\frac{\mu_+}{\mu_+ - 1}K\Big),$$

where

$$\mu_+ = \frac{-(r-d-\sigma^2/2) + \sqrt{(r-d-\sigma^2/2)^2 + 2r\sigma^2}}{\sigma^2} > 1.$$

Since the value of an American call equals its pay-off function $\psi(x) = (e^x - K)_+$ for $x > x_f(t)$, cf. (3.48), we choose $x_N > \log\left(\frac{\mu_+}{\mu_+ - 1}K\right)$ and set the semi-discrete solution $u_{\varepsilon h} = e^x - K$ on T_{N+1}. Finally we define the functions f_{x_0} and f_{x_N}. Since $u_{\varepsilon h} = 0$ on T_0 we choose $f_{x_0} = 1$ and $c_0(t) = 0$. On T_{N+1} the solution $u_{\varepsilon h} = e^x - K$, hence $f_{x_N}(x) = \frac{e^x - K}{e^{x_N} - K}$ and $c_N(t) = e^{x_N} - K$. Note that $f_{x_N} \in H^1_\eta(T_{N+1})$ and $f_{x_N}(x_N) = 1$.

5.2.3 Summary

Table 5.1 summaries the truncation points x_0 and x_N, the functions f_{x_0} and f_{x_N}, the coefficients $c_0(t)$ and $c_N(t)$, as well as the semidiscrete solution $u_{\varepsilon h}$ on $(-\infty, x_0) \cup (x_N, \infty)$ for American put and call options from Subsection 5.2.1 and 5.2.2, respectively.

	American put option	American call option
pay-off function $\psi(x)$	$(K - e^x)_+$	$(e^x - K)_+$
x_0	$x_0 < \log(\frac{\mu_-}{\mu_- - 1}K)$	x_0^C from Theorem 3.29
x_N	x_N^P from Theorem 3.20	$x_N > \log(\frac{\mu_+}{\mu_+ - 1}K)$
f_{x_0}	$\frac{K - e^x}{K - e^{x_0}}$	1
f_{x_N}	1	$\frac{e^x - K}{e^{x_N} - K}$
$c_0(t)$	$K - e^{x_0}$	0
$c_N(t)$	0	$e^{x_N} - K$
$u_{\varepsilon h}$ for $x < x_0$	$\psi(x) = K - e^x$	$\psi(x) = 0$
$u_{\varepsilon h}$ for $x > x_N$	$\psi(x) = 0$	$\psi(x) = e^x - K$

Table 5.1: Summary of the critical values for x_0 and x_N, the semi-discrete solution $u_{\varepsilon h}$ on the outer domain, and the corresponding basis functions of $\bar{V}_h$.

Let u be the solution of the variational inequality 5.4 and $u_{\varepsilon h}$ the semi-discrete solution of the penalised problem 5.12. Define $\tilde{\mathcal{A}}_{\varepsilon,h}$ on each interval T_j, $j = 0, \ldots, N+1$ by

$$\tilde{\mathcal{A}}_{\varepsilon,h}[u] := \dot{u} - \frac{\sigma^2}{2}u'' - \left(r - d - \frac{\sigma^2}{2}\right)u' + ru - \frac{1}{\varepsilon}u^+,$$

cf. (5.64), with $u^+ := (\psi - u)_+$, i.e. $u \geq \psi$ implies $u^+ = 0$. Note that the operator $\tilde{\mathcal{A}}_{\varepsilon,h}$ corresponds to the weak penalised formulation (5.6). Since $\psi(x) = 0$ for $x > \log K$ for a American put option and $\psi(x) = 0$ for $x < \log K$ for a American call option, $u^+_{\varepsilon h} = 0$ holds for $x > x_N$ and $x < x_0$, respectively, from Table 5.1. Consequently, $\tilde{\mathcal{A}}_{\varepsilon,h}[u_{\varepsilon h}] = 0$ for $x > x_N$ for American put options and $\tilde{\mathcal{A}}_{\varepsilon,h}[ueh] = 0$ for $x < x_0$ for American call options.
Recall the operator $\tilde{\mathcal{A}}$ from (5.86),

$$\tilde{\mathcal{A}}[u] := \dot{u} - \frac{\sigma^2}{2}u'' - \left(r - d - \frac{\sigma^2}{2}\right)u' + ru.$$

Note that there holds

$$\tilde{\mathcal{A}}[u] = \dot{u} + \mathcal{A}[u]$$

with the operator $\mathcal{A}$ of the Black-Scholes equation in forward time and log-prices (3.4) on page 16. Since u is also a solution of the FBVP (3.23) on page 25 and FBVP (3.48) on page 32 for American put and call options, u satisfies the Black-Scholes equation, i.e. $\tilde{\mathcal{A}}[u] = 0$ for $x > x_N$ and $x < x_0$, respectively. Recall that we choose x_0 for an American put option and x_N for an American call option, such that $x_0 < x_f(t)$ and $x_N > x_f(t)$, respectively. We summerise the results from above, which are required in the proof of Theorem 5.22, in Table 5.2.

	American put option	American call option
$u = \psi(x)$	for $x \le x_0$	for $x \ge x_N$
$u_{\varepsilon h} = \psi(x)$	for $x \le x_0$	for $x \ge x_N$
$u_{\varepsilon h} = 0$	for $x \ge x_N$	for $x \le x_0$
$e = u - u_{\varepsilon h} = 0$	for $x \le x_0$	for $x \ge x_N$
$\tilde{\mathcal{A}}_{\varepsilon,h}[u_{\varepsilon h}] = 0$	for $x \ge x_N$	for $x \le x_0$
$\tilde{\mathcal{A}}[u] = 0$	for $x \ge x_N$	for $x \le x_0$

(5.16)

Table 5.2: Some properties of u, $u_{\varepsilon h}$, $\tilde{\mathcal{A}}$, and $\tilde{\mathcal{A}}_{\varepsilon,h}$ required in the proof of Theorem 5.22.

5.3 Approximation in the Weighted Sobolev Space H^1_η

This section establishes an approximation error estimate in the L^2_η-norm of the form

$$\|f - \mathcal{I}f\|_{L^2_\eta} \le C \, \|f'\|_{L^2_\eta} \,. \tag{5.17}$$

Estimate (5.17) is required in the residual based a posteriori error estimates in Section 5.4. Especially the determination of the constant C (or an upper bound) is of great importance. We proceed as follows. First we extend a weighted Poincare inequality (for function with integral mean zero) for smooth weights from Payne and Weinberger [1960] to weights belonging to $W^{2,1}(a,b)$. Then we approximate our weights $p(x) = \exp(-2\eta|x|)$ by a weight $p_\delta \in W^{2,1}(a,b)$ satisfying all condition such that the extended Payne and Weinberger Theorem is applicable. By applying a reflection argument the estimate may be extended to functions $f \in H^1_0(a,b)$. The last step is to estimate the gradient $(f - \mathcal{I}f)'$ by the gradient f'. The next theorem from Payne and Weinberger [1960] gives a weighted Poincare inequality for smooth weights.

Theorem 5.5 (Payne & Weinberger). *Suppose that $p \in C^2(a,b)$ is strictly positive on (a,b) with $(p'/p)' \le 0$ on (a,b). Let $m(f) := \|p\|^{-1}_{L^1(a,b)} \int_a^b pf \, dx$ denote the weighted integral mean of $f \in H^1(a,b)$. Then*

$$\big\|\sqrt{p}\big(f - m(f)\big)\big\|_{L^2(a,b)} \le \frac{b-a}{\pi} \, \|\sqrt{p} f'\|_{L^2(a,b)} \,.$$

Proof. The proof of the theorem can be found in Payne and Weinberger [1960]. □

The assumption $p \in C^2(a,b)$ can be immediately weakened since in the proof of Theorem 5.5 is crucial that there holds for some $v \in H^1_0(a,b) \cap H^2(a,b)$

$$\int_a^b \left(\frac{p'}{p}\right)' \frac{v^2}{2}\, dx \leq 0. \tag{5.18}$$

Consequently it suffices that $p \in W^{2,1}(a,b)$ which leads to the next corollary.

Corollary 5.6. *Let $p \in W^{2,1}(a,b)$ with $(p'/p)' \leq 0$ almost everywhere. Then, there holds for all $f \in H^1(a,b)$*

$$\left\|\sqrt{p}\big(f - m(f)\big)\right\|_{L^2} \leq \frac{b-a}{\pi} \left\|\sqrt{p} f'\right\|_{L^2}. \tag{5.19}$$

Remark 5.11. We will sometimes refere to Corollary 5.6 as the extended Theorem of Payne & Weinberger.

Since $p = \exp(-2\eta|x|)$ does not belong to $W^{2,1}(a,b)$ we need to find a strictly positive approximation $p_\delta \in W^{1,2}(\Omega)$ which satisfies $(p'/p)' \leq 0$ almost everywhere. Note that $p = \exp(-2\eta|x|)$ satisfies $(p'/p)' = 0$ for all $x \in (a,b) \setminus \{0\}$. Since $p''/p - (p'/p)^2 \leq 0$ for concave functions p, we constructed p_δ as follows. Given a ball with centre at the y-axis, choose the radius $\delta > 0$ arbitrary small such that ball touches both branches of p. Let $M = (0, y_M)$ denote the centner, and $-\hat{x}$ and $\hat{x}$ the two points where the ball touches p. Then the ball k is given as $k : x^2 + (y - y_M)^2 = \delta^2$. By $\tilde{p}$ we refer to the upper half of the ball for $x \in (-\delta, \delta)$. Now define p_δ (illustrated in Figure 5.1) by,

$$p_\delta := \begin{cases} p & \text{for } x \in [a, -\hat{x}] \cup [\hat{x}, b], \\ \tilde{p} & \text{for } x \in (-\hat{x}, \hat{x}). \end{cases} \tag{5.20}$$

Note that $p_\delta \in C^1([a,b])$ and that $-\hat{x}$ and $\hat{x}$ satisfy $|\hat{x}| < \delta$ by construction. Since $p \in C^2([a,-\hat{x}] \cup [\hat{x}, b])$ and $\tilde{p} \in C^2([-\hat{x}, \hat{x}])$ it follows by means of Lemma 5.7 that $p_\delta \in W^{2,1}(a,b)$ holds true. Since $(p_\delta'/p_\delta)' \leq 0$ almost everywhere (note that $(p_\delta'/p_\delta)' = 0$ for $x \notin [-\hat{x}, \hat{x}]$ and that $\tilde{p}'' < 0$), (5.18) holds true. The next lemma justifies $p_\delta \in W^{2,1}(a,b)$.

Lemma 5.7. *Suppose $f \in C^0([a,b])$ is defined for $c \in (a,b)$ by*

$$f(x) = \begin{cases} g(x) & \text{for } x \geq c, \\ h(x) & \text{for } x < c, \end{cases} \tag{5.21}$$

with $g \in C^1([a,c])$ and $h \in C^1([c,b])$. Then, the weak derivative f' exists in $L^\infty(a,b)$, i.e. $f \in C^{0,1}([a,b])$.

Proof. For all $\varphi \in \mathcal{D}(a,b)$ holds

$$\begin{aligned} \int_a^b f\varphi'\, dx &= g(x)\varphi(x)\Big|_a^c - \int_a^c h'(x)\varphi(x)\, dx + h(x)\varphi(x)\Big|_c^b - \int_c^b g'(x)\varphi(x)\, dx \\ &= -\int_a^b \big(H(c-x)g'(x) + H(c+x)h'(x)\big)\varphi\, dx. \end{aligned}$$

Consequently, the weak derivative $f'(x) = H(c-x)g'(x) + H(c+x)h'(x) \in L^\infty(a,b)$. □

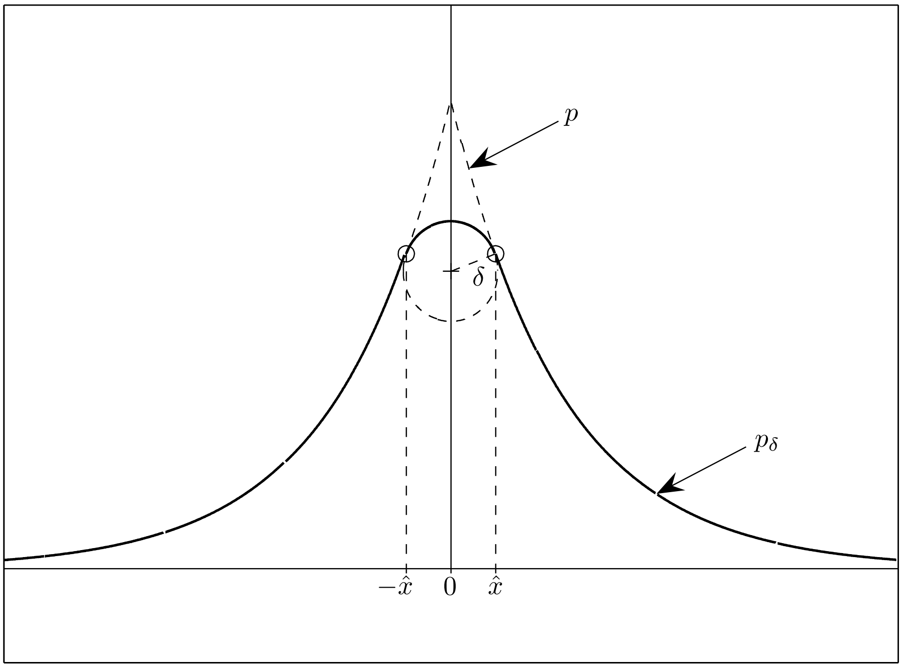

Figure 5.1: The smoothed weight function p_δ

Remark 5.12. If $h' \in L^q(0,a)$ and $g' \in L^q(0,b)$ it follows directly from the proof that $f' \in L^q(a,b)$ for $1 \le q \le \infty$.

Remark 5.13. Corollary (5.6) also holds for a smoothed version of $\hat{p}(\cdot) = \alpha \exp(-2\eta|\cdot - \beta|)$ for $\alpha > 0$ and $\beta \in \mathbb{R}$, (i.e. p is shifted by β and scaled with α) since the smoothed $\hat{p}_\delta$ also satisfies $(\hat{p}'_\delta/\hat{p}_\delta)' \le 0$.

The next step is to prove an estimate for functions $f \in H_0^1(a,b)$, which is done by using some reflexion trick, so that the extended theorem of Payne & Weinberger is applicable.

Proposition 5.8. *For $p = \exp(-2\eta|x|)$ with $\eta \ge 0$, $f \in H_0^1(a,b)$, (a,b) bounded, and $a \cdot b \ge 0$ there holds*

$$\|\sqrt{p} f\|_{L^2(a,b)} \le 2(b-a)/\pi \, \|\sqrt{p} f'\|_{L^2(a,b)} \, .$$

Proof. First we consider an interval (a,b) with $a, b \ge 0$. Let $b' = a - (b-a)$ be the reflexion of b in a. Reflect now the weight function p in a and define as $\hat{p}$ the extension of p in (b',b). Likewise reflect the function f in a and afterward at the x-axis. Let $\hat{f}$ in (b',b) denote the resulting extension of f illustrated in Figure 5.2. Since the extended weight function $\hat{p}$ has again a corner (this time in a), we apply the same trick with the smoothing ball as before. Let $\hat{p}_\delta$ denote the smoothed $\hat{p}$. With elementary geometric arguments one proves $\int_a^b \hat{p}_\delta \hat{f}\,dx = -\int_{b'}^a \hat{p}_\delta \hat{f}\,dx$, and consequently $\int_{b'}^b \hat{p}_\delta \hat{f}\,dx = 0$ holds true. Hence the weighted integral mean of $\hat{f}$ satisfies

$$m(\hat{f}) = \|\hat{p}_\delta\|_{L^1}^{-1} \int_{b'}^b \hat{p}_\delta \hat{f}\,dx = 0.$$

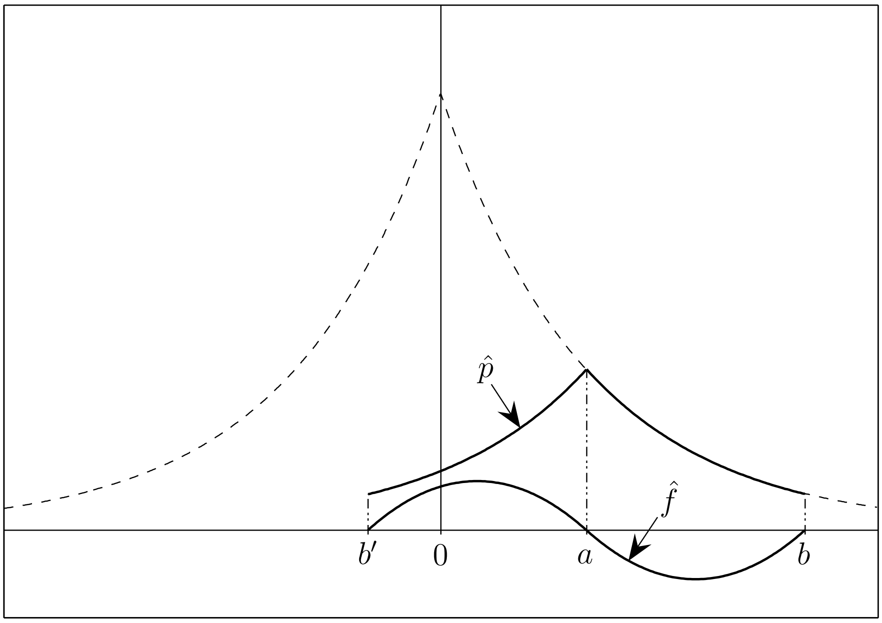

Figure 5.2: The weight function p and its extension $\hat{p}$ and $f \in H^1_0(a,b)$ with extension $\hat{f}$ as used in the proof of Proposition 5.8.

Note that $\hat{f} \in H^1_0(b',b)$ by means of Remark 5.12 for $q = 2$. Then there holds for $\hat{p}_\delta$ and for all $\hat{f} \in H^1_0(b',b)$, cf. Corollary 5.6,

$$\left\| \sqrt{\hat{p}_\delta} \hat{f} \right\|_{L^2(b',b)} \leq \frac{b-b'}{\pi} \left\| \sqrt{\hat{p}_\delta} \hat{f}' \right\|_{L^2(b',b)} . \tag{5.22}$$

Now we show that (5.22) also holds for the unsmoothed extension $\hat{p}$. Note that $\hat{p}_\delta(x) \leq \hat{p}(x)$ for all $x \in (b',b)$. Therefore there holds

$$\left\| \sqrt{\hat{p}_\delta} \hat{f}' \right\|_{L^2(b',b)} \leq \left\| \sqrt{\hat{p}} \hat{f}' \right\|_{L^2(b',b)} . \tag{5.23}$$

The next step is to prove that there holds

$$\left\| \sqrt{\hat{p}} \hat{f} \right\|_{L^2(b',b)} \leq \left\| \sqrt{\hat{p}_\delta} \hat{f} \right\|_{L^2(b',b)} . \tag{5.24}$$

Using the triangle inequality yields

$$\left\| \sqrt{\hat{p}} \hat{f} \right\|_{L^2(b',b)} \leq \left\| (\sqrt{\hat{p}} - \sqrt{\hat{p}_\delta}) \hat{f} \right\|_{L^2(b',b)} + \left\| \sqrt{\hat{p}_\delta} \hat{f} \right\|_{L^2(b',b)} . \tag{5.25}$$

Applying Hölders inequality on the first term of the right-hand-side of (5.25) yields

$$\left\| (\sqrt{\hat{p}} - \sqrt{\hat{p}_\delta}) \hat{f} \right\|_{L^2(b',b)} \leq \left\| (\sqrt{\hat{p}} - \sqrt{\hat{p}_\delta})^2 \right\|^{1/2}_{L^1(b',b)} \left\| \hat{f}^2 \right\|^{1/2}_{L^\infty(b',b)} .$$

With $b' \leq \hat{x}_1 < \hat{x}_2 \leq b$, where $\hat{x}_1$ and $\hat{x}_2$ denote the two points where the ball touches $\hat{p}$, there holds

$$\left\|(\sqrt{\hat{p}}-\sqrt{\hat{p}_\delta})^2\right\|_{L^1(b',b)}^{1/2} \leq \left\|\sqrt{\hat{p}}-\sqrt{\hat{p}_\delta}\right\|_{L^2(b',\hat{x}_1)} + \left\|\sqrt{\hat{p}}-\sqrt{\hat{p}_\delta}\right\|_{L^2(\hat{x}_1,\hat{x}_2)} + \left\|\sqrt{\hat{p}}-\sqrt{\hat{p}_\delta}\right\|_{L^2(\hat{x},b)}. \tag{5.26}$$

Since $\hat{p} = \hat{p}_\delta$ on $(b', \hat{x}_1) \cup (\hat{x}_2, b)$ the first and the third term of the right-hand-side of (5.26) vanish. For the second term there holds for any arbitrary $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$\left\|\sqrt{\hat{p}}-\sqrt{\hat{p}_\delta}\right\|_{L^2(\hat{x}_1,\hat{x}_2)} \leq \varepsilon. \tag{5.27}$$

As ε may be arbitrarily small we conclude that (5.24) holds. Combining (5.22), (5.23), and (5.24) yields the estimate for $\hat{p}$ and $\hat{f}$,

$$\left\|\sqrt{\hat{p}}\hat{f}\right\|_{L^2(b',b)} \leq \frac{b-b'}{\pi}\left\|\sqrt{\hat{p}}\hat{f}'\right\|_{L^2(b',b)}. \tag{5.28}$$

Recall the definition of $\hat{p}$ and $\hat{f}$ and note that $2\left\|\sqrt{p}f\right\|_{L^2(a,b)}^2 = \left\|\sqrt{\hat{p}}\hat{f}\right\|_{L^2(b',b)}^2$. Using (5.28) one obtains

$$\|\sqrt{p}f\|_{L^2(a,b)}^2 = 1/2\left\|\sqrt{\hat{p}}\hat{f}\right\|_{L^2(b',b)}^2 \leq 4(b-a)^2/(2\pi^2)\left\|\sqrt{\hat{p}}\hat{f}'\right\|_{L^2(b',b)}^2 = 4(b-a)^2/\pi^2\left\|\sqrt{p}f'\right\|_{L^2(a,b)}^2. \tag{5.29}$$

Taking the square root on both sides of inequality (5.29) concludes the proof. If $a, b \leq 0$, reflect f and p at b (instead of a) to obtain the extended functions $\hat{p}$ and $\hat{f}$ as before. Then, the same arguments as above apply. □

Up to now we have shown that there holds for $f \in H_0^1(a,b)$

$$\|\sqrt{p}f\|_{L^2(a,b)} \leq \frac{2(b-a)}{\pi}\|\sqrt{p}f'\|_{L^2(a,b)}. \tag{5.30}$$

With the definition of the nodal interpolation operator $\mathcal{I}$, i.e., $f - \mathcal{I}f$ vanishes at a and b, and the definition of the L^2_η-norm, (5.30) may be written

$$\|f-\mathcal{I}f\|_{L^2_\eta(a,b)} \leq \frac{2(b-a)}{\pi}\|(f-\mathcal{I}f)'\|_{L^2_\eta(a,b)}.$$

In order to obtain (5.17) we need to show that there exits a constant C such that

$$\|(f-\mathcal{I}f)'\|_{L^2_\eta} \leq C\|f'\|_{L^2_\eta}.$$

The next proposition determines this constant $C = C(h, \eta)$.

Proposition 5.9. *For $h := b - a$ and $a \cdot b \geq 0$ define*

$$C(\eta, h) := \sqrt{\frac{\cosh(2\eta h) - 1}{2\eta^2 h^2}}.$$

Then, there holds for all $f \in H^1_\eta(a,b)$ that

$$\|(f-\mathcal{I}f)'\|_{L^2_\eta(a,b)} \leq C(\eta,h)\|f'\|_{L^2_\eta(a,b)}. \tag{5.31}$$

Proof. Recall the definition of the weighted integral mean $m(f) := \|p\|^{-1}_{L^1(a,b)} \int_a^b pf\,dx$. Using the orthogonality of $(f' - m(f'))$ and $(m(f') - (\mathcal{I}f)')$ in the L^2_η sense, i.e.

$$\langle f' - m(f'), m(f') - (\mathcal{I}f)' \rangle_{L^2_\eta(a,b)} = \int_a^b \big(f' - m(f')\big)\big(m(f') - (\mathcal{I}f)'\big) \exp(-2\eta|x|))\,dx = 0$$

one obtains

$$\begin{aligned}\|(f - \mathcal{I}f)'\|^2_{L^2_\eta(a,b)} &= \big\|\big(f' - m(f')\big) + \big(m(f') - (\mathcal{I}f)'\big)\big\|^2_{L^2_\eta} \\ &= \|f' - m(f')\|^2_{L^2_\eta} + \|m(f') - (\mathcal{I}f)'\|^2_{L^2_\eta}.\end{aligned}$$

Note that $\|f' - m(f')\|_{L^2_\eta} \le \|f'\|_{L^2_\eta}$. So it remains to show that $\|m(f') - (\mathcal{I}f)'\|_{L^2_\eta(a,b)}$ is bounded above in terms of $C\,\|f'\|_{L^2_\eta(a,b)}$. Note that $(\mathcal{I}f)' = ⨍_a^b f'\,dx$ and set $\bar{p} := ⨍_a^b p\,dx$. Then,

$$\begin{aligned}\|m(f') - (\mathcal{I}f)'\|^2_{L^2_\eta(a,b)} &= \int_a^b \left(⨍_a^b f'(x)\,dx - \Big(\int_a^b p(x)\,dx\Big)^{-1} \int_a^b f'(x)p(x)\,dx\right)^2 p(y)\,dy \\ &= (b-a)\bar{p}\left(⨍_a^b f'(x)\,dx - \bar{p}^{-1} ⨍_a^b f'(x)p(x)\,dx\right)^2 \\ &= (b-a)^{-1}\left(\sqrt{\bar{p}}\int_a^b f'(x)\,dx - \bar{p}^{-1/2}\Big(\int_a^b f'(x)\big(p(x) - \bar{p}\big)\,dx + \bar{p}\int_a^b f'(x)\,dx\Big)\right)^2 \\ &= (b-a)^{-1}\left(\bar{p}^{-1/2}\int_a^b f'(x)\big(\bar{p} - p(x)\big)\,dx\right)^2 \\ &= \big(\bar{p}(b-a)\big)^{-1}\left(\int_a^b f'(x)\sqrt{p(x)}\Big(\frac{\bar{p}}{\sqrt{p(x)}} - \sqrt{p(x)}\Big)\,dx\right)^2 \\ &\le \big(\bar{p}(b-a)\big)^{-1}\left(\int_a^b \Big(\frac{\bar{p}}{\sqrt{p(x)}} - \sqrt{p(x)}\Big)^2\,dx\right)\left(\int_a^b f'(x)^2 p(x)\,dx\right) \\ &= C\,\|f'\|^2_{L^2_\eta(a,b)}.\end{aligned}$$

The constant C can be calculated for $b > a \ge 0$ as follows.

$$\begin{aligned}C = \big(\bar{p}(b-a)\big)^{-1}\int_a^b (\bar{p}p^{-1/2} - p^{1/2})^2\,dx &= \big(\bar{p}(b-a)\big)^{-1}\int_a^b p^{-1}(\bar{p}^2 - 2p\bar{p} + p^2)\,dx \\ &= \frac{\exp(-2\eta b) - \exp(-2\eta a)}{-2\eta(b-a)^2}\,\frac{\exp(2\eta b) - \exp(2\eta a)}{2\eta} - 1 \\ &= \frac{\cosh(2\eta(b-a)) - 1}{2\eta^2(b-a)^2} - 1.\end{aligned}$$

For $a < b \le 0$ the calculation holds as above and one obtains the same constant C. $\square$

Finally we can formulate the interpolation error estimate in the L^2_η norm by combining Proposition 5.8 on page 70 and 5.9 on page 72.

