Utility Maximization and Quadratic BSDEs under Exponential Moments

Existence, Uniqueness and Stability of Constrained Problems

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Abstract

In the present thesis we consider the problem of maximizing the power utility from terminal wealth where the stocks have continuous semimartingale dynamics and there are investment and information constraints on the agent's strategies. The main focus is on the backward stochastic differential equation (BSDE) that encodes the dynamic value process and on transferring new results on quadratic semimartingale BSDEs to the portfolio choice problem, in particular to its stability properties. This is accomplished under the assumption of finite exponential moments of the mean-variance tradeoff, generalizing previous results which require boundedness.

We first recall the precise relationship between the duality and BSDE approaches to solving the above problem and then study the quadratic semimartingale BSDE which arises in such a problem when the market price of risk is of BMO type. We show that even for a bounded mean-variance tradeoff there is always a continuum of distinct solutions to the BSDE with square-integrable martingale part (with only one of them being a bounded solution). For this we prove that in contrast to the classical Itô decomposition theorem an L^2-representation of random variables in terms of stochastic exponentials is not unique. We then provide a new sharp condition on the dynamic exponential moments of the mean-variance tradeoff which guarantees the boundedness of BSDE solutions in a general filtration. The main results are complemented by several interesting examples which illustrate their sharpness as well as important properties of the utility maximization BSDE.

In a subsequent step we establish existence, uniqueness and stability results for general quadratic continuous BSDEs under an exponential moments condition. An important additional result is that the martingale part of a solution does provide a true measure change even though the first component of the solution triple might not be bounded, equivalently, even though the martingale part might not be of BMO type. As a consequence, the verification argument for the utility maximization problem becomes feasible for unbounded mean-variance tradeoff variables which satisfy an appropriate exponential moments assumption.

We use these results to study the portfolio selection problem when there are conic investment constraints. Building on a decomposition result for the elements of the so-called dual domain we have the associated BSDE satisfied by the dynamic value process (the opportunity process) and show that by our moments assumptions this value process is contained in a specific space in which BSDE solutions are unique. This provides an argument for verification. A consequence of the stability result for BSDEs is then the continuity of the optimizers with respect to the input parameters of the model, i.e. the investor's relative risk aversion, the market price of risk process, the statistical probability measure and the constraints sets, in the semimartingale topology. This result integrates previous research into a unified BSDE framework.

Finally, we study the existence, uniqueness and stability of the optimal investment problem under exponential moments, compact constraints and restricted information. This is done by referring to BSDE results only and performing the verification argument using the previous measure change result, which is beyond the boundedness assumptions of recent literature.

Zusammenfassung

In der vorliegenden Dissertation befassen wir uns mit der Erwartungsnutzenmaximierung des Endvermögens für Potenznutzen, wenn die Aktienpreise stetigen Semimartingaldynamiken genügen und die Strategien des Agenten Investitions- und Informationsrestriktionen unterworfen sind. Hauptaugenmerk liegt dabei auf der stochastischen Rückwärtsdifferentialgleichung (BSDE) für den dynamischen Wertprozess und auf der Übertragung von neuen Ergebnissen zu quadratischen Semimartingal-BSDEs auf das Investitionsproblem, insbesondere auf dessen Stabilitätseigenschaften. Dieses gelingt unter der Annahme endlicher exponentiellen Momente des so genannten Mean-Variance Tradeoff und verallgemeinert frühere Resultate, die Beschränktheit fordern.

Wir betrachten dabei zunächst die genaue Beziehung zwischen den Dualitäts- und BSDE-Ansätzen zur Lösung des obigen Problems und gehen dann über zum Studium der quadratischen Semimartingal-BSDE, die in solch einem Problem auftritt, wenn der Marktpreis des Risikos vom BMO-Typ ist. Wir zeigen, dass es selbst für einen beschränkten Mean-Variance Tradeoff stets ein Kontinuum verschiedener BSDE-Lösungen mit quadratisch integrierbarem Martingalanteil gibt (wobei nur eine dieser Lösungen beschränkt ist). Hierfür beweisen wir, dass im Gegensatz zum Darstellungssatz von Itô eine L^2-Darstellung von Zufallsvariablen als stochastische Exponentiale nicht eindeutig ist. Wir stellen dann eine neue scharfe Bedingung an die dynamischen exponentiellen Momente des Mean-Variance Tradeoffs vor, die die Beschränktheit der BSDE-Lösungen in einer allgemeinen Filtration garantiert. Die Hauptergebnisse werden mit mehreren Beispielen vervollständigt, die ihre Schärfe sowie wichtige Eigenschaften der Nutzenmaximierungs-BSDE veranschaulichen.

In weiterer Folge weisen wir Existenz-, Eindeutigkeits- und Stabilitätsresultate für allgemeine quadratische stetige BSDEs unter exponentiellen Momenten nach. Ein wichtiges zusätzliches Ergebnis ist, dass der Martingalanteil einer Lösung einen Maßwechsel definiert, auch wenn die erste Komponente eines Lösungstripels nicht beschränkt bzw. der Martingalanteil nicht vom BMO-Typ ist. Folgerichtig lässt sich das Verifikationsargument für das Nutzenmaximierungsproblem auch für unbeschränkte Mean-Variance Tradeoff Variablen, die einer geeigneten Bedingung an die exponentiellen Momente genügen, durchführen.

Diese Ergebnisse verwenden wir, um das Investitionsproblem für den Fall konischer Investitionsrestriktionen zu untersuchen. Ausgehend von der Zerlegung von Elementen des dualen Gebietes erhalten wir die entsprechende BSDE für den dynamischen Wertprozess (den Opportunitätsprozess) und beweisen, dass dieser in einem Raum liegt, in welchem Lösungen quadratischer BSDEs eindeutig sind. Dies liefert ein Argument für den Verifikationsschritt. Als Folgerung aus dem Stabilitätsresultat für BSDEs erhalten wir die Stetigkeit der Optimierer in der Semimartingaltopologie in den Parametern des Modells, d. h. in der relativen Risikoaversion des Investors, dem Marktpreis des Risikos, dem statistischen Wahrscheinlichkeitsmaß und den Mengen, die die Restriktionen modellieren. Dieses Ergebnis vereinigt frühere Forschung in einem gemeinsamen BSDE-Rahmen.

Schließlich betrachten wir die Existenz, Eindeutigkeit und Stabilität des Investitionsproblems unter exponentiellen Momenten, kompakten Handelsrestriktionen und eingeschränkter Information. Hierbei benutzen wir ausschließlich BSDE-Resultate und führen das Verifikationsargument aus, indem wir uns nun auf die Aussage über den Maßwechsel beziehen, was wiederum außerhalb der Beschränktheitsannahmen neuerer Literatur liegt.

Contents

1 General Concepts and Preliminaries

1.1 Background and Introduction

Backward Stochastic Differential Equations

Since their introduction by Bismut [1973] within the Pontryagin maximum principle, backward stochastic differential equations (BSDEs) have attracted much attention in the mathematical literature. In a Brownian framework such equations are usually written

$$d\Psi_t = Z_t \, dW_t - F(t, \Psi_t, Z_t) \, dt, \quad \Psi_T = \xi, \tag{1.1.1}$$

where ξ is an $\mathcal{F}_T$-measurable random variable, the *terminal value*, F is the so-called *driver* or *generator* and $T > 0$ is a real number. Here, $(\mathcal{F}_t)_{t\in[0,T]}$ denotes the filtration generated by the one-dimensional Brownian motion $(W_t)_{t\in[0,T]}$. Solving such a BSDE corresponds to finding a *pair* of adapted processes (Ψ, Z) such that the integrated version of (1.1.1) holds. The presence of the *control process* Z stems from the requirement of adaptedness for Ψ together with the fact that Ψ must be driven into the random variable ξ at time T. One may think of Z as arising from the martingale representation theorem. To be more concrete, for a square-integrable ξ, consider the BSDE

$$d\Psi_t = Z_t \, dW_t, \quad \Psi_T = \xi,$$

which is solved explicitly by setting $\Psi_t := \mathbb{E}[\xi \,|\, \mathcal{F}_t]$, which gives a square-integrable martingale, and then applying the martingale representation theorem,

$$\Psi_t = \mathbb{E}[\xi |\, \mathcal{F}_t] = \mathbb{E}[\xi] + \int_0^t Z_s \, dW_s.$$

The above BSDE (1.1.1) is an extension of this example; it includes an additional driver F.

In a more general semimartingale framework, where the main source of randomness is encoded in a fixed continuous (one-dimensional) local martingale M on a filtration $(\mathcal{F}_t)_{t\in[0,T]}$ which is not necessarily generated by it, we have to add an extra orthogonal component N. The corresponding BSDE then takes the form

$$d\Psi_t = Z_t \, dM_t + dN_t - f(t, \Psi_t, Z_t) \, d\langle M\rangle_t - g_t \, d\langle N\rangle_t, \quad \Psi_T = \xi. \tag{1.1.2}$$

Solving (1.1.2) now corresponds to finding an adapted *triple* (Ψ, Z, N) of processes satisfying the integrated version of (1.1.2), where N is a (continuous) one-dimensional local martingale orthogonal to M, i.e. $\langle M, N\rangle \equiv 0$, where $\langle\cdot,\cdot\rangle$ denotes the quadratic covariation. We refer to $Z \cdot M + N$ as the *martingale part* of a solution to the BSDE (1.1.2). In a multidimensional setting we write this BSDE as

$$d\Psi_t = Z_t^{\intercal}\, dM_t + dN_t - F(t, \Psi_t, Z_t)\, dA_t - g_t\, d\langle N \rangle_t, \quad \Psi_T = \xi, \tag{1.1.3}$$

where A is a predictable increasing process that encodes the variation of the continuous d-dimensional local martingale M and N is now orthogonal to each component of M. Moreover, in all our applications, g equals the constant $1/2$ reflecting the fact that the drivers F of the BSDEs considered here are of quadratic type (in Z). In fact, it is well-known that such BSDEs can be solved by using an exponential transform and this fact will be a recurrent theme in the following exposition. To motivate, we state the prototype of a quadratic BSDE,

$$d\Psi_t = Z_t^{\intercal}\, dW_t - \frac{1}{2}\, \|Z_t\|^2\, dt, \quad \Psi_T = \xi.$$

Given suitable conditions, this is solved explicitly by setting $\Psi_t := \log\big(\mathbb{E}[\exp(\xi) |\, \mathcal{F}_t]\big)$ and applying the martingale representation theorem to the random variable $\exp(\xi)$, together with an appropriate transformation, in order to find Z. In the general case, assuming that M does not exhibit the representation property, an additional orthogonal martingale enters the formula and the exponential transformation requires that its quadratic variation appear with factor $1/2$.

As already mentioned, BSDEs of type (1.1.1) and (1.1.3) have found many fields of application in mathematical finance and the present thesis is one of the many contributions. The reader is directed to El Karoui et al. [1997] for a first survey. In view of the specific focus of the present work we remark that appropriate BSDEs have been derived in Hu et al. [2005] for the value processes of several constrained utility maximization problems. This article extends earlier work by Rouge and El Karoui [2000] as well as Sekine [2006] and motivated a subsequent extension of the Brownian framework to a continuous semimartingale setting by Morlais [2009]. Building on the work by Mania and Tevzadze [2003, 2008], the respective BSDE for a power utility function is investigated in Nutz [2011] in even more generality establishing a one-to-one correspondence between solutions to BSDEs and solutions to the so-called primal and dual problems of utility maximization. In Mania and Schweizer [2005] the authors use a BSDE to describe the dynamic indifference price for exponential utility and their approach is extended to robust utility in Bordigoni et al. [2007] and to an infinite time horizon in Hu and Schweizer [2009]. We also mention Becherer [2006] for further extensions to BSDEs with jumps and Mania and Tevzadze [2008] to backward stochastic partial differential equations (BSPDEs).

With regards to the theory of BSDEs, existence and uniqueness results were first provided in a Brownian setting by Pardoux and Peng [1990] under Lipschitz conditions. They were extended by Lepeltier and San Martín [1997] to continuous drivers with linear growth and by Kobylanski [2000] to generators which are quadratic as a function of the control variable Z. Corresponding results for a semimartingale framework may be found in Morlais [2009] and Tevzadze [2008]. In the situation when the generator has superquadratic growth, Delbaen et al. [2010] show that such BSDEs are essentially ill-posed.

A strong requirement present in the articles Kobylanski [2000], Morlais [2009] and Tevzadze [2008] is that the terminal condition as well as the processes which appear in the quadratic growth estimates be bounded. In a Brownian setting Briand and Hu [2006, 2008] have replaced this by the assumption that they need only have exponential moments but in addition, for uniqueness to hold, the driver is convex in the Z variable. More recently, by interpreting the Ψ component as the solution to a specific stochastic control problem, Delbaen et al. [2011] extend these results and show that one can reduce the order of exponential moments required. In addition, in the cited references as well as the recent articles by Frei [2009] and Barrieu and El Karoui [2011], the reader will also find – in various degrees of generality – stability theorems for such BSDEs.

Stability is meant here in the following sense. Suppose we are given a sequence $(F^n)_{n\in\mathbb{N}_0}$ of drivers and a sequence $(\xi^n)_{n\in\mathbb{N}_0}$ of terminal values such that in some fixed space of processes unique solutions $(\Psi^n, Z^n, N^n)_{n\in\mathbb{N}_0}$ to the BSDE (1.1.3) with $F = F^n$ and $\xi = \xi^n$ exist. In addition, under suitable conditions (typically that some specific boundedness or integrability assumptions are uniform in n), then, if

$$F^0 = \lim_{n\to+\infty} F^n \quad \text{and} \quad \xi^0 = \lim_{n\to+\infty} \xi^n$$

hold it is the case that

$$(\Psi^0, Z^0, N^0) = \lim_{n\to+\infty} (\Psi^n, Z^n, N^n)$$

where all these limits are taken with respect to appropriate norms.

The aim of this thesis is now to use the above mentioned link between BSDEs and the portfolio choice problem and to transfer the theorems that are proved within a thorough study of quadratic semimartingale BSDEs under an exponential moments assumption to results on the constrained investment problem under incomplete information. A particular focus is on the stability of the power utility maximization problem via BSDE methods alone. We now introduce this utility maximization problem.

The Portfolio Choice Problem

We assume that there is given a financial market of one bond and d stocks modelled on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, with the major sources of randomness encoded in a continuous d-dimensional local martingale M and a predictable, M-integrable process λ, the market price of risk, such that

$$dS_t = \mathrm{Diag}(S_t)\left(dM_t + d\langle M\rangle_t \lambda_t\right).$$

We write $X^{x,\nu}$ for the wealth process associated with an investment strategy ν when the initial capital is $x > 0$,

$$X^{x,\nu} := x\,\mathcal{E}(\nu \cdot M + \nu \cdot \langle M\rangle \lambda). \tag{1.1.4}$$

In particular, investment strategies represent the *proportion of wealth* invested in the stocks.

We then consider a risk averse agent trading in the above market and assume that their preferences are given by a power utility function U. In particular, their relative risk aversion is constant, they are *CRRA-investors*. We also assume that the agent is a price-taker, i.e. their actions do not affect the stock prices. Their goal is then to maximize the expected utility of terminal wealth. This leads to the following formulation of the primal optimization problem,

$$u(x) := \sup_{\nu \in \mathcal{A}} \mathbb{E}\Big[U\Big(X_T^{x,\nu}\Big)\Big] \tag{1.1.5}$$

where $\mathcal{A}$ denotes some family of admissible investment strategies.

The strategies available to the agent are those which are valued in either a convex cone (Chapter 4) or a compact set (Chapter 5), where we mention that using the notion of "compact" constraints is for ease of formulation. Such sets represent constraints like no short selling. We assume them to be *stochastic* and we can think of the following example where our framework is suited for application. Consider a market in which a regulator bans the short selling of certain stocks as soon as some reference levels are attained. For instance, if stock prices drop dramatically or some volatility index indicates a sharp spike, the regulatory framework might enforce a ban on short selling to discourage aggressive speculators. Similarly, the board of an investment, pension or hedge fund may impose rules on the fund manager's investment possibilities that come into effect when the market exhibits a specified behaviour. Clearly, both the unconstrained and the incomplete case can be included within such a framework. The former corresponds to the case in which the constraints sets equal the whole space $\mathbb{R}^d$. In the latter some of the stocks cannot be traded but rather stand for latent factors; formally, some components of the strategies are prescribed to be zero.

In addition to such *investment constraints*, in Chapter 5, we study the effects of *information constraints*. More precisely, we assume that both the agent and the regulator do not have access to the full information inherent in the economy but only to some restricted information flow modelled by a subfiltration of the main filtration. The most natural assumption in the context of such a partial information framework is that the market participants can observe the evolution of the stock prices or, more generally, of the stock returns. This situation has been studied in Lakner [1998], Pham and Quenez [2001] and Sass [2007] for instance. Although such a setting may serve as the major example we do not assume it in general but follow the ideas of Mania and Santacroce [2010] and Covello and Santacroce [2010], i.e. we consider a filtration which is possibly even smaller than the filtration generated by the stock prices or stock returns. We then extend the reasoning of these articles to the multidimensional as well as constrained case where the mean-variance tradeoff $\langle \lambda \cdot M \rangle$ is unbounded. The latter object is exactly that which enters the quadratic growth estimates of the associated BSDE to which we have alluded above.

The above portfolio selection problem (1.1.5) is classical in mathematical finance and has been extensively studied. It dates back to Merton [1969, 1971]. For general utility functions (not necessarily power) the main solution technique is convex duality, we mention Karatzas et al. [1991], Cvitanić and Karatzas [1992], Kramkov and Schachermayer [1999], Cvitanić et al. [2001], Karatzas and Žitković [2003] as well as the survey article of Schachermayer [2004] which gives an overview of the ideas involved together with many further references to which we refer the reader. The above list of references reflects the development and increasing complexity of the attacked problems, starting with nondegenerate incomplete Itô models (under constraints) and progressing to the semimartingale framework with random endowment, intertemporal consumption and fixed constraints. During this development the so-called dual domain evolved from the local martingale to the supermartingale and eventually to the finitely additive measures. Namely, one associates with the above primal problem (1.1.5) a dual problem involving the convex conjugate $\widetilde{U}$ of the utility function U. The dual problem consists of minimizing, over a suitable domain of, say, *supermartingale measures* for the stocks, the expectation of the $\widetilde{U}$-values of (the densities of) these measures. Here, the notion of supermartingale measures refers to the fact that all the admissible wealth processes are supermartingales under them. One can think of this approach as looking for a pricing measure, i.e. a measure under which the wealth process becomes a martingale (see condition (A) below). In the literature the duality approach is thus also called the "martingale method". The dual problem usually is easier to solve and by the conjugacy relations then yields a solution to the primal problem.

A second approach to tackling the problem (1.1.5) is via BSDEs using the factorization property of the value process when, as an example, the utility function is of power type. This allows one to apply the martingale optimality principle and, as shown in Hu et al. [2005], to describe the value process and optimal trading strategy completely via a BSDE. By the martingale optimality principle we mean the well-known paradigm of stochastic control theory to formulate an optimization problem in such a way that for all admissible strategies some controlled process is a supermartingale while it is a martingale for (only) one admissible strategy. The latter then optimizes the expectation value of the controlled process over the set of admissible strategies. As an alternative, one may attack the problem by pointwise minimization of the drivers of the BSDEs associated to the strategies. This is more in the spirit of the HJB equation in a Markov setting, but is based on the same principles.

From the above references we know that a major difficulty in the present topic consists of showing that a solution to the BSDE (assuming it exists) indeed provides the optimizers of the portfolio choice problem and, when appropriate, of the associated dual problem. In contrast to Nutz [2011], where for verification a condition is given that involves the martingale part of the BSDE solution, we here strive for criteria on the parameters of the model, mainly focussing on the market price of risk λ. Under an exponential moments condition on the mean-variance tradeoff, we study the existence and uniqueness of an optimal solution to problem (1.1.5), together with the solution to the associated dual problem. In Chapter 4, we show that under cone constraints, the dual domain can conveniently be restricted to be the family of supermartingale measures for the stock. We then describe the direct correspondence between the solutions of a quadratic BSDE and the primal and dual optimizers. The main tool for this is an extension of the decomposition of elements in the dual domain given in Karatzas and Žitković [2003] and Larsen and Žitković [2007] to the case of semimartingale dynamics with predictably measurable cone constraints, a contribution to the convex duality literature. We also mention that we derive our verification statement from BSDE comparison principles in the spirit of Hu et al. [2005] and Morlais [2009] but relax the boundedness and nondegeneracy assumptions there, instead building on the theoretical results for quadratic BSDEs proved in Chapter 3.

The Stability of the Portfolio Choice Problem

The most interesting application of the theoretical results on stability of BSDEs that we provide in Chapter 3 is in using the one-to-one correspondence between optimizers and BSDE solutions addressing the following question.

> "Do the components of the solution, such as the optimal wealth and investment strategy, depend continuously on the input parameters, i.e. utility function, asset price dynamics and investment constraints?"

Since we focus specifically on power utility our results are simultaneously more and less general than previous literature. Namely we are fixed within a class of utility functions but allow for a more general market model with continuous semimartingale dynamics. Moreover, using the link with BSDEs we can simultaneously consider continuity with respect to utility function, model dynamics, statistical probability measure and constraints, integrating previous research into one framework. Here, continuity with respect to the utility function is given in terms of the investor's relative risk aversion parameter. The continuity with respect to the model dynamics is formulated for the mean-variance tradeoff process. With regards to the stability in the statistical probability measure we consider convergence of the corresponding densities while for the constraints we rely on the notion of the closed set limit.

These modes of convergence are appropriate for an investigation of the stability of the utility maximization problem. For instance, in the motivating example above the regulator may well be concerned about the effect of having their decision taken under misspecifications of the market dynamics. As a second application no investor knows their risk aversion exactly, similarly the stock dynamics and the probability measure are based on historical estimation and thus it is necessary to investigate the impact of slight misspecifications on the optimal variables. Our results are to the stability of the portfolio choice problem, and we provide a mathematical framework to study this. We derive convergence of the optimal strategies, as well as convergence of the optimal wealth and dual processes in the semimartingale topology, which is stronger than convergence at terminal time in probability. We are thus able to improve the statements in Larsen and Žitković [2007], Larsen [2009] and Kardaras and Žitković [2011].

The BSDE approach to the study of the stability of the utility maximization problem is a new feature, in particular, the current literature in this area typically relies upon duality methods. More precisely, the previous research is divided into two themes, beginning with the article of Jouini and Napp [2004], the first analyzes continuity with respect to the preferences. A sequence $(U^n)_{n\in\mathbb{N}_0}$ of utility functions (not necessarily of power type) converging to $U = U^0$ is considered and the continuity of the corresponding optimizers investigated, for complete Itô-price models in Jouini and Napp [2004] and for incomplete markets with continuous semimartingale dynamics in Larsen [2009]. In the complete case, due to uniform boundedness of the market price of risk process, the authors of Jouini and Napp [2004] prove the L^ϱ, $\varrho \geq 1$, as well as pointwise convergence of the optimal wealth and consumption at each date, whereas in Larsen [2009] this is weakened to convergence in probability of only the optimal terminal wealth. We mention that both articles assume a regularity condition in the sense that all utility functions are dominated by a fixed one. More recently, in Kardaras and Žitković [2011] it is shown that such convergence in probability of the optimal terminal wealth also holds when there are illiquid assets which the investor may add to their portfolio and when the statistical probability measure simultaneously varies, modelled by a sequence of measures $(\mathbb{P}^n)_{n\in\mathbb{N}_0}$ converging in total variation norm. Again, there is a uniform integrability assumption. In fact, building on Larsen and Žitković [2007], one can provide an example which shows that convergence of the optimal wealth in probability may fail otherwise. Finally we mention the work of Nutz [2010a] who looks at risk aversion asymptotics for the power utility function, but also provides results on the continuity with respect to the risk aversion parameter. In addition, in Kramkov and Sîrbu [2006] the reader will find a related sensitivity analysis for the utility indifference prices.

The second theme, beginning with Larsen and Žitković [2007], relates to misspecifications in the model, i.e. the utility function is fixed (again, not necessarily of power type) and the asset price dynamics vary. Typically there is a continuous semimartingale S^λ, modelling the financial market, which is indexed by a market price of risk λ. A sequence λ^n is then chosen, appropriately convergent to some λ, and the convergence of the optimal terminal wealths $\hat{X}_T^{\lambda^n}$ is studied, again in probability. Continuity is shown under a suitable uniform integrability assumption (which is indispensable as an example shows). The results therein have recently been generalized to the conditional value functions and optimal wealth random variables $\hat{X}_\tau^{\lambda^n}$ for a stopping time τ valued in $[0, T]$, we refer to Bayraktar and Kravitz [2010] for further details.

The previous articles consider stability/continuity only in the situation when there are neither investment nor information constraints. As an exception, in the specific case when the utility function is the logarithm, appropriate stability results can be found in a recent article by Kardaras [2010]. The optimizing investment strategy is then called the numéraire portfolio and by using its known explicit formula it is shown to depend continuously on the filtration, probability measure as well as the investment constraints, which are modelled by a sequence of cones.

Summary of the First Chapter

Let us now give a summary of the content of each chapter and provide additional remarks about our contributions to the literature. Following this introduction we fix the framework and the notation which will be used throughout. We give an overview of the general concepts of utility maximization which we consider to be for the unconstrained problem in a first step. More precisely, along with the primal problem (1.1.5), we also introduce the dual optimization problem. From Kramkov and Schachermayer [1999] we then recall that, under suitable no arbitrage and nondegeneracy assumptions, there is a unique primal optimizer $\hat{X}$ together with a unique dual optimizer $\hat{Y}$. In particular, we show that these optimizers exist under the condition that the mean-variance tradeoff $\langle\lambda \cdot M\rangle_T$ have all exponential moments which is the *major* structural assumption within the entire thesis. As a matter of fact, finite exponential moments of some specific order are sufficient as we show in Chapter 5. This then improves the results from the literature which are for a bounded mean-variance tradeoff, see Hu et al. [2005] and Morlais [2009].

Two properties of the optimal pair $(\hat{X}, \hat{Y})$ turn out to be essential. Given that they satisfy the initial condition $\hat{Y}_0 = u'(\hat{X}_0)$, where u is the value function from (1.1.5), we have that

(A) the process $\hat{X}\hat{Y}$ is a martingale and

(B) it holds that $\hat{Y}_T = U'(\hat{X}_T)$.

In view of the relation (B) it is then natural to define

$$\hat{\Psi} := \log\left(\frac{\hat{Y}}{U'(\hat{X})}\right)$$

and to investigate the dynamics of this process which in this case are accompanied by the terminal condition $\hat{\Psi}_T = 0$. We are thus led to considering a BSDE for the process $\hat{\Psi}$, which turns out to be the (logarithmic transform of the) dynamic value process for the problem (1.1.5) and which is called *opportunity process* in Nutz [2010b]. We then obtain that

$$\begin{aligned} d\hat{\Psi}_t = \hat{Z}_t^\intercal\, dM_t + d\hat{N}_t - \frac{1}{2}\, d\langle\hat{N}\rangle_t \\ + \frac{q}{2}(Z_t + \lambda_t)^\intercal\, d\langle M\rangle_t(Z_t + \lambda_t) - \frac{1}{2}\, Z_t^\intercal\, d\langle M\rangle_t Z_t, \quad \hat{\Psi}_T = 0. \quad (1.1.6) \end{aligned}$$

The associated control process $\hat{Z}$ is defined in terms of the optimal strategy $\hat{\nu}$ and the market price of risk λ, i.e.

$$\hat{Z} = -\lambda + (1-p)\hat{\nu}$$

while $\hat{N}$ stems from representing $\hat{Y}$ as a stochastic exponential and is assumed continuous (which would hold in the case of a continuous filtration, for instance),

$$\hat{Y} = \hat{Y}_0\, \mathcal{E}(-\lambda \cdot M + \hat{N}).$$

In the equation (1.1.6) q stands for the dual number to $p \in (-\infty, 1)$, i.e. $q := \frac{p}{p-1}$, where $1-p$ is the investor's relative risk aversion. We observe that the BSDE is quadratic in the control variable Z and in the subsequent analysis we verify that the driver of (1.1.6) is convex in Z, which is due to our sign conventions and the fact that $q \in (-\infty, 1)$.

The above item (A) provides an argument for verification. Given a solution (Ψ, Z, N) to the BSDE in question and defining a wealth process X and a dual variable Y as suggested by the preceding remarks we find that (X, Y) is the pair of primal and dual optimizers if condition (A) is satisfied by $XY = e^{\Psi_0} x^p \, \mathcal{E}([(1-q)Z - q\lambda] \cdot M + N)$, see Nutz [2011] for a recent treatment of this martingale principle. The uniqueness of the optimal pair then may be used to derive uniqueness of solutions to the BSDE (1.1.6) for which condition (A) is satisfied. However, this condition involves the martingale part from a solution triple and hence may be difficult to check, given that the martingale part is derived from the martingale representation theorem and in general is only implicit. For verification we thus follow another idea. Namely, in Chapter 4, where the more general framework of the cone constrained problem is considered, we deduce that $\hat{\Psi}$ is contained in a specific space that we call $\mathfrak{E}$. It consists of all processes whose supremum has finite exponential moments of all orders. The proof of this fact relies crucially on us being able to choose the dual domain as a set of supermartingale measures. Indeed, we show that in the cone constrained setting of Chapter 4 this is feasible, so that we can conclude the verification argument from the theoretical results in Chapter 3. These results establish that uniqueness in the space $\mathfrak{E}$ holds for quadratic BSDEs with convex generators. The reader is referred to Table 1.1 in Section 1.3 for an overview of this discussion.

Summary of the Second Chapter

In Chapter 2 we then study the quadratic semimartingale BSDE (1.1.6) of power utility maximization in more detail, in particular when the market price of risk is of BMO type. We know that a solution to the BSDE provides candidates for the optimizers and that verification is the difficult part. It typically requires extra regularity of the BSDE solution which is guaranteed by the boundedness of the mean-variance tradeoff or, as we show in Chapter 3 in combination with Chapters 4 and 5, by the existence of specific exponential moments of the mean-variance tradeoff. In particular, when one can show the existence of a bounded solution, verification is feasible, see below for further discussion on this point. Motivated by the ease of verification given a bounded solution the main aim of Chapter 2 is to quantify in terms of assumptions on the mean-variance tradeoff process $\langle \lambda \cdot M \rangle$ when one can expect such a bounded solution. This natural question justifies the study.

The assumption that the mean-variance tradeoff process be bounded or have all exponential moments implies that the minimal martingale measure with density process $\mathcal{E}(-\lambda \cdot M)$ is a true probability measure, where $\mathcal{E}$ denotes the stochastic exponential. In particular, the set of equivalent martingale measures is nonempty, so that there is no arbitrage in the sense of NFLVR (*no free lunch with vanishing risk*), see Delbaen and Schachermayer [1998]. If the local martingale $\lambda \cdot M$ is instead assumed to be a BMO martingale then from Kazamaki [1994] Theorem 2.3 the minimal martingale measure is again a true probability measure and NFLVR holds. The acronym BMO stands for *bounded mean oscillation* and we refer to the monograph by Kazamaki [1994] for a major account of the theory involved. In the above case of a BMO martingale $\lambda \cdot M$ the mean-variance tradeoff now need not be bounded or have all exponential moments. A secondary objective of Chapter 2 is to study what happens to the solution of the BSDE in this situation. As discussed, such a condition on $\lambda \cdot M$ arises naturally from a no arbitrage point of view, additionally however there is a relation between boundedness of solutions to quadratic BSDEs and BMO martingales, see Mania and Schweizer [2005]. Under appropriate assumptions, this classical result states that when the generator of a BSDE has quadratic growth in the variable Z then for a solution triple (Ψ, Z, N) the process Ψ is bounded if and only if the martingale part $Z \cdot M + N$ is a BMO martingale, i.e. if and only if

$$\sup_{\tau} \left\| \mathbb{E}\left[\left(\int_{\tau}^{T} Z_s \, dM_s + N_T - N_\tau \right)^2 \middle| \mathcal{F}_\tau \right]^{1/2} \right\|_{L^\infty} < +\infty, \tag{1.1.7}$$

where the supremum is over all stopping times τ valued in $[0, T]$. In the setting where Ψ is assumed to satisfy an exponential moments condition only, such a correspondence is lost. However, we provide examples to show that whilst a solution triple $(\hat{\Psi}, \hat{Z}, \hat{N})$ to the BSDE (1.1.6) indeed provides the optimizers of the utility maximization problem, the respective martingale part $\hat{Z} \cdot M + \hat{N}$ need not be a BMO martingale, equivalently, the process $\hat{\Psi}$ need not be bounded. In short, there are BMO martingales $\lambda \cdot W$, where W denotes a one-dimensional Brownian motion, such that $\mathcal{E}\big([(1-q)\hat{Z} - q\lambda] \cdot W\big)$ is a martingale while $\hat{Z} \cdot W$ is not a BMO martingale. The question under which assumptions $Z \cdot M + N$ from a BSDE solution triple actually induces a measure change is thus also interesting from a mathematical standpoint and a further treatment is provided in Chapter 3.

The contributions of Chapter 2 are then as follows. Firstly in a Brownian setting, we provide a necessary and sufficient condition, related to the finiteness of the dual problem, which guarantees the existence of a unique solution to the BSDE (1.1.6) satisfying (an analogue to) the martingale condition in item (A) above. Here, we give the first component Ψ of a solution pair in explicit terms. The statement of this result is motivated by its applicability in a number of specific situations which arise in the subsequent analysis. Moreover, we construct an explicit example for which the BSDE fails to have a solution when the dual optimization problem is degenerate. This example serves us as the major building block in our further study of counterexamples to boundedness of BSDE solutions. Secondly we show that the BSDE always admits a continuum of distinct solutions with square-integrable martingale parts. We deduce this result from a lemma in which we show that, contrary to the classical Itô representation formula with square-integrable integrands, an analogous L^2-representation of random variables in terms of stochastic exponentials is not unique. More explicitly, in the Brownian framework, let ξ be a positive random variable bounded away from zero and infinity. From the classical martingale representation theorem we know that there is a constant k together with a predictable process β such that

$$\xi = k + (\beta \cdot W)_T \quad \text{with} \quad \mathbb{E}\left[\int_0^T |\beta_t|^2\,dt\right] < +\infty$$

and that the pair (k, β) is unique among all pairs with these properties. Let us now strive for an analogous "multiplicative" decomposition,

$$\xi = c\,\mathcal{E}(\alpha \cdot W)_T \quad \text{where} \quad \mathbb{E}\left[\int_0^T |\alpha_t|^2\,dt\right] < +\infty.$$

We prove that in order to deduce uniqueness of the pair (c, α) one has to add an extra assumption. For instance, uniqueness of the pair (c, α) holds if additionally $c = \mathbb{E}[\xi]$. Equivalently, one may also assume that additionally $\alpha \cdot W$ is a BMO martingale. Indeed, we prove that for *every* number $c \geq \mathbb{E}[\xi]$ there is a predictable square-integrable process α^c such that the above multiplicative decomposition holds. This shows that there is a continuum of distinct multiplicative L^2-decompositions. The intuition is that $\mathcal{E}(\alpha^c \cdot W)$ need not be a martingale so that increasing c may be offset by an appropriate choice of α^c. A consequence of this result is that it immediately carries over to nonuniqueness of solutions to quadratic BSDEs with *square-integrable* martingale parts. This is because the standard method of finding solutions to such equations involves an exponential transform. We point out that the type of nonuniqueness is nontrivial in the sense that the martingale parts are always square-integrable, in contrast to classical counterexamples that are only locally integrable. Indeed, it is well known that without square-integrability even the standard Itô decomposition is not unique. In fact, for every $k \in \mathbb{R}$ there exists a predictable process β^k such that

$$\xi = k + (\beta^k \cdot W)_T \quad \text{with} \quad \int_0^T |\beta_t^k|^2\,dt < +\infty \quad \mathbb{P}\text{-a.s.}$$

as we know from Émery et al. [1983] Proposition 1. We mention that the reader will find an example of a specific BSDE for which two distinct square-integrable solutions are constructed in Ankirchner et al. [2009] Section 2.2. However, we show that there is actually always a continuum of distinct square-integrable solutions for the class of quadratic BSDEs related to the power utility maximization problem. For a continuum of distinct solutions to the BSDE clearly only one of them can correspond to the optimal pair $(\hat{X}, \hat{Y})$.

We then proceed with a thorough investigation of when the BSDE admits a bounded solution. If the investor's relative risk aversion is greater than one and $\lambda \cdot M$ is a BMO martingale, we demonstrate that this is automatically satisfied. This relies on the fact that from Kazamaki [1994] we know that $\mathcal{E}(-\lambda \cdot W)$ defines an equivalent local martingale measure and that it satisfies a suitable reverse Hölder inequality. Here, we use that risk aversion greater than one corresponds to the case $q \in [0, 1)$, which is equivalent to $p < 0$. The reverse Hölder inequality is then known to be related to the boundedness of the first component of the optimal BSDE solution triple $(\hat{\Psi}, \hat{Z}, \hat{N})$. For a risk aversion smaller than one the picture is rather different and we provide an example to show that even when the mean-variance tradeoff has all exponential moments and the process $\lambda \cdot M$ is a BMO martingale, actually, even if it is a bounded martingale, then the solution to the utility maximization BSDE need not be bounded. We develop this example in several steps by using the major building block that we have alluded to above. The major difference is that we now do not aim at degeneracy, i.e. at $\Psi_0 \equiv +\infty$, but at unboundedness of the real-valued process Ψ. We hence add to the construction an $\mathcal{F}_{T/2}$-measurable random variable that diffuses on its image space $(0,1]$ so that $\Psi_{T/2}$, which involves the logarithm of this random variable, is unbounded. We then modify this construction by choosing a suitable stopping time which additionally ensures the finiteness of all exponential moments. We mention that we also provide explicit examples which are Markovian in M by referring to Azéma-Yor martingales.

Building on these examples our most important result is Theorem 2.5.10, which shows how to combine the BMO and exponential moment conditions so as to find a new minimal condition which guarantees, in a general filtration, that the BSDE admits a bounded solution. We point out that the result is for a situation in which $\hat{N}$ may exhibit jumps. Indeed, all the results which depend only on the specific continuous local martingale M also hold in this more general setting. The BSDE (1.1.6) is then replaced by

$$\begin{aligned} d\Psi_t = Z_t^\intercal \, dM_t + dN_t - \frac{1}{2}\, d\langle N^c\rangle_t + \log(1+\Delta N_t) - \Delta N_t \\ + \frac{q}{2}(Z_t+\lambda_t)^\intercal \, d\langle M\rangle_t (Z_t+\lambda_t) - \frac{1}{2}\, Z_t^\intercal \, d\langle M\rangle_t Z_t, \quad \Psi_T = 0, \quad (1.1.8) \end{aligned}$$

where N^c denotes the continuous part of N and ΔN its jump part. We note that in our setting $\hat{Y} > 0$, hence ΔN can be assumed to satisfy $\Delta N > -1$. In a first step we use the John-Nirenberg inequality to find a sufficient condition on the BMO_2 norm of $\lambda \cdot M$ which guarantees boundedness of the corresponding BSDE solution. This is done by choosing the sharpest possible Hölder type estimate and it gives us a critical value, denoted by k_q, for which subsequent analysis shows that this value cannot be improved. However, the feasibility of a uniform characterization of the boundedness property of solutions to the BSDE (1.1.8) in terms of the BMO_2 norm of $\lambda \cdot M$ is limited, see the discussion in Subsection 2.5.1. More precisely, the *property* of being bounded holds uniformly in $p \in (0,1)$ and λ, where $\lambda \cdot M$ is a BMO martingale, only if p is restricted to an interval truncated at 1 and the BMO_2 norm of $\lambda \cdot M$ is small enough.

As a consequence we investigate the implications of the finiteness of *dynamic* exponential moments of $\langle \lambda \cdot M\rangle$ of a specific order. After proving that their finiteness may be given in terms of a condition on a critical exponent b of $\lambda \cdot M$, we then fully characterize the boundedness of solutions to the quadratic BSDE arising in power utility maximization. The corresponding theorem also involves the construction of a market price of risk λ such that two conditions hold simultaneously, the first concerns the finiteness of exponential moments, i.e. the critical exponent $b(\lambda \cdot M)$, and the second relates to the (un)boundedness of the respective BSDE solution. Here, a major task consists of balancing these two conditions. More specifically, we show that $b(\lambda \cdot M) > k_q$ is a sufficient, but not a necessary, condition for the existence of a (then unique) BSDE solution (Ψ, Z, N) with Ψ bounded. However, the condition $b(\lambda \cdot M) > k$ where $k < k_q$ is not sufficient. As a result the value k_q cannot be improved in the sense that k_q *cannot* be chosen to be a smaller constant, which in turn would correspond to relaxing the requirements on the dynamic exponential moments of $\langle \lambda \cdot M \rangle$. Finally, we mention that the limiting case of risk aversion equal to one, i.e. the case of logarithmic utility, is covered by all these results.

Summary of the Third Chapter

Having studied when the BSDE that arises in power utility maximization has a bounded solution, we turn our attention to the unbounded case. More explicitly, in Chapter 3, we provide all the theoretical background material which is needed for the study of general continuous quadratic semimartingale BSDEs under an exponential moments condition. Having such results in greater generality increases the range of applications for BSDEs with the major practical application being the utility maximization problem with an unbounded mean-variance tradeoff.

Building on the theorems of Briand and Hu [2008] and Morlais [2009] we hence provide existence, uniqueness and stability results for those BSDEs whose drivers are Lipschitz continuous in Ψ, locally Lipschitz continuous, quadratic and convex in Z and where the processes that appear in the quadratic growth estimates as well as the terminal condition have finite exponential moments of some specific order. Since in this case we do not necessarily dispose of the more convenient boundedness properties, the method utilized in Morlais [2009] for the derivation of an a priori estimate cannot be directly applied within our setting. We thus have to include an additional assumption on the driver. Alternatively, we need to encode the quadratic variation $\langle M \rangle$ of M in a suitable way, see Section 3.3 for more details.

As usual, the derivation of an a priori estimate is key to deducing an existence result. The proof of existence relies on a double truncation procedure to not only obtain bounded solutions to the truncated BSDEs but to also apply a monotone stability result from Morlais [2009] for which growth estimates with *uniformly* bounded majorants are required. We also mention that the continuity of the filtration (in the sense that all local martingales are assumed to be continuous) is used directly for this existence result only.

We then move on to showing that if the drivers are convex in Z then solutions to the BSDEs under consideration with first component in the space $\mathfrak{E}$ are unique. The proof makes use of the so-called θ-technique and has to take care of the orthogonal martingale part N. Finally, under a uniform exponential moments assumption, we obtain that a stability theorem for the studied BSDEs holds as well. In contrast to the previous existence result, in its proof *all* exponential moments are required to be finite.

As a byproduct of establishing our results we are able to show via an example that the stability theorem as stated in Briand and Hu [2008] Proposition 7 needs a minor amendment to the mode of convergence assumed on the drivers and we include the appropriate formulation. In fact, unlike Frei [2009] Theorem 2.1 and Morlais [2009] Lemma 3.3 our growth estimates need not be uniform. Additionally, we do not assume that the respective majorants are uniformly bounded. As a consequence, in contrast to the setting of Frei [2009] Theorem 2.1, pointwise convergence of the generators is not sufficient for our stability result to hold.

Another main contribution of Chapter 3 is to address the question of measure change. We know that when the generator has quadratic growth in the control variable Z then the solution processes Ψ is bounded if and only if the corresponding martingale part $Z\cdot M+N$ is a BMO martingale. In the unbounded setting we do not dispose of this correspondence anymore. However, under the assumptions of Chapter 3, we are able to show that even if $Z\cdot M+N$ may not be a BMO martingale, the stochastic exponential $\mathcal{E}\big(\varrho(Z\cdot M+N)\big)$ is a true martingale for $|\varrho|$ sufficiently large. For the prototype of a quadratic BSDE above this is true for $|\varrho|>1/2$, in particular for $\varrho=1$. For the utility maximization BSDE (1.1.6) the relation $\varrho=1$ can be achieved by using a suitable generalized Young inequality, see Chapter 5 for the technical details involved. This result can then be used to perform a verification argument as suggested by condition (A) above. Hence the results of Hu et al. [2005], Morlais [2009] and Covello and Santacroce [2010] can all be extended beyond the case of a bounded mean-variance tradeoff, see also Heyne [2010] for a number of stochastic volatility models. Moreover such a theorem may be used in the partial equilibrium framework of Horst et al. [2010] where the market price of external risk present there is given by equilibrium considerations and is typically unbounded.

Summary of the Fourth Chapter

In the final two chapters we then apply all these results to the constrained utility maximization problem with a focus on its stability. In Chapter 4 we assume that the agent's strategies must take values in a predictably measurable closed and convex cone. Under an exponential moments condition on the mean-variance tradeoff, the existence of an optimal solution to the above problem (1.1.5) together with the solution to the associated dual problem is then obtained by arguments from convex duality. Here our line of reasoning relies on the fact that the dual domain can be defined as the family of supermartingale measures for the stocks. Contrary to the usual procedure of proving the existence of a solution to the portfolio choice problem via the dual problem we here consider the primal problem first and the dual problem second. In particular, we must guarantee that the dual optimizer obtained does not exhibit singular parts as in Westray [2009] from which we borrow a major part of the argument for showing the existence of the primal and dual optimizers. Indeed, we know from Cvitanić et al. [2001] that such singular parts operate on random endowments, which are not present here. We remark that an additional assumption on the cones is imperative, namely that they be polyhedral. Under such an assumption the family of wealth processes is closed with respect to the semimartingale topology, see Czichowsky et al. [2011]. We also note that previous literature, where a fixed constraints set is considered within an additive framework, see Karatzas and Žitković [2003], is covered by our approach. Here, "additive framework" is meant in the sense that the investment strategies, denoted by H, are defined to represent the amount of shares held in the portfolio so that wealth is written in additive format, $X=x+H\cdot S$. We refer to our formulation via equation (1.1.4) as the "multiplicative formulation". Such a formulation simplifies and generalizes to the constrained semimartingale case the proof of the decomposition of elements of the dual domain when compared to similar results from Karatzas and Žitković [2003] and Larsen and Žitković [2007]. We mention that the latter are in line with earlier results which are implicitly present in Cvitanić and Karatzas [1992] and Rouge and El Karoui [2000], for instance. The assumption that the cones be polyhedral remains crucial.

We then state the direct correspondence between solutions $(\hat{\Psi}, \hat{Z}, \hat{N})$ to the quadratic utility maximization BSDE and the primal and dual optimizers $(\hat{X}, \hat{Y})$ building on this decomposition. As a consequence we augment the results from previous literature by providing a simple decomposition of the optimal control process $\hat{Z}$ into a part with well defined properties related to the dual optimizer and the polar cone of the constraint set and another part related to the optimal strategy. In addition, in contrast to Chapter 5 and the results from Hu et al. [2005] and Morlais [2009], we use Moreau's decomposition theorem to avoid measurable selection arguments. In fact, nearest point projections onto closed and convex sets are uniquely defined (and continuous). We mention that the corresponding BSDE now contains the distance function in its driver which reflects the fact that there are trading constraints that have to be taken into account. Relying on the special choice of the dual domain we are able to define the so-called dual opportunity process and use its dynamic optimality properties. The dual opportunity process appears in Nutz [2010b] with the exception that the crucial properties needed here may not hold for the specific dual domain considered there in its full generality, but they *do* hold for our choice. As a result $\hat{\Psi}$ is contained in $\mathfrak{E}$ and therefore unique.

The main task is then to prove that the optimal wealth, strategy and dual variable all depend continuously on the input parameters of risk aversion, market price of risk, probability measure and constraints. We show that this convergence takes place in the semimartingale topology, hence directly on the level of processes as opposed to convergence in probability at terminal time. This extends the results in Larsen and Žitković [2007], Larsen [2009] and Kardaras and Žitković [2011]. A notable feature of our approach is that we rely on BSDE techniques rather than duality theory, which – given the main idea of integrating all the considered variations into one BSDE framework – is new in the literature in this area. The main difficulty consists of providing a unified treatment for the simultaneous investigation of all the variations in the input parameters. In particular, we have to encode these variations within some BSDE whose driving local martingale is the fixed M so that we can apply the stability result from Chapter 3. Via suitable transformations this BSDE then gives us the solution triples $(\hat{\Psi}^n, \hat{Z}^n, \hat{N}^n)$ of the portfolio choice problem when the market price of risk, the agent's relative risk aversion parameter, the statistical probability measure and the constraints sets vary, parameterized by $n \in \mathbb{N}_0$. From the stability result for these triples we then extract the stability (in n) of the corresponding optimal pairs $(\hat{X}^n, \hat{Y}^n)$.

Finally we give an example which shows that our conditions on the set convergence are well motivated. Namely, we have to take into consideration the so-called null-investments, i.e. consider set convergence modulo those strategies that do not contribute to the investor's wealth.

Summary of the Fifth Chapter

In Chapter 5 the main results consist of deriving the existence and uniqueness of solutions to the constrained utility maximization problem – now additionally taking into account a situation of restricted information – and relating them to the appropriate BSDE under weaker conditions than those present in the literature. Following the ideas provided in Mania and Santacroce [2010], see also Sass [2007], we transform the utility maximization problem under partial information into a related stochastic control problem which can be interpreted as a problem under full information. Unlike these two references we include trading constraints which are compact. Here, we write "compact" to ease the formulation and refer to Chapter 5 for the precise conditions. The inclusion of constraints is motivated by the riskiness of the optimal strategies which appear under limited information and which typically consist of large long or short positions, see Sass [2007] for further discussion.

We point out that in this chapter we rely purely on BSDE techniques. In particular, the existence of the solution to the portfolio selection problem is guaranteed by solving a specific BSDE which we identify to be the utility maximization BSDE in the context of incomplete information. The main difference is that now this BSDE is not for the local martingale M but for M^o, the optional projection of M onto the smaller filtration which encodes the investor's flow of partial information. We also mention that we extend the cited literature to the multidimensional framework. In doing so we overcome technical difficulties related to the covariations of M and M^o. More specifically, in the utility maximization BSDE both quadratic variations $\langle M\rangle$ and $\langle M^o\rangle$ appear although, as already stated, the BSDE is formulated with driving local martingale M^o. We mention that in Chapter 5 we assume that all local martingales (with respect to the smaller filtration) are continuous.

As an extension of the previous setting we also assume that at terminal time T there applies an additional *discount* D on the investor's accrued wealth. In contrast to Nutz [2010b], the variable D need not be bounded, but only satisfy an appropriate finite moments condition. One may think of D as arising from a stochastic tax rate or bonus. Alternatively, we can interpret D as triggering a measure change so that the optimization is under the investor's subjective beliefs determined by D.

As already explained above, when performing the verification argument we must show that some stochastic exponential of a process involving the martingale part of a BSDE solution defines a true probability measure. By means of the generalized Young inequality we hence consider growth estimates that are parameterized by $\varepsilon > 0$ and then fix a specific $\varepsilon = \varepsilon^*$ in such a way that a solution to the BSDE exists and that the corresponding martingale part indeed induces a true measure change. The martingale optimality argument then enables us to conclude that specific BSDE solutions indeed provide the optimizers of the incomplete information and constrained investment problem.

As an application of the stability result for quadratic BSDEs we finally show that the respective optimizers depend continuously on the investor's attitude towards risk, the market price of risk process, the constraints sets and the discount or measure change, similarly to the procedure in Chapter 4. In conclusion, our contribution lies in generalizing the approaches of the cited references in several directions as well as providing a common framework for the treatment of the various features that may be investigated within a portfolio choice problem. These comprise investment and information constraints, the effects of perturbations and – more importantly from a calibration perspective – the impact of misspecifications of the model parameters.

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1.2 Framework and Model Formulation

Throughout the entire thesis we work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ satisfying the usual conditions of right-continuity and completeness. We assume that the time horizon T is a finite number in $(0, \infty)$ and that $\mathcal{F}_0$ is the completion of the trivial σ-algebra. All semimartingales are assumed to be equal to their càdlàg modification. Later in the Chapters 3 and 5, in order to apply the techniques of BSDE theory, we will need a stronger assumption, referred to in the literature as *continuity of the filtration.*

Assumption 1.2.1. *All local martingales are continuous.*

For instance, if $(\mathcal{F}_t)_{0\leq t\leq T}$ is the augmentation of the natural filtration generated by a Brownian motion, this assumption is satisfied. Hence, in the following chapters, the Brownian setting will serve as a major framework for the construction of some specific (counter-)examples. More generally, $(\mathcal{F}_t)_{0\leq t\leq T}$ could be the augmented filtration generated by a continuous local martingale that allows for the representation property from Jacod and Shiryaev [2003] III.4c to hold.

The Market Model

We assume that on the given stochastic basis there is modelled a financial market consisting of one bond, assumed constant, and d stocks with (discounted) price process

$S = (S^1, \dots, S^d)^\intercal$, a d-dimensional continuous semimartingale, where we write $^\intercal$ for transposition. More precisely, our semimartingale S is assumed to have dynamics

$$dS_t = \mathrm{Diag}(S_t)\left(dM_t + d\langle M\rangle_t\lambda_t\right),$$

where $M = (M^1, \dots, M^d)^\intercal$ is a d-dimensional continuous local martingale with $M_0 = 0$, λ is a d-dimensional predictable process, the *market price of risk*, satisfying

$$\mathbb{P}\left(\int_0^T \lambda_t^\intercal\, d\langle M\rangle_t\lambda_t < +\infty\right) = 1$$

and $\mathrm{Diag}(S)$ denotes the $d \times d$ diagonal matrix whose diagonal elements are taken from S. Observe that $\cdot$ denotes stochastic integration and that we write $\langle M\rangle = \langle M, M\rangle$ for the quadratic (co-)variation matrix of M.

It is a consequence of Delbaen and Schachermayer [1995] Theorem 3.5 that any continuous, arbitrage free, numéraire denominated model of a market is of the above form so there is no loss of generality in the above framework in view of Assumption 1.2.3 below.

To precisely describe our model we need some further results on $\langle M\rangle$. We may use Jacod and Shiryaev [2003] Proposition II.2.9 and II.2.29 to write

$$\langle M\rangle = C \cdot A \tag{1.2.1}$$

where C is a predictable process valued in the space of symmetric positive semidefinite $d \times d$ matrices and A is a predictable increasing process. It is known that there are many such factorizations, see Jacod and Shiryaev [2003] Section III.4a. We can choose $A := \arctan\left(\sum_{i=1}^d \langle M^i\rangle\right)$ and then, following an application of the Kunita-Watanabe inequality, we may derive the absolute continuity of each $\langle M^i, M^j\rangle$ with respect to A to get C. From Karatzas and Shreve [1991] Theorem 3.4.2 it is known that there exist Borel measurable functions which diagonalize a symmetric positive semidefinite $d \times d$ matrix, in particular we deduce the existence of some processes P and Γ valued in the space of $d \times d$ orthogonal (resp. diagonal) matrices such that

$$\langle M\rangle = C \cdot A = P^\intercal \Gamma P \cdot A = B^\intercal B \cdot A, \tag{1.2.2}$$

where we set $B := \Gamma^{\frac{1}{2}} P$. The matrix Γ has nonnegative entries only, with the eigenvalues of C on its diagonal. We also point out that our results do not depend on the particular choice of A, but only on its *boundedness*. In particular, if $M = W$ is a d-dimensional Brownian motion we may choose $A_t = t$, $t \in [0, T]$, and B the identity matrix. The above processes A, B, C, P and Γ will be fixed throughout.

We let $\mathcal{P}$ denote the predictable σ-algebra on $[0, T] \times \Omega$, generated by all the left-continuous adapted processes. The process A induces a measure μ^A on $\mathcal{P}$, the *Doléans* *measure*, defined for $E \in \mathcal{P}$ by

$$\mu^A(E) := \mathbb{E}\left[\int_0^T \mathbf{1}_E(t)\, dA_t\right]. \tag{1.2.3}$$

In what follows, we use the abbreviation Υ for a process $(\Upsilon_t)_{0\le t\le T}$ and write "for all t" meaning "for all $t \in [0,T]$". A local martingale N is called *orthogonal* to M if $\langle M^i, N^c\rangle \equiv 0$ for all $i = 1, \dots, d$ where N^c denotes the *continuous part* of N. We refer the reader to Jacod and Shiryaev [2003] and Protter [2005] for unexplained terminology and background material.

The Portfolio Selection Problem

We consider an investor trading in the above market according to an admissible investment strategy ν that they choose. Here, a predictable d-dimensional process ν is called an *admissible* trading or investment strategy if it is M-integrable, i.e. $\int_0^T \nu_t^\intercal\, d\langle M\rangle_t \nu_t < +\infty$, $\mathbb{P}$-a.s. We write $\mathcal{A}$ for the family of such investment strategies ν and define the components ν^i to represent the *proportion of wealth* invested in each stock S^i, $i = 1, \dots, d$. In particular, for some initial capital $x > 0$ and an admissible strategy ν, the associated *wealth process* $X^{x,\nu}$ evolves as follows:

$$X^{x,\nu} := x\,\mathcal{E}(\nu \cdot M + \nu \cdot \langle M\rangle \lambda), \tag{1.2.4}$$

where $\mathcal{E}$ denotes the stochastic exponential. The family of all such wealth processes is denoted by $\mathcal{X}(x)$.