Theorem 5.10 (L^2_η Interpolation Error). *For $p = \exp(-2\eta|x|)$, $\eta \geq 0$, $f \in H^1_\eta(a,b)$, $\mathcal{I}$ the nodal interpolation operator, $h := b - a$, (a,b) bounded, and $a \cdot b \geq 0$ there holds*

$$\|f - \mathcal{I}f\|_{L^2_\eta(a,b)} \leq hC(\eta,h)\,\|f'\|_{L^2_\eta(a,b)} \tag{5.32}$$

with

$$C(\eta,h) := \frac{2}{\pi}\sqrt{\frac{\cosh(2\eta h)-1}{2\eta^2 h^2}}. \tag{5.33}$$

Proof. Proposition 5.8 and 5.9 yield (5.32). □

Remark 5.14. Note that there holds

$$\lim_{\eta\to 0} C(\eta,h) = \frac{2}{\pi} = \lim_{h\to 0} C(\eta,h).$$

The a priori error analysis in Section 5.5 requires an estimate of the form

$$\|f\|_{L^2_\eta(a,b)} \leq C\,\|f'\|_{L^2_\eta(a,b)}$$

for $f \in H^1_\eta(a,b)$ which vanishes at least at one point in (a,b). The proof is simular as the proof of Theorem 5.10. However, this time we need to split the interval (a,b) into (a,ξ) and (ξ,b) with $f(\xi) = 0$. If $0 \leq a < b$ it turns out that the reflexion technique only applies on (ξ,b). Hence, to estimate f on (a,ξ) another technique is required. For $a < b \leq 0$, the reflexion principle may be applied on (a,ξ), while on (ξ,b) another technique is required.

Corollary 5.11. *Let $p(x) = \exp(-2\eta|x|)$ with $\eta \geq 0$, $h := b-a$, $a \cdot b \geq 0$ and define*

$$C(\eta,h) := \sqrt{\left(\frac{1}{\pi} + \Big(\frac{\cosh(2\eta h)-1}{2\eta^2 h^2)}\Big)^{1/2}\right)^2 + \frac{4}{\pi^2}}. \tag{5.34}$$

Suppose $f \in H^1(a,b)$ vanishes at least at one point in $[a,b]$. Then there holds

$$\|\sqrt{p}f\|_{L^2(a,b)} \leq hC(\eta,h)\,\|\sqrt{p}f'\|_{L^2(a,b)}\,. \tag{5.35}$$

Proof. Assume $0 \leq a < b$, let $\xi \in (a,b)$ with $f(\xi) = 0$. Let us first consider the interval (ξ,b). Extend f point symmetric in ξ and p symmetric in ξ, denote $\hat{f}$ and $\hat{p}$ the extension of f and p, respectively and b' the reflexion of b in ξ, i.e $b' = \xi - (b - \xi)$. Since the extended weight $\hat{p}$ has a corner in ξ we apply the same trick with the smoothening ball as before to obtain the smooth extended weight function $\hat{p}_\delta$. By means of Remark 5.12 with $q = 2$ $\hat{f} \in H^1(b',b)$ holds true. Elementary geometric arguments prove

$$\left\|\hat{f}\sqrt{\hat{p}_\delta}\right\|^2_{L^2(b',b)} = 2\,\|f\sqrt{p_\delta}\|^2_{L^2(\xi,b)}\,.$$

Since

$$\int_{b'}^{\xi} \hat{f}\hat{p}_\delta\,dx = -\int_{\xi}^{b} \hat{f}\hat{p}_\delta\,dx$$

it follows

$$\int_{b'}^{b} \hat{f}\hat{p}_\delta \, dx = 0,$$

and therefore also the weighted integral mean $m(\hat{f}) = 0$. This allows to apply the extended Theorem of Payne and Weinberger to the smoothed weight function $\hat{p}_\delta$. Then,

$$\left\| \sqrt{\hat{p}_\delta}\hat{f} \right\|_{L^2(b',b)} \leq \frac{b-b'}{\pi} \left\| \sqrt{\hat{p}_\delta}\hat{f}' \right\|_{L^2(b',b)}. \tag{5.36}$$

As in the proof of Proposition 5.8, (5.23) and (5.24) hold true. This yields the desired estimate on (ξ, b), namely

$$\|f\sqrt{p}\|_{L^2(\xi,b)} = \frac{1}{\sqrt{2}} \left\| \hat{f}\sqrt{\hat{p}} \right\|_{L^2(b',b)} \leq \frac{1}{\sqrt{2}} \frac{2(b-\xi)}{\pi} \left\| \hat{f}'\sqrt{\hat{p}} \right\|_{L^2(b',b)} = \frac{2(b-\xi)}{\pi} \|f'\sqrt{p}\|_{L^2(\xi,b)}. \tag{5.37}$$

Considering the interval (a, ξ) we cannot apply the reflecting trick as before, because if we reflect p in b Payne and Weinberger cannot be applied to the smoothed weight function p_δ since the condition $\frac{p''}{p} - (\frac{p'}{p})^2 \leq 0$ is not satisfied. Instead, by adding and subtracting $m(f)\sqrt{p}$ and applying the triangle inequality on obtains

$$\|f\sqrt{p}\|_{L^2(a,\xi)} \leq \left\| \big(f - m(f)\big)\sqrt{p} \right\|_{L^2(a,\xi)} + \|m(f)\sqrt{p}\|_{L^2(a,\xi)}. \tag{5.38}$$

The first term can be estimated by directly applying the extended Theorem of Payne & Weinberger to the smoothed p. Then, together with (5.23) and (5.24) there holds

$$\|(f - m(f))\sqrt{p}\|_{L^2(a,\xi)} \leq \frac{\xi - a}{\pi} \|f'\sqrt{p}\|_{L^2(a,\xi)} \leq \frac{b-a}{\pi} \|f'\sqrt{p}\|_{L^2(a,b)}. \tag{5.39}$$

Now we consider the remaining term $\left\|m(f)\sqrt{p}\right\|_{L^2(a,\xi)}$. Using that $f(\xi) = 0$ one obtains $f(x) = -\int_x^\xi f'(y)\, dy$. Then,

$$\begin{aligned} &|f(x)| \\ &\leq \int_x^\xi \big|f'(y)\exp(-\eta y)\exp(\eta y)\big| \, dy \leq \Big(\int_x^\xi f'(y)^2 \exp(-2\eta y)\, dy\Big)^{1/2} \Big(\int_x^\xi \exp(2\eta y)\, dy\Big)^{1/2} \\ &\leq \|f'\sqrt{p}\|_{L^2(a,\xi)} \Big(\int_a^\xi \exp(2\eta y)\, dy\Big)^{1/2}. \end{aligned}$$

With the definition of the weighted integral mean $m(f) = \|p\|^{-1}_{L^1(a,b)} \int_a^b fp\, dx$ there holds

$$|m(f)| \leq \frac{\left\|f'\sqrt{p}\right\|_{L^2(a,\xi)}}{\|p\|_{L^1(a,b)}} \Big(\int_a^\xi \exp(2\eta y)\, dy\Big)^{1/2} \int_a^b p\, dx = \Big(\int_a^\xi \exp(2\eta y)\, dy\Big)^{1/2} \|f'\sqrt{p}\|_{L^2(a,\xi)},$$

and consequently

$$\begin{aligned} \|m(f)\sqrt{p}\|_{L^2(a,\xi)} &= |m(f)| \Big(\int_a^\xi p\, dx\Big)^{1/2} \leq \Big(\int_a^b \exp(2\eta y)\, dy\Big)^{1/2} \Big(\int_a^b p\, dx\Big)^{1/2} \|f'\sqrt{p}\|_{L^2(a,b)} \\ &= \sqrt{\frac{\cosh(2\eta(b-a)) - 1}{2\eta^2(b-a)^2}} (b-a) \|f'\sqrt{p}\|_{L^2(a,b)}. \end{aligned} \tag{5.40}$$

Using the estimates (5.39) and (5.40) in (5.38) yields

$$\|f\sqrt{p}\|_{L^2(a,\xi)} \leq \left(\frac{1}{\pi} + \sqrt{\frac{\cosh(2\eta(\xi-a))-1}{2\eta^2(\xi-a)^2}}\right)(\xi-a)\,\|f'\sqrt{p}\|_{L^2(a,\xi)}\,. \tag{5.41}$$

Using $\|f\sqrt{p}\|^2_{L^2(a,b)} = \|f\sqrt{p}\|^2_{L^2(a,\xi)} + \|f\sqrt{p}\|^2_{L^2(\xi,b)}$, (5.37) and (5.41) yields

$$\|f\sqrt{p}\|^2_{L^2(a,b)} \leq \left(\frac{4}{\pi^2} + \left(\frac{1}{\pi} + \sqrt{\frac{\cosh(2\eta(b-a))-1}{2\eta^2(b-a)^2}}\right)^2\right)(b-a)^2\,\|f'\sqrt{p}\|^2_{L^2(a,b)}\,,$$

□

which proves 5.35. If $a < b \leq 0$ holds, let ξ be a root in (a,b). Then, take the interval (a,ξ) and apply the reflection and extension principle as before, while for the interval (ξ,b) one has to consider the estimate with adding and subtracting $m(f)$ and applying the triangle inequality, cf. (5.38).

The proof of Theorem 5.22 requires for $f \in H^2_\eta$ approximation error estimates of $\|f-\mathcal{I}f\|_{L^2_\eta}$ and $\|(f-\mathcal{I}f)'\|_{L^2_\eta}$. The next Corollary establishes such estimates using Theorem 5.8 on page 70 and Corollary 5.11 on page 74.

Corollary 5.12. *Let the constant $C_1 = C_1(\eta,h)$ and $C_2 = C_2(\eta,h)$ be defined by*

$$C_1(\eta,h) := \frac{2}{\pi}C(\eta,h), \quad C_2(\eta,h) := C(\eta,h)\sqrt{\frac{4}{\pi}h^2+1} \tag{5.42}$$

where $C(\eta,h)$ is defined in (5.34). Assume that there holds $a\cdot b \geq 0$ and set $h := b-a$. Suppose $f \in H^2_\eta(a,b)$. Then,

$$\|f-\mathcal{I}f\|_{L^2_\eta(a,b)} \leq C_1(\eta,h)h^2\,\|f''\|_{L^2_\eta(a,b)} \tag{5.43}$$

and

$$\|f-\mathcal{I}f\|_{H^1_\eta(a,b)} \leq C_2(\eta,h)h\,\|f''\|_{L^2_\eta(a,b)}\,. \tag{5.44}$$

Proof. In Theorem 5.8 we proof that there holds for $f \in H^1_\eta(a,b)$

$$\|f-\mathcal{I}f\|_{L^2_\eta(T_j)} \leq {}^2\!/\!_\pi h_j\,\|(f-\mathcal{I}f)'\|_{L^2_\eta(T_j)}\,. \tag{5.45}$$

Since

$$\int_a^b (f-\mathcal{I}f)'dx = (f-\mathcal{I}f)(b) - (f-\mathcal{I}f)(a) = 0$$

$(f-\mathcal{I}f)'$ contains at least a root in (a,b), which allows to apply Corollary 5.11 with C from (5.34), i.e.

$$\|(f-\mathcal{I}f)'\|_{L^2_\eta(a,b)} \leq hC(\eta,h)\,\|f''\|_{L^2_\eta(a,b)}\,. \tag{5.46}$$

Combining the estimates (5.45) and (5.46) yields (5.43) and (5.44). □

Up to now we always assumed that $0 \notin (a,b)$. The trick with adding and subtracting $m(f)$ to f as in the last proof also applies for some $f \in H^1(a,b)$ which vanishes at a or b. The next corollary gives the Poincare type inequality for such functions f.

Corollary 5.13. *Let $f \in H^1(a,b)$ with $a < 0 < b$, vanish either at a or b. Then,*

$$\|f\sqrt{p}\|_{L^2(a,b)} \leq \left(\frac{1}{\pi}+\sqrt{\frac{2\big(\cosh(2\eta b)+\cosh(2\eta a)\big)-\cosh(2\eta(b+a))-3}{2\eta^2(b-a)^2}}\right)(b-a)\,\|f'\sqrt{p}\|_{L^2(a,b)}\,.$$

Proof. We proceed as in the second part of the proof of Corollary 5.11. Using the triangle inequality

$$\|f\sqrt{p}\|_{L^2(a,b)} \leq \|(f-m(f))\sqrt{p}\|_{L^2(a,b)} + \|m\sqrt{p}\|_{L^2(a,b)}\,. \tag{5.47}$$

We consider the second term $\left\|m(f)\sqrt{p}\right\|_{L^2(a,b)}$. Using the same calculations as in the second part of the proof of Corollary 5.11, just note that this time $a < 0 < b$, one obtains

$$\begin{aligned}\|m(f)\sqrt{p}\|_{L^2(a,b)} &\leq \Big(\int_a^b \exp(2\eta|x|)\,dx\Big)^{1/2}\Big(\int_a^b \exp(-2\eta|x|)\,dx\Big)^{1/2}\|f'\sqrt{p}\|_{L^2(a,b)}\\ &= \sqrt{\frac{2\big(\cosh(2\eta b)+\cosh(2\eta a)\big)-\cosh(2\eta(b+a))-3}{2\eta^2(b-a)^2}}(b-a)\,\|f'\sqrt{p}\|_{L^2(a,b)}\,.\end{aligned} \tag{5.48}$$

The first term of (5.47) can be estimated by directly applying the extended Theorem of Payne & Weinberger to the smoothed p which yields with (5.23) and (5.24) from the proof of Proposition 5.8

$$\|(f-m(f))\sqrt{p}\|_{L^2(a,b)} \leq \frac{b-a}{\pi}\,\|f'\sqrt{p}\|_{L^2(a,b)}\,. \tag{5.49}$$

The use of the estimates (5.48) and (5.49) in (5.47) proves the corollary. □

5.4 A posteriori Error Estimates

This section establishes an a posteriori error estimate of the solution u of the variational inequality (5.4) and the semidiscrete solution $u_{\varepsilon h}$ of the penalised problem (5.12). The finite element discretisation is based on the basis functions from Definition 5.6 on page 64. Recall that the finite element space $V_h = \operatorname{span}\{\varphi_1,\dots,\varphi_{N-1}\} \subset H_0^1(x_0,x_N) \subset V := H_\eta^1(\mathbb{R})$. First we define the residual R originating from the semidiscrete formulation (5.12).

Definition 5.15. With $a_\eta(\cdot,\cdot)$ and $c_\eta^+(\cdot,\cdot)$ from (5.1) and (5.5) on page 62, the linear functional $R \in V^*$, called residuum, is defined by

$$\langle R,\cdot\rangle := -\int_0^T \Big((\dot u_{\varepsilon h},\cdot)_{L^2_\eta} + a_\eta(u_{\varepsilon h},\cdot) + c_\eta^+(u_{\varepsilon h},\cdot)\Big)\,dt. \tag{5.50}$$

The next proposition shows the Galerkin orthogonality of the residual.

Lemma 5.14. *Let $\mathcal{I} : V \to V_h$ be a nodal interpolation operator defined in Definition 5.9 on page 65. Then, there holds for all $v \in V$ that*

$$\langle R,v\rangle = \langle R, v-\mathcal{I}v\rangle = -\int_0^T \Big((\dot u_{\varepsilon h}, v-\mathcal{I}v)_{L^2_\eta} + a_\eta(u_{\varepsilon h}, v-\mathcal{I}v) + c_\eta^+(u_{\varepsilon h}, v-\mathcal{I}v)\Big)\,dt. \tag{5.51}$$

Proof. Set $v_h := \mathcal{I}v$ in (5.12) and integrate the equation from 0 to T. This yields

$$\langle R, \mathcal{I}v\rangle = -\int_0^T \Big((\dot{u}_{\varepsilon h}, \mathcal{I}v)_{L^2_\eta} + a_\eta(u_{\varepsilon h}, \mathcal{I}v) + c^+_\eta(u_{\varepsilon h}, \mathcal{I}v)\Big)\, dt = 0. \qquad \square$$

The linear functional ρ defined in the next definition plays an important part in the proof of the a posteriori error bound.

Definition 5.16. Let $\rho \in V^*$ denote the linear functional defined by

$$\langle \rho, \cdot\rangle := \int_0^T \big((\dot{u}, \cdot)_{L^2_\eta} + a_\eta(u, \cdot)\big)\, dt. \tag{5.52}$$

The next lemma formulates an error representation, which is the starting point for the derivation of the a posteriori error estimate.

Lemma 5.15 (Error Representation). *Set $e := u - u_{\varepsilon h}$ where u and $u_{\varepsilon h}$ denotes the solution of variational inequality (5.4) and the semi-discrete penalised problem (5.12), respectively. With R and ρ from (5.51) and (5.52) there holds for all $v_h \in V_h$*

$$\int_0^T (\dot{e}, e)_{L^2_\eta}\, dt + \int_0^T a_\eta(e, e)\, dt + \int_0^T \big(c^+_\eta(u, e) - c^+_\eta(u_{\varepsilon h}, e)\big)\, dt = \langle R, e - v_h\rangle + \langle \rho, e\rangle\,. \tag{5.53}$$

Proof. The definition of $e = u - u_{\varepsilon h}$, the fact that $c^+_\eta(u, \cdot) = 0$, and the Galerkin orthogonality of the residual (5.51) followed by elementary calculation yields

$$\begin{aligned}
&\int_0^T \big((\dot{e}, e)_{L^2_\eta} + a_\eta(e, e)\big)\, dt \\
&\qquad = \langle \rho, e\rangle + \langle R, u - u_{\varepsilon h}\rangle + \int_0^T c^+_\eta(u_{\varepsilon h}, u - u_{\varepsilon h})\, dt \\
&\qquad = \langle \rho, e\rangle + \langle R, e - v_h\rangle - \int_0^T \Big(c^+_\eta(u, u - u_{\varepsilon h}) - c^+_\eta(u_{\varepsilon h}, u - u_{\varepsilon h})\Big)\, dt. \qquad \square
\end{aligned}$$

Remark 5.17. Let u be a solution of the variational inequality (5.4). Then there holds for all $v \in \mathcal{K}_\psi$ that $\rho(v - u) \geq 0$.

The following two lemmas show important properties of the linear functional ρ. They play a crucial part in the proof of Proposition 5.18.

Lemma 5.16. *The linear functional ρ from (5.52) is positive, i.e.,*

$$\langle \rho, w\rangle \geq 0 \quad \textit{for all } w \geq 0.$$

Proof. For brevity define the bilinear form $\tilde{a}_\eta(u, v) := (\dot{u}, v)_\eta + a_\eta(u, v)$. Let $u \in \mathcal{K}$ be a solution of the variational inequality (5.4), i.e., there holds

$$\tilde{a}_\eta(u, v - u) \geq 0 \quad \text{for all } v \in \mathcal{K}_\psi. \tag{5.54}$$

Note that by definition $\langle \rho, \cdot \rangle = \int_0^T \tilde{a}_\eta(u, \cdot)\, dt$. For $\tilde{v} \in \mathcal{K}_\psi$ set $v = \tilde{v} + u - \psi \in \mathcal{K}_\psi$. Inserting this in (5.54) yields

$$\tilde{a}_\eta(u, \tilde{v} - \psi) \geq 0 \quad \text{for all } \tilde{v} \in \mathcal{K}_\psi. \tag{5.55}$$

□

With $w = \tilde{v} - \psi \geq 0$ (since $\tilde{v} \in \mathcal{K}_\psi$) there holds $\rho(w) \geq 0$.

Lemma 5.17. *With ρ defined in (5.52), u the solution of (5.4), $u_{\varepsilon h}$ the solution of (5.12), and $e = u - u_{\varepsilon h}$ there holds*

$$\rho(e) \leq \rho(u_{\varepsilon h}^+).$$

Proof. With $v = \psi \in \mathcal{K}_\psi$ in (5.54), $\tilde{a}_\eta(u, \psi - u) \geq 0$, holds true. Since $u \in \mathcal{K}_\psi$ it follows from (5.55) that $\tilde{a}_\eta(u, u - \psi) \geq 0$ and consequently there holds $\tilde{a}_\eta(u, \psi - u) = 0$, i.e., $\rho(u) = \rho(\psi)$. This and $\rho \geq 0$, cf. Lemma 5.16, yield

$$\rho(e) = \rho(u - u_{\varepsilon h}) = \rho(\psi - u_{\varepsilon h}) \leq \rho((\psi - u_{\varepsilon h})_+) = \rho(u_{\varepsilon h}^+).$$

□

Definition 5.18 (Jump Terms). The jumps of the derivative of $u_{\varepsilon h}$ at some node x of the triangulation is defined by

$$[u'_{\varepsilon h}]_x := u'_{\varepsilon h}(x^+) - u'_{\varepsilon h}(x^-) \tag{5.56}$$

where

$$u'_{\varepsilon h}(x^\pm) := \lim_{y \to x^\pm} u'_{\varepsilon h}(y). \tag{5.57}$$

The next proposition provides estimates for the error terms in (5.53) and bounds the residual R and the functional ρ from above. This estimates immediately lead to the a posteriori error estimate in Theorem 5.20. For brevity set $L^2(V) := L^2(0, T; V)$ and $L^2_\eta := L^2_\eta(\mathbb{R})$. Recall the the definitions of $T_j := (x_{j-1}, x_j)$ and $h_j := x_j - x_{j-1}$ for $j = 1, \ldots, N$. We use the following notation.

Notation 5.19. Let $x_{0,N}$ be defined by

$$x_{0,N} := \begin{cases} x_0 & \text{for an American call option,} \\ x_N & \text{for an American put option.} \end{cases} \tag{5.58}$$

In other words, $x_{0,N}$ denotes the point, where the truncation error occurs.
Define $x_{1,N-1}$ by

$$x_{1,N-1} := \begin{cases} x_1 & \text{for an American call option,} \\ x_{N-1} & \text{for an American put option;} \end{cases} \tag{5.59}$$

the interval $T_{1,N}$ by

$$T_{1,N} := \begin{cases} (x_0, x_1) & \text{for an American call option,} \\ (x_{N-1}, x_N) & \text{for an American put option.} \end{cases} \tag{5.60}$$

Then, $h_{1,N}$, the length of the length of the interval $T_{1,N}$, reads

$$h_{1,N} := \begin{cases} h_1 & \text{for an American call option,} \\ h_N & \text{for an American put option.} \end{cases} \tag{5.61}$$

Finally, define the index set J by

$$J := \begin{cases} \{2,\dots,N\} & \text{for an American call option,} \\ \{1,\dots,N-1\} & \text{for an American put option.} \end{cases} \tag{5.62}$$

The constants α, λ, and M originate from the Gårding inequality and the continuity estimate and are given in Proposition 5.1 on page 63.

Proposition 5.18. *Let u be the solution of the variation inequality (5.4) on page 63, $u_{\varepsilon h}$ the discrete solution of the penalised problem (5.12) on page 65 , and $v_h \in V_h$ arbitrary. Then,*

$$\frac{1}{2}\|e(T)\|^2_{L^2_\eta} + \alpha\|e\|^2_{L^2(H^1_\eta)} - \lambda\|e\|^2_{L^2(L^2_\eta)} \leq \frac{1}{2}\|e(0)\|^2_{L^2_\eta} + \langle R, e - v_h\rangle + \langle \rho, e\rangle + \int_0^T c^+_\eta(u_{\varepsilon h}, u^+_{\varepsilon h})\,dt. \tag{5.63}$$

Given the discrete operator $\tilde{\mathcal{A}}_{\varepsilon,h}$ operating on each space interval T_j, $j = 0,\dots,N+1$, defined as

$$\tilde{\mathcal{A}}_{\varepsilon,h}[u] := \dot{u} - \frac{\sigma^2}{2}u'' - \big(r - d - \frac{\sigma^2}{2}\big)u' + ru - \frac{1}{\varepsilon}u^+, \tag{5.64}$$

and the constant $C(\eta, h)$ from (5.33) on page 74

$$C(h) = C(\eta, h) = \frac{2}{\pi}\sqrt{\frac{\cosh(2\eta h) - 1}{2\eta^2 h^2}}, \tag{5.65}$$

the residual R is bounded above by

$$\begin{aligned} \langle R, e - v_h\rangle \leq{} & \frac{\alpha}{6}\|e'\|^2_{L^2(L^2_\eta(\mathbb{R}))} + \frac{3}{2\alpha}\sum_{j\in J} h_j^2 C(h_j)^2 \left\|\tilde{\mathcal{A}}_{\varepsilon,h}[u_{\varepsilon h}]\right\|^2_{L^2(L^2_\eta(T_j))} \\ & + \Big(1 + \frac{1}{\sqrt{3}}\Big)\sqrt{h_{1,N}}\,|u(x_{1,N-1})|\,\exp\big(-\eta|x_{1,N-1}|\big)\left\|\tilde{\mathcal{A}}_{\varepsilon,h}[u_{\varepsilon h}]\right\|_{L^2(L^2_\eta(T_{1,N}))} \\ & + \frac{\sigma^2}{2}\int_0^T \big|[u'_{\varepsilon h}]_{x_{0,N}}\big|\,|u(x_{0,N})|\exp(-2\eta|x_{0,N}|)\,dt. \end{aligned} \tag{5.66}$$

The functional ρ is bounded by

$$\begin{aligned} \langle \rho, e\rangle \leq{} & 1/4\,\|e(T)\|^2_{L^2_\eta(\mathbb{R})} + \left\|u^+_{\varepsilon h}(T)\right\|^2_{L^2_\eta(\mathbb{R})} - \big(e(0), u^+_{\varepsilon h}(0)\big)_\eta + \frac{\alpha}{6}\|e\|^2_{L^2(L^2_\eta(\mathbb{R}))} \\ & + \frac{3}{2\alpha}\left\|(u^+_{\varepsilon h})^{\cdot}\right\|^2_{L^2(L^2_\eta(\mathbb{R}))} + \frac{\alpha}{6}\|e\|^2_{L^2(H^1_\eta(\mathbb{R}))} + \Big(\frac{3}{2\alpha}M^2 + 1\Big)\left\|u^+_{\varepsilon h}\right\|^2_{L^2(H^1_\eta(\mathbb{R}))} \\ & + \sum_{j=1}^N h_j^2 C(h_j)^2\left\|\tilde{\mathcal{A}}_{\varepsilon,h}[u_{\varepsilon h}]\right\|^2_{L^2(L^2_\eta(T_j))} - \int_0^T c^+_\eta(u_{\varepsilon h}, u^+_{\varepsilon h})\,dt. \end{aligned} \tag{5.67}$$

The proof of Proposition 5.18 requires estimates of $\|e - \mathcal{I}e\|_{L^2_\eta(T_j)}$, $j = 0, \dots, N+1$. Since the interpolation error $\mathcal{I}$ maps onto V_h, five cases are under consideration, namely, the interior intervals T_j, $j = 2, \dots, N-1$, the intervals (x_0, x_1) and (x_{N-1}, x_N), as well as the unbounded intervals $(-\infty, x_0)$ and (x_N, ∞).

Lemma 5.19. *Let u be a solution of the variational inequality 5.4 and $u_{\varepsilon h}$ a solution of the semi-discrete problem (5.12). Then, the term $e - \mathcal{I}e$ is bounded above in the following way.*

interval T_j	American put option	American call option
$j=0$	$\|e-\mathcal{I}e\|_{L^2_\eta(T_j)} = 0$	$\|e-\mathcal{I}e\|_{L^2_\eta(T_j)} = \|u\|_{L^2_\eta(T_j)}$
$j=1$	$\|e-\mathcal{I}e\|_{L^2_\eta(T_j)} \lesssim h_j \, \|e'\|_{L^2_\eta(T_j)}$	$\|e-\mathcal{I}e\|_{L^2_\eta(T_j)} \lesssim \sqrt{h_j}\|u(x_j)\|e^{-\eta\|x_j\|}$
$j=2,\dots,N-1$	$\|e-\mathcal{I}e\|_{L^2_\eta(T_j)} \lesssim h_j \, \|e'\|_{L^2_\eta(T_j)}$	$\|e-\mathcal{I}e\|_{L^2_\eta(T_j)} \lesssim h_j \, \|e'\|_{L^2_\eta(T_j)}$
$j=N$	$\|e-\mathcal{I}e\|_{L^2_\eta(T_j)} \lesssim \sqrt{h_j}\|u(x_{j-1})\|e^{-\eta\|x_{j-1}\|}$	$\|e-\mathcal{I}e\|_{L^2_\eta(T_j)} \lesssim h_j \, \|e'\|_{L^2_\eta(T_j)}$
$j=N+1$	$\|e-\mathcal{I}e\|_{L^2_\eta(T_j)} = \|u\|_{L^2_\eta(T_j)}$	$\|e-\mathcal{I}e\|_{L^2_\eta(T_j)} = 0$

Remark 5.20. The terms $\|u\|_{L^2_\eta(T_N)}$ and $u(x_{N-1})$ for American put options and $\|u\|_{L^2_\eta(T_1)}$ and $u(x_1)$ for American call options are bounded by means of the truncation error estimates of Section 3.4.

Proof. First we consider American put options. Since $u(x) = u_{\varepsilon h}(x)$ for $x \in (-\infty, x_0)$ for American put options, cf. Table 5.2, $(e - \mathcal{I}e) = 0$ on $(-\infty, x_0)$. By definition of $\mathcal{I}$ and $u(x_0) = u_{\varepsilon h}(x_0)$ there holds $(e - \mathcal{I}e)(x_j) = 0$ for $j = 0, \dots N-1$. Hence, the interpolation error estimate 5.32 on page 74 is applicable, i.e.,

$$\|e - \mathcal{I}e\|_{L^2_\eta(T_j)} \lesssim h_j \, \|e'\|_{L^2_\eta(T_j)} \quad \text{for } j = 1, \dots, N-1.$$

Taking into account that $u_{\varepsilon h} = 0$ on (x_N, ∞), there holds $\|e - \mathcal{I}e\|_{L^2_\eta(x_N,\infty)} = \|u\|_{L^2_\eta(x_N,\infty)}$. Finally, we consider the interval (x_{N-1}, x_N), where $x_{N-1} > 0$. Recall that $u_{\varepsilon h}(x_N) = 0$, consequently there holds $u_{\varepsilon h} = \mathcal{I}u_{\varepsilon h}$. Taking into account that

$$\|u\|^2_{L^2_\eta(T_N)} = \int_{T_N} u^2 \exp(-2\eta|x|)\, dx \le h_N |u(x_{N-1})|^2 \exp(-2\eta|x_{N-1}|), \tag{5.68}$$

and

$$\|\mathcal{I}u\|^2_{L^2_\eta(T_N)} \le \frac{|u(x_{N-1})|^2}{h_N^2} \exp(-2\eta|x_{N-1}|) \int_{T_N} (x_N - x)^2\, dx \le \frac{h_N}{3} |u(x_{N-1})|^2 \exp(-2\eta|x_{N-1}|), \tag{5.69}$$

yields

$$\|e - \mathcal{I}e\|_{L^2_\eta(T_N)} \le \big(1 + \frac{1}{\sqrt{3}}\big) \sqrt{h_N} |u(x_{N-1})| \exp(-\eta|x_{N-1}|). \tag{5.70}$$

This finishes the proof for American put options.
For American call options there holds $u(x) = u_{\varepsilon h}(x)$ for $x \ge x_N$, consequently $e - \mathcal{I}e = 0$

on (x_N,∞). By definition of $\mathcal{I}$ and $u(x_N) = u_{\varepsilon h}(x_N)$ there holds $(e-\mathcal{I}e)(x_j) = 0$ for $j = 1,\dots N$. Hence, the interpolation error estimate 5.32 on page 74 is applicable, i.e.,

$$\|e-\mathcal{I}e\|_{L^2_\eta(T_j)} \lesssim h_j\,\|e'\|_{L^2_\eta(T_j)} \quad \text{for } j=2,\dots,N.$$

Since $u_{\varepsilon h} = 0$ on $(-\infty,x_0)$, $\|e-\mathcal{I}e\|_{L^2_\eta(-\infty,x_0)} = \|u\|_{L^2_\eta(-\infty,x_0)}$. Finally, we consider the interval (x_0,x_1), where $x_1<0$. Recall that $u_{\varepsilon h}(x_0) = 0$, consequently there holds $u_{\varepsilon h} = \mathcal{I}u_{\varepsilon h}$. Taking into account that

$$\|u\|^2_{L^2_\eta(T_1)} = \int_{T_1} u^2 \exp(-2\eta|x|)\,dx \le h_N|u(x_1)|^2\exp(-2\eta|x_1|), \tag{5.71}$$

and

$$\|\mathcal{I}u\|^2_{L^2_\eta(T_1)} \le \frac{|u(x_1)|^2}{h_1^2}\exp(-2\eta|x_1|)\int_{T_1}(x-x_1)^2\,dx \le \frac{h_1}{3}|u(x_1)|^2\exp(-2\eta|x_1|), \tag{5.72}$$

yields

$$\|e-\mathcal{I}e\|_{L^2_\eta(T_1)} \le \big(1+\frac{1}{\sqrt{3}}\big)\sqrt{h_1}|u(x_1)|\exp(-\eta|x_1|). \tag{5.73}$$

□

Proof (of Proposition 5.18). First we prove inequality (5.63) starting from the error representation (5.53) on page 78. An integration by part in time in the first term of (5.53) and the Gårding Inequality of a_η shows that, the left-hand side of (5.53) is greater or equal to

$$1/2\,\|e(T)\|^2_{L^2_\eta} - 1/2\,\|e(0)\|^2_{L^2_\eta} + \alpha\,\|e\|^2_{L^2(H^1_\eta)} - \lambda\,\|e\|^2_{L^2(L^2_\eta)}\,. \tag{5.74}$$

With the monotony (5.7) of c^+_η and $u^+ = 0$ there holds

$$-\int_0^T \big(c^+_\eta(u,e) - c^+_\eta(u_{\varepsilon h},e)\big)\,dt \le -\frac{1}{\varepsilon}\int_0^T\int_{\mathbb{R}}(u^+-u^+_{\varepsilon h})^2 e^{-2\eta|x|}\,dx\,dt = \int_0^T c^+_\eta(u_{\varepsilon h},u^+_{\varepsilon h})\,dt.$$

which proves inequality (5.63).

The next step is to show that the residual $\langle R, e-v_h\rangle$ is bounded above, i.e. inequality (5.66) holds true. Using the Galerkin orthogonality of the residual (5.51), and elementwise integration by part yields

$$\begin{aligned}
\langle R, e-v_h\rangle = \langle R, e-\mathcal{I}e\rangle &= -\int_0^T \Big((\dot u_{\varepsilon h}, e-\mathcal{I}e)_{L^2_\eta} + a_\eta(u_{\varepsilon h}, e-\mathcal{I}e) + c^+_\eta(u_{\varepsilon h}, e-\mathcal{I}e)\Big)\,dt\\
&= \int_0^T\sum_{j=1}^N\int_{x_{j-1}}^{x_j}\big(-\tilde{\mathcal{A}}_{\varepsilon,h}[u_{\varepsilon h}]\big)(e-\mathcal{I}e)\exp(-2\eta|x|)\,dx\,dt\\
&\quad+\int_0^T\int_{-\infty}^{x_0}\big(-\tilde{\mathcal{A}}_{\varepsilon,h}[u_{\varepsilon h}]\big)(e-\mathcal{I}e)\exp(-2\eta|x|)\,dx\,dt\\
&\quad+\int_0^T\int_{x_N}^{\infty}\big(-\tilde{\mathcal{A}}_{\varepsilon,h}[u_{\varepsilon h}]\big)(e-\mathcal{I}e)\exp(-2\eta|x|)\,dx\,dt\\
&\quad-\frac{\sigma^2}{2}\int_0^T\sum_{j=0}^N[u'_{\varepsilon h}]_{x_j}(e-\mathcal{I}e)(x_j)\exp(-2\eta|x_j|)\,dt.
\end{aligned} \tag{5.75}$$

Note that $(e - \mathcal{I}e)(x_j) = 0$ for all $j = 1, \ldots, N-1$, hence all jump-terms on the inner nodes $x_1 \ldots x_{N-1}$ vanish in the last term in (5.75). Moreover, $e(x_0) = 0$ for American put options and $e(x_N) = 0$ for American call options, cf. Table 5.2, hence the only jump term arises at $x_{0,N}$. We consider now the second and third integral in (5.75). Take into account that for American call and put options the following hold true, cf. Table 5.1,

$$\tilde{\mathcal{A}}_{\varepsilon,h}[u_{\varepsilon h}](e - \mathcal{I}e) = 0 \quad \text{on } (-\infty, x_0) \cup (x_N, \infty).$$

Hence, the second and third integral in (5.75) vanish.
Using Cauchy inequality on each space interval and then applying the interpolation error estimate from Theorem 5.10 on page 74, (5.70), and (5.73) yields

$$\begin{aligned}
\langle R, e\rangle \leq & \int_0^T \sum_{j=1}^N \left\| \tilde{\mathcal{A}}_{\varepsilon,h}(u_{\varepsilon h}) \right\|_{L^2_\eta(T_j)} \| e - \mathcal{I}e \|_{L^2_\eta(T_j)} \, dt \\
& + \frac{\sigma^2}{2} [u'_{\varepsilon h}]_{x_{0,N}} |u(x_{0,N})| \exp(-2\eta |x_{0,N}|) \, dt \\
\leq & \int_0^T \sum_{j\in J} \Big(2h_j C(\eta, h_j) \left\| \tilde{\mathcal{A}}_{\varepsilon,h}[u_{\varepsilon h}] \right\|_{L^2_\eta(T_j)} \|e'\|_{L^2_\eta(T_j)} \Big) \, dt \\
& + \Big(1 + \frac{1}{\sqrt{3}}\Big) \sqrt{h_{1,N}} \int_0^T |u(x_{1,N-1})| \exp\big(-\eta |x_{1,N-1}|\big) \left\| \tilde{\mathcal{A}}_{\varepsilon,h}[u_{\varepsilon h}] \right\|_{L^2_\eta(T_{1,N})} dt \\
& + \frac{\sigma^2}{2} \int_0^T \big| [u'_{\varepsilon h}]_{x_{0,N}} \big| |u(x_{0,N})| \exp(-2\eta |x_{0,N}|) \, dt.
\end{aligned}$$

Using Young's inequality with ε and Cauchy's inequality in time one obtains

$$\begin{aligned}
\langle R, e\rangle \leq & \frac{\alpha}{6} \|e'\|^2_{L^2(L^2_\eta(\mathbb{R}))} + \frac{3}{2\alpha} \sum_{j\in J} h_j^2 C(h_j)^2 \left\| \tilde{\mathcal{A}}_{\varepsilon,h}[u_{\varepsilon h}] \right\|^2_{L^2(L^2_\eta(T_j))} \\
& + \Big(1 + \frac{1}{\sqrt{3}}\Big) \sqrt{h_{1,N}} \int_0^T |u(x_{1,N-1})| \exp\big(-\eta |x_{1,N-1}|\big) \left\| \tilde{\mathcal{A}}_{\varepsilon,h}[u_{\varepsilon h}] \right\|_{L^2_\eta(T_{1,N})} dt \\
& + \frac{\sigma^2}{2} \int_0^T \big| [u'_{\varepsilon h}]_{x_{0,N}} \big| |u(x_{0,N})| \exp(-2\eta |x_{0,N}|) \, dt.
\end{aligned} \tag{5.76}$$

This proves (5.66).
Finally we show that inequality (5.67) holds. Since $\langle \rho, e\rangle \leq \rho(u^+_{\varepsilon h})$, cf. Lemma 5.17 on page 79 and $u_{\varepsilon h}$ solves the semi-discrete equation 5.89, there holds

$$\begin{aligned}
\langle \rho, e\rangle \leq \big\langle \rho, u^+_{\varepsilon h} \big\rangle = & \int_0^T \big((\dot{e}, u^+_{\varepsilon h})_\eta + a_\eta(e, u^+_{\varepsilon h}) + (\dot{u}_{\varepsilon h}, u^+_{\varepsilon h})_\eta + a_\eta(u_{\varepsilon h}, u^+_{\varepsilon h}) \big) \, dt \\
= & \int_0^T \big((\dot{e}, u^+_{\varepsilon h})_\eta + a_\eta(e, u^+_{\varepsilon h}) \big) \, dt - \int_0^T c^+_\eta(u_{\varepsilon h}, \mathcal{I}u^+_{\varepsilon h}) \, dt \\
& + \int_0^T \big((\dot{u}_{\varepsilon h}, u^+_{\varepsilon h} - \mathcal{I}u^+_{\varepsilon h})_\eta + a_\eta(\dot{u}_{\varepsilon h}, u^+_{\varepsilon h} - \mathcal{I}u^+_{\varepsilon h}) \big) \, dt \\
= & \int_0^T \big((\dot{e}, u^+_{\varepsilon h})_\eta + a_\eta(e, u^+_{\varepsilon h}) \big) \, dt - \big\langle R, u^+_{\varepsilon h} - \mathcal{I}u^+_{\varepsilon h} \big\rangle - \int_0^T c^+_\eta(u_{\varepsilon h}, u^+_{\varepsilon h}) \, dt.
\end{aligned} \tag{5.77}$$

An integration by parts in time in the first term on the right-hand-side of (5.77) yields

$$\int_0^T \left((\dot{e}, u_{\varepsilon h}^+)_\eta \, dt = \left(e(T), u_{\varepsilon h}^+(T)\right)_\eta - \left(e(0), u_{\varepsilon h}^+(0)\right)_\eta - \int_0^T (e, (u_{\varepsilon h}^+)\dot{})_\eta \, dt.$$

Using the continuity of a_η, Cauchy's and Young's Inequality with ε lead to

$$\begin{aligned}\int_0^T \left((\dot{e}, u_{\varepsilon h}^+)_\eta + a_\eta(e, u_{\varepsilon h}^+)\right) dt \leq \frac{1}{4}\|e(T)\|_{L^2_\eta}^2 + \left\|u_{\varepsilon h}^+(T)\right\|_{L^2_\eta}^2 - (e(0), u_{\varepsilon h}^+(0))_\eta + \frac{\alpha}{6}\|e\|_{L^2(L^2_\eta)}^2 \\ + \frac{3}{2\alpha}\left\|(u_{\varepsilon h}^+)\dot{}\right\|_{L^2(L^2_\eta)}^2 + \frac{\alpha}{6}\|e\|_{L^2(H^1_\eta)}^2 + \frac{3}{2\alpha}M^2\left\|u_{\varepsilon h}^+\right\|_{L^2(H^1_\eta)}^2 .\end{aligned} \tag{5.78}$$

An element-wise integration by parts in the residual in (5.77) (note that $u_{\varepsilon h}^+(x_0) = u_{\varepsilon h}^+(x_N) = 0$), using the interpolation error estimate (5.32) on page 74, and Young's inequality yields

$$\begin{aligned} -\left\langle R, u_{\varepsilon h}^+ - \mathcal{I}u_{\varepsilon h}^+\right\rangle &= \int_0^T \left((\dot{u}_{\varepsilon h}, u_{\varepsilon h}^+ - \mathcal{I}u_{\varepsilon h}^+)_\eta + a_\eta(u_{\varepsilon h}, u_{\varepsilon h}^+ - \mathcal{I}u_{\varepsilon h}^+) + c_\eta^+(u_{\varepsilon h}, u_{\varepsilon h}^+ - \mathcal{I}u_{\varepsilon h}^+)\right) dt \\ &= \int_0^T \sum_{j=1}^N \int_{x_{j-1}}^{x_j} \left(\tilde{\mathcal{A}}_{\varepsilon,h}[u_{\varepsilon h}](u_{\varepsilon h}^+ - \mathcal{I}u_{\varepsilon h}^+)\exp(-2\eta|x|)\right) dx\, dt \\ &\leq \int_0^T \sum_{j=1}^N \left(2h_j C(h_j)\left\|\tilde{\mathcal{A}}_{\varepsilon,h}[u_{\varepsilon h}]\right\|_{L^2_\eta(T_j)} \left\|(u_{\varepsilon h}^+)'\right\|_{L^2_\eta(T_j)}\right) dt \\ &\leq \sum_{j=1}^N h_j^2 C(h_j)^2 \left\|\tilde{\mathcal{A}}_{\varepsilon,h}[u_{\varepsilon h}]\right\|_{L^2(L^2_\eta(T_j))}^2 + \left\|(u_{\varepsilon h}^+)'\right\|_{L^2(L^2_\eta)}^2 .\end{aligned} \tag{5.79}$$

Inserting estimate (5.78) and (5.79) in (5.77) proves (5.67), which finishes the proof. □

Remark 5.21. Note that the last estimate in (5.79) could be sharpend by only taking the sum of elements where the semi-discrete solution $u_{\varepsilon h}$ is below the obstacle, i.e., where $u_{\varepsilon h}^+ > 0$ instead of taking the sum over all elements. However, the sum over all elements already appears in the estimate of the residual, cf. (5.76) on page 83, so the effect on the error estimate would only be minimal.

Finally we can formulate an a posteriori error estimate by means of Proposition 5.18.

Theorem 5.20 (A posteriori estimate for $e = u - u_{\varepsilon h}$). *Let u be the solution of the variation inequality (5.4), $u_{\varepsilon h}$ the discrete solution of the penalised problem (5.12). Then,* *an a posteriori error bound for the error $e = u - u_{\varepsilon h}$ is given by*

$$
\begin{aligned}
&\frac{1}{4}\|e(T)\|^2_{L^2_\eta(\mathbb{R})} + \frac{\alpha}{2}\|e\|^2_{L^2(0,T;H^1_\eta(\mathbb{R}))} - \lambda\|e\|^2_{L^2(0,T;L^2_\eta(\mathbb{R}))} + \big(e(0), u^+_{\varepsilon h}(0)\big)_{L^2_\eta} \\
&\quad\le \frac{1}{2}\|e(0)\|^2_{L^2_\eta(\mathbb{R})} + \big\|u^+_{\varepsilon h}(T)\big\|^2_{L^2_\eta(\mathbb{R})} + \frac{1}{2}\big\|u^+_{\varepsilon h}(0)\big\|^2_{L^2_\eta(\mathbb{R})} + \frac{3}{2\alpha}\big\|(u^+_{\varepsilon h})^{\cdot}\big\|^2_{L^2(0,T;L^2_\eta(\mathbb{R}))} \\
&\qquad + \big(\frac{3}{2\alpha}M^2 + 1\big)\big\|u^+_{\varepsilon h}\big\|^2_{L^2(0,T;H^1_\eta(\mathbb{R}))} + \sum_{j=1}^{N} h_j^2 C(\eta, h_j)^2 \Big\|\tilde{\mathcal{A}}_{\varepsilon,h}[u_{\varepsilon h}]\Big\|^2_{L^2(0,T;L^2_\eta(T_j))} \\
&\qquad + \frac{3}{2\alpha}M^2 \sum_{j\in J} h_j^2 C(\eta, h_j)^2 \Big\|\tilde{\mathcal{A}}_{\varepsilon,h}[u_{\varepsilon h}]\Big\|^2_{L^2(0,T;L^2_\eta(T_j))} \\
&\qquad + \Big(1 + \frac{1}{\sqrt{3}}\Big)\sqrt{h_{1,N}} \int_0^T |u(x_{1,N-1})| \exp\big(-\eta|x_{1,N-1}|\big) \Big\|\tilde{\mathcal{A}}_{\varepsilon,h}[u_{\varepsilon h}]\Big\|_{L^2_\eta(T_{1,N})} dt \\
&\qquad + \frac{\sigma^2}{2}\int_0^T \big|[u'_{\varepsilon h}]_{x_{0,N}}\big| |u(x_{0,N})| \exp(-2\eta|x_{0,N}|)\, dt.
\end{aligned}
\tag{5.80}
$$

Remark 5.22. Note that the terms on the right-hand side in (5.80) are only supported in the computational domain (x_0, x_N), since on the outer domain there holds $u_{\varepsilon h}(x) = \psi(x)$.

Proof. Inserting the estimates of $\langle R, e\rangle$, (5.66), and $\langle \rho, e\rangle$, (5.67), in (5.63), and absorbing the error terms yields the a posteriori error estimate (5.80). □

Remark 5.23. For an problem with H^1-elliptic, continuous bilinear form $a(\cdot,\cdot)$ and $f \in L^2$ of the form

$$a(u, v) = \langle f, v\rangle \quad \text{for all } v \in H^1_0(\Omega), \tag{5.81}$$

L^2-error estimates can be proved using duality arguments. The Aubin-Nitsche Lemma guarantees the following error estimate for $e = u - u_h$ with mesh-size h,

$$\|e\|_{L^2} \lesssim h\,\|e\|_{H^1}\,. \tag{5.82}$$

For variational inequalities, however, such error estimates are much more involved. Natterer [1976] proved such an estimate (5.82) for variational inequalities of the form

$$a(u, v-u) = \langle f, v-u\rangle \quad \text{for all } v \in K, \tag{5.83}$$

where $a(\cdot,\cdot)$ is a symmetric, continuous and elliptic bilinear form. According to the author it is unclear how to weaken these assumptions. On the other hand we deal with parabolic non-symmetric variational inequalities on unbounded domains. Hence, these standard method does not apply to bound $\alpha/2\,\|e\|^2_{H^1_\eta} - \lambda\,\|e\|^2_{L^2_\eta}$ on the left-hand side of (5.80) from below. This would allow for suitable mesh-sizes h_j to obtain a strictly positive bound of the left-hand side in (5.80).

Hence we need to bound $\alpha/2\,\|e\|^2_{H^1_\eta} - \lambda\,\|e\|^2_{L^2_\eta}$ from below by other means. Using the L^∞-norm in time instead of the L^2-norm yields an lower bound for the left-hand side of (5.80) which is strictly positive for sufficient small final time T, cf. the subsequent remark.

Remark 5.24. Up to now we have shown that (5.80) holds for an arbitrary $T > 0$ Choose now some $\tilde{T} \in (0, T]$ such that $\left\| e(\tilde{T}) \right\|_{L^2_\eta}^2$ gets maximal. Then

$$\left\| e(\tilde{T}) \right\|_{L^2_\eta}^2 = \|e\|^2_{L^\infty(0,T;L^2_\eta)}$$

and therefore

$$\begin{aligned}
\frac{1}{4} \|e\|^2_{L^\infty(0,T;H^1_\eta)} &+ \frac{\alpha}{2} \|e\|^2_{L^2(0,\tilde{T};H^1_\eta)} - \lambda \|e\|^2_{L^2(0,\tilde{T};L^2_\eta)} + \big(e(0), u^+_{\varepsilon h}(0)\big)_\eta \\
&\leq \frac{1}{2} \|e(0)\|^2_{L^2_\eta(\mathbb{R})} + \left\| u^+_{\varepsilon h}(\tilde{T}) \right\|^2_{L^2_\eta(\mathbb{R})} + \frac{1}{2} \left\| u^+_{\varepsilon h}(0) \right\|^2_{L^2_\eta(\mathbb{R})} + \frac{3}{2\alpha} \left\| (u^+_{\varepsilon h})^{\cdot} \right\|^2_{L^2(0,T;L^2_\eta(\mathbb{R}))} \\
&\quad + \big(\frac{3}{2\alpha} M^2 + 1\big) \left\| u^+_{\varepsilon h} \right\|^2_{L^2(0,T;H^1_\eta(\mathbb{R}))} + \sum_{j=1}^{N} h_j^2 C(\eta, h_j)^2 \left\| \tilde{\mathcal{A}}_{\varepsilon,h}[u_{\varepsilon h}] \right\|^2_{L^2(0,T;L^2_\eta(T_j))} \\
&\quad + \frac{3}{2\alpha} M^2 \sum_{j \in J} h_j^2 C(\eta, h_j)^2 \left\| \tilde{\mathcal{A}}_{\varepsilon,h}[u_{\varepsilon h}] \right\|^2_{L^2(0,T;L^2_\eta(T_j))} \\
&\quad + \Big(1 + \frac{1}{\sqrt{3}}\Big) \sqrt{h_{1,N}} \int_0^{\tilde{T}} |u(x_{1,N-1})| \, \exp\big(-\eta |x_{1,N-1}|\big) \left\| \tilde{\mathcal{A}}_{\varepsilon,h}[u_{\varepsilon h}] \right\|_{L^2_\eta(T_{1,N})} \, dt \\
&\quad + \frac{\sigma^2}{2} \int_0^{\tilde{T}} \big|[u'_{\varepsilon h}]_{x_{0,N}}\big| |u(x_{0,N})| \exp(-2\eta |x_{0,N}|) \, dt.
\end{aligned}$$

Note that there holds

$$\|e\|^2_{L^2(0,\tilde{T};L^2_\eta)} \leq \|e\|^2_{L^2(0,T;L^2_\eta)} \leq T \, \|e\|^2_{L^\infty(0,T;L^2_\eta)} \, . \tag{5.84}$$

Taking into account that $\big(e(0), u^+_{\varepsilon h}(0)\big)_{L^2_\eta} \geq 0$ there holds

$$\begin{aligned}
\Big(\frac{1}{4} - \lambda T\Big) &\|e\|^2_{L^\infty(0,T;L^2_\eta)} \\
&\leq \frac{1}{2} \|e(0)\|^2_{L^2_\eta(\mathbb{R})} + \left\| u^+_{\varepsilon h} \right\|^2_{L^\infty(0,T;L^2_\eta(\mathbb{R}))} + \frac{1}{2} \left\| u^+_{\varepsilon h}(0) \right\|^2_{L^2_\eta(\mathbb{R})} + \frac{3}{2\alpha} \left\| (u^+_{\varepsilon h})^{\cdot} \right\|^2_{L^2(0,T;L^2_\eta(\mathbb{R}))} \\
&\quad + \big(\frac{3}{2\alpha} M^2 + 1\big) \left\| u^+_{\varepsilon h} \right\|^2_{L^2(0,T;H^1_\eta(\mathbb{R}))} + \sum_{j=1}^{N} h_j^2 C(\eta, h_j)^2 \left\| \tilde{\mathcal{A}}_{\varepsilon,h}[u_{\varepsilon h}] \right\|^2_{L^2(0,T;L^2_\eta(T_j))} \\
&\quad + \frac{3}{2\alpha} M^2 \sum_{j \in J} h_j^2 C(\eta, h_j)^2 \left\| \tilde{\mathcal{A}}_{\varepsilon,h}[u_{\varepsilon h}] \right\|^2_{L^2(0,T;L^2_\eta(T_j))} \\
&\quad + \Big(1 + \frac{1}{\sqrt{3}}\Big) \sqrt{h_{1,N}} \int_0^{T} |u(x_{1,N-1})| \, \exp\big(-\eta |x_{1,N-1}|\big) \left\| \tilde{\mathcal{A}}_{\varepsilon,h}[u_{\varepsilon h}] \right\|_{L^2_\eta(T_{1,N})} \, dt \\
&\quad + \frac{\sigma^2}{2} \int_0^{T} \big|[u'_{\varepsilon h}]_{x_{0,N}}\big| |u(x_{0,N})| \exp(-2\eta |x_{0,N}|) \, dt.
\end{aligned} \tag{5.85}$$

Consequently, this a posteriori error estimator makes perfectly sense if $\lambda \leq 0$ or $T < (4\lambda)^{-1}$.