Our agent has preferences modelled by a utility function U, which is throughout assumed to be of power type,

$$U(x) = \frac{x^p}{p}, \quad \text{for } p \in (-\infty, 0) \cup (0,1).$$

We also include the case $p = 0$, in which the utility function becomes logarithmic,

$$U(x) = \log(x).$$

Starting with initial capital $x > 0$, they choose admissible strategies ν and aim to maximize the expected utility of terminal wealth. This leads to the following formulation of the primal optimization problem,

$$u(x) := \sup_{\nu\in\mathcal{A}} \mathbb{E}\Big[U\Big(X_T^{x,\nu}\Big)\Big]. \tag{1.2.5}$$

Remark 1.2.2. A key property arising under power and logarithmic utility and to be used in the sequel is the factorization property of the value process, more precisely we may write

$$u(x) = \begin{cases} x^p \sup_{\nu \in \mathcal{A}} \mathbb{E}\Big[U\big(X_T^{1,\nu}\big)\Big] & \text{if } p \neq 0, \\ \log(x) + \sup_{\nu \in \mathcal{A}} \mathbb{E}\Big[U\big(X_T^{1,\nu}\big)\Big] & \text{if } p = 0. \end{cases}$$
$$= \begin{cases} U(x)\, c_p & \text{if } p \neq 0, \\ U(x) + c_0 & \text{if } p = 0. \end{cases}$$

for some constants c_p, $p \in (-\infty, 1)$, to be identified below under suitable assumptions. A well known corollary of this is that the optimal investment strategy $\hat{\nu}$, when it exists, is independent of x and the primal optimizer $\hat{X} = \hat{X}^{x,\hat{\nu}}$ has a simple linear dependence on x.

Utility Maximization and Duality Theory

Related to the above primal problem is a dual problem which we now describe. For $y > 0$ we introduce the set of adapted càdlàg processes

$$\mathcal{Y}(y) := \{Y \geq 0 \,|\, Y_0 = y \text{ and } XY \text{ is a supermartingale for all } X \in \mathcal{X}(1)\},$$

as well as the minimization problem

$$\widetilde{u}(y) := \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}\Big[\widetilde{U}(Y_T)\Big]\,, \tag{1.2.6}$$

where $\widetilde{U}$ is the *conjugate* (or *dual*) of U given by

$$\widetilde{U}(y) = \sup_{x>0} \Big\{U(x) - xy\Big\}, \quad y > 0.$$

In the present setting there is an explicit formula for $\widetilde{U}$,

$$\widetilde{U}(y) = \begin{cases} \Big(\frac{1}{p} - 1\Big)\, y^q = -\frac{y^q}{q} & \text{if } p \neq 0, \\ -\log(y) - 1 & \text{if } p = 0, \end{cases}$$

where here and throughout q is the *dual exponent* to p,

$$q := \frac{p}{p-1}.$$

There is a bijection between p and q so that in what follows we often state the results for q rather than for p. Note that the set $\mathcal{Y}(y)$ has the following factorization property,

$\mathcal{Y}(y) = y\mathcal{Y}(1)$. Similarly to u we then see the factorization property for $\widetilde{u}$,

$$\widetilde{u}(y) = \inf_{Y \in \mathcal{Y}(1)} \mathbb{E}\Big[\widetilde{U}(yY_T)\Big] = \begin{cases} y^q \inf_{Y\in\mathcal{Y}(1)} \mathbb{E}\Big[\widetilde{U}(Y_T)\Big] & \text{if } p \neq 0, \\ -\log(y) + \inf_{Y\in\mathcal{Y}(1)} \mathbb{E}\Big[\widetilde{U}(Y_T)\Big] & \text{if } p = 0, \end{cases}$$
$$= \begin{cases} \widetilde{U}(y)\, \widetilde{c}_p & \text{if } p \neq 0, \\ \widetilde{U}(y) + \widetilde{c}_0 & \text{if } p = 0. \end{cases}$$

The relationship between $\widetilde{c}_p$ and c_p is provided in Theorem 1.2.4.

It is shown in Kramkov and Schachermayer [1999, 2003] (among others) that for general utility functions (not necessarily power) the following assumption is the weakest possible for well posedness of the market model and the utility maximization problem.

Assumption 1.2.3.

(i) The set $\mathcal{M}^e(S)$ of equivalent local martingale measures for S is non-empty.

(ii) If $p \geq 0$, there is an $x > 0$ such that $u(x) < +\infty$.

When $p \geq 0$, we then derive that $u(x) < +\infty$ for all $x > 0$, because the initial condition factors. Conversely, if $p < 0$, $u(x) \leq 0 < +\infty$ automatically holds for all $x > 0$.

The Main Results in the Unconstrained Case

Summarizing the results of Kramkov and Schachermayer [1999] we then have the following theorem where the additional claim in item (iii) follows from a calculation using the conjugacy of u and $\widetilde{u}$.

Theorem 1.2.4 (Kramkov and Schachermayer [1999] Theorem 2.2)**.** *Suppose Assumption 1.2.3 holds, then*

(i) There exists a strategy $\hat{\nu} \in \mathcal{A}$ which is optimal for the primal problem. That is, given $x > 0$,

$$u(x) = \mathbb{E}\big[U(\hat{X}_T)\big], \text{ where } \hat{X} = X^{x,\hat{\nu}}.$$

In addition, $\hat{\nu}$ is unique in the following sense. Any other strategy $\bar{\nu} \in \mathcal{A}$ which is also optimal for the primal problem satisfies

$$\mathbb{E}\left[\int_0^T (\hat{\nu}_t - \bar{\nu}_t)^\intercal \, d\langle M\rangle_t (\hat{\nu}_t - \bar{\nu}_t)\right] = 0,$$

so that $X^{x,\hat{\nu}}$ and $X^{x,\bar{\nu}}$ are indistinguishable.

(ii) Given $y > 0$, there exists an optimal $\hat{Y}^y \in \mathcal{Y}(y)$ for the dual problem, unique up to indistinguishability, i.e.

$$\widetilde{u}(y) = \mathbb{E}\Big[\widetilde{U}(\hat{Y}_T)\Big], \text{ where } \hat{Y} = \hat{Y}^y.$$

(iii) *The functions u and $\widetilde{u}$ are finite, continuously differentiable and conjugate, moreover, u' and $-\widetilde{u}'$ are strictly decreasing. If $y = u'(x)$ then, adopting the notation from (i) and (ii), we have the following relations,*

$$\mathbb{E}[\hat{X}_T \hat{Y}_T] = xy, \quad \hat{Y}_T = U'(\hat{X}_T), \quad u(x) = \widetilde{u}(y) + xy.$$

More explicitly, there are constants c_p, $p \in (-\infty, 1)$, such that when $\widetilde{c}_p := c_p^{\frac{1}{1-p}}$,

$$u(x) = \begin{cases} U(x)\, c_p & \text{if } p \neq 0, \\ U(x) + c_0 & \text{if } p = 0, \end{cases} \qquad \widetilde{u}(y) = \begin{cases} \widetilde{U}(y)\, \widetilde{c}_p & \text{if } p \neq 0, \\ \widetilde{U}(y) + \widetilde{c}_0 & \text{if } p = 0. \end{cases}$$

(iv) *If $y = u'(x)$ then the process $\hat{X}\hat{Y}$ is a martingale on $[0, T]$.*

Remark 1.2.5. In Kramkov and Schachermayer [1999] the authors work in the additive formulation where strategies represent the *number* of shares of each stock held in the portfolio and wealth remains (only) nonnegative. However, for power utility maximization, the additive formulation and the setting here are equivalent. Our motivation for writing wealth in exponential format stems from the fact that the dual domain of the portfolio choice problem will (and for reasons of convenience should) be a family of supermartingale measures, hence stochastic exponentials. Since we want to relate the utility maximization problem straigthly to a BSDE and we dispose of a relationship between primal and dual optimizer at terminal time, see Theorem 1.2.4 (iii), it turns out to be most convenient to write wealth as a stochastic exponential as well. Moreover, as in our setting the optimal wealth $\hat{X}$ exists and satisfies $\hat{X}_T > 0$ we may, without loss of generality, choose to optimize over the family of strictly positive wealth processes $\mathcal{X}(x)$. As a byproduct this simplifies and generalizes the proof of the decomposition of the elements of the dual domain. Also, in Kramkov and Schachermayer [1999] it is stated only that there exists a unique primal and dual optimizer, however the precise interpretation of "uniqueness" is not directly discussed. We refer the reader to Chapter 4 for further details on these issues.

The almost explicit form of the value function u already yields a useful characterization of the optimal processes $\hat{X}$ and $\hat{Y}$. We provide it in the following theorem which collects together necessary and sufficient conditions for two candidate processes to be optimal and whose statement is assumed to be part of common knowledge.

Theorem 1.2.6. *For $x, y > 0$ let $X \in \mathcal{X}(x)$ and $Y \in \mathcal{Y}(y)$ satisfy the terminal condition $Y_T = U'(X_T) = X_T^{p-1}$. Then, under Assumption 1.2.3, the following are equivalent.*

(i) *The relation $y = u'(x)$ holds.*

(ii) *The equality $\mathbb{E}[X_T Y_T] = xy$ holds.*

(iii) *The process XY is a martingale on $[0, T]$.*

(iv) *The processes X and Y are optimal for x and y respectively, i.e. X gives equality in (1.2.5) and Y gives equality in (1.2.6).*

Remark 1.2.7. One may in fact use Theorem 1.2.6 as a method of finding the optimal pair $(\hat{X}, \hat{Y})$. This is the approach favoured in Kallsen and Muhle-Karbe [2010] and successfully employed there for a number of affine stochastic volatility models. It also provides the justification for using the so-called martingale optimality principle in deriving a suitable BSDE, as performed in Hu et al. [2005] for instance. Namely, one can find the optimizers by imposing a condition at terminal time T together with a condition on some stochastic dynamics on $[0, T]$. The latter is given in item (iii) above and corresponds to property (A) of the introduction. This naturally leads to a specific BSDE which is the main objects of the present study.

Proof. Let us start with the implication (i) $\Rightarrow$ (ii) and suppose that $p < 0$. Then, XY being a supermartingale,

$$xy \geq \mathbb{E}[X_T Y_T] = p\,\mathbb{E}[U(X_T)] \geq p\,u(x) = c_p\,x^p = xu'(x) = xy.$$

If $p = 0$, we have $X_T Y_T = X_T \log'(X_T) = 1 = x \log'(x) = xu'(x) = xy$, see the Appendix 6.1. If $p \in (0, 1)$, then $q < 0$ so that

$$\begin{aligned} xy \geq \mathbb{E}[X_T Y_T] &= \mathbb{E}\Big[Y_T^{\frac{1}{p-1}} Y_T\Big] = \mathbb{E}\Big[Y_T^{q-1} Y_T\Big] = -q\,\mathbb{E}\Big[\widetilde{U}(Y_T)\Big] \\ &\geq -q\,\widetilde{u}(y) = y^q\,c_p^{\frac{1}{1-p}} = c_p^q\,x^{q(p-1)}\,c_p^{\frac{1}{1-p}} = c_p\,x^p = u'(x)x = xy. \end{aligned}$$

The implication (ii) $\Rightarrow$ (iii) follows from the fact that XY is a càdlàg supermartingale with constant expectation xy.

If XY is a martingale, then from $Y_T = U'(X_T)$ and Theorem 1.2.4 (iii)

$$\begin{aligned} u(x) \geq \mathbb{E}[U(X_T)] &= \mathbb{E}[\widetilde{U}(Y_T) + X_T Y_T] = \mathbb{E}[\widetilde{U}(Y_T)] + xy \\ &\geq \widetilde{u}(y) + xy \geq u(x) - xy + xy = u(x). \end{aligned}$$

Therefore, $u(x) = \mathbb{E}[U(X_T)]$ and $\widetilde{u}(y) = \mathbb{E}[\widetilde{U}(Y_T)]$, i.e. X and Y are optimal, so that (iv) follows from (iii).

Finally, assume that X and Y are optimal. Set $\hat{y} := u'(x)$ and take the dual optimizer $\hat{Y} \in \mathcal{Y}(\hat{y})$. Then, by assertion (iii) of Theorem 1.2.4, $\hat{Y}_T = U'(X_T) = Y_T$, from which $x\hat{y} = \mathbb{E}[X_T \hat{Y}_T] = \mathbb{E}[X_T Y_T] \leq xy$. In particular, $u'(x) = \hat{y} \leq y = u'((u')^{-1}(y)) =: u'(\hat{x})$ where $x \geq \hat{x} > 0$, since u' is strictly decreasing and continuous, $u'(0) = +\infty$ and $u'(+\infty) = 0$ by Kramkov and Schachermayer [1999] Theorem 2.2. Now take a primal optimizer $\hat{X} \in \mathcal{X}(\hat{x})$. From $U'(X_T) = Y_T = U'(\hat{X}_T)$ we derive that $X_T = \hat{X}_T$, which in turn yields $\hat{x}\hat{y} \geq \mathbb{E}[\hat{X}_T \hat{Y}_T] = \mathbb{E}[X_T \hat{Y}_T] = x\hat{y}$, so that $\hat{x} \geq x$, from which $y = u'(\hat{x}) = u'(x)$. $\square$

Remark 1.2.8. Observe that we made use of the explicit form of the value function only in the proof of the first implication.

We now wish to derive more structure of the dual optimizer. In Larsen and Žitković [2007] the following characterization theorem is shown when S is one-dimensional. We state here the multidimensional analogue whose proof we delegate to Chapter 4. However, we mention that our proof given there builds on, simplifies and extends that of Larsen and Žitković [2007] as we consider utility maximization under constraints there.

Proposition 1.2.9. *Let Assumption 1.2.3 hold. If $\hat{Y} \in \mathcal{Y}(y)$ denotes the dual optimizer then there exists a local martingale $\hat{N}$ such that $\hat{N}$ is orthogonal to M and*

$$\hat{Y} = \hat{Y}_0\, \mathcal{E}(-\lambda \cdot M + \hat{N}).$$

1.3 Power Utility Maximization and Quadratic BSDEs

Let us now concentrate on the case $p \in (-\infty, 0) \cup (0, 1)$. The logarithmic case is fully dealt with in Appendix 6.1. We here recall the BSDE satisfied by the so-called *opportunity process* from Nutz [2010b], more precisely by its log-transform. We start with the solutions $\hat{X}$ and $\hat{Y}$ to the above primal and dual problem (when $\hat{Y}_0 = y = u'(x) = u'(\hat{X}_0)$) and derive the BSDE satisfied by the following process

$$\hat{\Psi} := \log\left(\frac{\hat{Y}}{U'(\hat{X})}\right).$$

The logic is now very similar to the procedure in Mania and Schweizer [2005], we apriori obtained the existence of the object $\hat{\Psi}$ of interest. Imposing a suitable assumption we show that $\hat{\Psi}$ lies in a certain space in which solutions to (a special type of) quadratic semimartingale BSDE are unique. This approach of using BSDE comparison principles in utility maximization may also be found in Hu et al. [2005] and Morlais [2009]. We observe that in these references the *mean-variance tradeoff* $\langle \lambda \cdot M \rangle_T$ is bounded. In what follows we extend their reasoning to the unbounded case under exponential moments. Hence our assumption is

Assumption 1.3.1. *For all $\varrho > 0$ we have that*

$$\mathbb{E}\Big[\exp\big(\varrho\, \langle \lambda \cdot M \rangle_T\big)\Big] = \mathbb{E}\left[\exp\left(\varrho \int_0^T \lambda_s^\top\, d\langle M \rangle_s \lambda_s\right)\right] < +\infty.$$

We describe this by saying that the mean-variance tradeoff $\langle \lambda \cdot M \rangle_T$ has exponential moments of all orders.

For instance, the above assumption is satisfied in a model of a one-dimensional Brownian motion $M = W$ with stock price dynamics given by

$$\frac{dS_t}{S_t} = dW_t - \operatorname{sgn}(W_t)\sqrt{|W_t|}\, dt,$$

see Subsection 2.4.1 for more details. Here, "$-\operatorname{sgn}$" reflects a return reverting behaviour of the stock prices. Clearly, this example is the prototype from a whole class of similar ones. In Chapter 2 we also provide some more, rather sophisticated, examples.

The preceding assumption is compatible with Assumption 1.2.3. However, clearly, it is not the minimal one which is sufficient for the latter. As a matter of fact, for the next lemma to hold, a finite exponential moment of $\langle \lambda \cdot M \rangle_T$ of order

$$\max\Big(\frac{1}{2}, \Big(q^2 - \frac{q}{2} - q\sqrt{q^2 - q}\Big)\mathbf{1}_{\{q<0\}}\Big)$$

is sufficient. We will investigate such minimal sufficient conditions in Chapter 2 to which we refer for more details. We have

Lemma 1.3.2. *Assumption 1.3.1 implies Assumption 1.2.3, more precisely,*

(i) The process $Y^\lambda := \mathcal{E}(-\lambda \cdot M)$ is a martingale on $[0, T]$, hence defines an equivalent local martingale measure for S, the so-called minimal martingale measure*.*

(ii) The function u is finite on all of $(0, +\infty)$.

Proof. Item (i) follows from Novikov's criterion and the product rule using the continuity of M. For item (ii) we need only consider the case of $p \in (0, 1)$. Let $x > 0$ and observe that from the definition of $\widetilde{U}$, Hölder's inequality, $q < 0$ and the exponential moments assumption

$$\begin{aligned} u(x) &= \sup_{\nu\in\mathcal{A}} \mathbb{E}\Big[U(X_T^{x,\nu})\Big] \le \mathbb{E}\Big[\widetilde{U}(Y_T^\lambda)\Big] + \sup_{\nu\in\mathcal{A}} \mathbb{E}\Big[X_T^{x,\nu} Y_T^\lambda\Big] \le -\frac{1}{q}\,\mathbb{E}\Big[(Y_T^\lambda)^q\Big] + x \\ &\le -\frac{1}{q}\,\mathbb{E}\bigg[\exp\bigg(q(2q-1)\int_0^T \lambda_t^\intercal\, d\langle M\rangle_t \lambda_t\bigg)\bigg]^{1/2} + x < +\infty, \end{aligned}$$

which completes the proof. □

Before we discuss properties of the process $\hat{\Psi}$ we first fix some notation.

Definition 1.3.3. Let $\mathfrak{E}$ denote the space of all processes Υ on $[0, T]$ whose supremum $\Upsilon^* := \sup_{0\le t\le T} |\Upsilon_t|$ has finite exponential moments of all orders, i.e. those processes Υ such that for all $\varrho > 0$

$$\mathbb{E}\left[\exp\left(\varrho \Upsilon^*\right)\right] < +\infty.$$

In the Chapter 4 we show that if Assumption 1.3.1 holds and if $(\hat{X}, \hat{Y})$ is the solution to the primal and dual optimization problem, then $\hat{\Psi} \in \mathfrak{E}$. With regards to the derivation of the BSDE satisfied by $\hat{\Psi}$, we note that, using the formulae for $\hat{X}$ and $\hat{Y}$,

$$\hat{\Psi} = \log\Big(y\,\mathcal{E}(-\lambda \cdot M + \hat{N})\Big(x\,\mathcal{E}\left(\hat{\nu} \cdot M + \hat{\nu} \cdot \langle M\rangle\lambda\right)\Big)^{1-p}\Big).$$

Assuming that $\hat{N}$ is continuous and after the change of variables

$$\hat{Z} := -\lambda + (1-p)\hat{\nu}$$

a calculation shows that we have found a solution $(\hat{\Psi}, \hat{Z}, \hat{N})$ to the following quadratic semimartingale BSDE (written in the generic variables (Ψ, Z, N)),

$$\begin{aligned} d\Psi_t = Z_t^\intercal \, dM_t + dN_t - \frac{1}{2}\, d\langle N\rangle_t \\ + \frac{q}{2}(Z_t + \lambda_t)^\intercal \, d\langle M\rangle_t (Z_t + \lambda_t) - \frac{1}{2}\, Z_t^\intercal \, d\langle M\rangle_t Z_t, \quad \Psi_T = 0, \end{aligned} \tag{1.3.1}$$

where we call BSDE *solution* a triple (Ψ, Z, N) of processes valued in $\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}$ satisfying (1.3.1) $\mathbb{P}$-a.s. such that:

(i) The function $t \mapsto \Psi_t$ is continuous $\mathbb{P}$-a.s.

(ii) The process Z is predictable and M-integrable, in particular $\int_0^T Z_t^\intercal \, d\langle M\rangle_t Z_t < +\infty$ $\mathbb{P}$-a.s.

(iii) The local martingale N is continuous and orthogonal to M.

(iv) We have that $\mathbb{P}$-a.s.

$$\int_0^T \Big((Z_t + \lambda_t)^\intercal \, d\langle M\rangle_t (Z_t + \lambda_t) + Z_t^\intercal \, d\langle M\rangle_t Z_t \Big) + \langle N\rangle_T < +\infty.$$

We summarize these findings in the following theorem noting that it is uniqueness that requires the stronger Assumption 1.3.1, existence is guaranteed under Assumption 1.2.3.

Theorem 1.3.4. *Let Assumption 1.3.1 hold and $(\hat{X}, \hat{Y})$ be the solution pair to the primal and dual optimization problem, i.e. for $x > 0$*

$$\hat{X} = x\, \mathcal{E}(\hat{\nu} \cdot M + \hat{\nu} \cdot \langle M\rangle \lambda) \text{ and } \hat{Y} = u'(x)\, \mathcal{E}(-\lambda \cdot M + \hat{N}).$$

Assume that $\hat{N}$ is continuous and set

$$\hat{\Psi} := \log\left(\frac{\hat{Y}}{U'(\hat{X})}\right) \qquad \text{and} \qquad \hat{Z} := -\lambda + (1-p)\hat{\nu}.$$

Then

(i) The triple $(\hat{\Psi}, \hat{Z}, \hat{N})$ is the unique solution (Ψ, Z, N) to the BSDE (1.3.1) where $\Psi \in \mathfrak{E}$ and $Z \cdot M$ and N are two square-integrable (continuous) martingales.

(ii) In terms of the BSDE we may write $\hat{Y}$ as

$$\hat{Y} = \exp(\hat{\Psi})\, U'(\hat{X}) = e^{\hat{\Psi}_0} x^{p-1}\, \mathcal{E}(-\lambda \cdot M + \hat{N}) \in \mathcal{Y}\Big(c_p\, x^{p-1}\Big)$$

where $c_p = \exp(\hat{\Psi}_0)$, $\mathbb{P}$-a.s. Here, c_p is the constant from Theorem 1.2.4.

(iii) The process $\mathcal{E}\Big([(1-q)\hat{Z} - q\lambda] \cdot M + \hat{N}\Big)$ is a martingale on $[0, T]$.

Proof. The content of item (i), i.e. the uniqueness part, follows from Theorem 3.2.6 and Corollary 3.4.3 (ii) in Chapter 3. A calculation yields the alternative formula for $\hat{Y}$ in item (ii) and the relation

$$e^{\hat{\Psi}_0} x^p \, \mathcal{E}\Big([(1-q)\hat{Z} - q\lambda] \cdot M + \hat{N}\Big) \equiv \hat{X}\hat{Y}$$

gives the remaining assertion in item (iii). □

For the convenience of the reader we summarize the previous discussion and the reasoning of the present thesis in the following table, which also includes a hint at an important application of the theory, namely the one concerned with stability.

Duality Approach	**BSDE Approach**
Under the **exponential moments** assumption 1.3.1,	
• There **exist** primal and dual optimizers $\hat{X}$ and $\hat{Y}$.	• *Quadratic BSDEs allow for solutions* (Ψ, Z, N) *with* $\Psi \in \mathfrak{E}$ *(if in addition Assumption 1.2.1 holds).*
• The process $\hat{\Psi} := \log(\hat{Y}/U'(\hat{X}))$ is part of a triple $(\hat{\Psi}, \hat{Z}, \hat{N})$ that satisfies a **quadratic BSDE**.	
• We have that $\hat{\Psi} \in \mathfrak{E}$.	• Solutions with $\Psi \in \mathfrak{E}$ are **unique**. • Solutions with $\Psi \in \mathfrak{E}$ hence **coincide** with those from the duality approach. • Quadratic BSDEs allow for **stability** results.

Table 1.1: Link between the Duality and the BSDE Approach to the Study of the Existence, Uniqueness and Stability of the Utility Maximization Problem under Exponential Moments

The statement of the above theorem is essentially known. In Hu et al. [2005] and Morlais [2009] the boundedness of the mean-variance tradeoff is used to ensure uniqueness. In Chapter 4 we are going to extended this argument to the unbounded case with exponential moments. Building on previous work by Mania and Tevzadze [2003, 2008] the article of Nutz [2011] shows that in a general setting the opportunity process $\exp(\hat{\Psi})$ satisfies a BSDE which reduces to (1.3.1) under the additional assumption of continuity of the filtration. In particular, $\exp(\hat{\Psi})$ is identified as the *minimal* solution to this BSDE. Having identified candidate optimizers from the BSDE, a difficult task is then verification, i.e. showing that a solution to the BSDE indeed provides the primal and dual optimizers. A sufficient condition is that $\mathcal{E}\big([(1-q)Z - q\lambda] \cdot M + N\big)$ is a martingale as can be derived from Nutz [2011] Theorems 5.2 and 5.15 or the above Theorem 1.2.6. For a Brownian framework the reader will find a more explicit result in Theorem 2.2.1 below. However, given a solution (Ψ, Z, N) to the BSDE (1.3.1), the above martingale condition need not be satisfied, hence a solution to the BSDE (1.3.1) need *not* yield the optimizers even when $Z \cdot M$ and N are square-integrable, as we demonstrate in detail in Subsection 2.2.2. For completeness and improved exposition of the material below we state the following verification result as a theorem.

Theorem 1.3.5 (Nutz [2011] Theorems 5.2 and 5.15)**.** *Let Assumption 1.2.3 hold. Then:*

(i) Given the solution pair $(\hat{X}, \hat{Y})$ *to the primal/dual problem, where* $\hat{X}$ *and* $\hat{Y}$ *together with* $\hat{\Psi}$, $\hat{Z}$ *and* $\hat{N}$ *are as in Theorem 1.3.4, then the triple* $(\hat{\Psi}, \hat{Z}, \hat{N})$ *solves the BSDE* (1.3.1)*.*

(ii) Suppose a triple (Ψ, Z, N) *solves the BSDE* (1.3.1) *and is such that*

$$\mathcal{E}\Big([(1-q)Z-q\lambda]\cdot M+N\Big) = \mathcal{E}(-\lambda\cdot M+N)\,\mathcal{E}\Big(\frac{Z+\lambda}{1-p}\cdot M + \frac{Z+\lambda}{1-p}\cdot\langle M\rangle\lambda\Big) \quad (1.3.2)$$

is a martingale on $[0, T]$*. If we define* $\nu := \frac{Z+\lambda}{1-p}$ *and, for* $x > 0$,

$$X := x\,\mathcal{E}(\nu\cdot M + \nu\cdot\langle M\rangle\lambda), \quad Y := \exp(\Psi)\,U'(X), \quad (1.3.3)$$

then the pair (X, Y) *is optimal for the primal/dual problem. In particular,*

$$u(x) = \exp(\Psi_0)\,U(x) < +\infty.$$

(iii) Suppose there exist two solutions to the BSDE (1.3.1) (Ψ^i, Z^i, N^i), $i = 1, 2$, *with the property that the stochastic exponential from* (1.3.2) *is a martingale for each* $i = 1, 2$*. Then* $(\Psi^1, Z^1\cdot M, N^1) \equiv (\Psi^2, Z^2\cdot M, N^2)$ *up to indistinguishability.*

Proof. Only item (ii) needs (little) consideration. We observe that under the given transformations,

$$XY = \exp(\Psi)X^p = \exp(\Psi_0)\,x^p\,\mathcal{E}\Big([(1-q)Z - q\lambda]\cdot M + N\Big).$$

The proof is then immediate, either from Theorem 1.2.6 or the stated reference. □

Hence, if a solution triple (Ψ, Z, N) exists, then under some conditions it provides the solution $(\hat{X}, \hat{Y})$ to the primal and dual problem and we have uniqueness to the BSDE within a certain class. As already noted, this is in the spirit of Mania and Tevzadze [2008] Theorem 1.3.2, Nutz [2011] Theorems 5.2 and 5.13 and Theorem 1.2.6. However, above and in these theorems, the requirement imposed via the exponential in (1.3.2) is not on the model. In contrast, our goal is to study which conditions on the model, i.e. on λ and M, ensure such a BSDE characterization, the regularity of its solution, for instance in terms of a bounded dynamic value process and, most importantly, the stability of its solution in the input parameters. While we examine the latter in Chapter 4 and look at the second one in Sections 2.3 to 2.5, the former motivates our Assumption 1.3.1 under which we can give a unified treatment, i.e. where both methods, duality and

BSDEs, can be compared alongside one another, see Theorem 1.3.4. The result imposes exponential moment conditions on the mean-variance tradeoff as well as on the BSDE solution component Ψ. However, as the uniqueness results of Delbaen et al. [2011] suggest, Assumption 1.3.1 is not the minimal one.

In the Subsection 2.2.1 we provide a necessary condition for this correspondence between duality and BSDEs. This condition is formulated in terms of the dual problem and is made explicit via a counterexample. We mention that the Assumption 1.2.3 (i) is avoided, hence we need the following result for the calculations involved.

Lemma 1.3.6. *For a triple* (Ψ, Z, N) *to satisfy the BSDE* (1.3.1) *it is equivalent that* $\mathbb{P}$*-a.s. for all* t*,*

$$\exp\big((1-q)\Psi_t\big)\,\mathcal{E}(-\lambda\cdot M+N)_t^q = e^{(1-q)\Psi_0}\,\mathcal{E}\big([(1-q)Z-q\lambda]\cdot M+N\big)_t, \quad \Psi_T=0.$$

Proof. If (Ψ, Z, N) solves the BSDE (1.3.1), then an application of the partial integration formula yields that

$$\begin{aligned} &d\Big(\exp\big((1-q)\Psi_t\big)\,\mathcal{E}(-\lambda\cdot M+N)_t^q\Big) \\ &\qquad\qquad = \exp\big((1-q)\Psi_t\big)\,\mathcal{E}(-\lambda\cdot M+N)_t^q\,\big([(1-q)Z_t-q\lambda_t]\,dM_t+dN_t\big). \end{aligned}$$

The converse statement follows from a calculation after taking logarithms of the expression in the lemma. $\square$

2 BSDEs in Utility Maximization with BMO Market Price of Risk

2.1 Introduction

The aim of this chapter is to study the specific quadratic semimartingale BSDE (1.3.1) arising in (unconstrained) power utility maximization concentrating on a market price of risk λ for which $\lambda \cdot M$ is a BMO martingale. This is a natural condition as it ensures that the minimal martingale measure with density process $\mathcal{E}(-\lambda \cdot M)$ is a true probability measure, see Kazamaki [1994] Theorem 2.3. In particular, it ensures that there is no arbitrage in the sense of NFLVR, see Delbaen and Schachermayer [1998]. In a Brownian setting we provide a necessary and sufficient condition for the existence of a solution to (1.3.1) but show that uniqueness fails to hold in the sense that there exists a continuum of distinct square-integrable solutions. This feature occurs since, contrary to the classical Itô representation theorem, an L^2-representation of random variables in terms of stochastic exponentials is not unique. We then study in detail when the BSDE has a bounded solution and derive a new *dynamic* exponential moments condition which is shown to be the minimal sufficient condition in a general filtration. We point out that this result is for the BSDE (1.1.8) which includes jump terms of N.

The study of bounded solutions is motivated by the ease of the verification argument. Namely, when one can show the existence of a bounded solution to (1.3.1), verification is feasible. This is due to the fact that the martingale part of the corresponding BSDE solution then is a BMO martingale. Moreover, such an argument involves only the first component of a BSDE solution which in concrete situations usually is a conditional expectation of a known object. Thus, it may be more accessible than the martingale part. As a direct consequence we obtain additional regularity of the optimal strategy which we then find to be of BMO type as well. The main results of the present chapter are complemented by several interesting examples which illustrate their sharpness as well as important properties of the utility maximization BSDE. In particular, we first motivate the new dynamic exponential moments condition by showing that the ordinary exponential moments assumption together with the BMO property are not sufficient for boundedness. Then we give this dynamic exponential moments condition in terms of a critical exponent and show that it cannot be improved.

The chapter is based on joint work with Christoph Frei and Nicholas Westray, see Frei et al. [2011]. It is organized as follows. In the next section we analyze the questions of existence and uniqueness of BSDE solutions and in Section 2.3 turn our attention to the interplay between boundedness of solutions and the BMO property of $\lambda \cdot M$ and the martingale parts. In Section 2.4 we develop some related counterexamples and then provide the characterization of boundedness in Section 2.5.

2.2 Existence, Uniqueness and Optimality for Quadratic BSDEs

From Theorem 1.3.4 we see that under Assumption 1.3.1 one can connect the duality and BSDE approaches to solving the utility maximization problem. This relies on the results for quadratic BSDEs under such an exponential moments condition as presented in the next Chapter 3. To analyze the above connection in further detail, we consider in the present section a setting where the BSDE (1.3.1) is explicitly solvable. Proposition 2.2.1 gives a sufficient condition for a solution to the BSDE (1.3.1) to exist and provides an expression for $\hat{\Psi}$ in terms of $Y^\lambda := \mathcal{E}(-\lambda \cdot M)$.

We then go on to study uniqueness and show in Theorem 2.2.5 that in general there are infinitely many distinct solutions with a square-integrable martingale part. This is a consequence of the fact that a multiplicative L^2-representation of random variables as stochastic exponentials need not be unique, which is the content of Lemma 2.2.3. Finally, one aim in the present chapter is to study the boundedness of solutions to the BSDE (1.3.1) under the exponential moments and BMO conditions. This involves constructing counterexamples and some of the key techniques and ideas used for this are introduced in the current section.

Therefore, in the present section, we restrict ourselves to the *Brownian setting*, which we assume to be one-dimensional for notational simplicity. So let $M = W$ be a one-dimensional Brownian motion under $\mathbb{P}$ and $(\mathcal{F}_t)_{t\in[0,T]}$ its augmented natural filtration. In particular, $N \equiv 0$ is the unique local martingale orthogonal to M. A generalization of the following results to the multidimensional Brownian framework is immediate.

2.2.1 Necessary Conditions for the Existence of Solutions to Quadratic BSDEs

In this subsection we provide a condition that is sufficient and necessary for the BSDE (1.3.1) to allow for a (then unique) solution in some specific space. This space is defined precisely via the condition that stems from the utility maximization problem (the condition (A) in the introduction) so that solving the BSDE (1.3.1) leads to the primal and dual optimizers.

Proposition 2.2.1. *For $q \in [0,1)$ the BSDE* (1.3.1) *always admits a solution. For $q < 0$ the BSDE* (1.3.1) *admits a solution if and only if*

$$\mathbb{E}\Big[\big(Y_T^\lambda\big)^q\Big] = \mathbb{E}\big[\mathcal{E}(-\lambda \cdot W)_T^q\big] < +\infty. \tag{2.2.1}$$

If there exists a solution, there is a unique solution $(\hat{\Psi}, \hat{Z})$ with $\mathcal{E}\big([(1-q)\hat{Z} - q\lambda] \cdot W\big)$ a martingale. Its first component is given by

$$\hat{\Psi}_t = \frac{1}{1-q} \log\Big(\mathbb{E}\big[\mathcal{E}(-\lambda \cdot W)_{t,T}^q \,\big|\, \mathcal{F}_t\big]\Big), \quad t \in [0,T], \quad \mathbb{P}\text{-}a.s. \tag{2.2.2}$$

In particular, solving (1.3.1) *and setting* (X, Y) *as suggested by Theorem 1.3.4 gives the pair of primal and dual optimizer.*

As a result, condition (2.2.1) is sufficient for the existence and uniqueness of the optimizers and we mention that it corresponds to condition (10) in Kramkov and Schachermayer [2003]. It says that the dual problem is finite, however, the Assumption 1.2.3 (i) is avoided. Hence, the utility maximization problem is well-defined even if NFLVR (*no free lunch with vanishing risk*) does not hold. This is because FLVR strategies cannot be used beneficially by the CRRA-investor due to the requirement of having a positive wealth at any time.

Proof. Let us first show that the BSDE (1.3.1) admits a solution if (2.2.1) holds. Observe that from Jensen's inequality, for $q \in [0, 1)$,

$$\mathbb{E}\big[\mathcal{E}(-\lambda \cdot W)_T^q\big] \le \mathbb{E}\big[\mathcal{E}(-\lambda \cdot W)_T\big]^q \le 1,$$

so that (2.2.1) automatically holds in this case. For $t \in [0, T]$ consider

$$\overline{M}_t := \mathbb{E}\Bigg[\mathcal{E}(-q\lambda \cdot W)_T \exp\bigg(\frac{q(q-1)}{2}\int_0^T \lambda_s^2\, ds\bigg)\,\bigg|\, \mathcal{F}_t\Bigg]. \tag{2.2.3}$$

Since we have

$$\mathbb{E}\Bigg[\mathcal{E}(-q\lambda \cdot W)_T \exp\bigg(\frac{q(q-1)}{2}\int_0^T \lambda_s^2\, ds\bigg)\Bigg] = \mathbb{E}\big[\mathcal{E}(-\lambda \cdot W)_T^q\big] < +\infty,$$

$\overline{M}$ is a positive martingale so that by Itô's representation theorem there exists a predictable process $\overline{Z}$ with $\int_0^T \overline{Z}_t^2\, dt < +\infty$ $\mathbb{P}$-a.s. such that $\frac{1}{\overline{M}} \cdot \overline{M} \equiv \overline{Z} \cdot W$. We set

$$\hat{Z} := \frac{\overline{Z} + q\lambda}{1 - q}$$

and $\hat{\Psi}$ as in (2.2.2). A calculation then shows that $(\hat{\Psi}, \hat{Z})$ solves the BSDE (1.3.1) with $\mathcal{E}\big([(1-q)\hat{Z} - q\lambda] \cdot W\big) \equiv \mathcal{E}\big(\overline{Z} \cdot W\big) \equiv \frac{1}{\overline{M}_0}\overline{M}$ a martingale.

We now turn our attention to uniqueness. Let us assume that (Ψ, Z) is a solution to (1.3.1) such that $\mathcal{E}\big([(1-q)Z - q\lambda] \cdot W\big)$ is a martingale. For $t \in [0, T]$ a calculation gives, see Lemma 1.3.6,

$$\begin{aligned}\exp\big(-(1-q)\Psi_t\big)\,\mathcal{E}\big(-\lambda \cdot W\big)_{t,T}^q &= \exp\big((1-q)(\Psi_T - \Psi_t)\big)\,\mathcal{E}\big(-\lambda \cdot W\big)_{t,T}^q \\ &= \mathcal{E}\big([(1-q)Z - q\lambda] \cdot W\big)_{t,T} \quad \mathbb{P}\text{-a.s.}\end{aligned} \tag{2.2.4}$$

so that we obtain

$$\Psi_t = \frac{1}{1-q}\,\log\Big(\mathbb{E}\big[\mathcal{E}(-\lambda \cdot W)_{t,T}^q \,\big|\, \mathcal{F}_t\big]\Big) \quad \mathbb{P}\text{-a.s.}$$

We derive that Ψ and $\hat{\Psi}$ are indistinguishable which is due to continuity. From (2.2.4) we then obtain that $\mathcal{E}\big([(1-q)Z-q\lambda]\cdot W\big)$ is uniquely determined, from which it follows that $\hat{Z}\cdot W \equiv Z\cdot W$.

Finally, we show that the condition (2.2.1) is also necessary. Assume that a solution (Ψ, Z) to (1.3.1) exists but $\mathbb{E}\big[\mathcal{E}(-\lambda\cdot W)_T^q\big]=+\infty$. Then, together with the supermartingale property of $\mathcal{E}\big([(1-q)Z-q\lambda]\cdot W\big)$, the equality (2.2.4) shows that $\mathbb{P}$-a.s.

$$\begin{aligned}\exp\big((1-q)\Psi_0\big) &\geq \mathbb{E}\Big[\mathcal{E}\big([(1-q)Z-q\lambda]\cdot W\big)_T\Big]\exp\big((1-q)\Psi_0\big)\\ &= \mathbb{E}\Big[\mathcal{E}(-\lambda\cdot W)_T^q\Big]=+\infty,\end{aligned}$$

from which $\Psi_0=+\infty$ $\mathbb{P}$-a.s. in contradiction to the existence of Ψ. □

The above theorem thus provides a condition that is sufficient and necessary for the existence of solutions to a class of quadratic BSDEs. We now provide an explicit market price of risk for which condition (2.2.1) fails to hold, hence for which the BSDE (1.3.1) has no solution.

Proposition 2.2.2. *For every $q<0$ there exists λ such that $\lambda\cdot W$ is a bounded martingale and $\mathbb{E}\big[(Y_T^\lambda)^q\big]=\mathbb{E}\big[\mathcal{E}(-\lambda\cdot W)_T^q\big]=+\infty$.*

Proof. For $t\in[0,T]$ define

$$\lambda_t := \frac{\pi}{2\sqrt{-q(T-t)}}\,\mathbf{1}_{]\!]T/2,\tau]\!]}(t,\cdot), \tag{2.2.5}$$

where τ is the stopping time

$$\tau := \inf\left\{t>\frac{T}{2}\,\middle|\,\left|\int_{T/2}^t \frac{1}{\sqrt{T-s}}\,dW_s\right|\geq 1\right\}.$$

Here, we define λ from time $T/2$ onwards to be consistent with the construction in Subsection 2.4.3. For the present proof, we could equally well replace $T/2$ by 0 in the definitions of λ and τ. Observe that we have that $\mathbb{P}(T/2<\tau<T)=1$ due to continuity and the relation

$$\left\langle \int_{T/2}^{\cdot}\frac{1}{\sqrt{T-t}}\,dW_t\right\rangle_T=\int_{T/2}^T\frac{1}{T-t}\,dt=+\infty.$$

By construction, $\lambda\cdot W$ is bounded by $\frac{\pi}{2\sqrt{-q}}$. We obtain, using Kazamaki [1994] Lemma 1.3 similarly to the proof of Frei and dos Reis [2011] Lemma A.1,

$$\begin{aligned}\mathbb{E}\Big[\mathcal{E}(-\lambda\cdot W)_T^q\Big] &= \mathbb{E}\left[\exp\left(-q(\lambda\cdot W)_T-\frac{q}{2}\int_0^T\lambda_t^2\,dt\right)\right]\\ &\geq e^{-\frac{\pi\sqrt{-q}}{2}}\,\mathbb{E}\left[\exp\left(\frac{\pi^2}{8}\int_{T/2}^{\tau}\frac{1}{T-t}\,dt\right)\right]=+\infty,\end{aligned}$$

from which the statement follows immediately. □

Let us make three points concerning the above example, firstly that when $q \in [0,1)$ such a degeneracy cannot occur, as shown in Proposition 2.2.1. In fact for a BMO martingale $\lambda \cdot W$ there actually always exists a (then unique) *bounded* solution as Theorem 2.3.4 shows. We recall from Kazamaki [1994] that a continuous martingale $\overline{M}$ on the compact interval $[0,T]$ with $\overline{M}_0 = 0$ is a *BMO martingale* if

$$\|\overline{M}\|_{\mathrm{BMO}_2} := \sup_{\tau} \left\| \mathbb{E}\Big[(\overline{M}_T - \overline{M}_\tau)^2 \Big| \mathcal{F}_\tau \Big]^{1/2} \right\|_{L^\infty} < +\infty, \tag{2.2.6}$$

where the supremum is over all stopping times τ valued in $[0,T]$.

Secondly we point out that the martingale in the above Propostion 2.2.2 is bounded. Indeed, it is a leitmotiv of the present chapter that requiring (in addition) the martingale $\lambda \cdot M$ to be bounded does not improve the situation with respect to finiteness of a BSDE solution. This is because the key estimates are all on the quadratic variation process $\langle \lambda \cdot M \rangle$ which in general does not inherit such properties.

Thirdly, we elaborate further on the construction of τ, which is a first hitting time of the set $\mathbb{R}\backslash(-1,1)$ for some continuous local martingale. As we know from the general theory, this local martingale is a time-changed Brownian motion so that we may write τ as a time change of the first time that a Brownian motion leaves $(-1,1)$. More explicitly, by the above construction, we obtain that $\int_{T/2}^{\tau} \frac{1}{T-t}\,dt$ is this passage time. From Kazamaki [1994] Lemma 1.3 we then derive that for $c \in \mathbb{R}$, $|c| < 1$,

$$\mathbb{E}\left[\exp\left(\frac{c^2\pi^2}{8} \int_{T/2}^{\tau} \frac{1}{T-t}\,dt \right) \right] = \frac{\cos(0)}{\cos\left(\frac{|c|\pi}{2}\right)} = \frac{1}{\cos\left(\frac{|c|\pi}{2}\right)}. \tag{2.2.7}$$

As $|c|$ tends to one, this expectation tends to $+\infty$, a fact which we used in the above proof and which is due to $\cos\left(\frac{\pi}{2}\right) = 0$. In conclusion, there is a dichotomy in the behaviour of the above expectation; either it is finite (for $|c| < 1$) or infinite (for $|c| \geq 1$), and we repeatedly exploit variants of this dichotomy in the sequel, see the Section 2.4 for more examples.

2.2.2 Nonoptimality of BSDE Solutions

If a solution to the BSDE (1.3.1) does exist, it does *not* automatically lead to an optimal pair for the utility maximization problem. This is because it may fail to be in the right space (e.g. with respect to which uniqueness for BSDE solutions holds). We now provide a theoretical result to illustrate the problem. More precisely, in contrast to the classical Itô representation theorem with square-integrable integrands, an analogous representation of random variables in terms of stochastic exponentials is not necessarily unique. We have the following result.

Lemma 2.2.3. *Let ξ be a random variable bounded away from zero and infinity, i.e. there are constants $L, \ell > 0$ such that $\ell \leq \xi \leq L$ $\mathbb{P}$-a.s. Then, for every real number*

$c \geq \mathbb{E}[\xi]$, there exists a predictable process α^c such that

$$\xi = c\,\mathcal{E}(\alpha^c \cdot W)_T, \quad \mathbb{E}\left[\int_0^T |\alpha_t^c|^2\, dt\right] < +\infty. \tag{2.2.8}$$

However, there is only one pair (c, α) consisting of a real constant c and a predictable W-integrable process α satisfying $\xi = c\,\mathcal{E}(\alpha \cdot W)_T$ with $\alpha \cdot W$ a BMO martingale or, equivalently, with $c = \mathbb{E}[\xi]$.

Remark 2.2.4. Comparing the multiplicative representation (2.2.8) with the classical one, see Karatzas and Shreve [1991] Theorem 4.15, namely

$$\xi = k + (\beta \cdot W)_T, \quad \mathbb{E}\left[\int_0^T |\beta_t|^2\, dt\right] < +\infty,$$

we see that existence holds in both cases, whereas there is no uniqueness of (c, α^c) in (2.2.8) despite the fact that $\mathbb{E}\left[\int_0^T |\alpha_t^c|^2\, dt\right] < +\infty$ and in contrast to the uniqueness of (k, β). While in the standard Itô representation theorem for L^2-random variables the square-integrability of β and the martingale property of $\beta \cdot W$ are equivalent, our result shows that in the multiplicative form $\mathbb{E}\left[\int_0^T |\alpha_t^c|^2\, dt\right] < +\infty$ does not guarantee uniqueness. The intuition for the difference between β in the additive and α^c in the multiplicative form is the following. Since β is a square-integrable process, $\beta \cdot W$ is a martingale, hence it *must* be the case that $k = \mathbb{E}[\xi]$. In contrast, the square-integrability of α^c is not sufficient for $\mathcal{E}(\alpha^c \cdot W)$ to be a martingale. It can be that $\mathbb{E}[\mathcal{E}(\alpha^c \cdot W)_T] < 1$ so that increasing $c \geq \mathbb{E}[\xi]$ may be offset by an appropriate choice of α^c such that (2.2.8) still holds. A consequence of this is that uniqueness of the decomposition $\xi = c\,\mathcal{E}(\alpha \cdot W)_T$ holds if $\alpha \cdot W$ is a BMO martingale or equivalently (see Kazamaki [1994] Theorem 3.4, using the boundedness of ξ) if $\mathcal{E}(\alpha \cdot W)$ is a martingale.

One could argue that a more natural condition in (2.2.8) is to assume that $\mathcal{E}(\alpha^c \cdot W)$ be a true martingale, however our aim is a characterization in terms of $\alpha^c \cdot W$ itself and thus we do not pursue this. Note that it is not possible to find $c < \mathbb{E}[\xi]$ such that (2.2.8) holds, because $\mathcal{E}(\alpha^c \cdot W)$ is always a positive local martingale, hence a supermartingale.

Proof. We first define $\overline{M}_t := \mathbb{E}[\xi|\mathcal{F}_t]$, $t \in [0, T]$, and apply Itô's representation theorem to the stochastic logarithm of $\overline{M}$, which is a BMO martingale by Kazamaki [1994] Theorem 3.4 since $\overline{M}$ is bounded away from zero and infinity. This application yields a predictable process $\overline{\alpha}$ such that $\overline{\alpha} \cdot W$ is a BMO martingale and $\xi = \mathbb{E}[\xi]\,\mathcal{E}(\overline{\alpha} \cdot W)_T$. The uniqueness part of the statement is then immediate; if $\alpha \cdot W$ is a BMO martingale, we have $c = \mathbb{E}[\xi]$ and $\alpha \cdot W \equiv \overline{\alpha} \cdot W$ since $\mathcal{E}(\alpha \cdot W)$ is a martingale. Conversely, if $c = \mathbb{E}[\xi]$ the process $\mathcal{E}(\alpha \cdot W)$ is a supermartingale with constant expectation, hence a martingale. Again we have that

$$\mathcal{E}(\alpha \cdot W) \equiv \mathbb{E}[\mathcal{E}(\alpha \cdot W)_T |\, \mathcal{F}_\cdot] \equiv \frac{1}{\mathbb{E}[\xi]}\,\mathbb{E}[\xi |\, \mathcal{F}_\cdot] \equiv \mathbb{E}[\mathcal{E}(\overline{\alpha} \cdot W)_T |\, \mathcal{F}_\cdot] \equiv \mathcal{E}(\overline{\alpha} \cdot W)$$

and thus $\alpha \cdot W \equiv \overline{\alpha} \cdot W$, which is the BMO martingale from above.

We now fix $c \geq \mathbb{E}[\xi] > 0$ and construct α^c. For this we first define the stopping time

$$\tau_c := \inf\left\{ t \geq 0 \;\middle|\; \int_0^t \frac{1}{T-s}\, dW_s \leq \frac{t}{2T(T-t)} + \log \frac{\overline{M}_t}{c} \right\}.$$

We argue that $\tau_c < T$ $\mathbb{P}$-a.s. To this end consider

$$\overline{\tau}_c := \inf\left\{ t \geq 0 \;\middle|\; \int_0^t \frac{1}{T-s}\, dW_s \leq \frac{t}{2T(T-t)} + \log \frac{\ell}{c} \right\}$$

and observe that $\tau_c \leq \overline{\tau}_c$. If we define the time change $\rho : [0,T] \to [0,+\infty]$ by $\rho(t) := \frac{t}{T(T-t)}$, then it follows from Revuz and Yor [1999] II.3.14. that

$$\mathbb{E}\left[\exp\left(\frac{1}{8}\,\rho(\overline{\tau}_c)\right)\right] = \exp\left(-\frac{1}{2}\log\frac{\ell}{c}\right) = \frac{c^{1/2}}{\ell^{1/2}} < +\infty. \tag{2.2.9}$$

We deduce that $\mathbb{E}[\rho(\overline{\tau}_c)] < +\infty$, hence $\rho(\overline{\tau}_c) < +\infty$ $\mathbb{P}$-a.s. and thus $\overline{\tau}_c < T$ $\mathbb{P}$-a.s. from which it follows that indeed $\tau_c < T$ $\mathbb{P}$-a.s.

We now set

$$\alpha_t^c := \frac{1}{T-t}\,\mathbf{1}_{[\![0,\tau_c]\!]}(t,\cdot) + \overline{\alpha}\,\mathbf{1}_{]\!]\tau_c,T]\!]}(t,\cdot) \tag{2.2.10}$$

and observe that it satisfies

$$c\,\mathcal{E}(\alpha^c \cdot W)_T = c\,\frac{\overline{M}_T}{\overline{M}_{\tau_c}}\,\mathcal{E}(\alpha^c \cdot W)_{\tau_c} = c\,\frac{\overline{M}_T}{\overline{M}_{\tau_c}}\,\frac{\overline{M}_{\tau_c}}{c} = \overline{M}_T = \xi,$$

where the second equality is due to the specific definition of the stopping time τ_c. Moreover, we have

$$\mathbb{E}\left[\int_0^T |\alpha_t^c|^2\, dt\right] \leq \mathbb{E}\left[\int_0^{\tau_c} \frac{1}{(T-t)^2}\, dt\right] + \mathbb{E}\left[\int_0^T |\overline{\alpha}_t|^2\, dt\right] = \mathbb{E}[\rho(\tau_c)] + \mathbb{E}\left[\int_0^T |\overline{\alpha}_t|^2\, dt\right],$$

which is finite because $\mathbb{E}[\rho(\tau_c)] \leq \mathbb{E}[\rho(\overline{\tau}_c)] < +\infty$ and $\overline{\alpha} \cdot W$ is a BMO martingale. □

The standard method of finding solutions to quadratic BSDEs involves an exponential change of variables. A consequence of the preceding lemma is that the above type of nonuniqueness carries over to the corresponding BSDE solutions, in particular to those of the utility maximization problem. Observe that for each c the process $\alpha^c \cdot W$ is square-integrable in contrast to classical locally integrable counterexamples. Indeed it is well known that without square-integrability even the standard Itô decomposition is not unique. In fact, for every $k \in \mathbb{R}$ there exists β^k such that

$$\xi = k + (\beta^k \cdot W)_T, \quad \int_0^T |\beta_t^k|^2\, dt < +\infty \quad \mathbb{P}\text{-a.s.}$$

by Émery et al. [1983] Proposition 1. We are hence able to construct distinct solutions (Ψ, Z, N) to the BSDE (1.3.1) which are nontrivial in the above sense. This amounts to some of these solutions being nonoptimal by Proposition 2.2.1. Actually, all except one of these solutions are nonoptimal. Alternatively, uniqueness of the multiplicative decomposition $\xi = c\,\mathcal{E}(\alpha \cdot W)_T$ holds under an additional BMO assumption which then implies the uniqueness of Ψ via Lemma 2.3.1 below. We summarize these comments in the following theorem.

Theorem 2.2.5. *For all $p \in (-\infty, 1)$ and all predictable processes λ with $\langle \lambda \cdot W \rangle_T = \int_0^T \lambda_t^2\,dt$ bounded, there exists a continuum of distinct solutions $(\Psi^b, Z^b, N^b \equiv 0)$ to the BSDE* (1.3.1)*, parameterized by $b \geq 0$, satisfying the following properties:*

(i) The martingale part $Z^b \cdot W$ is square-integrable for all $b \geq 0$.

(ii) The process $\mathcal{E}\Big([(1-q)Z^b - q\lambda] \cdot W\Big)$ is a martingale if and only if $b = 0$.

(iii) Defining ν^b and X^b as suggested by the formulae in Theorem 1.3.4, the admissible process ν^b is the optimal strategy and the associated wealth process $X^b = X^{\nu^b}$ is the primal optimizer if and only if $b = 0$.

It is known from Ankirchner et al. [2009] Section 2.2 that quadratic BSDEs need not have unique square-integrable solutions. These authors present a specific example of a quadratic BSDE with a particular terminal condition which allows for two distinct solutions with square-integrable martingale parts. In contrast, Lemma 2.2.3 shows that every BSDE related to power utility maximization with bounded mean-variance tradeoff has no unique square-integrable solution, independently of the value of p. This underlines the importance of being able to find a solution to the BSDE (1.3.1) with $Z \cdot W$ a BMO martingale in Hu et al. [2005] and Morlais [2009] (as is done by means of BSDE theory there).