Remark 5.25. The a posteriori estimate (5.85) indicates, that not only the truncation points x_0 for American call options and x_N for American put options must be located beyond the thresholds x_0^C and x_N^P given in Table 5.1, rather the whole interval $T_{1,N}$ must be located beyond the corresponding treshold. Consequently the exact solution u satisfies the Black-Scholes equation not only for $x > x_N$ (American put options) and $x < x_0$ (American call options), cf. Table 5.2 on page 68 but for $x > x_{N-1}$ (American put options) and $x < x_1$ (American call options).

Remark 5.26. No efficient a posteriori error bounds error are known for penalised stationary variational inequalities. Even for simpler time-dependent problems lower bounds has been established only in the sense of sharp a priori bounds, cf. Eriksson et al. [1996].

5.5 A priori Estimates

5.5.1 Penalisation Error $e = u - u_\varepsilon$

In this subsection we suppose that the solution u of the variational inequality (5.4) satisfies $u \in L^2(0,T;H^2_\eta) \cap H^1(0,T;L^2_\eta)$ and that u_ε is a solution of the penalised problem (5.6). The main result is that the penalisation error $e = u - u_\varepsilon$ is proportional to $\sqrt{\varepsilon}$ as $\varepsilon \to 0$. Again, we abbreviate $L^2(H) := L^2(0,T;H)$. The constants α, λ, and M originate from the Gårding inequality and the continuity estimate and are given in Proposition 5.1 on page 63.

Theorem 5.21 (Penalisation error). *Suppose $u \in L^2(0,T;H^2_\eta) \cap H^1(0,T;L^2_\eta)$ solves the variational inequality (5.4) and u_ε the penalised problem (5.6). Set $e := u - u_\varepsilon$ and*

$$\tilde{\mathcal{A}}[u] := \dot{u} - \frac{\sigma^2}{2}u'' - \Big(r - d - \frac{\sigma^2}{2}\Big)u' + ru. \tag{5.86}$$

Then,

$$\frac{1}{2}\|e(T)\|^2_{L^2_\eta(\mathbb{R})} + \alpha\|e\|^2_{L^2(H^1_\eta(\mathbb{R}))} - \lambda\|e\|^2_{L^2(L^2_\eta(\mathbb{R}))} + \frac{1}{2\varepsilon}\left\|u_\varepsilon^+\right\|^2_{L^2(L^2_\eta(\mathbb{R}))} \le \frac{\varepsilon}{2}\left\|\tilde{\mathcal{A}}[u]\right\|^2_{L^2(L^2_\eta(\mathbb{R}))}.$$

Proof. Recall that $V := H^1_\eta$ and that u_ε solves

$$(\dot{u}_\varepsilon, v)_\eta + a_\eta(u_\varepsilon, v) + c^+_\eta(u_\varepsilon, v) = 0 \quad \text{for all } v \in V. \tag{5.87}$$

As $e = u - u_\varepsilon \in V$ and $c^+_\eta(u,v) = 0$ for all $v \in V$ one obtains

$$\int_0^T (\dot{u} - \dot{u}_\varepsilon, u - u_\varepsilon)_\eta + a_\eta(u - u_\varepsilon, u - u_\varepsilon) + c^+_\eta(u, u - u_\varepsilon) - c^+_\eta(u_\varepsilon, u - u_\varepsilon)\,dt$$
$$= \int_0^T (\dot{u}, u - u_\varepsilon)_\eta + a_\eta(u, u - u_\varepsilon)\,dt.$$

Integration by parts in time of the first term in the left hand side, using the Gårding inequality for a_η, and the monotony of c^+_η, cf. (5.1) and (5.7), respectively, and integration by part in the second term of the right hand side yields

$$1/2\,\|e(T)\|^2_{L^2_\eta} + \alpha\|e\|^2_{L^2(H^1_\eta)} - \lambda\|e\|^2_{L^2(L^2_\eta)} + 1/\varepsilon\left\|u^+_\varepsilon\right\|^2_{L^2(L^2_\eta)} \le \int_0^T \int_{\mathbb{R}} \tilde{\mathcal{A}}[u](u - u_\varepsilon)e^{-2\eta|x|}\,dx\,dt.$$

Using the complementary conditions from (3.18) on page 22 we obtain

$$\int_0^T \int_{\mathbb{R}} \tilde{\mathcal{A}}[u](u-u_\varepsilon)e^{-2\eta|x|}\,dx\,dt = \int_0^T \int_{\mathbb{R}} \tilde{\mathcal{A}}[u](\psi-u_\varepsilon)e^{-2\eta|x|}\,dx\,dt.$$

Using Cauchys inequality, Youngs inequality, and $\psi - u_\varepsilon \le (\psi-u_\varepsilon)_+ = u_\varepsilon^+$ yields

$$\begin{aligned}\frac{1}{2}\|e(T)\|^2_{L^2_\eta} + \alpha\|e\|^2_{L^2(H^1_\eta)} - \lambda\|e\|^2_{L^2(L^2_\eta)} + \frac{1}{\varepsilon}\left\|u_\varepsilon^+\right\|^2_{L^2(L^2_\eta)} \\ \le \int_0^T \int_{\mathbb{R}} \tilde{\mathcal{A}}[u]\,u_\varepsilon^+ e^{-2\eta|x|}\,dx\,dt \le \frac{\varepsilon}{2}\left\|\tilde{\mathcal{A}}[u]\right\|^2_{L^2(L^2_\eta)} + \frac{1}{2\varepsilon}\left\|u_\varepsilon^+\right\|^2_{L^2(L^2_\eta)}\end{aligned}$$

which proves the result. □

Remark 5.27. The same arguments as in Remark 5.24 yields

$$\left(\frac{1}{2} - \lambda T\right)\|e\|^2_{L^\infty(0,T;L^2_\eta)} \le \frac{\varepsilon}{2}\left\|\tilde{\mathcal{A}}[u]\right\|^2_{L^2(0,T;L^2_\eta)}. \tag{5.88}$$

5.5.2 Discretisation Error $e = u - u_{\varepsilon h}$

In this subsection we prove an a priori error estimate for the the error $e := u - u_{\varepsilon h}$ of the the exact solution u of the variational inequality (5.4) and the semi-discrete solution $u_{\varepsilon h}$ of the penalised problem (5.12). Since $V_h \subset H^1_0(x_0, x_N)$ the extended solution $u_{\varepsilon h} \in \bar{V}_h$ solves the weak formulation

$$(\dot{u}_{\varepsilon h}, v_h)_{L^2_\eta} + a_\eta(u_{\varepsilon h}, v_h) + c^+_\eta(u_{\varepsilon h}, v_h) = 0 \quad \text{for all } v_h \in V_h. \tag{5.89}$$

Recall the definition of $x_{0,N}$, $x_{1,N-1}$, $h_{1,N}$, $T_{1,N}$, and J from Notation 5.19 on page 79. Further, define $T_{0,N+1}$ by

$$T_{0,N+1} := \begin{cases} T_0 & \text{for an American call option,} \\ T_{N+1} & \text{for an American put option.} \end{cases} \tag{5.90}$$

As before, we abbreviate $L^2(H) := L^2(0,T;H)$ and $L^2_\eta := L^2_\eta(\mathbb{R})$ and use the constants α, λ, and M from Proposition 5.1 on page 63. The operator $\tilde{\mathcal{A}}$ is defined in Theorem 5.21 on page 87.

Theorem 5.22. *Suppose $u \in L^2(H^2_\eta) \cap H^1(H^1_\eta)$ solves the variational inequality (5.4) and*

$u_{\varepsilon h} \in H^1(\bar{V}_h)$ solves the semi-discrete problem (5.12). Then ,

$$
\begin{aligned}
&\frac{1}{4}\|e(T)\|^2_{L^2_\eta} + \frac{\alpha}{2}\|e\|^2_{L^2(H^1_\eta)} - \lambda\|e\|^2_{L^2(L^2_\eta)} + \frac{1}{2\varepsilon}\left\|u^+_{\varepsilon h}\right\|^2_{L^2(L^2_\eta)} + \frac{1}{2}\|e(0)\|^2_{L^2_\eta(\mathbb{R})} \\
&\quad \le \varepsilon\left\|\tilde{\mathcal{A}}[u]\right\|^2_{L^2(L^2_\eta)} + \frac{3}{2\alpha}\sum_{j\in J} C_2(h_j)^2 h_j^2\|u''\|^2_{L^2(L^2_\eta(T_j))} + \frac{1}{\varepsilon}\sum_{j\in J} C_1(h_j)^2 h_j^4\|u''\|^2_{L^2(L^2_\eta(T_j))} \\
&\qquad + \sum_{j\in J} h_j^4 C_1(h_j)^2\|u''(T)\|^2_{L^2_\eta(T_j)} + \frac{3}{2\alpha}\sum_{j\in J} h_j^2 C_1(h_j)^2\|(\dot{u})'\|^2_{L^2(L^2_\eta(T_j))} \\
&\qquad + \int_0^T \sum_{j\in J} C_1(h_j) h_j^2\left\|\tilde{\mathcal{A}}[u]\right\|_{L^2_\eta(T_j)}\|u''\|_{L^2_\eta(T_j)}\,dt \\
&\qquad + \Big(\frac{3}{2\alpha}M^2 + \frac{1}{\varepsilon}\Big)\Big(1 + \frac{1}{\sqrt{3}}\Big)^2 h_{1,N}\|u(x_{1,N-1})\|^2_{L^2(0,T)}\exp\big(-2\eta|x_{1,N-1}|\big) \\
&\qquad + \Big(1 + \frac{1}{\sqrt{3}}\Big)^2 h_{1,N}|u(x_{1,N-1},T)|^2\exp\big(-2\eta|x_{1,N-1}|\big) \\
&\qquad + \frac{3}{2\alpha}M^2\|u\|^2_{L^2(H^1_\eta(T_{0,N+1}))} + \|u(T)\|^2_{L^2_\eta(T_{0,N+1})} + \frac{3}{2\alpha}\|\dot{u}\|^2_{L^2(L^2_\eta(T_{0,N+1}))} \\
&\qquad + \frac{3}{2\alpha}M^2\big(h_{1,N} + h_{1,N}^{-1}\big)\max\big\{\|u(x_{1,N-1})'\|^2_{L^2(0,T)}, \|u(x_{1,N-1})\|^2_{L^2(0,T)}\big\}\exp(-2\eta|x_{1,N-1}|).
\end{aligned}
\tag{5.91}
$$

with

$$C_1(h_j) = C_1(h_j,\eta) := \frac{2}{\pi}C(h_j,\eta), \qquad C_2(h_j) = C_2(h_j,\eta) := \sqrt{1 + \frac{4}{\pi^2}h_j^2\,C(h_j,\eta)^2}.$$

The constant $C(h,\eta)$ is given in (5.34) on page 74.

Remark 5.28. The terms on the right-hand side of (5.91) can be subdivided into three groups. The first term is the penalisation error, the next seven terms are all discretisation errors, and the last four terms control to the truncation error. An estimation of the truncation error is given in Section 3.4.

Remark 5.29. In the last term in (5.91) the term $h_{1,N}+\frac{1}{h_{1,N}}$ represents a valid upper bound which does not reflect the different decay of u and u'. In fact, one may expect that a refined analysis of the constants in Section 3.4 discovers an improved upper bound in terms of the interval length $h_{1,N}$.

Remark 5.30. The last error term in (5.91) indicates that not only the truncation point x_0 for American call options and x_N for American put options determines the truncation error, rather the whole ‘critical’ interval $T_{1,N}$ is important. In other words, there is a trade-off between the length $h_{1,N}$ of the interval $T_{1,N}$ and its position on the real line. If the length of $T_{1,N}$ gets to small, the interval $T_{1,N}$ need to be shifted further out to reduce the truncation error of $u(x_{N-1})$ so that the last error term decreases.

Remark 5.31. The same arguments as in Remark 5.24 yield

$$
\begin{aligned}
&\Big(\frac{1}{4}-\lambda T\Big)\,\|e\|^2_{L^\infty(0,T;L^2_\eta)}\\
&\quad\le \varepsilon\,\Big\|\tilde{\mathcal{A}}[u]\Big\|^2_{L^2(L^2_\eta)}+\frac{3}{2\alpha}\sum_{j\in J}C_2(h_j)^2h_j^2\,\|u''\|^2_{L^2(L^2_\eta(T_j))}+\frac{1}{\varepsilon}\sum_{j\in J}C_1(h_j)^2h_j^4\,\|u''\|^2_{L^2(L^2_\eta(T_j))}\\
&\qquad+\sum_{j\in J}h_j^4C_1(h_j)^2\,\|u''\|^2_{L^\infty(L^2_\eta(T_j))}+\frac{3}{2\alpha}\sum_{j\in J}h_j^2C_1(h_j)^2\,\|(\dot u)'\|^2_{L^2(L^2_\eta(T_j))}\\
&\qquad+\int_0^T\sum_{j\in J}C_1(h_j)h_j^2\,\Big\|\tilde{\mathcal{A}}[u]\Big\|_{L^2_\eta(T_j)}\,\|u''\|_{L^2_\eta(T_j)}\,dt\\
&\qquad+\Big(\frac{3}{2\alpha}M^2+\frac{1}{\varepsilon}\Big)\Big(1+\frac{1}{\sqrt{3}}\Big)^2h_{1,N}\,\|u(x_{1,N-1})\|^2_{L^2(0,T)}\exp\big(-2\eta|x_{1,N-1}|\big)\\
&\qquad+\Big(1+\frac{1}{\sqrt{3}}\Big)^2h_{1,N}\,\|u(x_{1,N-1})\|^2_{L^\infty(0,T)}\exp\big(-2\eta|x_{1,N-1}|\big)\\
&\qquad+\frac{3}{2\alpha}M^2\,\|u\|^2_{L^2(H^1_\eta(T_{0,N+1}))}+\|u\|^2_{L^\infty(L^2_\eta(T_{0,N+1}))}+\frac{3}{2\alpha}\,\|\dot u\|^2_{L^2(L^2_\eta(T_{0,N+1}))}\\
&\qquad+\frac{3}{2\alpha}M^2\big(h_{1,N}+h_{1,N}^{-1}\big)\max\big\{\,\|u(x_{1,N-1})'\|^2_{L^2(0,T)}\,,\|u(x_{1,N-1})\|^2_{L^2(0,T)}\,\big\}\exp(-2\eta|x_{1,N-1}|).
\end{aligned}
\tag{5.92}
$$

In the proof of Theorem 5.22 the term $\|e-\mathcal{I}e\|_{L^2_\eta(T_j)}$ for $j=0,\dots,N+1$ need to be bounded above by suitable a priori terms. Since the interpolation error $\mathcal{I}$ maps onto V_h, five cases are under consideration, namely, the interior intervals T_j, $j=2,\dots,N-1$, the intervals (x_0,x_1) and x_{N-1},x_N, as well as the unbounded intervals $(-\infty,x_0)$ and (x_N,∞).

Lemma 5.23. *Suppose $u\in L^2(H^2_\eta)\cap H^1(H^1_\eta)$ solves the variational inequality 5.4 and $u_{\varepsilon h}\in H^1(\bar V_h)$ solves the semi-discrete problem (5.12). Then, the error $e=u-u_{\varepsilon h}$ satisfies*

interval T_j	American put option	American call option
$j=0$	$\|e-\mathcal{I}e\|_{L^2_\eta(T_j)}=0$	$\|e-\mathcal{I}e\|_{L^2_\eta(T_j)}=\|u\|_{L^2_\eta(T_j)}$
$j=1$	$\|e-\mathcal{I}e\|_{L^2_\eta(T_j)}\lesssim h_j^2\,\|u''\|_{L^2_\eta(T_j)}$	$\|e-\mathcal{I}e\|_{L^2_\eta(T_j)}\lesssim\sqrt{h_j}\|u(x_1)\|e^{-\eta\|x_1\|}$
$j=2,\dots,N-1$	$\|e-\mathcal{I}e\|_{L^2_\eta(T_j)}\lesssim h_j^2\,\|u''\|_{L^2_\eta(T_j)}$	$\|e-\mathcal{I}e\|_{L^2_\eta(T_j)}\lesssim h_j^2\,\|u''\|_{L^2_\eta(T_j)}$
$j=N$	$\|e-\mathcal{I}e\|_{L^2_\eta(T_j)}\lesssim\sqrt{h_j}\|u(x_{j-1})\|e^{-\eta\|x_{j-1}\|}$	$\|e-\mathcal{I}e\|_{L^2_\eta(T_j)}\lesssim h_j^2\,\|u''\|_{L^2_\eta(T_j)}$
$j=N+1$	$\|e-\mathcal{I}e\|_{L^2_\eta(T_j)}=\|u\|_{L^2_\eta(T_j)}$	$\|e-\mathcal{I}e\|_{L^2_\eta(T_j)}=0$

Proof. Taking into account that $u(x_0)=u_{\varepsilon h}(x_0)$ for American put options and the definition of the nodal interpolation operator $\mathcal{I}$ there holds

$$(e-\mathcal{I}e)\in H^1_0(T_j)\cap H^2_\eta(T_j)\qquad\text{for } j=1,\dots N-1,$$

Hence, the interpolation error estimate (5.43) on page 76 is applicable, i.e.,

$$\|e-\mathcal{I}e\|_{L^2_\eta(T_j)}\lesssim h_j^2\,\|u''\|_{L^2_\eta(T_j)}\qquad\text{for } j=1,\dots N-1$$

.
The estimates for $j = 0$, $j = N$, and $j = N + 1$ were already proved in Lemma 5.19 on page 81. This finishes the proof for American put options.
Since $u(x_N) = u_{\varepsilon h}(x_N)$ for American call options there holds

$$(e - \mathcal{I}e) \in H_0^1(T_j) \cap H_\eta^2(T_j) \quad \text{for } j = 2, \dots N,$$

Hence, the interpolation error estimate (5.43) on page 76 is applicable, i.e.,

$$\|e - \mathcal{I}e\|_{L^2_\eta(T_j)} \lesssim h_j^2 \, \|u''\|_{L^2_\eta(T_j)} \quad \text{for } j = 2, \dots N$$

.
The estimates for $j = 0$, $j = 1$, and $j = N + 1$ were already proved in Lemma 5.19 on page 81. □

Proof (of Theorem 5.22). Step (i) aims at an error representation for all $v \in \mathcal{K}, v_h \in V_h$:

$$\begin{aligned}\int_0^T \Big(a_\eta(e,e) + (\dot{e},e)_{L^2_\eta} + c_\eta^+(u,e) - c_\eta^+(u_{\varepsilon h},e)\Big)\,dt + \langle \rho, v-u\rangle = \int_0^T \Big\langle \tilde{\mathcal{A}}[u], v - u_{\varepsilon h}\Big\rangle_\eta \Big)\,dt \\ + \int_0^T \Big(\Big\langle \tilde{\mathcal{A}}[u], v_h - e\Big\rangle_\eta + a_\eta(u - u_{\varepsilon h}, e - v_h) + (\dot{u} - \dot{u}_{\varepsilon h}, e - v_h)_\eta + c_\eta^+(u_{\varepsilon h}, v_h - e)\Big)\,dt.\end{aligned} \tag{5.93}$$

In fact, with ρ from (5.52) and (5.89) there holds

$$\begin{aligned}&\int_0^T \Big(a_\eta(e,e) + (\dot{e},e)_\eta\Big)\,dt \\ &\quad= \int_0^T \Big(a_\eta(u,e) - a_\eta(u_{\varepsilon h},e) + (\dot{u},e)_\eta + (\dot{u}_{\varepsilon,h}, v_h)_\eta + a_\eta(u_{\varepsilon h}, v_h) + c_\eta^+(u_{\varepsilon h}, v_h)\Big)\,dt \\ &\quad= -\langle \rho, v-u\rangle + \int_0^T \Big(a_\eta(u,v) + (\dot{u},v)_\eta - (\dot{u}, u_{\varepsilon h})_\eta - a_\eta(u, u_{\varepsilon h})\Big)\,dt \\ &\qquad+ \int_0^T \Big((\dot{u}_{\varepsilon,h}, v_h - e) + a_\eta(u_{\varepsilon h}, v_h - e) + c_\eta^+(u_{\varepsilon h}, v_h)\Big)\,dt \\ &\quad= \int_0^T \Big(\Big\langle \tilde{\mathcal{A}}[u], v - u_{\varepsilon h}\Big\rangle_\eta + \Big\langle \tilde{\mathcal{A}}[u], v_h - e\Big\rangle_\eta + (\dot{e}, e - v_h)_\eta + a_\eta(e, e - v_h)\Big)\,dt \\ &\qquad+ \int_0^T c_\eta^+(u_{\varepsilon h}, v_h)\,dt - \langle \rho, v - u\rangle\,.\end{aligned}$$

Taking into account $c_\eta^+(u,\cdot) = 0$ there holds

$$c_\eta^+(u_{\varepsilon h}, v_h) = c_\eta^+(u_{\varepsilon h}, v_h - e) + c_\eta^+(u_{\varepsilon h}, u - u_{\varepsilon h}) - c_\eta^+(u, u - u_{\varepsilon h}),$$

which proves (5.93) .
Step (ii) consists of bounding the left-hand side of (5.93) from below. Note that $v \in \mathcal{K}$ and $v_h \in V_h$ can be chosen arbitrarily in the error representation (5.93). Set $v := u$ and $v_h = \mathcal{I}u$ the nodal interpolation operator from (5.14). Then, the monotony of c_η^+ (5.7), the Gårding

inequality for a_η (5.2), $\rho(v-u) \geq 0$, and an integration by parts in time on the left-hand side of (5.93) yield

$$
\begin{aligned}
\frac{1}{2}\|e(T)\|_{L^2_\eta}^2 &+ \alpha\|e\|_{L^2(H^1_\eta)}^2 - \lambda\|e\|_{L^2(L^2_\eta)}^2 + \frac{1}{\varepsilon}\left\|u^+_{\varepsilon h}\right\|_{L^2(L^2_\eta)}^2 \\
&\leq \frac{1}{2}\|e(0)\|_{L^2_\eta}^2 + \int_0^T \left\langle \tilde{\mathcal{A}}[u], u-u_{\varepsilon h}\right\rangle_\eta + \left\langle \tilde{\mathcal{A}}[u], \mathcal{I}e - e\right\rangle_\eta + a_\eta(e, e-\mathcal{I}e)\,dt \\
&\quad + \int_0^T \Big((\dot{e}, e-\mathcal{I}e)_\eta + c^+_\eta(u_{\varepsilon h}, \mathcal{I}e - e)\Big)\,dt \\
&=: \frac{1}{2}\|e(0)\|_{L^2_\eta}^2 + I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}
\tag{5.94}
$$

Step (iii) aims to bound each term I_1 to I_5 from above.
To bound $I_1 := \int_0^T \left\langle \tilde{\mathcal{A}}[u], u-u_{\varepsilon h}\right\rangle_\eta dt$ proceed as in the proof of Theorem 5.21 by using the complimentary conditions (3.18), $\psi - u_{\varepsilon h} \leq u^+_{\varepsilon h}$, and Youngs inequality. In fact,

$$
\begin{aligned}
I_1 = \int_0^T \left\langle \tilde{\mathcal{A}}[u], u-u_{\varepsilon h}\right\rangle_\eta dt &= \int_0^T \int_{\mathbb{R}} \tilde{\mathcal{A}}[u](u-u_{\varepsilon h})e^{-2\eta|x|}\,dx\,dt \\
&= \int_0^T \int_{\mathbb{R}} \tilde{\mathcal{A}}[u](\psi-u_{\varepsilon h})e^{-2\eta|x|}\,dx\,dt \\
&\leq \varepsilon\left\|\tilde{\mathcal{A}}[u]\right\|_{L^2(L^2_\eta)}^2 + (4\varepsilon)^{-1}\left\|u^+_{\varepsilon h}\right\|_{L^2(L^2_\eta)}^2.
\end{aligned}
$$

In the upcoming estimates for I_2, I_3, and I_4 we use that for American put options there holds for $x \geq x_{N-1}$ that $\tilde{\mathcal{A}}[u] = 0$ and for $x \leq x_0$ that $e = 0$. For American call options there holds for $x \leq x_1$ that $\tilde{\mathcal{A}}[u] = 0$ and for $x \geq x_N$ that $e = 0$, cf. Remark 5.25 on page 86 and Table 5.2 on page 68. By using Chauchys inequality and the interpolation error estimate (5.43) the term $I_2 := \int_0^T \left\langle \tilde{\mathcal{A}}[u], \mathcal{I}e - e\right\rangle_\eta dt$ is bounded by

$$
\begin{aligned}
I_2 = \int_0^T \left\langle \tilde{\mathcal{A}}[u], \mathcal{I}e - e\right\rangle_\eta dt &= \int_0^T \int_{\mathbb{R}} \tilde{\mathcal{A}}[u](\mathcal{I}e - e)e^{-2\eta|x|}\,dx\,dt \\
&\leq \int_0^T \sum_{j=0}^{N+1} \left\|\tilde{\mathcal{A}}[u]\right\|_{L^2_\eta(T_j)} \|e - \mathcal{I}e\|_{L^2_\eta(T_j)}\,dt \\
&\leq \int_0^T \sum_{j\in J} C_1(h_j)h_j^2 \left\|\tilde{\mathcal{A}}[u]\right\|_{L^2_\eta(T_j)} \|u''\|_{L^2_\eta(T_j)}\,dt.
\end{aligned}
$$

Using the continuity of $a_\eta(\cdot,\cdot)$ (5.2), Youngs inequality, and the interpolation error estimate

(5.44), (5.70), and (5.73), the term $I_3 := \int_0^T a_\eta(e, e - \mathcal{I}e)\,dt$ is bounded by

$$
\begin{aligned}
I_3 = \int_0^T a_\eta(e, e-\mathcal{I}e)\,dt &\le M \int_0^T \Big(\sum_{j=0}^{N+1} \|e\|_{H^1_\eta(T_j)} \|e - \mathcal{I}e\|_{H^1_\eta(T_j)} \Big)\,dt \\
&\le \frac{\alpha}{6} \|e\|^2_{L^2(H^1_\eta(\mathbb{R}))} + \frac{3}{2\alpha} M^2 \sum_{j=1}^{N} \|e - \mathcal{I}e\|^2_{L^2(H^1_\eta(T_j))} \\
&\le \frac{\alpha}{6} \|e\|^2_{L^2(H^1_\eta(\mathbb{R}))} + \frac{3}{2\alpha} M^2 \|u\|^2_{L^2(H^1_\eta(T_{0,N+1}))} + \frac{3}{2\alpha} M^2 \sum_{j\in J} C_2(h_j) h_j^2 \|u''\|^2_{L^2(L^2_\eta(T_j))} \\
&+ \frac{3}{2\alpha} M^2 \Big(1 + \frac{1}{\sqrt{3}}\Big)^2 h_{1,N} \|u(x_{1,N-1})\|^2_{L^2(0,T)} \exp\big(-2\eta |x_{1,N-1}|\big) \\
&+ \frac{3}{2\alpha} M^2 \|(u - \mathcal{I}u)'\|^2_{L^2(L^2_\eta(T_{1,N}))}.
\end{aligned}
$$