Proof. We set $\xi := \exp\Big(\frac{q(q-1)}{2}\int_0^T \lambda_t^2\,dt\Big)$ and define the measure change

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} := \mathcal{E}(-q\lambda \cdot W)_T,$$

so that $\widetilde{\mathbb{P}}$ is an equivalent probability measure under which $\widetilde{W}$ is a Brownian motion on $[0, T]$ where

$$\widetilde{W}_t := W_t + q\int_0^t \lambda_s\,ds.$$

This measure change is implicitly already present in the proof of Proposition 2.2.1, see (2.2.3). We now apply Lemma 2.2.3 to the triple $\big(\widetilde{W}, \widetilde{\mathbb{P}}, (\mathcal{F}_t)_{t\in[0,T]}\big)$ noting that in its proof we may use Itô's representation theorem in the form of Karatzas and Shreve [1998] Theorem 1.6.7, i.e. we can write any $\widetilde{\mathbb{P}}$-martingale as a stochastic integral with respect to $\widetilde{W}$, although $\widetilde{W}$ may not generate the whole filtration $(\mathcal{F}_t)_{t\in[0,T]}$. For every real number $c \geq \mathbb{E}_{\widetilde{\mathbb{P}}}[\xi] > 0$ we then derive the existence of a predictable process α^c such that

$$\xi = c\,\mathcal{E}(\alpha^c \cdot \widetilde{W})_T, \quad \mathbb{E}_{\widetilde{\mathbb{P}}}\left[\int_0^T |\alpha_t^c|^2\,dt\right] < +\infty.$$

For $t \in [0,T]$ we then set

$$\widetilde{\Psi}^c_t := \log(c) + \int_0^t \alpha^c_s \, d\widetilde{W}_s - \frac{1}{2}\int_0^t |\alpha^c_s|^2 \, ds - \frac{q(q-1)}{2}\int_0^t \lambda^2_s \, ds,$$

so that $\widetilde{\Psi}^c$ solves the BSDE

$$d\widetilde{\Psi}^c_t = \alpha^c_t \, d\widetilde{W}_t - \frac{1}{2}|\alpha^c_t|^2 \, dt - \frac{q(q-1)}{2}\lambda^2_t \, dt, \quad \widetilde{\Psi}^c_T = 0. \tag{2.2.11}$$

Using the transformations $b := c - \mathbb{E}_{\widetilde{\mathbb{P}}}[\xi] \geq 0$, $\widetilde{\Psi}^c =: (1-q)\Psi^b$ and $\alpha^c =: (1-q)Z^b$ we arrive at the BSDE (1.3.1),

$$d\Psi^b_t = Z^b_t \, dW_t + \frac{q}{2}(Z^b_t + \lambda_t)^2 \, dt - \frac{1}{2}\left(Z^b_t\right)^2 dt, \quad \Psi^b_T = 0,$$

which admits a continuum of distinct solutions, parameterized by $b \geq 0$, as we will see shortly. We first show that each martingale part is additionally square-integrable under $\mathbb{P}$. This follows from the inequality

$$\mathbb{E}\left[\int_0^T |\alpha^c_t|^2 \, dt\right] \leq \mathbb{E}_{\widetilde{\mathbb{P}}}\Big[\mathcal{E}(q\lambda \cdot \widetilde{W})^2_T\Big]^{1/2} \mathbb{E}_{\widetilde{\mathbb{P}}}\left[\left(\int_0^T |\alpha^c_t|^2 \, dt\right)^2\right]^{1/2}.$$

Note that the second term on the right hand side is finite since from (2.2.9) in the proof of Lemma 2.2.3 we have that $\mathbb{E}_{\widetilde{\mathbb{P}}}\big[\rho(\overline{\tau}_c)^2\big] < +\infty$. Moreover, using $\overline{\alpha}$ from this proof, $\overline{\alpha} \cdot \widetilde{W}$ is a BMO martingale (under $\widetilde{\mathbb{P}}$), hence $\int_0^T |\overline{\alpha}_t|^2 \, dt$ has an exponential $\widetilde{\mathbb{P}}$-moment of some order by Kazamaki [1994] Theorem 2.2, see also Lemma 2.5.1 and the comments thereafter. To derive that the first term is finite we use that

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\Big[\mathcal{E}(q\lambda \cdot \widetilde{W})^2_T\Big] \leq \mathbb{E}_{\widetilde{\mathbb{P}}}\left[\exp\left(6q^2 \int_0^T \lambda^2_t \, dt\right)\right]^{1/2} < +\infty.$$

We now observe that the Assumption 1.3.1 is satisfied, hence our previous analysis applies. However, there is a continuum of distinct solutions (Ψ^b, Z^b) to the BSDE (1.3.1) since for every $b = c - \mathbb{E}_{\widetilde{\mathbb{P}}}[\xi] \geq 0$ we have that $\Psi^b_0 = \frac{\log(c)}{1-q}$. From Kazamaki [1994] Theorem 3.6 we have that $\alpha^c \cdot \widetilde{W}$ is a BMO martingale under $\widetilde{\mathbb{P}}$ if and only if $\alpha^c \cdot W$ is a BMO martingale under $\mathbb{P}$. This last condition holds if and only if $Z^b \cdot W$ is a BMO martingale under $\mathbb{P}$. We conclude that $\mathcal{E}\big(\big[(1-q)Z^b - q\lambda\big] \cdot W\big)$ is a martingale if $b = 0$ and it cannot be a martingale for $b > 0$ since otherwise (Ψ^b, Z^b) would coincide with $(\hat{\Psi}, \hat{Z}) \equiv (\Psi^0, Z^0)$. The last assertion in item (iii) is then immediate. $\square$

2.3 Boundedness of BSDE Solutions and the BMO Property

Thus far we have worked under an exponential moments assumption on the mean-variance tradeoff which provides us with the existence of the primal and dual optimizers as well as a link between these optimizers and a special quadratic BSDE. We now connect the above study to the boundedness of solutions to quadratic BSDEs which we show to be intimately related to the BMO property of the martingale part and the mean-variance tradeoff. In fact, a sufficient condition for showing that $Y^\lambda = \mathcal{E}(-\lambda \cdot M)$ indeed defines a true measure change, is that $\lambda \cdot M$ be a BMO martingale, see Kazamaki [1994] Theorem 2.3.

To motivate, suppose that $\lambda \cdot M$ is a BMO martingale and that we can find a solution (Ψ, Z, N) to the BSDE (1.3.1) with Ψ *bounded*. Due to the quadratic form of the driver of (1.3.1), we may find a positive constant $\bar{c}$ such that

$$\int_0^T (Z_t + \lambda_t)^\intercal \, d\langle M\rangle_t (Z_t + \lambda_t) + Z_t^\intercal \, d\langle M\rangle_t Z_t \leq \bar{c} \int_0^T Z_t^\intercal \, d\langle M\rangle_t Z_t + \lambda_t^\intercal \, d\langle M\rangle_t \lambda_t.$$

Thus, we can apply Mania and Schweizer [2005] Proposition 7 to see that the process $Z \cdot M + N$ is a BMO martingale. We derive that $\big[(1-q)Z - q\lambda\big] \cdot M + N$ is a BMO martingale which, by Kazamaki [1994] Theorem 2.3, shows that $\mathcal{E}\big(\big[(1-q)Z - q\lambda\big] \cdot M + N\big)$ is a true martingale. We then deduce that solving the BSDE with a bounded Ψ gives rise to an optimal pair for the primal and dual problem.

Conversely, given $\lambda \cdot M$ a BMO martingale, suppose that $Z \cdot M$ and N are BMO martingales, then Ψ is bounded. This follows by taking the conditional t-expectation in the integrated version of (1.3.1) and estimating the remaining finite variation parts with the help of the BMO_2 norms of $\lambda \cdot M$, $Z \cdot M$ and N, uniformly in t. To summarize, we deduce the following equivalence regarding solutions (Ψ, Z, N) to the BSDE (1.3.1) of the utility maximization problem with BMO market price of risk λ.

Lemma 2.3.1. *Suppose that $\lambda \cdot M$ is a BMO martingale and that the triple (Ψ, Z, N) solves the BSDE (1.3.1). Then Ψ is bounded if and only if $Z \cdot M$ and N are BMO martingales. If this is the case, setting (X, Y) as suggested by Theorem 1.3.4 gives the pair of primal and dual optimizer.*

Remark 2.3.2. If the pair $(\hat{X}, \hat{Y})$ of primal and dual optimizer is *known* to exist then Nutz [2010a] Corollary 5.12 shows that $\hat{\Psi} := \log(\hat{Y}/U'(\hat{X}))$ is bounded if and only if $\lambda \cdot M$, $Z \cdot M$ and N are BMO martingales.

Regarding our previous remarks on placing sufficient conditions on the model, when $\langle \lambda \cdot M\rangle_T$ is bounded, Hu et al. [2005] and Morlais [2009] show by BSDE techniques that there is a unique solution (Ψ, Z, N) to (1.3.1) with Ψ bounded. The existence of such a solution allows one to deduce the existence of a solution to the utility maximization problem and the argument relies crucially upon the boundedness of Ψ.

Proceeding differently, if $\langle \lambda \cdot M\rangle_T$ is bounded, we again have $Y^\lambda = \mathcal{E}(-\lambda \cdot M) \in \mathcal{Y}(1)$ and observe that it satisfies the following reverse Hölder inequality R_q; there is a constant

$c_{rH,p} > 0$ (which depends on p) such that for all stopping times τ valued in $[0,T]$,

$$\mathbb{E}\left[\left(Y_T^\lambda/Y_\tau^\lambda\right)^q \,\middle|\, \mathcal{F}_\tau\right] \leq c_{rH,p} \text{ in the case of } p \in (0,1), \tag{2.3.1}$$

$$\mathbb{E}\left[\left(Y_T^\lambda/Y_\tau^\lambda\right)^q \,\middle|\, \mathcal{F}_\tau\right] \geq c_{rH,p} \text{ in the case of } p \in (-\infty,0). \tag{2.3.2}$$

We may then refer to Nutz [2010b] Proposition 4.5 for boundedness of the *special* solution $(\hat{\Psi}, \hat{Z}, \hat{N})$ to the BSDE when it is assumed to exist (e.g. under Assumption 1.2.3). Conversely, given the solution $(\hat{X}, \hat{Y})$ to the primal and dual optimization problem we find from Nutz [2010b] Propositions 4.3 and 4.4 and the above definition that

$$\exp\left(\sup_{0\leq t\leq T} |\hat{\Psi}_t|\right) = \sup_{0\leq t\leq T} \mathbb{E}\left[\left(\hat{Y}_T/\hat{Y}_t\right)^q \,\middle|\, \mathcal{F}_t\right]^{\operatorname{sgn}(p)(1-p)}.$$

The above discussion then amounts to the following lemma.

Lemma 2.3.3 (Nutz [2010b] Proposition 4.5)**.** *Assume that the pair $(\hat{X}, \hat{Y})$ of the primal and dual optimizer exists and consider $\hat{\Psi} := \log(\hat{Y}/U'(\hat{X}))$. Then $\hat{\Psi}$ is bounded if and only if the reverse Hölder inequality* (2.3.1)*, respectively* (2.3.2)*, holds for the dual optimizer $\hat{Y}$, if and only if it holds for* some $Y \in \mathcal{Y}(1)$.

From Remark 2.3.2 we already know that the BMO property of $\lambda \cdot M$ is necessary for the boundedness of $\hat{\Psi}$. The following result shows that for nonpositive risk aversion parameters $p \in (-\infty, 0]$ it is also *sufficient*.

Theorem 2.3.4. *Assume that $q \in [0,1)$ and that $\lambda \cdot M$ is a BMO martingale. Then the pair $(\hat{X}, \hat{Y})$ of the primal and dual optimizer exists and $\hat{\Psi} := \log(\hat{Y}/U'(\hat{X}))$ is* bounded*. In particular, solving the BSDE* (1.3.1) *with bounded first component gives rise to the pair of primal and dual optimizer.*

Proof. Since $\lambda \cdot M$ is a BMO martingale, $Y^\lambda := \mathcal{E}(-\lambda \cdot M)$ defines an equivalent local martingale measure for S so that Assumption 1.2.3 is satisfied, where we use a calculation similar to the proof of Lemma 1.3.2 to extend its item (ii) to $p = 0$. By Kazamaki [1994] Corollary 3.4 $Y^\lambda \in \mathcal{Y}(1)$ also satisfies the reverse Hölder inequality (2.3.2) if $q \in (0,1)$. The assertion then follows from Lemma 2.3.3 and from the explicit formula $\hat{\Psi} \equiv 0$ that holds in the case $q = 0$. □

Proposition 2.2.1 shows that in a Brownian framework the BSDE (1.3.1) always admits a solution if $q \in [0,1)$. In view of the above theorem this property extends to the general framework under the condition that $\lambda \cdot M$ is a BMO martingale. In particular, there is a unique solution which is bounded and it is given by the opportunity process of the utility maximization problem.

Let us now contrast this with the situation when $q < 0$. The example in Subsection 2.2.1 provides a bounded BMO martingale $\lambda \cdot M$ such that the corresponding BSDE admits no solution. For this example the utility maximization problem satisfies $u(1) = +\infty$, i.e. the utility maximization problem is degenerate. In fact, $\Psi_0 \equiv +\infty$ in this case.

The question now becomes whether, given an arbitrary λ such that $\lambda \cdot M$ is a BMO martingale and given some $q < 0$, we can still guarantee a bounded solution Ψ to the BSDE (1.3.1) when the utility maximization problem is *nondegenerate*. We settle this question negatively in the next section providing an example for which Assumption 1.3.1 as well as the BMO (even the boundedness) property of $\lambda \cdot M$ hold, but for which the BSDE (1.3.1) does not have a bounded solution.

To counterbalance this negative result, in Section 2.5 we provide via the John-Nirenberg inequality a condition on the order of the *dynamic* exponential moments of the mean-variance tradeoff process that guarantees boundedness of $\hat{\Psi}$. This is accompanied by a further example showing that this condition cannot be improved. To conclude, Theorems 2.3.4 and 2.5.10 provide a full characterization of the boundedness of solutions to the BSDE (1.3.1) in terms of the dynamic exponential moments of $\langle \lambda \cdot M \rangle$ for a BMO martingale $\lambda \cdot M$.

2.4 Counterexamples to the Boundedness of BSDE Solutions

We know that an optimal pair for the utility maximization problem gives rise to a triple $(\hat{\Psi}, \hat{Z}, \hat{N})$ solving the BSDE (1.3.1). Conversely, under suitable conditions, BSDE theory based on Briand and Hu [2008], Kobylanski [2000] or the results given in Chapter 3, provides solutions to the BSDE with $\hat{\Psi}$ bounded (in $\mathfrak{E}$), with uniqueness in the class of bounded processes (in the class $\mathfrak{E}$). We now present an example of a BMO martingale $\lambda \cdot M$ which satisfies Assumption 1.3.1 and for which the BSDE (1.3.1) related to the utility maximization problem has an unbounded solution for a given p.

We develop this example in three steps. Firstly, we show that Assumption 1.3.1 alone (rather unsurprisingly) is not sufficient to guarantee a bounded solution. The corresponding $\lambda \cdot M$ involved is however not a BMO martingale. The second example is of BMO type, but lacks finite exponential moments of a sufficiently high order. It resembles the example provided in Subsection 2.2.1. Finally, we combine these two examples to construct a BMO martingale $\lambda \cdot M$ such that $\langle \lambda \cdot M \rangle_T$ has all exponential moments, but for which the BSDE does not allow for a bounded solution. Although this last step leaves the first two obsolete, we believe that the outlined presentation helps the reader in gaining insight into the nature of the degeneracy. In addition it hints at the minimal sufficient condition in Theorem 2.3.4. Namely, instead of simply requiring both the BMO and the exponential moments properties, they should be combined into a dynamic condition. Since in the present section we construct suitable counterexamples let $M = W$ be again a one-dimensional $\mathbb{P}$-Brownian motion in its augmented natural filtration.

2.4.1 Unbounded Solutions under All Exponential Moments

Let us assume that the market price of risk is given by $\lambda := -\operatorname{sgn}(W)\sqrt{|W|}$ so that the stock price dynamics read as follows,

$$\frac{dS_t}{S_t} = dW_t - \operatorname{sgn}(W_t)\sqrt{|W_t|}\,dt.$$

Note that in the above definition "$-\operatorname{sgn}$" is motivated by economic rationale, to simulate a certain reverting behaviour of the returns. Assumption 1.3.1 is satisfied since

$$\int_0^T \lambda_t^2\,d\langle M\rangle_t = \int_0^T |W_t|\,dt \le T \cdot \sup_{0\le t\le T} |W_t|$$

and by Doob's inequality, for $\varrho > 1$,

$$\begin{aligned}\mathbb{E}\left[\exp\left(\varrho \sup_{0\le t\le T} |W_t|\right)\right] &= \mathbb{E}\left[\sup_{0\le t\le T} \exp(\varrho|W_t|)\right] \le \left(\frac{\varrho}{\varrho-1}\right)^{\varrho} \mathbb{E}[\exp(\varrho|W_T|)] \\ &\le 2\left(\frac{\varrho}{\varrho-1}\right)^{\varrho} e^{\varrho^2 T/2} < +\infty.\end{aligned}$$

Now let $p \in (0,1)$ so that $q < 0$ and let $(\hat{X}, \hat{Y})$ be the optimizers of the utility maximization problem, where $\hat{X}_0 = x > 0$ and $\hat{Y}_0 = y := u'(x)$. Since we are in a complete Brownian framework we have that $\hat{Y} = y\,\mathcal{E}(\operatorname{sgn}(W)\sqrt{|W|}\cdot W)$. If $\hat{\nu}$ denotes the optimal investment strategy we derive from Theorem 1.3.4 that $(\hat{\Psi}, \hat{Z}, 0)$ is the unique solution to the BSDE (1.3.1) where $\hat{\Psi} := \log(\hat{Y}/\,U'(\hat{X})) \in \mathfrak{E}$ and $\hat{Z} := \operatorname{sgn}(W)\sqrt{|W|} + (1-p)\hat{\nu}$. According to Lemma 2.3.3 $\hat{\Psi}$ is bounded if and only if $\hat{Y}$ satisfies the reverse Hölder inequality (2.3.1) for some positive constant $c_{rH,p}$ and all stopping times τ valued in $[0,T]$, which is not the case as we now show.

The family $\{|W_t| \,|\, t \in [0,T]\}$ is uniformly integrable since $\mathbb{E}[W_t^2] = t \le T$, so we may apply the stochastic Fubini theorem (see Becherer [2006] Lemma A.1) to get, for some $t \in (0,T)$, via Jensen's inequality,

$$\begin{aligned}\mathbb{E}\left[\left(\hat{Y}_T/\hat{Y}_t\right)^q \,\middle|\, \mathcal{F}_t\right] &= \mathbb{E}\left[\exp\left(q\int_t^T \operatorname{sgn}(W_s)\sqrt{|W_s|}\,dW_s - \frac{q}{2}\int_t^T |W_s|\,ds\right) \,\middle|\, \mathcal{F}_t\right] \\ &\ge \exp\left(\mathbb{E}\left[q\int_t^T \operatorname{sgn}(W_s)\sqrt{|W_s|}\,dW_s - \frac{q}{2}\int_t^T |W_s|\,ds \,\middle|\, \mathcal{F}_t\right]\right) \\ &= \exp\left(-\frac{q}{2}\mathbb{E}\left[\int_t^T |W_s|\,ds \,\middle|\, \mathcal{F}_t\right]\right) = \exp\left(-\frac{q}{2}\int_t^T \mathbb{E}[|W_s| \,|\, \mathcal{F}_t]\;ds\right) \\ &\ge \exp\left(-\frac{q}{2}\int_t^T |W_t|\,ds\right) = \exp\left(-\frac{q(T-t)}{2}\,|W_t|\right).\end{aligned}$$

Since the last random variable is unbounded it cannot be the case that (2.3.1) holds, hence $\hat{\Psi}$ cannot be bounded.

However, $\lambda \cdot W$ from this example is not a BMO martingale since for $t \in (0,T)$,

$$\mathbb{E}\left[\int_t^T |W_s|\,ds \,\middle|\, \mathcal{F}_t\right] = \int_t^T \mathbb{E}[|W_s| \,|\, \mathcal{F}_t]\,ds \geq (T-t)\,|W_t|,$$

which shows that $\|\lambda \cdot M\|_{\mathrm{BMO}_2} = \|\lambda \cdot W\|_{\mathrm{BMO}_2}$ cannot be finite.

2.4.2 Unbounded Solutions under the BMO Property

We continue with a BMO example for which the solution to the BSDE (1.3.1) satisfying the (analogue of) condition (A) from the introduction is unbounded. The idea is the following, from Proposition 2.2.2, for $q < 0$, there exists λ with $\lambda \cdot W$ a BMO martingale such that the BSDE (1.3.1) has no solution (in any class of possible solutions). Replacing this λ by $c\,\lambda$ for a constant c, it follows from (2.5.9) below that the BSDE has either no solution (for $|c| \geq 1$) or has a solution which is bounded and fulfills a BMO property (for $|c| < 1$), see also the remarks following the proof of Proposition 2.2.2. This dichotomy is in line with the fact that for a BMO martingale $\overline{M}$ the set of all $\varrho < 0$ such that $\mathcal{E}(\overline{M})$ satisfies the reverse Hölder inequality R_ϱ is open; compare Kazamaki [1994] Corollary 3.2. The insight then is to make c a *random* variable in order to construct λ such that the BSDE (1.3.1) has a solution which is not bounded. More precisely, we have the following result.

Proposition 2.4.1. *For every $q < 0$ there exists a predictable W-integrable process λ with $\lambda \cdot W$ a bounded (hence a BMO) martingale such that,*

(i) The BSDE (1.3.1) *has a unique solution $(\hat{\Psi}, \hat{Z}, \hat{N} \equiv 0)$ with $\mathcal{E}([(1-q)\hat{Z} - q\lambda] \cdot W)$ a martingale. In particular, solving* (1.3.1) *and setting (X, Y) as suggested by Theorem 1.3.4 gives the pair of primal and dual optimizer.*

(ii) There does not *exist a solution (Ψ, Z) to* (1.3.1) *with $Z \cdot W$ a BMO martingale or* Ψ bounded.

Proof. For $t \in [0,T]$ we set

$$\lambda_t := \frac{\pi\alpha}{2\sqrt{-q(T-t)}}\,\mathbf{1}_{]\!]T/2,\tau]\!]}(t,\cdot), \tag{2.4.1}$$

where

$$\alpha := \frac{2}{\pi}\arccos\sqrt{\Phi\Big(\sqrt{2/T}\,W_{T/2}\Big)}$$

for Φ the standard normal cumulative distribution function and τ the stopping time from the proof of Proposition 2.2.2,

$$\tau := \inf\left\{t > \frac{T}{2}\,\middle|\,\left|\int_{T/2}^t \frac{1}{\sqrt{T-s}}\,dW_s\right| \geq 1\right\}. \tag{2.4.2}$$

Note that $\Phi\big(\sqrt{2/T}\,W_{T/2}\big)$ is uniformly distributed on $(0,1]$ and that α is valued in $[0,1)$ $\mathbb{P}$-a.s. It follows immediately that $\lambda\cdot W$ is bounded by $\frac{\pi}{2\sqrt{-q}}$, in particular it is a BMO martingale. This is due to Kazamaki [1994] Corollary 2.1 which states that the BMO_1 and BMO_2 norms are equivalent for uniformly integrable martingales. Here, for a continuous martingale $(\overline{M}_t)_{t\in[0,T]}$ with $\overline{M}_0=0$,

$$\|\overline{M}\|_{\mathrm{BMO}_1} := \sup_\tau \left\|\mathbb{E}\Big[|\overline{M}_T-\overline{M}_\tau|\,\Big|\,\mathcal{F}_\tau\Big]\right\|_{L^\infty} < +\infty,$$

where the supremum is over all stopping times τ valued in $[0,T]$.

Using Kazamaki [1994] Lemma 1.3 in the same way as in the proof of Frei and dos Reis [2011] Lemma A.1, see the remarks after the proof of Proposition 2.2.2, we obtain that

$$\begin{aligned}\mathbb{E}\Big[\mathcal{E}(-\lambda\cdot W)_T^q\Big] &\le e^{\frac{\pi\sqrt{-q}}{2}}\,\mathbb{E}\left[\exp\left(\frac{\pi^2\alpha^2}{8}\int_{T/2}^{\tau}\frac{1}{T-t}\,dt\right)\right] = e^{\frac{\pi\sqrt{-q}}{2}}\,\mathbb{E}\left[\frac{1}{\cos(\pi\alpha/2)}\right]\\ &= e^{\frac{\pi\sqrt{-q}}{2}}\,\mathbb{E}\left[\frac{1}{\sqrt{\Phi\big(\sqrt{2/T}\,W_{T/2}\big)}}\right] = 2e^{\frac{\pi\sqrt{-q}}{2}} < +\infty,\end{aligned}$$

so that Proposition 2.2.1 gives the first assertion. Due to the $\mathcal{F}_{T/2}$-measurability of α and the $\mathcal{F}_{T/2}$-independence of τ, we have

$$\begin{aligned}\exp\big((1-q)\hat{\Psi}_{T/2}\big) &\ge e^{\frac{-\pi\sqrt{-q}}{2}}\,\mathbb{E}\left[\exp\left(\frac{\pi^2\alpha^2}{8}\int_{T/2}^{\tau}\frac{1}{T-t}\,dt\right)\middle|\,\mathcal{F}_{T/2}\right]\\ &= e^{\frac{-\pi\sqrt{-q}}{2}}\,\mathbb{E}\left[\exp\left(\frac{\pi^2x^2}{8}\int_{T/2}^{\tau}\frac{1}{T-t}\,dt\right)\right]\Bigg|_{x=\alpha} = \frac{e^{\frac{-\pi\sqrt{-q}}{2}}}{\cos(\pi\alpha/2)}.\end{aligned}$$

This shows that

$$(1-q)\hat{\Psi}_{T/2} \ge -\frac{\pi\sqrt{-q}}{2} - \frac{1}{2}\log\Big(\Phi\Big(\sqrt{2/T}\,W_{T/2}\Big)\Big),$$

which is unbounded by the uniform distribution of $\Phi\big(\sqrt{2/T}\,W_{T/2}\big)$ on $(0,1]$.

For item (ii) assume that there exists a solution (Ψ,Z) to (1.3.1) with $Z\cdot W$ a BMO martingale or Ψ bounded. By Lemma 2.3.1 we can restrict ourselves to assuming that Ψ is bounded, which implies that $Z\cdot W$ is a BMO martingale so that $\mathcal{E}\big([(1-q)Z-q\lambda]\cdot W\big)$ is a martingale. By uniqueness, (Ψ,Z) coincides with $(\hat{\Psi},\hat{Z})$ above, in contradiction to the unboundedness of $\hat{\Psi}$. $\square$

2.4.3 Unbounded Solutions under All Exponential Moments *and* the BMO Property

The two previous subsections raise the question whether we can find a BMO martingale $\lambda \cdot M$ such that its quadratic variation has all exponential moments and the BSDE (1.3.1) has only an unbounded solution. Roughly speaking, the idea is to combine the above two examples by translating the crucial distributional properties of $|W|$ and α into the corresponding properties of a suitable stopping time σ to be constructed. This guarantees that the BMO property and the exponential moments condition are satisfied simultaneously, while we can also achieve the unboundedness of the BSDE solution by using independence. Table 2.1 summarizes the key properties.

	Form of λ_t^2	**Crucial Properties**
Example in 2.4.1	$\lvert W_t\rvert$	$\lvert W_t\rvert$ is unbounded, has all exponential moments
Example in 2.4.2	$\frac{\pi^2\alpha^2}{4(-q)}\frac{1}{T-t}\mathbf{1}_{]\!]T/2,\tau]\!]}(t,\cdot)$	$\alpha^2 \in [0,1)$, $\mathbb{P}(\alpha^2 \geq \varrho) > 0\,\forall \varrho < 1$, $\mathbb{E}\big[1/\cos(\alpha\pi/2)\big] < +\infty$
Combination	$\frac{\pi^2}{4(-q)}\frac{1}{T-t}\mathbf{1}_{]\!]T/2,\tau\wedge\sigma]\!]}(t,\cdot)$	$\sigma \in (T/2,T]$, $\mathbb{P}(\sigma \geq \varrho) > 0\,\forall \varrho < T$, $\int_0^\sigma \frac{1}{T-t}\,dt$ has all exp. moments

Table 2.1: Comparison of the BSDE Examples from the Present Section 2.4

Theorem 2.4.2. *For every $q < 0$, there exists a predictable W-integrable process λ such that,*

(i) The process $\lambda \cdot W$ is a bounded (hence a BMO) martingale.

(ii) For all $\varrho > 0$ we have $\mathbb{E}\Big[\exp\big(\varrho\int_0^T \lambda_t^2\,dt\big)\Big] < +\infty$.

(iii) The BSDE (1.3.1) *has a unique solution $(\hat\Psi, \hat Z, \hat N \equiv 0)$ with $\mathcal{E}\big([(1-q)\hat Z - q\lambda]\cdot W\big)$ a martingale. In particular, solving* (1.3.1) *and setting (X,Y) as suggested by Theorem 1.3.4 gives the pair of primal and dual optimizer.*

(iv) There does not *exist a solution (Ψ, Z) to* (1.3.1) *with $Z\cdot W$ a BMO martingale or Ψ bounded.*

Proof. Let us first construct σ with the desired distributional properties. We define the nonnegative continuous function $f : (T/2,T] \to \mathbb{R}$, $f(t) := c_0 \cdot e^{-\frac{1}{T-t}}$, where $c_0 > 0$ is a constant such that $\int_{T/2}^T f(t)\,dt = 1$. We then consider the strictly increasing function $F : (T/2,T] \to (0,1]$, $F(t) := \int_{T/2}^t f(s)\,ds$ and its inverse $F^{-1} : (0,1] \to (T/2,T]$. We set

$$\sigma := (F^{-1}\circ\Phi)\Big(\sqrt{2/T}\,W_{T/2}\Big),$$

so that σ is an $\mathcal{F}_{T/2}$-measurable random variable with values in $(T/2, T]$ and cumulative distribution function F. Now define for $t \in [0, T]$,

$$\lambda_t := \frac{\pi}{2\sqrt{-q(T-t)}} \mathbf{1}_{]\!]T/2, \tau\wedge\sigma]\!]}(t, \cdot),$$

where τ is the stopping time from (2.4.2). It follows immediately that $\lambda \cdot W$ is bounded by $\frac{\pi}{2\sqrt{-q}}$, hence a BMO martingale, see the argument given in the previous proof.

Let us now show that $\int_0^T \lambda_t^2 \, dt$ has all exponential moments. Take $\varrho > 0$ and fix an integer $\bar{\varrho} \geq \frac{\pi^2 \varrho}{4(-q)} \vee 2$. We derive

$$\begin{aligned}
\mathbb{E}\left[\exp\left(\varrho \int_0^T \lambda_t^2 \, dt\right)\right] &\leq \mathbb{E}\left[\exp\left(\bar{\varrho} \int_{T/2}^{\sigma} \frac{1}{T-t} \, dt\right)\right] \\
&= \mathbb{E}\big[\exp\big(\bar{\varrho}\,[\log(T/2) - \log(T-\sigma)]\big)\big] = (T/2)^{\bar{\varrho}}\, \mathbb{E}\left[\frac{1}{(T-\sigma)^{\bar{\varrho}}}\right] \\
&= c_0 (T/2)^{\bar{\varrho}} \int_{T/2}^{T} \left(\frac{1}{T-t}\right)^{\bar{\varrho}} e^{-\frac{1}{T-t}} \, dt = c_0 (T/2)^{\bar{\varrho}} \int_{2/T}^{+\infty} s^{\bar{\varrho}-2} e^{-s} \, ds \\
&= c_0 (T/2)^{\bar{\varrho}} (\bar{\varrho} - 2)!\, e^{-2/T} \sum_{k=0}^{\bar{\varrho}-2} \frac{(2/T)^k}{k!} < +\infty,
\end{aligned}$$

where in the last equality we used the representation of the incomplete gamma function at integer points (or, directly, integration by parts),

$$\Gamma(n, x) := \int_x^{+\infty} s^{n-1} e^{-s} \, ds = (n-1)!\, e^{-x} \sum_{k=0}^{n-1} \frac{x^k}{k!} \quad \text{for } n \in \mathbb{N},\ x \in \mathbb{R}.$$

A standard argument then shows that $\mathbb{E}\big[\mathcal{E}(-\lambda \cdot W)_T^q\big] < +\infty$, see the proof of Lemma 1.3.2. It follows from Proposition 2.2.1 that the BSDE (1.3.1) has a unique solution $(\hat{\Psi}, \hat{Z})$ with $\mathcal{E}([(1-q)\hat{Z} - q\lambda] \cdot W)$ a martingale where the first component is given by

$$\hat{\Psi}_t = \frac{1}{1-q} \log\Big(\mathbb{E}\big[\mathcal{E}(-\lambda \cdot W)_{t,T}^q \,\big|\, \mathcal{F}_t\big]\Big), \quad t \in [0, T], \quad \mathbb{P}\text{-a.s.}$$

We deduce that $\mathbb{P}$-a.s.

$$\begin{aligned}
\exp\big((1-q)\hat{\Psi}_{T/2}\big) &\geq e^{\frac{-\pi\sqrt{-q}}{2}} \mathbb{E}\left[\exp\left(\frac{\pi^2}{8} \int_{T/2}^{\tau\wedge\sigma} \frac{1}{T-t} \, dt\right) \middle| \mathcal{F}_{T/2}\right] \\
&= e^{\frac{-\pi\sqrt{-q}}{2}} \mathbb{E}\left[\exp\left(\frac{\pi^2}{8} \int_{T/2}^{\tau\wedge s} \frac{1}{T-t} \, dt\right)\right]\Bigg|_{s=\sigma}
\end{aligned}$$

because σ is $\mathcal{F}_{T/2}$-measurable and τ is independent from $\mathcal{F}_{T/2}$. From monotone convergence it follows that

$$\lim_{s\uparrow T}\mathbb{E}\left[\exp\left(\frac{\pi^2}{8}\int_{T/2}^{\tau\wedge s}\frac{1}{T-t}\,dt\right)\right]=\mathbb{E}\left[\exp\left(\frac{\pi^2}{8}\int_{T/2}^{\tau}\frac{1}{T-t}\,dt\right)\right]=+\infty, \tag{2.4.3}$$

the last equality being a consequence of (2.2.7). We now fix $K>0$ and take $s_0\in(T/2,T)$ such that

$$\mathbb{E}\left[\exp\left(\frac{\pi^2}{8}\int_{T/2}^{\tau\wedge s}\frac{1}{T-t}\,dt\right)\right]\geq e^{(1-q)K+\frac{\pi\sqrt{-q}}{2}}\quad\text{for all } s\in[s_0,T),$$

which is possible by (2.4.3). This implies $\mathbb{P}(\hat{\Psi}_{T/2}\geq K)\geq\mathbb{P}(\sigma\geq s_0)=1-F(s_0)>0$ since $s_0<T$, in particular $\hat{\Psi}$ is unbounded. The assertion of item (iv) then follows as in the previous proof. □

Remark 2.4.3. It is interesting to compare, for different constants $c\in\mathbb{R}$, the above different definitions of λ regarding the behaviour of the solution to the BSDE

$$d\Psi_t=Z_t\,dW_t+\frac{q}{2}\,(Z_t+c\lambda_t)^2\,dt-\frac{1}{2}\,Z_t^2\,dt,\quad \Psi_T=0. \tag{2.4.4}$$

In the example of Proposition 2.2.2 λ_t^2 is of the form $\frac{\pi^2}{4(-q)}\frac{1}{T-t}\mathbf{1}_{]\!]T/2,\tau]\!]}(t,\cdot)$, while in Subsection 2.4.2 λ_t^2 equals $\frac{\pi^2\alpha^2}{4(-q)}\frac{1}{T-t}\mathbf{1}_{]\!]T/2,\tau]\!]}(t,\cdot)$, which we modified to $\frac{\pi^2}{4(-q)}\frac{1}{T-t}\mathbf{1}_{]\!]T/2,\tau\wedge\sigma]\!]}(t,\cdot)$ in the above discussion. Table 2.2 shows that by introducing additional random variables in the construction of λ, the BSDE (2.4.4) becomes solvable for bigger values of $|c|$, but the solution for $|c|\geq 1$ is unbounded. The assertions of Table 2.2 can be deduced from the arguments in the above proofs together with the additional calculation given in (2.5.11) below.

	Form of $\frac{4(-q)}{\pi^2}\lambda_t^2$	**Solution to the BSDE** (2.4.4) $\lvert c\rvert\in[0,1)$	$\lvert c\rvert=1$	$\lvert c\rvert>1$
Example in 2.2.1	$\frac{1}{T-t}\mathbf{1}_{]\!]T/2,\tau]\!]}(t,\cdot)$	bounded	no solution	
Example in 2.4.2	$\alpha^2\frac{1}{T-t}\mathbf{1}_{]\!]T/2,\tau]\!]}(t,\cdot)$	bounded	unbounded	no solution
Example in 2.4.3	$\frac{1}{T-t}\mathbf{1}_{]\!]T/2,\tau\wedge\sigma]\!]}(t,\cdot)$	bounded	unbounded	

Table 2.2: Description of Solutions to the BSDE (2.4.4)

2.4.4 Markovian Examples via Azéma-Yor Martingales

In the Subsections 2.2.1, 2.4.2 and 2.4.3 we constructed several explicit counterexamples to the boundedness of solutions to the BSDE (1.3.1). The aim of this subsection is to show that such examples can also be given in Markovian form. More precisely, for a specific choice of M we find market price of risk processes $\lambda_t = \lambda(M_t)$, $t \in [0,T]$, which are functions of M and such that the corresponding assumptions from the Subsections 2.2.1, 2.4.2 and 2.4.3 are satisfied. In particular, the corresponding results carry over to such a Markovian framework. The construction of M involves Azéma-Yor martingales for which we recall the following result.

Lemma 2.4.4. *Let $(X_t)_{t\geq 0}$ be a one-dimensional (continuous) local martingale with running maximum $\overline{X}$, let $g : \mathbb{R} \to \mathbb{R}$ be a locally integrable function and $G(x) := \int_0^x g(y)\,dy$. Then*

$$G(\overline{X}) - g(\overline{X})(\overline{X} - X) = g(\overline{X}) \cdot X,$$

which defines a (continuous) local martingale. The corresponding statement for the running minimum $\underline{X}$ is immediate upon using $\underline{X} = -\overline{(-X)}$.

The construction of M now takes the following familiar form. Let $(W_t)_{t\in[0,T]}$ be a standard one-dimensional Brownian motion and $(\mathcal{F}_t)_{t\in[0,T]}$ its natural augmented filtration. For $t \in [0,T)$ let

$$X_t := \int_0^t \frac{1}{\sqrt{T-s}}\,\mathbf{1}_{(T/2,T]}(s)\,dW_s,$$

which defines a continuous martingale X on $[0,T)$. In analogy to the previous examples consider the stopping time

$$\tau := \inf\left\{ t > \frac{T}{2} \,\middle|\, \left| \int_{T/2}^t \frac{1}{\sqrt{T-s}}\,dW_s \right| \geq 1 \right\} = \inf\{t \geq 0 \,|\, |X_t| \geq 1\}.$$

By the above Lemma 2.4.4, applied to X and the identity function $g(x) = x$,

$$U := -X\underline{X} + \frac{1}{2}\underline{X}^2 \quad \text{and} \quad V := X\overline{X} - \frac{1}{2}\overline{X}^2$$

are continuous local martingales on $[0,T)$. In particular, setting

$$\check{M} := (X, U, V)$$

we obtain that $\check{M}$ is a three-dimensional continuous local martingale on $[0,T)$. Our goal being the construction of a local martingale M which (by the general assumptions) is continuous on the whole interval $[0,T]$, it remains to close the continuity gap at $t = T$. To this end we first define the stopping time

$$\check{\tau} := \inf\{t \geq 0 \,|\, |X_t| \geq 2\}$$

and derive that $T/2 < \tau < \check{\tau} < T$, $\mathbb{P}$-a.s. Then $M := \check{M}^{\check{\tau}}$ is a continuous local martingale on $[0,T]$, as required.

Lemma 2.4.5. *There are measurable functions $\underline{g}$, $\overline{g} : \mathbb{R}^2 \to \mathbb{R}$ such that*

$$\underline{X} = \underline{g}(X, U) \quad \textit{and} \quad \overline{X} = \overline{g}(X, V),$$

where X, U and V are from the above construction. In particular,

$$\mathbf{1}_{[\![0,\tau]\!]} = \mathbf{1}_{\left\{\underline{X}\geq -1\right\}}\mathbf{1}_{\left\{\overline{X}\leq 1\right\}} = \mathbf{1}_{\left\{\underline{g}(X,U)\geq -1\right\}}\mathbf{1}_{\left\{\overline{g}(X,V)\leq 1\right\}}$$

is a function of $\check{M}$ and hence also of M.

Proof. We have that $\frac{1}{2}\underline{X}^2 - X\underline{X} - U = 0$. Since $X^2 + 2U = X^2 - 2X\underline{X} + \underline{X}^2 = (X-\underline{X})^2 \geq 0$ this is equivalent to $\underline{X} = X \pm \sqrt{X^2+2U}$. Due to $\underline{X} \leq X$ we find that $\underline{X} = X - \sqrt{X^2+2U} = \underline{g}(X,U)$ for $\underline{g}(x,u) := (x - \sqrt{x^2+2u})\mathbf{1}_{\{x^2\geq -2u\}}$. The proof for $\overline{X}$ is similar and therefore omitted. Finally,

$$\mathbf{1}_{\left\{\underline{g}(X_t^{\check{\tau}},U_t^{\check{\tau}})\geq -1\right\}}\mathbf{1}_{\left\{\overline{g}(X_t^{\check{\tau}},V_t^{\check{\tau}})\leq 1\right\}} = \begin{cases}\mathbf{1}_{\left\{\underline{g}(X_t,U_t)\geq -1\right\}}\mathbf{1}_{\left\{\overline{g}(X_t,V_t)\leq 1\right\}} = \mathbf{1}_{[\![0,\tau]\!]}(t) & t \leq \check{\tau},\\ \mathbf{1}_{\left\{\underline{X}_{\check{\tau}}\geq -1\right\}}\mathbf{1}_{\left\{\overline{X}_{\check{\tau}}\leq 1\right\}} = 0 = \mathbf{1}_{[\![0,\tau]\!]}(t) & t > \check{\tau}.\end{cases}$$

□

As a consequence, defining λ in the usual way, i.e. by using an indicator function of τ we find that it can be chosen as a function of M. We now provide some more details as to how to adapt the examples in the Subsections 2.2.1, 2.4.2 and 2.4.3.

First Example in Subsection 2.2.1

Our goal is to prove the analogue of Proposition 2.2.2 with a Markovian λ such that the BSDE does not allow for a solution. We could take the three-dimensional M from above (as we do in the next subsection), but actually the one-dimensional local martingale $M^1 = X^{\check{\tau}}$ turns out to be sufficient for our purposes. We recall that we are in the standard one-dimensional Brownian framework.

Lemma 2.4.6. *For every $q<0$ there exists a predictable process λ which is a function of the local martingale $M^1 = X^{\check{\tau}}$, i.e. $\lambda_t = \lambda(M_t^1)$, such that $\lambda\cdot M^1$ is a bounded martingale with $\mathbb{E}\big[\mathcal{E}(-\lambda\cdot M^1)_T^q\big] = +\infty$. In particular, the BSDE* (1.3.1) *with driving local martingale M^1 does not have a solution.*

Proof. We set $\lambda := \frac{\pi}{2\sqrt{-q}}\mathbf{1}_{\{|M^1|\leq 1\}}$ which is predictable since M^1 is adapted and continuous. Moreover, $\lambda\cdot M^1$ is a martingale since it is bounded by $\frac{\pi}{2\sqrt{-q}}$. We then derive from $M^1\equiv 0$ on $[0,T/2]$ and independence that $\mathbb{P}$-a.s.

$$\begin{aligned}\mathbb{E}\Big[\mathcal{E}(-\lambda\cdot M^1)_T^q\,\Big|\,\mathcal{F}_{T/2}\Big] = \mathbb{E}\Big[\mathcal{E}(-\lambda\cdot M^1)_T^q\Big] &\geq e^{-\frac{\pi\sqrt{-q}}{2}}\,\mathbb{E}\left[\exp\left(\frac{\pi^2}{8}\int_{T/2}^{\check{\tau}}\frac{\mathbf{1}_{\{|X_t|\leq 1\}}}{T-t}\,dt\right)\right]\\ &\geq e^{-\frac{\pi\sqrt{-q}}{2}}\,\mathbb{E}\left[\exp\left(\frac{\pi^2}{8}\int_{T/2}^{\tau}\frac{1}{T-t}\,dt\right)\right] = +\infty.\end{aligned}$$

Now let us assume that a solution (Ψ,Z,N) to the BSDE (1.3.1) exists. Considering it on $[T/2,T]$, we observe that from time $T/2$ up to time $\check{\tau}$ the filtration generated by M^1

coincides with the filtration generated by a Brownian motion, hence $dN \equiv 0$ on $[\![T/2, \check{\tau}]\!]$. As for (2.2.4) we then derive that $\mathbb{P}$-a.s. for $t \in [\![T/2, \check{\tau}]\!]$,

$$\exp\big((1-q)(\Psi_{\check{\tau}} - \Psi_t)\big) = \mathcal{E}\big([(1-q)Z - q\lambda] \cdot M^1\big)_{t,T}\, \mathcal{E}\big(-\lambda \cdot M^1\big)_{t,T}^{-q},$$

where we use that M^1 is constant after time $\check{\tau}$. This fact also shows that

$$\exp(\Psi_{\check{\tau}}) = \exp\big(-(\Psi_T - \Psi_{\check{\tau}})\big) = -\,\mathcal{E}(N)_{\check{\tau},T}.$$

In fact, the BSDE (1.3.1) degenerates to $d\Psi_t = dN_t - \frac{1}{2}\, d\langle N\rangle_t$ on $[\![\check{\tau}, T]\!]$. After taking $\mathcal{F}_{\check{\tau}}$-conditional expectation we obtain that $\Psi_{\check{\tau}} \geq 0$, $\mathbb{P}$-a.s. As a consequence,

$$\begin{aligned}
\exp\big((1-q)\Psi_{T/2}\big) &\geq \mathbb{E}\Big[\mathcal{E}\big([(1-q)Z - q\lambda] \cdot M^1\big)_{T/2,T} \,\Big|\, \mathcal{F}_{T/2}\Big] \exp\big((1-q)\Psi_{T/2}\big) \\
&= \mathbb{E}\Big[\mathbb{E}\Big[\mathcal{E}\big([(1-q)Z - q\lambda] \cdot M^1\big)_{T/2,T} \exp\big((1-q)\Psi_{T/2}\big) \,\Big|\, \mathcal{F}_{T/2}\Big] \,\Big|\, \mathcal{F}_{\check{\tau}}\Big] \\
&\geq \mathbb{E}\Big[\mathbb{E}\Big[\mathcal{E}\big([(1-q)Z - q\lambda] \cdot M^1\big)_{T/2,T} \exp\big((1-q)(\Psi_{T/2} - \Psi_{\check{\tau}})\big) \,\Big|\, \mathcal{F}_{T/2}\Big] \,\Big|\, \mathcal{F}_{\check{\tau}}\Big] \\
&= \mathbb{E}\Big[\mathcal{E}\big(-\lambda \cdot M^1\big)_{T/2,T}^q \,\Big|\, \mathcal{F}_{T/2}\Big] = +\infty \quad \mathbb{P}\text{-a.s.}
\end{aligned}$$

from which $\Psi_{T/2} = +\infty$ $\mathbb{P}$-a.s. This contradicts the existence of Ψ. □

Second and Third Example in Subsections 2.4.2 and 2.4.3

For the examples from the Subsections 2.4.2 and 2.4.3 we consider the three-dimensional local martingale M constructed above and extend it to a fourth dimension by setting $M^4 := W\mathbf{1}_{[0,T/2]} + W_{T/2}\mathbf{1}_{(T/2,T]}$. This fourth component serves for writing α and σ from the Subsections 2.4.2 and 2.4.3 as a function of M, see their specific definitions there. The process λ is then chosen to satisfy $\lambda = (\lambda^1, 0, 0, 0)$, where

$$\lambda^1 := \frac{\pi\alpha}{2\sqrt{-q}}\,\mathbf{1}_{[\![0,\tau]\!]} \quad \text{or} \quad \lambda^1 := \frac{\pi}{2\sqrt{-q}}\,\mathbf{1}_{[\![0,\tau\wedge\sigma]\!]}.$$

Then λ is a function of M.

We now concentrate on transferring the proof of Theorem 2.4.2 to the present context, noting that we can construct a solution $(\hat{\Psi}, \hat{Z}, \hat{N})$ with orthogonal component $\hat{N}$ satisfying $\hat{N} \equiv 0$ on $[0,T]$. In fact, the BSDE (1.3.1) degenerates to $d\Psi_t = dN_t - \frac{1}{2}\, d\langle N\rangle_t$ on $[\![\check{\tau}, T]\!]$ hence we can set $(\hat{\Psi}, \hat{N}) \equiv (0,0)$ there in view of the terminal condition $\Psi_T = 0$. On $[\![T/2, \check{\tau}]\!]$ we have that the filtration generated by M^1 coincides with the filtration generated by a Brownian motion, hence $\hat{N} \equiv 0$ due to continuity. It follows that $\mathbb{P}$-a.s.

$$\exp\big((1-q)\hat{\Psi}_{T/2}\big) = \mathbb{E}\Big[\mathcal{E}\big(-\lambda \cdot M\big)_{T/2,T}^q \,\Big|\, \mathcal{F}_{T/2}\Big] = \mathbb{E}\Big[\mathcal{E}\big(-\lambda \cdot M\big)_T^q\Big] < +\infty,$$

which is finite and a constant, using $\lambda \cdot M \equiv \lambda^1 \cdot M^1$, its independence from $\mathcal{F}_{T/2}$ and $M^1 \equiv 0$ on $[0,T/2]$. On $[0,T/2]$ the BSDE degenerates again, hence for continuity reasons we may choose $(\hat{\Psi}, \hat{N}) \equiv (\hat{\Psi}_{T/2}, 0)$. This gives a solution $(\hat{\Psi}, \hat{Z}, \hat{N} \equiv 0)$ with

$\mathcal{E}\big([(1-q)\hat{Z} - q\lambda] \cdot M + \hat{N}\big)$ a martingale. Conversely, if (Ψ, Z, N) is a solution with $\mathcal{E}\big([(1-q)Z - q\lambda] \cdot M + N\big)$ a martingale we derive from Theorem 1.3.5 (iii), which can be applied due to the finiteness of the exponential moments of $\langle \lambda \cdot M \rangle_T$, that $\Psi \equiv \hat{\Psi}$. In analogy to Lemma 2.4.6 we obtain the following Markovian result.

Theorem 2.4.7. *For every $q < 0$ there exists a predictable process λ which is a function of the continuous local martingale M constructed above, i.e. $\lambda_t = \lambda(M_t)$, such that,*

(i) *The process $\lambda \cdot M$ is a bounded (hence a BMO) martingale.*

(ii) *For all $\varrho > 0$ we have $\mathbb{E}\Big[\exp\big(\varrho \langle \lambda \cdot M \rangle_T\big)\Big] < +\infty$.*

(iii) *The BSDE* (1.3.1) *has a unique solution $(\hat{\Psi}, \hat{Z}, \hat{N})$ with $\mathcal{E}\big([(1-q)\hat{Z} - q\lambda] \cdot M + \hat{N}\big)$ a martingale. Here, $\hat{N} \equiv 0$. In particular, setting (X, Y) as suggested by Theorem 1.3.4 gives the pair of primal and dual optimizer.*

(iv) *There does* not *exist a solution (Ψ, Z, N) to* (1.3.1) *with $Z \cdot M + N$ a BMO martingale or Ψ* bounded*.*

2.5 Characterization of Boundedness of BSDE Solutions

We have already shown that for a BMO martingale $\lambda \cdot M$ and $q \in [0, 1)$ the BSDE (1.3.1) allows for a unique bounded solution given by the utility maximization problem. In the previous section we gave some examples to show that for $q < 0$ the situation is rather different. In this section we complete the analysis by developing and finally providing a sufficient condition that guarantees (necessarily unique) bounded solutions to (1.3.1). It is also shown that this particular condition cannot be improved. More precisely, we consider here a more general situation where the orthogonal local martingale N may exhibit jumps. We assume that the local martingale M is still continuous. In this case the BSDE (1.3.1) is replaced by

$$\begin{aligned} d\Psi_t = Z_t^\intercal \, dM_t + dN_t - \frac{1}{2}\, d\langle N^c \rangle_t + \log(1 + \Delta N_t) - \Delta N_t \\ + \frac{q}{2}(Z_t + \lambda_t)^\intercal \, d\langle M \rangle_t (Z_t + \lambda_t) - \frac{1}{2}\, Z_t^\intercal \, d\langle M \rangle_t Z_t, \quad \Psi_T = 0, \quad (2.5.1) \end{aligned}$$

where ΔN denotes the jump part of N and satisfies $\Delta N > -1$, recalling that $\hat{Y} > 0$. We mention that all the results which depend only on the specific continuous local martingale M also hold in this more general setting. In particular, the statements of Lemma 2.3.3 and Theorem 2.3.4 continue to hold for the BSDE (2.5.1) in place of the BSDE (1.3.1).

2.5.1 Bounded Solutions via the John-Nirenberg Inequality

We exploit here the John-Nirenberg inequality and show that if $\lambda \cdot M$ is a BMO martingale with BMO_2 norm in a specific range then the utility maximization problem admits a unique solution $(\hat{X}, \hat{Y})$ with $\hat{\Psi} = \log(\hat{Y}/U'(\hat{X}))$ bounded.

The Critical Exponent

We recall the John-Nirenberg inequality for the convenience of the reader. In what follows $\overline{M}$ is an arbitrary continuous one-dimensional martingale on $[0,T]$ with $\overline{M}_0 = 0$.

Lemma 2.5.1 (Kazamaki [1994] Theorem 2.2)**.** *If* $\|\overline{M}\|_{\mathrm{BMO}_2} < 1$ *then for every stopping time* τ *valued in* $[0,T]$*,*

$$\mathbb{E}\Big[\exp\Big(\langle \overline{M}\rangle_T - \langle \overline{M}\rangle_\tau\Big)\Big|\,\mathcal{F}_\tau\Big] \leq \frac{1}{1 - \|\overline{M}\|^2_{\mathrm{BMO}_2}}. \tag{2.5.2}$$

Define the *critical exponent* b via

$$b(\overline{M}) := \sup\bigg\{b \geq 0 \,\bigg|\, \sup_\tau \Big\|\mathbb{E}\Big[\exp\Big(b\Big(\langle \overline{M}\rangle_T - \langle \overline{M}\rangle_\tau\Big)\Big)\,\Big|\,\mathcal{F}_\tau\Big]\Big\|_{L^\infty} < +\infty\bigg\}, \tag{2.5.3}$$

where the supremum inside the brackets is over all stopping times τ valued in $[0,T]$. This inner supremum is called a *dynamic exponential moment* of $\langle \overline{M}\rangle$. We observe that if $b(\overline{M}) > 0$ for a continuous *local* martingale $\overline{M}$ then $\overline{M}$ is already a true martingale. A consequence of Lemma 2.5.1 is that a martingale $\overline{M}$ is a BMO martingale if and only if $b(\overline{M}) > 0$. In addition, the following lemma shows that the supremum in (2.5.3) is never attained.

Lemma 2.5.2. *Let* $b > 0$ *and* $\overline{M}$ *be a continuous martingale with* $\overline{M}_0 = 0$ *and*

$$\sup_\tau \Big\|\mathbb{E}\Big[\exp\Big(b\Big(\langle \overline{M}\rangle_T - \langle \overline{M}\rangle_\tau\Big)\Big)\,\Big|\,\mathcal{F}_\tau\Big]\Big\|_{L^\infty} < +\infty, \tag{2.5.4}$$

where the supremum is over all stopping times τ *valued in* $[0,T]$*. Then there exists* $\tilde{b} > b$ *such that also*

$$\sup_\tau \Big\|\mathbb{E}\Big[\exp\Big(\tilde{b}\Big(\langle \overline{M}\rangle_T - \langle \overline{M}\rangle_\tau\Big)\Big)\,\Big|\,\mathcal{F}_\tau\Big]\Big\|_{L^\infty} < +\infty.$$

Proof. Motivated by Kazamaki [1994] Corollary 3.2 we aim to apply Gehring's inequality. To this end, fix a stopping time τ and set $\Theta_t := \exp(b(\langle \overline{M}\rangle_t - \langle \overline{M}\rangle_\tau))$ for $t \in [0,T]$. For each $\mu > 1$, we then define the stopping time $\tau_\mu := \inf\{t \in [\![\tau, T]\!] \,|\, \Theta_t > \mu\}$. It follows from $\mu > 1$ and the continuity of Θ that

$$\Theta_{\tau_\mu} = \mu \quad \text{on} \quad \{\tau_\mu < +\infty\}. \tag{2.5.5}$$

Since Θ is nondecreasing, we have that $\{\Theta_T > \mu\} = \{\tau_\mu < +\infty\}$ and this event is $\mathcal{F}_{\tau_\mu}$-measurable. Therefore, we obtain

$$\begin{aligned}
\mathbb{E}[\mathbf{1}_{\{\Theta_T > \mu\}}\Theta_T] &= \mathbb{E}[\mathbf{1}_{\{\Theta_T > \mu\}}\mathbb{E}[\Theta_T|\,\mathcal{F}_{\tau_\mu}]] \\
&= \mathbb{E}\Big[\mathbf{1}_{\{\Theta_T > \mu\}}\Theta_{\tau_\mu}\mathbb{E}\Big[\exp\Big(b\left(\langle \overline{M}\rangle_T - \langle \overline{M}\rangle_{\tau_\mu}\right)\Big)\Big|\,\mathcal{F}_{\tau_\mu}\Big]\Big] \\
&\leq c_b\,\mathbb{E}[\mathbf{1}_{\{\Theta_T > \mu\}}\Theta_{\tau_\mu}],
\end{aligned}$$

where we used (2.5.4) and denoted its left-hand side by c_b. Now fix $\varepsilon \in (0,1)$. Using (2.5.5), we derive

$$\mathbb{E}\big[\mathbf{1}_{\{\Theta_T>\mu\}}\Theta_{\tau_\mu}\big] = \mu^\varepsilon\,\mathbb{E}\big[\mathbf{1}_{\{\Theta_T>\mu\}}\Theta_{\tau_\mu}^{1-\varepsilon}\big] \le \mu^\varepsilon\,\mathbb{E}\big[\mathbf{1}_{\{\Theta_T>\mu\}}\Theta_T^{1-\varepsilon}\big]$$

and conclude that

$$\mathbb{E}\big[\mathbf{1}_{\{\Theta_T>\mu\}}\Theta_T\big] \le c_b\,\mu^\varepsilon\,\mathbb{E}\big[\mathbf{1}_{\{\Theta_T>\mu\}}\Theta_T^{1-\varepsilon}\big].$$

It follows from the probabilistic version of Gehring's inequality given in Kazamaki [1994] Theorem 3.5, however see Remark 2.5.6 below, that there exist $r>1$ and $K>0$ (depending on ε and c_b only) such that

$$\mathbb{E}\big[\Theta_T^r\big] \le K\,\mathbb{E}[\Theta_T]^r.$$

To obtain the conditional version, we take $A \in \mathcal{F}_\tau$ and derive from the same argument and Jensen's inequality that

$$\mathbb{E}\big[\Theta_T^r\mathbf{1}_A\big] \le K\,\mathbb{E}[\Theta_T\mathbf{1}_A]^r \le K\,\mathbb{E}\big[\mathbb{E}[\Theta_T|\mathcal{F}_\tau]^r\mathbf{1}_A\big]$$

so that

$$\mathbb{E}\big[\Theta_T^r|\mathcal{F}_\tau\big] \le K\,\mathbb{E}[\Theta_T|\mathcal{F}_\tau]^r \le c_b^r\,K \quad \mathbb{P}\text{-a.s.}$$

Since this holds for any stopping time τ, we conclude the proof by setting $\tilde{b} = rb$. $\square$

A direct consequence of Lemma 2.5.2 is the following result.