To estimate the term $\|(u - \mathcal{I}u)'\|^2_{L^2_\eta(T_{1,N})}$ note that on $T_{1,N}$

$$
\big|\mathcal{I}u(x)\big| = \Big| u(x_{1,N-1}) \frac{(x - x_{1,N-1})}{h_{1,N}} \Big| \qquad \text{and} \qquad \big|(\mathcal{I}u)'(x)\big| = \Big| \frac{u(x_{1,N-1})}{h_{1,N}} \Big|.
$$

Then,

$$
\begin{aligned}
\|(u - \mathcal{I}u)'\|^2_{L^2_\eta(T_{1,N})} &\le \exp(-2\eta|x_{1,N-1}|)\, \|(u - \mathcal{I}u)'\|^2_{L^2(T_{1,N})} \\
&\le \exp(-2\eta|x_{1,N-1}|)\big(\|u'\|^2_{L^2(T_{1,N})} + \|(\mathcal{I}u)'\|^2_{L^2(T_{1,N})} \big) \\
&\le \exp(-2\eta|x_{1,N-1}|) \max\big\{ |u(x_{1,N-1})'|^2, |u(x_{1,N-1}|) \big\} \big(h_{1,N} + h_{1,N}^{-1}\big).
\end{aligned}
$$

An integration by parts in time yields

$$
\begin{aligned}
I_4 = \int_0^T (\dot{u} - \dot{u}_{\varepsilon h}, e - \mathcal{I}e)_\eta\,dt &= \int_0^T \int_{\mathbb{R}} \dot{e}(e - \mathcal{I}e) e^{-2\eta|x|}\,dt \\
&= (e(T), (e - \mathcal{I}e)(T))_\eta - (e(0), (e - \mathcal{I}e)(0))_\eta - \int_0^T \int_{\mathbb{R}} e(\dot{e} - \mathcal{I}\dot{e}) e^{-2\eta|x|}\,dt.
\end{aligned}
$$

By using Cauchys inequality, Youngs inequality and (5.43) the first term on the right-hand side is bounded by

$$
\begin{aligned}
(e(T), (e - \mathcal{I}e)(T))_\eta &= \int_{\mathbb{R}} e(T)\big((e - \mathcal{I}e)(T)\big) e^{-2\eta|x|}\,dx \\
&\le \frac{1}{4} \|e(T)\|^2_{L^2_\eta} + \|u(T))\|^2_{L^2_\eta(T_{0,N+1})} + \sum_{j\in J} h_j^4 C_1(h_j)^2 \|u''(T)\|^2_{L^2_\eta(T_j)} \\
&+ \Big(1 + \frac{1}{\sqrt{3}}\Big)^2 h_{1,N} |u(x_{1,N-1}, T)|^2 \exp\big(-2\eta|x_{1,N-1}|\big).
\end{aligned}
$$

Similarly, $-\int_0^T \int_{\mathbb{R}} e(\dot{e} - \mathcal{I}\dot{e}) e^{-2\eta|x|}\, dt$ is bounded by

$$\begin{aligned}
-\int_0^T \int_{\mathbb{R}} e(\dot{e} - \mathcal{I}\dot{e}) e^{-2\eta|x|}\, dt &\leq \int_0^T \sum_{j=1}^{N+1} \|e\|_{L^2_\eta(T_j)} \|\dot{e} - \mathcal{I}\dot{e}\|_{L^2_\eta(T_j)}\, dt \\
&\leq \frac{\alpha}{6} \|e\|^2_{L^2_\eta} + \frac{3}{2\alpha} \|\dot{u}\|^2_{L^2_\eta(T_{0,N+1})} + \frac{3}{2\alpha} \sum_{j\in J} h_j^2 C_1(h_j) \|(\dot{u})'\|^2_{L^2_\eta(T_j)} \\
&\quad + \Big(1 + \frac{1}{\sqrt{3}}\Big)^2 h_{1,N} \|\dot{u}(x_{1,N-1})\|^2_{L^2(0,T)} \exp\big(-2\eta|x_{1,N-1}|\big).
\end{aligned}$$

Note that $\mathcal{I}u(0) = u_{\varepsilon h}(0)$, and therefore $\langle e(0), (u - \mathcal{I}u)(0)\rangle = \|e(0)\|^2_{L^2_\eta}$.
Using Cauchys inequality, $u^+_{\varepsilon h} = 0$ on $(-\infty, x_0) \cup (x_N, \infty)$, Youngs inequality, and the interpolation error estimate (5.43), the penalty term $I_5 := -\int_0^T c^+_\eta(u_{\varepsilon h}, e - \mathcal{I}e)\, dt$ is bounded by

$$\begin{aligned}
I_5 = -\int_0^T c^+_\eta(u_{\varepsilon h}, e - \mathcal{I}e)\, dt &= \frac{1}{\varepsilon} \int_0^T \int_{\mathbb{R}} (u^+_{\varepsilon h})(e - \mathcal{I}e) e^{-2\eta|x|}\, dx\, dt \\
&\leq \frac{1}{\varepsilon} \int_0^T \sum_{j=1}^{N} \left\|u^+_{\varepsilon h}\right\|_{L^2_\eta(T_j)} \|e - \mathcal{I}e\|_{L^2_\eta(T_j)}\, dt \\
&\leq \frac{1}{4\varepsilon} \left\|u^+_{\varepsilon h}\right\|^2_{L^2(L^2_\eta(\mathbb{R}))} + \frac{1}{\varepsilon} \sum_{j\in J} C_1(h_j)^2 h_j^4 \|u''\|^2_{L^2(L^2_\eta(T_j))} \\
&\quad + \frac{1}{\varepsilon}\Big(1 + \frac{1}{\sqrt{3}}\Big)^2 h_{1,N} \|u(x_{1,N-1})\|^2_{L^2(0,T)} \exp\big(-2\eta|x_{1,N-1}|\big).
\end{aligned}$$

Inserting all estimates for $I_1 - I_5$ in (5.94), absorbing the error terms yields (5.91). $\square$

Chapter 6

Numerics

This chapter establishes a finite element (FE) implementation of the semi-discrete solution of the non-linear problem (5.12) on page 65. Further, we introduce an algorithm for adaptive mesh refinement based on the a posteriori error estimator from Theorem 5.20 on page 84. Numerous numerical experiments close this chapter.

For the FE implementation we assume, that we already determined the computational interval (x_0, x_N), as well as the discrete solution $u_{\varepsilon h}$ on the outer domain according to Subsection 5.2.3. Hence, we deal with standard finite-elements in space and the method of lines in time. The implementation only differ in using weighted scaler products from standard finite-element implementation. The code is general in the sense that we allow a non-linearity depending on $u_{\varepsilon h}$ (but not of its derivatives), space dependend coefficients $a(x)$, $b(x)$ and $c(x)$ of the bilinear form $a_\eta(\cdot,\cdot)$. However, we assume that the coefficients and the Dirichlet boundary are constant in time. The implementation in Matlab is given in the Appendix B. In Section 6.1 we derive the system of ODEs which describe the semi-discrete solution. An algorithm for adaptive mesh refinement is introduced in Section 6.2. Numerical experiments indicate the convergence of the adaptive algorithm, compare adaptive versus mesh uniform refinement, and investigate the effect of the truncation error, and finish this chapter.

6.1 Method of Lines

Since the solution $u_{\varepsilon h}$ equals the obstacle ψ from Table 5.1 on page 67 on $(-\infty, x_0]$ and $[x_N, \infty)$, we only need to calculate the discrete solution in (x_0, x_N). Therefore, we define the nodal basis functions $\varphi_1, \ldots, \varphi_{N-1}$ on (x_0, x_N), in contrast to Chapter 5, Definition 5.6, where the basis functions are defined on $\mathbb{R}$.

Definition 6.1 (V_h). With $-\infty < x_0 < x_1 < \cdots < x_N < \infty$ the P_1-FEM nodal basis functions $\varphi_0, \ldots, \varphi_N$ are defined on (x_0, x_N) for $k = 0, \cdots N$ by

$$\varphi_k(x) := \begin{cases} (x - x_{k-1})/h_k & \text{for } x \in (x_{k-1}, x_k], \\ (x_{k+1} - x)/h_{k+1} & \text{for } x \in (x_k, x_{k+1}], \\ 0 & \text{elsewhere.} \end{cases} \tag{6.1}$$

Define the length of each space interval by $h_j := x_j - x_{j-1}$ and each element by $T_j := (x_{j-1}, x_j)$ for $j = 1, \dots, N$. Let $V_h = \operatorname{span}\{\varphi_1, \dots, \varphi_{N-1}\}$.

We consider the bilinear form $a(\cdot,\cdot)_{\Omega,\eta}$ defined on $\Omega := (x_0, x_N)$ with $p := p(x) = e^{-2\eta|x|}$, and the space dependent coefficients $a \in L^\infty(\Omega)$, $b \in L^2(\Omega)$, and $c \in L^1(\Omega)$ by

$$a(u,v)_{\Omega,\eta} := (au', v')_{\Omega,\eta} + (bu', v)_{\Omega,\eta} + (cu, v)_{\Omega,\eta}, \tag{6.2}$$

where

$$(u,v)_{\Omega,\eta} := \int_\Omega u(x)v(x)p(x)\,dx. \tag{6.3}$$

For brevity we set $(\cdot,\cdot) := (\cdot,\cdot)_{\Omega,\eta}$ and formulate the semi-discrete problem on the bounded interval (x_0, x_N).

Problem 6.2 ($P_{\varepsilon h}$). The semi discrete problem $(P_{\varepsilon h})$ reads: Find $u_{\varepsilon h} \in H^1(0,T;V_h)$ such that $u_{\varepsilon h}(\cdot,0) = \psi(\cdot)$, $u_{\varepsilon h}(x_0,t) = \psi(x_0)$, $u_{\varepsilon h}(x_N,t) = \psi(x_N)$ and

$$(\dot{u}_{\varepsilon h}, v_h)_{\Omega,\eta} + a(u_{\varepsilon h}, v_h)_{\Omega,\eta} + (g(u_{\varepsilon h}), v_h)_{\Omega,\eta} = 0 \quad \text{for all } v_h \in V_h. \tag{6.4}$$

The Lipschitz continuous function $g(u_{\varepsilon h}) = -\varepsilon^{-1}u_{\varepsilon h}^+$, i.e. $g \in C^{0,1}(\Omega)$ denotes the penalisation term.

Finally we define the semi-discrete solution $u_{\varepsilon h}$ in terms of the basis functions φ_j and time-dependent coefficients $c_j(t)$.

Definition 6.3. Given the time-dependent coefficients $c_0(t), \dots, c_N(t) \in H^1(0,T)$ the semi-discrete solution $u_{\varepsilon h} \in \bar{V}_h$ is written by

$$u_{\varepsilon h}(x,t) = \sum_{j=0}^{N} c_j(t)\varphi_j(x). \tag{6.5}$$

The time independent coefficients c_0 and c_N and the basis functions φ_0 and φ_N are given in Table 5.1 on page 67.

The next step is to derive the system of ODEs which yields the semi-discrete solution $u_{\varepsilon h}$.

With (6.5), (6.4) reads for $k = 1, \dots N-1$

$$\sum_{j=0}^{N} \dot{c}_j(t)\,(\varphi_j, \varphi_k) + \sum_{j=0}^{N} c_j(t)\,\big(a(x)\varphi_j', \varphi_k'\big) + \sum_{j=0}^{N} c_j(t)\,\big(b(x)\varphi_j', \varphi_k\big) + \sum_{j=0}^{N} c_j(t)\big(c(x)\varphi_j, \varphi_k\big) + \Big(g\big(\sum_{j=0}^{N} c_j(t)\varphi_j\big), \varphi_k\Big) = 0. \tag{6.6}$$

Since c_0 and c_N are given Dirichlet data, we rewrite (note that $\dot{c}_0(t) = \dot{c}_N(t) = 0$) (6.6) as

$$\begin{aligned}\sum_{j=1}^{N-1} \dot{c}_j(t)\,(\varphi_j,\varphi_k) + \sum_{j=1}^{N-1} c_j(t)\,\big(a(x)\varphi_j',\varphi_k'\big) + \sum_{j=1}^{N-1} c_j(t)\,\big(b(x)\varphi_j',\varphi_k\big) + \sum_{j=1}^{N-1} c_j(t)\big(c(x)\varphi_j,\varphi_k\big)\\ + \Big(g\big(\sum_{j=0}^{N} c_j(t)\varphi_j\big),\varphi_k\Big) + c_0\big(\,(a(x)\varphi_0',\varphi_k') + (b(x)\varphi_0',\varphi_k) + \big(c(x)\varphi_0,\varphi_k\big)\big)\\ + c_N\big(\,(a(x)\varphi_N',\varphi_k') + (b(x)\varphi_N',\varphi_k) + \big(c(x)\varphi_N,\varphi_k\big)\big) = 0.\end{aligned} \tag{6.7}$$

Define the matrices M, A, B, and $C \in \mathbb{R}^{(N-1)\times(N-1)}$ for $j,k = 1,\dots,N-1$ by

$$\begin{aligned} M_{kj} &:= (\varphi_j,\varphi_k),\\ A_{kj} &:= (a(x)\varphi_j',\varphi_k'),\\ B_{kj} &:= (b(x)\varphi_j',\varphi_k),\\ C_{kj} &:= (c(x)\varphi_j,\varphi_k).\end{aligned} \tag{6.8}$$

With the solution vector $c(t) := \big(c_1(t), c_2(t),\dots,c_{N-1}(t)\big)^T$ the components $k = 1,\dots,N-1$ of the vector G are defined by

$$G_k(c(t)) := \Big(g\big(\sum_{j=0}^{N} c_j(t)\varphi_j\big),\varphi_k\Big) + \sum_{j=0,N} c_j\big(\,\big(a(x)\varphi_j',\varphi_k'\big) + \big(b(x)\varphi_j',\varphi_k\big) + \big(c(x)\varphi_j,\varphi_k\big)\big). \tag{6.9}$$

Hence, we can write Problem (6.4) as: seek $c(t)$ with $[c_0(0),\dots,c_N(0)] = [u_0(x_0),\dots,u_0(x_N)]$ such that

$$M\dot{c}(t) + (A+B+C)c(t) + G(c(t)) = 0 \tag{6.10}$$

with $\dot{c}(t) := \big(\dot{c}_1(t),\dot{c}_2(t),\dots,\dot{c}_{N-1}(t)\big)^T$

Remark 6.4. Note that the mass-matrix M is regular and that we assumed $g \in C^{0,1}(\Omega)$. Hence, the Theorem of Picard-Lindelöf, cf. Heuser [1989] yields the existence of a unique solution

$$c \in [C^{1,1/2}(0,T)]^{N-1}$$

of Problem (6.10).

6.2 Adaptive Mesh Refinement

This section introduces an adaptive finite element method to solve the semidiscrete problem 5.89 on page 88. The a posteriori error estimator from Theorem 5.20 on page 84 motivates local refinement indicators for adaptive mesh refinement.

6.2.1 Refinement Indicator

The adaptive mesh refinement is based on the a posteriori error estimator for the error $e = u - u_{\varepsilon h}$ from Theorem 5.20 on page 84 given by

$$
\begin{aligned}
&(\frac{1}{4} - \lambda T)\, \|e\|^2_{L^\infty(0,T;L^2_\eta(\mathbb{R})} \\
&\qquad \le \frac{1}{2}\, \|e(0)\|^2_{L^2_\eta(\mathbb{R})} + \frac{1}{2}\, \big\|u^+_{\varepsilon h}(0)\big\|^2_{L^2_\eta(\mathbb{R})} + \big\|u^+_{\varepsilon h}\big\|^2_{L^\infty(0,T;L^2_\eta(\mathbb{R}))} + \frac{3}{2\alpha}\, \big\|(u^+_{\varepsilon h})^{\cdot}\big\|^2_{L^2(0,T;L^2_\eta(\mathbb{R}))} \\
&\qquad + \Big(\frac{3}{2\alpha} M^2 + 1\Big)\, \big\|u^+_{\varepsilon h}\big\|^2_{L^2(0,T;H^1_\eta(\mathbb{R}))} + \sum_{j=1}^{N} h_j^2 C(\eta, h_j)^2 \, \Big\|\tilde{\mathcal{A}}_{\varepsilon,h}[u_{\varepsilon h}]\Big\|^2_{L^2(0,T;L^2_\eta(T_j))} \\
&\qquad + \frac{3}{2\alpha} M^2 \sum_{j\in J} h_j^2 C(\eta, h_j)^2 \, \Big\|\tilde{\mathcal{A}}_{\varepsilon,h}[u_{\varepsilon h}]\Big\|^2_{L^2(0,T;L^2_\eta(T_j))} \\
&\qquad + \Big(1 + \frac{1}{\sqrt{3}}\Big) \sqrt{h_{1,N}} \int_0^T |u(x_{1,N-1})| \, \exp\big(-\eta|x_{1,N-1}|\big)\, \Big\|\tilde{\mathcal{A}}_{\varepsilon,h}[u_{\varepsilon h}]\Big\|_{L^2_\eta(T_{1,N})} \, dt \\
&\qquad + \frac{\sigma^2}{2} \int_0^T [u'_{\varepsilon h}]_{x_{0,N}} |u(x_{0,N})| \exp(-2\eta|x_{0,N}|)\, dt.
\end{aligned}
\tag{6.11}
$$

The first two terms on the right-hand side are errors regarding the interpolation of the initial condition. The next three terms measure the penalisation error, i.e., if the semi-discrete solutions $u_{\varepsilon h}$ lies below the obstacle. The next three terms are residual terms. The last of the three looks different to standard residual terms, because of the truncation of the unbounded domain. The last term incorporates the truncation error. For brevity we denote by μ_N the truncation error estimator

$$\mu_N := \frac{\sigma^2}{2} \int_0^T [u'_{\varepsilon h}]_{x_{0,N}} |u(x_{0,N})| \exp(-2\eta|x_{0,N}|)\, dt. \tag{6.12}$$

The elementwise error estimators μ_{T_j} are defined for $j \in J$ by

$$
\begin{aligned}
\mu_{T_j} := {}& \frac{1}{2}\, \|e(0)\|^2_{L^2_\eta(T_j)} + \frac{1}{2}\, \big\|u^+_{\varepsilon h}(0)\big\|^2_{L^2_\eta(T_j)} + \big\|u^+_{\varepsilon h}\big\|^2_{L^\infty(L^2_\eta(T_j))} + \frac{3}{2\alpha}\, \big\|(u^+_{\varepsilon h})^{\cdot}\big\|^2_{L^2(L^2_\eta(T_j))} \\
&+ \Big(\frac{3}{2\alpha} M^2 + 1\Big)\, \big\|u^+_{\varepsilon h}\big\|^2_{L^2(H^1_\eta(T_j))} + \Big(\frac{3}{2\alpha} M^2 + 1\Big) h_j^2 C(\eta, h_j)^2 \, \Big\|\tilde{\mathcal{A}}_{\varepsilon,h}[u_{\varepsilon h}]\Big\|^2_{L^2(L^2_\eta(T_j))},
\end{aligned}
\tag{6.13}
$$

and for $T_{1,N}$, i.e., $j = 1$ for American call options and $j = N$ for American put options by

$$
\begin{aligned}
\mu_{T_j} := {}& \frac{1}{2}\, \|e(0)\|^2_{L^2_\eta(T_j)} + \frac{1}{2}\, \big\|u^+_{\varepsilon h}(0)\big\|^2_{L^2_\eta(T_j)} + \big\|u^+_{\varepsilon h}\big\|^2_{L^\infty(L^2_\eta(T_j))} + \frac{3}{2\alpha}\, \big\|(u^+_{\varepsilon h})^{\cdot}\big\|^2_{L^2(L^2_\eta(T_j))} \\
&+ \Big(\frac{3}{2\alpha} M^2 + 1\Big)\, \big\|u^+_{\varepsilon h}\big\|^2_{L^2(H^1_\eta(T_j))} + h_j^2 C(\eta, h_j)^2 \, \Big\|\tilde{\mathcal{A}}_{\varepsilon,h}[u_{\varepsilon h}]\Big\|^2_{L^2(L^2_\eta(T_j))} \\
&+ \Big(1 + \frac{1}{\sqrt{3}}\Big) \sqrt{h_{1,N}} \int_0^T |u(x_{1,N-1})| \, \exp\big(-\eta|x_{1,N-1}|\big)\, \Big\|\tilde{\mathcal{A}}_{\varepsilon,h}[u_{\varepsilon h}]\Big\|_{L^2_\eta(T_{1,N})} \, dt.
\end{aligned}
\tag{6.14}
$$

The sum over all elementwise (or local) error estimator is called discretisation error estimator and denoted by μ_d, i.e.,

$$\mu_d := \sum_{j=1}^{N} \mu_{T_j}. \tag{6.15}$$

Finally, we define the total error estimator $\mu = \mu_d + \mu_N$, so that there holds (6.11)

$$(1/4 - \lambda T) \|e\|^2_{L^\infty(0,T;L^2_\eta(\mathbb{R}))} \leq \mu. \tag{6.16}$$

Remark 6.5. In this chapter we always consider the error estimator μ which bounds

$$(1/4 - \lambda T) \|e\|^2_{L^\infty(0,T;L^2_\eta(\mathbb{R}))}$$

from above as well as its components μ_d and μ_N. Hence in all figures and tables, the error estimators are *not* divided by $(1/4 - \lambda T)$.

6.2.2 Adaptive Finite Element Method

A typical adaptive finite element method (abbreviated by AFEM in the following) after Dörfler [1996], Morin et al. [2002], Brenner and Carstensen [2004] consists of consecutive loops of the steps

$$\texttt{SOLVE} \longrightarrow \texttt{ESTIMATE} \longrightarrow \texttt{MARK} \longrightarrow \texttt{REFINE}.$$

The proof of convergence of AFEM is neither trivial nor implied by the convergence of the finite element method since, in general, the mesh-size in the automatic mesh-refinement is not guaranteed to shrink to zero. A first proof of the convergence of AFEM is given in Babuška and Vogelius [1984], where the authors prove convergence for a simple 1D elliptic boundary value problem using the maximum criterion for mesh refinement. The extension to higher dimensions and the convergence speed, however, is still unclear with some work in progress according to an Oberwolfach Miniworkshop in September 2005. By introducing a new marking strategy, namely the so called bulk criterion, and assumptions on the 'finess' of the mesh, Dörfler [1996] proved the convergence of AFEM for the Poisson problem. With the concept of small data oscillation, Morin et al. [2002] guarantee convergence of adaptive algorithm without any further assumptions on the data. For the non-linear Laplacian the convergence of AFEM is guaranteed in Veeser [2002]. For a large class of degenerated convex minimisation problems convergence of local mesh refinement is analysed in Carstensen [2006]. All convergence proofs so far relied on energy principles which are not available in our case due to the non-symmetric bilinear form. Another important tool in the convergence proofs are discrete local efficency estimates and the concept of data oscillation, cf. Morin et al. [2002], Carstensen and Hoppe [2006a,b]. Data oscillations are weighted norms of the difference of data functions and their approximation through piecewise polynomials. Within this setting, one may expect that in the problem of this thesis the pay-off function, namely the obstacle, possibly generates oscillations terms. In the current literature of spatial discretisation errors one key argument is a local discrete efficiency estimate. Even simpler global efficiency estimates are unknown for time-dependent problems, cf. Verfürth [2003]. The subsequent Algorithm 6.6 adapts the adaptive algorithm for Problem 5.89 on page 88. Although Algorithm 6.6 is not a priori known to converge, there is numerical evidence that the adaptive algorithm does so in Subsection 6.3.2.

parameter	abrivation	value
expiry date	T	1
strike price	K	10
risk-less interest rate	r	0.25
dividend yield	d	0
volatility	σ	0.6

Table 6.1: Parameters for the American put option

Algorithm 6.6 (AFEM). Input: A corse triangulation $\mathcal{T}_0$.
For $\ell = 0, 1, 2, \ldots$ (until termination) do `SOLVE, ESTIMATE, MARK, REFINE`:

`SOLVE`: Compute the semi-discrete solution $u_{\varepsilon h}$ in $V_h^{(\ell)}$ with the method of lines.

`ESTIMATE`: Compute μ_{T_j} defined in (6.13), (6.14) for each $T_j \in \mathcal{T}_\ell$.

`MARK`: Sort T_j such that there holds $\mu_{T_1} \leq \cdots \leq \mu_{T_N}$ and choose minimal k such that

$$\frac{1}{2}\sqrt{\mu_d^{(\ell)}} \leq \Big(\sum_{j=k}^{N} \mu_{T_j}\Big)^{1/2},$$

where $\mu_d^{(\ell)}$ is the discretisation error estimate defined in (6.15) for current level ℓ. Set the set of marked elements $\mathcal{M}_\ell := \{T_k, \ldots, T_N\}$.

`REFINE`: Generate refined triangulation $\mathcal{T}_{\ell+1}$ with subordinated finite element space $V_h^{(\ell+1)} = \mathcal{P}_1(\mathcal{T}_{\ell+1}) \cap V \supset V_h^{(\ell)}$ such that every marked interval T_k is bisected.

6.3 Numerical Experiments

In the subsequent numerical experiments we consider an American put option on non-dividend paying share with strike price $K = 10$ and expiry date $T = 1$. The risk-less interest rate is $r = 0.25$ and the volatility $\sigma = 0.6$. These values are summarised in Table 6.1. Since the a posteriori error analysis relies on the constant C from the pointwise truncation error (3.98) on page 42, which we explicitely determined for American put options, cf. Remark (3.84), (3.89), and (3.96) we restrict our numerical experiments to American put options. We choose the exponent of the weight function $\eta = 0.0001$. Recall that for American put options $\eta > 0$ guarantees the existence of a unique solution on $\mathbb{R}$.
The value of the parameter λ from the Gårding inequality in Proposition 5.1 on page 63 in this setting reads $\lambda = -0.14637$, hence the error estimate (6.16) is reasonable for estimating the a posteriori error for all $T > 0$. The following numerical experiments are accomplished in this section.

- In Subsection 6.3.1 the pointwise truncation error is under consideration. Using the truncation error estimates from Section 3.4 we explicitely calculate the truncation error for different truncation points.

x_N	trunc. error	x_N	trunc. error
2.68	2.625	4.0	0.445e-02
3.0	0.592	4.1	0.245e-02
3.1	0.387	4.2	0.132e-02
3.2	0.251	4.3	0.696e-03
3.3	0.161	4.4	0.358e-03
3.4	0.102	4.5	0.179e-03
3.5	0.636e-01	4.6	0.879e-04
3.6	0.389e-01	4.7	0.420e-04
3.7	0.234e-01	4.8	0.196e-04
3.8	0.137e-01	4.9	0.890e-05
3.9	0.791e-02	5.0	0.396e-05

Table 6.2: Pointwise truncation error

- Subsection 6.3.2 examines uniform versus adaptive refinement. We compare the convergence rate of discretisation error estimator μ_d, i.e., we exclude the truncation error estimator μ_N. This makes perfectly sense, since in the adaptive algorithm only the discretisation error estimator is used as refinement indicator. This subsection gives numerical evidence of the convergence of the adaptive Algorithm 6.6.

- The main concern of Subsection 6.3.3 is to analyse the total error estimator μ, i.e., the truncation error estimator μ_N is considered. We test two different refinement strategies and compare the influence of the discretisation error μ_d and the truncation error μ_N for different truncation points x_N.

- Since the parameter $\eta > 0$ can be chosen arbitrarily, Subsection 6.3.4 considers different values of η and compares the outcome of the numerical experiments.