Corollary 2.5.3. *For $k>0$ and a continuous martingale $\overline{M}$, $\overline{M}_0 = 0$, the following two items are equivalent,*

(i) $\sup_\tau \Big\|\mathbb{E}\Big[\exp\Big(k\Big(\langle\overline{M}\rangle_T - \langle\overline{M}\rangle_\tau\Big)\Big)\,\Big|\,\mathcal{F}_\tau\Big]\Big\|_{L^\infty} < +\infty,$

(ii) $b\big(\overline{M}\big) > k$,

where the supremum in (i) is over all stopping times τ valued in $[0,T]$.

Boundedness of BSDE Solutions

The John-Nirenberg inequality (2.5.2) can now be used to provide a first sufficient condition for the boundedness of solutions to the BSDE (2.5.1).

Proposition 2.5.4. *For $p \in (0,1)$, i.e. $q<0$, set*

$$k_q := q^2 - \frac{q}{2} - q\sqrt{q^2-q} = \frac{1}{2}\left(q - \sqrt{q^2-q}\right)^2 > 0 \tag{2.5.6}$$

and consider a martingale $\lambda \cdot M$ with $\|\lambda \cdot M\|_{BMO_2} < 1/\sqrt{k_q}$. Then we have,

(i) *The reverse Hölder inequality* (2.3.1) *holds for* $Y^\lambda := \mathcal{E}(-\lambda \cdot M)$, *i.e. for all stopping times valued in* $[0,T]$ *we have that*

$$\mathbb{E}\Big[\Big(Y_T^\lambda / Y_\tau^\lambda\Big)^q \,\Big|\, \mathcal{F}_\tau\Big] \leq c_{rH,p}$$

for some positive constant $c_{rH,p}$ *depending on* p *and the* BMO_2 *norm of* $\lambda \cdot M$.

(ii) *Assumption 1.2.3 holds and hence the solution pair* $(\hat{X}, \hat{Y})$ *to the primal and dual problem exists.*

(iii) *If* $\hat{X}$, $\hat{Y}$, $\hat{\Psi}$, $\hat{Z}$ *and* $\hat{N}$ *are as in Theorem 1.3.4 (*$\hat{N}$ *not necessarily continuous), then the triple* $(\hat{\Psi}, \hat{Z}, \hat{N})$ *is the unique solution to the BSDE* (2.5.1) *with* $\hat{\Psi}$ bounded.

Proof. For item (i) we proceed similarly to the proof of Lemma 1.3.2 by choosing the sharpest possible version of Hölder's inequality in the sense that the condition on the BMO_2 norm of $\lambda \cdot M$ is the least restrictive; this is how k_q is selected. We set $\beta := 1 - \frac{1}{q}\sqrt{q^2 - q} > 1$, then with $\varrho := \beta/(\beta - 1) > 1$, the dual number to β, we have that for any stopping time τ valued in $[0,T]$,

$$\begin{aligned}
\mathbb{E}\Big[\Big(Y_T^\lambda / Y_\tau^\lambda\Big)^q \,\Big|\, \mathcal{F}_\tau\Big] &\leq \mathbb{E}\bigg[\mathcal{E}(-\beta q \lambda \cdot M)_{\tau,T}^{1/\beta} \exp\bigg(\frac{\varrho q}{2}(\beta q - 1)\int_\tau^T \lambda_s^\intercal \, d\langle M\rangle_s \lambda_s\bigg)^{1/\varrho} \,\bigg|\, \mathcal{F}_\tau\bigg] \\
&\leq \mathbb{E}\bigg[\exp\bigg(k_q \int_\tau^T \lambda_s^\intercal \, d\langle M\rangle_s \lambda_s\bigg) \,\bigg|\, \mathcal{F}_\tau\bigg]^{1/\varrho} \qquad (2.5.7) \\
&\leq \bigg(\frac{1}{1 - k_q \|\lambda \cdot M\|_{\mathrm{BMO}_2}^2}\bigg)^{1/\varrho} =: c_{rH,p} < +\infty,
\end{aligned}$$

where we used Hölder's inequality, the supermartingale property of $\mathcal{E}(-\beta q \lambda \cdot M)$, the definition of the constants and the John-Nirenberg inequality (2.5.2). For item (ii) we remark that $Y^\lambda := \mathcal{E}(-\lambda \cdot M)$ is a martingale by Kazamaki [1994] Theorem 2.3. Moreover, using $x > 0$ and $\tau \equiv 0$ in the previous calculation, we obtain

$$\begin{aligned}
0 \leq u(x) = \sup_{\nu \in \mathcal{A}} \mathbb{E}\Big[U\big(X_T^{x,\nu}\big)\Big] &\leq \mathbb{E}\Big[\widetilde{U}\big(Y_T^\lambda\big)\Big] + \sup_{\nu \in \mathcal{A}} \mathbb{E}\Big[X_T^{x,\nu} Y_T^\lambda\Big] \leq -\frac{1}{q}\,\mathbb{E}\Big[\big(Y_T^\lambda\big)^q\Big] + x \\
&\leq -\frac{1}{q}\, c_{rH,p} + x < +\infty.
\end{aligned}$$

For the last statement we assume that $\hat{X}$, $\hat{Y}$, $\hat{\Psi}$, $\hat{Z}$ and $\hat{N}$ are as in Theorem 1.3.4. Then $(\hat{\Psi}, \hat{Z}, \hat{N})$ is a solution to the BSDE (2.5.1) where the process $\hat{\Psi}$ is bounded. This is due to Lemma 2.3.3. Conversely, if the triple (Ψ, Z, N) is a solution to the BSDE (2.5.1) with Ψ bounded, we can identify it with $(\hat{\Psi}, \hat{Z}, \hat{N})$ by Nutz [2011] Corollary 5.6 and the following lemma, which we apply to $\overline{M} = -\lambda \cdot M$. Namely, the cited result implies that if the utility maximization problem is finite for some $\tilde{p} \in (p, 1)$ then a solution triple (Ψ, Z, N) with Ψ bounded coincides with $(\hat{\Psi}, \hat{Z}, \hat{N})$. □

Lemma 2.5.5. *Let $q < 0$ and $\overline{M}$ be a continuous BMO martingale such that the reverse Hölder inequality* (2.3.1) *holds for $\mathcal{E}(\overline{M})$. Then there exists $\tilde{q} < q$ such that $\mathcal{E}(\overline{M})$ satisfies the reverse Hölder inequality* (2.3.1) *with $\tilde{q}$.*

Proof. We note that the reverse Hölder inequality R_q for $q < 0$ is equivalent to the Muckenhoupt inequality A_ϱ with $\varrho = 1 - 1/q > 1$. Indeed, the inequality

$$\mathbb{E}\Big[(Y_T/Y_\tau)^q \,\Big|\, \mathcal{F}_\tau\Big] \le c_{rH,p}$$

is equivalent to the estimate

$$\mathbb{E}\Big[(Y_\tau/Y_T)^{\frac{1}{\varrho-1}} \,\Big|\, \mathcal{F}_\tau\Big] \le c_{rH,p}$$

where $-q = \frac{1}{\varrho-1}$ and τ is any stopping time valued in $[0,T]$. This second inequality is the Muckenhoupt inequality A_ϱ, see Kazamaki [1994] Definition 2.2. Therefore, the statement of Lemma 2.5.5 follows from Kazamaki [1994] Corollary 3.3 which states that if $\mathcal{E}(\overline{M})$ satisfies A_ϱ, $\varrho > 1$, then it also satisfies $A_{\varrho-\varepsilon}$ for some $\varepsilon \in (0, \varrho - 1)$. □

Remark 2.5.6. We mention that in the formulation of Kazamaki [1994] Theorem 3.5 as well as the proof of Kazamaki [1994] Corollary 3.2 there is a small gap which can be easily filled. Namely, for a nonnegative random variable U and positive constants K, β and $\varepsilon \in (0,1)$ the author requires Gehring's condition

$$\mathbb{E}\big[\mathbf{1}_{\{U>\mu\}}U\big] \le K\mu^\varepsilon\,\mathbb{E}\big[\mathbf{1}_{\{U>\beta\mu\}}U^{1-\varepsilon}\big] \tag{2.5.8}$$

to hold for *all* $\mu > 0$, which cannot be satisfied for $U \in L^1_+(\mathbb{P})$ unless $U = 0$, $\mathbb{P}$-a.s. This is because for $U \in L^1_+(\mathbb{P})$, $U \neq 0$, the right-hand side tends to zero as $\mu \downarrow 0$ whereas the left-hand side tends to $\mathbb{E}[U] > 0$. However, an inspection of the proof of Kazamaki [1994] Theorem 3.5 reveals that (2.5.8) is needed only for $\mu > \mathbb{E}[U]$. In conclusion, Kazamaki [1994] Theorem 3.5 should be stated for $\mu > \mathbb{E}[U]$ instead of $\mu > 0$. If this is the case it then can be applied in the proof of Kazamaki [1994] Corollary 3.2, where for $\mu > 0$ the following stopping time is considered, $\tau_\mu = \inf\{t \ge 0 \,|\, \mathcal{E}(\overline{M})^p_t > \mu\}$ for a continuous martingale $\mathcal{E}(\overline{M})$, see also the proof of Lemma 2.5.2. Then, the desired estimate $\mathcal{E}(\overline{M})^p_{\tau_\mu} \le \mu$ is derived, but the latter holds for $\mu \ge 1$ only, since for $\mu \in (0,1)$ we obtain that $\tau_\mu = 0$ which in turn gives $\mathcal{E}(\overline{M})^p_{\tau_\mu} = \mathcal{E}(\overline{M})^p_0 = 1 > \mu$.

Let us now continue our investigation of boundedness of solutions to the BSDE (2.5.1). Setting $k_q := 0$ for $q \in [0,1)$ and using the convention $1/0 := +\infty$, then in view of Theorem 2.3.4, the above Proposition 2.5.4 may be formulated for *all* $q \in (-\infty, 1)$.

The following figure depicts the upper bound of $\|\lambda \cdot M\|_{\mathrm{BMO}_2}$ as a function of p which guarantees that the statements of Proposition 2.5.4 hold. In particular, it indicates that as $p \to 0+$ there are no constraints on $\|\lambda \cdot M\|_{\mathrm{BMO}_2}$. In fact, $\lim_{p\to 0+}(1/\sqrt{k_q}) = +\infty$ which is consistent with Theorem 2.3.4.

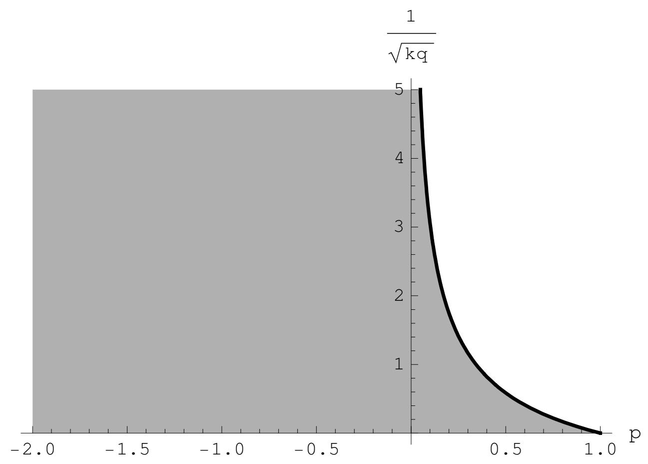

Figure 2.1: Range in p of BMO_2 Norms of $\lambda \cdot M$ for which Proposition 2.5.4 Holds.

Before providing a *sharp sufficient condition* for boundedness of solutions to the BSDE (2.5.1), we investigate the implications of $\lim_{p\to 1-}\big(1/\sqrt{k_q}\big) = 0$. Since $1/\sqrt{k_q}$ is decreasing in p we can give the following corollary.

Corollary 2.5.7. *Fix a $\tilde{p} \in (-\infty, 1)$ with dual exponent $\tilde{q} = \frac{\tilde{p}}{\tilde{p}-1}$. Then choosing*

$$k := k(\tilde{p}) := 1\Big/\sqrt{k_{\tilde{q}}} \in (0, +\infty]$$

the following property holds:
If $\lambda \cdot M$ is a martingale with $\|\lambda \cdot M\|_{BMO_2} < k$ then for all $p \in (-\infty, \tilde{p}]$ *we have:*

(i) The solution pair $(\hat{X}, \hat{Y})$ to the primal and dual optimization problem exists.

(ii) If $\hat{X}$, $\hat{Y}$, $\hat{\Psi}$, $\hat{Z}$ and $\hat{N}$ are as in Theorem 1.3.4 ($\hat{N}$ not necessarily continuous), then the triple $(\hat{\Psi}, \hat{Z}, \hat{N})$ is the unique solution to the BSDE (2.5.1) *with $\hat{\Psi}$* bounded.

Remark 2.5.8. Observe that in Corollary 2.5.7 (ii), for $\tilde{p}$ fixed, $\hat{\Psi}$ is bounded but the bound itself may well depend on $p \in (-\infty, \tilde{p}]$, especially as $p \to -\infty$. In fact, by Nutz [2010a] Theorem 6.6 (ii), we have

$$\lim_{p\to-\infty} \hat{\Psi}_t = \log\big(L_t^{\exp}\big) \leq 0 \ \text{ for all } t \in [0, T],\ \mathbb{P}\text{-a.s.}$$

and this limit is from above. Here $L^{\exp}$ is the opportunity process for the exponential utility maximization problem and is uniformly bounded away from zero if and only if the density process of the minimal entropy martingale measure $\mathbb{Q}^E$ (assumed to exist) satisfies the so-called reversed Hölder inequality $R_{L\log(L)}$, i.e.

$$\mathbb{E}\Big[\big(Y_T^E/Y_\tau^E\big)\log\big(Y_T^E/Y_\tau^E\big)\Big|\,\mathcal{F}_\tau\Big] \leq c_{rH,-\infty}$$

for all stopping times τ valued in $[0, T]$, a positive constant $c_{rH,-\infty}$ and with Y^E the

$\mathbb{P}$-density process of $\mathbb{Q}^E$. If $R_{L\log(L)}$ is satisfied, or we restrict ourselves to $p \in [\overline{p}, \tilde{p}]$ for a fixed $\overline{p} \in (-\infty, \tilde{p}]$, then the bound on $\hat{\Psi}$ is uniform in p and depends on $\tilde{p}$ and either on $c_{rH,-\infty}$ or on $\overline{p}$.

Let us now investigate the feasibility of a classification of the boundedness (property) of solutions to the BSDE (2.5.1) given a BMO assumption. We interpret the BSDE as being parameterized by $p \in (-\infty, 1)$ and λ, where $\lambda \cdot M$ is a BMO martingale. Theorem 2.3.4 shows that if p is restricted to $(-\infty, 0]$, $\hat{\Psi} = \hat{\Psi}(p, \lambda)$ is a bounded process, with the boundedness *property* holding uniformly in (p, λ). Corollary 2.5.7 shows that this is true for $p > 0$ if p is restricted to an interval truncated at 1 and the BMO_2 norm of $\lambda \cdot M$ is small enough.

We show that one cannot extend this property to hold for the whole interval $(-\infty, 1)$ of values for p. This degeneracy is suggested by the observation that $\lim_{\tilde{p}\to 1-} k(\tilde{p}) = 0$ in the above corollary. However, if we drop the assumption that the boundedness property of $\hat{\Psi} = \hat{\Psi}(p, \lambda)$ be uniform in λ, there is an example for which the described extension is indeed possible. More precisely, we have the following result.

Proposition 2.5.9.

(i) *There does* not *exist a finite $k > 0$ with the following property:*
If (under the Assumption 1.2.3 (ii)) $\lambda \cdot M$ is a (local) martingale which satisfies $\|\lambda \cdot M\|_{BMO_2} < k$ then the process Ψ from the unique solution (Ψ, Z, N) to (2.5.1) *with $\mathcal{E}\big([(1-q)Z - q\lambda] \cdot M + N\big)$ a martingale is bounded for* all $p \in (-\infty, 1)$.

(ii) *However, there* does *exist an unbounded BMO martingale $\lambda \cdot M$, with unbounded $\langle \lambda \cdot M\rangle_T$, such that for all $p \in (-\infty, 1)$ the process Ψ from the solution (Ψ, Z, N) to the BSDE* (2.5.1) *with $\mathcal{E}\big([(1-q)Z - q\lambda] \cdot M + N\big)$ a martingale is bounded.*

Proof. For item (i) assume to the contrary that such a k exists and let $M = W$ be a one-dimensional Brownian motion. Set $\overline{\lambda}_t := \frac{\pi\alpha}{2\sqrt{T-t}} \mathbf{1}_{]\!]T/2,\tau]\!]}$, $t \in [0, T]$, where α and τ are as in the proof of Proposition 2.4.1. This defines a bounded, hence a BMO, martingale $\overline{\lambda} \cdot W$. Let us now consider an arbitrary $q < -\frac{\|\overline{\lambda}\cdot W\|^2_{\mathrm{BMO}_2}}{k^2} < 0$ and define $\lambda := \frac{\overline{\lambda}}{\sqrt{-q}}$. Then λ is precisely that of Proposition 2.4.1 for the chosen q and we have that only unbounded solutions to (2.5.1), which coincides with (1.3.1) here, can (and do) exist. However, $\|\lambda \cdot W\|_{\mathrm{BMO}_2} = \frac{\|\overline{\lambda}\cdot W\|_{\mathrm{BMO}_2}}{\sqrt{-q}} < k$ by the choice of $q < 0$, which in turn corresponds to some $p \in (0, 1)$.

For item (ii) we observe that from the estimate (2.5.7) in the proof of Proposition 2.5.4, we are done if we find an example of an unbounded martingale $\lambda \cdot M$ with unbounded $\langle \lambda \cdot M\rangle_T$ and $b(\lambda \cdot M) = +\infty$, where we use the notation of the critical exponent from (2.5.3). An explicit martingale with these properties is provided by Example 3.1 of Schachermayer [1996]. While that example is defined on $\mathbb{R}_+$, we can apply an increasing bijection (e.g. $t \mapsto \rho(t) := \frac{Tt}{1+t}$ from $[0, \infty]$ to $[0, T]$) in order to obtain such a martingale on $[0, T]$. □

Using the specific construction in the above proof the boundedness property of a solution to (2.5.1) as $p \to 1-$ can be illustrated as follows; observe that $p \to 1-$ is equivalent to $q \to -\infty$. Then, to avoid the degeneracy of the above counterexample, for all q it must hold that $q \geq -\frac{\|\overline{\lambda} \cdot W\|^2_{\mathrm{BMO}_2}}{k^2}$ from which it follows that $k \to 0+$. Hence a *fixed* $k > 0$ implying the desired boundedness property of a solution to (2.5.1) cannot exist. The above item (ii) is not against the intuition that the investor becomes risk neutral as $p \to 1-$. It only states that for each p the corresponding $\hat{\Psi}$ is bounded, but it is not uniformly bounded in p.

The results above summarize and show the limitations of a BMO characterization of the boundedness property of the process Ψ from a solution to the BSDE (2.5.1). In particular, if b denotes the critical exponent from (2.5.3), we see from (2.5.2) and the estimate in (2.5.7) that if $\lambda \cdot M$ is a martingale with

$$b(\lambda \cdot M) > k_q$$

then Proposition 2.5.4 continues to hold. Contrary to this result, the specific example of a BMO martingale that does not yield a bounded solution to the BSDE in Subsection 2.4.2 exhibits

$$b(\lambda \cdot M) = -\frac{q}{2} < q^2 - \frac{q}{2} - q\sqrt{q^2 - q} = k_q,$$

recalling that $q < 0$ and where the first equality can be shown using (2.5.11) below. Hence the following questions arise,

- Which boundedness properties do hold for solutions to the BSDE (2.5.1) for those λ with $b(\lambda \cdot M) \in \left(-\frac{q}{2},\, k_q\right)$?
- Can we use the critical exponent b to characterize boundedness of solutions to the BSDE (2.5.1)?

We answer these questions in the next subsection by showing that the bound k_q is indeed the maximal one which guarantees boundedness in the general case and that it cannot be improved. In doing so we provide a full description of the boundedness of solutions to the quadratic BSDE (2.5.1) with $\lambda \cdot M$ a BMO martingale in terms of the critical exponent b.

2.5.2 Boundedness under Dynamic Exponential Moments

We have seen that neither the BMO property of $\lambda \cdot M$ nor an exponential moments condition guarantees the boundedness of a BSDE solution. While a counterexample showed that a simple combination of the two conditions does not suffice, we next see that a dynamic combination provides the required characterization. In particular, while the existence of all exponential moments of the mean-variance tradeoff is sufficient for the existence of a unique solution $(\hat{\Psi}, \hat{Z}, \hat{N})$ to (1.3.1) with $\hat{\Psi} \in \mathfrak{E}$, the existence of all *dynamic* exponential moments is sufficient for the existence of a unique solution with $\hat{\Psi}$ bounded, and in general this requirement cannot be dropped. We recall that by

Corollary 2.5.3 any requirement on the dynamic exponential moments may be written in terms of a condition on the critical exponent b.

Theorem 2.5.10. *Fix $p \in (0,1)$, i.e. $q < 0$, and define k_q as in* (2.5.6). *Then,*

(i) If $\lambda \cdot M$ is a martingale with $b(\lambda \cdot M) > k_q$ then the solution pair $(\hat{X}, \hat{Y})$ to the primal and dual problem exists and if $\hat{\Psi}$, $\hat{Z}$ and $\hat{N}$ are as in Theorem 1.3.4 ($\hat{N}$ not necessarily continuous), then the triple $(\hat{\Psi}, \hat{Z}, \hat{N})$ is the unique solution to the BSDE (2.5.1) *with $\hat{\Psi}$* bounded.

(ii) For a one-dimensional Brownian motion $M = W$ and every $k < k_q$, there exists a BMO martingale $\lambda \cdot M$ with $b(\lambda \cdot M) > k$ such that the solutions to the primal and dual problem exist and the corresponding triple $(\hat{\Psi}, \hat{Z}, \hat{N} \equiv 0)$ is a solution to the BSDE (2.5.1) *with $\hat{\Psi}$* unbounded.

(iii) For a one-dimensional Brownian motion $M = W$, there exists a BMO martingale $\lambda \cdot M$ with $b(\lambda \cdot M) = k_q$ such that the solutions to the primal and dual problem exist and the corresponding triple $(\hat{\Psi}, \hat{Z}, \hat{N} \equiv 0)$ is the unique solution to the BSDE (2.5.1) *with $\hat{\Psi}$* bounded.

We can summarize this result as follows: Item (i) gives a sufficient condition for boundedness of solutions to the BSDE (2.5.1) in terms of dynamic exponential moments, which is less restrictive than a bound on the BMO_2 norm. Item (ii) shows that this condition is sharp in the sense that it cannot be improved. In particular, the critical exponent b from (2.5.3) characterizes the boundedness property of solutions to the BSDE (2.5.1) that stem from the utility maximization problem. Item (iii) gives information about the critical point k_q of the interval $(k_q, +\infty)$. It yields that the converse of item (i) does not hold.

The following Figure 2.2 provides a visualization of this discussion, it depicts the value k_q as a function of p. Let us now discuss it briefly, fix $p \in (0,1)$ and assume that we are on the critical black line, i.e. we have a specific $\lambda \cdot M$ with $b(\lambda \cdot M) > k_q$. Note that the black line is included in the area that ensures boundedness because a finite dynamic exponential moment of order k_q is equivalent to $b(\lambda \cdot M) > k_q$ by Corollary 2.5.3. Now choosing $\tilde{q} < 0$ such that $b(\lambda \cdot M) > k_{\tilde{q}} > k_q$ we can derive the statement of Theorem 2.5.10 (i) for the corresponding $\tilde{p} > p$. However, $\tilde{p}$ depends on the specific choice of λ and therefore it is not possible to shift the whole black line uniformly for all processes λ.

A main difference between Figure 2.1 and Figure 2.2 (apart from the fact that Figure 2.1 depicts only a sufficient condition) is that in the former the boundary line is not included in the area that ensures boundedness. For Figure 2.1, we fix again $p \in (0,1)$ and consider all $\lambda \cdot M$ whose BMO_2 norm is bounded by some $c < 1/\sqrt{k_q}$. In this case, the boundedness result holds for some $\tilde{p} > p$ (depending on c), uniformly for all λ. Hence the line determined by c can be moved slightly to the right which is in line with the fact that the grey area in Figure 2.1 is open.

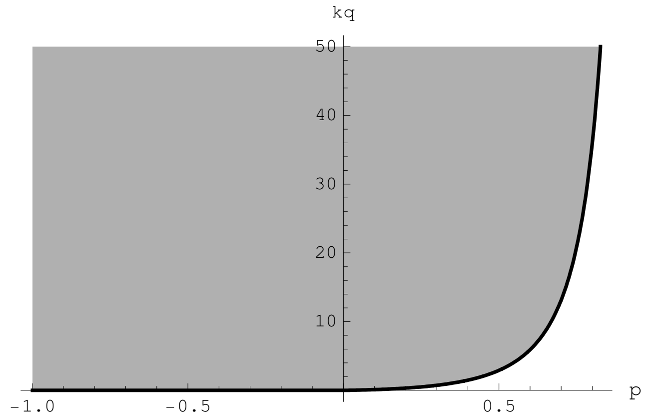

Figure 2.2: Dynamic Exponential Moments of $\langle\lambda\cdot M\rangle$ Sharply Sufficient for the Boundedness of $\hat{\Psi}$.

Items (ii) and (iii) of the above theorem rely on the construction of a specific example which we first provide in the following auxiliary lemma.

Lemma 2.5.11. *Let W be a one-dimensional Brownian motion. Then, for every $b\in\mathbb{R}$, there exists a predictable process $\widetilde{\lambda}$ such that $\widetilde{\lambda}\cdot W$ is a BMO martingale and*

$$\sup_{\substack{\tau\text{ stopping time}\\ \text{valued in }[0,T]}}\left\|\mathbb{E}_{\widetilde{\mathbb{P}}}\left[\exp\left(c^2\int_\tau^T\widetilde{\lambda}_t^2\,dt\right)\middle|\,\mathcal{F}_\tau\right]\right\|_{L^\infty}\begin{cases}<+\infty & \text{if } |c|<1,\\ =+\infty & \text{if } |c|\geq 1.\end{cases}\tag{2.5.9}$$

where $\widetilde{\mathbb{P}}$ is the probability measure given by $\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}:=\mathcal{E}\big(-b\widetilde{\lambda}\cdot W\big)_T$.

Proof. We proceed similarly to the example from Subsection 2.4.2 and define for $t\in[0,T]$,

$$\widetilde{\lambda}_t:=\frac{\pi\alpha}{\sqrt{8(T-t)}}\mathbf{1}_{\rrbracket T/2,\widetilde{\tau}\rrbracket}(t,\cdot),\tag{2.5.10}$$

where α is as in the proof of Proposition 2.4.1 and where $\widetilde{\tau}$ is now the stopping time

$$\widetilde{\tau}:=\inf\left\{t>\frac{T}{2}\,\middle|\,\left|\int_{T/2}^t\frac{1}{\sqrt{T-s}}\left(dW_s+\frac{b\pi\alpha}{\sqrt{8(T-s)}}\,ds\right)\right|\geq 1\right\},$$

for which again $\mathbb{P}(T/2<\widetilde{\tau}<T)=1$. Then $\int_0^\cdot\widetilde{\lambda}_t\big(dW_t+b\widetilde{\lambda}_t\,dt\big)$ is bounded by $\frac{\pi}{\sqrt{8}}$. If $b<0$ we derive from

$$\widetilde{\lambda}\cdot W=\int_0^\cdot\widetilde{\lambda}_t\big(dW_t+b\widetilde{\lambda}_t\,dt\big)-b\int_0^\cdot\widetilde{\lambda}_t^2\,dt\geq-\frac{\pi}{\sqrt{8}}$$

that the continuous local martingale $\widetilde{\lambda}\cdot W$ is bounded from below, hence a supermartingale. It then follows from the Optional Sampling Theorem, see Karatzas and Shreve

[1991] Theorem 1.3.22, that for any stopping time τ valued in $[0,T]$,

$$\mathbb{E}\bigg[\int_\tau^T \widetilde{\lambda}_t^2\,dt\,\bigg|\,\mathcal{F}_\tau\bigg] = \frac{1}{b}\,\mathbb{E}\bigg[\int_\tau^T \widetilde{\lambda}_t(dW_t + b\widetilde{\lambda}_t\,dt)\,\bigg|\,\mathcal{F}_\tau\bigg] - \frac{1}{b}\,\mathbb{E}\bigg[\int_\tau^T \widetilde{\lambda}_t\,dW_t\,\bigg|\,\mathcal{F}_\tau\bigg] \le \frac{\pi}{\sqrt{2}\,|b|}.$$

In particular, $\widetilde{\lambda}\cdot W$ is a BMO martingale. A similar reasoning applies if $b > 0$ and the claim is immediate for $b = 0$. We hence may consider the measure $\widetilde{\mathbb{P}}$ given by $\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} := \mathcal{E}\big(-b\widetilde{\lambda}\cdot W\big)_T$ under which $\widetilde{W} := W + b\int_0^\cdot \widetilde{\lambda}_t\,dt$ is a Brownian motion. Now, for a stopping time τ valued in $[T/2,T]$ and for $u \in \mathbb{R}$ and $v \in [0,T]$, we set

$$\begin{aligned}\widetilde{\tau}_{u,v}(\tau) &:= v + \inf\bigg\{t \ge 0\,\bigg|\,\bigg|u + \int_0^t \frac{1}{\sqrt{T-s-v}}\,d\widetilde{W}_{\tau+s}\bigg| \ge 1\bigg\}\\ &= \inf\bigg\{t \ge v\,\bigg|\,\bigg|u + \int_v^t \frac{1}{\sqrt{T-s}}\,d\widetilde{W}_{\tau+s-v}\bigg| \ge 1\bigg\},\end{aligned}$$

where we extend the $\widetilde{\mathbb{P}}$-Brownian motion $\widetilde{W}$ to $[0,2T]$.

Let $|c| < 1$. Since $\widetilde{\lambda}$ vanishes on $[0,T/2]$ and $\exp\big(c^2\int_\tau^T \widetilde{\lambda}_t^2\,dt\big) = 1$ on $\{\tau = T\}$, for the first assertion of (2.5.9), it is enough to consider stopping times τ valued in $(T/2,T)$. Then, using the $\mathcal{F}_\tau$-measurable random variable $U := \int_{T/2}^\tau \frac{1}{\sqrt{T-s}}\,d\widetilde{W}_s$ we have that $\widetilde{\tau} \le \widetilde{\tau}_{U,\tau}(\tau)$ $\mathbb{P}$-a.s. Moreover, $\widetilde{\tau}_{u,v}(\tau)$ is $\widetilde{\mathbb{P}}$-independent of $\mathcal{F}_\tau$ since it is $\sigma\big(\widetilde{W}_{\tau+s} - \widetilde{W}_\tau, s \ge 0\big)$-measurable. We thus obtain

$$\begin{aligned}\mathbb{E}_{\widetilde{\mathbb{P}}}\bigg[\exp\bigg(c^2\int_\tau^T \widetilde{\lambda}_t^2\,dt\bigg)\bigg|\,\mathcal{F}_\tau\bigg] &\le \mathbb{E}_{\widetilde{\mathbb{P}}}\bigg[\exp\bigg(\frac{c^2\pi^2}{8}\int_\tau^{\widetilde{\tau}_{U,\tau}(\tau)} \frac{1}{T-t}\,dt\bigg)\bigg|\,\mathcal{F}_\tau\bigg] \qquad (2.5.11)\\ &= \mathbb{E}_{\widetilde{\mathbb{P}}}\bigg[\exp\bigg(\frac{c^2\pi^2}{8}\int_v^{\widetilde{\tau}_{u,v}(\tau)} \frac{1}{T-t}\,dt\bigg)\bigg]\bigg|_{u=U,v=\tau}\\ &= \mathbf{1}_{\{|U|\ge 1\}} + \frac{\cos(c\pi U/2)}{\cos(c\pi/2)}\,\mathbf{1}_{\{|U|<1\}} \le \frac{1}{\cos(c\pi/2)} < +\infty,\end{aligned}$$

where we applied Kazamaki [1994] Lemma 1.3 in a similar way as in the proof of Frei and dos Reis [2011] Lemma A.1 and used that $\widetilde{\tau}_{u,v}(\tau)$ and $\widetilde{\tau}_{u,v}(0)$ have the same distribution under $\widetilde{\mathbb{P}}$. This gives an upper bound for (2.5.9) in the case $|c| < 1$.

If $|c| \ge 1$, we note that from $\widetilde{\tau} = \widetilde{\tau}_{0,T/2}(T/2)$ $\mathbb{P}$-a.s. and the definition of α,

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\bigg[\exp\bigg(c^2\int_{T/2}^T \widetilde{\lambda}_t^2\,dt\bigg)\bigg|\,\mathcal{F}_{T/2}\bigg] \ge \mathbb{E}_{\widetilde{\mathbb{P}}}\bigg[\exp\bigg(\frac{\pi^2\alpha^2}{8}\int_{T/2}^{\widetilde{\tau}} \frac{1}{T-t}\,dt\bigg)\bigg|\,\mathcal{F}_{T/2}\bigg] = \frac{1}{\cos(\pi\alpha/2)}, \qquad (2.5.12)$$

which is *unbounded* and this concludes the proof of Lemma 2.5.11. □

We are now ready to provide the proof of Theorem 2.5.10.

Proof of Theorem 2.5.10. Item (i) follows from the proof of Proposition 2.5.4, see the estimate in (2.5.7) and recall Corollary 2.5.3.

For item (ii) observe that since

$$k < k_q := q^2 - \frac{q}{2} - q\sqrt{q^2 - q},$$

there exists an $a > 0$ such that

$$k < q^2 - \frac{q}{2} - q\sqrt{q^2 - q - 2a^2}. \tag{2.5.13}$$

Choose such an a and then set $b := \frac{1}{a}\,(q - \sqrt{q^2 - q - 2a^2}) < \frac{q}{a} < 0$. We mention that the need for two parameters a and b stems from the fact that we have two conditions which must both be satisfied, the first concerns the finiteness of exponential moments and the second relates to the (un)boundedness of $\hat{\Psi}$. We then define $\widetilde{\lambda}$ and $\widetilde{\mathbb{P}}$ as in Lemma 2.5.11 and observe that contrary to the previous examples the measure change is now part of the construction. Finally, we set $\lambda := \frac{1}{a}\widetilde{\lambda}$ and deduce for $t \in [0, T]$ that,

$$\begin{aligned}\mathbb{E}\big[\mathcal{E}(-\lambda \cdot W)_{t,T}^q \,\big|\, \mathcal{F}_t\big] &= \mathbb{E}_{\widetilde{\mathbb{P}}}\Big[\exp\Big(\Big(b - \frac{q}{a}\Big)\int_t^T \widetilde{\lambda}_s\, d\widetilde{W}_s + \Big(\frac{qb}{a} - \frac{q}{2a^2} - \frac{b^2}{2}\Big)\int_t^T \widetilde{\lambda}_s^2\, ds\Big)\,\Big|\, \mathcal{F}_t\Big] \\ &\begin{cases} \le e^{\frac{(q/a-b)\pi}{\sqrt{2}}}\,\mathbb{E}_{\widetilde{\mathbb{P}}}\Big[\exp\Big(\int_t^T \widetilde{\lambda}_s^2\, ds\Big)\Big|\, \mathcal{F}_t\Big], \\ \ge e^{\frac{(b-q/a)\pi}{\sqrt{2}}}\,\mathbb{E}_{\widetilde{\mathbb{P}}}\Big[\exp\Big(\int_t^T \widetilde{\lambda}_s^2\, ds\Big)\Big|\, \mathcal{F}_t\Big], \end{cases}\end{aligned}$$

where we used the boundedness of $\widetilde{\lambda} \cdot \widetilde{W}$ and $\frac{qb}{a} - \frac{q}{2a^2} - \frac{b^2}{2} = 1$, together with $b < q/a$. By the inequality (2.5.12), this shows that $\mathbb{E}\big[\mathcal{E}(-\lambda \cdot W)_{T/2,T}^q \,\big|\, \mathcal{F}_{T/2}\big]$ is unbounded, whereas $\mathbb{E}\big[\mathcal{E}(-\lambda \cdot W)_T^q\big] < +\infty$ since $\mathbb{E}_{\widetilde{\mathbb{P}}}\big[\exp(\int_0^T \widetilde{\lambda}_t^2\, dt)\big] = 2$, see the proof of Proposition 2.4.1. Proposition 2.2.1 now yields the existence of a solution $(\hat{\Psi}, \hat{Z}, \hat{N} \equiv 0)$ and the identification with the primal and dual problems. The conclusion is that $\hat{\Psi}$ is unbounded. Moreover, using the boundedness of $\widetilde{\lambda} \cdot \widetilde{W}$ again, we have

$$\sup_\tau \Big\|\mathbb{E}\Big[\exp\Big(k\int_\tau^T \lambda_t^2\, dt\Big)\Big|\, \mathcal{F}_\tau\Big]\Big\|_{L^\infty} \le e^{\frac{|b|\pi}{\sqrt{2}}} \sup_\tau \Big\|\mathbb{E}_{\widetilde{\mathbb{P}}}\Big[\exp\Big(\int_\tau^T \Big(\frac{k}{a^2} - \frac{b^2}{2}\Big)\widetilde{\lambda}_t^2\, dt\Big)\Big|\, \mathcal{F}_\tau\Big]\Big\|_{L^\infty}.$$

This is finite by (2.5.9) since the relation $\frac{k}{a^2} - \frac{b^2}{2} < 1$ is equivalent to

$$k < a^2 + \frac{a^2 b^2}{2} = qab - \frac{q}{2} = q^2 - \frac{q}{2} - q\sqrt{q^2 - q - 2a^2},$$

which is inequality (2.5.13).

The proof of item (iii) is similar to that of item (ii). We use the same definitions subject to the modification that now we must choose $a > 0$ and $b \in \mathbb{R}$ such that

$$\frac{qb}{a} - \frac{q}{2a^2} - \frac{b^2}{2} < 1 \quad \text{and} \quad \frac{k_q}{a^2} - \frac{b^2}{2} = 1.$$

This choice ensures the existence of the optimizers and guarantees the boundedness of $\hat{\Psi}$, again thanks to Proposition 2.2.1 and (2.5.9). Note that now a dynamic exponential moment of order k_q will not exist.

The above equation is satisfied for $b := \sqrt{\frac{2k_q}{a^2} - 2} > 0$ if $a^2 < k_q$, and then the inequality reads as $\frac{q}{a}\sqrt{\frac{2k_q}{a^2} - 2} - \frac{q}{2a^2} - \frac{k_q}{a^2} < 0$. This last relation holds for any choice of $a \in \big(0, \sqrt{k_q}\,\big)$ since we have $k_q > -\frac{q}{2} > 0$. □

A consequence of Theorem 2.5.10 is the following result.

Corollary 2.5.12.

(i) If $\lambda \cdot M$ is a martingale that satisfies $b(\lambda \cdot M) = +\infty$, then for all $p \in (0,1)$ the solution pair $(\hat{X}, \hat{Y})$ to the primal and dual problem exists. If $\hat{\Psi}$, $\hat{Z}$ and $\hat{N}$ are as in Theorem 1.3.4 ($\hat{N}$ not necessarily continuous), then the triple $(\hat{\Psi}, \hat{Z}, \hat{N})$ is the unique solution to the BSDE (2.5.1) *with* $\hat{\Psi}$ bounded.

(ii) The converse statement, however, is not true. More precisely, if $\lambda \cdot M$ is a BMO martingale such that for all $p \in (0,1)$ the solutions to the primal and dual problem exist with $\hat{\Psi}$ bounded, the critical exponent need not *satisfy $b(\lambda \cdot M) = +\infty$.*

Proof. The first part is an immediate consequence of item (i) of Theorem 2.5.10. For the second part, we proceed similarly to the proof of item (ii) of the Theorem 2.5.10. Taking a one-dimensional Brownian motion $M = W$, we define λ via (2.5.10) with $b = 1/2$ and $\lambda = \widetilde{\lambda}$. By construction, $\int_0^{\cdot} \lambda_t (dW_t + \frac{\lambda_t}{2}\, dt)$ is bounded by $\frac{\pi}{\sqrt{8}}$ so that for $q < 0$

$$\sup_{\substack{\tau \text{ stopping time} \\ \text{valued in } [0,T]}} \left\| \mathbb{E}\left[\exp\left(-q \int_\tau^T \lambda_t \, dW_t - \frac{q}{2} \int_\tau^T \lambda_t^2 \, dt \right) \middle| \mathcal{F}_\tau \right] \right\|_{L^\infty} \leq e^{\frac{-q\pi}{\sqrt{2}}} < +\infty.$$

Hence, for all $p \in (0,1)$, the solutions to the primal and dual problem exist and the corresponding triple $(\hat{\Psi}, \hat{Z}, \hat{N} \equiv 0)$ is the unique solution to the BSDE (2.5.1) with $\hat{\Psi}$ bounded. For the estimate on the process $\langle \lambda \cdot W \rangle$ we have

$$\sup_\tau \left\| \mathbb{E}\left[\exp\left(k \int_\tau^T \lambda_t^2 \, dt \right) \middle| \mathcal{F}_\tau \right] \right\|_{L^\infty} \geq e^{\frac{-\pi}{\sqrt{8}}} \sup_\tau \left\| \mathbb{E}_{\widetilde{\mathbb{P}}}\left[\exp\left(\left(k - \frac{1}{8}\right) \int_\tau^T \lambda_t^2 \, dt \right) \middle| \mathcal{F}_\tau \right] \right\|_{L^\infty}.$$

The right hand side is $+\infty$ when $k - \frac{1}{8} \geq 1$ by (2.5.9), this implies $b(\lambda \cdot W) \leq \frac{9}{8} < +\infty$ (actually, $b(\lambda \cdot W) = \frac{9}{8}$) despite the fact that $\hat{\Psi}$ is bounded for arbitrary $p \in (0,1)$. □

Remark 2.5.13. Corollary 6.7 is based on the fact that $b(\lambda \cdot M) = +\infty$ is stronger than requiring that $\mathcal{E}(-\lambda \cdot M)$ to satisfy the reverse Hölder inequality R_q for all $q < 0$. However, there exists an equivalence between $b(\lambda \cdot M) = +\infty$ and a *strengthened* reverse Hölder condition. It follows from Theorem 4.2 of Delbaen and Tang [2010] that $b(\lambda \cdot M) = +\infty$ holds if and only if for some (or equivalently, all) $\varrho \in [1, +\infty)$ and all

$a \in \mathbb{C}$ there exists $c_{\varrho,a} > 0$ such that

$$\mathbb{E}\left[\left|\frac{\mathcal{E}(a\lambda \cdot M)_\sigma}{\mathcal{E}(a\lambda \cdot M)_\tau}\right|^\varrho \middle| \mathcal{F}_\tau\right] \leq c_{\varrho,a}$$

for all stopping times $\tau \leq \sigma$ valued in $[0, T]$.

3 Quadratic Semimartingale BSDEs under an Exponential Moments Condition

3.1 Introduction

In the present chapter we provide all the background material that is needed for a thorough study of the utility maximization problem under exponential moments via BSDE methods. We recall that in a (one-dimensional) Brownian framework BSDEs are usually written

$$d\Psi_t = Z_t \, dW_t - F(t, \Psi_t, Z_t) \, dt, \quad \Psi_T = \xi, \tag{3.1.1}$$

where ξ is an $\mathcal{F}_T$-measurable random variable, the *terminal value*, and F is the so-called *driver* or *generator*. Solving such an equation consists of finding a *pair* of adapted processes (Ψ, Z) such that the integrated version of (3.1.1) holds. The presence of the *control process* Z stems from the requirement of adaptedness for Ψ together with the fact that Ψ must be driven into the random variable ξ at time T. One may think of Z as arising from the martingale representation theorem, see the introduction for more on this topic.

In the semimartingale framework where the main source of randomness is encoded in a given (continuous) local martingale M on a filtration $(\mathcal{F}_t)_{t\in[0,T]}$ that is not necessarily generated by M, we have to add an extra orthogonal component N. The corresponding BSDE then takes the form

$$d\Psi_t = Z_t \, dM_t + dN_t - f(t, \Psi_t, Z_t) \, d\langle M\rangle_t - g_t \, d\langle N\rangle_t, \quad \Psi_T = \xi, \tag{3.1.2}$$

and solving (3.1.2) now corresponds to finding an adapted *triple* (Ψ, Z, N) of processes that satisfies the integrated version of (3.1.2), where N is a (continuous) local martingale orthogonal to M.

As we already know, BSDEs of type (3.1.1) and (3.1.2) have found many fields of application in mathematical finance and the reader is directed to El Karoui et al. [1997] for a first survey. Moreover, we find a BSDE derived in Hu et al. [2005] for the value process of the utility maximization problem, being in line with work by Rouge and El Karoui [2000] as well as Sekine [2006]. In Mania and Schweizer [2005] the authors used a BSDE to describe the dynamic indifference price for exponential utility and their approach was extended to robust utility in Bordigoni et al. [2007] and to an infinite time horizon in Hu and Schweizer [2009]. We also mention Becherer [2006] for further extensions to BSDEs with jumps and Mania and Tevzadze [2008] to backward stochastic partial differential equations.

With regards to the theory of BSDEs, existence and uniqueness results were first provided in a Brownian setting by Pardoux and Peng [1990] under Lipschitz conditions. They were extended by Lepeltier and San Martín [1997] to continuous drivers with linear growth and by Kobylanski [2000] to generators which are quadratic as a function of the control variable Z. Corresponding results for the semimartingale case may be found in Morlais [2009] and Tevzadze [2008]. In addition some stability results for quadratic BSDEs are also found in the recent articles by Frei [2009] and Barrieu and El Karoui [2011]. In the situation when the generator has superquadratic growth, Delbaen et al. [2010] show that such BSDEs are essentially ill-posed.

A strong requirement present in the articles Kobylanski [2000], Morlais [2009] and Tevzadze [2008] is that the terminal condition be bounded. In a Brownian setting Briand and Hu [2006, 2008] have replaced this by the assumption that it need only have exponential moments but in addition the driver is convex in the Z variable. More recently, Delbaen et al. [2011] show that one can reduce the order of exponential moments required.

The present chapter has two main contributions, the first is to extend the existence, uniqueness and stability theorems of Briand and Hu [2008] and Morlais [2009] to the unbounded continuous semimartingale case. The motivation is that having results in greater generality increases the range of possible applications for BSDEs. The main practical application for the results derived in this chapter is the utility maximization problem with an unbounded mean-variance tradeoff though. This provides a second, if not the more important, motivation for the present work.

In order to prove the respective results in the unbounded semimartingale framework technical difficulties related to an a priori estimate must be overcome. This requires an additional assumption when compared to Briand and Hu [2008] and Morlais [2009]. As a byproduct of establishing our results we are able to show via an example that the stability theorem as stated in Briand and Hu [2008] Proposition 7 needs a minor amendment to the mode of convergence assumed on the drivers and we include the appropriate formulation.

Our second contribution is to address the question of measure change. We know from Lemma 2.3.1 that when the generator has quadratic growth in Z then the solution processes Ψ is bounded if and only if the martingale part $Z\cdot M+N$ is a BMO martingale. In the present setting, where Ψ is assumed to satisfy an exponential moments condition only, such a correspondence is lost. However, we are able to show that whilst $Z\cdot M+N$ need not be a BMO martingale, see Chapter 2 for further discussion and some examples, the stochastic exponential $\mathcal{E}\big(\varrho\,(Z\cdot M+N)\big)$ is still a true martingale for $|\varrho|$ large enough. It is thus not only mathematically interesting to be able to describe the properties of the martingale part of the BSDE but also relevant for applications. For instance, the above result can be used to extend the BSDE approaches of Hu et al. [2005] and Morlais [2009] to utility maximization, see Chapter 5 for further details as well as Heyne [2010] for some explicit stochastic volatility models. Moreover such a theorem may be used in the partial equilibrium framework of Horst et al. [2010] where the market price of external risk is given by equilibrium considerations and is typically unbounded.

The chapter is based on the working paper Mocha and Westray [2011a] and organized as follows. In the next section we lay out the notation and the assumptions and state the main results. The subsequent sections contain the proofs. Section 3.3 gives the a priori estimates together with some remarks on the necessity of an additional assumption, Section 3.4 deals with existence and Section 3.5 includes the comparison and uniqueness results. In Section 3.6 we prove the stability property as well as providing an appropriate counterexample. In Section 3.7, we turn our attention to the measure change problem and finally, in Section 3.8, we point at interesting applications of our results to constrained utility maximization and partial equilibrium models. The constrained portfolio choice problem will then be the main concern of the Chapters 4 and 5.

3.2 Framework and Statement of Results

We work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ satisfying the usual conditions of right-continuity and completeness. We also assume that $\mathcal{F}_0$ is the completion of the trivial σ-algebra. The time horizon T is a finite number in $(0,\infty)$ and all semimartingales are considered equal to their càdlàg modification.

Notation and Problem Formulation

Throughout this chapter $M = (M^1, \dots, M^d)^\intercal$ stands for a continuous d-dimensional local martingale, where $^\intercal$ denotes transposition. We refer the reader to Jacod and Shiryaev [2003] and Protter [2005] for further details on the general theory of stochastic integration.

The objects of study in the present chapter will be continuous semimartingale BSDEs considered on $[0,T]$. In the d-dimensional case such a BSDE may be written

$$d\Psi_t = Z_t^\intercal\, dM_t + dN_t - \mathbf{1}^\intercal\, d\langle M\rangle_t f(t,\Psi_t,Z_t) - g_t\, d\langle N\rangle_t, \quad \Psi_T = \xi. \tag{3.2.1}$$

Here ξ is an $\mathbb{R}$-valued $\mathcal{F}_T$-measurable random variable and f and g are random predictable functions $[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^d$ and $[0,T]\times\Omega\to\mathbb{R}$, respectively. We set $\mathbf{1} := (1,\dots,1)^\intercal \in \mathbb{R}^d$. Moreover, N is a continuous local martingale that is orthogonal to M, i.e. $\langle M^i, N\rangle \equiv 0$ for all $i = 1,\dots,d$.

The format in which the BSDE (3.2.1) encodes its finite variation parts is not so tractable from the point of view of analysis. Therefore we write semimartingale BSDEs by factorizing the matrix-valued process $\langle M\rangle = \langle M^i, M^j\rangle_{i,j=1,\dots,d}$. This separates its matrix property from its nature as measure.

For $i,j \in \{1,\dots,d\}$ we may write $\langle M^i, M^j\rangle = C^{ij}\cdot A$ where C^{ij} are the components of a predictable process C valued in the space of symmetric positive semidefinite $d\times d$ matrices and A is a predictable increasing process. There are many such factorizations (see Jacod and Shiryaev [2003] Section III.4a). We may choose $A := \arctan\Big(\sum_{i=1}^d \langle M^i\rangle\Big)$ so that A is uniformly bounded by $K_A = \pi/2$ and derive the absolute continuity of all the $\langle M^i, M^j\rangle$ with respect to A from the Kunita-Watanabe inequality. This together with the Radon-Nikodým theorem provides C. Furthermore, we can factorize C as $C = B^\intercal B$ for a predictable process B valued in the space of $d\times d$ matrices. We note that all the results below do not rely on the specific choice of A, but only on its *boundedness*.

In particular, if $M = W$ is a d-dimensional Brownian motion we may choose $A_t = t$, $t \in [0,T]$, and B the identity matrix. Then A is bounded by $K_A = T$.

We recall that $\mathcal{P}$ denotes the predictable σ-algebra on $[0,T] \times \Omega$ and that μ^A stands for the *Doléans measure*, defined by

$$\mu^A(E) := \mathbb{E}\left[\int_0^T \mathbf{1}_E(t)\, dA_t\right], \quad E \in \mathcal{P}.$$

Given the above discussion the equation (3.2.1) may be rewritten as

$$d\Psi_t = Z_t^\intercal\, dM_t + dN_t - F(t, \Psi_t, Z_t)\, dA_t - g_t\, d\langle N\rangle_t, \quad \Psi_T = \xi, \tag{3.2.2}$$

where again ξ is an $\mathbb{R}$-valued $\mathcal{F}_T$-measurable random variable, the *terminal condition*, and F and g are random predictable functions $[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ and $[0,T]\times\Omega \to \mathbb{R}$ respectively, called *generators* or *drivers*. This formulation of the BSDE is very flexible, allowing for various applications and being amenable to analysis. Starting with (3.2.1) and setting $F(t,\psi,z) := \mathbf{1}^\intercal C_t f(t,\psi,z) = \mathbf{1}^\intercal B_t^\intercal B_t f(t,\psi,z)$ we get (3.2.2). A reversion of this procedure is not relevant in our applications as we will see.

Under boundedness assumptions, existence of solutions to (3.2.2) is provided in Morlais [2009] via an exponential transformation that makes the $d\langle N\rangle$ term disappear. A necessary condition for this kind of transformation to work properly is $dg = 0$. In the sequel we thus consider the above BSDE to be given in the form

$$d\Psi_t = Z_t^\intercal\, dM_t + dN_t - F(t, \Psi_t, Z_t)\, dA_t - \frac{1}{2}\, d\langle N\rangle_t, \quad \Psi_T = \xi, \tag{3.2.3}$$

except in specific situations where a solution is assumed to exist.

Definition 3.2.1. A *solution* to the BSDE (3.2.2), or (3.2.3), is a triple (Ψ, Z, N) of processes valued in $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$ satisfying (3.2.2), or (3.2.3), $\mathbb{P}$-a.s. such that:

(i) The function $t \mapsto \Psi_t$ is continuous $\mathbb{P}$-a.s.

(ii) The process Z is predictable and M-integrable, in particular $\int_0^T Z_t^\intercal\, d\langle M\rangle_t Z_t < +\infty$ $\mathbb{P}$-a.s.

(iii) The local martingale N is continuous and orthogonal to each component of M, i.e. $\langle M^i, N\rangle \equiv 0$ for all $i = 1, \dots, d$.

(iv) We have that $\mathbb{P}$-a.s.

$$\int_0^T |F(t, \Psi_t, Z_t)|\, dA_t + \langle N\rangle_T < +\infty.$$

As in the introduction we call $Z \cdot M + N$ the *martingale part* of a solution.

Our goal is now to address the questions under which assumptions the BSDE (3.2.3) allows for a solution, under which conditions and in which spaces uniqueness and stability of solutions to (3.2.3) hold and finally to derive the so-called measure change property of appropriate solutions to (3.2.3).

The Model Assumptions

In what follows we collect together the assumptions that allow for all the assertions of this chapter to hold simultaneously. However we want to point out that not all of our results require that every item of Assumption 3.2.2 be satisfied, as will be indicated in appropriate remarks.

Assumption 3.2.2. *There exist nonnegative constants β and $\overline{\beta}$, positive numbers β_f and $\gamma \geq \max(1, \beta)$ together with a predictable M-integrable $\mathbb{R}^d$-valued process λ so that writing*

$$\alpha := \|B\lambda\|^2 \text{ and } |\alpha|_1 := \int_0^T \alpha_t \, dA_t = \int_0^T \lambda_t^T \, d\langle M\rangle_t \lambda_t$$

we have $\mathbb{P}$-a.s.

(i) The random variable $|\xi| + |\alpha|_1$ has exponential moments of all orders, i.e. for all $\varrho > 1$

$$\mathbb{E}\Big[\exp\Big(\varrho\,[|\xi| + |\alpha|_1]\Big)\Big] < +\infty. \tag{3.2.4}$$

(ii) For all $t \in [0, T]$ the driver $(\psi, z) \mapsto F(t, \psi, z)$ is continuous in (ψ, z), convex in z and Lipschitz continuous in ψ with Lipschitz constant $\overline{\beta}$, i.e. for all ψ_1, ψ_2 and z we have

$$|F(t, \psi_1, z) - F(t, \psi_2, z)| \leq \overline{\beta}\,|\psi_1 - \psi_2|. \tag{3.2.5}$$

(iii) The generator F satisfies a quadratic growth condition in z, i.e. for all t, ψ and z we have

$$|F(t, \psi, z)| \leq \alpha_t + \alpha_t\beta|\psi| + \frac{\gamma}{2}\|B_t z\|^2. \tag{3.2.6}$$

(iv) The function F is locally Lipschitz in z, i.e. for all t, ψ, z_1 and z_2

$$|F(t, \psi, z_1) - F(t, \psi, z_2)| \leq \beta_f\Big(\|B_t\lambda_t\| + \|B_t z_1\| + \|B_t z_2\|\Big)\|B_t(z_1 - z_2)\|.$$

(v) The constant β in (iii) equals zero and then we set $c_A := 0$. Alternatively, $\beta > 0$, but additionally assume that for all t, ψ and z we have

$$|F(t, \psi, z) - F(t, 0, z)| \leq \overline{\beta}\,|\psi| \quad \text{and} \quad A_t \leq c_A \cdot t$$

for a positive constant c_A.