6.3.1 Truncation Error

In Section 3.4 we derived an truncation error for American call and put option. Recall that we proved that there exists some threshold x_N (fully determined by given financial data) so that there holds for all $x > x_N$ and $\kappa > 2\sigma^2 t$ that

$$|u(x,t)| \leq C \exp(-x^2/\kappa).$$

The constant C can be explicitely determined by means of Remark 3.39, 3.40, and 3.40 of page 39, 40, and 41, and the representation of the solution of an American put option (3.74) of page 38. Note that the smallest truncation point and the constant C depend on κ and t. In our example the smallest possible value for x_N is $x_N > 2.673$ for $T = 1$. However, the truncation error is not acceptable. Table 6.2 gives the truncation error for different values x_N for an American put option with the financial input data from Table 6.1. Figure 6.1 illustrates the pointwise truncation error. The truncation error bounds are calculated with maple. The corresponding program is listed in Appendix C.

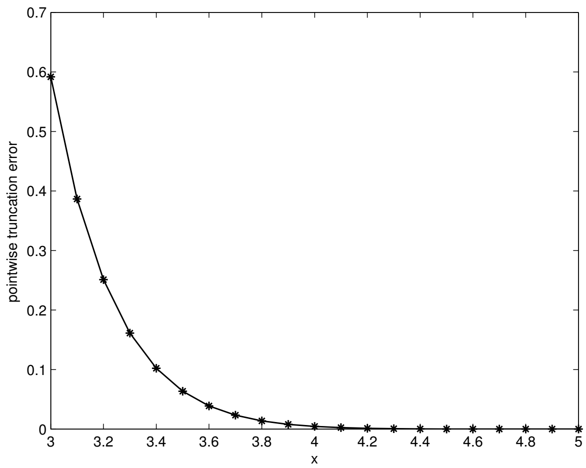

Figure 6.1: Pointwise truncation error.

level	no. elements	number time steps	total no. dof	μ_d
1	100	1,400	$1.386 \cdot 10^5$	0.1433
2	200	2,211	$4.400 \cdot 10^5$	0.0865
3	400	3,583	$1.430 \cdot 10^6$	0.0509
4	800	6,205	$4.958 \cdot 10^6$	0.0279
5	1600	11,386	$1.821 \cdot 10^7$	0.0152

Table 6.3: Convergence of the discretisation error estimator μ_d using uniform refinement with truncation point $x_N = 3.5$.

6.3.2 Convergence of adaptive versus uniform mesh refinement

This subsection establishes the experimental convergence of the adaptive algorithm defined in Algorithm 6.6 on page 100. Since the adaptive algorithm only uses the elementwise discretisation error μ_{T_j} as a refinement indicator, the truncation error is excluded in this subsection. The effect of the truncation error on the total error estimator is investigated in the subsequent subsection. Then we compare the convergence rate of adaptive mesh refinement with uniform mesh refinement. Since we only consider adaptivity in space and use the method of lines for the time integration, it is interesting to compare

- the convergence of the error against the spatial degrees of freedon and
- the convergence of the error against the total number of degree of freedom.

By the total number of degree of freedom we mean the number of spatial degrees of freedom times the number of time steps used by the Matlab routine `ode15s` to solve the system of ODEs.

According to the bounds of the free boundary in Subsection 2.2.4 the truncation point is chosen as $x_0 = 1.6$. The pointwise truncation error in Table 6.2 is acceptable for the

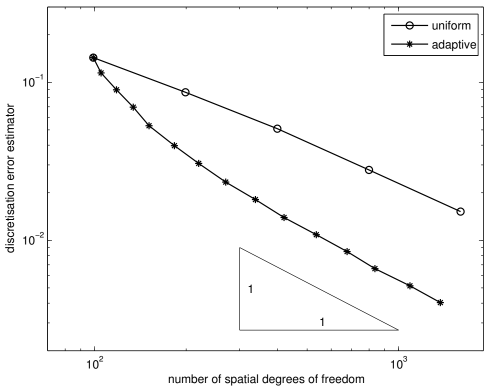

Figure 6.2: Convergence of the discretisation error estimator μ_d versus spatial degrees of freedom with $x_N = 3.5$. Adaptive refinement (—$*$—), uniform refinement (—o—).

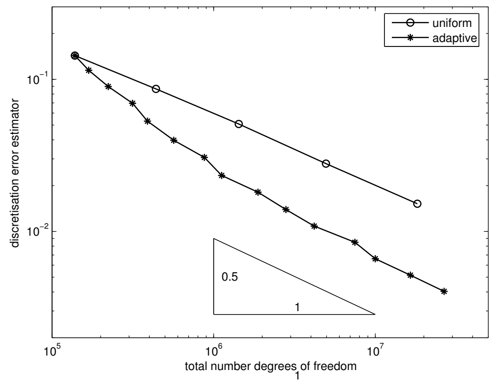

Figure 6.3: Convergence of the discretisation error estimator μ_d versus total number degrees of freedom with $x_N = 3.5$. Adaptive refinement (—$*$—), uniform refinement (—o—).

level	no. elements	no. time steps	total no. dof	μ_d
1	100	1400	$1.386 \cdot 10^5$	0.1433
2	106	1606	$1.686 \cdot 10^5$	0.1146
3	119	1890	$2.230 \cdot 10^5$	0.0897
4	135	2351	$3.150 \cdot 10^5$	0.0697
5	152	2581	$3.897 \cdot 10^5$	0.0531
6	184	3095	$5.664 \cdot 10^5$	0.0397
7	221	3984	$8.765 \cdot 10^5$	0.0306
8	271	4151	$1.121 \cdot 10^6$	0.0233
9	339	5568	$1.882 \cdot 10^6$	0.0181
10	421	6676	$2.804 \cdot 10^6$	0.0139
11	537	7822	$4.193 \cdot 10^6$	0.0108
12	678	11057	$7.486 \cdot 10^6$	0.0085
13	837	11972	$1.001 \cdot 10^7$	0.0066
14	1091	15150	$1.651 \cdot 10^7$	0.0051
15	1373	19439	$2.667 \cdot 10^7$	0.0040

Table 6.4: Convergence of the discretisation error estimator μ_d using adaptive refinement with truncation point $x_N = 3.5$.

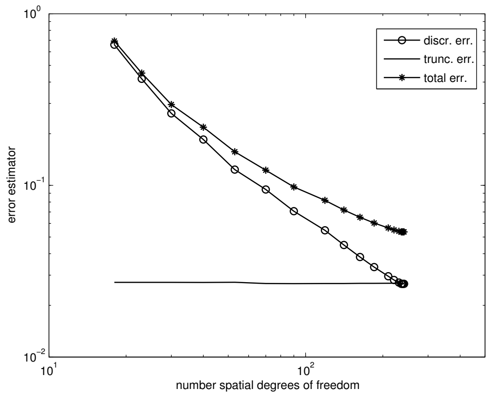

Figure 6.4: Truncation error estimator μ_N (——), the discretisation error estimator μ_d (—o—), and the total error estimator μ (—$*$—). Adaptive refinement with Refinement Strategy 1 and truncation point is $x_N = 3.5$.

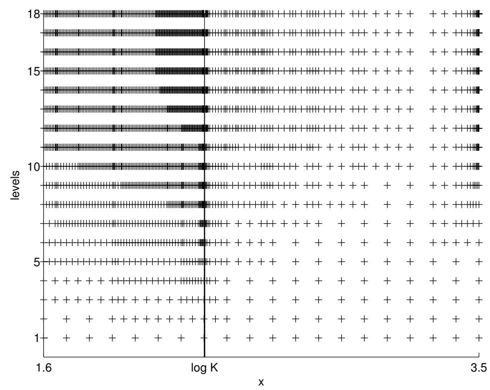

Figure 6.5: Spatial mesh for each level. Adaptive refinement using Refinement Strategy 1 and truncation point $x_N = 3.5$. As soon as the discretisation error reaches the order of magnitude of the truncation error estimator, the algorithm only refines at the right interval boundary.

truncation point $x_N = 3.5$; used in this numerical experiment. Different truncation points are considered in the subsequent subsections.

The a priori estimator in Theorem 5.22 on page 88 proposes the choice $\varepsilon \propto h_j^2$, in order to obtain optimal convergence. Hence, we set the penalisation parameter $\varepsilon = 0.1h_j^2$. We start the calculations with 100 spatial elements and refined until 1373 elements in the adaptive case and 1600 in the uniform case. Table 6.3 and 6.4 show the number of spatial elements, the number of time steps, the total number of degrees of freedom (total no. dof), and the discretisation error estimator μ_d (discr. err. est.) for uniform and adaptive mesh refinement. Figure 6.2 shows a loglog-plot of the spatial degrees of freedom versus the global discretisation error μ_d. Using adaptive mesh refinement yields the optimal convergence rate of one, whereas uniform mash-refinement yields a slightly worse convergence rate, illustrated in Figure 6.3. With uniform refinement, 1600 elements are necessary to obtain an error of less than 0.0152, using adaptive refinement, however, this error bound is reach with 421 elements, illustrated in Table 6.3 and 6.4. Figure 6.3 shows the discretisation error estimate versus the total number of degrees of freedom (no. dof). The total number of degrees of freedom is the product of the number of spatial freenodes and the number of time steps used by the ODE solver. It turns out that the convergence rate in the loglog plot is one half, illustrated in Figure 6.3. The ODE solver requires more time steps when using adaptive mesh refinement than uniform refinement, cf. the column 'no. time steps' in Table 6.3 and 6.4. It is reasonable to assume that the unstructured mesh causes this effect.

6.3.3 Discretisation Error versus Truncation Error

In this subsection we investigate the influence of the truncation error estimator μ_N on the total error estimator μ. Note that the pointwise truncation error is a part of the a posteriori

level	no. freenodes	no. time steps	total no. dof	μ_d	μ_N	μ
1	18	482	8676	0.65941	0.03331	0.69273
2	23	898	20654	0.41769	0.03331	0.45100
3	30	1142	34260	0.26244	0.03329	0.29573
4	40	1441	57640	0.18485	0.03329	0.21815
5	53	1600	84800	0.12349	0.03337	0.15687
6	70	2118	148260	0.09463	0.02801	0.12264
7	90	2190	197100	0.07071	0.02721	0.09795
8	119	2517	299523	0.05469	0.02707	0.08176
9	141	3058	431178	0.04493	0.02694	0.07188
10	163	3404	554852	0.03823	0.02696	0.06519
11	185	3411	631035	0.03342	0.02692	0.06034
12	210	3609	757890	0.02956	0.02691	0.05647
13	221	3831	846651	0.02810	0.02689	0.05500
14	231	3892	899052	0.02714	0.02689	0.05403
15	236	4060	958160	0.02670	0.02689	0.05359
16	238	4003	952714	0.02670	0.02689	0.05359
17	240	4005	961200	0.02670	0.02689	0.05359
18	242	4034	976228	0.02670	0.02689	0.05359

Table 6.5: Convergence of the error discretisation error estimator μ_d with adaptive mesh refinement using Refinement Strategy 1. After level twelve the truncation error estimator μ_N and the discretisation error μ_d are of the same order of magnitude and the algorithm only refines elements on the right boundary, cf. the column with the total error estimator μ. The truncation point is $x_N = 3.5$

error estimator (6.11) on page 98, where it bounds $|u(x_{0,N})|$ and $|u(x_{1,N-1})|$ from above.

This subsection is organised as follows. We start with discussing if the truncation error estimator μ_N should be used as a part of the refinement indicator μ_{T_N} of the interval $T_N = (x_{N-1}, x_N)$ or not. It turns out, that the truncation error estimator μ_N should *not* be added to the refinement indicator μ_{T_N}. Then, we investigate the effect of the truncation error estimator μ_N on the total error estimator μ for different truncation points x_N.

The question arises, if the truncation error estimate μ_N should be added to the local error estimator μ_{T_N}. Since the truncation error estimate evolves trough integration by parts on the element T_N, cf. the proof of Theorem 5.20 on page 84, it is consequent to consider it as a part of the local error estimator of the interval T_N which leads to the following refinement strategy.

Refinement Strategy 1. The refinement indicator for the interval T_N is defined as

$$\begin{aligned}\mu_{T_N} := {}& \frac{1}{2}\left\|e(0)\right\|^2_{L^2_\eta(T_j)} + \frac{1}{2}\left\|u^+_{\varepsilon h}(0)\right\|^2_{L^2_\eta(T_N)} + \left\|u^+_{\varepsilon h}\right\|^2_{L^\infty(L^2_\eta(T_N))} + \frac{3}{2\alpha}\left\|(u^+_{\varepsilon h})^{\cdot}\right\|^2_{L^2(L^2_\eta(T_N))}\\ &+ \Big(\frac{3}{2\alpha}M^2 + 1\Big)\left\|u^+_{\varepsilon h}\right\|^2_{L^2(H^1_\eta(T_N))} + h_j^2 C(\eta, h_j)^2 \left\|\tilde{\mathcal{A}}_{\varepsilon,h}[u_{\varepsilon h}]\right\|^2_{L^2(L^2_\eta(T_N))}\\ &+ \Big(1 + \frac{1}{\sqrt{3}}\Big)\sqrt{h_{1,N}} \int_0^T |u(x_{N-1})| \exp\big(-\eta|x_{N-1}|\big) \left\|\tilde{\mathcal{A}}_{\varepsilon,h}[u_{\varepsilon h}]\right\|_{L^2_\eta(T_N)} dt\\ &+ \mu_N.\end{aligned} \tag{6.17}$$

Numerical experiments, however, show that an adaptive mesh refinement based on Refinement Strategy 1, does not work properly, since the adaptive algorithm only refines elements on the right boundary if the discretisation error reaches the order of magnitude of the truncation error estimator. This effect is shown in Table 6.5, Figure 6.4, and Figure 6.5. To be precise, Figure 6.4 shows that as soon as the discretisation error reaches the order of magnitude of the truncation error estimator, the algorithm only refines elements on the right boundary of the computational boundary. Table 6.5 reveals that after level 15 only two freenodes per refinement level are added. Figure 6.5 indicates that the new intervals are added on the right boundary of the computational domain. Hence, the truncation error should *not* be included in the error indicator of the element T_N which leads the the following refinement strategy which excludes this term from the refinement indicator.

Refinement Strategy 2. The refinement indicator for the interval T_N is defined as

$$\begin{aligned}\mu_{T_N} := {}& \frac{1}{2}\left\|e(0)\right\|^2_{L^2_\eta(T_j)} + \frac{1}{2}\left\|u^+_{\varepsilon h}(0)\right\|^2_{L^2_\eta(T_N)} + \left\|u^+_{\varepsilon h}\right\|^2_{L^\infty(L^2_\eta(T_N))} + \frac{3}{2\alpha}\left\|(u^+_{\varepsilon h})^{\cdot}\right\|^2_{L^2(L^2_\eta(T_N))}\\ &+ \Big(\frac{3}{2\alpha}M^2 + 1\Big)\left\|u^+_{\varepsilon h}\right\|^2_{L^2(H^1_\eta(T_N))} + h_j^2 C(\eta, h_j)^2 \left\|\tilde{\mathcal{A}}_{\varepsilon,h}[u_{\varepsilon h}]\right\|^2_{L^2(L^2_\eta(T_N))}\\ &+ \Big(1 + \frac{1}{\sqrt{3}}\Big)\sqrt{h_{1,N}} \int_0^T |u(x_{N-1})| \exp\big(-\eta|x_{N-1}|\big) \left\|\tilde{\mathcal{A}}_{\varepsilon,h}[u_{\varepsilon h}]\right\|_{L^2_\eta(T_N)} dt\end{aligned} \tag{6.18}$$

Figure 6.6 and Table 6.6 show that Refinement Strategy 2 works satisfactorily. The total error converges to the truncation error, which is not reduced, since the truncation point x_N is fixed. Figure 6.7 shows the mesh for each refinement level. Note that in contrast to the first refinement strategy, not only elements on the right boundary are refined. The

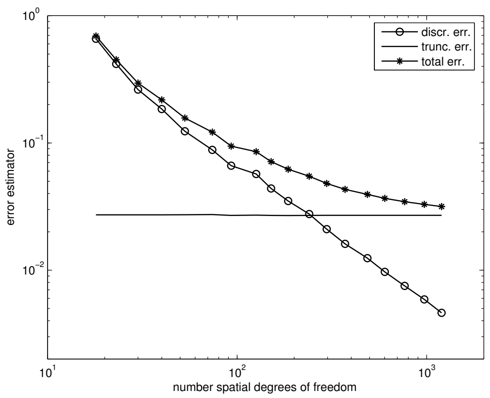

Figure 6.6: Truncation error estimator μ_N (——), the discretisation error estimator μ_d (—o—), and the total error estimator μ (—$*$—). Adaptive mesh refinement using Refinement Strategy 2 and truncation point $x_N = 3.5$.

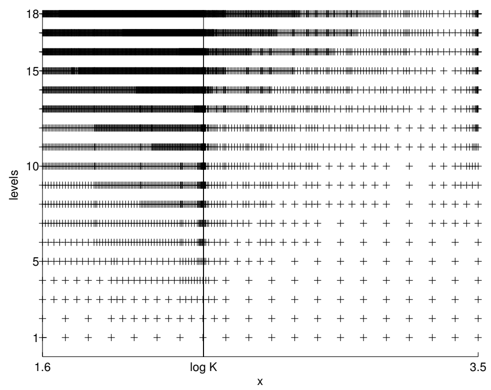

Figure 6.7: Spatial mesh for each level for adaptive mesh refinement using Refinement Strategy 2 and truncation point $x_N = 3.5$.

effect, that there is a cluster point of refined intervals on the right boundary is caused by the artificial truncation of the domain: Note that in the proof of the a posteriori error estimator in Theorem 5.20 on page 84 this non-standard residuum term, cf. the last line in (6.14), enters the error bound.

From now on we always use Refinement Strategy 2 and turn our attention to the influence of the truncation point x_N on the total error estimator μ. Choosing the truncation point $x_N = 4$ yields an pointwise truncation error less than $0.45 \cdot 10^{-2}$, cf. Table 6.2 on page 101. Figure 6.8 shows that the total error is only influenced slightly by the truncation error estimator which is of magnitude 10^{-3}. Figure 6.9 shows the spatial meshes for each refinement level. Note that in level twelve, the adaptive algorithm starts refining the elements on the right-hand side of the computational domain. Table 6.7 shows the convergence of the discretisation error estimator μ_d.

The pointwise truncation error for a truncation point $x_N = 5$ is less than $0.4 \cdot 10^{-5}$, cf. Table 6.2 on page 101. One may expect, that the truncation error is neglectable in the error estimate μ. Indeed, a numerical experiment showed, that also the truncation error estimator μ_N is of magnitude 10^{-6}, cf. Table 6.8. Figure 6.10 shows the convergence of the total error estimator which is almost equal to the discretisation error estimator. Figure 6.11 shows that no refinement is necessary on the right-hand side of the computational interval.

level	no.freenodes	no. time steps	total no. dof	μ_d	μ_N	μ
1	18	482	8676	0.65941	0.03331	0.69273
2	23	898	20654	0.41769	0.03331	0.45100
3	30	1142	34260	0.26244	0.03330	0.29573
4	40	1441	57640	0.18486	0.03329	0.21815
5	53	1600	84800	0.12349	0.03337	0.15687
6	74	2177	161098	0.08814	0.03352	0.12166
7	93	2140	199020	0.06628	0.02812	0.09439
8	126	2840	357840	0.05706	0.02826	0.08532
9	151	3298	497998	0.04384	0.02738	0.07123
10	186	3351	623286	0.03497	0.02713	0.06210
11	240	4176	1002240	0.02761	0.02710	0.05471
12	297	5516	1638252	0.02102	0.02704	0.04805
13	371	5595	2075745	0.01611	0.02701	0.04312
14	487	7591	3696817	0.01242	0.02699	0.03942
15	601	9183	5518983	0.00968	0.02699	0.03667
16	765	10329	7901685	0.00752	0.02699	0.03451
17	972	14264	13864608	0.00589	0.02699	0.03288
18	1199	15413	18480187	0.00462	0.02699	0.03162

Table 6.6: Convergence of the error discretisation error estimator μ_d using adaptive refinement which excludes the truncation error estimator μ_N from the refinement indicator of T_N. The truncation error estimator μ_N slightly decreases for finer meshes. The truncation point is $x_N = 3.5$.

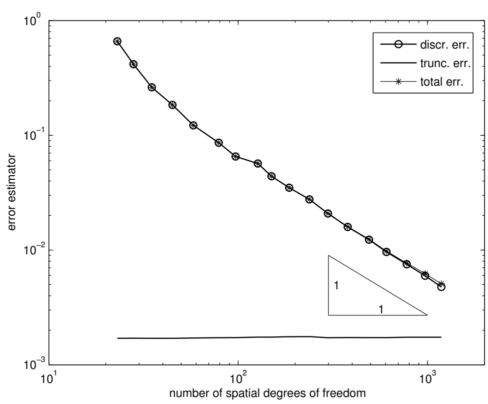

Figure 6.8: The discretisation error estimator μ_d (—o—), and the total error estimator μ (—$*$—). Since the truncation error estimator ($x_N = 4$) is of magnitude 10^{-3}, cf. Table 6.7 both estimators are almost on the same line.

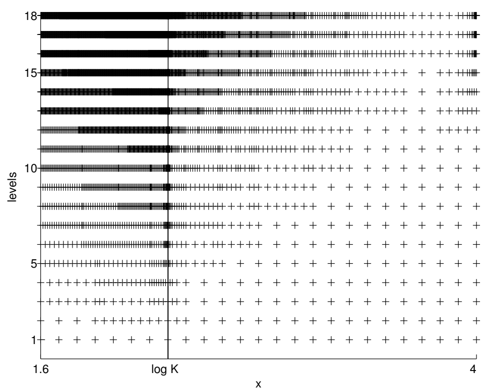

Figure 6.9: Spatial mesh for each refinement level for truncation point $x_N = 4$. In level twelve the refinement on the right-hand side of the interval becomes necessary.

level	no.freenodes	no. time steps	total no. dof	μ_d	μ_N	μ
1	23	495	11385	0.65951	0.00170	0.65952
2	28	877	24556	0.41699	0.00170	0.41700
3	35	1137	39795	0.26176	0.00170	0.26177
4	45	1438	64710	0.18388	0.00170	0.18389
5	58	1608	93264	0.12206	0.00171	0.12207
6	79	2216	175064	0.08609	0.00172	0.08612
7	97	2113	204961	0.06529	0.00172	0.06531
8	127	2826	358902	0.05680	0.00174	0.05683
9	150	3255	488250	0.04398	0.00174	0.04402
10	186	3365	625890	0.03489	0.00176	0.03494
11	238	4205	1000790	0.02761	0.00176	0.02767
12	298	5598	1668204	0.02079	0.00171	0.02087
13	379	5698	2159542	0.01589	0.00173	0.01599
14	491	7592	3727672	0.01230	0.00172	0.01242
15	607	9415	5714905	0.00962	0.00172	0.00977
16	777	10381	8066037	0.00751	0.00174	0.00771
17	971	14551	14129021	0.00597	0.00174	0.00622
18	1183	15351	18160233	0.00477	0.00173	0.00508

Table 6.7: Convergence of the discretisation error estimator μ_d using adaptive refinement. The truncation error estimator μ_N is almost independent of the refinement level. The truncation point is $x_N = 4$.

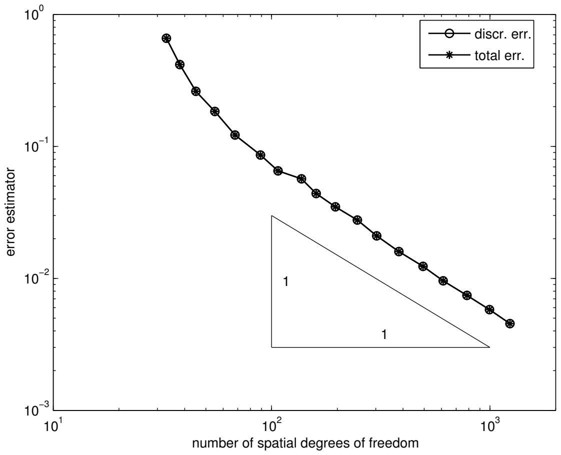

Figure 6.10: The discretisation error estimator μ_d (—o—), and the total error estimator μ (—∗—). Since the truncation error estimator ($x_N = 5$) is of magnitude 10^{-6}, cf. Table 6.8 both estimators are on the same line.

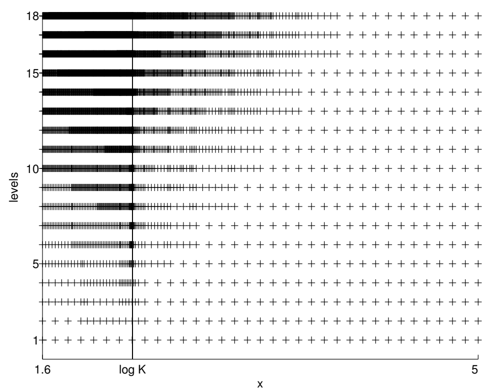

Figure 6.11: Spatial mesh for each refinement level. No refinement near the truncation point $x_N = 5$ is necessary for all the 18 levels displayed.

level	no.freenodes	no. time steps	total no. dof	μ_d	μ_N	μ
1	33	486	16038	0.65928	9.659e-07	0.65928
2	38	876	33288	0.41699	9.677e-07	0.41699
3	45	1135	51075	0.26176	9.655e-07	0.26176
4	55	1431	78705	0.18388	9.649e-07	0.18388
5	68	1629	110772	0.12206	9.892e-07	0.12206
6	89	2196	195444	0.08608	1.005e-06	0.08608
7	107	2147	229729	0.06527	1.009e-06	0.06526
8	137	2793	382641	0.05678	1.047e-06	0.05678
9	160	3246	519360	0.04395	1.049e-06	0.04395
10	196	3362	658952	0.03485	1.078e-06	0.03485
11	247	4137	1021839	0.02769	1.085e-06	0.02769
12	303	5587	1692861	0.02099	1.085e-06	0.02099
13	383	5583	2138289	0.01598	1.132e-06	0.01598
14	494	7657	3782558	0.01233	1.133e-06	0.01233
15	611	9324	5696964	0.00959	1.137e-06	0.00959
16	785	10410	8171850	0.00743	1.183e-06	0.00743
17	995	14186	14115070	0.00581	1.183e-06	0.00581
18	1235	15526	19174610	0.00454	1.187e-06	0.00455

Table 6.8: Convergence of the discretisation error estimator μ_d using adaptive refinement. The truncation error estimator μ_N is almost independent of the refinement level and has no impact on the total error estimator μ. The truncation point is $x_N = 5$.

6.3.4 The Influence of the parameter η

Recall that the parameter η is chosen such that the pay-off function belongs to H^1_η to guarantee the existence of a unique solution. For American put options $\eta > 0$ is sufficient. Note that in the case of constant coefficients we proved the exponential decay of the exact solution by using the Fourier transform . Since the solution equals the pay-off function beyond a treshold x_0, it is sufficient to consider it on the interval (x_0, ∞) for numerical computation. This together with the exponential decay yields that $\eta = 0$, i.e., standard Sobolev spaces, are applicable. However, the Fourier transform does not work for space dependent coefficients, hence it is not possible to prove the exponential decay of the solution via the Fourier transform. Although in the case of constant coefficients the weighted spaces are not necessary we used them to be as general as possible. In the previous experiments we choose $\eta = 0.0001$. This subsection investigates the influence of the parameter η on the error estimator.

In this subsection we choose $x_N = 4$ since the influence of the truncation error is neglectable, cf. Figure 6.8. We compare $\eta = 0.5$, $\eta = 0.1$, $\eta = 0.0001$, and $\eta = 0$. Recall that the constant λ from the Gårding inequality depends on η. The corresponding λ for different η and $1/4 - \lambda T$ for $T = 1$ are listed in the tabular below.