If this assumption is satisfied we refer to (3.2.3) *as BSDE(F, ξ) with the set of parameters $(\alpha, \beta, \overline{\beta}, \beta_f, \gamma)$.*

Remark 3.2.3. The above items (i)-(iv) correspond to the assumptions made in Briand and Hu [2008] and Morlais [2009]. In particular, the BSDEs under consideration are of *quadratic type* (in the control variable z) and of Lipschitz type in ψ. Item (v) is new and arises from the fact that the methods used in Morlais [2009] to derive an a priori estimate may no longer be directly applied so that an additional assumption is required. We elaborate further on this topic in Section 3.3. Observe that in the key application of power utility maximization the associated driver is independent of ψ and hence $\beta = 0$ applies. In particular, this shows that we can indeed allow for more general models; in contrast to Hu et al. [2005] and Morlais [2009] we need not assume that the mean-variance tradeoff of the underlying market be bounded.

Notice that items (ii) and (iii) from above provide

$$|F(t,\psi,z)| \le \alpha_t + \overline{\beta}|\psi| + \frac{\gamma}{2}\,\|B_t z\|^2, \tag{3.2.7}$$

for all t, ψ and z, $\mathbb{P}$-a.s. This is an inequality which does not involve α in the $|\psi|$ term on the right hand side and which is used repeatedly throughout the proofs. We also define the constant

$$\beta^* := c_A \cdot \overline{\beta}. \tag{3.2.8}$$

Statement of the Main Results

Before giving the main results of the chapter let us introduce some notation. For $\varrho > 0$, $\mathcal{S}^\varrho$ denotes the set of $\mathbb{R}$-valued, adapted and continuous processes Υ on $[0,T]$ such that

$$\mathbb{E}\left[\sup_{0\le t\le T}|\Upsilon_t|^\varrho\right] < +\infty.$$

The space $\mathcal{S}^\infty$ consists of the continuous bounded processes. An $\mathbb{R}$-valued, adapted and continuous process Υ belongs to $\mathfrak{E}$ if the random variable

$$\Upsilon^* := \sup_{t\in[0,T]}|\Upsilon_t|$$

has exponential moments of all orders. We also recall that Υ is called *of class D* if the family $\{\Upsilon_\tau \,|\, \tau \in [0,T] \text{ stopping time}\}$ is uniformly integrable. The set of (equivalence classes of) $\mathbb{R}^d$-valued predictable processes Z on $[0,T]\times\Omega$ satisfying

$$\mathbb{E}\left[\left(\int_0^T Z_t^\intercal\, d\langle M\rangle_t Z_t\right)^{\varrho/2}\right] < +\infty$$

is denoted by $\mathfrak{M}^\varrho$. Finally, $\mathcal{M}^\varrho$ stands for the set of $\mathbb{R}$-valued continuous local martingales N on $[0,T]$, such that

$$\|N\|_{\mathcal{M}^\varrho} := \mathbb{E}\Big[\langle N\rangle_T^{\varrho/2}\Big] < +\infty.$$

In order to deduce the existence of solutions to BSDEs, which we assume to be continuous by definition, the following assumption is needed.

Assumption 3.2.4. *The filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is a continuous filtration in the sense that all local $(\mathcal{F}_t)_{t\in[0,T]}$-martingales are continuous.*

The following four theorems constitute the main results of the present chapter. We mention that only the existence result requires the assumption of the continuity of the filtration. We also recall that their statements may hold under weaker conditions which we provide in the subsequent detailed analysis.

Theorem 3.2.5 (Existence)**.** *If Assumptions 3.2.2 and 3.2.4 hold there exists a solution (Ψ, Z, N) to the BSDE* (3.2.3) *such that $\Psi \in \mathfrak{E}$ and $Z \cdot M + N \in \mathcal{M}^\varrho$ for all $\varrho > 0$.*

Theorem 3.2.6 (Uniqueness)**.** *Suppose that Assumption 3.2.2 holds. Then any two solutions (Ψ, Z, N) and (Ψ', Z', N') in $\mathfrak{E} \times \mathfrak{M}^2 \times \mathcal{M}^2$ to the BSDE* (3.2.3) *coincide in the sense that Ψ and Ψ', $Z \cdot M$ and $Z' \cdot M$, and N and N' are indistinguishable.*

Theorem 3.2.7 (Stability)**.** *Consider a family of BSDEs(F^n, ξ^n) indexed by the extended natural numbers $n \geq 0$ for which Assumption 3.2.2 holds true with parameters $(\alpha^n, \beta^n, \overline{\beta}, \beta_f, \gamma)$. Assume that the exponential moments assumption* (3.2.4) *holds uniformly in n, i.e. for all $\varrho > 1$,*

$$\sup_{n\geq 0} \mathbb{E}\Big[e^{\varrho\,(|\xi^n|+|\alpha^n|_1)}\Big] < +\infty.$$

If for $n \geq 0$ (Ψ^n, Z^n, N^n) is the solution in $\mathfrak{E} \times \mathfrak{M}^2 \times \mathcal{M}^2$ to the BSDE(F^n, ξ^n) and if

$$|\xi^n - \xi^0| + \int_0^T \left|F^n - F^0\right|(s, \Psi_s^0, Z_s^0)\, dA_s \longrightarrow 0 \quad \textit{in probability, as } n \to +\infty, \tag{3.2.9}$$

then for each $\varrho > 0$ as $n \to +\infty$

$$\mathbb{E}\left[\left(\exp\left(\sup_{0\leq t\leq T} \left|\Psi_t^n - \Psi_t^0\right|\right)\right)^\varrho\right] \longrightarrow 1 \quad \textit{and} \quad Z^n \cdot M + N^n \longrightarrow Z^0 \cdot M + N^0 \ \textit{in}\ \mathcal{M}^\varrho.$$

Theorem 3.2.8 (Exponential Martingales)**.** *Suppose that Assumption 3.2.2 holds, let $|\varrho| > \gamma/2$ and let $(\Psi, Z, N) \in \mathfrak{E} \times \mathfrak{M}^2 \times \mathcal{M}^2$ be a solution to the BSDE* (3.2.3)*. Then $\mathcal{E}\big(\varrho\,(Z \cdot M + N)\big)$ is a true martingale on* $[0, T]$*.*

Remark 3.2.9. The preceding theorems generalize the results of Briand and Hu [2008] and Morlais [2009]. For their proofs we combine the localization and θ-technique from Briand and Hu [2008] together with the existence and stability results for BSDEs with bounded solutions found in Morlais [2009]. Similar ideas are used in Hu and Schweizer [2009] on a specific quadratic BSDE arising in a robust utility maximization problem where the authors also investigate the measure change problem for their special BSDE, however here we pursue the general theory. As we know from Proposition 2.2.2, if $|\xi| + |\alpha|_1$ does not have sufficiently large exponential moments then the BSDE may fail to have a solution. In particular, we here present all the theoretical background for the study of power utility maximization under exponential moments, see the remaining chapters, as well as of partial equilibrium, see Horst et al. [2010].

3.3 A Priori Estimates

In this section we show that, under appropriate conditions, solutions to the BSDE (3.2.2) satisfy some a priori norm bounds. After giving an important result used in the subsequent sections we motivate Assumption 3.2.2 (v) by showing that without such an assumption the method utilized in Morlais [2009] for the purpose of deriving appropriate a priori bounds fails in the present unbounded case.

Let (Ψ, Z, N) be a solution to (3.2.2), suppose that Assumption 3.2.2 (iii) and (v) hold and that g is uniformly bounded by $\gamma/2$. Recall β^* from (3.2.8), fix $s \in [0,T]$ and set, for $t \in [s,T]$,

$$\widetilde{H}_t := \exp\biggl(\gamma e^{\beta^*(t-s)}|\Psi_t| + \gamma \int_s^t e^{\beta^*(r-s)}\, d\langle \lambda \cdot M\rangle_r\biggr).$$

where we have written $\langle \lambda \cdot M\rangle_t := \int_0^t \lambda_r^\intercal\, d\langle M\rangle_r \lambda_r = \int_0^t \alpha_r\, dA_r$. First we show that $\widetilde{H}$ is, up to integrability, a local submartingale.

From Tanaka's formula,

$$d|\Psi_t| = \operatorname{sgn}(\Psi_t)(Z_t^\intercal\, dM_t + dN_t) - \operatorname{sgn}(\Psi_t)\bigl(F(t,\Psi_t,Z_t)\, dA_t + g_t\, d\langle N\rangle_t\bigr) + d\ell_t, \quad (3.3.1)$$

where ℓ is the local time of Ψ at 0. Itô's formula then yields

$$\begin{aligned} d\widetilde{H}_t = \gamma \widetilde{H}_t\, e^{\beta^*(t-s)} \biggl[& \operatorname{sgn}(\Psi_t)(Z_t^\intercal\, dM_t + dN_t) + \overline{\beta}|\Psi_t|(c_A\, dt - dA_t) \\ & + \Bigl(-\operatorname{sgn}(\Psi_t)F(t,\Psi_t,Z_t) + \alpha_t + \overline{\beta}|\Psi_t| + \frac{\gamma}{2}\, e^{\beta^*(t-s)}\|B_t Z_t\|^2\Bigr)\, dA_t \\ & + \Bigl(-\operatorname{sgn}(\Psi_t)\, g_t + \frac{\gamma}{2}\, e^{\beta^*(t-s)}\Bigr)\, d\langle N\rangle_t + d\ell_t\biggr]. \end{aligned} \quad (3.3.2)$$

An inspection of the finite variation parts shows that under the present assumptions they are nonnegative. In particular, the semimartingale $\widetilde{H}$ is a local submartingale, which leads to the following result.

Proposition 3.3.1 (A Priori Estimate)**.** *Suppose Assumption 3.2.2 (iii) and (v) hold and assume that the function g is uniformly bounded by $\gamma/2$, $\mathbb{P}$-a.s. Let (Ψ, Z, N) be a solution to the BSDE* (3.2.2) *and let the family*

$$\biggl(\exp\biggl(\gamma e^{\beta^* T}|\Psi_t| + \gamma \int_0^T e^{\beta^* r}\, d\langle \lambda \cdot M\rangle_r\biggr)\biggr)_{t \in [0,T]}$$

be of class D. Then $\mathbb{P}$-a.s. for all $s \in [0,T]$,

$$|\Psi_s| \le \frac{1}{\gamma} \log \mathbb{E}\biggl[\exp\biggl(\gamma e^{\beta^*(T-s)}|\xi| + \gamma \int_s^T e^{\beta^*(r-s)}\, d\langle \lambda \cdot M\rangle_r\biggr)\biggm| \mathcal{F}_s\biggr]. \quad (3.3.3)$$

Proof. Fix $s \in [0,T]$ and set $\widetilde{H}$ as above. Since $\widetilde{H}$ is a local submartingale there exists a sequence of stopping times $(\tau_n)_{n\geq 1}$ valued in $[s,T]$, which converges $\mathbb{P}$-a.s. to T, such that $\widetilde{H}^{\tau_n}$ is a submartingale for each $n \geq 1$. We then derive

$$\exp(\gamma|\Psi_s|) \leq \mathbb{E}[\widetilde{H}_{T\wedge\tau_n} | \mathcal{F}_s] \leq \mathbb{E}\left[\exp\left(\gamma e^{\beta^*(T-s)}|\Psi_{T\wedge\tau_n}| + \gamma \int_s^T e^{\beta^*(r-s)}\, d\langle\lambda\cdot M\rangle_r\right)\middle| \mathcal{F}_s\right].$$

Letting $n \to +\infty$ the claim follows from the class D assumption. $\square$

On the Additional Assumption 3.2.2 (v)

Proposition 3.3.1 provides the appropriate a priori estimate, indeed suppose that $|\xi|$ and $|\alpha|_1$ are bounded random variables and (Ψ, Z, N) is a solution to (3.2.3). If the current assumptions hold and $\exp(\gamma e^{\beta^* T}|\Psi|)$ is of class D, then Ψ satisfies

$$|\Psi| \leq \left\| e^{\beta^* T}(|\xi| + |\alpha|_1)\right\|_\infty. \tag{3.3.4}$$

Comparing with (3.3.3) this indicates that the inclusion of Assumption 3.2.2 (v) allows us to prove similar estimates to the bounded case which enables us to establish existence for the BSDE (3.2.3) when $|\xi| + |\alpha|_1$ has exponential moments of all orders, to be more precise, an order of at least $\gamma e^{\beta^* T}$.

Contrary to the above let us investigate the method utilized in Morlais [2009] under Assumption 3.2.2 (iii) only, supposing that g is bounded by $\gamma/2$. We set

$$H_t := \exp\left(\gamma e^{\beta\langle\lambda\cdot M\rangle_{s,t}}|\Psi_t| + \gamma \int_s^t e^{\beta\langle\lambda\cdot M\rangle_{s,r}}\, d\langle\lambda\cdot M\rangle_r\right), \tag{3.3.5}$$

where $\langle\lambda\cdot M\rangle_{s,t} := \langle\lambda\cdot M\rangle_t - \langle\lambda\cdot M\rangle_s = \int_s^t \alpha_r\, dA_r$. We derive from Itô's formula that

$$\begin{aligned} dH_t = \gamma H_t\, e^{\beta\langle\lambda\cdot M\rangle_{s,t}} \Bigg[& \operatorname{sgn}(\Psi_t)(Z_t^\intercal\, dM_t + dN_t) \\ & + \left(-\operatorname{sgn}(\Psi_t)F(t,\Psi_t,Z_t) + \alpha_t + \alpha_t\beta|\Psi_t| + \frac{\gamma}{2}\, e^{\beta\langle\lambda\cdot M\rangle_{s,t}}\|B_t Z_t\|^2\right) dA_t \\ & + \left(-\operatorname{sgn}(\Psi_t)\, g_t + \frac{\gamma}{2}\, e^{\beta\langle\lambda\cdot M\rangle_{s,t}}\right) d\langle N\rangle_t + d\ell_t \Bigg]. \end{aligned}$$

Once again, the finite variation parts are nonnegative. We conclude in the same way as for Proposition 3.3.1 that the corresponding a priori result holds for H as well. To sum up, we have that under a similar class D assumption, now on

$$\exp\left(\gamma e^{\beta\langle\lambda\cdot M\rangle_T}|\Psi| + \gamma \int_0^T e^{\beta\langle\lambda\cdot M\rangle_r}\, d\langle\lambda\cdot M\rangle_r\right),$$

$\mathbb{P}$-a.s. for all $s \in [0,T]$,

$$|\Psi_s| \le \frac{1}{\gamma} \log \mathbb{E}\left[\exp\left(\gamma e^{\beta\langle\lambda\cdot M\rangle_{s,T}}|\xi| + \gamma \int_s^T e^{\beta\langle\lambda\cdot M\rangle_{s,r}}\, d\langle\lambda\cdot M\rangle_r\right)\middle|\, \mathcal{F}_s\right]. \tag{3.3.6}$$

If $\beta = 0$, then $\widetilde{H}$ from above equals H and there is no difference with the statement of Proposition 3.3.1. However when $\beta > 0$ the estimate (3.3.6) is not sufficient for our purposes. We aim at using the a priori estimate to show the existence of solutions to the BSDE (3.2.3) in $\mathfrak{E} \times \mathfrak{M}^2 \times \mathcal{M}^2$ using an appropriate approximating procedure. If $|\xi|$ and $|\alpha|_1$ are bounded random variables there exists a solution (Ψ, Z, N) to (3.2.3) with Ψ bounded, see Morlais [2009]. With (3.3.6) at our disposal we then have the estimate

$$|\Psi| \le \left\| e^{\beta|\alpha|_1}\left(|\xi| + |\alpha|_1\right)\right\|_\infty . \tag{3.3.7}$$

Our goal is to remove the boundedness assumption and to replace it with the assumption on the existence of exponential moments of $|\xi|+|\alpha|_1$ in the spirit of Briand and Hu [2008]. However a closer inspection of the a priori estimate from (3.3.6) together with (3.3.7) already indicates that more restrictive assumptions are necessary. More specifically, when $\beta > 0$ we cannot deduce any integrability of $\exp\Big(\gamma e^{\beta|\alpha|_1}\left(|\xi| + |\alpha|_1\right)\Big)$ when $|\xi|$ and $|\alpha|_1$ have only exponential moments. This motivates Assumption 3.2.2 (v), which is sufficient for the present study as we have seen using the formula in (3.3.2). Note that we could opt for deriving the existence result under the weaker assumption that the above random variable $\exp\Big(\gamma e^{\beta|\alpha|_1}\left(|\xi| + |\alpha|_1\right)\Big)$ be integrable. In this case, describing the space in which a solution to the BSDE exists is more technical, as would be a statement of uniqueness.

3.4 Existence and Norm Bounds

In the present section we establish Theorem 3.2.5 together with some related results on norm bounds of the solution. The proof of existence follows the following recipe. Firstly we truncate $\langle\lambda\cdot M\rangle$ to get approximate solutions. Then by using the estimate from Proposition 3.3.1 we localize and work on a random time interval so that the approximations are uniformly bounded and we can apply a stability result. Finally we glue together on $[0,T]$ to construct a solution. The a priori estimates ensure that we may take all limits in the described procedure.

The Existence Result

Theorem 3.4.1 (Existence)**.** *Let Assumptions 3.2.2 (ii)-(v) and 3.2.4 hold and let* $|\xi| + |\alpha|_1$ *have an exponential moment of order* $\gamma e^{\beta^* T}$. *Then the BSDE* (3.2.3) *has a solution*

(Ψ, Z, N) *such that*

$$|\Psi_t| \leq \frac{1}{\gamma} \log \mathbb{E}\left[\exp\left(\gamma\, e^{\beta^*(T-t)}|\xi| + \gamma \int_t^T e^{\beta^*(r-t)}\, d\langle\lambda \cdot M\rangle_r\right) \middle| \mathcal{F}_t\right]. \tag{3.4.1}$$

Proof. Exactly as in Briand and Hu [2008] we first assume that F and ξ are nonnegative. For each integer $n \geq 1$, we set

$$\sigma_n := \inf\left\{t \in [0,T] \,\middle|\, \langle\lambda \cdot M\rangle_t := \int_0^t \alpha_s\, dA_s \geq n\right\} \wedge T,$$

$\xi^n := \xi \wedge n$, $\lambda_t^n := \mathbf{1}_{\{t \leq \sigma_n\}}\lambda_t$ and $F^n(t,\psi,z) := \mathbf{1}_{\{t\leq\sigma_n\}}F(t,\psi,z)$. Then F^n satisfies Assumption 3.2.2 (ii)-(v) with the same constants, but with the processes λ^n and α^n where

$$\alpha_t^n := \|B_t\lambda_t^n\|^2 = \mathbf{1}_{\{t\leq\sigma_n\}}\|B_t\lambda_t\|^2 = \mathbf{1}_{\{t\leq\sigma_n\}}\alpha_t.$$

In particular, $|\alpha^n|_1 = \int_0^{\sigma_n} \alpha_s\, dA_s \leq n$ and

$$\int_0^T (\lambda_t^n)^\intercal\, d\langle M\rangle_t \lambda_t^n = \int_0^T \|B_t\lambda_t^n\|^2\, dA_t = |\alpha^n|_1 \leq n,$$

so we may apply Morlais [2009] Theorem 2.5 and Theorem 2.6 to conclude that there exists a unique solution $(\Psi^n, Z^n, N^n) \in \mathcal{S}^\infty \times \mathfrak{M}^2 \times \mathcal{M}^2$ to the BSDE (3.2.3), where F is replaced by F^n and ξ is replaced by ξ^n. From Proposition 3.3.1 we derive

$$\begin{aligned}
|\Psi_t^n| &\leq \frac{1}{\gamma}\log\mathbb{E}\left[\exp\left(\gamma e^{\beta^*(T-t)}|\xi^n| + \gamma\int_t^T e^{\beta^*(r-t)}\, d\langle\lambda^n \cdot M\rangle_r\right)\middle|\mathcal{F}_t\right] \\
&\leq \frac{1}{\gamma}\log\mathbb{E}\left[\exp\left(\gamma e^{\beta^*(T-t)}|\xi| + \gamma\int_t^T e^{\beta^*(r-t)}\, d\langle\lambda \cdot M\rangle_r\right)\middle|\mathcal{F}_t\right] \\
&\leq \frac{1}{\gamma}\log\mathbb{E}\left[\exp\left(\gamma e^{\beta^* T}(|\xi| + |\alpha|_1)\right)\middle|\mathcal{F}_t\right] =: X_t.
\end{aligned} \tag{3.4.2}$$

Let $n \leq m$ so that we have $\sigma_n \leq \sigma_m$ and $\mathbf{1}_{\{t\leq\sigma_n\}} \leq \mathbf{1}_{\{t\leq\sigma_m\}}$. In particular, $\xi^n \leq \xi^m$ and $F^n \leq F^m$, from which we deduce that the Assumptions 3.2.2 (ii)-(v), hence the corresponding assumptions in Morlais [2009], hold for both F^n and F^m with the same set of parameters $(\alpha^m, \beta, \overline{\beta}, \beta_f, \gamma)$ where the additional c_θ in Morlais [2009] is equal to m. An application of Theorem 2.7 therein now shows that $\Psi^n \leq \Psi^m$ so that $(\Psi^n)_{n\geq1}$ is an increasing sequence of bounded continuous processes.

The next step would be to send n to infinity, however, we do not dispose of a suitable stability result. Indeed we have only Morlais [2009] Lemma 3.3 which applies for bounded processes under uniform growth assumptions on the drivers, hence we introduce an additional truncation. Let $k \geq 1$ be a fixed integer and

$$\tau_k := \inf\left\{t \in [0,T] \,\middle|\, X_t \geq k \text{ or } \langle\lambda \cdot M\rangle_t \geq k\right\} \wedge T.$$

Thanks to the continuity of the filtration the martingale $\exp(\gamma X)$ is continuous so that the random variable

$$V := \max_{t\in[0,T]} (X_t) \vee \langle \lambda \cdot M\rangle_T$$

is finite $\mathbb{P}$-a.s. We derive that $\mathbb{P}$-a.s. $\tau_k = T$ for large k. Due to (3.4.2) the sequence $(\Psi^{n,k})_{n\geq 1}$ given by

$$\Psi^{n,k}_t := \Psi^n_{t\wedge\tau_k},$$

is uniformly bounded by k. For the martingale parts we define

$$Z^{n,k}_t := \mathbf{1}_{\{t\leq\tau_k\}} Z^n_t \quad\text{and}\quad N^{n,k}_t := \mathbf{1}_{\{t\leq\tau_k\}} N^n_t.$$

An inspection of the respective cases shows that

$$\Psi^{n,k}_t = \Psi^n_{\tau_k} - \int_t^T (Z^{n,k}_s)^\intercal\, dM_s - \int_t^T dN^{n,k}_s \\ + \int_t^T \mathbf{1}_{\{s\leq\tau_k\wedge\sigma_n\}} F(s,\Psi^{n,k}_s, Z^{n,k}_s)\, dA_s + \frac{1}{2}\int_t^T d\langle N^{n,k}\rangle_s.$$

Moreover, $\Psi^n_{\tau_k} \xrightarrow{n\uparrow+\infty} \sup_{n\geq 1}\Psi^n_{\tau_k} =: \xi_k$, where ξ_k is bounded by k. Next we appeal to the stability result stated in Morlais [2009] Lemma 3.3, noting Remark 3.4 therein. Note that this result requires estimates that are uniform in n which is accomplished by the specific choice of the stopping time τ_k. Hence $(\Psi^{n,k}, Z^{n,k}, N^{n,k})$ converges to $(\Psi^{\infty,k}, Z^{\infty,k}, N^{\infty,k})$ in the sense that

$$\lim_{n\to+\infty} \mathbb{E}\left[\sup_{0\leq t\leq T} \left|\Psi^{n,k}_t - \Psi^{\infty,k}_t\right|\right] = 0,$$

$$\lim_{n\to+\infty} \mathbb{E}\left[\int_0^T \left(Z^{n,k}_s - Z^{\infty,k}_s\right)^\intercal d\langle M\rangle_s \left(Z^{n,k}_s - Z^{\infty,k}_s\right)\right] = 0$$

and

$$\lim_{n\to+\infty} \mathbb{E}\left[\left|N^{n,k}_T - N^{\infty,k}_T\right|^2\right] = 0,$$

where the triples $(\Psi^{\infty,k}, Z^{\infty,k}, N^{\infty,k})$ solve the BSDE

$$d\Psi^{\infty,k}_t = \left(Z^{\infty,k}_t\right)^\intercal dM_t + dN^{\infty,k}_t \\ - \mathbf{1}_{\{t\leq\tau_k\}} F(t,\Psi^{\infty,k}_t, Z^{\infty,k}_t)\, dA_t - \frac{1}{2}\, d\langle N^{\infty,k}\rangle_t, \quad \Psi^{\infty,k}_{\tau_k} = \xi_k,$$

on the random horizon $[\![0,\tau_k]\!] \subset [0,T]$. The stopping times τ_k are monotone in k and therefore it follows that

$$\Psi^{n,k+1}_{\cdot\wedge\tau_k} \equiv \Psi^{n,k}, \quad \mathbf{1}_{[\![0,\tau_k]\!]} Z^{n,k+1} \equiv Z^{n,k} \quad\text{and}\quad \mathbf{1}_{[\![0,\tau_k]\!]} N^{n,k+1} \equiv N^{n,k},$$

so that the above convergence yields (for the two last objects in $\mathcal{M}^2$)

$$\Psi^{\infty,k+1}_{\cdot\wedge\tau_k} \equiv \Psi^{\infty,k}, \quad \left(\mathbf{1}_{[\![0,\tau_k]\!]} Z^{\infty,k+1}\right) \cdot M \equiv Z^{\infty,k} \cdot M \;\text{ and }\; \mathbf{1}_{[\![0,\tau_k]\!]} N^{\infty,k+1} \equiv N^{\infty,k}.$$

To finish the proof, we define the processes

$$\begin{aligned} \Psi_t &:= \mathbf{1}_{\{t\le\tau_1\}} \Psi_t^{\infty,1} + \sum_{k\ge2} \mathbf{1}_{\{t\in]\!]\tau_{k-1},\tau_k]\!]\}} \Psi_t^{\infty,k}, \\ Z_t &:= \mathbf{1}_{\{t\le\tau_1\}} Z_t^{\infty,1} + \sum_{k\ge2} \mathbf{1}_{\{t\in]\!]\tau_{k-1},\tau_k]\!]\}} Z_t^{\infty,k} \\ \text{and }\; N_t &:= \mathbf{1}_{\{t\le\tau_1\}} N_t^{\infty,1} + \sum_{k\ge2} \mathbf{1}_{\{t\in]\!]\tau_{k-1},\tau_k]\!]\}} N_t^{\infty,k}. \end{aligned}$$

By construction this gives a solution to the BSDE

$$d\Psi_t = Z_t^\intercal\, dM_t + dN_t - F(t,\Psi_t,Z_t)\, dA_t - \frac{1}{2}\, d\langle N\rangle_t, \quad \Psi_T = \xi,$$

since $\mathbf{1}_{\{t\le\tau_1\}} + \sum_{k\ge2} \mathbf{1}_{\{t\in]\!]\tau_{k-1},\tau_k]\!]\}} = \mathbf{1}_{\{t\in[0,T]\!]\}}$ $\mathbb{P}$-a.s. More precisely, there is a $\mathbb{P}$-null set $\mathfrak{N}$ such that for all $\omega\in\mathfrak{N}^c$ there is a minimal $k_0(\omega)$ with $\tau_{k_0(\omega)}(\omega) = T$ and such that $\Psi^{\infty,k}_{\tau_k(\omega)}(\omega) = \xi_k(\omega)$ for all k, which yields that (possibly after another modification of $\mathfrak{N}$)

$$\Psi_T(\omega) = \Psi_T^{\infty,k_0(\omega)}(\omega) = \xi_{k_0(\omega)}(\omega) = \sup_{n\ge1} \Psi_T^n(\omega) = \xi(\omega).$$

The bound in (3.4.1) holds as we have it for all n and k from (3.4.2).
In the case when ξ and f are not necessarily nonnegative, we reduce the problem to using a double truncation procedure defined by $\xi^{n,m} := \xi^+\wedge n - \xi^-\wedge m$, $\lambda^{n,m} := \mathbf{1}_{\{t\le\sigma_n\}}\lambda^+ - \mathbf{1}_{\{t\le\sigma_m\}}\lambda^-$ and $F^{n,m} := \mathbf{1}_{\{t\le\sigma_n\}}F^+ - \mathbf{1}_{\{t\le\sigma_m\}}F^-$. □

Remark 3.4.2. Let us recall the specific counterexample to the existence of a BSDE solution provided in Proposition 2.2.2. More precisely, in a Brownian framework, for every $q<0$, we can construct a predictable W-integrable process λ which lacks sufficient integrability. The BSDE (1.3.1) now is amenable to the analysis in this chapter. For instance, the quadratic growth estimate reads

$$|F(t,z)| \le \frac{|q|}{2\varepsilon_0}\,\|\lambda_t\|^2 + \frac{\gamma(\varepsilon_0)}{2}\,\|z\|^2,$$

where F denotes the driver of (1.3.1) and $\gamma(\varepsilon_0) := 1 + \left(1+|q|/2\right)\varepsilon_0$ for $\varepsilon_0>0$, by the generalized Young inequality, see also Proposition 4.6.3. Hence, in the setting of Proposition 2.2.2, $|\alpha|_1 := \frac{|q|}{2\varepsilon_0}\int_0^T \lambda_t^2\, dt = \frac{\pi^2}{8\varepsilon_0}\int_{T/2}^T \frac{1}{T-t}\, dt$ has an exponential moment of order $\varrho>0$ if and only if $\varrho<\varepsilon_0$, see the remarks that follow Proposition 2.2.2. As we have seen above, a sufficient condition for the existence of a BSDE solution is that $\varrho \ge \gamma(\varepsilon_0) = 1 + \left(1+|q|/2\right)\varepsilon_0 > \varepsilon_0$ which is indeed incompatible with the deficient integrability of the specific λ considered in Proposition 2.2.2.

Norm Bounds Results

As a corollary of the previous theorem we deduce

Corollary 3.4.3 (Norm Bounds for the Components of BSDE Solution Triples)**.**

(i) Let the Assumptions 3.2.2 (ii)-(v) and 3.2.4 hold and let $|\xi| + |\alpha|_1$ have an exponential moment of order $\delta^* > \gamma e^{\beta^* T}$. Then the BSDE (3.2.3) *has a solution (Ψ, Z, N) such that* $\exp(\gamma\Psi) \in \mathcal{S}^{\varrho^*}$ *for* $\varrho^* := \frac{\delta^*}{\gamma e^{\beta^* T}} > 1$.
When additionally $|\xi| + |\alpha|_1$ has exponential moments of all orders, i.e. Assumption 3.2.2 (i) holds, this solution is such that $\Psi \in \mathfrak{E}$. In particular, for each $\varrho > 1$ we have the estimate

$$\mathbb{E}\big[\exp(\varrho\gamma\Psi^*)\big] \leq \left(\frac{\varrho}{\varrho-1}\right)^{\varrho} \mathbb{E}\Big[\exp\Big(\varrho\gamma e^{\beta^* T}\Big(|\xi| + |\alpha|_1\Big)\Big)\Big]. \tag{3.4.3}$$

(ii) Let the Assumption 3.2.2 (i)-(iii) and (v) hold and suppose there exists a solution (Ψ, Z, N) to the BSDE (3.2.3) *such that $\Psi \in \mathfrak{E}$. Then $(Z, N) \in \mathfrak{M}^{\varrho} \times \mathcal{M}^{\varrho}$ for all $\varrho > 0$, more precisely*

$$\mathbb{E}\left[\left(\int_0^T Z_s^\top \, d\langle M\rangle_s Z_s + d\langle N\rangle_s\right)^{\varrho/2}\right] \leq c_{\varrho,\gamma}\, \mathbb{E}\Big[\exp\Big(4\varrho\gamma e^{\beta^* T}\Big(|\xi| + |\alpha|_1\Big)\Big)\Big], \tag{3.4.4}$$

where $c_{\varrho,\gamma}$ is a positive constant depending on ϱ and γ. The estimate (3.4.3) *then holds as well.*

Proof. (i) Let (Ψ, Z, N) be the solution to (3.2.3) obtained in Theorem 3.4.1. As in the previous section set

$$\widetilde{H}_t := \exp\left(\gamma e^{\beta^* t}|\Psi_t| + \gamma \int_0^t e^{\beta^* r}\, d\langle \lambda \cdot M\rangle_r\right), \tag{3.4.5}$$

which is a local submartingale. Moreover, from the estimate (3.4.1), Jensen's inequality and the adaptedness of $\int_0^\cdot e^{\beta^* r}\, d\langle\lambda\cdot M\rangle_r$ we deduce that

$$\begin{aligned}\widetilde{H}_t &= \Big[\exp(\gamma|\Psi_t|)\Big]^{\exp(\beta^* t)} \exp\left(\gamma\int_0^t e^{\beta^* r}\, d\langle\lambda\cdot M\rangle_r\right)\\ &\leq \mathbb{E}\left[\exp\left(\gamma e^{\beta^*(T-t)}|\xi| + \gamma\int_t^T e^{\beta^*(r-t)}\, d\langle\lambda\cdot M\rangle_r\right)\middle|\, \mathcal{F}_t\right]^{\exp(\beta^* t)} \exp\left(\gamma\int_0^t e^{\beta^* r}\, d\langle\lambda\cdot M\rangle_r\right)\\ &\leq \mathbb{E}\left[\exp\Big(\gamma e^{\beta^* T}\Big(|\xi| + |\alpha|_1\Big)\Big)\middle|\, \mathcal{F}_t\right].\end{aligned}$$

Observe that this upper estimate is a uniformly integrable martingale, in particular it is of class D and therefore $\widetilde{H}$ is a true submartingale. Then, via the Doob maximal inequality, we find that for $\varrho > 1$

$$\mathbb{E}\big[\exp(\varrho\gamma\Psi^*)\big] \le \mathbb{E}\left[\sup_{0\le t\le T}\widetilde{H}_t^{\varrho}\right] \le \left(\frac{p}{p-1}\right)^{\varrho}\mathbb{E}[\widetilde{H}_T^{\varrho}]$$
$$\le \left(\frac{p}{p-1}\right)^{\varrho}\mathbb{E}\left[\exp\left(\varrho\gamma e^{\beta^* T}\Big(|\xi|+|\alpha|_1\Big)\right)\right], \qquad (3.4.6)$$

provided the right hand side is finite. In particular, $\exp(\gamma\Psi) \in \mathcal{S}^{\varrho^*}$ and $\Psi \in \mathfrak{E}$ as soon as $|\xi| + |\alpha|_1$ has exponential moments of all orders, in which case (3.4.3) holds.

(ii) We first verify that (3.4.3) continues to hold when (Ψ, Z, N) is a solution to (3.2.3) with $\Psi \in \mathfrak{E}$. First observe that we may reformulate the result of Proposition 3.3.1 under the condition that

$$\exp\left(\gamma e^{\beta^* T}|\Psi_\cdot| + \gamma\int_0^{\cdot} e^{\beta^* r}\, d\langle\lambda\cdot M\rangle_r\right)$$

be of class D. Then, repeating the argument from item (i) above, using (3.3.3) instead of (3.4.1), leads to the same conclusion, since we have the relation

$$\exp\left(\gamma e^{\beta^* t}|\Psi_t| + \gamma\int_0^t e^{\beta^* r}\, d\langle\lambda\cdot M\rangle_r\right) \le \mathbb{E}\left[\exp\left(\gamma e^{\beta^* T}\Big(\Psi^* + |\alpha|_1\Big)\right)\middle|\,\mathcal{F}_t\right],$$

where the right hand side is indeed a process of class D. For the remaining claim, i.e. relation (3.4.4), define the functions $u, v : \mathbb{R} \to \mathbb{R}_+$ via $u(x) := \frac{1}{\gamma^2}(e^{\gamma x} - 1 - \gamma x)$ and $v(x) := u(|x|)$. We have that v is a $\mathcal{C}^2$-function, so we use Itô's formula to see that for a stopping time τ (to be chosen later),

$$\begin{aligned}v(\Psi_0) = v(\Psi_{t\wedge\tau}) &- \int_0^{t\wedge\tau} u'(|\Psi_s|)\operatorname{sgn}^*(\Psi_s)(Z_s^\intercal\, dM_s + dN_s)\\ &+ \int_0^{t\wedge\tau} u'(|\Psi_s|)\operatorname{sgn}^*(\Psi_s)\Big(F(s,\Psi_s,Z_s)\, dA_s + \frac{1}{2}\, d\langle N\rangle_s\Big)\\ &- \frac{1}{2}\int_0^{t\wedge\tau} u''(|\Psi_s|)\Big(Z_s^\intercal\, d\langle M\rangle_s Z_s + d\langle N\rangle_s\Big),\end{aligned}$$

where use the notation $\operatorname{sgn}^*(x) := -\mathbf{1}_{\{x\le 0\}} + \mathbf{1}_{\{x>0\}}$ and observe that $u'(0) = 0$. Assumption 3.2.2 (iii) yields

$$\begin{aligned}v(\Psi_0) \le v(\Psi_{t\wedge\tau}) &- \int_0^{t\wedge\tau} u'(|\Psi_s|)\operatorname{sgn}^*(\Psi_s)(Z_s^\intercal\, dM_s + dN_s)\\ &+ \int_0^{t\wedge\tau} u'(|\Psi_s|)\Big(\alpha_s + \alpha_s\beta|\Psi_s|\Big)dA_s + \frac{1}{2}\int_0^{t\wedge\tau}\Big(\gamma u'(|\Psi_s|) - u''(|\Psi_s|)\Big)Z_s^\intercal\, d\langle M\rangle_s Z_s\\ &+ \frac{1}{2}\int_0^{t\wedge\tau}\Big(u'(|\Psi_s|)\operatorname{sgn}^*(\Psi_s) - u''(|\Psi_s|)\Big)\, d\langle N\rangle_s,\end{aligned}$$

since $u'(x) = \frac{1}{\gamma}(e^{\gamma x} - 1) \geq 0$ for $x \geq 0$. Using the relation $\gamma u'(x) - u''(x) = -1$ together with $\gamma \geq 1$ it follows that

$$0 \leq v(\Psi_0) \leq v(\Psi_{t\wedge\tau}) - \int_0^{t\wedge\tau} u'(|\Psi_s|)\,\mathrm{sgn}^*(\Psi_s)(Z_s^\intercal\, dM_s + dN_s)$$
$$+ \int_0^{t\wedge\tau} u'(|\Psi_s|)\Big(\alpha_s + \alpha_s\beta|\Psi_s|\Big)\, dA_s - \frac{1}{2}\int_0^{t\wedge\tau} Z_s^\intercal\, d\langle M\rangle_s Z_s + d\langle N\rangle_s. \quad (3.4.7)$$

Suppose first that $\varrho \geq 2$. Then (3.4.4) can be proved using the Burkholder-Davis-Gundy inequalities as follows. From (3.4.7) we deduce that

$$\frac{1}{2}\int_0^\tau Z_s^\intercal\, d\langle M\rangle_s Z_s + d\langle N\rangle_s \leq \frac{1}{\gamma^2}\, e^{\gamma\Psi^*} + \frac{1}{\gamma}\int_0^T e^{\gamma|\Psi_s|}\Big(\alpha_s + \alpha_s\beta|\Psi_s|\Big)\, dA_s$$
$$+ \sup_{0\leq t\leq T}\left|\int_0^{t\wedge\tau} u'(|\Psi_s|)\,\mathrm{sgn}^*(\Psi_s)(Z_s^\intercal\, dM_s + dN_s)\right|,$$

where we used the estimates $u'(x) \leq e^{\gamma x}/\gamma$ and $v(x) \leq e^{\gamma x}/\gamma^2$, valid for $x \geq 0$. From the inequalities $x \leq e^x - 1$ and $\beta \leq \gamma$ we derive

$$\left(\int_0^\tau Z_s^\intercal\, d\langle M\rangle_s Z_s + d\langle N\rangle_s\right)^{\varrho/2} \leq 2^{3\varrho/2-2}\Bigg(\frac{1}{\gamma^\varrho}\, e^{\varrho/2\,\gamma\Psi^*} + \frac{1}{\gamma^{\varrho/2}}\, e^{\varrho\gamma\Psi^*}|\alpha|_1^{\varrho/2}$$
$$+ \sup_{0\leq t\leq T}\left|\int_0^{t\wedge\tau} u'(|\Psi_s|)\mathrm{sgn}^*(\Psi_s)(Z_s^\intercal\, dM_s + dN_s)\right|^{\varrho/2}\Bigg),$$

which yields, after taking expectation and applying the estimate $|x|^{\varrho/2} < e^{\varrho/2\,|x|}$ and the Burkholder-Davis-Gundy inequality,

$$\mathbb{E}\left[\left(\int_0^\tau Z_s^\intercal\, d\langle M\rangle_s Z_s + d\langle N\rangle_s\right)^{\varrho/2}\right] \leq c_{\varrho,\gamma}\,\mathbb{E}\Big[e^{\varrho/2\,\gamma\Psi^*} + e^{\varrho\gamma\Psi^*}e^{\varrho/2\,\gamma|\alpha|_1}\Big]$$
$$+ c_{\varrho,\gamma}\,\mathbb{E}\left[\left(\int_0^\tau e^{2\gamma|\Psi_s|}\Big(Z_s^\intercal\, d\langle M\rangle_s Z_s + d\langle N\rangle_s\Big)\right)^{\varrho/4}\right],$$

where we used the estimate $u'(x) \leq e^{\gamma x}/\gamma$ for $x \geq 0$. Note that in the above and in what follows $c_{\varrho,\gamma} > 0$ is a generic constant depending on ϱ and γ that may change from line to line. We apply the generalized Young inequality, $|ab| \leq \frac{\varepsilon}{2}\, a^2 + \frac{b^2}{2\varepsilon}$, for $\varepsilon := 1$ and for

$\varepsilon := c_{\varrho,\gamma}$. Then, after an adjustment of $c_{\varrho,\gamma}$,

$$\begin{aligned}
&\mathbb{E}\bigg[\Big(\int_0^\tau Z_s^\intercal\, d\langle M\rangle_s Z_s + d\langle N\rangle_s\Big)^{\varrho/2}\bigg] \\
&\qquad \le c_{\varrho,\gamma}\,\Big(\mathbb{E}\Big[e^{\varrho/2\,\gamma\Psi^*}\Big] + \frac{1}{2}\,\mathbb{E}\Big[e^{2\varrho\gamma\Psi^*}\Big] + \frac{1}{2}\,\mathbb{E}\Big[e^{\varrho\gamma|\alpha|_1}\Big]\Big) \\
&\qquad + c_{\varrho,\gamma}\,\mathbb{E}\Big[e^{\varrho\gamma\Psi^*}\Big] + \frac{1}{2}\,\mathbb{E}\bigg[\Big(\int_0^\tau Z_s^\intercal\, d\langle M\rangle_s Z_s + d\langle N\rangle_s\Big)^{\varrho/2}\bigg] \\
&\qquad \le c_{\varrho,\gamma}\,\Big(\mathbb{E}\Big[e^{2\varrho\gamma\Psi^*}\Big] + \mathbb{E}\Big[e^{2\varrho\gamma|\alpha|_1}\Big]\Big) + \frac{1}{2}\,\mathbb{E}\bigg[\Big(\int_0^\tau Z_s^\intercal\, d\langle M\rangle_s Z_s + d\langle N\rangle_s\Big)^{\varrho/2}\bigg].
\end{aligned}$$

Next define, for each integer $n \ge 1$, the stopping time

$$\tau_n := \inf\Big\{t \in [0,T]\,\Big|\,\int_0^t e^{2\gamma|\Psi_s|}\Big(Z_s^\intercal\, d\langle M\rangle_s Z_s + d\langle N\rangle_s\Big) \ge n\Big\} \wedge T.$$

Inserting τ_n into the above calculation and using $e^a + e^b \le 2e^{a+b}$ for $a, b \ge 0$ together with (3.4.3), we may rewrite the last estimate as

$$\mathbb{E}\bigg[\Big(\int_0^{\tau_n} Z_s^\intercal\, d\langle M\rangle_s Z_s + d\langle N\rangle_s\Big)^{\varrho/2}\bigg] \le c_{\varrho,\gamma}\,\mathbb{E}\Big[\exp\Big(2\varrho\gamma e^{\beta^* T}\Big(|\xi| + |\alpha|_1\Big)\Big)\Big]. \tag{3.4.8}$$

By Fatou's lemma, since $\tau_n \to T$ as $n \to +\infty$,

$$\mathbb{E}\bigg[\Big(\int_0^{T} Z_s^\intercal\, d\langle M\rangle_s Z_s + d\langle N\rangle_s\Big)^{\varrho/2}\bigg] \le c_{\varrho,\gamma}\,\mathbb{E}\Big[\exp\Big(2\varrho\gamma e^{\beta^* T}\Big(|\xi| + |\alpha|_1\Big)\Big)\Big]$$

and (3.4.4) follows. In the situation where $\varrho < 2$, $\varsigma := 2/\varrho > 1$ and we may combine Jensen's inequality with (3.4.8), which is valid for $\varrho = 2$, to get

$$\begin{aligned}
&\mathbb{E}\bigg[\Big(\int_0^{\tau_n} Z_s^\intercal\, d\langle M\rangle_s Z_s + d\langle N\rangle_s\Big)^{\varrho/2}\bigg]^{\varsigma} \\
&\qquad\qquad \le \mathbb{E}\bigg[\int_0^{\tau_n} Z_s^\intercal\, d\langle M\rangle_s Z_s + d\langle N\rangle_s\bigg] \le c_{2,\gamma}\,\mathbb{E}\Big[\exp\Big(4\gamma e^{\beta^* T}\Big(|\xi| + |\alpha|_1\Big)\Big)\Big]
\end{aligned}$$

from which (3.4.4) follows after another application of Fatou's lemma together with the fact that the right hand side in the inequality above is greater or equal one while $1/\varsigma = \varrho/2 < 1$. □

Remark 3.4.4. We point out that the results of this section do not require that F be convex in z, but only that F be continuous in (ψ, z). The reader may have noticed that the continuity of F is not used directly in the proofs. However in Theorem 3.4.1 we rely on the results of Morlais [2009] where continuity is a technical condition needed for an application of Dini's theorem. In addition our results also apply to the BSDE (3.2.2) if g is identically equal to a nonzero constant $\gamma_g/2$, in which case we assume without loss of generality that $\gamma \geq |\gamma_g|$.

3.5 Comparison Principle and Uniqueness

We now provide a comparison theorem that yields uniqueness of a BSDE solution triple in a specific space. The proof makes use of the θ-technique applied in the context of second order Bellman-Isaacs equations by Da Lio and Ley [2006] and subsequently adapted to the framework of Brownian BSDEs in Briand and Hu [2008]. We extend these ideas to take into account the orthogonal part of the BSDE solution.

Theorem 3.5.1 (Comparison Principle)**.** *Let* (Ψ, Z, N) *and* (Ψ', Z', N') *be solutions to the BSDE* (3.2.3) *with drivers* F *and* F' *and terminal conditions* ξ *and* ξ'*, respectively. Suppose in addition that* $\Psi \in \mathfrak{E}$ *and* $\Psi' \in \mathfrak{E}$*. If* $\mathbb{P}$*-a.s. for all* $t \in [0, T]$*,*

$$\xi \leq \xi' \quad \text{and} \quad F(t, \Psi'_t, Z'_t) \leq F'(t, \Psi'_t, Z'_t),$$

and if (F, ξ) *satisfies Assumption 3.2.2 (i)-(iii) then* $\mathbb{P}$*-a.s. for each* $t \in [0, T]$

$$\Psi_t \leq \Psi'_t.$$

Proof. Let θ be a real number in $(0, 1)$ and set $U := \Psi - \theta\Psi'$, $V := Z - \theta Z'$ and $W := N - \theta N'$. We first collect together some helpful estimates concerning the drivers F and F'. Consider the process

$$\rho_s := \begin{cases} \frac{F(s,\Psi_s,Z_s)-F(s,\theta\Psi'_s,Z_s)}{U_s} & \text{if } U_s \neq 0, \\ \overline{\beta} & \text{if } U_s = 0. \end{cases}$$

By Assumption 3.2.2 (ii) ρ is bounded by $\overline{\beta}$. We define $R_s := \int_0^s \rho_r \, dA_r$ and notice that by the boundedness of A we have that $|R| \leq \overline{\beta}\, A_T \leq \overline{\beta} K_A$. From Itô's formula we deduce

$$e^{R_t} U_t = e^{R_T} U_T - \int_t^T e^{R_s} (V_s^\intercal \, dM_s + dW_s) + \int_t^T e^{R_s} \Big(F_s^\theta \, dA_s + \frac{1}{2} \Big(d\langle N \rangle_s - \theta \, d\langle N' \rangle_s \Big) \Big),$$

where we define $F_s^\theta := F(s, \Psi_s, Z_s) - \theta F'(s, \Psi'_s, Z'_s) - \rho_s U_s$. We also set

$$\Delta F(s) := (F - F')(s, \Psi'_s, Z'_s) \leq 0,$$

where the inequality is due to the assumption of the theorem, and observe that from the convexity of F in z together with (3.2.7) we get

$$F(s,\Psi'_s,Z_s) - \theta F(s,\Psi'_s,Z'_s) = F\Big(s,\Psi'_s,\theta Z'_s + (1-\theta)\frac{Z_s-\theta Z'_s}{1-\theta}\Big) - \theta F(s,\Psi'_s,Z'_s)$$
$$\le (1-\theta)\, F\Big(s,\Psi'_s,\frac{Z_s-\theta Z'_s}{1-\theta}\Big) \le (1-\theta)\alpha_s + (1-\theta)\overline{\beta}|\Psi'_s| + \frac{\gamma}{2(1-\theta)}\,\|B_sV_s\|^2. \tag{3.5.1}$$

Another application of the Lipschitz assumption 3.2.2 (ii), yields

$$\begin{aligned} F(s,\Psi_s,Z_s) - F(s,\Psi'_s,Z_s) &= F(s,\Psi_s,Z_s) - F(s,\theta\Psi'_s,Z_s) + F(s,\theta\Psi'_s,Z_s) - F(s,\Psi'_s,Z_s) \\ &= \rho_sU_s + F(s,\theta\Psi'_s,Z_s) - F(s,\Psi'_s,Z_s) \\ &\le \rho_sU_s + (1-\theta)\overline{\beta}|\Psi'_s|. \end{aligned} \tag{3.5.2}$$

Combining (3.5.1) and (3.5.2) we see that

$$\begin{aligned} F^\theta_s &= F(s,\Psi_s,Z_s) - \theta F(s,\Psi'_s,Z'_s) + \theta\,\Delta F(s) - \rho_sU_s \\ &= [F(s,\Psi_s,Z_s) - F(s,\Psi'_s,Z_s)] + [F(s,\Psi'_s,Z_s) - \theta F(s,\Psi'_s,Z'_s)] + \theta\,\Delta F(s) - \rho_sU_s \\ &\le (1-\theta)\Big(\alpha_s + 2\overline{\beta}|\Psi'_s|\Big) + \frac{\gamma}{2(1-\theta)}\,\|B_sV_s\|^2 + \theta\,\Delta F(s). \end{aligned} \tag{3.5.3}$$

Let us now work towards an estimate for $U = \Psi - \theta\Psi'$. Set $\kappa := \frac{\gamma\exp(\overline{\beta}K_A)}{1-\theta} > 0$ and $\overline{P}_t := \exp\Big(\kappa e^{R_t}U_t\Big) > 0$. In what follows the logic is similar to how we derived the a priori estimates, namely, to show that by removing an appropriate drift $\overline{P}$ is a (local) submartingale. By Itô's formula, for $t \in [0,T]$,

$$\overline{P}_t = \overline{P}_T - \int_t^T \kappa\overline{P}_se^{R_s}(V_s^\intercal\,dM_s + dW_s) + \int_t^T \kappa\overline{P}_se^{R_s}\left(F^\theta_s - \frac{\kappa e^{R_s}}{2}\,\|B_sV_s\|^2\right)dA_s \tag{3.5.4}$$

$$+ \int_t^T \kappa\overline{P}_se^{R_s}\Big(-\frac{\kappa e^{R_s}}{2}\,d\langle W\rangle_s + \frac{1}{2}\Big(d\langle N\rangle_s - \theta\,d\langle N'\rangle_s\Big)\Big). \tag{3.5.5}$$

To simplify notation set

$$G := \kappa\overline{P}e^R\left(F^\theta - \frac{\kappa e^R}{2}\,\|BV\|^2\right) \quad \text{and} \tag{3.5.6}$$

$$H := \int_0^\cdot \kappa\overline{P}_se^{R_s}\Big(-\frac{\kappa e^{R_s}}{2}\,d\langle W\rangle_s + \frac{1}{2}\Big(d\langle N\rangle_s - \theta\,d\langle N'\rangle_s\Big)\Big). \tag{3.5.7}$$

Let us first investigate the finite variation process H. We claim that H is decreasing, indeed for all $r, u \in [0,T]$, $r \le u$, we have

$$\int_r^u d\langle W\rangle_s = \int_r^u d\langle N\rangle_s - 2\theta\, d\langle N, N'\rangle_s + \theta^2\, d\langle N'\rangle_s.$$

Applying the Kunita-Watanabe and Young inequalities,

$$\begin{aligned}\int_r^u d\langle W\rangle_s &\ge \int_r^u d\langle N\rangle_s + \int_r^u \theta^2\, d\langle N'\rangle_s - 2\theta \left(\int_r^u d\langle N\rangle_s\right)^{1/2} \left(\int_r^u d\langle N'\rangle_s\right)^{1/2} \\ &\ge \int_r^u d\langle N\rangle_s + \int_r^u \theta^2\, d\langle N'\rangle_s - \theta \left(\int_r^u d\langle N\rangle_s + \int_r^u d\langle N'\rangle_s\right) \\ &= (1-\theta)\left(\int_r^u d\langle N\rangle_s - \theta\, d\langle N'\rangle_s\right).\end{aligned}$$

In particular, since $\gamma \ge 1$ and $|R| \le \overline{\beta} K_A$ we have,

$$\int_r^u \kappa e^{R_s} d\langle W\rangle_s \ge \frac{\gamma}{1-\theta}\int_r^u d\langle W\rangle_s \ge \int_r^u d\langle N\rangle_s - \theta\, d\langle N'\rangle_s,$$

which shows that the process H is decreasing and hence the integral in (3.5.5) is non-positive.

Next we consider the finite variation integral in (3.5.4). Combining (3.5.3), $\Delta F \le 0$ and the boundedness of R we have

$$G = \kappa\overline{P}e^R\left(F^\theta - \frac{\kappa e^R}{2}\|BV\|^2\right) \le \kappa\overline{P}e^R\Big((1-\theta)\Big(\alpha + 2\overline{\beta}|\Psi'|\Big)\Big) \le \overline{P}J, \tag{3.5.8}$$

where

$$J := \gamma e^{2\overline{\beta}K_A}\Big(\alpha + 2\overline{\beta}|\Psi'|\Big) \ge 0.$$

We set

$$D_t := \exp\left(\int_0^t J_s\, dA_s\right) \qquad \text{and} \qquad \widetilde{P}_t := D_t\overline{P}_t.$$

Partial integration yields

$$\begin{aligned} d\widetilde{P}_t &= D_t\Big(-G_t\, dA_t - dH_t + \kappa\overline{P}_t e^{R_t}(V_t^\intercal\, dM_t + dW_t)\Big) + \overline{P}_t D_t J_t\, dA_t \\ &= D_t\Big((\overline{P}_t J_t - G_t)\, dA_t - dH_t + \kappa\overline{P}_t e^{R_t}(V_t^\intercal\, dM_t + dW_t)\Big) \end{aligned} \tag{3.5.9}$$

and we conclude that the finite variation parts in the last expression are nonnegative. We can now use the following stopping time argument to derive that $\mathbb{P}$-a.s.

$$\overline{P}_t \le \mathbb{E}\left[\frac{D_T}{D_t}\,\overline{P}_T \,\middle|\, \mathcal{F}_t\right]. \tag{3.5.10}$$

Namely, consider the stopping time

$$\tau_n := \inf\Big\{u \in [t,T] \,\Big|\, \int_t^u \kappa^2 \widetilde{P}_s^2 \, e^{2R_s}\Big(V_s^\top \, d\langle M\rangle_s V_s + d\langle W\rangle_s\Big) \geq n\Big\} \wedge T,$$

where $n \geq 1$ is an integer. Observe that $\tau_n \to T$ as $n \to +\infty$ due to the integrability assumptions on α, Ψ and Ψ', as well as the boundedness of A. Then (3.5.9) provides the estimate

$$\overline{P}_t \leq \mathbb{E}\Big[\exp\Big(\int_t^{\tau_n} J_s \, dA_s\Big)\, \overline{P}_{\tau_n}\Big|\, \mathcal{F}_t\Big] = \mathbb{E}\Big[\exp\Big(\int_t^{\tau_n} \gamma e^{2\overline{\beta}K_A}\Big(\alpha_s + 2\overline{\beta}|\Psi'_s|\Big)\, dA_s\Big)\overline{P}_{\tau_n}\Big|\, \mathcal{F}_t\Big].$$

In view of the current integrability and boundedness assumptions we can send n to infinity and deduce (3.5.10). This last inequality is the relation that we need for estimating $U = \Psi - \theta\Psi'$.

Notice that we also have $\xi - \theta\xi' \leq (1-\theta)|\xi| + \theta\Delta\xi$, where $\Delta\xi := \xi - \xi' \leq 0$. Then together with the definition of $\overline{P}$ the inequality (3.5.10) shows that

$$\begin{aligned}
&\exp\Big(\frac{\gamma e^{\overline{\beta}K_A + R_t}}{1-\theta}\,(\Psi_t - \theta\Psi'_t)\Big) \\
&\qquad \leq \mathbb{E}\Big[\exp\Big(\int_t^T \gamma e^{2\overline{\beta}K_A}\Big(\alpha_s + 2\overline{\beta}|\Psi'_s|\Big)\, dA_s\Big)\exp\Big(\kappa e^{R_T}(\xi - \theta\xi')\Big)\Big|\, \mathcal{F}_t\Big] \\
&\qquad \leq \mathbb{E}\Big[\exp\Big(\gamma e^{2\overline{\beta}K_A}\int_t^T \Big(\alpha_s + 2\overline{\beta}|\Psi'_s|\Big)\, dA_s\Big)\exp\Big(\gamma e^{2\overline{\beta}K_A}|\xi|\Big)\Big|\, \mathcal{F}_t\Big].
\end{aligned}$$

Thus, we can derive the estimate

$$\Psi_t - \theta\Psi'_t \leq \frac{1-\theta}{\gamma}\log\mathbb{E}\Big[\exp\Big(\gamma e^{2\overline{\beta}K_A}\Big(|\xi| + \int_t^T \Big(\alpha_s + 2\overline{\beta}|\Psi'_s|\Big)\, dA_s\Big)\Big)\Big|\, \mathcal{F}_t\Big],$$

which follows from the above by checking the cases $\Psi_t - \theta\Psi'_t \geq 0$ and $\Psi_t - \theta\Psi'_t < 0$ separately, noting that $R + \overline{\beta}K_A \geq 0$. Once again, by the integrability assumptions on ξ, α and Ψ' and the boundedness of A, the conditional expectation on the right hand side is finite, $\mathbb{P}$-a.s. Taking $\theta \uparrow 1$ then gives $\Psi_t \leq \Psi'_t$ and the continuity of Ψ and Ψ' yields the claim. □

The following corollary is then immediate.

Corollary 3.5.2 (Uniqueness)**.** *Let the Assumption 3.2.2 (i)-(iii) hold and let* (Ψ, Z, N) *and* (Ψ', Z', N') *be two solutions to the BSDE* (3.2.3) *with* $\Psi \in \mathfrak{E}$ *and* $\Psi' \in \mathfrak{E}$*. Then* Ψ *and* Ψ'*,* $Z \cdot M$ *and* $Z' \cdot M$*, and* N *and* N' *are indistinguishable. Under the additional Assumption 3.2.2 (v) both* $(Z \cdot M, N)$ *and* $(Z' \cdot M, N')$ *belong to* $\mathcal{M}^\varrho \times \mathcal{M}^\varrho$ *for all* $\varrho > 0$*.*

Proof. By Theorem 3.5.1 and Corollary 3.4.3 (ii) only the assertion regarding the indistinguishability of the martingale part remains. Itô's formula gives $\mathbb{P}$-a.s.

$$0 = (\Psi_T - \Psi'_T)^2 = (\Psi_0 - \Psi'_0)^2 + 2\int_0^T (\Psi_t - \Psi'_t)\, d(\Psi_t - \Psi'_t)$$
$$+ \int_0^T (Z_t - Z'_t)^\intercal\, d\langle M\rangle_t (Z_t - Z'_t) + d\langle N - N'\rangle_t$$
$$= \int_0^T (Z_t - Z'_t)^\intercal\, d\langle M\rangle_t (Z_t - Z'_t) + d\langle N - N'\rangle_t,$$

from which $Z \cdot M \equiv Z' \cdot M$ and $N \equiv N'$. □

3.6 Stability

It follows from the previous results that the BSDE (3.2.3) has a unique solution with first component in $\mathfrak{E}$ under appropriate Lipschitz and convexity assumptions on the driver F and under an exponential moments condition on the terminal value ξ and the process α. In the present section we show that a stability result for such BSDEs holds as well. More precisely, given a sequence of terminal values and a sequence of drivers such that the exponential moments condition is fulfilled uniformly and such that they converge to a fixed terminal value and a fixed generator in a suitable sense, then we gain convergence on the level of the respective BSDE solutions. This is as in the Brownian framework of Briand and Hu [2008], however see Remark 3.6.8 and the following subsection for a discussion of the appropriate mode of convergence of the drivers.

Theorem 3.6.1 (Stability)**.** *Let $(F^n)_{n\geq 0}$ be a sequence of generators for the BSDE* (3.2.3) *such that Assumption 3.2.2 (ii)-(iii) and (v) hold for each F^n with the set of parameters $(\alpha^n, \beta^n, \overline{\beta}, \beta_f, \gamma)$. If $(\xi^n)_{n\geq 0}$ are the associated random terminal values then suppose that, for each $\varrho > 1$,*

$$\sup_{n\geq 0} \mathbb{E}\Big[e^{\varrho\,(|\xi^n| + |\alpha^n|_1)}\Big] < +\infty. \tag{3.6.1}$$

Let (Ψ^n, Z^n, N^n) be the solution to the BSDE (3.2.3) *with driver F^n and terminal condition ξ^n such that $\Psi^n \in \mathfrak{E}$ for all $n \geq 0$. If*

$$|\xi^n - \xi^0| + \int_0^T \left|F^n - F^0\right| (s, \Psi^0_s, Z^0_s)\, dA_s \longrightarrow 0 \quad \text{in probability, as } n \to +\infty, \tag{3.6.2}$$

then for each $\varrho > 0$,

$$\lim_{n\to+\infty} \mathbb{E}\left[\exp\left(\varrho \sup_{0\leq t\leq T} \left|\Psi^n_t - \Psi^0_t\right|\right)\right] = 1 \quad \textit{and}$$
$$\lim_{n\to+\infty} \mathbb{E}\left[\left(\int_0^T (Z^n_s - Z^0_s)^\intercal\, d\langle M\rangle_s (Z^n_s - Z^0_s) + d\langle N^n - N^0\rangle_s\right)^{\varrho/2}\right] = 0.$$

Remark 3.6.2. Let us briefly indicate how the above stability theorem differs from those in the literature, Frei [2009] Theorem 2.1 and Morlais [2009] Lemma 3.3. The key points are that firstly in our conditions the parameters α^n and β^n are allowed to depend on n, whereas in Frei [2009] and Morlais [2009] they are assumed independent of n. Secondly, we assume a uniform exponential moments condition, as opposed to a uniform boundedness condition in the cited references. Finally, in the unbounded setting we require the mode of convergence assumed above, this is in contrast to the setting of Frei [2009] Theorem 2.1 where the weaker notion of pointwise convergence is sufficient for a stability result to hold (due to the uniform growth and boundedness estimates).

Proof. Note that Assumption 3.2.2 (i) holds for each n thanks to (3.6.1). Exactly as in the statement of Corollary 3.4.3 we deduce that for each $\varrho \geq 1$

$$\sup_{n\geq 0} \mathbb{E}\left[\exp\left(\varrho \sup_{0\leq t\leq T} |\Psi^n_t|\right) + \left(\int_0^T (Z^n_s)^\intercal \, d\langle M\rangle_s Z^n_s + d\langle N^n\rangle_s\right)^{\varrho/2}\right] < +\infty.$$

Hence the sequences in n of random variables

$$\exp\left(\varrho \sup_{0\leq t\leq T} |\Psi^n_t|\right) \quad \text{and} \quad \left(\int_0^T (Z^n_s)^\intercal \, d\langle M\rangle_s Z^n_s + d\langle N^n\rangle_s\right)^{\varrho/2}$$

are uniformly integrable for all $\varrho \geq 1$. By the Vitali convergence theorem, it is thus sufficient to prove that

$$\sup_{0\leq t\leq T} |\Psi^n_t - \Psi^0_t| + \int_0^T (Z^n_s - Z^0_s)^\intercal \, d\langle M\rangle_s (Z^n_s - Z^0_s) + d\langle N^n - N^0\rangle_s \longrightarrow 0$$

in probability when n tends to infinity.

We split the proof of the last statement into four steps. The first two steps construct one-sided estimates for the difference of Ψ^n and Ψ^0 proceeding very similarly to the proof of the comparison result. In the third step we combine the aforementioned estimates to show that $\Psi^n - \Psi^0$ converges to zero uniformly on $[0, T]$ in probability, i.e. in *ucp*. Finally, in Step 4, we use this result to show the required convergence of the martingale parts.

Step 1. First fix $\theta \in (0, 1)$ and $n \geq 1$ and proceed in the same way as in the proof of Theorem 3.5.1 by defining the same objects U, V, W, ρ, R, F^θ, κ, $\overline{P}$, G, and H, subject to the following modification. All the objects Ξ', $\Xi \in \{\Psi, Z, N, F\}$, with a prime $'$ are replaced by the respective object Ξ^0 with a superscript 0. All the objects $\Xi \in \{\Psi, Z, N, F, \alpha\}$ without a prime are replaced by the respective object Ξ^n with a superscript n, e.g. $U := \Psi^n - \theta\Psi^0$. We observe that the above objects U, V, W, $\ldots$ depend on n however we omit this dependence for notational brevity. In addition set

$\Delta^n F(s) := (F^n - F^0)(s, \Psi^0_s, Z^0_s)$. From (3.5.3) and (3.5.6),

$$
\begin{aligned}
G &\le \kappa \overline{P} e^R \Big((1-\theta)(\alpha^n + 2\overline{\beta}|\Psi^0|) + \theta\, \Delta^n F \Big) \\
&\le \gamma e^{2\overline{\beta}K_A} \overline{P} \left(\alpha^n + 2\overline{\beta}|\Psi^0| + \frac{|\Delta^n F|}{1-\theta} \right) = \overline{P} J^n + \gamma e^{2\overline{\beta}K_A} \overline{P} \frac{|\Delta^n F|}{1-\theta},
\end{aligned}
$$

where, consistent with our modification,

$$
J^n := \gamma e^{2\overline{\beta}K_A} \Big(\alpha^n + 2\overline{\beta}|\Psi^0| \Big) \ge 0.
$$

Observe that in contrast to the proof of the comparison theorem, the difference $\Delta^n F$ of the drivers cannot be bounded above by zero here. Considering

$$
D^n_t := \exp\left(\int_0^t J^n_s \, dA_s \right) \qquad \text{and} \qquad \widetilde{P}^n_t := D^n_t \overline{P}_t
$$

and applying the partial integration formula we obtain

$$
\begin{aligned}
d\widetilde{P}^n_t + \gamma e^{2\overline{\beta}K_A} \widetilde{P}^n_t \frac{|\Delta^n F|}{1-\theta}\, dA_t &= D^n_t \, d\overline{P}_t + \overline{P}_t \, dD^n_t + \gamma e^{2\overline{\beta}K_A} \widetilde{P}^n_t \frac{|\Delta^n F|}{1-\theta}\, dA_t \\
= D^n_t \left[\left(\overline{P}_t J^n_t + \gamma e^{2\overline{\beta}K_A} \overline{P}_t \frac{|\Delta^n F|}{1-\theta} - G_t \right) dA_t \right. & \left. - dH_t + \kappa \overline{P}_t e^{R_t} (V_t^{\intercal}\, dM_t + dW_t) \right],
\end{aligned}
$$

noting that the orthogonal terms also have been dealt with in the proof of the comparison theorem. Again, we conclude that the finite variation parts in the last expression are nonnegative. We then use the stopping time argument from the proof of Theorem 3.5.1 to derive the inequality

$$
\overline{P}_t \le D^n_t \overline{P}_t \le \mathbb{E}\left[D^n_T \overline{P}_T + \frac{\gamma e^{2\overline{\beta}K_A}}{1-\theta} \int_t^T D^n_s \overline{P}_s |\Delta^n F(s)| \, dA_s \,\middle|\, \mathcal{F}_t \right]. \tag{3.6.3}
$$

From the boundedness of ρ and the definition $\overline{P} = \exp\Big(\kappa e^R U\Big)$ we deduce, for $s \in [0,T]$,

$$
\begin{aligned}
\overline{P}_s &\le \sup_{0\le t\le T} \left[\exp\left(\frac{\gamma e^{2\overline{\beta}K_A}}{1-\theta} \Big(|\Psi^0_t| + |\Psi^n_t| \Big) \right) \right] =: \Upsilon^n(\theta) \quad \text{and} \\
\overline{P}_T &\le \exp\left(\frac{\gamma e^{2\overline{\beta}K_A}}{1-\theta} |\xi^n - \theta\xi^0| \right) \le \exp\left(\frac{\gamma e^{2\overline{\beta}K_A}}{1-\theta} \Big(|\xi^n - \theta\xi^0| \vee |\xi^0 - \theta\xi^n| \Big) \right) =: \chi^n(\theta),
\end{aligned}
$$

where the definitions of $\Upsilon^n(\theta)$ and $\chi^n(\theta)$ are in anticipation of a converse inequality to be derived in Step 2. Using the boundedness of ρ, the inequalities $\log(x) \le x$, (3.6.3) and $1 \le D^n_s \le D^n_T$ we then find that

$$\begin{aligned}\Psi^n_t - \Psi^0_t &\le (1-\theta)|\Psi^0_t| + \Psi^n_t - \theta\Psi^0_t = (1-\theta)|\Psi^0_t| + U_t \\ &= (1-\theta)|\Psi^0_t| + \frac{1-\theta}{\gamma}\exp\Big(-\overline{\beta}K_A - R_t\Big)\log(\overline{P}_t) \\ &\le (1-\theta)|\Psi^0_t| + \frac{1-\theta}{\gamma}\,\mathbb{E}\left[D^n_T\chi^n(\theta) + \frac{\gamma e^{2\overline{\beta}K_A}}{1-\theta}\,D^n_T\Upsilon^n(\theta)\int_t^T |\Delta^n F(s)|\,dA_s \,\middle|\, \mathcal{F}_t\right].\end{aligned} \tag{3.6.4}$$

Step 2. With regards to the converse inequality we proceed as in the proof of Briand and Hu [2008] Proposition 7. More specifically, recalling the setting of the proof of Theorem 3.5.1 we define the same objects U, V, W, R, F^θ, κ, $\overline{P}$, G, and H but now subject to the following modification. All the objects Ξ', $\Xi \in \{\Psi, Z, N, F\}$, with a prime $'$ are replaced by the respective object Ξ^n with a superscript $n \ge 1$. All the objects $\Xi \in \{\Psi, Z, N, F, \alpha\}$ without a prime are replaced by the respective object Ξ^0 with a superscript 0, e.g. $U := \Psi^0 - \theta\Psi^n$. Moreover, here, we define ρ differently, namely,

$$\rho_s := \frac{F^n(s, \Psi^0_s, Z^n_s) - F^n(s, \Psi^n_s, Z^n_s)}{\Psi^0_s - \Psi^n_s}\,\mathbf{1}_{\{|\Psi^0_s - \Psi^n_s| > 0\}}.$$

This ensures that $|\rho| \le \overline{\beta}$ still holds and that

$$\theta F^n(s, \Psi^0_s, Z^n_s) - \theta F^n(s, \Psi^n_s, Z^n_s) = \rho(\theta\Psi^0 - \theta\Psi^n) \le \overline{\beta}(1-\theta)|\Psi^0_s| + \rho_s U_s.$$

Using the convexity of F^n, the estimate (3.2.7) and the *same* definition of $\Delta^n F$ as in Step 1 we derive

$$F^\theta \le |\Delta^n F| + (1-\theta)(\alpha^n + 2\overline{\beta}|\Psi^0|) + \frac{\gamma}{2(1-\theta)}\,\|BV\|^2$$

so that the following inequality holds

$$G \le \gamma e^{2\overline{\beta}K_A}\overline{P}\left(\alpha^n + 2\overline{\beta}|\Psi^0| + \frac{|\Delta^n F|}{1-\theta}\right).$$

Observe that this is the *same* estimate on G as that obtained in Step 1. Thus we may rewrite (3.6.4) as

$$\Psi^0_t - \Psi^n_t \le (1-\theta)|\Psi^n_t| + \frac{1-\theta}{\gamma}\,\mathbb{E}\left[D^n_T\chi^n(\theta) + \frac{\gamma e^{2\overline{\beta}K_A}}{1-\theta}\,D^n_T\Upsilon^n(\theta)\int_t^T |\Delta^n F(s)|\,dA_s \,\middle|\, \mathcal{F}_t\right], \tag{3.6.5}$$

where J^n and thus D^n, Υ^n and $\chi^n(\theta)$ are as in Step 1.

Step 3. Let us now prove that $\left(\sup_{0\le t\le T}|\Psi^n_t - \Psi^0_t|\right)_{n\ge1}$ converges to zero in probability. Summing up (3.6.4) and (3.6.5) we deduce

$$\begin{aligned}|\Psi_t^n - \Psi_t^0| \le (1-\theta)\left(|\Psi_t^0| + |\Psi_t^n|\right) + \frac{1-\theta}{\gamma}\,\mathbb{E}\Big[D_T^n\chi^n(\theta)\Big|\,\mathcal{F}_t\Big] \\ + e^{2\overline{\beta}K_A}\,\mathbb{E}\bigg[D_T^n\Upsilon^n(\theta)\int_t^T |\Delta^n F(s)|\,dA_s\bigg|\,\mathcal{F}_t\bigg].\end{aligned}$$

We note that by the usual assumptions on the filtration and by the continuity of Ψ^n and Ψ^0 this holds for all t, $\mathbb{P}$-a.s. Applying the Doob, Markov and Hölder inequalities, we deduce the existence of some positive constants c_1 and c_2, which are independent of θ, as well as of a positive constant $c_3(\theta)$ such that for $\varepsilon > 0$,

$$\begin{aligned}&\mathbb{P}\bigg(\sup_{0\le t\le T}|\Psi_t^n - \Psi_t^0| \ge \varepsilon\bigg) \\ &\le \mathbb{P}\bigg((1-\theta)\sup_{0\le t\le T}\left(|\Psi_t^0| + |\Psi_t^n|\right) \ge \frac{\varepsilon}{3}\bigg) + \mathbb{P}\bigg(\frac{1-\theta}{\gamma}\sup_{0\le t\le T}\mathbb{E}\Big[D_T^n\chi^n(\theta)\Big|\,\mathcal{F}_t\Big] \ge \frac{\varepsilon}{3}\bigg) \\ &\qquad + \mathbb{P}\bigg(e^{2\overline{\beta}K_A}\sup_{0\le t\le T}\mathbb{E}\bigg[D_T^n\Upsilon^n(\theta)\int_t^T |\Delta^n F(s)|\,dA_s\bigg|\,\mathcal{F}_t\bigg] \ge \frac{\varepsilon}{3}\bigg) \\ &\le \frac{3(1-\theta)}{\varepsilon}\,\mathbb{E}\bigg[\sup_{0\le t\le T}\left(|\Psi_t^0| + |\Psi_t^n|\right)\bigg] + \frac{3(1-\theta)}{\varepsilon\gamma}\,\mathbb{E}[D_T^n\chi^n(\theta)] \\ &\qquad + \frac{3e^{2\overline{\beta}K_A}}{\varepsilon}\,\mathbb{E}\bigg[D_T^n\Upsilon^n(\theta)\int_0^T |\Delta^n F(s)|\,dA_s\bigg] \\ &\le \frac{c_1(1-\theta)}{\varepsilon} + \frac{c_2(1-\theta)}{\varepsilon}\,\mathbb{E}\Big[\chi^n(\theta)^2\Big]^{1/2} + \frac{c_3(\theta)}{\varepsilon}\,\mathbb{E}\bigg[\bigg(\int_0^T |\Delta^n F(s)|\,dA_s\bigg)^2\bigg]^{1/2},\end{aligned} \tag{3.6.6}$$

where the last inequality is due to the fact that by our assumptions the sequences $\left(\sup_{0\le t\le T}\left(|\Psi_t^0| + |\Psi_t^n|\right)\right)_{n\ge1}$, $(D_T^n)_{n\ge1}$ and $(\Upsilon^n(\theta))_{n\ge1}$ are bounded in all $L^\varrho(\mathbb{P})$ spaces, $\varrho \ge 1$. In addition, for the application of Doob's inequality, we used that A is bounded together with

$$\begin{aligned}|\Delta^n F(s)| \le |F^n(s,\Psi_s^0,Z_s^0) - F^n(s,0,Z_s^0)| + |F^n(s,0,Z_s^0)| + |F^0(s,0,Z_s^0)| \\ + |F^0(s,\Psi_s^0,Z_s^0) - F^0(s,0,Z_s^0)| \le 2\overline{\beta}|\Psi_s^0| + \alpha_s^n + \alpha_s^0 + \gamma\|B_sZ_s^0\|^2,\end{aligned} \tag{3.6.7}$$

which in turn now also implies that for all $\varrho \ge 1$ the family of random variables

$$\left(\left(\int_0^T |\Delta^n F(s)|\,dA_s\right)^\varrho\right)_{n\ge1}$$

is uniformly integrable due to Corollary 3.4.3 and (3.6.1). Here, for reasons explained in Subsection 3.6.1, we deviate from Briand and Hu [2008]. The Vitali convergence theorem and (3.6.2) then imply that

$$\int_0^T |\Delta^n F(s)|\, dA_s \to 0 \ \text{ in all } L^\varrho(\mathbb{P}) \text{ spaces.}$$

Furthermore, the sequence $(\chi^n(\theta))_{n\geq 1}$ converges in probability to $\exp\Big(\gamma e^{2\overline{\beta} K_A}|\xi^0|\Big)$ as n goes to infinity. This convergence is also in all $L^\varrho(\mathbb{P})$ spaces because of the uniform integrability assumption on $(\xi^n)_{n\geq 1}$. More precisely, for all $\varrho \geq 1$, we have

$$\begin{aligned}\sup_{n\geq 1}\mathbb{E}[\chi^n(\theta)^\varrho] &\leq \sup_{n\geq 1}\mathbb{E}\left[\exp\left(\frac{\varrho\gamma e^{2\overline{\beta}K_A}}{1-\theta}\Big(|\xi^n|+|\xi^0|\Big)\right)\right]\\ &\leq \sup_{n\geq 1}\mathbb{E}\left[\exp\left(\frac{2\varrho\gamma e^{2\overline{\beta}K_A}}{1-\theta}\,|\xi^n|\right)\right]^{1/2}\mathbb{E}\left[\exp\left(\frac{2\varrho\gamma e^{2\overline{\beta}K_A}}{1-\theta}\,|\xi^0|\right)\right]^{1/2} < +\infty.\end{aligned}$$

From (3.6.6) we then deduce that for all $\theta \in (0,1)$,

$$\limsup_{n\to+\infty}\mathbb{P}\left(\sup_{0\leq t\leq T}|\Psi^n_t - \Psi^0_t| \geq \varepsilon\right) \leq \frac{c_1(1-\theta)}{\varepsilon} + \frac{c_2(1-\theta)}{\varepsilon}\,\mathbb{E}\Big[\exp\Big(2\gamma e^{2\overline{\beta}K_A}|\xi^0|\Big)\Big]^{1/2}.$$

We then send θ to 1 to conclude that

$$\lim_{n\to+\infty}\mathbb{P}\left(\sup_{0\leq t\leq T}|\Psi^n_t - \Psi^0_t| \geq \varepsilon\right) = 0.$$

Step 4. Let us now turn to the last assertion of the theorem. We derive from Itô's formula that

$$\begin{aligned}&\mathbb{E}\left[\int_0^T (Z^n_s - Z^0_s)^\intercal\, d\langle M\rangle_s (Z^n_s - Z^0_s) + d\langle N^n - N^0\rangle_s\right]\\ &\quad\leq \mathbb{E}\left[(\xi^n - \xi^0)^2 + 2\left(\sup_{0\leq t\leq T}|\Psi^n_t - \Psi^0_t|\right)\int_0^T |F^n(s,\Psi^n_s,Z^n_s) - F^0(s,\Psi^0_s,Z^0_s)|\, dA_s\right]\\ &\qquad\qquad + \mathbb{E}\left[\left(\sup_{0\leq t\leq T}|\Psi^n_t - \Psi^0_t|\right)\left|\int_0^T d\langle N^n\rangle_s - d\langle N^0\rangle_s\right|\right]\end{aligned}$$

after observing that the local martingale arising therein is in fact a true martingale thanks to the present integrability assumptions, see Corollary 3.4.3. By (3.2.7),

$$|F^n(s,\Psi^n_s,Z^n_s) - F^0(s,\Psi^0_s,Z^0_s)| \leq \alpha^n_s + \alpha^0_s + \overline{\beta}|\Psi^n_s| + \overline{\beta}|\Psi^0_s| + \frac{\gamma}{2}\,\|B_s Z^n_s\|^2 + \frac{\gamma}{2}\,\|B_s Z^0_s\|^2.$$

Clearly, $\left|\int_0^T d\langle N^n\rangle_s - d\langle N^0\rangle_s\right| \leq \langle N^n\rangle_T + \langle N^0\rangle_T$, so that applying Hölder's inequality, the formula (3.4.4) and the condition (3.6.1) we recognize (the expectation of the squares of) the integrals from the right hand side above as uniformly bounded (in n). The result then follows from the fact that $\xi^n \to \xi^0$ in $L^2(\mathbb{P})$ and that by the Steps 1-3 also $\left(\sup_{0\le t\le T}|\Psi_t^n - \Psi_t^0|\right) \to 0$ in $L^2(\mathbb{P})$. To sum up, we conclude that

$$\int_0^T (Z_s^n - Z_s^0)^\intercal \, d\langle M\rangle_s (Z_s^n - Z_s^0) + d\langle N^n - N^0\rangle_s \xrightarrow{\mathbb{P}} 0 \quad \text{as} \quad n \to +\infty,$$

which completes the proof. □

Remark 3.6.3. As previously mentioned the sense of convergence given here differs from that in Briand and Hu [2008] where the pointwise convergence of the drivers is assumed, namely

$$\mu^A\text{-a.e. for all } \psi \text{ and } z \text{ we have } \lim_{n\to+\infty} F^n(\cdot,\psi,z) = F^0(\cdot,\psi,z). \tag{3.6.8}$$

We provide an example in the next section showing that this condition is not sufficient in the present setting so that the statement of Briand and Hu [2008] Proposition 7 needs a small modification.

3.6.1 Stability Counterexample

Suppose our filtration is the augmentation of the filtration generated by a one-dimensional Brownian motion W so that we may set $A_t = t$ and $B \equiv 1$. The measure μ^A now becomes the product of $\mathbb{P}$ and the Lebesgue measure on $[0,T]$. In this setting BSDEs take the form

$$d\Psi_t = Z_t \, dW_t - F(t,\Psi_t,Z_t)\, dt, \quad \Psi_T = \xi, \tag{3.6.9}$$

and solutions consist of pairs (Ψ, Z) such that Ψ has continuous paths, Z is a predictable process with $\int_0^T Z_t^2 \, dt < +\infty$ $\mathbb{P}$-a.s., $\int_0^T |F(t,\Psi_t,Z_t)|\, dt < +\infty$ $\mathbb{P}$-a.s. and such that the integrated version of (3.6.9) holds, $\mathbb{P}$-a.s.

Suppose our condition $\int_0^T |F^n - F^0|\,(s,\Psi_s^0,Z_s^0)\, ds \xrightarrow{\mathbb{P}} 0$ as $n \to +\infty$ is replaced by (3.6.8), i.e. F^n converges to F^0 pointwise (t,ω)-almost everywhere on $[0,T]\times\Omega$, where the μ^A-null set does not depend on (ψ,z). One may ask whether this is sufficient for Theorem 3.6.1 to hold, in particular if

$$\sup_{0\le t\le T} |\Psi_t^n - \Psi_t^0| \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \to +\infty. \tag{3.6.10}$$

We now present an example to show that this is in fact not the case. The example resembles the standard counterexample to the dominated convergence theorem and shows that such a stability statement (under the present assumptions) already fails to hold in an essentially deterministic situation.

Consider $T > 1$ together with parameters $F^0 \equiv \alpha^0 \equiv \xi^0 \equiv 0$. Then all the assumptions in Briand and Hu [2008] and in the present chapter are satisfied and the unique solution to the BSDE (3.6.9) with parameters (F^0, ξ^0) is given by $(\Psi^0, Z^0) \equiv (0, 0)$, up to appropriate null sets.

Furthermore, for integers $n \geq 1$, define the terminal values $\xi^n \equiv 0$ and drivers

$$F^n \equiv \alpha^n \equiv n\,\mathbf{1}_{[0,\frac{1}{n}]\times\Omega} \geq 0.$$

Observe that F^n does not depend on ψ or on z and is constant in ω, hence deterministic. In particular $|\alpha^n|_1 = \int_0^T \alpha_s^n\, ds = 1$, independently of ω and n, which shows that again all the assumptions in Briand and Hu [2008] and in the present chapter are satisfied by each pair (F^n, ξ^n), $n \geq 1$.

The unique solution to the BSDE (3.6.9) with parameters (F^n, ξ^n) is given $\mathbb{P}$-a.s. by $Z^n \equiv 0$, more precisely the zero element in $L^2([0, T] \times \Omega)$, and

$$\Psi_t^n = (1 - nt)\,\mathbf{1}_{[0,\frac{1}{n}]\times\Omega}(t, \cdot).$$

We deduce that Ψ^n is nonnegative, nonincreasing and that $\Psi_0^n = 1$, independently of n, $\mathbb{P}$-a.s. It follows that $\mathbb{P}$-a.s. for all $n \geq 1$, $\sup_{0\leq t\leq T} |\Psi_t^n - \Psi_t^0| = \Psi_0^n = 1$, from which

$$\lim_{n\to+\infty} \left(\sup_{0\leq t\leq T} |\Psi_t^n - \Psi_t^0| \right) = 1 \quad \mathbb{P}\text{-a.s.} \tag{3.6.11}$$

However, by construction, $\lim_{n\to+\infty} F^n = 0 = F$ on $(0, T] \times \Omega$, hence μ^A-a.e. independently of ψ and z, so that (3.6.8) holds. Since (3.6.10) and (3.6.11) cannot hold simultaneously, the condition in (3.6.8) is not sufficient for a stability theorem to hold under the present assumptions. Clearly, this phenomenon also occurs in a situation in which the paths of the Ψ^n are differentiable since one can choose F^n to be arbitrarily smooth in t. Indeed, independently of ω, take a smooth nonnegative function on $[0, T]$ that is identically zero on $(\frac{1}{n}, T]$ and integrates to one over $[0, T]$. The corresponding Ψ^n in the BSDE solution is smooth in t and we derive the same contradiction.

The problem arising in the proof of Briand and Hu [2008] Proposition 7 can be observed from equations (3.6.6) and (3.6.7). More specifically, the authors require $L^2(\mathbb{P})$-convergence of the random variables $\int_0^T |\Delta^n F(s)|\, ds$ however they only dispose of an estimate on the product space $[0, T] \times \Omega$ of the form

$$|\Delta^n F| \leq 2\overline{\beta}|\Psi^0| + \alpha^n + \alpha^0 + \gamma \|Z^0\|^2,$$

together with uniform integrability assumptions that are on the level of Ω, *with the t-component integrated away.* There is no guarantee that the pointwise convergence of $|\Delta^n F|$ on the product space $[0, T] \times \Omega$ will transform to pointwise convergence of the integrals $\int_0^T |\Delta^n F(s)|\, ds$ on Ω, which is necessary to utilize the uniform integrability assumptions. This is the insight behind the present example and motivates the modified condition.

We now move on to look at whether the martingale part of our BSDE solution determines a change of measure.

3.7 Change of Measure

In this section we show that under the exponential moments assumption the martingale part of a solution (Ψ, Z, N) to the BSDE (3.2.2) defines a measure change. In particular, we need not show that $Z \cdot M + N$ is a BMO martingale, which is a stronger statement that may indeed not hold, see Chapter 2 for some examples and related discussion. Here, we do not require that the driver F be convex in z. Our proof is based upon Kazamaki [1994] Lemma 1.6 and Lemma 1.7 which we state here for continuous local martingales on compact time intervals.

Lemma 3.7.1 (Kazamaki [1994] Lemma 1.6 and 1.7)**.** *If $\overline{M}$ is a continuous local martingale on $[0,T]$ such that*

$$\sup_{\substack{\tau\,\text{stopping time}\\ \text{valued in}\,[0,\mathrm{T}]}} \mathbb{E}\left[\exp\left(\eta\,\overline{M}_\tau + \left(\frac{1}{2} - \eta\right)\langle \overline{M}\rangle_\tau\right)\right] < +\infty, \tag{3.7.1}$$

for a real number $\eta \neq 1$, then $\mathcal{E}\left(\eta\,\overline{M}\right)$ is a true martingale on $[0,T]$. Moreover, if condition (3.7.1) *holds for* some $\eta^* > 1$ *then it holds for* all $\eta \in (1, \eta^*)$.

We deduce the following result.

Theorem 3.7.2. *Let Assumption 3.2.2 (iii) hold, $|\alpha|_1$ have all exponential moments, ϱ be a real number with $|\varrho| > \gamma/2$ and (Ψ, Z, N) be a solution to the BSDE* (3.2.2) *where g is bounded by $\gamma/2$ and $\Psi \in \mathfrak{E}$. If $\beta > 0$ we also require that $\Psi^*|\alpha|_1$ has exponential moments of all orders or that* (3.2.5) *holds with fixed $\psi_2 = 0$. Then $\mathcal{E}\big(\varrho\,(Z \cdot M + N)\big)$ is a true martingale on $[0,T]$. In particular, when $\gamma < 2$, $\mathcal{E}(Z \cdot M + N)$ is a true martingale.*

Remark 3.7.3. In Mania and Schweizer [2005] Proposition 7 the authors show that the martingale part of solutions to the BSDE (3.2.1) with bounded first component and $\lambda \cdot M$ a BMO martingale also belongs to the class of BMO martingales so that it yields a measure change. Our theorem may thus be seen as a generalization of this result to the case in which Ψ is not necessarily bounded. We mention that it follows from the proof of this theorem that the assumption of all exponential moments may be weakened to requiring exponential moments of some specific order, see the proof of Proposition 5.3.6 (iii) for more details.

Proof. We apply Lemma 3.7.1 with $\overline{M} := \tilde{\varrho}(Z \cdot M + N)$ for some fixed $|\tilde{\varrho}| > \gamma/2$. Firstly, we assume that $\beta > 0$ and that $\Psi^*|\alpha|_1$ has exponential moments of all orders. Considering

$$\log G_\eta(t) := \tilde{\varrho}\eta\left[(Z \cdot M)_t + N_t\right] + \tilde{\varrho}^2\left(\frac{1}{2} - \eta\right)\langle Z \cdot M + N\rangle_t$$

for $\eta > 0$ we get from the BSDE (3.2.2) and the growth condition in (3.2.6),

$$\log G_\eta(t) = \tilde{\varrho}\eta\Big(\Psi_t - \Psi_0 + \int_0^t F(s, \Psi_s, Z_s)\, dA_s + \int_0^t g_s\, d\langle N\rangle_s\Big) + \tilde{\varrho}^2\Big(\frac{1}{2} - \eta\Big)\langle Z \cdot M + N\rangle_t$$
$$\leq |\tilde{\varrho}|\eta(\Psi^* + |\Psi_0|) + |\tilde{\varrho}|\eta|\alpha|_1 + |\tilde{\varrho}|\eta\beta\, \Psi^*|\alpha|_1 + |\tilde{\varrho}|\eta\left(\frac{\gamma}{2} + \frac{|\tilde{\varrho}|}{\eta}\left(\frac{1}{2} - \eta\right)\right)\langle Z \cdot M + N\rangle_t.$$

Noting that

$$\frac{\gamma}{2} + \frac{|\tilde{\varrho}|}{\eta}\left(\frac{1}{2} - \eta\right) \leq 0 \iff \eta \geq \frac{|\tilde{\varrho}|}{2|\tilde{\varrho}| - \gamma} =: \varrho_0,$$

we have that $\mathbb{P}$-a.s. for all $t \in [0, T]$,

$$G_\eta(t) \leq \exp\Big(|\tilde{\varrho}|\eta(\Psi^* + |\Psi_0|)\Big)\exp\Big(|\tilde{\varrho}|\eta|\alpha|_1 + |\tilde{\varrho}|\eta\beta\, \Psi^*|\alpha|_1\Big), \tag{3.7.2}$$

for all $\eta \geq \varrho_0$. By the exponential moments assumption on Ψ^*, $|\alpha|_1$ and $\Psi^*|\alpha|_1$, we conclude from Hölder's inequality that

$$\sup_{\substack{\tau \text{ stopping time} \\ \text{valued in } [0,T]}} \mathbb{E}\big[G_\eta(\tau)\big] < +\infty \tag{3.7.3}$$

for all $\eta \geq \varrho_0 > 1/2$. It now follows from Lemma 3.7.1 that $\mathcal{E}(\tilde{\varrho}\eta(Z \cdot M + N))$ is a true martingale for all $\eta \in [\varrho_0, +\infty)\backslash\{1\}$. The second part of this lemma ensures that in fact $\mathcal{E}(\tilde{\varrho}\eta(Z \cdot M + N))$ is a true martingale for all $\eta > 1$. Thus, if $|\varrho| > \gamma/2$ we apply this result for some fixed $|\tilde{\varrho}| \in (\gamma/2, |\varrho|)$ and $\eta := \varrho/\tilde{\varrho} = |\varrho/\tilde{\varrho}| > 1$ to conclude that indeed $\mathcal{E}(\varrho\,(Z \cdot M + N))$ is a true martingale.
Now if $\beta > 0$ and (3.2.5) holds with fixed $\psi_2 = 0$, we use (3.2.7) to derive, similarly to the above,

$$\log G_\eta(t) \leq |\tilde{\varrho}|\eta(\Psi^* + |\Psi_0|) + |\tilde{\varrho}|\eta|\alpha|_1 + |\tilde{\varrho}|\eta\overline{\beta}\, \Psi^* A_T + |\tilde{\varrho}|\eta\left(\frac{\gamma}{2} + \frac{|\tilde{\varrho}|}{\eta}\left(\frac{1}{2} - \eta\right)\right)\langle Z \cdot M + N\rangle_t$$

so that the claim follows from the boundedness of A_T using exactly the same arguments. The reasoning from above also applies when Assumption 3.2.2 (iii) holds with $\beta = 0$, without any further conditions. □

3.8 Possible Applications

In this final section we explore two applications of the theoretical results from this chapter, specifically focusing on utility maximization and partial equilibrium. We find that the standard results continue to hold when the usual boundedness assumptions are replaced by appropriate exponential moments conditions, allowing for more generality.

3.8.1 Constrained Utility Maximization under Exponential Moments

In the context of the *constrained* utility maximization problem with power utility the following BSDE appears, see Morlais [2009] Section 4.2.1 and the next two Chapters 4 and 5,

$$d\Psi_t = Z_t^\intercal \, dM_t + dN_t - F(t, Z_t)\, dA_t - \frac{1}{2}\, d\langle N\rangle_t, \quad \Psi_T = 0,$$

where the generator is given by

$$F(t,z) = -\frac{p(1-p)}{2} \inf_{\nu\in\mathcal{K}} \left\| B_t\Big(\nu - \frac{z-\lambda_t}{1-p}\Big)\right\|^2 + \frac{p(1-p)}{2} \left\| B_t\Big(\frac{z-\lambda_t}{1-p}\Big)\right\|^2 + \frac{1}{2}\|B_t z\|^2.$$

In the above $1-p \in (0,+\infty)$ is the investor's relative risk aversion and ν refers to investment strategies (in a stock whose returns are driven by the continuous local martingale M under the market price of risk process λ) which must be valued in the *closed constraint set* $\mathcal{K}$. This $\mathcal{K}$ is constant in Morlais [2009] – a condition that we are going to weaken in the Chapters 4 and 5. Writing the infimum in terms of the distance function, which is Lipschitz continuous, one can show that the driver F satisfies Assumption 3.2.2 (ii)-(v), see the Propositions 4.6.3 and 5.3.4, so that there exist constants c_λ and c_z such that

$$|F(t,z)| \le c_\lambda \|B_t\lambda_t\|^2 + c_z\|B_t z\|^2.$$

When we enforce that the *mean-variance tradeoff* $\langle \lambda\cdot M\rangle_T = \int_0^T \lambda_t^\intercal \langle M\rangle_t \lambda_t$ has all exponential moments, an assumption weaker than that of boundedness given in the cited literature, we are in the current framework and see that the BSDE admits a unique solution in $\mathfrak{E}\times\mathfrak{M}^2\times\mathcal{M}^2$. The crucial step in Hu et al. [2005] and Morlais [2009] is, given a solution triple (Ψ, Z, N), to construct the relevant optimizers; this is the process of verification. Building on Theorem 3.7.2 and not relying on BMO arguments such a verification is performed in Chapter 5. For the Brownian framework, in Heyne [2010] the reader will find additional illustration given via a class of stochastic volatility models. Hence, using the theorems of the present chapter, it is possible to show that one can repeat the reasoning of Hu et al. [2005] and Morlais [2009] and that similar results continue to hold for more general classes of market price of risk processes under appropriate trading constraints such as bounded short-selling and borrowing.

We explore further implications in the subsequent two chapters where a detailed study of the stability of the utility maximization problem is undertaken for the cone constrained problem in Chapter 4 and in Chapter 5 for the constrained portfolio choice problem under incomplete information.

3.8.2 Partial Equilibrium and Market Completion under Exponential Moments

We now briefly describe the partial equilibrium framework of Horst et al. [2010] in which structured securities that are written on nontradable assets are priced via a market clearing condition.

The agents in this economy have preferences which are given by dynamic convex risk measures. The risk they are exposed to is given by two sources. The first is encoded in a financial market in which frictionless trading in a stock S is possible. The second is a non-financial risk factor R that can only be dealt with via a derivative written on this external factor. It is assumed that this derivative completes the market, in fact it is shown that in equilibrium the market is complete.

More specifically, while the market price λ^S of financial risk is given exogenously the market price λ^R of external risk is determined via an equilibrium condition. This states that when the derivative is priced according to the pricing rule arising from (λ^S, λ^R) the agents' aggregated demand matches the fixed supply. The demand is in this setting given by the solutions to the agents' individual risk minimization problems and is a function of λ^R.

To ease the exposition we put ourselves in a representative agent setting where the agent's preferences are of entropic type, i.e. their utility function is exponential. Then the following BSDE for the agent's dynamic risk Ψ appears

$$d\Psi_t = Z_t^\intercal \, dW_t - \frac{1}{2}\Big((\lambda_t^S)^2 - 2\lambda_t^S Z_t^1 - (Z_t^2)^2\Big)\, dt, \quad \Psi_T = H,$$

where W is a two dimensional Brownian motion representing the two sources of risk, $Z = (Z^1, Z^2)$ is the corresponding control process of the pair (Ψ, Z) and H is the agent's endowment. Under suitable exponential moments assumptions the present study provides the existence of a unique solution $(\hat{\Psi}, \hat{Z})$ to the above BSDE. Once we check that $\hat{Z}^2$ defines a valid pricing rule, i.e. that $\mathcal{E}\big((\lambda^S, \hat{Z}^2)\cdot W\big)$ is a true martingale, we know that the equilibrium market price λ^R of external risk is given by $\lambda^R \equiv \hat{Z}^2$. In conclusion we can apply the above results in order to generalize the approach of Horst et al. [2010].

4 A BSDE Approach to the Stability of the Cone Constrained Utility Maximization Problem

4.1 Introduction

In this chapter we study the optimal investment problem for an agent whose aim is to maximize the expected power utility of terminal wealth when the admissible strategies are those which are valued in a closed convex cone. They represent constraints like no short selling. The focus in the present chapter is on stability, addressing the question when the components of the solution, such as the optimal wealth and investment strategy, depend continuously on the input parameters. These input parameters concern the utility function, the asset price dynamics and the investment constraints.

This research is motivated by both practical applications as well as theory. Consider a situation where the optimal investment portfolio is implemented, typically there will be small errors in the calibration of input parameters. In order that the usefulness of performing such an optimization is not diminished it is necessary to show that such errors do not largely affect the optimizers, at least locally, which ties in with the above question.

There is a huge volume of literature related to utility maximization going back as far as Merton [1969, 1971], for an overview of the case where there are no investment constraints we refer to the survey article of Schachermayer [2004] as well as the references therein. The situation where there are cone or closed convex constraints has been studied more recently. We refer the interested reader to the articles of Cvitanić and Karatzas [1993] and Cuoco [1997] for Itô price dynamics and Mnif and Pham [2001], Karatzas and Žitković [2003] and Westray [2009] for the case of semimartingale dynamics. The modern solution approach for both constrained and unconstrained problems is via the duality or martingale method, where the convexity of the problem as well as the link between (a generalization of) martingale measures and replicable wealths is exploited.

With regards to stability, it is by the study of this dual problem that the literature also has proceeded thus far. For instance, continuity with respect to the preferences is investigated in Jouini and Napp [2004] for complete Itô price models and in Larsen [2009] for incomplete markets with continuous semimartingale dynamics. In the complete case of Jouini and Napp [2004], due to greater structure of the problem and uniform boundedness assumptions, the authors prove the L^{ϱ} and the pointwise convergence of the optimal wealth at each date, whereas in Larsen [2009] this is weakened to only convergence in probability of the optimal terminal wealth. More recently, Kardaras and

Žitković [2011] show that such convergence in probability of the optimal terminal wealth also holds when there are illiquid assets which the investor may add to their portfolio and when the statistical probability measure simultaneously varies. Finally we mention the work by Nutz [2010a] who looks at risk aversion asymptotics and also provides results on the continuity with respect to the risk aversion parameter.

Another theme, beginning with Larsen and Žitković [2007], relates to misspecifications in the model, i.e. when the asset price dynamics vary. Continuity of the optimal wealth is then shown under an additional uniform integrability assumption, again at terminal time T only.

The previous articles consider stability in the situation when there are no investment constraints. In the specific case when the utility function is the logarithm, this can be generalized as shown in a recent article by Kardaras [2010]. The optimizing investment strategy is then the numéraire portfolio and one may use its known explicit formula.

As we know, BSDEs provide an alternative framework for tackling the utility maximization problem, even in the presence of constraints. It is this fact which is exploited in the present chapter, showing that questions of sensitivity for the optimal wealth process, investment strategy and dual optimizer are directly related to stability results for semimartingale BSDEs established in Chapter 3. This is under an exponential moments condition which is weaker than the boundedness assumptions from the cited references and hence allows for more generality in the model.

In this chapter we investigate the investor's portfolio choice problem under cone constraints which we assume to be *stochastic* and we refer to the introduction in Chapter 1 for some examples of possible real-life interpretations. The first part of the chapter now focuses on the existence and uniqueness of optimal solutions to the primal and dual problems under an exponential moments condition on the mean-variance tradeoff. A byproduct of our results is that the family of *supermartingale measures* for the stock can serve as the dual domain for the cone constrained problem. Then, extending the decomposition of elements in the dual domain given in Karatzas and Žitković [2003] and Larsen and Žitković [2007] to the case of semimartingale dynamics with predictably measurable cone constraints, we are led to the utility maximization BSDE from Nutz [2011]. However, we derive our verification statement from BSDE comparison principles in the spirit of Hu et al. [2005] and Morlais [2009] by showing that within our setting the dual opportunity process exists and exhibits specific properties that are well suited for the calculations. We mention that this is done under less strict assumptions on the mean-variance tradeoff when compared to the above two references.

The second part of this chapter uses the one-to-one correspondence between optimizers and solutions to the BSDE to study the continuity. Using this link with BSDEs we can simultaneously consider continuity with respect to utility function, model dynamics, statistical probability measure and cone constraints, integrating previous research into one framework. To be more precise, stability with respect to the utility function is formulated in terms of the agent's relative risk aversion. The continuity with respect to the model dynamics is based on variations of the mean-variance tradeoff. For the statistical probability measure we consider convergence of the corresponding densities while for the constraints we refer to the so-called closed set limit. The stability result then is for the semimartingale topology, i.e. on the level of processes in contrast to convergence at terminal time from the cited literature. Our final contribution is an example which clarifies the compatibility of the so-called null-investments and the constraints as well as provides the right notion of convergence of these constraints.

The present chapter is based on the working paper Mocha and Westray [2011b]. Its structure is as follows, the modelling framework and main results are described in Section 4.2. In Section 4.3 we then consider the primal optimization problem. Sections 4.4 and 4.5 discuss the description of the dual domain and relationship between the utility maximization problem and the solution to an appropriate BSDE. The connection with continuity is then shown in Section 4.6. We note that related results concerning set-valued analysis whose proof would interrupt the flow of the text are given in the Appendix 6.2.

4.2 Framework and Main Results

For the convenience of the reader let us briefly recall the utility maximization framework from Section 1.2 which holds throughout. We work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ satisfying the usual conditions of right-continuity and completeness. The time horizon T is a finite number in $(0,\infty)$ and $\mathcal{F}_0$ is the completion of the trivial σ-algebra. All semimartingales are càdlàg.

The Market Model

There is a market consisting of one bond paying zero interest and d stocks with price process $S = (S^1, \ldots, S^d)^\intercal$ which is assumed to have dynamics

$$dS_t = \mathrm{Diag}(S_t)\left(dM_t + d\langle M\rangle_t\lambda_t\right),$$

where $M = (M^1, \ldots, M^d)^\intercal$ is a d-dimensional continuous local martingale with $M_0 = 0$ and λ is a d-dimensional predictable process, the *market price of risk*, satisfying

$$\mathbb{P}\left(\int_0^T \lambda_t^\intercal\, d\langle M\rangle_t\lambda_t < +\infty\right) = 1.$$

Moreover, we recall the decomposition of the quadratic variation of M,

$$\langle M\rangle = C\cdot A \tag{4.2.1}$$

where C is a predictable process valued in the space of symmetric positive semidefinite $d\times d$ matrices and A is a predictable increasing process. We are not restricted to a specific choice of A as long as it remains uniformly bounded, see the remarks in Section 3.2. We then deduce the existence of predictable processes P and Γ valued in the space of $d\times d$ orthogonal (resp. diagonal) matrices such that

$$\langle M\rangle = C\cdot A = P^\intercal\Gamma P\cdot A = B^\intercal B\cdot A, \tag{4.2.2}$$

where we set $B := \Gamma^{\frac{1}{2}} P$. The matrix Γ has nonnegative entries only, with the eigenvalues of C on its diagonal.

In the present chapter we require that the Assumption 1.3.1 on the existence of *exponential moments of all orders* of the mean-variance tradeoff $\langle \lambda \cdot M \rangle_T$ holds.

Assumption 4.2.1. *For all $\varrho > 0$ we have that $\mathbb{E}\big[\exp(\varrho \langle \lambda \cdot M \rangle_T)\big] < +\infty$.*

Remark 4.2.2. Assumption 4.2.1 allows us to provide a unified presentation of the duality and the BSDE approach to solving the utility maximization problem; it ensures the existence of an equivalent local martingale measure for S as well as implying finiteness of the primal and dual problems, see also Lemma 1.3.2. Since our analysis involves semimartingale BSDEs the above condition allows us to apply the existence, uniqueness and stability results from Chapter 3. We observe that our setting extends the framework of Hu et al. [2005] and Morlais [2009] beyond the case of a bounded mean-variance tradeoff. The reader may find a summary of the ideas behind these comments in Table 1.1 of Section 1.3.

The Portfolio Selection Problem

In contrast to the previous chapters, in the sequel we assume that trading in the above market is subject to *constraints* which we now describe. Recall that an $\mathbb{R}^d$-valued *multivalued mapping* G is a function $G : [0,T] \times \Omega \to 2^{\mathbb{R}^d}$ (the power set of $\mathbb{R}^d$). It is called *predictably measurable* if, for all closed subsets Q of $\mathbb{R}^d$,

$$G^{-1}(Q) := \{(t,\omega) \in [0,T] \times \Omega \mid G(t,\omega) \cap Q \neq \emptyset\} \in \mathcal{P},$$

where $\mathcal{P}$ is the predictable σ-algebra on $[0,T] \times \Omega$. The function G is called closed (convex) if $G(t,\omega)$ is a closed (convex) set for all $(t,\omega) \in [0,T] \times \Omega$. We refer to Wagner [1977] and Rockafellar [1976] for further details. The constraints are modelled by a predictably multivalued mapping $\mathcal{K}$ and we assume it satisfies the following assumption.

Assumption 4.2.3. *The mapping $(t,\omega) \mapsto \mathcal{K}(t,\omega) \subset \mathbb{R}^d$ is closed, convex, and polyhedral in the following sense. There is an integer $m \geq 1$, independent of (t,ω), together with corresponding predictable M-integrable processes $K^1, \dots, K^m$ such that $\mathbb{P}$-a.s. for all $t \in [0,T]$*

$$\mathcal{K}(t,\omega) = \left\{ \sum_{j=1}^{m} c_j\, K^j_t(\omega) \,\middle|\, c_j \geq 0, j = 1, \dots, m \right\}.$$

Further discussion and explanation on the importance of the above assumption on $\mathcal{K}$ from the point of view of existence of optimal strategies can be found in Czichowsky and Schweizer [2011] as well as Czichowsky et al. [2011]. Namely, it ensures that the family of the corresponding wealth processes is closed in the semimartingale topology. We point at another important property. More explicitly, since the cone $\mathcal{K}$ is polyhedral, the image $B\mathcal{K}$ of $\mathcal{K}$ under B is *closed*. Such a closedness property is not true in general and the reader may find explicit counterexamples in Pataki [2007] along with a discussion of sufficient conditions. In Hu et al. [2005] the closedness of $B\mathcal{K}$ is ensured by nondegeneracy assumptions on B. Since B is determined by the exogenous stock dynamics and the choice of A we rather prefer to impose conditions on $\mathcal{K}$ which we regard as being given by a regulator. A similar reasoning will lead to the consideration of compact constraints in Chapter 5.

Clearly, the unconstrained case is covered by setting $\mathcal{K} \equiv \mathbb{R}^d$. Other special cases include a constant polyhedral cone in $\mathbb{R}^d$ as in Karatzas and Žitković [2003], as well as $\mathcal{K} \equiv \mathbb{R}^{d_1} \times \{0\}^{d_2}$ with $d = d_1 + d_2$ in which we face a model where the processes $S^{d_1+1}, \dots, S^d$ are *nontradable* latent factors. Further constraints on the tradable stocks $S^1, \dots, S^{d_1}$ may then still be imposed.

We are now ready to introduce the notion of a trading strategy.

Definition 4.2.4. A predictable d-dimensional process ν is called *admissible* and we write $\nu \in \mathcal{A}_{\mathcal{K}}$, if

(i) It is M-integrable, i.e.

$$\mathbb{P}\left(\int_0^T \nu_t^\intercal \, d\langle M\rangle_t \nu_t < +\infty\right) = 1.$$

(ii) We have that $\nu \in \mathcal{K}$, μ^A-a.e. Here, μ^A is the *Doléans measure* on $\mathcal{P}$.

Exactly as in Chapter 1, an admissible process ν will be interpreted as an investment strategy and its components ν^i represent the *proportion of wealth* invested in each stock S^i, $i = 1, \dots, d$, now subject to *investment constraints* that are determined by $\mathcal{K}$. In particular, for some initial capital $x > 0$ and an admissible strategy ν, the associated *wealth process* $X^{x,\nu}$ evolves as follows

$$X^{x,\nu} := x\,\mathcal{E}(\nu \cdot M + \nu \cdot \langle M\rangle\lambda), \tag{4.2.3}$$

where $\mathcal{E}$ denotes the stochastic exponential. The family of all wealth processes arising from admissible strategies will be denoted by $\mathcal{X}(x)$, where we notationally suppress the dependence on $\mathcal{K}$. Furthermore, we will omit writing explicitly the dependence of $X^{x,\nu}$ on the initial capital, when no ambiguity arises we just write X^ν.

Remark 4.2.5. We know that the wealth equation is often written in additive format, $X = x + H \cdot S$ for a predictable S-integrable process H specifying the amount of the asset held in the portfolio and chosen such that it is valued in some constraint set and the resulting wealth process remains (only) *nonnegative*. We write $\mathcal{X}^{add}(x)$ for such wealth processes and observe that $\mathcal{X}(x) \subset \mathcal{X}^{add}(x)$. In the case that $X_T > 0$, which implies $X > 0$ since X is a supermartingale under some equivalent measure (assumed to exist, e.g. by the Assumption 4.2.1 the minimal martingale measure with density process $\mathcal{E}(-\lambda \cdot M)$ is such a measure), the correspondence between H and ν is given by $H^i S^i = \nu^i X$ for $i = 1, \dots, d$. The cone constraint in the additive formulation consists of the requirement that $H \in \mathcal{L}$ where

$$\mathcal{L}(t,\omega) := \left\{ \sum_{j=1}^{m} c_j \, L_t^j(\omega) \,\middle|\, c_j \geq 0 \right\}$$

with $\mathbb{R}^d$-valued predictable S-integrable processes $L^1, \ldots, L^m$ and such that the ith component of each K^j equals S^i times the ith component of L^j. Here, K^j, $j = 1, \ldots, m$ denote the processes from the Assumption 4.2.3. In particular the framework of Karatzas and Žitković [2003], where $\mathcal{L}$ is *constant*, can be embedded into ours since we allow for a *predictably measurable* multivalued mapping $\mathcal{K}$. We thus allow for more generality in the model, e.g. cover constraints that are determined by a stochastic process.

Our motivation for writing wealth in exponential format stems from the fact that the dual domain of the portfolio choice problem will (and should) be a family of supermartingale measures, hence stochastic exponentials. It then turns out that to describe the primal and dual optimizers and a BSDE it is most convenient to write wealth as a stochastic exponential as well. An additional byproduct of this parameterization is that it simplifies the proof of the decomposition of the elements of the dual domain.