η	λ	$1/4 - \lambda$
0	-0.14639	0.39639
0.0001	-0.14637	0.39637
0.1	-0.12879	0.37879
0.5	0.013611	0.23639
1	0.35361	-0.10361

Note that for all η up to $\eta = 0.5$ the corresponding λ satisfies $1/4 - \lambda T > 0$ for $T = 1$ which means, that the error estimator (6.16) on page 99 makes sense. For $\eta = 1$, however, $T < (4\lambda)^{-1} = 0.707$ need to be satisfied.

It turns out that $\eta = 0$ and $\eta = 0.0001$ yield almost the same results. The discretisation error μ_d cannot be distinguished in Figure 6.12 and comparing Table 6.7 for $\eta = 0.0001$ and Table 6.9 for $\eta = 0$ reveals that the number of spatial degrees of freedom are the same in each refinement level. Figure 6.14 for $\eta = 0.0001$ and Figure 6.15 for $\eta = 0$ show that the refined meshes look the same. Note that the total error estimator μ, the discretisation error estimator μ_d and the truncation error estimator μ_N are almost equal, cf. Table 6.7 and Table 6.9. Only the number of time steps vary a little.

To show the effect of bigger changes of η, we set $\eta = 0.1$ and $\eta = 0.5$. Figure 6.12 and Figure 6.13 compare the different convergence curves for different η. As mentioned before, $\eta = 0.0001$ and $\eta = 0$ can not be distinguished; for $\eta = 0.1$ the difference is visable but not too big. For $\eta = 0.5$, however, the value of the error estimator is considerable lower, but the order of convergence is the same. In contrast to $\eta = 0$, $\eta = 0.1$, and $\eta = 0.0001$, where at level twelve the intervals on the right boundary of the interval are refined, this is not the case for $\eta = 0.5$, illustrated in Figure 6.14, Figure 6.15, Figure 6.16, and Figure 6.17.

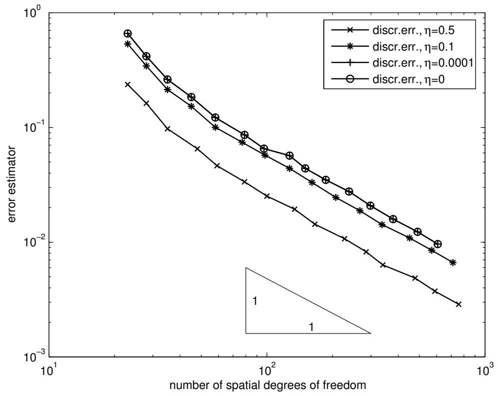

Figure 6.12: The discretisation error estimator μ_d for $\eta = 0.5$ (—×—), $\eta = 0.1$ (—∗—), $\eta = 0.0001$ (—+—) and $\eta = 0$ (—o—). Note that the curves for $\eta = 0$ and $\eta = 0.0001$ overlap.

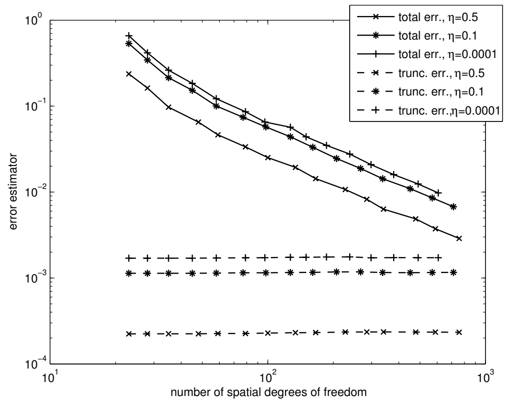

Figure 6.13: The total error estimator μ for $\eta = 0.5$ (—×—), $\eta = 0.1$ (—∗—), and $\eta = 0.0001$ (—+—) and the truncation error estimator μ_N for $\eta = 0.5$ (- -×- -), $\eta = 0.1$ (- -∗- -), and $\eta = 0.0001$ (- -+- -).

level	no.freenodes	no. time steps	total no. dof	μ_d	μ_N	μ
1	23	484	11132	0.65936	0.00170	0.65935
2	28	878	24584	0.41710	0.00170	0.41710
3	35	1143	40005	0.26184	0.00170	0.26184
4	45	1420	63900	0.18389	0.00170	0.18390
5	58	1628	94424	0.12209	0.00171	0.12210
6	79	2183	172457	0.08612	0.00172	0.08613
7	97	2138	207386	0.06530	0.00172	0.06532
8	127	2805	356235	0.05681	0.00174	0.05684
9	150	3270	490500	0.04399	0.00174	0.04403
10	186	3398	632028	0.03489	0.00176	0.03494
11	238	4195	998410	0.02761	0.00176	0.02767
12	298	5473	1630954	0.02079	0.00172	0.02087
13	379	5695	2158405	0.01589	0.00173	0.01599
14	491	7820	3839620	0.01231	0.00172	0.01242
15	607	9159	5559513	0.00962	0.00172	0.00977

Table 6.9: Convergence of the discretisation error estimator μ_d using adaptive refinement. The truncation error estimator μ_N is almost independent of the refinement level. The truncation point is $x_N = 4$ and $\eta = 0$.

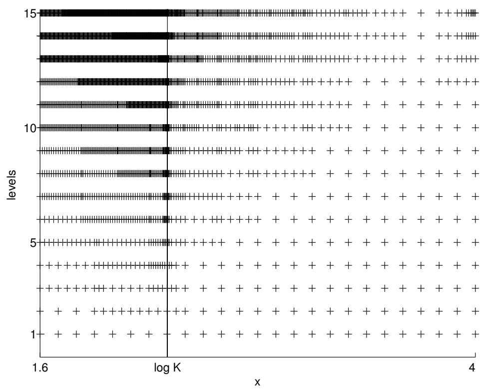

Figure 6.14: Spatial mesh for each refinement level for $\eta = 0.0001$ and truncation point $x_N = 4$.

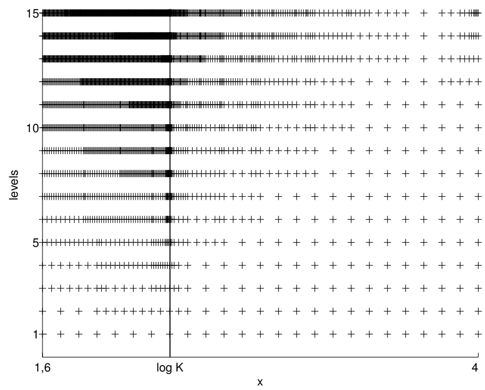

Figure 6.15: Spatial mesh for each refinement level for $\eta = 0$ and truncation point $x_N = 4$.

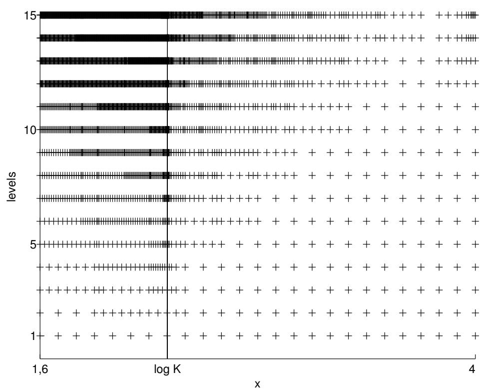

Figure 6.16: Spatial mesh for each refinement level for $\eta = 0.1$ and truncation point $x_N = 4$.

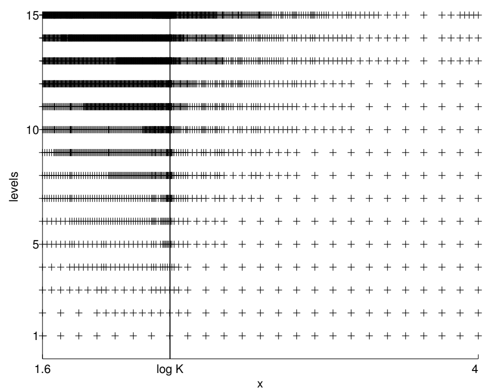

Figure 6.17: Spatial mesh for each refinement level for $\eta = 0.5$ and truncation point $x_N = 4$.

Outlook

The numerical analysis in this thesis considers spatial discretisation and truncation errors. A next step is to analyse a fully discretized problem, including time discretisation errors and develop error estimates and adaptive mesh refinement in time and space.

So far we implemented an adaptive algorithm for adaptive mesh-refinement for a fixed truncation point x_N. We analysed different choices of x_N and compered the influence of the truncation error estimator to the total error estimator. An fully adaptive algorithm, which enlarges the computational interval if the order of magnitude of the discretiastion error estimate reaches the order of the truncation error estimates requires

- a automatisation of the maple code given in Appendix C and
- a coupling of the matlab code given in Appendix B and the maple code.

Since there are no reference solutions known, the error $e = u - u_{\varepsilon h}$ is unknown. It would be interesting to approximate an reference solution using the integral representation derived in Section 3.3. This requires

- solving the non-linear integral equations which yields the free boundary and
- evaluation the integral representation including the error function.

Both need to be done in an almost exact way, to give an reliable approximation of the exact solution and to compare the exact error with the derived error estimator and truncation error.

Bibliography

Y. Achdou and O. Pironneau. *Computational methods for option pricing*, volume 30 of *Frontiers in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2005. ISBN 0-89871-573-3.

M. Ainsworth and J. T. Oden. *A posteriori error estimation in finite element analysis*. Pure and Applied Mathematics (New York). Wiley-Interscience [John Wiley & Sons], New York, 2000. ISBN 0-471-29411-X.

W. Allegretto, Y. Lin, and H. Yang. Finite element error estimates for a nonlocal problem in American option valuation. *SIAM J. Numer. Anal.*, 39(3):834–857 (electronic), 2001. ISSN 1095-7170.

I. Babuška and T. Strouboulis. *The finite element method and its reliability*. Numerical Mathematics and Scientific Computation. The Clarendon Press Oxford University Press, New York, 2001. ISBN 0-19-850276-1.

I. Babuška and M. Vogelius. Feedback and adaptive finite element solution of one-dimensional boundary value problems. *Numer. Math.*, 44(1):75–102, 1984. ISSN 0029-599X.

C. Baiocchi and A. Capelo. *Variational and quasivariational inequalities*. A Wiley-Interscience Publication. John Wiley & Sons Inc., New York, 1984. ISBN 0-471-90201-2. Applications to free boundary problems, Translated from the Italian by Lakshmi Jayakar.

W. Bangerth and R. Rannacher. *Adaptive finite element methods for differential equations*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2003. ISBN 3-7643-7009-2.

A. Bensoussan and J.-L. Lions. *Applications of variational inequalities in stochastic control*, volume 12 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, 1982. ISBN 0-444-86358-3. Translated from the French.

F. Black and M. Scholes. The pricing of options and corporate liabilities. *J. Polit. Econ.*, 81:637–659, 1973.

S. Brenner and C. Carstensen. Finite element methods. In *Encyclopedia of Computational Mechanics*. John Wiley and Sons, 2004.

C. Carstensen. On the convergence of adaptive fem for uniformely convex minimization problems. *Submitted.*, 2006.

C. Carstensen and R. Hoppe. Convergence analysis of an adaptive nonconforming finite element method. *Numer. Math.*, 103:251–266, 2006a.

C. Carstensen and R. Hoppe. Error reduction and convergence for an adaptive mixed finite element method. *To appear in Math. Comp.*, 2006b.

C. Carstensen, O. Scherf, and P. Wriggers. Adaptive finite elements for elastic bodies in contact. *SIAM J. Sci. Comput.*, 20(5):1605–1626 (electronic), 1999. ISSN 1095-7197.

R. Dautray and J.-L. Lions. *Mathematical analysis and numerical methods for science and technology. Vol. 5.* Springer-Verlag, Berlin, 1992. ISBN 3-540-50205-X; 3-540-66101-8. Evolution problems. I, With the collaboration of Michel Artola, Michel Cessenat and Hélène Lanchon, Translated from the French by Alan Craig.

J. N. Dewynne, S. D. Howison, I. Rupf, and P. Wilmott. Some mathematical results in the pricing of American options. *European J. Appl. Math.*, 4(4):381–398, 1993. ISSN 0956-7925.

W. Dörfler. A convergent adaptive algorithm for Poisson's equation. *SIAM J. Numer. Anal.*, 33(3):1106–1124, 1996. ISSN 0036-1429.

G. Duvaut and J.-L. Lions. *Inequalities in mechanics and physics.* Springer-Verlag, Berlin, 1976. ISBN 3-540-07327-2. Translated from the French by C. W. John, Grundlehren der Mathematischen Wissenschaften, 219.

M. Ehrhardt and R. Mickens. Discrete artificial boundary conditions for the black-scholes equation of american options. *Submitted.*, 2006.

K. Eriksson, D. Estep, P. Hansbo, and C. Johnson. *Computational differential equations.* Cambridge University Press, Cambridge, 1996. ISBN 0-521-56312-7; 0-521-56738-6.

A. Friedman. *Variational principles and free-boundary problems.* Pure and Applied Mathematics. John Wiley & Sons Inc., New York, 1982. ISBN 0-471-86849-3. A Wiley-Interscience Publication.

J. Goodman and D. N. Ostrov. On the early exercise boundary of the American put option. *SIAM J. Appl. Math.*, 62(5):1823–1835 (electronic), 2002. ISSN 1095-712X.

H. Han and X. Wu. A fast numerical method for the Black-Scholes equation of American options. *SIAM J. Numer. Anal.*, 41(6):2081–2095 (electronic), 2003. ISSN 1095-7170.

H. Heuser. *Gewöhnliche Differentialgleichungen.* Mathematische Leitfäden. [Mathematical Textbooks]. B. G. Teubner, Stuttgart, 1989. ISBN 3-519-02227-3. Einführung in Lehre und Gebrauch. [Introduction to theory and use].

M. Holtz. Konstruktion B-Spline-basierter monotoner Mehrgitterverfahren zur Bewertung Amerikanischer Optionen. Diplomarbeit, Institut für Angewandte Mathematik, Universität Bonn, Bonn, Germany, 2004.

P. Jaillet, D. Lamberton, and B. Lapeyre. Variational inequalities and the pricing of American options. *Acta Appl. Math.*, 21(3):263–289, 1990. ISSN 0167-8019.

C. Johnson. *Numerical solution of partial differential equations by the finite element method.* Cambridge University Press, Cambridge, 1987. ISBN 0-521-34514-6; 0-521-34758-0.

R. Kangro and R. Nicolaides. Far field boundary conditions for Black-Scholes equations. *SIAM J. Numer. Anal.*, 38(4):1357–1368 (electronic), 2000. ISSN 0036-1429.

I. Karatzas and S. E. Shreve. *Methods of mathematical finance*, volume 39 of *Applications of Mathematics (New York).* Springer-Verlag, New York, 1998. ISBN 0-387-94839-2.

I. Karatzas and S. E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1991. ISBN 0-387-97655-8.

R. Kornhuber. *Adaptive monotone multigrid methods for nonlinear variational problems.* Advances in Numerical Mathematics. B. G. Teubner, Stuttgart, 1997. ISBN 3-519-02722-4.

Y. K. Kwok. *Mathematical models of financial derivatives.* Springer Finance. Springer-Verlag Singapore, Singapore, second edition, 1998. ISBN 981-3083-25-5.

D. Lamberton and B. Lapeyre. *Introduction to stochastic calculus applied to finance.* Chapman & Hall, London, 1996. ISBN 0-412-71800-6. Translated from the 1991 French original by Nicolas Rabeau and Francois Mantion.

A.-M. Matache, T. von Petersdorff, and C. Schwab. Fast deterministic pricing of options on Lévy driven assets. *M2AN Math. Model. Numer. Anal.*, 38(1):37–71, 2004. ISSN 0764-583X.

H. P. McKean, Jr. Appendix: A free boundary problem for the heat equation arising from a problem of mathematical economics. *Industrial Management Review*, 6:32–39, 1965.

R. C. Merton. Theory of rational option pricing. *Bell J. Econom. and Management Sci.*, 4: 141–183, 1973. ISSN 0741-6261.

P. Morin, R. H. Nochetto, and K. G. Siebert. Convergence of adaptive finite element methods. *SIAM Rev.*, 44(4):631–658 (electronic) (2003), 2002. ISSN 0036-1445. Revised reprint of "Data oscillation and convergence of adaptive FEM" [SIAM J. Numer. Anal. **38** (2000), no. 2, 466–488 (electronic); MR1770058 (2001g:65157)].

F. Natterer. Optimale L_2-Konvergenz finiter Elemente bei Variationsungleichungen. In *Finite Elemente (Tagung, Inst. Angew. Math., Univ. Bonn, Bonn, 1975)*, pages 1–12. Bonn. Math. Schrift., No. 89. Inst. Angew. Math. Univ. Bonn, Bonn, 1976.

L. E. Payne and H. F. Weinberger. An optimal Poincaré inequality for convex domains. *Arch. Rational Mech. Anal.*, 5:286–292 (1960), 1960.

C. Reisinger and G. Wittum. On multigrid for anisotropic equations and variational inequalities: pricing multi-dimensional European and American options. *Comput. Vis. Sci.*, 7(3-4):189–197, 2004. ISSN 1432-9360.

R. Seydel. *Tools for computational finance.* Universitext. Springer-Verlag, Berlin, second edition, 2004. ISBN 3-540-40604-2.

R. Underwood and J. Wang. An integral representation and computation for the solution of American options. *Nonlinear Anal. Real World Appl.*, 3(2):259–274, 2002. ISSN 1468-1218.

P. Van Moerbeke. On optimal stopping and free boundary problems. *Arch. Ration. Mech. Anal.*, 60:101–148, 1976.

A. Veeser. Convergent adaptive finite elements for the nonlinear Laplacian. *Numer. Math.*, 92(4):743–770, 2002. ISSN 0029-599X.

R. Verfürth. A posteriori error estimates for finite element discretizations of the heat equation. *Calcolo*, 40(3):195–212, 2003. ISSN 0008-0624.

P. Wilmott, J. Dewynne, and S. Howison. *Option pricing: mathematical models and computation.* Oxford: Financial Press. xii , 1995.

K. Yosida. *Functional analysis.* Classics in Mathematics. Springer-Verlag, Berlin, 1995. ISBN 3-540-58654-7. Reprint of the sixth (1980) edition.

E. Zeidler. *Nonlinear functional analysis and its applications. II/A.* Springer-Verlag, New York, 1990. ISBN 0-387-96802-4. Linear monotone operators, Translated from the German by the author and Leo F. Boron.

E. Zeidler. *Nonlinear functional analysis and its applications. III.* Springer-Verlag, New York, 1985. ISBN 0-387-90915-X. Variational methods and optimization, Translated from the German by Leo F. Boron.

Appendix A

Notation

Standard notations and abbreviations frequently used in this thesis are listed for convenient reading in the tables below.

$\mathbb{R}_{>0}$	$=$	$\{x \in \mathbb{R} \mid x > 0\}$
$\mathbb{R}_{<0}$	$=$	$\{x \in \mathbb{R} \mid x < 0\}$
$\mathrm{erf}(x)$	$=$	$\frac{2}{\pi} \int_0^x e^{-t^2}\, dt$
$(f)_+$	$=$	$\max(f, 0)$
$\hat{f}(\xi)$	$=$	$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\, dx$
$\tilde{f}(x)$	$=$	$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) e^{i\xi x}\, dx$
u'	$=$	spatial derivative
$\dot{u}$	$=$	temporal derivative

Table A.1: Mathematical symbols

PDE	partial differential equation
VI	variational inequality
FBVP	free boundary value problem
LCF	linear complementary formulation
ODE	odenary differential equation
FE	finite elements
FEM	finite element method
AFEM	adaptive finite element method

Table A.2: Some abbreviations

parameter	description
t	time
S	price of underlying asset
$V(S,t)$	value of an option
T	expiry date
K	strike price
r	risk-less interest rate
d	dividend yield
σ	volatility
ψ	pay-off function

Table A.3: Notation for options

$L^1_{loc}(\Omega)$	$=$	space of locally integrable functions on Ω
$L^2(\Omega)$	$=$	space of square integrable functions on Ω
$H^1(\Omega)$	$=$	$\{f \in L^1_{loc}(\Omega) \mid f, f' \in L^2(\Omega)\}$
$\mathcal{D}(\mathbb{R})$	$=$	class of smooth functions with compact support
$\mathcal{S}(\mathbb{R})$	$=$	Schwartz class of rapidly decreasing smooth functions
$\mathcal{D}'(\mathbb{R})$	$=$	Schwartz distributions, topological dual of $\mathcal{D}(\mathbb{R})$
$\mathcal{S}'(\mathbb{R})$	$=$	tempered distributions, topological dual of $\mathcal{S}(\mathbb{R})$
$H^s(\mathbb{R})$	$=$	$\{f \in \mathcal{S}'(\mathbb{R}) \mid (1+\lvert\xi\rvert^2)^{s/2}\hat{f}(\xi) \in L^2(\mathbb{R})\}$
$L^2_\eta(\mathbb{R})$	$=$	$\big\{v \in L^1_{loc}(\mathbb{R}) \mid v e^{-\eta\lvert x\rvert} \in L^2(\mathbb{R})\big\}$
$H^1_\eta(\mathbb{R})$	$=$	$\big\{v \in L^1_{loc}(\mathbb{R}) \mid v e^{-\eta\lvert x\rvert}, v' e^{-\eta\lvert x\rvert} \in L^2(\mathbb{R})\big\}$
$(u,v)_\eta$	$=$	$\int_{\mathbb{R}} uv\, e^{-2\eta\lvert x\rvert}\, dx$
$\lVert u\rVert_{L^2_\eta}$	$=$	$(u,u)_\eta^{1/2}$
$\lVert u\rVert_{H^1_\eta}$	$=$	$\left(\lVert u\rVert^2_{L^2_\eta} + \lVert u'\rVert^2_{L^2_\eta}\right)^{1/2}$
$\lVert u\rVert_{L^2(a,b;X)}$	$=$	$\Big(\int_a^b \lVert u(\cdot,t)\rVert_X^2\, dt\Big)^{1/2}$
$\lVert u\rVert_{L^\infty(a,b;X)}$	$=$	$\operatorname{ess.sup}_{a\le t\le b} \lVert u(\cdot,t)\rVert_X$

Table A.4: Function spaces and norms

u	$=$	solution of the VI
u_ε	$=$	solution of the penalised problem
$u_{\varepsilon h}$	$=$	solution of the semi-discrete penalised problem
u^+	$=$	$\max(u-\psi, 0)$
$\mathcal{A}[u]$	$=$	$-\frac{\sigma^2}{2}u'' - \big(r - d - \frac{\sigma^2}{2}\big)u' + ru$
$\tilde{\mathcal{A}}[u]$	$=$	$\dot{u} - \S 2u'' - \big(r - d - \S 2\big)u' + ru$
$\tilde{\mathcal{A}}_{\varepsilon,h}[u]$	$=$	$\dot{u} - \S 2u'' - \big(r - d - \S 2\big)u' + ru - \frac{1}{\varepsilon}u^+$
$\mathcal{I}$	$=$	nodal interpolation operator

Table A.5: Notation for Chapter 5

Appendix B

Matlab Implementation

This chapter presents an implementation in Matlab of the finite element solution of (6.10). Section B.1 outlines the data structure of the FE implementation while Section B.2 gives a brief overview over the program structure. In Section B.3 all `m.files` are listed.

B.1 Data Structures

All required data is stored in the structure `p`, which is organised as follows. In `p.problem.*` the data required to define the problem is stored. In `p.params.*` the number of initial nodes, the number of refinement levels, and θ, the parameter from the bulk criteria, is stored. In `p.level(j).*` all data for the j-th level is stored. It contains two substructures `p.level(j).geom.*` and `p.level(j).enum.*`. In `p.level(j).geom.*` data regarding the mesh geometry of each level is stored. In `p.level(j).enum.*` constants that are often required are stored. The subsequent tables give a detailed description of these data structures.

`p.params.initialNodes`	number of initial modes in space
`p.params.nrLevels`	number of refinement levels
`p.params.theta`	$\theta \in [0, 1]$

Table B.1: `p.params.*`

p.problem.x0	x_0	truncation point x_0
p.problem.xN	x_N	truncation point x_N
p.problem.T	T	final time
p.problem.K	K	strike price
p.problem.r	r	riskless interest rate
p.problem.sigma	σ	volatility
p.problem.d	d	dividend yield
p.problem.epsilon	ε	penalisation parameter
p.problem.a	$a(x)$	coefficient a as in (6.6) on page 96
p.problem.b	$b(x)$	coefficient b as in (6.6) on page 96
p.problem.c	$c(x)$	coefficient c as in (6.6) on page 96
p.problem.m	$m = 1$	coefficient of the mass matrix
p.problem.weight	$p(x) = e^{-2\eta\|x\|}$	weight function
p.problem.eta	η	exponent of the weight function
p.problem.ux0	$u(x_0)$	Dirichlet boundary at $u(x_0)$
p.problem.uxN	$u(x_N)$	Dirichlet boundary at $u(x_N)$
p.problem.psi	$\psi(x)$	obstacle
p.problem.penalty	$g(x)$	non-linear term
p.problem.truncEstxN	$\max_{0\leq t\leq T} \|u(x_N)\|$	truncation error at x_N
p.problem.knownValues	$\max_{0\leq t\leq T} \|u(x_N)\|$	matrix of truncation errors for several x_N

Table B.2: p.problem.*

p.level(j).geom	data regarding the mesh geometry
p.level(j).enum	several constants
p.level(j).error4e	refinement indicator η_{T_j} as in (5.80)
p.level(j).error4timeStep	error estimator for each time step
p.level(j).U	discret solution $u_{\varepsilon h}$
p.level(j).refineElems	refined elements
p.level(j).markedElems	marked elements
p.level(j).D	$D = A + B + C$, global FE-matrix from (6.8)
p.level(j).M	global mass matrix
p.level(j).totalError	global error estimator

Table B.3: p.level(j).*

p.level(j).geom.c4n	coordinates for nodes
p.level(j).geom.n4e	nodes for element
p.level(j).geom.Db	Dirichlet nodes
p.level(j).geom.timesteps	time steps

Table B.4: p.level(j).geom.*

`p.level(j).enum.alpha`	constant α from the Gårding inequality (3.15)
`p.level(j).enum.const2`	constant C from (3.13)
`p.level(j).enum.constEll`	continuity constant M from (3.14)
`p.level(j).enum.lambda`	constant λ from the Gårding inequality (3.15)
`p.level(j).enum.nrFreenodes`	number of freenodes
`p.level(j).enum.nrElems`	number of elements
`p.level(j).enum.length4e`	length of each element
`p.level(j).enum.midPoint4e`	midpoints of each element
`p.level(j).enum.freeNodes`	free nodes
`p.level(j).enum.length4timeStep`	length of each timestep
`p.level(j).enum.const`	interpolation constant $C(\eta, h)$ from (5.33)
`p.level(j).enum.nrTimeSteps`	number of time steps
`p.level(j).enum.nrDoF`	total degrees of freedom
`p.level(j).enum.e4n`	element for nodes
`p.level(j).enum.eps4e`	element dependent penalisation parameter ε
`p.level(j).enum.newNode4e`	nodes of new elements

Table B.5: `p.level(j).enum.*`

B.2 Short Progamme Description

This sections gives a short description of the `m.files` listed in the next section. The main file is `fem1d.m`. It calls the other procedures and contains the loop for the adaptive mesh refinement. The procedure `solve.m` solves the non-linear system of ODEs using the Matlab routine `ode15s`. In `problemDefinition.m` the users enters all the necessary problem data described in Table B.2 on page 123. The procedure `initializeProblem.m` configurates the geometry and boundary data from the given initial data. The procedure `mark.m` realises the `MARK`ing procedure from Algorithm 6.6 on page 100. The procedure `refine.m` bisects all marked elements. The routine `enum.m` calculates all necessary constants and data structures that are required in the program. The procedure `createSystem.m` determines the finite element matrices A, B, C, and M from (6.8) on page 97. The procedure `estimate.m` implements the element-wise error estimator (6.12) (6.13) (6.14) on page 98. The code of these file are in the following section.