Since in our setting the optimal wealth $\hat{X}$ exists and satisfies $\hat{X}_T > 0$ we may, without loss of generality, choose to optimize over the family of (strictly) positive wealth processes $\mathcal{X}(x)$, see Lemma 4.3.1. A consequence of this is that for our definitions of $\mathcal{K}$ and $\mathcal{L}$, one can switch freely between the two formulations.

Our agent has preferences modelled by a power utility function U,

$$U(x) = \frac{x^p}{p},$$

for $p \in (-\infty, 0) \cup (0, 1)$. They start with initial capital $x > 0$ and choose admissible strategies ν so as to maximize the expected utility of terminal wealth. We hence derive the following primal optimization problem

$$u(x) := \sup_{\nu \in \mathcal{A}_{\mathcal{K}}} \mathbb{E}\Big[U\Big(X_T^{x,\nu}\Big)\Big]. \tag{4.2.4}$$

Remark 4.2.6. Similarly to Remark 1.2.2 we point out that for power utility the value function factors, i.e. we can write

$$u(x) = x^p \sup_{\nu \in \mathcal{A}_{\mathcal{K}}} \mathbb{E}\Big[U\Big(X_T^{1,\nu}\Big)\Big] = U(x)\, c_p,$$

for some constant c_p, $p \in (-\infty, 0) \cup (0, 1)$, to be identified below. Again as a corollary the optimal investment strategy $\hat{\nu}$, when it exists, is independent of x and the primal optimizer $\hat{X}$ has a simple linear dependence on x.

The Dual Problem

In analogy to the procedure in Chapter 1 we introduce the set of adapted càdlàg processes

$$\mathcal{Y}(y) := \{Y \geq 0 \,|\, Y_0 = y \text{ and } XY \text{ is a supermartingale for all } X \in \mathcal{X}(1)\}\,,\ y > 0,$$

and consider the minimization problem

$$\widetilde{u}(y) := \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}\Big[\widetilde{U}(Y_T)\Big]\,, \tag{4.2.5}$$

where $\widetilde{U}$ is the *conjugate* (or *dual*) of U given for $y > 0$ by

$$\widetilde{U}(y) := \sup_{x>0} \{U(x) - xy\} = -\tfrac{y^q}{q},$$

with $q := \frac{p}{p-1}$ the dual exponent to p. From the relation $\mathcal{Y}(y) = y\mathcal{Y}(1)$ we see the factorization property for $\widetilde{u}$,

$$\widetilde{u}(y) = \inf_{Y \in \mathcal{Y}(1)} \mathbb{E}\Big[\widetilde{U}(yY_T)\Big] = y^q \inf_{Y \in \mathcal{Y}(1)} \mathbb{E}\Big[\widetilde{U}(Y_T)\Big] = \widetilde{U}(y)\,\widetilde{c}_p.$$

The relationship between $\widetilde{c}_p$ and c_p is provided in Theorem 4.2.8.

The Main Result for the Cone Constrained Problem

The utility maximization problem with semimartingale dynamics and general utility functions has been studied under *constant* constraints, see Karatzas and Žitković [2003] for the case with intertemporal consumption as well as Westray [2009]. The next lemma shows that the assumptions necessary to apply the results from the second reference hold in our setting.

Lemma 4.2.7. *Let Assumption 4.2.1 hold then there exists an equivalent local martingale measure for S and* $\max(u(x), \widetilde{u}(y)) < +\infty$ *for all $x, y > 0$.*

Proof. As for Lemma 1.3.2 we have that $Y^{\lambda} := \mathcal{E}(-\lambda \cdot M)$ defines an equivalent local martingale measure (the so-called *minimal martingale measure*) and for the second part, considering the case $p \in (0,1)$ only and from the definition of $\widetilde{U}$, we have

$$\max(u(x), \widetilde{u}(y)) \leq \mathbb{E}\Big[\widetilde{U}(yY_T^{\lambda})\Big] + \sup_{\nu \in \mathcal{A}_{\mathcal{K}}} \mathbb{E}\Big[X_T^{x,\nu} y Y_T^{\lambda}\Big] \leq -\tfrac{y^q}{q}\,\mathbb{E}\Big[(Y_T^{\lambda})^q\Big] + xy.$$

Observing $q < 0$ and using the standard estimates, see (2.5.7) or the proof of Lemma 1.3.2, completes the proof. $\square$

The following theorem states the existence and uniqueness results that are pertinent for our study. These results are precisely the counterparts to the known results stated in Theorem 1.2.4. However, we note that it is essential for our study that the dual optimizer indeed is contained in the specific dual domain defined above. We provide the details in Section 4.3 where we require that the set of wealth processes be closed in the semimartingale topology, which is guaranteed by the assumption on $\mathcal{K}$ to be polyhedral. Contrary to the usual procedure of proving the existence of a solution to the portfolio choice problem via duality we consider the primal problem first and the dual problem afterwards, having to guarantee that the dual optimizer obtained does not exhibit singular parts as it possibly does in the literature which also covers random endowment, see Cvitanić et al. [2001] or Westray [2009].

Theorem 4.2.8. *Suppose Assumptions 4.2.1 and 4.2.3 hold and let $x, y > 0$. Then:*

(i) There exists an admissible strategy $\hat{\nu} \in \mathcal{A}_{\mathcal{K}}$ which is optimal for the primal problem,

$$u(x) = \mathbb{E}\big[U(\hat{X}_T)\big], \text{ where } \hat{X} = X^{x,\hat{\nu}}.$$

In addition, $\hat{\nu}$ is unique in the sense that for any other optimal strategy $\bar{\nu} \in \mathcal{A}_{\mathcal{K}}$ the wealth processes $X^{x,\hat{\nu}}$ and $X^{x,\bar{\nu}}$ are indistinguishable.

(ii) There exists an optimal $\hat{Y}^y \in \mathcal{Y}(y)$ for the dual problem, unique up to indistinguishability,

$$\widetilde{u}(y) = \mathbb{E}\Big[\widetilde{U}(\hat{Y}_T)\Big], \text{ where } \hat{Y} = \hat{Y}^y.$$

(iii) The functions u and $\widetilde{u}$ are continuously differentiable and conjugate. If $y = u'(x)$ then, adopting the notation from (i) and (ii), we have the relation $\hat{Y}_T = U'(\hat{X}_T)$ and $\hat{X}\hat{Y}$ is a martingale on $[0,T]$. More explicitly, there are constants c_p, $p \in (-\infty, 0) \cup (0,1)$, such that with $\widetilde{c}_p := (c_p)^{\frac{1}{1-p}}$,

$$u(x) = U(x)\, c_p, \quad \widetilde{u}(y) = \widetilde{U}(y)\, \widetilde{c}_p.$$

BSDEs and Semimartingale Background

Our aim is now to analyze the above problems and their stability by relating them directly to the solution of a continuous semimartingale BSDE of the following type:

$$d\Psi_t = Z_t^{\mathsf{T}}\, dM_t + dN_t - F(t, Z_t)\, dA_t - \frac{1}{2}\, d\langle N\rangle_t, \quad \Psi_T = 0, \tag{4.2.6}$$

where F is a predictable function $[0,T] \times \Omega \times \mathbb{R}^d \to \mathbb{R}$ called the *generator* or *driver*. For the notion of a *solution* to the BSDE (4.2.6) we refer to the Definition 3.2.1.

In view of the results from Chapter 3 we shall be especially interested in solution triples (Ψ, Z, N) with $\Psi \in \mathfrak{E}$, where $\mathfrak{E}$ denotes the space of processes Υ such that

$$\mathbb{E}[\exp(\varrho \Upsilon^*)] < +\infty \text{ for all } \varrho > 0,$$

i.e. those processes whose supremum, $\Upsilon^* := \sup_{0 \leq t \leq T} |\Upsilon_t|$, possesses exponential moments of all orders. In particular we can rely on a uniqueness result for BSDE solutions with $\Psi \in \mathfrak{E}$, providing us with the desired straight link. Moreover, we then can also refer to the stability results from Section 3.2.7. As we will see later, showing that $\hat{\Psi} \in \mathfrak{E}$

relies on the dynamic optimality properties of the so-called primal and dual opportunity processes. While such results may not hold for the dual domain considered in the general setting of Nutz [2010b], they *do* hold for the dual domain considered here, which is another motivation for defining the dual domain as above.

For $\varrho \geq 1$ we write $\mathcal{M}^\varrho$ for the space of continuous $\mathbb{P}$-local martingales $\overline{M}$ satisfying $\overline{M}_0 = 0$ and

$$\mathbb{E}\Big[\langle \overline{M}\rangle_T^{\varrho/2}\Big] < +\infty.$$

More generally for an arbitrary continuous semimartingale Υ we shall indirectly use the $\mathcal{H}^\varrho$ norm. Given the canonical decomposition $\Upsilon = \Upsilon_0 + M^\Upsilon + A^\Upsilon$ where M^Υ is a (continuous) local martingale and A^Υ a (continuous) process of finite variation, it is defined via

$$\|\Upsilon\|_{\mathcal{H}^\varrho} := |\Upsilon_0| + \Big\|\langle M^\Upsilon\rangle_T^{1/2}\Big\|_{L^\varrho(\mathbb{P})} + \Big\|\int_0^T |dA_s^\Upsilon|\Big\|_{L^\varrho(\mathbb{P})}.$$

The stability result that we are going to derive involves the notion of convergence in the semimartingale topology for which we refer the reader to Émery [1979] and Mémin [1980] for more details. The following proposition collects together the key results needed in the present study.

Proposition 4.2.9 (Émery [1979] Lemma 6, Nutz [2010a] Appendix A)**.** *Let $(\Upsilon^n)_{n\in\mathbb{N}_0}$ be a family of continuous semimartingales and $\varrho \geq 1$, then*

(i) The sequence $(\Upsilon^n)_{n\in\mathbb{N}}$ converges to Υ^0 in the semimartingale topology if and only if every subsequence of $(\Upsilon^n)_{n\in\mathbb{N}}$ has a subsequence converging locally to Υ in $\mathcal{H}^\varrho$.

(ii) If $(\Upsilon^n)_{n\in\mathbb{N}}$ converges to Υ^0 in the semimartingale topology then $\big(\mathcal{E}(\Upsilon^n)\big)_{n\in\mathbb{N}}$ converges to $\mathcal{E}(\Upsilon^0)$ in the semimartingale topology.

(iii) Convergence in the semimartingale topology implies convergence uniformly on compacts in probability, ucp in short, see Protter [2005] Section II.4.

The Decomposition of the Dual Domain and the BSDE

One final notation we shall need is that of the *polar cone*. Given the conic predictably measurable multivalued mapping $\mathcal{K}$ we define (and easily derive)

$$\begin{aligned}\mathcal{K}^\circ(t,\omega) :&= \Big\{l \in \mathbb{R}^d \,\Big|\, k^\intercal l \leq 1 \text{ for all } k \in \mathcal{K}(t,\omega)\Big\}\\ &= \Big\{l \in \mathbb{R}^d \,\Big|\, k^\intercal l \leq 0 \text{ for all } k \in \mathcal{K}(t,\omega)\Big\}\\ &= \bigcap_{j=1}^m \Big\{l \in \mathbb{R}^d \,\Big|\, (K_t^j(\omega))^\intercal l \leq 0\Big\},\end{aligned}$$

where the $K^j, j = 1, \dots, m$ are from Assumption 4.2.3. Under the present assumptions on $\mathcal{K}$ we have that $\mathcal{K}^\circ$ is again a closed convex predictably measurable multivalued mapping. The above definition and second equality apply to any given cone, the third characterization holds for polyhedral cones.

Having described our framework and relevant background, we can state the first of the main results. Specifically, we provide a more precise structure of the elements in the dual domain as well as the optimizer $\hat{Y}$. A version of this result may be found in Karatzas and Žitković [2003] Proposition 4.1 for the case where one has nondegenerate Itô dynamics for S and a polyhedral cone $\mathcal{K}$ which is independent of (t, ω), see also Larsen and Žitković [2007] Proposition 3.2 for the one-dimensional unconstrained case. These findings are in line with earlier results which are implicitly present in Cvitanić and Karatzas [1992] and Rouge and El Karoui [2000], for instance. We extend them to the case of semimartingale dynamics and predictably measurable constraint sets. Due to writing wealth in exponential format the proof becomes simpler and slightly more general when compared to Karatzas and Žitković [2003] and Larsen and Žitković [2007].

Theorem 4.2.10. *Let Assumption 4.2.3 hold.*

(i) Let $Y \in \mathcal{Y}(1)$ with $Y_T > 0$. Then there exist a predictable M-integrable process κ^Y with

$$B(\lambda - \kappa^Y) \in (B\mathcal{K})^\circ, \quad \mu^A\text{-a.e.}$$

as well as a local martingale N^Y orthogonal to M and a predictable decreasing càdlàg process D^Y with $D_0^Y = 1$ and $D_T^Y > 0$ $\mathbb{P}$-a.s. such that

$$Y = D^Y \mathcal{E}(-\kappa^Y \cdot M + N^Y).$$

(ii) For the optimizer $\hat{Y}^y \in \mathcal{Y}(y)$ (assumed to exist) we have the representation,

$$\hat{Y}^y = y\, \mathcal{E}(-\hat{\kappa} \cdot M + \hat{N}),$$

for processes $\hat{\kappa} := \kappa^{\hat{Y}^1}$ and $\hat{N} := N^{\hat{Y}^1}$ which are independent of y. In particular the decreasing process from (i) satisfies $D^{\hat{Y}^1} \equiv 1$.

The next proposition relates the optimizers to the solution of a quadratic semimartingale BSDE, similarly to Mania and Tevzadze [2008] for the unconstrained case and Nutz [2011] for the constrained case. Under boundedness and nondegeneracy assumptions the BSDE also appears in Hu et al. [2005] and Morlais [2009]. We mention that we prefer to require the continuity of the orthogonal martingale part separately, with the main application of a continuous filtration in mind.

Proposition 4.2.11. *Let the Assumptions 4.2.1 and 4.2.3 hold.*

(i) Let $\hat{\nu}$ denote the optimal strategy and $\hat{N}$ the local martingale from Theorem 4.2.10 which we assume to be continuous. Then for every $x > 0$ the triple $(\hat{\Psi}, \hat{Z}, \hat{N})$, where

$$\hat{\Psi} := \log\left(\frac{u'(x)\hat{Y}^1}{U'(\hat{X})}\right) \quad \text{and} \quad \hat{Z} := -\hat{\kappa} + (1-p)\hat{\nu},$$

is the unique solution to the BSDE (4.2.6) *with* $\hat{\Psi} \in \mathfrak{E}$ *where*

$$F(\cdot, z) = \frac{1}{2} \left\| Bz \right\|^2 - \frac{q}{2} \left\| \Pi_{B\mathcal{K}}\big(B(z+\lambda)\big) \right\|^2.$$

We write Π *for the nearest point or projection operator onto the indicated cone.*

(ii) Given the unique solution $(\hat{\Psi}, \hat{Z}, \hat{N})$ *from (i) we can write the optimizers, for the initial values* $x = 1$, $y = 1$, *up to indistinguishability as*

$$\hat{X}^1 = \mathcal{E}(\widetilde{\nu} \cdot M + \widetilde{\nu} \cdot \langle M \rangle \lambda), \qquad \hat{Y}^1 = \mathcal{E}\big(- \widetilde{\kappa} \cdot M + \hat{N}\big),$$

where the predictable integrands $\widetilde{\nu}$ *and* $\widetilde{\kappa}$ *are defined via*

$$\widetilde{\nu} := \frac{1}{1-p} P^{\intercal} \widetilde{\Gamma}^{\frac{1}{2}} \Pi_{B\mathcal{K}}\big(B(\hat{Z}+\lambda)\big), \quad \widetilde{\kappa} := P^{\intercal} \widetilde{\Gamma}^{\frac{1}{2}} \Big[B\lambda - \Pi_{(B\mathcal{K})^\circ}\big(B(\hat{Z}+\lambda)\big)\Big]$$

and satisfy, μ^A*-a.e.* $B\widetilde{\nu} = B\hat{\nu}$ *and* $B\widetilde{\kappa} = B\hat{\kappa}$*. The process* $(\widetilde{\Gamma}^{i,j})_{i,j=1,\dots,d}$ *is chosen to be a predictable process valued in the space of* $d \times d$ *diagonal matrices such that it satisfies*

$$\widetilde{\Gamma}^{ij} = \begin{cases} 1/\Gamma^{ii} & \text{if } i = j \text{ and } \Gamma^{ii} \neq 0 \\ 0 & \text{if } i \neq j. \end{cases}$$

Remark 4.2.12. The content of the above proposition is essentially known, cf. Nutz [2011] Corollaries 3.12 and 5.18, although it is stated differently there. Define the process $L := \exp(\hat{\Psi})$, then it is easy to see that L is the *opportunity process* of Nutz [2011] where the author shows that the Galtchouk-Kunita-Watanabe decomposition of L satisfies an appropriate BSDE. In contrast via our Theorem 4.2.10, we can apply Itô's formula directly to $\hat{\Psi}$ to get the BSDE. As a consequence we augment the results of Nutz [2011] by providing a simple additive decomposition of the process $\hat{Z}$ into a part $\hat{\kappa}$ with well defined properties related to the dual problem and polar cone $(B\mathcal{K})^\circ$ and a part related to the optimal strategy $\hat{\nu}$. We also note that the unique correspondence above is derived via a BSDE comparison theorem under exponential moments, rather than a verification theorem as in Nutz [2011].

Proceeding as above we use a reasoning similar to Hu et al. [2005] and Morlais [2009] but allow for more generality in the market dynamics. No boundedness or nondegeneracy assumptions are required. To be more precise, nondegeneracy has been partially replaced by the assumption on the cones to be polyhedral. The formula for the optimal strategy is as in the cited references, however, due to Moreau's decomposition theorem, which is available for cones, no measurable selection argument is involved.

Remark 4.2.13. We point out a consequence of item (ii) above. Whilst the wealth process is unique in the space of càdlàg processes, the representation of the strategies is not unless C is invertible or the strategy is considered in the image of B; similar remarks apply to $\hat{\kappa}$. This is related to the discussion of what is known as *null-investments* in the literature and the above proposition identifies their components.

The Stability Result

The main idea of the present chapter is to use the link from Proposition 4.2.11 to study the continuity of the utility maximization problem with respect to its inputs via BSDE methods. More explicitly, we are interested in the dependence of the optimal objects with respect to the market price of risk process λ, the probability measure $\mathbb{P}$, the investor's relative risk aversion parameter p and the constraint set $\mathcal{K}$. As pointed out above, the dependence on the initial wealth is a simple linear one, due to the factorization property. Hence we vary only the four inputs λ, $\mathbb{P}$, p and $\mathcal{K}$ by means of sequences

$$(\lambda^n)_{n\in\mathbb{N}},\ (\mathbb{P}^n)_{n\in\mathbb{N}},\ (p^n)_{n\in\mathbb{N}} \text{ and } (\mathcal{K}^n)_{n\in\mathbb{N}}$$

of parameters that converge to $\lambda =: \lambda^0$, $\mathbb{P} =: \mathbb{P}^0$, $p =: p^0$ and $\mathcal{K} =: \mathcal{K}^0$ in an appropriate sense.

Fix $n \in \mathbb{N}$, now we have that λ^n is a predictable M-integrable process and $\mathbb{P}^n$ is assumed to be a measure equivalent to $\mathbb{P}$ with Radon-Nikodým derivative

$$\frac{d\mathbb{P}^n}{d\mathbb{P}} = \mathcal{E}(-\beta^n \cdot M + L^n)_T.$$

Here $(\beta^n)_{n\in\mathbb{N}}$ is a sequence of predictable M-integrable processes, $(L^n)_{n\in\mathbb{N}}$ a sequence of continuous $\mathbb{P}$-local martingales orthogonal to M and $\beta^0 \cdot M :\equiv L^0 :\equiv 0$. Due to the Girsanov theorem the process $M^n := M + \langle M\rangle \cdot \beta^n$ is a (continuous) $\mathbb{P}^n$-local martingale. This leads to dynamics for the asset $S = S^n$ under $\mathbb{P}^n$, of the form

$$dS_t^n = \mathrm{Diag}(S_t^n)\Big(dM_t^n + d\langle M^n\rangle_t(\lambda_t^n - \beta_t^n)\Big),$$

where we have used the continuity to deduce $\langle M^n\rangle = \langle M\rangle = C \cdot A$. Each risk aversion parameter p^n is valued in $(-\infty, 0) \cup (0, 1)$ and corresponds to a utility function

$$U^n(x) := \frac{1}{p^n}\, x^{p^n}, \quad x > 0.$$

The cone $\mathcal{K}^n$ is assumed to satisfy Assumption 4.2.3 so that we can consider the primal problem as a function of the inputs

$$u^n(x) := \sup_{\nu\in\mathcal{A}_{\mathcal{K}^n}} \mathbb{E}_{\mathbb{P}^n}\Big[U^n\Big(X_T^{n,x,\nu}\Big)\Big],$$

where $X^{n,x,\nu}$ represents the wealth acquired from an investment in S^n and considered under $\mathbb{P}^n$, so that we have

$$X^{n,x,\nu} = x\,\mathcal{E}(\nu \cdot M + \nu \cdot \langle M\rangle\lambda^n) = x\,\mathcal{E}(\nu \cdot M^n + \nu \cdot \langle M^n\rangle(\lambda^n - \beta^n)). \tag{4.2.7}$$

The definition of $\mathcal{A}_{\mathcal{K}^n}$ is invariant under changes of equivalent probability measures so that the above maximization is well defined under suitable assumptions on the parameters.

Assumption 4.2.14. *Each $\mathcal{K}^n, n \in \mathbb{N}_0$, satisfies Assumption 4.2.3 and for all $\varrho > 0$*

$$\sup_{n\in\mathbb{N}_0} \mathbb{E}_{\mathbb{P}}\Big[\exp\Big(\varrho\left(\langle\lambda^n \cdot M\rangle_T + \langle\beta^n \cdot M\rangle_T + \langle L^n\rangle_T\right)\Big)\Big] < +\infty.$$

The previous assumption ensures that for fixed $n \in \mathbb{N}_0$ and all $\varrho > 0$

$$\mathbb{E}_{\mathbb{P}^n}\Big[\exp\Big(\varrho\left\langle(\lambda^n - \beta^n)\cdot M\right\rangle_T\Big)\Big] < +\infty,$$

which can be shown similarly to the proof of Lemma 4.2.7 and is omitted. In particular we may apply Theorem 4.2.8 for each $n \in \mathbb{N}_0$ to deduce the existence of a primal optimizer and corresponding optimal portfolio,

$$\hat{X}^n := \hat{X}(\lambda^n, \mathbb{P}^n, p^n, \mathcal{K}^n), \quad \hat{\nu}^n := \hat{\nu}(\lambda^n, \mathbb{P}^n, p^n, \mathcal{K}^n), \tag{4.2.8}$$

where we write the optimizers as a function of the parameters $(\lambda^n, \mathbb{P}^n, p^n, \mathcal{K}^n)$. A similar convention holds for the value function $u^n := u(\lambda^n, \mathbb{P}^n, p^n, \mathcal{K}^n)$. We also use the notation $\hat{X} := \hat{X}^0$ and $\hat{\nu} := \hat{\nu}^0$ and observe that due to the integrability assumption above L^n is actually a true martingale for every $n \in \mathbb{N}_0$.

The main result shows that under suitable assumptions the optimizers are continuous with respect to these inputs. Note that in the following assumption, since each $\mathcal{K}^n$ is polyhedral, the projection $B\mathcal{K}^n$ is closed.

Assumption 4.2.15. *The preferences and markets converge in the following sense*

$$\lim_{n\to+\infty} p^n = p,$$
$$\lim_{n\to+\infty}\Big(\left\langle(\lambda^n - \lambda)\cdot M\right\rangle_T + \left\langle\beta^n \cdot M\right\rangle_T + \left\langle L^n\right\rangle_T\Big) = 0$$

in $\mathbb{P}$-probability. We assume that

$$\operatorname*{Lim}_{n\to+\infty} B\mathcal{K}^n = B\mathcal{K} \ \mu^A\text{-a.e.}$$

where Lim *denotes the closed set limit and we refer to Appendix 6.2 for more details.*

Observe that the above assumption on $(p^n)_{n\in\mathbb{N}}$ implies pointwise convergence of the investor's utilities as considered in Jouini and Napp [2004], Kardaras and Žitković [2011] and Larsen [2009]. Stability with respect to the statistical measure is investigated in Kardaras and Žitković [2011] and with respect to the market price of risk in Larsen and Žitković [2007]. We also note that the sensitivity with respect to the constraints is considered in Kardaras [2010] for the numéraire portfolio, i.e. for $p = 0$, as well as in Frei [2009] for the exponential indifference value under boundedness assumptions.

Theorem 4.2.16. *Let the Assumptions 4.2.14 and 4.2.15 hold and $\hat{X}^n$ and $\hat{\nu}^n$ be as in* (4.2.8)*, with each $\hat{N}^n$ continuous. Then as $n \to +\infty$:*

(i) The sequence of processes $(\hat{\nu}^n - \hat{\nu})\cdot M$ converges to zero in $\mathcal{M}^2$.

(ii) The family of wealth processes $\hat{X}^n \in \mathcal{X}(x)$, $n \in \mathbb{N}$, converges to $\hat{X} \in \mathcal{X}(x)$ in the semimartingale topology.

(iii) The functions u^n and their derivatives $(u^n)'$ converge pointwise to u and u' respectively.

Remark 4.2.17. We can establish *identical* results for the corresponding sequence of dual problems and their optimizers. However since these are not the main objects of interest we do not state them now but pursue them further in Section 4.6. We also note that the above theorem is best formulated for a continuous filtration.

Remark 4.2.18. We discuss here in more detail how the theorem above relates to others in the literature. When $\beta^n \cdot M \equiv L^n \equiv 0$, $p^n = p$ and $\mathcal{K}^n \equiv \mathbb{R}^d$ we are in the setting of Larsen and Žitković [2007]. Observe that our Assumption 4.2.14 is more restrictive than the notion of “V-relative compactness” introduced therein. Thus by fixing the utility and imposing stricter conditions on the λ^n we get convergence of the whole path process together with the convergence of the optimal strategies, strengthening the main results of Larsen [2009] as well as Larsen and Žitković [2007] where one gets convergence in probability of the optimal *terminal* values $\hat{X}^n_T$. From the convergence in the semimartingale topology we then deduce the corresponding continuity results given in Bayraktar and Kravitz [2010] where T is replaced by a stopping time τ.

When $\beta^n \cdot M \equiv L^n \equiv 0$, $\lambda^n \equiv \lambda$ and $\mathcal{K}^n \equiv \mathbb{R}^d$ we recover Nutz [2010a] Corollary 5.7. Therein the process S need not be continuous. The reason for this is that when only the risk aversion parameter varies one can compare the opportunity processes directly via Jensen's inequality. When $\mathbb{P}, \lambda$ and $\mathcal{K}$ also vary such an approach seems not to be feasible, hence our reliance on BSDE methods alone which necessitates more stringent assumptions.

For $\lambda^n \equiv \lambda$ and $\mathcal{K}^n \equiv \mathbb{R}^d$ observe that under our assumptions $(\mathbb{P}^n)_{n\in\mathbb{N}}$ converges to $\mathbb{P}$ in total variation. Thus we recover Kardaras and Žitković [2011] Theorem 1.5 in the case where there is no random endowment and the utility is power. Similarly to the case of Larsen and Žitković [2007] above, our Assumption 4.2.14 implies the Assumption (UI) therein. As a consequence we partially extend their results to convergence of the optimal wealth process in the semimartingale topology in a setting without random endowment.

When $p = 0$, $\lambda^n \equiv \lambda$ and, in addition to the cones and measure, the information structure is also allowed to vary, Kardaras [2010] obtains results similar to ours for the numéraire portfolio using the explicit formula for it. The problem there differs from ours as it is “myopic” and as such there is no opportunity process and corresponding BSDE (in the sense above, however see Appendix 6.1), so that one cannot directly compare the two approaches. We note however that in both cases the convergence of the cones is cast in terms of the closed set limit, see however Remark 4.2.19.

An approach similar to ours was given for the exponential indifference value in Frei [2009]. Since $\lambda^n \equiv \lambda$ and $\beta^n \cdot M \equiv L^n \equiv 0$ hold there, the quadratic growth and locally Lipschitz assumptions on the respective BSDEs are uniform in n so that a corresponding stability result can be used in the setting of a bounded mean-variance tradeoff that is present in that article. Moreover, the conditions required for stability to hold are on the drivers of the BSDEs and not in terms of the input parameters.

As a final remark, when the utility function is allowed to vary, one needs to assume that the sequence converges pointwise *and* satisfies a uniform growth condition, see Jouini and Napp [2004], Kardaras and Žitković [2011] and Larsen [2009]. This is implied by our Assumption 4.2.15 so that we are consistent with the literature in this respect.

Remark 4.2.19. Here we elaborate further on the type of convergence assumed on the cones. Together with Proposition 6.2.3, Assumption 4.2.15 implies that the projections $\Pi_{B\mathcal{K}^n}$ converge pointwise to $\Pi_{B\mathcal{K}}$ which is the key property in showing the convergence of the drivers of the related BSDEs. Define the set

$$\mathfrak{N}(t,\omega) := \operatorname{Ker}\big(C(t,\omega)\big) = \operatorname{Ker}\big(B(t,\omega)\big),$$

a closed predictably measurable multivalued mapping. This is the set of null-investments described in Karatzas and Kardaras [2007]. In Kardaras [2010] the author replaces Assumption 4.2.15 with

$$\mathfrak{N} \subset \mathcal{K}^n \text{ for all } n \in \mathbb{N}_0 \text{ and } \operatorname*{Lim}_{n\to+\infty} \mathcal{K}^n = \mathcal{K} \quad \mu^A\text{-a.e.}$$

Proposition 6.2.4 in the appendix shows that this is sufficient to imply $\operatorname{Lim}_{n\to+\infty} B\mathcal{K}^n = B\mathcal{K}$ μ^A-a.e. so that the results of the present chapter remain valid under this alternative assumption. The requirement $\mathfrak{N} \subset \mathcal{K}^n$ for all $n \in \mathbb{N}_0$ means that although the investor faces investment constraints imposed on their portfolio, e.g. by regulators or the market structure, these constraints must be compatible with the null-investments in the sense that simultaneously the agent must be allowed to choose null-investment strategies. When $\operatorname{Ker}(C)$ has a complicated structure this can be difficult to check and thus we prefer Assumption 4.2.15.

The Compatibility of the Set Convergence with the Null-investments

Note that $\operatorname{Lim}_{n\to\infty} \mathcal{K}^n = \mathcal{K}$ alone is *not* sufficient for the stability result to hold as is illustrated by a simple counterexample in which the investment constraints are not compatible with a redundant structure of the market.

Namely, consider a standard one-dimensional Brownian motion W and set $M := (W,0)^\intercal$. Taking a constant $\lambda = (\lambda_1, 0)^\intercal \in \mathbb{R}^2$, $\lambda_1 > 0$, completes the description of the market. We may choose $A_t \equiv t$ so that the process B becomes

$$B \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The sequence of (deterministic) constraint sets is defined by setting

$$\mathcal{K}^n := \{(x,y) \in \mathbb{R}^2 \,|\, y = nx,\ x \geq 0\},\ \mathcal{K} := \{(x,y) \in \mathbb{R}^2 \,|\, x = 0,\ y \geq 0\}.$$

One can see that these cones are polyhedral and that we have $\{(0,0)\} = B\mathcal{K} \neq$

$\mathrm{Lim}_{n\to+\infty} B\mathcal{K}^n = [0,+\infty) \times \{0\}$. Note though that we do have $\mathcal{K} = \mathrm{Lim}_{n\to+\infty} \mathcal{K}^n = \{0\} \times [0,+\infty)$. From this description we immediately have that in the limiting case the agent is only allowed to invest in stocks that do not yield any extra profit when compared to the bond while for $n \geq 1$ they can choose an optimal strategy $\hat{\nu}^n$ and it does not matter that $\hat{\nu}_2^n = n\hat{\nu}_1^n$ may become arbitrarily large since it can be offset by a position in the bond, whose evolution is the same as the one of S^2. Indeed, the optimal position in the first stock is $\hat{\nu}_1^n = \lambda_1/(1-p)$ which clearly does not converge to zero, the only possible position in the first stock in the limiting case, by assumption. The optimal wealth for $n \geq 1$ is given by

$$\hat{X}_t^n = x \exp\left(\frac{\lambda_1}{1-p} W_t + \frac{\lambda_1^2(1-2p)}{2(1-p)^2} t\right),$$

which does not equal $\hat{X} \equiv x$, the optimal wealth process for the constraint set $\mathcal{K}$. Correspondingly, the value functions u^n do not converge to u, since for $x > 0$

$$u^n(x) = \frac{1}{p} x^p \exp\left(\frac{p\lambda_1^2 T}{2(1-p)}\right) \quad \text{and} \quad u(x) = \frac{1}{p} x^p.$$

Remark 4.2.20. The reader may ask whether it is necessary to vary λ *and* $\mathbb{P}$ or whether by a sensible choice of the Girsanov transform this can be reduced to simply varying $\mathbb{P}$. In certain cases this is indeed the case, typically when $M = W$ is a Brownian motion. However in general not so as the following example illustrates. Set $M := W \cdot W$ for a one-dimensional Brownian motion. Thus the asset has dynamics

$$dS_t = S_t(W_t\, dW_t + \lambda_t W_t^2\, dt) \quad \text{under } \mathbb{P}.$$

If λ is allowed to vary, say to $\widetilde{\lambda}$, all models can be achieved such that

$$dS_t = S_t(W_t\, dW_t + \widetilde{\lambda}_t W_t^2\, dt) \quad \text{under } \mathbb{P}.$$

However, if only $\mathbb{P}$ can be varied, we have $d\widetilde{\mathbb{P}}/d\mathbb{P} := \mathcal{E}(-\beta \cdot W)$ and the process S has dynamics

$$dS_t = S_t(W_t\, d\widetilde{W}_t + (\lambda_t - \beta_t) W_t^2\, dt) \quad \text{under } \widetilde{\mathbb{P}},$$

where $\widetilde{W}$ is a $\widetilde{\mathbb{P}}$-Brownian motion. In particular we will find it impossible to recreate the first dynamics as W is not a Brownian motion under $\widetilde{\mathbb{P}}$.

4.3 Cone Constrained Utility Maximization

The utility maximization problem under polyhedral cone constraints is studied in detail in Karatzas and Žitković [2003] and Westray [2009] for the additive representation. We hence work also in the *additive framework* here and refer the reader to Remark 4.2.5 for more details. We note that in the mentioned articles the constraint set $\mathcal{L}$ is independent of (t, ω). In this section we show how the results of Westray [2009] can be extended to give Theorem 4.2.8. The key result for our analysis above consists of having the dual optimizer as an element of our specific dual domain of supermartingale measures. A careful reading of the proof of Westray [2009] Theorem 3.4.2 on existence and uniqueness shows that one needs one specific property of the cone $\mathcal{K}$ ($\mathcal{L}$ respectively), namely, provided that the set

$$\mathcal{X}^{add}(1) \text{ is closed in the semimartingale topology}$$

then the main existence result Westray [2009] Theorem 3.4.2 continues to hold with a predictably measurable, non-empty, closed convex multi-valued mapping $\mathcal{K}$ ($\mathcal{L}$ respectively).

Lemma 4.3.1. *Suppose that $\mathcal{K}$ satisfies Assumption 4.2.3 then $\mathcal{X}^{add}(1)$ is closed in the semimartingale topology.*

Proof. Since $\mathcal{K}(t,\omega)$ (and hence $\mathcal{L}(t,\omega)$) is a polyhedral cone for all $t \in [0,T]$ $\mathbb{P}$-a.s. we see that Czichowsky and Schweizer [2011] Corollary 4.6 applies. This guarantees the result. ☐

We now adapt the results of Westray [2009] which are in the context of the utility maximization with a random endowment and begin with the primal problem.

Lemma 4.3.2. *Suppose that Assumptions 4.2.1 and 4.2.3 hold. Then:*

(i) There exists an optimal terminal wealth $\hat{X}_T$, $\hat{X} \in \mathcal{X}^{add}(1)$, such that

$$\mathbb{E}[U(\hat{X}_T)] = \sup_{X \in \mathcal{X}^{add}(1)} \mathbb{E}[U(X_T)] \,.$$

Moreover, any two such primal optimizers $\hat{X}$ and $\bar{X}$ are indistinguishable.

(ii) We have that $\hat{X}_T > 0$ $\mathbb{P}$-a.s. so there is an optimal strategy $\hat{\nu} \in \mathcal{A}_{\mathcal{K}}$ with $\hat{X} = X^{1,\hat{\nu}} \in \mathcal{X}(1)$.

(iii) The optimal strategy $\hat{\nu}$ is unique in the sense that given any other admissible strategy $\bar{\nu}$ with corresponding wealth process $X_T^{1,\bar{\nu}}$ which is optimal for the primal problem we have

$$\mathbb{E}\big[\langle(\hat{\nu} - \bar{\nu}) \cdot M\rangle_T\big] = 0.$$

Proof. From Westray [2009] Theorem 3.4.2 (iii) we see that there is an admissible $\hat{H}$ such that with $\hat{X}_T := 1 + (\hat{H} \cdot S)_T$

$$\mathbb{E}[U(\hat{X}_T)] = \sup_{X \in \mathcal{X}^{add}(1)} \mathbb{E}[U(X_T)] \,.$$

For the notion of admissibility of $\hat{H}$ we refer to the cited thesis. Since U is strictly concave a standard argument involving convex combinations gives the uniqueness at terminal time, see also Kramkov and Schachermayer [1999] Lemma 3.3. For completeness we now derive the uniqueness on the level of processes. Let $\hat{X}$ and $\bar{X}$ be two primal optimizers, for which we know that $\hat{X}_T = \bar{X}_T$. Now suppose there is a $t \in [0,T)$ and a set $A \in \mathcal{F}_t$ such that $\hat{X}_t > \bar{X}_t$ on A and $\mathbb{P}(A) > 0$. Define the integrand

$$H := \hat{H}\mathbf{1}_{[0,t]} + \bar{H}\mathbf{1}_{(t,T]}\mathbf{1}_A + \hat{H}\mathbf{1}_{(t,T]}\mathbf{1}_{A^c},$$

where $\hat{H}$ and $\bar{H}$ are the integrands for $\hat{X}$ and $\bar{X}$. Observe that $X := 1 + H \cdot S \in \mathcal{X}^{add}(1)$ as we have $(H \cdot S)_u = (\bar{X}_u + \hat{X}_t - \bar{X}_t)\,\mathbf{1}_A + \hat{X}_u\,\mathbf{1}_{A^c}$ for $u \geq t$ and this is nonnegative by assumption (recall that $\hat{X}_t > \bar{X}_t$ on A). Now we note that $\hat{X}_T = \bar{X}_T$ $\mathbb{P}$-a.s. and write

$$\mathbb{E}[U(X_T)] = \mathbb{E}\Big[\mathbf{1}_{A^c}\mathbb{E}\Big[U(\hat{X}_T)\Big|\,\mathcal{F}_t\Big] + \mathbf{1}_A\mathbb{E}\Big[U(\bar{X}_T + \hat{X}_t - \bar{X}_t)\Big|\,\mathcal{F}_t\Big]\Big] > \mathbb{E}\Big[U(\hat{X}_T)\Big].$$

This is a contradiction and the result in (i) follows from the continuity of the wealth processes.

For item (ii) observe from Westray [2009] Theorem 3.4.2 (iv) that $\hat{X}_T = -\widetilde{U}'\Big(\frac{d\hat{\zeta}_c}{d\mathbb{P}}\Big)$ where $\hat{\zeta}_c$ is a finite, nonnegative and countably additive measure that is absolutely continuous with respect to $\mathbb{P}$. Since $-\widetilde{U}'(y) = 0$ if and only if $y = +\infty$ for $y \geq 0$ we cannot have that $\hat{X}_T$ is zero on a set of nonzero $\mathbb{P}$-measure, this would contradict the finiteness of $\hat{\zeta}_c$.

For item (iii) we have the equality $\mathcal{E}(\hat{\nu} \cdot M + \hat{\nu} \cdot \langle M\rangle\lambda) \equiv \mathcal{E}(\bar{\nu} \cdot M + \bar{\nu} \cdot \langle M\rangle\lambda)$. By the uniqueness of the stochastic logarithm we derive that $\hat{\nu} \cdot M + \hat{\nu} \cdot \langle M\rangle\lambda \equiv \bar{\nu} \cdot M + \bar{\nu} \cdot \langle M\rangle\lambda$ and thus it follows that $(\hat{\nu} - \bar{\nu}) \cdot M$ is a continuous local martingale of finite variation and is hence constant and equal to zero, which proves the last assertion. □

In Westray [2009], following Cvitanić et al. [2001], the dual domain is a subset of $L^\infty(\mathbb{P})^*$, the bounded, finitely additive measures that are absolutely continuous with respect to $\mathbb{P}$. It contains $\mathcal{Y}^{add}(y)$ where

$$\mathcal{Y}^{add}(y) := \Big\{Y \geq 0 \,|\, Y_0 = y \text{ and } XY \text{ is a supermartingale for all } X \in \mathcal{X}^{add}(1)\Big\} \subset \mathcal{Y}(y).$$

Note that $\mathcal{Y}^{add}(y)$ depends on $\mathcal{L}$ (respectively $\mathcal{K}$). The next lemma which shows that the dual minimizer of Westray [2009] can be related to an element of $\mathcal{Y}^{add}(y)$ is key.

Lemma 4.3.3. *Let the assumptions of the previous lemma hold. Then, given $y > 0$, there is a $\hat{Y}^y \in \mathcal{Y}^{add}(y) \subset \mathcal{Y}(y)$ which is optimal for the dual problem* (4.2.5)*, unique up to indistinguishability and with $\hat{Y}^y_T > 0$, $\mathbb{P}$-a.s.*

Proof. Define the sets

$$\mathcal{C} := \{\xi \in L^0(\mathbb{P}) \,|\, 0 \leq \xi \leq X_T,\, X \in \mathcal{X}^{add}(1)\}$$
$$\mathcal{D} := \{\eta \in L^0(\mathbb{P}) \,|\, 0 \leq \eta \leq Y_T,\, Y \in \mathcal{Y}^{add}(1)\}.$$

By construction and the above lemma we have

$$u(1) = \mathbb{E}[U(\hat{X}_T)] = \sup_{\xi \in \mathcal{C}} \mathbb{E}[U(\xi)]$$

and thus, using the Calculus of Variations argument from the proof of Bouchard and Pham [2004] Lemma 5.7, one can show that with $\widetilde{\eta} := U'(\hat{X}_T) > 0$ we obtain that $\mathbb{E}[\widetilde{\eta}\,(\hat{X}_T - \xi)] \geq 0$ for all $\xi \in \mathcal{C}$. We set $y := \mathbb{E}[\widetilde{\eta}\hat{X}_T] = \mathbb{E}[(\hat{X}_T)^p] > 0$ and observe that $\mathbb{E}[\widetilde{\eta}\,\xi] \leq y$ for all $\xi \in \mathcal{C}$. Hence $\widetilde{\eta}/y \in \mathcal{C}^\circ$, where we write $\mathcal{C}^\circ$ for the polar of the cone $\mathcal{C}$,

$$\mathcal{C}^\circ := \{\eta \in L^0_+(\mathbb{P}) \,|\, \mathbb{E}[\xi\eta] \leq 1 \text{ for all } \xi \in \mathcal{C}\}.$$

Observing from Westray [2009] Lemma 3.5.7 that $\xi \in \mathcal{C}$ if and only if $\xi \geq 0$ and $\mathbb{E}_{\mathbb{Q}}[\xi] \leq 1$ for all $\mathbb{Q} \in \mathcal{M}^{\text{sup}}$, we derive that $\mathcal{C} = (\mathcal{M}^{\text{sup}})^\circ$. Here,

$$\mathcal{M}^{\text{sup}} := \{\mathbb{Q} \sim \mathbb{P} \,|\, X \text{ is a } \mathbb{Q}\text{-supermartingale for all } X \in \mathcal{X}^{add}(1)\}.$$

Applying the same reasoning as in the proof of Kramkov and Schachermayer [1999] Lemma 4.1 we derive that $\mathcal{D}^{\circ\circ} = \mathcal{D}$ and equating measures $\mathbb{Q}$ with their densities $Z^{\mathbb{Q}}$ we are led to conclude that $\mathcal{M}^{\text{sup}} \subset \mathcal{D}$. Hence $\mathcal{C}^\circ = (\mathcal{M}^{\text{sup}})^{\circ\circ} \subset \mathcal{D}^{\circ\circ} = \mathcal{D}$ from which $\widetilde{\eta}/y \in \mathcal{D}$. Thus there is a $\hat{Y} \in \mathcal{Y}^{add}(1)$ with $0 < \widetilde{\eta}/y \leq \hat{Y}_T$ and such that

$$1 = \mathbb{E}[\hat{X}_0\hat{Y}_0] \geq \mathbb{E}[\hat{X}_T\hat{Y}_T] \geq \mathbb{E}[\hat{X}_T\widetilde{\eta}/y] = 1.$$

In particular $\hat{X}\hat{Y}$ is a martingale. We conclude that $\hat{Y}^y := y\hat{Y} \in \mathcal{Y}^{add}(y)$ is a dual optimizer. More explicitly, since $\widetilde{\eta} = U'(\hat{X}_T)$,

$$\begin{aligned}\mathbb{E}[\widetilde{U}(\hat{Y}^y_T)] &\geq \inf_{Y \in \mathcal{Y}^{add}(y)} \mathbb{E}[\widetilde{U}(Y)] \geq \inf_{Y \in \mathcal{Y}^{add}(y)} \mathbb{E}[U(\hat{X}_T) - \hat{X}_T Y] \geq \mathbb{E}[U(\hat{X}_T)] - y \\ &= \mathbb{E}[U(\hat{X}_T)] - \mathbb{E}[\hat{X}_T\widetilde{\eta}] = \mathbb{E}[\widetilde{U}(\widetilde{\eta})] \geq \mathbb{E}[\widetilde{U}(\hat{Y}^y_T)].\end{aligned}$$

For uniqueness we again suppose that there exists a $t \in [0, T)$ and a set $A \in \mathcal{F}_t$ such that $\hat{Y}_t > \overline{Y}_t$ on A and $\mathbb{P}(A) > 0$, where $\hat{Y}$ and $\overline{Y}$ are two optimal dual processes that are necessarily equal at terminal time T. Since the dual function is strictly decreasing we have that the following inequality holds on A,

$$\mathbb{E}\Big[\widetilde{U}\Big(\tfrac{\hat{Y}_t}{\overline{Y}_t}\,\overline{Y}_T\Big)\,\Big|\,\mathcal{F}_t\Big] < \mathbb{E}\Big[\widetilde{U}(\overline{Y}_T)\Big|\,\mathcal{F}_t\Big] = \mathbb{E}\Big[\widetilde{U}(\hat{Y}_T)\Big|\,\mathcal{F}_t\Big].$$

Note that $\overline{Y}$ being a supermartingale $\overline{Y}_T > 0$ implies that $\overline{Y} > 0$. We then define the process

$$Y := \hat{Y}\,\mathbf{1}_{[0,t]} + \tfrac{\hat{Y}_t}{\overline{Y}_t}\,\overline{Y}\,\mathbf{1}_A\,\mathbf{1}_{(t,T]} + \hat{Y}\,\mathbf{1}_{A^c}\,\mathbf{1}_{(t,T]}.$$

It is now essential to show that $Y \in \mathcal{Y}^{add}(1)$ which holds by separately checking the respective cases thanks to the choice of $\mathcal{Y}^{add}(1)$ as a family of supermartingale measures for S, more precisely, XY is a supermartingale for any admissible wealth process X. Note that here it is also important that the Y constructed above is right-continuous at t. A computation similar to that for the uniqueness of $\hat{X}$ leads to

$$\mathbb{E}\Big[\widetilde{U}(Y_T)\Big] < \mathbb{E}\Big[\mathbf{1}_{A^c}\mathbb{E}\Big[\widetilde{U}(\hat{Y}_T)\Big|\,\mathcal{F}_t\Big] + \mathbf{1}_A\mathbb{E}\Big[\widetilde{U}(\hat{Y}_T)\Big|\,\mathcal{F}_t\Big]\Big] = \mathbb{E}\Big[\widetilde{U}(\hat{Y}_T)\Big] = \widetilde{u}(y),$$

which is a contradiction. The processes $\hat{Y}$ and $\overline{Y}$ are càdlàg and they satisfy $\hat{Y}_t = \overline{Y}_t$ $\mathbb{P}$-a.s. for each $t \in [0,T]$. We then conclude that they are indistinguishable. $\square$

The remaining items from Theorem 4.2.8, if not already implicitly contained in the previous proofs, can be deduced in a standard fashion so we omit the details.

4.4 The Dual Domain in the Presence of Cone Constraints

This section is devoted to a proof of Theorem 4.2.10, a full description of the dual domain in the cone constrained problem. Due to our choice of writing wealth in exponential format the proof becomes simpler when compared to Karatzas and Žitković [2003] and Larsen and Žitković [2007]. The theorem is slightly more general because it covers the "multiplicative" dual domain $\mathcal{Y}(1)$ which contains the "additive" dual domain from the cited references. Since in this chapter the filtration is not assumed to be continuous we note that dual elements themselves need not be continuous. However, the polyhedral nature of the cones is essential for the results of this section.

Proposition 4.4.1. *Let Assumption 4.2.3 hold and $Y \in \mathcal{Y}(1)$ with $Y_T > 0$. Then there exist:*

(i) A predictable M-integrable process κ^Y with $B(\lambda - \kappa^Y) \in (B\mathcal{K})^\circ$, μ^A-a.e.

(ii) A local martingale N^Y orthogonal to M.

(iii) A predictable decreasing càdlàg process D^Y with $D^Y_0 = 1$ and $D^Y_T > 0$ $\mathbb{P}$-a.s. such that with the above
$$Y = D^Y\,\mathcal{E}(-\kappa^Y \cdot M + N^Y).$$

Proof. Since $0 \in \mathcal{K}$ μ^A-a.e. we may proceed as in Larsen and Žitković [2007] Proposition 3.2. to deduce that a given $Y \in \mathcal{Y}(1)$ with $Y_T > 0$ admits a multiplicative decomposition which we can write as
$$Y = D^Y\,\mathcal{E}(-\kappa^Y \cdot M + N^Y),$$
where D^Y is a positive, predictable, nonincreasing process with $D^Y_0 = 1$, κ^Y is a predictable M-integrable process and N^Y a local martingale orthogonal to M, see Jacod and Shiryaev [2003] Theorem II.8.21 and Lemma III.4.24. It thus remains to show that $B(\lambda - \kappa^Y) \in (B\mathcal{K})^\circ$ μ^A-a.e. and we drop the superscripts in the remainder of the proof to ease the exposition.

Set $\widetilde{D} := \log(D)$. By Delbaen and Schachermayer [1995] Theorem 2.1 there exists a predictable μ^A-null set E together with a nonnegative predictable process η such that
$$\widetilde{D}_t = -\int_0^t \eta_s\,dA_s + \int_0^t \mathbf{1}_E(s)\,d\widetilde{D}_s =: -\int_0^t \eta_s\,dA_s + \overline{D}_t.$$

From Itô's formula we derive that for any admissible investment strategy ν with corresponding wealth process $X^\nu \in \mathcal{X}(1)$ we have

$$d(X_t^\nu Y_t) = X_t^\nu Y_{t-}\Big((\nu_t - \kappa_t)^\top dM_t + dN_t + d[\overline{D}, N]_t + \Big(\nu_t^\top B_t^\top B_t(\lambda_t - \kappa_t) - \eta_t\Big)\, dA_t + d\overline{D}_t \Big).$$

Observe that by Yoeurp's lemma $(\nu-\kappa)^\top dM + dN + d[\overline{D}, N]$ is the differential of a local martingale, M being continuous. Since the product $X^\nu Y$ is a supermartingale, we hence must have that the differential

$$\Big(\nu^\top B^\top B(\lambda - \kappa) - \eta\Big)\, dA + d\overline{D}$$

generates a nonpositive measure on the predictable σ-algebra $\mathcal{P}$. Since $\mu^A(E) = 0$ we conclude, using the cone property of $\mathcal{K}$, that the following inequality must hold

$$(B\nu)^\top B(\lambda - \kappa) = \nu^\top B^\top B(\lambda - \kappa) \leq 0 \tag{4.4.1}$$

μ^A-a.e. for each $\nu \in \mathcal{A}_\mathcal{K}$. To conclude we have to show that arbitrary elements of $B\mathcal{K}$ can be realized as trading strategies, μ^A-a.e. This is where the assumption that the constraints be polyhedral is needed.

Choosing $\nu = K^1, \ldots, K^m$ it now follows that there exists a single μ^A-null set (also denoted E) such that for all $(t,\omega) \in E^c$ and all $j \in \{1, \ldots, m\}$

$$(B_t(\omega) K_t^j(\omega))^\top B_t(\omega)(\lambda_t(\omega) - \kappa_t(\omega)) \leq 0.$$

In particular we have $B(\lambda - \kappa) \in (B\mathcal{K})^\circ$, μ^A-a.e. as for fixed (t,ω) any $k \in B_t(\omega)\mathcal{K}(t,\omega)$ may be written μ^A-a.e. as

$$k = \sum_{j=1}^m c_j B_t(\omega) K_t^j(\omega)$$

with some $c_j \geq 0$ for $j \in \{1, \ldots, m\}$. □

Remark 4.4.2. Suppose that $\mathcal{K} \equiv \mathbb{R}^d$ then in (4.4.1), given a Y and corresponding κ^Y, we can directly insert $\nu = \lambda - \kappa^Y$. Integrating the resulting expression over $[0,T]$ with respect to μ^A we derive that the stochastic integrals $\lambda \cdot M$ and $\kappa^Y \cdot M$ are indistinguishable and thus we deduce the multidimensional version of Larsen and Žitković [2007] Proposition 3.2.

Corollary 4.4.3. *Under the Assumptions 4.2.1 and 4.2.3 there exist a predictable M-integrable process $\hat{\kappa}$ as well as a local martingale $\hat{N}$ orthogonal to M, such that $\hat{Y}^1 = \mathcal{E}\big(-\hat{\kappa}\cdot M + \hat{N}\big)$ for the dual optimizer $\hat{Y}^1$ where $y = 1$ and $B(\lambda - \hat{\kappa}) \in (B\mathcal{K})^\circ$ μ^A-a.e. If $\hat{Y}^y$ denotes the dual optimizer for $y > 0$ we have that $\hat{Y}^y = y\hat{Y}^1 = y\,\mathcal{E}\big(-\hat{\kappa}\cdot M + \hat{N}\big)$.*

Proof. In view of Proposition 4.4.1 the key is to show that $\hat{Y}_T^1 > 0$ which we derive from Lemma 4.3.3. We may then proceed as in the proof of Larsen and Žitković [2007] Corollary 3.3. The independence of y follows from the factorization property. □

Corollary 4.4.4. *Under the Assumptions 4.2.1 and 4.2.3 the optimal portfolio $\hat{\nu}$ satisfies μ^A-a.e. for all admissible strategies ν,*

$$\hat{\nu}^\intercal B^\intercal B(\lambda - \hat{\kappa}) = 0 \quad \text{and} \quad (\nu - \hat{\nu})^\intercal B^\intercal B(\lambda - \hat{\kappa}) \leq 0.$$

Proof. Due to factorization we may suppose that $x = 1$. Then for the optimizers we know from Theorem 4.2.8 (iii) that the process $\hat{X}\hat{Y}^y$ is a martingale when $y = u'(1)$. We derive

$$d(\hat{X}_t\hat{Y}^y_t) = \hat{X}_t\hat{Y}^y_{t-}\Big((\hat{\nu}_t - \hat{\kappa}_t)^\intercal dM_t + d\hat{N}_t + \Big(\hat{\nu}_t^\intercal B_t^\intercal B_t(\lambda_t - \hat{\kappa}_t)\Big)dA_t\Big).$$

Thanks to Assumption 4.2.3 it must hold that $\hat{\nu}^\intercal B^\intercal B(\lambda - \hat{\kappa}) = 0$ for all $\nu \in \mathcal{A}_\mathcal{K}$, μ^A-a.e. The second statement of the corollary now follows upon addition of (4.4.1). □

4.5 Relationship with Quadratic Semimartingale BSDEs

Having established a representation for elements of the dual domain, in this section we use this to connect the optimizers $(\hat{X}, \hat{Y})$ with the solution triple of a specific BSDE proving Proposition 4.2.11. As noted before, admitting Theorem 4.2.10, one may find some of the results in Nutz [2011] Corollaries 3.12 and 5.18, however we provide here a complete proof as it illustrates the interplay between $\hat{\kappa}$ and $\hat{\nu}$. Moreover, the verification argument is via uniqueness of BSDEs building on the following lemma.

Lemma 4.5.1. *In the setting of Theorem 4.2.8 let $\hat{\Psi} := \log\Big(\frac{u'(x)\hat{Y}^1}{U'(\hat{X})}\Big)$. Then $\hat{\Psi} \in \mathfrak{E}$.*

The proof of this lemma relies on our special choice of the dual domain which allows us to define the so-called dual opportunity process and to use its dynamic optimality properties. From Nutz [2010b] we first recall the following result concerning the primal opportunity process and which also holds in the cone constrained utility maximization framework.