B.3 Matlab Files

`fem1d.m`

```
clear p;
clc

% problem initialization
p = problemDefinition;
p = parameters(p);
p = initializeProblem(p);
```

```
% Loop for each refinement levels
nrLevels = p.params.nrLevels;
for curLevel = 1 : nrLevels
        curLevel
        p = mark(p);
        p = refine(p);
        p = enum(p);
        p = createSystem(p);
        p = solve(p);
        p = estimate(p);
end

% update the data of the final level
c4n = p.level(end).geom.c4n;
timeSteps = p.level(end).geom.timeSteps;
U = p.level(end).U;
psi = p.problem.psi;
nrTimeSteps = p.level(end).enum.nrTimeSteps;
```

fem1d.m

solve.m

```
function p = solve(p)
    % load data
    nrTimeSteps = p.level(end).enum.nrTimeSteps;
    nrNodes = p.level(end).enum.nrNodes;
    ux0 = p.problem.ux0;
    uxN = p.problem.uxN;
    freeNodes = p.level(end).enum.freeNodes;
    T = p.problem.T;
    c4n = p.level(end).geom.c4n;
    u0 = p.problem.u0;
    Db = p.level(end).geom.Db;
    nrFreeNodes = p.level(end).enum.nrFreeNodes;
    M = p.level(end).M;
    freeNodes = p.level(end).enum.freeNodes;

    % initialize U
    U = zeros(nrNodes,nrTimeSteps);

    % set initial condition
    U(:,1) = u0(c4n,p);

    % ODE solver
    options=odeset('Mass', M(freeNodes, freeNodes), ...
        'MStateDependence', 'none', 'MassSingular', 'no',...
        'RelTol',1e-8,'AbsTol',1e-9);
    [t2, X]=ode15s(@rhs, [0,T], U(freeNodes,1), options, p);
    X = X';
    t = t2';
    p.level(end).geom.timeSteps = t;
    p = enum(p);
    nrTimeSteps = p.level(end).enum.nrTimeSteps;
    U = zeros(nrNodes,nrTimeSteps);
```

```
    U(freeNodes,:) = X;
    U(Db,:) = [ux0(t,p); uxN(t,p)];

    % save solution
    p.level(end).U = U;
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function val = rhs(t,U,p)
    % load data
    D = p.level(end).D;
    freeNodes = p.level(end).enum.freeNodes;
    length4e = p.level(end).enum.length4e;
    nrNodes = p.level(end).enum.nrNodes;
    midPoint4e = p.level(end).enum.midPoint4e;
    ux0 = p.problem.ux0;
    uxN = p.problem.uxN;
    epsilon =p.level(end).enum.eps4e;
    Db = p.level(end).geom.Db;
    n4e = p.level(end).geom.n4e;

    % set dirichlet data
    dummy = zeros(nrNodes,1);
    dummy(freeNodes) = U;
    dummy(Db) = [ux0(t,p);uxN(t,p)];
    U = dummy;

    % one point gauss integration to evaluate the non-linear term g
    penalty4e = penalty(midPoint4e',U,p);
    dummy = 1./epsilon.*length4e.*penalty4e';
    S = [dummy;dummy];
    I = n4e(:);
    G = accumarray(I,S);

    % right hand side
    val = -D(freeNodes,:)*U + G(freeNodes);
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function val = penalty(x, U,p)
    %load data
    n4e = p.level(end).geom.n4e;
    c4n = p.level(end).geom.c4n;
    psi = p.problem.psi; ;
    eta = p.problem.eta;
    K = p.problem.K;

    U = U(n4e);
    U1 = U(:,1)';
    U2 = U(:,2)';
    psiX = psi(x,p);
    vertices = c4n(n4e);
    val1 = psiX-(U1.*(vertices(:,2)'-x)./(vertices(:,2)'-vertices(:,1)')...
        +U2.*(x-vertices(:,1)')./(vertices(:,2)'-vertices(:,1)')) ;
    val = max(val1',0)';
```

```
end
```

solve.m

problemDefinition.m

```
function p = problemDefinition
    % set problem data
    p.problem.x0 = 1.6;
    p.problem.xN = 3;
    p.problem.T = 1;
    p.problem.K = 10;
    p.problem.eta = 0.0001;
    p.problem.truncEstxN = 0.00636;
    p.problem.r = 0.25;
    p.problem.sigma = 0.6;
    p.problem.d = 0;
    p.problem.epsilon  = 0.1;
    p.problem.a = @a;
    p.problem.b = @b;
    p.problem.c = @c;
    p.problem.m = @m;
    p.problem.weight = @weight;
    p.problem.ux0 = @ux0;
    p.problem.uxN = @uxN;
    p.problem.u0 = @u0;
    p.problem.psi = @psi;
    p.problem.penalty = @penalty;
    p.problem.knownValues =    [2.9            0.90721; ...
                                2.95           0.770748; ...
                                2.975          0.709669;...
                                2.9925         0.669476;...
                                3              0.591722;...
                                3.1            0.386659;...
                                3.2            0.251089;...
                                3.3            0.16126;...
                                3.4            0.10211;...
                                3.45           0.08248;...
                                3.475          0.07398;...
                                3.4875         0.06722;...
                                3.49375        0.06539;...
                                3.496875       0.6449;...
                                3.4984375      0.064047;...
                                3.49921875     0.063825;...
                                3.5            0.06360;...
                                3.6            0.038911;...
                                3.7            0.02335;...
                                3.8            0.01373;...
                                3.9            0.0079066;...
                                3.95           0.00602189;...
                                3.975          0.0052436;...
                                3.9875         0.00489; ...
                                4              0.0044548;...
                                4.1            0.0024546;...
                                4.2            0.001322;...
                                4.3            0.0006957;...
```

```
                                4.4             0.0003576;...
                                4.5             0.00017948;...
                                4.6             0.00008793;...
                                4.7             0.000042036;...
                                4.8             0.000019607;...
                                4.9             0.0000089;...
                                5               0.000003959];
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function val = a(x,p)
    sigma = p.problem.sigma;
    val = sigma^2/2;
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function val = b(x,p)
    sigma = p.problem.sigma;
    eta = p.problem.eta;
    r = p.problem.r;
    K = p.problem.K;
    val = -r+sigma^2/2-eta*sigma^2*sign(x);
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function val = c(x,p)
    r = p.problem.r;
    val = r;
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function val = m(x,p)
    val = ones(size(x));
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function val = weight(x,p)
    K = p.problem.K;
    eta = p.problem.eta;
    val = exp(-2*eta*abs(x));
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function val = ux0(t,p)
    K = p.problem.K;
    r = p.problem.r;
    x0 =  p.problem.x0;
    val = K-exp(x0)+0.*t;
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
function val = uxN(t,p)
    K = p.problem.K;
    r = p.problem.r;
    xN =  p.problem.xN;
    d = p.problem.d;
    val = 0.*t;
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function val = u0(x,p)
    val = psi(x',p);
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function val = psi(x,p)
    K = p.problem.K;
    val = max((K-exp(x)),0);
end
```

problemDefinition.m

parameters.m

```
function p = parameters(p)
% set proplem paramters
p.params.nrInitialNodes =20;
p.params.nrLevels =8;
p.params.theta=0.5;
```

parameters.m

initializeProblem.m

```
function p = initializeProblem(p)
% initialize Problem
u0 = p.problem.u0;
T=p.problem.T;

% space discretisation
c4n = linspace(p.problem.x0, p.problem.xN, p.params.nrInitialNodes)'
p.level(1).geom.c4n = c4n;
nrNodes = length(c4n);
n4e = [1:nrNodes-1;2:nrNodes]';
p.level(1).geom.n4e = n4e;
Db = [1, nrNodes];
p.level(1).geom.Db = Db;
p.level(1).geom.timeSteps =  linspace(0, T, 0.1) ;

% calculation of serval constants
p = enum(p);

%set the error to 0
```

```
p.level(1).error4e = zeros(p.level(end).enum.nrElems,1);
p.level(1).error4timeStep = zeros(length(p.level(end).geom.timeSteps)-1,1);

% Initializing U
U = zeros(length(c4n), length(linspace(0, T, 0.1)));
U(:,1) = u0(c4n,p);
p.level(1).U = U;
```

initializeProblem.m

mark.m

```
function p = mark(p)
    % load data
    error4e = p.level(end).error4e;
    nrElems = p.level(end).enum.nrElems;
    refineElems = false(nrElems,1);
    theta=p.params.theta;

    % mark elements with Bulk criteria
     [sortedError4e,I] = sort(error4e,'descend');
     sumError4e = cumsum(sortedError4e);
     J = find(sumError4e > theta * sum(error4e),1,'first');
     refineElems(I(1:J)) = true;
     p.level(end).refineElems = refineElems;

    % closure algorithm
    p = closure(p);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function p = closure(p)

    nrElems = p.level(end).enum.nrElems;
    e4n = p.level(end).enum.e4n;
    n4e = p.level(end).geom.n4e;
    refineElems = p.level(end).refineElems ;

    markedNodes = n4e(refineElems,:);
    markedNodes = markedNodes(:);
    dummy = e4n(markedNodes,:);

    markedElems = false(nrElems,1);
    markedElems(dummy(:)) = true;

    p.level(end).markedElems = markedElems;
```

mark.m

refine.m

```
function p = refine(p)
    % load data
        c4n = p.level(end).geom.c4n;
```

```
        n4e = p.level(end).geom.n4e;
        Db = p.level(end).geom.Db;
        level = length(p.level);
        markedElems = p.level(end).markedElems;
        midPoint4e = p.level(end).enum.midPoint4e;
        nrElems = p.level(end).enum.nrElems;
        nrNodes = p.level(end).enum.nrNodes;
        timeSteps = p.level(end).geom.timeSteps;

        newC4n = midPoint4e(markedElems);
        c4n = [c4n;newC4n];
        newNode4e = zeros(nrElems,1);
        newNode4e(markedElems) = (nrNodes + 1):(nrNodes + nnz(markedElems));

        n4e = [        n4e(~markedElems,:);
                        [n4e(markedElems,1),newNode4e(markedElems)];
                        [newNode4e(markedElems),n4e(markedElems,2)]];

        [Y,I] = min(c4n);
        Db(1) = I;

        [Y,I] = max(c4n);
        Db(2) = I;

        % update n4e, c4n
        p.level(level+1).geom.c4n = c4n;
        p.level(level+1).geom.n4e = n4e;
        p.level(level+1).geom.Db = Db;
        p.level(level).enum.newNode4e = newNode4e;
        p.level(level+1).geom.timeSteps = timeSteps;
end
```

refine.m

enum.m

```
function p = enum(p)
    % load data
    c4n = p.level(end).geom.c4n;
    n4e = p.level(end).geom.n4e;
    timeSteps = p.level(end).geom.timeSteps;
    eta = p.problem.eta;
    Db = p.level(end).geom.Db;
    sigma = p.problem.sigma;
    d = p.problem.d;
    r = p.problem.r;
    eps= p.problem.epsilon;

    % determination of constants or data structures
    nrNodes = size(c4n,1);
    length4e = c4n(n4e(:,2))-c4n(n4e(:,1));
    const = 2*length4e/pi.*((cosh(2*eta*length4e)-1)./(2*eta^2.*length4e.^2)).^0.5;
    alpha = sigma^2/4;
    const2 = max(abs(r-d-sigma^2/2+eta*sigma^2),abs(r-d-sigma^2/2-eta*sigma^2));
    constEll = max(sigma^2/2,r)+const2;
    lambda = r-const2^2-sigma^2/4;
```

```
    nrElems = size(n4e,1);
    midPoint4e = (c4n(n4e(:,1)) + c4n(n4e(:,2)))/2;
    length4timeStep =  timeSteps(2:end)-timeSteps(1:end-1);
    freeNodes = setdiff(1:nrNodes, unique(Db));
    nrFreeNodes = length(freeNodes);
    nrTimeSteps = length(timeSteps);
    nrDoF = nrTimeSteps * nrFreeNodes;
    e4n = zeros(nrNodes,2);
    e4n(n4e(:,1),1) = 1:nrElems;
    e4n(n4e(:,2),2) = 1:nrElems;
    e4n = sort(e4n,2);
    [I,dontUse] = find(e4n == 0);
    e4n(I,1) = e4n(I,2);
    eps4e=eps.*length4e.^2;

    % save data
    p.level(end).enum.alpha = alpha;
    p.level(end).enum.const2 = const2;
    p.level(end).enum.constEll = constEll;
    p.level(end).enum.lambda = lambda;
    p.level(end).enum.nrFreeNodes = nrFreeNodes;
    p.level(end).enum.nrElems = nrElems;
    p.level(end).enum.nrNodes = nrNodes;
    p.level(end).enum.length4e = length4e;
    p.level(end).enum.midPoint4e = midPoint4e;
    p.level(end).enum.freeNodes = freeNodes;
    p.level(end).enum.length4timeStep = length4timeStep;
    p.level(end).enum.const = const;
    p.level(end).enum.nrTimeSteps = nrTimeSteps;
    p.level(end).enum.nrDoF = nrDoF;
    p.level(end).enum.e4n = e4n;
    p.level(end).enum.eps4e=eps4e;
end
```

enum.m

createSystem.m

```
function  p = createSystem(p)
% load data
    nrNodes = p.level(end).enum.nrNodes;
    nrElems = p.level(end).enum.nrElems;
    c4n = p.level(end).geom.c4n;
    n4e = p.level(end).geom.n4e;

    % (phi_k, phi_j)
    M = sparse(nrNodes,nrNodes);

    % d = a(gradPhi_k, gradPhi_j) + b(gradPhi_k, phi_j) + c(phi_k, phi_j)
    D = sparse(nrNodes,nrNodes);

    % determination of the  FE-matrices A, B, C, and M
    for curElem = 1:nrElems
        curNodes = n4e(curElem,:);
        curCoords = c4n(curNodes,:);
        M(curNodes,curNodes) = M(curNodes,curNodes) + coeffM(curCoords,p);
```

```
        D(curNodes,curNodes) = D(curNodes,curNodes) + coeffA(curCoords,p) ...
                               + coeffB(curCoords,p) + coeffC(curCoords,p);
    end

    p.level(end).D = D;
    p.level(end).M = M;
    end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function val = coeffA(x,p)
    a = p.problem.a;
    weight = p.problem.weight;
    dummy = quad(@integrand, x(1), x(2), [],[],a, weight, 0, 1,0,1,p);
    val = [1 -1;-1 1] .* dummy /(x(2)-x(1))^2;
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function val = coeffB(x,p)
    b = p.problem.b;
    weight = p.problem.weight;
    dummy1 = quad(@integrand, x(1), x(2), [],[],b, weight, 1, -x(2),0,1,p);
    dummy2 = quad(@integrand, x(1), x(2), [],[],b, weight, 1, -x(1),0,1,p);
    val = [1 -1;-1 1] .* [dummy1, dummy1; dummy2, dummy2]/(x(2)-x(1))^2;
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function val = coeffC(x,p)
    c = p.problem.c;
    weight = p.problem.weight;
    dummy1 = quad(@integrand, x(1), x(2), [],[],c, weight, 1, -x(2),1,-x(2),p);
    dummy2 = quad(@integrand, x(1), x(2), [],[],c, weight, 1, -x(2),1,-x(1),p);
    dummy3 = quad(@integrand, x(1), x(2), [],[],c, weight, 1, -x(1),1,-x(2),p);
    dummy4 = quad(@integrand, x(1), x(2), [],[],c, weight, 1, -x(1),1,-x(1),p);
    val = [1 -1;-1 1] .* [dummy1, dummy2; dummy3, dummy4]/(x(2)-x(1))^2;
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function val = coeffM(x,p)
    m = p.problem.m;
    weight = p.problem.weight;
    dummy1 = quad(@integrand, x(1), x(2), [],[],m, weight, 1, -x(2),1,-x(2),p);
    dummy2 = quad(@integrand, x(1), x(2), [],[],m, weight, 1, -x(2),1,-x(1),p);
    dummy3 = quad(@integrand, x(1), x(2), [],[],m, weight, 1, -x(1),1,-x(2),p);
    dummy4 = quad(@integrand, x(1), x(2), [],[],m, weight, 1, -x(1),1,-x(1),p);
    val = [1 -1;-1 1] .* [dummy1, dummy2; dummy3, dummy4]/(x(2)-x(1))^2;
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function val=integrand(x, func, weight, c1, c0, d1, d0, p)
    val = func(x,p).*(c1*x+c0).*(d1*x+d0).*weight(x,p);
end
```

createSystem.m

estimate.m

```
function p = estimate(p)
    % Implementation of the residual based error estimate

    % load data
    n4e = p.level(end).geom.n4e;
    c4n = p.level(end).geom.c4n;
    length4timeStep = p.level(end).enum.length4timeStep;
    const = p.level(end).enum.const;
    nrTimeSteps = p.level(end).enum.nrTimeSteps
    nrElems = p.level(end).enum.nrElems;
    Res = zeros(nrTimeSteps-1,nrElems)';
    U = p.level(end).U;
    weight = p.problem.weight;
    epsilon=p.level(end).enum.eps4e;
    sigma = p.problem.sigma;
    truncEstxN = p.problem.truncEstxN;
    % initialize elementwise error
    error4e = zeros(nrElems,nrTimeSteps-1);

    % loop over all spatial elements
    for curElem = 1:nrElems
        n1 = n4e(curElem,1);
        n2 = n4e(curElem,2);
        x1 = c4n(n1);
        x2 = c4n(n2);

        % midpoint of each time step
        U1L = ( U(n1,1:end-1) + U(n1,2:end) )/2;
        U1R = ( U(n2,1:end-1) + U(n2,2:end) )/2;
        U1 = [U1L; U1R];

        % approximated time derivative
        U2L = ( U(n1,2:end) - U(n1,1:end-1) )./length4timeStep;
        U2R = ( U(n2,2:end) - U(n2,1:end-1) )./length4timeStep;
        U2 = [U2L; U2R;];

        % values of U at x1 and x2
        U3=[U(n1,:);U(n2,:)];

        % U at time 0
        U0=[U(n1,1);U(n2,1)];

        % epsilon on current element
        curEps=epsilon(curElem);

        % integration over current element, for all time steps simultanuously
        integral = ...
            quadv(@integrandfehler,x1,x2,[],[],U1,[x1 x2],U2,U3,curEps,curElem,p);

        % integration over current element, for time 0
        integral2 = ...
            quad(@integrandfehler2,x1,x2,[],[],U0,[x1 x2],curEps,curElem,p);

        error4e(curElem,:) = integral;
```

```
        error4e(curElem,1) = error4e(curElem,1)+integral2;
     end


    % truncation error
    TruncNode=find(c4n==xN); % find truncation point
    [TruncElem, tmp] = find(n4e==TruncNode);
    % find element belonging to truncation point
    curElem = TruncElem ; % set current  element
    n1 = n4e(curElem,1);
    n2 = n4e(curElem,2);
    x1 = c4n(n1)
    x2 = c4n(n2)

    % find truncation error at x_{N-1}
    knownValues_x=p.problem.knownValues(:,1);
    knownValues_u=p.problem.knownValues(:,2);
    truncEstxN_1= knownValues_u(max(find(knownValues_x<=x1)))
    p.level(end).enum.xn_1=x1;
    p.level(end).enum.u_xn_1=truncEstxN_1;

     % midpoint of each time step
    U1L = ( U(n1,1:end-1) + U(n1,2:end) )/2;
    U1R = ( U(n2,1:end-1) + U(n2,2:end) )/2;
    U1 = [U1L; U1R];

     % approximated time derivative
    U2L = ( U(n1,2:end) - U(n1,1:end-1) )./length4timeStep;
    U2R = ( U(n2,2:end) - U(n2,1:end-1) )./length4timeStep;
    U2 = [U2L; U2R;];

     % values of U at x1 and x2
    U3=[U(n1,:);U(n2,:)];

     % slope
    slope = (U1(2,:)-U1(1,:))./(x2-x1);

     % calculation of the truncation error terms
    truncErr1=sigma^2/2*abs(slope)*truncEstxN*gewicht(x2,p).*length4timeStep;
    integral1= ...
       quadv(@integrandfehler4,x1,x2,[],[],U1,[x1 x2],U2,U3, curEps,curElem,p);
    integral = ...
       quadv(@integrandfehler3,x1,x2,[],[],U1,[x1 x2],U2, curEps,curElem,p);
    L2Res = sqrt(integral);
    truncErr2= sqrt(x2-x1)*(1+3^(-0.5))*abs(truncEstxN_1)*sqrt(gewicht(x1,p))...
                 .*L2Res.*length4timeStep;
    ResErr= integral1;

     % calculate and save the different error terms
    error4e(curElem,:) = truncErr2+ResErr;
    p.level(end).discrError = sqrt(sum(sum(error4e,2)));
    error4e(curElem,:) = truncErr1+truncErr2+ResErr;
    p.level(end).totalError = sqrt(sum(sum(error4e,2)));
    p.level(end).error4e = sum(error4e,2);
    p.level(end).error4timeStep = sum(error4e,1)';
    p.level(end).truncErrorxN = sqrt(sum(truncErr1));
end
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function val = integrandfehler(x, U, vertices, U2, U3, curEpsilon,curElem,p)
    % error terms that are evaluated on (0,T), time integration via one-point
    % Gauss quadrature

    eta = p.problem.eta;
    sigma = p.problem.sigma;
    alpha = p.level(end).enum.alpha;
    r = p.problem.r;
    weight = p.problem.weight;
    psi = p.problem.psi;
    d = p.problem.d;
    u0 = p.problem.u0;
    length4timeStep = p.level(end).enum.length4timeStep;
    const = p.level(end).enum.const;
    constEll = p.level(end).enum.constEll;

    x1 = vertices(1);
    x2 = vertices(2);
    slope = (U(2,:)-U(1,:))./(vertices(2)-vertices(1));
    uhU1 = slope.*x+U(1,:)-slope*vertices(1);
    uhU2 = slope.*x+U2(1,:)-slope*vertices(1);
    slopeU3 = (U3(2,:)-U3(1,:))./(vertices(2)-vertices(1));
    uhU3 = slopeU3.*x+U3(1,:)-slopeU3*vertices(1);
    uhPrime = slope;
    uhPlus = max(psi(x,p)-uhU1, 0);
    weight = weight(x,p);

    h = 1e-9;
    uhU1dx = slope.*(x+h)+U(1,:)-slope*vertices(1);
    Udx = max(psi(x+h,p)-uhU1dx, 0);
    z = diff([uhPlus;Udx],1,1)/h;
    uhPlusH1 = uhPlus.^2.*weight + z.^2.*weight;
    uhPlusH1 = length4timeStep.*uhPlusH1;

    uhPlusDot = ( max(psi(x,p)-uhU3(2:end), 0)-max(psi(x,p)-uhU3(1:end-1), 0));
    uhPlusDot =  uhPlusDot.^2.*weight;

    uhPlusInfinity = max(uhPlus).^2.*weight;

    resError = ( r.*uhU1-(r-d-sigma^2).*uhPrime - 1./curEpsilon.*uhPlus + uhU2 )...
        .^2.*weight;
    resError = const(curElem).^2.*length4timeStep.*resError;

    val = 3/(2*alpha)*(uhPlusDot+constEll^2*(uhPlusH1+resError))+uhPlusInfinity;
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function val = integrandfehler2(x, U, vertices,   curEpsilon,curElem,p)
    %  terms of the error estimate that are considered at time t=0

    eta = p.problem.eta;
    sigma = p.problem.sigma;
    alpha = p.level(end).enum.alpha;
```

```
    r = p.problem.r;
    weight = p.problem.weight;
    psi = p.problem.psi;
    d = p.problem.d;
    u0 = p.problem.u0;
    length4timeStep = p.level(end).enum.length4timeStep;
    const = p.level(end).enum.const;
    constEll = p.level(end).enum.constEll;

    x1 = vertices(1);
    x2 = vertices(2);
    slope = (U(2)-U(1))./(vertices(2)-vertices(1));
    uhU1 = slope.*x+U(1)-slope*vertices(1);
    uhPlus = max(psi(x,p)-uhU1, 0);
    weight = weight(x,p);

    initialError = (u0(x',p) - uhU1).^2.*weight;
    initialPenaltyError = uhPlus.^2.*weight;

    val= 1/2*(initialError+initialPenaltyError) ;
end
```

estimate.m

Appendix C

Maple Code

This chapter displays the maple code which is used to calculate the pointwise truncation errors.

maple file

```
> r:=0.25; K:=10; sigma:=0.6; T:=1; d:=0; t:=T; xN:=3;
> a:=2*sigma^2; b:=r-d-sigma^2/2; c:=r-d; e:=log(K); kappa:=10^20;
> alpha1:=(kappa-a*t)/(a*kappa); beta1:=min(-2*e/a,2*(b*t-e)/a);
> gamma1:=min(e^2/a, (b*t-e)^2/a+r*t^2);
> xp1:=evalf(1/(2*alpha1)*(-beta1+sqrt(beta1^2-4*alpha1*gamma1)));
> if (evalf(xp1) > xN) then print('Warnung') else print('ok') end if;
> tildef:=x->alpha1+beta1/x+gamma1/x^2;
> xstar:=-2*gamma1/beta1;
> evalf(xstar);
> if (evalf(beta1)>=0) then C1:=alpha1 end if;
> if (evalf(beta1)<0 and evalf(tildef(xstar)) <=0) then C1:=tildef(xN) end if;
> if (evalf(beta1)<0 and evalf(tildef(xstar)) >0 and evalf(xN)<=evalf(xstar))
      then C1:=tildef(xstar) end if;
> if (evalf(beta1)<0 and evalf(tildef(xstar)) >0 and evalf(xN)>evalf(xstar))
      then C1:=tildef(xN) end if;
> evalf(C1);
> constI1:=(x)->2*sqrt(t)*exp(-C1*x^2/t)+2*x*sqrt(C1*Pi)*(erf(sqrt(C1)*x/sqrt(t))-1);
> evalf(constI1(xN));
> CI1:=evalf(constI1(xN)*exp(-xN^2/kappa));
> eps:=0.6;
> xtest:=evalf(e+abs(b)*t+eps);
> if (xN<evalf(xtest)) then print('Warnung') else print('ok') end if;
> constI2:=(x)->2*sqrt(a)/(eps*Pi)*(2/3*sqrt(t)*exp(-C1*x^2/t)*(t-2*C1*x^2)
      +4/3*x^3*sqrt(C1^3*Pi)*(1-erf(sqrt(C1)*x/sqrt(t))));
> CI2:=evalf(constI2(xN)*exp(-xN^2/kappa));
> constU:=CI1/(sigma*sqrt(2*Pi))+CI2*r*K/2;
> evalf(CI1); evalf(CI2);
> evalf(constU);
```

maple file

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Selbständigkeitserklärung

Hiermit erkläre ich, die vorliegende Arbeit selbstständig ohne fremde Hilfe verfaßt und nur die angegebene Literatur und Hilfsmittel verwendet zu haben.

Karin Mautner
Berlin, 22. Mai 2006