Proposition 4.5.2 (Nutz [2010b] Proposition 3.1)**.** *There is a unique càdlàg semimartingale L, the opportunity process, such that for any admissible strategy $\nu \in \mathcal{A}_\mathcal{K}$ and $t \in [0, T]$*

$$L_t\, U(X^\nu_t) = \operatorname*{ess\,sup}_{\check{\nu} \in \mathcal{A}_{\mathcal{K},\nu}} \mathbb{E}\Big[U\Big(X^{\check{\nu}}_T\Big)\Big|\, \mathcal{F}_t\Big]\,, \tag{4.5.1}$$

where the optimization is over all the continuation strategies $\check{\nu} \in \mathcal{A}_{\mathcal{K},\nu}$ for ν, i.e. over all the admissible strategies $\check{\nu}$ that are equal to ν on $[0, t]$. If $(\hat{X}, \hat{Y})$ denotes the optimal pair for the utility maximization problem satisfying $\hat{Y}_0 = u'(\hat{X}_0)$ $\mathbb{P}$-a.s. then $\hat{Y} = L\, U'(\hat{X})$. In particular, $\hat{\Psi} = \log(L)$.

We now turn our attention to the dual counterpart. To this end first define the domain

$$\mathcal{Y}^*(y) := \mathcal{Y}^{add}(y) \cap \{Y > 0\},$$

which in view of $\hat{Y}^1 \in \mathcal{Y}^*(1)$ does not affect the optimizers[1], in particular $\mathcal{Y}^*(y) \neq \emptyset$ (under the Assumptions of Theorem 4.2.8). This set exhibits two important properties which we state in the following lemma. These properties are related to the so-called fork-convexity of a set.

Lemma 4.5.3.

(i) Let $\bar{Y}, \widetilde{Y}$ and $\check{Y}$ be in $\mathcal{Y}^*(1)$ with $\widetilde{Y} = \check{Y}$ on $[0,t]$ for some $t \in [0,T]$, then

$$Y := \bar{Y}\mathbf{1}_{[0,t]} + \frac{\bar{Y}_t}{\check{Y}_t}\check{Y}\mathbf{1}_{(t,T]} \in \mathcal{Y}^*(1).$$

(ii) Let $\bar{Y}, \widetilde{Y}$ and $\check{Y}$ be in $\mathcal{Y}^*(1)$ with $\bar{Y} = \widetilde{Y} = \check{Y}$ on $[0,t]$ for some $t \in [0,T]$. If $A \in \mathcal{F}_t$, then

$$Y := \bar{Y}\mathbf{1}_A + \widetilde{Y}\mathbf{1}_{A^c} \in \mathcal{Y}^*(1).$$

The above properties of $\mathcal{Y}^*(1)$ can be proved easily using the definition of $\mathcal{Y}^{add}(1)$ as a family of "supermartingale measures" for S, see the proof of uniqueness of the dual optimizer. They are also the only two properties that are needed in Nutz [2010b] to derive the existence, uniqueness and characterization of the *dual opportunity process* for the unconstrained case. Hence, although not necessarily true for the dual domain considered there, we can prove the following result which is the counterpart to Proposition 4.5.2 and in which we use the notion of so-called dual continuation strategies. Namely, for $t \in [0,T]$, define

$$\mathcal{Y}^*(Y,t) = \{\widetilde{Y} \in \mathcal{Y}^*(y) : \widetilde{Y} = Y \text{ on } [0,t]\}.$$

Then, mimicking the proof of Nutz [2010b] Proposition 4.3, we deduce from Lemma 4.5.3 that the following result holds.

Proposition 4.5.4. *There exists a unique càdlàg process $\widetilde{L}$, the dual opportunity process, such that for any $Y \in \mathcal{Y}^*(y)$ and $t \in [0,T]$*

$$\widetilde{L}_t\,\widetilde{U}(Y_t) = \operatorname*{ess\,inf}_{\check{Y} \in \mathcal{Y}^*(Y,t)} \mathbb{E}\Big[\widetilde{U}(\check{Y}_T)\Big|\,\mathcal{F}_t\Big], \tag{4.5.2}$$

Moreover, the infimum is attained at $Y = \hat{Y}$ and we have that $\widetilde{L} = L^{\frac{1}{1-p}}$.

The previous two propositions allow us to prove the required estimates on $\hat{\Psi}$.

Proof of Lemma 4.5.1. Let $p \in (0,1)$ so that $q = \frac{p}{p-1} \in (-\infty, 0)$ and $L \geq 1$. The last inequality follows from (4.5.1) by using the strategy $\nu \equiv 0$. In particular $\hat{\Psi} \geq 0$ and we

[1] Alternatively, one could keep $\mathcal{Y}^{add}(y)$. Then in the calculations and definitions involved, whenever a Y_t appears in the denominator of some expression, there is also a related $\tilde{Y}$ in the numerator, e.g. Y itself, such that $\tilde{Y} = 0$. This relies on the fact that $Y \in \mathcal{Y}^{add}(y)$ with $Y_t = 0$ implies $Y_s = 0$ for $s \geq t$. Thus, all the possible expressions like $\frac{0}{0}$ and $\frac{\pm\infty}{+\infty}$ could be given a meaning such that the assertions, established via the $Y \in \mathcal{Y}^*(1)$ where no problems arise, are valid. However, we opt for avoiding all these technicalities.

notice that for all $\delta > 0$

$$\mathbb{E}\Big[\exp\Big(\delta\hat{\Psi}^*\Big)\Big] = \mathbb{E}\Bigg[\sup_{0\le t\le T}\Big(\exp\Big(\delta\hat{\Psi}_t\Big)\Big)\Bigg] = \mathbb{E}\Big[\Big(L^\delta\Big)^*\Big]. \tag{4.5.3}$$

In what follows the constant $c_{p,\delta} > 0$ is generic, depends on p and δ and may change from line to line. Let us consider an exponential moment of $\langle\lambda\cdot M\rangle_T$ of order $k > k_q := q^2 - \frac{q}{2} - q\sqrt{q^2-q}$. We now set $\beta := 1 - \frac{1}{q}\sqrt{q^2-q} > 1$, $\varrho := \beta/(\beta-1) > 1$ and $\delta := k\varrho/k_q > 1$. After defining $Y^\lambda := \mathcal{E}(-\lambda\cdot M)$ we deduce from (4.5.1) that for a fixed strategy $\nu \in \mathcal{A}_\mathcal{K}$, denoting $\check{\nu}$ a time-t continuation strategy of ν,

$$\begin{aligned}
L_t^\delta &\le p^\delta \operatorname*{ess\,sup}_{\check{\nu}\in\mathcal{A}_{\mathcal{K},\nu}}\bigg(\mathbb{E}\Big[\widetilde{U}\Big(Y_T^\lambda/Y_t^\lambda\Big)\Big|\,\mathcal{F}_t\Big] + \mathbb{E}\Big[\Big(X_T^{\check{\nu}}/X_t^{\check{\nu}}\Big)\Big(Y_T^\lambda/Y_t^\lambda\Big)\Big|\,\mathcal{F}_t\Big]\bigg)^\delta \\
&\le c_{p,\delta}\,\mathbb{E}\bigg[\mathcal{E}(-\beta q\lambda\cdot M)_{t,T}^{1/\beta}\,\exp\Big(k_q\langle\lambda\cdot M\rangle_{t,T}\Big)^{1/\varrho}\bigg|\,\mathcal{F}_t\bigg]^\delta + c_{p,\delta} \\
&\le c_{p,\delta}\,\mathbb{E}\bigg[\exp\Big(k_q\langle\lambda\cdot M\rangle_T\Big)\,\bigg|\,\mathcal{F}_t\bigg]^{\delta/\varrho} + c_{p,\delta} =: c_{p,\delta}\left(\chi_t^{\delta/\varrho}+1\right),
\end{aligned}$$

by making use of the definition of $\widetilde{U}$, the supermartingale property of $Y^\lambda X^{\check{\nu}}$ and of $\mathcal{E}(-\beta q\lambda\cdot M)$, Hölder's inequality and the positiveness of $-1/q$ and of k_q. Thanks to the assumption on the exponential moment of $\langle\lambda\cdot M\rangle_T$, the process χ is a (nonnegative) martingale on $[0,T]$ and thus amenable to Doob's inequality from which the result follows.

Let us now turn to the case of $p < 0$, i.e. when $q = \frac{p}{p-1} \in (0,1)$ and $0 < L \le 1$. Take an exponential moment of $\langle\lambda\cdot M\rangle_T$ of order $k > (1-p)k_q > k_q := q^2 + \frac{q}{2} + \sqrt{q^2+q}$. We define $\delta := \frac{k\varrho}{(1-p)k_q} > 1$ where $\beta := 1 + \frac{1}{q}\sqrt{q^2+q} > 1$ and $\varrho := \beta/(\beta-1) > 1$. Then

$$\begin{aligned}
\mathbb{E}\Big[\exp\Big(\delta\hat{\Psi}^*\Big)\Big] &= \mathbb{E}\Bigg[\Bigg(\exp\Bigg(\delta\sup_{0\le t\le T}\Big(-\hat{\Psi}_t\Big)\Bigg)\Bigg)\Bigg] = \mathbb{E}\Big[\Big(\widetilde{L}^{-\delta(1-p)}\Big)^*\Big] \\
&\le \mathbb{E}\bigg[\exp\Big(k_q\langle\lambda\cdot M\rangle_T\Big)\,\bigg|\,\mathcal{F}_t\bigg]^{\delta(1-p)/\varrho},
\end{aligned}$$

where $\widetilde{L}$ is the dual opportunity process. The claim can then again be deduced from Doob's inequality. □

We can now derive

Proposition 4.5.5. *Under the Assumptions 4.2.1 and 4.2.3 let $\hat{\nu}$ denote the optimal strategy, $\hat{X}$ the optimal wealth process and $\hat{Y}^1$ the optimal dual minimizer with decomposition $\hat{Y}^1 = \mathcal{E}\big(-\hat{\kappa}\cdot M + \hat{N}\big)$ which we assume to be continuous. If we set $\hat{\Psi} := \log(u'(x)\hat{Y}^1/U'(\hat{X}))$ and $\hat{Z} := -\hat{\kappa} + (1-p)\hat{\nu}$, then the triple $(\hat{\Psi},\hat{Z},\hat{N})$ is the* *unique solution to the BSDE* (4.2.6) *with* $\hat{\Psi} \in \mathfrak{E}$ *where*

$$F(\cdot, z) = \frac{1}{2} \|Bz\|^2 - \frac{q}{2} \|\Pi_{B\mathcal{K}}(B(z+\lambda))\|^2.$$

Remark 4.5.6. In analogy to Chapter 2, if $\hat{N}$ is not continuous but exhibits jumps then the BSDE reads

$$d\Psi_t = Z_t^\intercal \, dM_t + dN_t - F(t, Z_t)\, dA_t - \frac{1}{2} d\langle N^c\rangle_t + \log(1 + \Delta N_t) - \Delta N_t, \quad \Psi_T = 0,$$

where the driver F is from the above proposition. The triple $(\hat{\Psi}, \hat{Z}, \hat{N})$ is a solution to this BSDE with $\Delta\hat{N} > -1$ since $\hat{Y} > 0$.

Proof. An application of Itô's formula to the process $\hat{\Psi}$ gives

$$d\hat{\Psi}_t = \hat{Z}_t^\intercal \, dM_t + d\hat{N}_t - \frac{1}{2} d\langle \hat{N}\rangle_t + \left[(1-p)\hat{\nu}_t^\intercal B_t^\intercal B_t \left(\lambda_t - \frac{\hat{\nu}_t}{2}\right) - \frac{1}{2} \hat{\kappa}_t^\intercal B_t^\intercal B_t \hat{\kappa}_t\right] dA_t. \tag{4.5.4}$$

It remains to show that the generator in the previous equation corresponds to that given in the statement of the proposition. Using the relation

$$\hat{\nu}^\intercal B^\intercal B \lambda = \hat{\nu}^\intercal B^\intercal B \hat{\kappa}$$

implied by Corollary 4.4.4 we end up with the following form for the generator of (4.5.4),

$$\frac{1}{2}\|B\hat{Z}\|^2 + \frac{p(1-p)}{2} \left\| B\left(\frac{\hat{Z}+\lambda}{1-p}\right)\right\|^2 - \frac{p(1-p)}{2} \left\| B\left(\hat{\nu} - \frac{\hat{Z}+\lambda}{1-p}\right)\right\|^2.$$

Now from the definition of $\hat{Z}$ together with Corollary 4.4.4 the following equation holds μ^A-a.e. for all admissible ν

$$(\nu - \hat{\nu})^\intercal B^\intercal B \left[(1-p)\hat{\nu} - (\hat{Z}+\lambda)\right] \geq 0.$$

This equation can be understood as the subgradient condition for the convex function

$$\mathbb{R}^d \ni \eta \mapsto \frac{1-p}{2} \left\| B\left(\eta - \frac{\hat{Z}+\lambda}{1-p}\right)\right\|^2$$

to have a minimum over $\mathcal{K}$ at $\hat{\nu}$ holding μ^A-a.e. In particular, μ^A-a.e.

$$\frac{1-p}{2} \left\| B\left(\hat{\nu} - \frac{\hat{Z}+\lambda}{1-p}\right)\right\|^2 = \frac{1-p}{2} \inf_{\eta\in\mathcal{K}} \left\| B\left(\eta - \frac{\hat{Z}+\lambda}{1-p}\right)\right\|^2.$$

Since it coincides with the generator of (4.5.4) μ^A-a.e. F is hence of the claimed form, paying attention to the signs and using the Pythagorean rule (see Theorem 4.5.7).

Since $\hat{N}$ is assumed continuous we have constructed a solution $(\hat{\Psi}, \hat{Z}, \hat{N})$ to the BSDE (4.2.6) with $\hat{\Psi} \in \mathfrak{E}$. The claimed uniqueness then follows from Theorem 3.2.6 noting that Proposition 4.6.3 implies the required Assumption 3.2.2 on the driver F. $\square$

To write the processes $\hat{X}$ and $\hat{Y}$ in terms of the solution $(\hat{\Psi}, \hat{Z}, \hat{N})$ to the above BSDE we first recall a classical result.

Theorem 4.5.7 (Moreau Orthogonal Decomposition)**.** *Let $Q \subset \mathbb{R}^d$ be a closed convex cone and $Q^\circ \subseteq \mathbb{R}^d$ its polar cone. Then for all $q, r, u \in \mathbb{R}^d$ the following statements are equivalent:*

(i) $u = q + r$, $q \in Q$, $r \in Q^\circ$ *and* $q^\intercal r = 0$,

(ii) $q = \Pi_Q(u)$ *and* $r = \Pi_{Q^\circ}(u)$,

where Π *denotes the projection or nearest point operator onto the indicated set.*

Proposition 4.5.8. *Suppose that the Assumptions 4.2.1 and 4.2.3 hold and that $y = u'(x)$ for some $x > 0$. Given $(\hat{\Psi}, \hat{Z}, \hat{N})$, the unique solution to the BSDE* (4.2.6) *with $\hat{\Psi} \in \mathfrak{E}$ and the above driver F, we can write the optimizers, up to indistinguishability, as*

$$\hat{X}^x = x\,\mathcal{E}(\widetilde{\nu} \cdot M + \widetilde{\nu} \cdot \langle M \rangle \lambda), \qquad \hat{Y}^y = y\,\mathcal{E}\big(-\widetilde{\kappa} \cdot M + \hat{N}\big),$$

where the predictable integrands $\widetilde{\nu}$ and $\widetilde{\kappa}$ are defined via

$$\widetilde{\nu} := \frac{1}{1-p} P^\intercal \widetilde{\Gamma}^{\frac{1}{2}}\, \Pi_{B\mathcal{K}}\big(B(\hat{Z} + \lambda)\big), \quad \widetilde{\kappa} := P^\intercal \widetilde{\Gamma}^{\frac{1}{2}} \Big[B\lambda - \Pi_{(B\mathcal{K})^\circ}\big(B(\hat{Z} + \lambda)\big)\Big]$$

and satisfy, μ^A-a.e. $B\widetilde{\nu} = B\hat{\nu}$ and $B\widetilde{\kappa} = B\hat{\kappa}$. The process $(\widetilde{\Gamma}^{i,j})_{i,j=1,\dots,d}$ is chosen to be a predictable process valued in the space of $d \times d$ diagonal matrices such that

$$\widetilde{\Gamma}^{ij} = \begin{cases} 1/\Gamma^{ii} & \text{if } i = j \text{ and } \Gamma^{ii} \neq 0 \\ 0 & \text{if } i \neq j. \end{cases}$$

Proof. The formulae for $\hat{X}$ and $\hat{\nu}$ are given (up to null-investments) in Nutz [2011] Corollary 3.12, see also Hu et al. [2005] Theorem 14 and Morlais [2009] Theorem 4.4. To derive the result for $\hat{Y}$ observe that from Proposition 4.5.5 and the uniqueness result in Theorem 3.2.6 (ii) we have the relation $\hat{Z} \equiv -\hat{\kappa} + (1-p)\hat{\nu}$ which is equivalent to

$$B(\hat{Z} + \lambda) = B(\lambda - \hat{\kappa}) + (1-p)B\hat{\nu}.$$

Since $\mathcal{K}$ is a cone we see that $(1-p)B\hat{\nu} \in B\mathcal{K}$, combining this with Corollary 4.4.4 and using Theorem 4.5.7 we deduce that up to a μ^A-null set

$$(1-p)B\hat{\nu} = \Pi_{B\mathcal{K}}\big(B(\hat{Z} + \lambda)\big) \quad \text{and} \quad B(\lambda - \hat{\kappa}) = \Pi_{(B\mathcal{K})^\circ}\big(B(\hat{Z} + \lambda)\big). \tag{4.5.5}$$

We then use the relation $B = \Gamma^{1/2} P$ to write

$$\Gamma^{1/2} P \hat{\kappa} = B\lambda - \Pi_{(B\mathcal{K})^\circ}\big(B(\hat{Z} + \lambda)\big).$$

The matrix valued process Γ may have some zero diagonal elements and so we may not be able to invert the above relation uniquely. However, by the construction of the process $\widetilde{\kappa}$ we have that $B\hat{\kappa} = B\widetilde{\kappa}$ holds μ^A-a.e. Integrating the difference over $[0,T] \times \Omega$ with respect to μ^A shows

$$\mathbb{E}\Big[\big\langle(\hat{\kappa} - \widetilde{\kappa}) \cdot M\big\rangle_T\Big] = \int_{[0,T]\times\Omega} \|B(\hat{\kappa} - \widetilde{\kappa})\|^2 \, d\mu^A = 0.$$

In particular the stochastic integrals $\hat{\kappa} \cdot M$ and $\widetilde{\kappa} \cdot M$ are indistinguishable so that the representation for $\hat{Y}$ now follows. $\square$

With regards to the nonunique representation of the optimal strategy we refer to the Remarks 4.2.13 and 4.2.19.

4.6 Continuity of the Optimizers

In this section we prove Theorem 4.2.16 on the continuity of the optimizers

$$\hat{X}^n := \hat{X}(\lambda^n, \mathbb{P}^n, p^n, \mathcal{K}^n) \text{ and } \hat{\nu}^n := \hat{\nu}(\lambda^n, \mathbb{P}^n, p^n, \mathcal{K}^n)$$

for the problem

$$u^n(x) := \sup_{\nu \in \mathcal{A}_{\mathcal{K}^n}} \mathbb{E}_{\mathbb{P}^n}\big[U^n(X_T^{n,x,\nu})\big]$$

discussed in Section 4.2, to which we refer for any unexplained notation. For instance, L^n now stands for a continuous local martingale that is orthogonal to M and which appears in the density process of $\mathbb{P}^n$. We assume throughout that the Assumptions 4.2.14 and 4.2.15 hold and that $x = 1$ which, due to the factorization property, is no loss of generality. Moreover, we assume that every local martingale $\hat{N}^n$ arising in the decomposition of the corresponding dual optimizer is *continuous*. This is the case under the Assumption 1.2.1, for instance.

Continuity for an Auxiliary BSDE

The first result is an immediate consequence of the standing assumptions.

Lemma 4.6.1. *Under the Assumptions 4.2.14 and 4.2.15 the sequence of random variables $(\zeta^n)_{n\in\mathbb{N}}$ defined via*

$$\zeta^n := (L^n)^* + \langle L^n \rangle_T,$$

satisfies $\sup_{n\in\mathbb{N}} \mathbb{E}[\exp(\varrho\zeta^n)] < +\infty$ *for all* $\varrho > 0$ *and converges to zero in* $\mathbb{P}$*-probability.*

Proof. Let $\varrho > 0$ and $n \in \mathbb{N}$, then we have

$$\mathbb{E}\big[\exp(\varrho\zeta^n)\big] \leq 4\,\mathbb{E}\big[\exp(2\varrho L_T^n)\big]^2 + \mathbb{E}\big[\exp(2\varrho\langle L^n\rangle_T)\big]$$

by Doob's inequality. The second term on the right hand side is finite due to Assumption 4.2.14, uniformly in n. For the first term, adding and subtracting suitable multiples of $\langle L^n \rangle_T$, then using the Hölder inequality together with the fact that the stochastic exponential is always a supermartingale we see that it is finite, uniformly in n, as well. The convergence to zero in probability is again a consequence of Assumption 4.2.15. Actually, we can even deduce the L^ϱ convergence, $\varrho \geq 1$, from the Vitali convergence theorem. □

Given the optimizers $(\hat{X}^n, \hat{Y}^n)$ Proposition 4.5.5 describes the link to the solution triple $(\hat{\Psi}^n, \hat{Z}^n, \hat{N}^n)$ of the following BSDE under $\mathbb{P}^n$ for $n \in \mathbb{N}_0$ (written in generic variables (Ψ, Z, N)),

$$d\Psi_t = Z_t^\intercal \, dM_t^n + dN_t - F_1^n(t, Z_t)\, dA_t - \frac{1}{2}\, d\langle N \rangle_t, \quad \Psi_T = 0. \tag{4.6.1}$$

Here,

$$F_1^n(\cdot, z) = \frac{1}{2} \left\| Bz \right\|^2 - \frac{q^n}{2} \left\| \Pi_{B\mathcal{K}^n}(B(z + \lambda^n - \beta^n)) \right\|^2,$$

$M^n := M + \langle M \rangle \cdot \beta^n$ and N are continuous $\mathbb{P}^n$-local martingales which are orthogonal and the necessary integrability conditions are satisfied with respect to the measure $\mathbb{P}^n$. To deduce the convergence of $(\hat{X}^n, \hat{Y}^n)$ we shall show first that $(\hat{\Psi}^n, \hat{Z}^n, \hat{N}^n)$ converges to $(\hat{\Psi}, \hat{Z}, \hat{N})$. In order to do this it is necessary to perform a change of variables related to considering the BSDE (4.6.1) under $\mathbb{P}$ rather than $\mathbb{P}^n$. This is the content of the next proposition.

Proposition 4.6.2. *Let $(\hat{\Psi}^n, \hat{Z}^n, \hat{N}^n)$ be as above then the triple*

$$(\hat{\Xi}^n, \hat{V}^n, \hat{O}^n) := \Big(\hat{\Psi}^n + L^n - \tfrac{1}{2} \langle L^n \rangle, \hat{Z}^n, \hat{N}^n + \langle \hat{N}^n, L^n \rangle + L^n \Big)$$

is the unique solution to the BSDE under $\mathbb{P}$

$$d\Psi_t = Z_t^\intercal \, dM_t + dN_t - F^n(t, Z_t)\, dA_t - \frac{1}{2}\, d\langle N \rangle_t, \quad \Psi_T = L_T^n - \tfrac{1}{2} \langle L^n \rangle_T, \tag{4.6.2}$$

with $\Psi \in \mathfrak{E}$ where the generator is given by

$$F^n(\cdot, z) = \frac{1}{2} \left\| Bz \right\|^2 - \frac{q^n}{2} \left\| \Pi_{B\mathcal{K}^n}(B(z + \lambda^n - \beta^n)) \right\|^2 - (Bz)^\intercal (B\beta^n), \tag{4.6.3}$$

q^n is the dual number corresponding to p^n and the process N is a (continuous) $\mathbb{P}$-local martingale orthogonal to M.

Proof. The Girsanov theorem implies that $\hat{O}^n$ is a $\mathbb{P}$-local martingale. To see this, define the $\mathbb{P}^n$-local martingale $\widetilde{L}^n := L^n - \langle L^n \rangle$ and observe that $\frac{d\mathbb{P}}{d\mathbb{P}^n} = \mathcal{E}\big(\beta^n \cdot M^n - \widetilde{L}^n\big)$. It follows that $\hat{N}^n + \langle \hat{N}^n, L^n \rangle$ is a $\mathbb{P}$-local martingale. The orthogonality of $\hat{O}^n$ to M follows from the fact that $\langle \hat{N}^n, M^n \rangle \equiv 0$ and $\langle M, L^n \rangle \equiv 0$. Thanks to (4.6.1) the triple

$(\hat{\Xi}^n, \hat{V}^n, \hat{O}^n)$ then solves (4.6.2) with driver (4.6.3). Moreover, once we show that $\hat{\Xi}^n \in \mathfrak{E}$ then Theorem 3.2.6 provides the claimed uniqueness. Via Hölder's inequality, using the notation of Lemma 4.6.1, we have the estimate

$$\mathbb{E}\Big[\exp\big(\varrho(\hat{\Xi}^n)^*\big)\Big] \le \mathbb{E}\left[\left(\frac{d\mathbb{P}}{d\mathbb{P}^n}\right)^2\right]^{1/2} \mathbb{E}_{\mathbb{P}^n}\Big[\exp\big(4\varrho(\hat{\Psi}^n)^*\big)\Big]^{1/2} + \mathbb{E}[\exp(2\varrho\zeta^n)] < +\infty$$

for all $\varrho > 0$. This completes the proof in view of Assumption 4.2.14. $\square$

The BSDE (under $\mathbb{P} = \mathbb{P}^0$) satisfied by $(\hat{\Psi}, \hat{Z}, \hat{N}) = (\hat{\Psi}^0, \hat{Z}^0, \hat{N}^0)$ related to the optimizers $(\hat{X}, \hat{Y}) = (\hat{X}^0, \hat{Y}^0)$ is given by

$$d\Psi_t = Z_t^\intercal\, dM_t + dN_t - F(t, Z_t)\, dA_t - \frac{1}{2}\, d\langle N\rangle_t, \quad \Psi_T = 0, \tag{4.6.4}$$

where the driver $F = F^0$ satisfies

$$F(\cdot, z) = \frac{1}{2}\left\|Bz\right\|^2 - \frac{q}{2}\left\|\Pi_{B\mathcal{K}}(B(z+\lambda))\right\|^2.$$

Our goal is continuity of the optimizers, which we prove via the stability result in Theorem 3.2.7. We show that this theorem implies convergence of $(\hat{\Xi}^n, \hat{V}^n, \hat{O}^n)$ to $(\hat{\Psi}, \hat{Z}, \hat{N})$ in an appropriate sense and then deduce the result for $(\hat{\Psi}^n, \hat{Z}^n, \hat{N}^n)$. We first collect some properties of the drivers F^n.

Proposition 4.6.3. *The following items (i)-(iii) hold for each $n \in \mathbb{N}_0$, $\mathbb{P}$-a.s.*

(i) For all t the driver $F^n(t, \cdot)$ is continuously differentiable and convex (in z).

(ii) It satisfies a quadratic growth condition in z. More precisely, for all $t \in [0, T]$, $z \in \mathbb{R}^d$ and $\varepsilon_0 > 0$,

$$|F^n(t, z)| \le \frac{1}{2\varepsilon_0}\,\|B_t\beta_t^n\|^2 + \frac{|q^n|}{4\varepsilon_0}\,\|B_t(\lambda_t^n - \beta_t^n)\|^2 + \frac{\gamma(\varepsilon_0)}{2}\,\|B_t z\|^2,$$

where $\gamma(\varepsilon_0) := 1 + \varepsilon_0(1 + |q^n|/2)$.

(iii) For all t the function $F^n(t, \cdot)$ is locally Lipschitz continuous, i.e. for $z_1, z_2 \in \mathbb{R}^d$,

$$\begin{aligned}&|F^n(t, z_1) - F^n(t, z_2)|\\ &\le \left(\|B_t\beta_t^n\| + \frac{1+|q^n|}{2}\Big(\|B_t z_1\| + \|B_t z_2\|\Big) + |q^n|\cdot\|B_t(\lambda_t^n - \beta_t^n)\|\right)\|B_t(z_1 - z_2)\|.\end{aligned}$$

Moreover, if $\hat{Z} = \hat{Z}^0$ denotes the process from above, then,

(iv) Under the Assumptions 4.2.14 and 4.2.15 we have that

$$\lim_{n\to+\infty}\int_0^T |F^n(t, \hat{Z}_t) - F(t, \hat{Z}_t)|\, dA_t = 0$$

in $L^1(\mathbb{P})$ and hence in $\mathbb{P}$-probability.

Proof. Items (ii) and (iii) follow from the explicit form of the driver together with the generalized Young inequality and the Lipschitz property of the nearest point operator. Items (i) and (iv) are a little more involved and we provide a proof, suppressing the argument (t,ω) for brevity. Starting with item (i) we recall from Borwein and Lewis [2006] Section 3.3 that for the function $\theta : \mathbb{R}^d \to \mathbb{R}$,

$$\theta(z) := \left\|Bz - \Pi_{B\mathcal{K}}(Bz)\right\|^2,$$

we have

$$D_z\theta(z_0)(\cdot) = 2\left\langle Bz_0 - \Pi_{B\mathcal{K}}(Bz_0), B(\cdot)\right\rangle$$

where $D_z\theta(z_0)$ denotes the differential of θ (with respect to z) at a point $z_0 \in \mathbb{R}^d$ and which is a linear functional on $\mathbb{R}^d$. Here, $\langle\cdot,\cdot\rangle$ stands for the inner product on $\mathbb{R}^d$.

We now can show differentiability of F^n. From Theorem 4.5.7,

$$\left\|B(z+\lambda^n-\beta^n)\right\|^2 = \left\|B(z+\lambda^n-\beta^n) - \Pi_{B\mathcal{K}^n}(B(z+\lambda^n-\beta^n))\right\|^2 + \left\|\Pi_{B\mathcal{K}^n}(B(z+\lambda^n-\beta^n))\right\|^2$$

and we conclude that $D_zF^n(z_0)(\cdot) = \langle Bz_0 - q^n\Pi_{B\mathcal{K}^n}(B(z_0+\lambda^n-\beta^n)) - B\beta^n, B(\cdot)\rangle$.

As to convexity we then derive from the Lipschitz property of $\Pi_{B\mathcal{K}^n}$ and the Cauchy-Schwarz inequality, that for $q^n \in (0,1)$ and for all $z_1, z_2 \in \mathbb{R}^d$,

$$\Big(D_zF^n(z_1) - D_zF^n(z_2)\Big)(z_1 - z_2) \geq (1-q^n)\left\|B(z_1-z_2)\right\|^2 \geq 0.$$

This is the multidimensional version of monotonicity of the derivatives and it is equivalent to the convexity property, see Borwein and Lewis [2006] Section 3.1. For $q^n \in (-\infty, 0)$ we use the representation

$$F^n(z) = \frac{1}{2}\left\|Bz\right\|^2 - (Bz)^\intercal(B\beta^n) + \frac{q^n}{2}\inf_{\eta\in B\mathcal{K}^n}\Big(\|\eta\|^2 - 2\langle\eta, B(z+\lambda^n-\beta^n)\rangle\Big).$$

An infimum of affine functions (in z) is concave (in z), hence the last term is convex in z due to the sign of q^n. Thus F^n is convex as a sum of two convex functions.

We continue with item (iv). Using the definition of the drivers one can derive the following inequality

$$\begin{aligned}
|F^n(t,\hat{Z}_t) - F(t,\hat{Z}_t)| \leq{}& \frac{|q|}{2}\cdot\left|\left\|\Pi_{B_t\mathcal{K}^n_t}(B_t(\hat{Z}_t+\lambda_t))\right\|^2 - \left\|\Pi_{B_t\mathcal{K}_t}(B_t(\hat{Z}_t+\lambda_t))\right\|^2\right| \\
&+ \|B_t\hat{Z}_t\|\cdot\|B_t\beta^n_t\| \\
&+ \left|\frac{q-q^n}{2}\right|\cdot\left\|\Pi_{B_t\mathcal{K}^n_t}(B_t(\hat{Z}_t+\lambda_t))\right\|^2 \\
&+ \frac{|q^n|}{2}\cdot\left|\left\|\Pi_{B_t\mathcal{K}^n_t}(B_t(\hat{Z}_t+\lambda_t))\right\|^2 - \left\|\Pi_{B_t\mathcal{K}^n_t}(B_t(\hat{Z}_t+\lambda^n_t-\beta^n_t))\right\|^2\right| \\
=:{}& G^n_t + H^n_t + I^n_t + J^n_t.
\end{aligned}$$

We have to show that

$$\lim_{n\to+\infty} \mathbb{E}\left[\int_0^T (G_t^n + H_t^n + I_t^n + J_t^n)\, dA_t\right] = 0,$$

for which we work term by term, beginning with G^n. By Proposition 6.2.3 $(G^n)_{n\in\mathbb{N}}$ then converges to zero μ^A-a.e. and is dominated by $|q| \cdot \|B(\hat{Z} + \lambda)\|^2$. In particular thanks to the dominated convergence theorem we have

$$\lim_{n\to+\infty} \mathbb{E}\left[\int_0^T G_t^n\, dA_t\right] = \lim_{n\to+\infty} \int_{[0,T]\times\Omega} G^n\, d\mu^A = 0.$$

For the second term we apply the Cauchy-Schwarz inequality to get

$$\mathbb{E}\left[\int_0^T H_t^n\, dA_t\right]^2 \le \mathbb{E}\Big[\langle \hat{Z} \cdot M\rangle_T\Big]\, \mathbb{E}\Big[\langle \beta^n \cdot M\rangle_T\Big].$$

The convergence to zero now follows from Assumption 4.2.14 and the condition on β^n. For the I^n term we apply the contraction property of the projection map to deduce

$$\mathbb{E}\left[\int_0^T I_t^n\, dA_t\right] \le \frac{|q - q^n|}{2}\, \mathbb{E}\Big[\langle (\hat{Z} + \lambda) \cdot M\rangle_T\Big],$$

from which the convergence follows. For the final term we first derive, similarly to item (iii), the local Lipschitz estimate

$$J_t^n \le |q^n|\Big(2\|B_t\hat{Z}_t\| + \|B_t\lambda_t\| + \|B_t\lambda_t^n\| + \|B_t\beta_t^n\|\Big)\|B_t(\lambda_t - \lambda_t^n + \beta_t^n)\|.$$

Applying the Cauchy-Schwarz and Young inequalities we derive the existence of a constant $\hat{c}$, independent of n (due to the convergence assumptions the sequences appearing in the estimates are bounded), such that

$$\mathbb{E}\left[\int_0^T J_t^n\, dA_t\right]^2 \le \hat{c}\, \mathbb{E}\Big[\langle (\lambda - \lambda^n + \beta^n) \cdot M\rangle_T\Big].$$

Letting n go to infinity and using Assumptions 4.2.14 and 4.2.15 the result follows. □

The Convergence Results

Theorem 4.6.4. *Under the Assumptions 4.2.14 and 4.2.15 let the triple* $(\hat{\Psi}^n, \hat{Z}^n, \hat{N}^n)$ *denote the unique solution to the BSDE* (4.6.1) *with* $\hat{\Psi}^n \in \mathfrak{E}$ *(and* $\hat{N}^n$ *continuous). Then*

$$\lim_{n\to+\infty} \mathbb{E}\Big[\exp\Big(\varrho\,(\hat{\Psi}^n - \hat{\Psi})^*\Big)\Big] = 1,$$

$$\lim_{n\to+\infty} \mathbb{E}\Big[\Big(\langle (\hat{Z}^n - \hat{Z}) \cdot M\rangle_T + \langle \hat{N}^n - \hat{N}\rangle_T\Big)^{\varrho/2}\Big] = 0,$$

for all $\varrho \geq 1$, where $(\hat{\Psi}, \hat{Z}, \hat{N})$ denotes the unique solution triple of the BSDE (4.2.6) *with $\hat{\Psi} \in \mathfrak{E}$ (and $\hat{N}$ continuous).*

Proof. Using the notation of Lemma 4.6.1 and Proposition 4.6.2 we can write

$$0 \leq (\hat{\Psi}^n - \hat{\Psi})^* \leq (\hat{\Xi}^n - \hat{\Psi})^* + (\zeta^n)^*.$$

Hence the sequence $\big(\exp\big(\varrho(\hat{\Psi}^n - \hat{\Psi})^*\big)\big)_{n\in\mathbb{N}}$ is uniformly integrable and converges to zero in $\mathbb{P}$-probability. Both these claims are consequences of Lemma 4.6.1 and Theorem 3.2.7, whose conditions are guaranteed by Proposition 4.6.3 and Assumption 4.2.14. Since $\hat{Z}^n \equiv \hat{V}^n$ and

$$\langle \hat{N}^n - \hat{N}\rangle_T \leq 2\,\langle \hat{O}^n - \hat{N}\rangle_T + 2\,\langle L^n\rangle_T,$$

we derive the second convergence in a similar fashion. □

We now show how this implies convergence of the objects of interest and begin with the primal variables.

Theorem 4.6.5. *Under the assumptions of the previous theorem we have that for all $\varrho \geq 1$*

$$\lim_{n\to+\infty} \mathbb{E}\left[\langle(\hat{\nu}^n - \hat{\nu})\cdot M\rangle_T^{\varrho/2}\right] = 0.$$

In particular, $(\hat{\nu}^n - \hat{\nu}) \cdot M$ converges to zero in $\mathcal{M}^2$ and hence in the semimartingale topology.

Proof. Using the definitions, it follows that the relation

$$\lim_{n\to+\infty} \mathbb{E}\left[\langle(\hat{\nu}^n - \hat{\nu})\cdot M\rangle_T^{\varrho/2}\right] = 0$$

is equivalent to

$$\lim_{n\to+\infty} \mathbb{E}\left[\left(\int_0^T \left\| \frac{\Pi_{B_t\mathcal{K}_t^n}\big(B_t(\hat{Z}_t^n + \lambda_t^n - \beta_t^n)\big)}{(1-p^n)} - \frac{\Pi_{B_t\mathcal{K}_t}\big(B_t(\hat{Z}_t + \lambda_t)\big)}{(1-p)} \right\|^2 dA_t\right)^{\varrho/2}\right] = 0.$$

To establish this we may proceed similarly to the proof of Proposition 4.6.3 (iv) so that Proposition 4.2.9 (i) then yields the assertion. □

Theorem 4.6.6. *Under the assumptions of Theorem 4.6.4 the sequence of processes $\hat{X}^n \in \mathcal{X}(x)$, $n \in \mathbb{N}$, converges to $\hat{X} \equiv \hat{X}^0 \in \mathcal{X}(x)$ in the semimartingale topology.*

Proof. We note the dynamics of the optimal wealth processes given by (4.2.7) and set

$$\Upsilon^n := \hat{\nu}^n \cdot M + \hat{\nu}^n \cdot \langle M\rangle\lambda^n,$$

for $n \in \mathbb{N}_0$. We show the convergence in $\mathcal{H}^2$ of the sequence $(\Upsilon^n)_{n\in\mathbb{N}}$ so that the result of the theorem will follow via Proposition 4.2.9 (ii) since $\hat{X}^n = \mathcal{E}(\Upsilon^n)$ and $\hat{X} = \mathcal{E}(\Upsilon^0)$.

Observe from Theorem 4.6.5 that $(\hat{\nu}^n - \hat{\nu}) \cdot M$ converges to zero in $\mathcal{M}^2$ so that we need only show the convergence of the finite variation parts, namely that

$$\lim_{n \to +\infty} \mathbb{E}\left[\left(\int_0^T | \, d(\langle \hat{\nu}^n \cdot M, \lambda^n \cdot M \rangle - \langle \hat{\nu} \cdot M, \lambda \cdot M \rangle) | \right)^2 \right] = 0.$$

Adding and subtracting $\langle \hat{\nu} \cdot M, \lambda^n \cdot M \rangle$ and then applying the Kunita-Watanabe inequality, we see that the above holds due to Theorem 4.6.5 together with the convergence of $\langle (\lambda^n - \lambda) \cdot M \rangle_T$ to zero in all $L^\varrho(\mathbb{P})$ spaces. $\square$

Theorem 4.6.7. *Under the standing assumptions of this section the value functions u^n converge pointwise to u. Their derivatives converge pointwise to u'.*

Proof. From the BSDE (4.6.4) the reader may verify the relation,

$$d(\exp(\hat{\Psi}) U'(\hat{X}))_t = \exp(\hat{\Psi}_t) U'(\hat{X}_t)(-\hat{\kappa}_t \, dM_t + d\hat{N}_t)$$

which implies that

$$\hat{Y} = u'(x) \hat{Y}^1 = \exp(\hat{\Psi}) U'(\hat{X}) = e^{\hat{\Psi}_0} x^{p-1} \hat{Y}^1 \quad \mathbb{P}\text{-a.s.}$$

It follows that $c_p = e^{\hat{\Psi}_0}$ $\mathbb{P}$-a.s. which shows that

$$u^n(x) = U^n(x) c^n_{p^n} = U^n(x) e^{\hat{\Psi}^n_0} \quad \mathbb{P}\text{-a.s.}$$

From Theorem 4.6.4 we have that $\lim_{n \to +\infty} |\hat{\Psi}^n_0 - \hat{\Psi}_0| = 0$ in probability. Hence for an arbitrary $\varepsilon > 0$, $\lim_{n \to +\infty} \mathbb{P}(|\hat{\Psi}^n_0 - \hat{\Psi}_0| > \varepsilon) = 0$ which means that for n large enough,

$$\mathbb{P}(|\hat{\Psi}^n_0 - \hat{\Psi}_0| > \varepsilon) \le \tfrac{1}{2}.$$

Since $\mathcal{F}_0$ consists of the $\mathbb{P}$-null sets and their complements only, we thus derive that there exists some $m_0 \in \mathbb{N}$ such that $\mathbb{P}(|\hat{\Psi}^n_0 - \hat{\Psi}_0| > \varepsilon) = 0$ for all $n \in \mathbb{N}$ with $n \ge m_0$. In particular,

$$\lim_{m \to +\infty} \mathbb{P}\left(\left\{ \sup_{n \ge m} |\hat{\Psi}^n_0 - \hat{\Psi}_0| > \varepsilon \right\} \right) = \lim_{\substack{m \to +\infty \\ m \ge m_0}} \mathbb{P}\left(\bigcup_{n \ge m} \{ |\hat{\Psi}^n_0 - \hat{\Psi}_0| > \varepsilon \} \right) = 0$$

which is a well-known criterion for almost sure convergence. Hence $\lim_{n \to +\infty} \hat{\Psi}^n_0 = \hat{\Psi}_0$ $\mathbb{P}$-a.s. which implies the convergence of $u^n(x)$ to $u(x)$. The convergence of $(u^n)'(x)$ to $u'(x)$ is then immediate. $\square$

Similar arguments can be used to study the dual variables and we collect the results together in the following theorem.

Theorem 4.6.8. *Suppose that the assumptions of Theorem 4.6.4 hold. Then,* *(i) The processes* $(\hat{\kappa}^n - \hat{\kappa}) \cdot M$, $n \in \mathbb{N}$, *converge to zero in* $\mathcal{M}^2$.

(ii) The processes $\hat{Y}^n \in \mathcal{Y}(y)$, $n \in \mathbb{N}$, *converge to* $\hat{Y} \equiv \hat{Y}^0 \in \mathcal{Y}(y)$ *in the semimartingale topology.*

(iii) The functions $\widetilde{u}^n$, $n \in \mathbb{N}$, *converge pointwise to* $\widetilde{u}$. *Their derivatives converge pointwise to* $\widetilde{u}'$.

Proof. Item (i) follows from the decomposition

$$\hat{\kappa}^n = (1 - p^n)\hat{\nu}^n - \hat{Z}^n$$

together with Theorems 4.6.4 and 4.6.5. Item (i) and Theorem 4.6.4 provides the convergence of $\Upsilon^n := \hat{\kappa}^n \cdot M + \hat{N}^n$ to $\hat{\kappa} \cdot M + \hat{N}$ in $\mathcal{H}^\varrho$ for all $\varrho \geq 1$. Convergence in the semimartingale topology then follows from Proposition 4.2.9 (i) and (ii). For the last item observe that from Theorem 4.2.8 we may write

$$\widetilde{u}^n(y) = \widetilde{U}^n(y)\widetilde{c}^n_{p^n}, \quad \widetilde{c}^n_{p^n} = (c^n_{p^n})^{\frac{1}{1-p^n}} = e^{\frac{1}{1-p^n}\hat{\Psi}^n_0} \quad \mathbb{P}\text{-a.s.}$$

so that the claim is again a corollary of Theorem 4.6.4, as in the proof of Theorem 4.6.7. $\square$

5 Utility Maximization under Compact Constraints and Partial Information

5.1 Introduction

In this final chapter we study the existence, uniqueness and stability of the constrained power utility maximization problem under incomplete information. In contrast to Chapter 4 we now rely purely on the BSDE results from Chapter 3 without any reference to duality theory. As already mentioned, given the inclusion of investment constraints, the second theme of the exposition is the inclusion of incomplete information to the power utility maximization problem. This is meant in the following sense. We assume that the agent as well as the regulator cannot access all the information which is present in the market but only some restricted information which we model by a subfiltration of the main filtration. For instance, it is natural to assume that they can observe the stock prices or the stock returns only. Such a partial information framework has been studied in Lakner [1998], Pham and Quenez [2001] and Sass [2007], amongst others. Here, we analyze an even more restrictive setting. Following the ideas of Mania and Santacroce [2010] and Covello and Santacroce [2010] we consider a filtration which may be smaller than the filtration generated by the stock prices or the stock returns.

The main other features are now as follows. Firstly, following the exposition in the previous chapter, we generalize the underlying model beyond the case of a bounded mean-variance tradeoff by requiring appropriate finite exponential moments only. Secondly, we assume that at terminal time there is an additional bonus or penalty D which applies to the investor's accrued wealth. This is as in Nutz [2010b] but again we relax the boundedness assumptions. As an example we may think of D as arising from a stochastic tax rate whose level depends on the state of the economy at terminal time T. Another interpretation of D is that it defines a measure change so that the portfolio choice problem is under the agent's subjective beliefs. Thirdly, trading in the market is subject to constraints that are not necessarily convex but merely closed. This is exactly as in Hu et al. [2005] with the exception that we do not require any nondegeneracy of the stock dynamics in which case we additionally have to make use of the closure operator to derive the appropriate results. However, we mention that the use of the closure operator becomes superfluous if we restrict ourselves to compact constraints, hence the title of this chapter. Since, contrary to Hu et al. [2005], Mania and Santacroce [2010] and Morlais [2009], we do not dispose of any BMO properties, the verification argument becomes more involved. It consists of applying a variant of Theorem 3.7.2, see also Heyne [2010] for explicit Brownian stochastic volatility models.

The main results thus consist of deriving the existence and uniqueness of solutions to the constrained utility maximization problem under partial information and of relating them to the appropriate BSDE. This is done under weaker conditions than those which are present in the relevant literature. As an application of the stability result for quadratic BSDEs we are then able to show that the optimizers depend continuously on the investor's risk aversion parameter, the market price of risk, the constraints and the bonus at terminal time, alternatively, the measure change.

The present chapter hence is in line with the previous results and to some extent of concluding character. It is organized as follows. In the next section we lay out the framework of partial information as well as providing the investment constraints together with the statement of the main result. We then relate the original problem to an auxiliary one which can be interpreted as a problem under full information and hence can be studied by using the BSDE results from Chapter 3. Finally, we provide an analysis of the stability of the primal problem.

5.2 Framework and Main Result

Let us again recall the utility maximization framework from the Sections 1.2 and 4.2, adapted to the present setting of partial information. We work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ satisfying the usual conditions. The time horizon T is a positive number and $\mathcal{F}_0$ is the completion of the trivial σ-algebra. All semimartingales are càdlàg.

In what follows, our aim is to develop and to analyze the utility maximization problem within the setting of partial information which will be modelled by using a subfiltration $(\mathcal{G}_t)_{t\in[0,T]}$ of $(\mathcal{F}_t)_{t\in[0,T]}$, i.e.

$$\mathcal{G}_t \subset \mathcal{F}_t \quad \text{ for all } t \in [0, T],$$

satisfying the usual conditions. In particular, we have to be more precise about to which filtration the processes in question are adapted. The filtration $(\mathcal{G}_t)_{t\in[0,T]}$ represents the information which is available to the investor and the regulator or to which they restrict themselves in choosing their strategies. In contrast, $(\mathcal{F}_t)_{t\in[0,T]}$ stands for the full information of the economy.

The Market Model

The market consists of one bond paying zero interest and d stocks whose price process $S = (S^1, \dots, S^d)^\intercal$ has dynamics

$$dS_t = \mathrm{Diag}(S_t)\left(dM_t + d\langle M\rangle_t \lambda_t\right),$$

where $M = (M^1, \dots, M^d)^\intercal$ is a d-dimensional continuous $(\mathcal{F}_t)_{t\in[0,T]}$-local martingale with $M_0 = 0$ and λ is a d-dimensional $(\mathcal{F}_t)_{t\in[0,T]}$-predictable process, the *market price* *of risk*, satisfying

$$\mathbb{P}\left(\int_0^T \lambda_t^\intercal \, d\langle M\rangle_t \lambda_t < +\infty\right) = 1.$$

For notational reasons it turns out to be convenient to also introduce the *stock returns process* R, the d-dimensional continuous semimartingale whose components evolve according to

$$dR_t^i = \frac{dS_t^i}{S_t^i} = dM_t^i + \sum_{j=1}^d d\langle M^i, M^j\rangle_t \lambda_t^j, \quad i = 1, \dots, d,$$

so that

$$S = S_0 \, \mathcal{E}(R).$$

The most natural assumption within this framework is that the investor can observe the stock prices, i.e. $\mathcal{F}_t^S \subset \mathcal{G}_t$ for all $t \in [0,T]$, where $(\mathcal{F}_t^S)_{t\in[0,T]}$ is the augmentation of the filtration generated by the process S. A sligthly more restrictive but still natural assumption is that they can observe the stock returns, i.e. $\mathcal{F}_t^R \subset \mathcal{G}_t$ for all $t \in [0,T]$, where $(\mathcal{F}_t^R)_{t\in[0,T]}$ is the augmentation of the filtration generated by the process R and for which we have that $\mathcal{F}_t^S \subset \mathcal{F}_t^R$ for all $t \in [0,T]$. If the evolution of the stock returns indeed is part of the agent's information flow, we conclude that R is a $(\mathcal{G}_t)_{t\in[0,T]}$-semimartingale (see below) and that $\langle M\rangle$ is $(\mathcal{G}_t)_{t\in[0,T]}$-predictable which motivates one of our structural conditions below.

However, in what follows, we consider a slightly more general situation. More precisely, we *do not* require that $(\mathcal{G}_t)_{t\in[0,T]}$ contains $(\mathcal{F}_t^R)_{t\in[0,T]}$. In this case, R is not necessarily a $(\mathcal{G}_t)_{t\in[0,T]}$-semimartingale. This leads us to considering an additional filtration $(\mathcal{H}_t)_{t\in[0,T]}$ which is the augmentation of the filtration generated by $(\mathcal{G}_t)_{t\in[0,T]}$ and $(\mathcal{F}_t^R)_{t\in[0,T]}$, made right-continuous. We then have

$$dR_t = dM_t + d\langle M\rangle_t(\lambda_t - \lambda_t^{\mathcal{H}}) + d\langle M\rangle_t \lambda_t^{\mathcal{H}} = d\widetilde{M}_t + d\langle \widetilde{M}\rangle_t \lambda_t^{\mathcal{H}},$$

where $\lambda^{\mathcal{H}}$ is the $(\mathcal{H}_t)_{t\in[0,T]}$-predictable projection of λ and

$$\widetilde{M}_t := M_t + \int_0^t d\langle M\rangle_s(\lambda_s - \lambda_s^{\mathcal{H}})$$

defines an $(\mathcal{H}_t)_{t\in[0,T]}$-local martingale. This can be seen as follows. Note that we have $\langle \widetilde{M}\rangle \equiv \langle M\rangle \equiv \langle R\rangle$ and that this is $(\mathcal{F}_t^R)_{t\in[0,T]}$-, hence $(\mathcal{H}_t)_{t\in[0,T]}$-predictable. More precisely, we should refer to a common version of these quadratic variations since they are defined in the different filtrations in the first place. We then need to show that the localizing sequence of stopping times which is necessary for the local martingale property of $\widetilde{M}$ can indeed be chosen $(\mathcal{H}_t)_{t\in[0,T]}$-adapted, see Kohlmann et al. [2007] Lemma 2.2.

Remark 5.2.1. If $\langle M\rangle$ is $(\mathcal{G}_t)_{t\in[0,T]}$-predictable (as it will be assumed in this chapter), then the introduction of the auxiliary filtration $(\mathcal{H}_t)_{t\in[0,T]}$ is superfluous under the additional assumption that λ be $(\mathcal{G}_t)_{t\in[0,T]}$-predictable. If this is the case we choose $\widetilde{M}$ to be identical to M, which is not necessarily a $(\mathcal{G}_t)_{t\in[0,T]}$-local martingale, but for which all the crucial calculations below remain true.

The Model Assumptions

Since we do not opt for assuming that λ be $(\mathcal{G}_t)_{t\in[0,T]}$-predictable, we put the following *structural* conditions on the above model. This is in analogy to Mania and Santacroce [2010]. As in the previous chapters μ^A denotes the Doléans measure associated to M.

Assumption 5.2.2.

(i) The process $\langle M\rangle$ is $(\mathcal{G}_t)_{t\in[0,T]}$-predictable and we have that $\lambda^o := \lambda^{\mathcal{G}} = \lambda^{\mathcal{H}}$, μ^A-a.e.

(ii) For any $(\mathcal{G}_t)_{t\in[0,T]}$-local martingale $\overline{M}$ the process $\langle \widetilde{M}, \overline{M}\rangle$ is $(\mathcal{G}_t)_{t\in[0,T]}$-predictable.

(iii) All $(\mathcal{G}_t)_{t\in[0,T]}$-martingales are $(\mathcal{H}_t)_{t\in[0,T]}$-local martingales.

(iv) All $(\mathcal{G}_t)_{t\in[0,T]}$-local martingales are continuous.

Remark 5.2.3. If $(\mathcal{F}^R_t)_{t\in[0,T]}$ is contained in $(\mathcal{G}_t)_{t\in[0,T]}$ then $(\mathcal{H}_t)_{t\in[0,T]}$ coincides with $(\mathcal{G}_t)_{t\in[0,T]}$ and items (i)-(iii) from the above assumption are satisfied. In fact, this is the *main example* for the present study. We also mention that if $\langle M\rangle$ is $(\mathcal{G}_t)_{t\in[0,T]}$-predictable then it follows from $R = \log(S) - \log(S_0) + \frac{1}{2}\langle R\rangle = \log(S) - \log(S_0) + \frac{1}{2}\langle M\rangle$ that $(\mathcal{F}^R_t)_{t\in[0,T]}$ can be replaced by $(\mathcal{F}^S_t)_{t\in[0,T]}$ in the definition of $(\mathcal{H}_t)_{t\in[0,T]}$.

We now recall the decomposition of the quadratic variation of M,

$$\langle M\rangle = C \cdot A = P^{\intercal}\Gamma P \cdot A = B^{\intercal} B \cdot A, \tag{5.2.1}$$

where C is a process valued in the space of symmetric positive semidefinite $d\times d$ matrices, A is an increasing process and P and Γ are valued in the space of $d\times d$ orthogonal (resp. diagonal) matrices and $B := \Gamma^{\frac{1}{2}}P$. In what follows below we also need the Moore-Penrose pseudoinverse $B^\dagger$ of B. Namely, after defining

$$\widetilde{\Gamma} := \mathrm{Diag}(\Gamma^{ii}\cdot \mathbf{1}_{\{\Gamma^{ii}>0\}}, i = 1,\dots,d)$$

we set $B^\dagger := P^{\intercal}\widetilde{\Gamma}^{\frac{1}{2}}$. Note that due to Assumption 5.2.2 (i) all these processes are $(\mathcal{G}_t)_{t\in[0,T]}$-predictable.

In order to precisely describe the utility maximization problem under partial information let us now consider the stock dynamics under the smaller filtration $(\mathcal{G}_t)_{t\in[0,T]}$, using ^{o} to denote $(\mathcal{G}_t)_{t\in[0,T]}$-optional projection and noting that under the assumptions of the present chapter this optional projection coincides with the $(\mathcal{G}_t)_{t\in[0,T]}$-predictable projection for all the processes in question. More explicitly, under the Assumption 5.2.2,

$$dR^o_t = dM^o_t + d\langle M\rangle_t \lambda^o_t$$

where the prescription $M^o_t := \mathbb{E}[\widetilde{M}_t|\,\mathcal{G}_t] = \mathbb{E}[M_t|\,\mathcal{G}_t], t\in[0,T]$, indeed defines a $(\mathcal{G}_t)_{t\in[0,T]}$-local martingale. From Mania et al. [2008] we derive that for all $i = 1,\dots,d$ the process

$\langle M^i \rangle - \langle (M^i)^o \rangle$ is increasing so that $\langle (M^i)^o \rangle \leq \langle M^i \rangle$. It follows that we can find an analogous decomposition of the quadratic variation of M^o, *with A from above*, i.e. $\breve{A} \equiv A$ and

$$\langle M^o \rangle = \breve{C} \cdot A = \breve{P}^\intercal \breve{\Gamma} \breve{P} \cdot A = \breve{B}^\intercal \breve{B} \cdot A,$$

where $\breve{C}$, $\breve{P}$, $\breve{\Gamma}$ and $\breve{B}$ are matrix-valued $(\mathcal{G}_t)_{t\in[0,T]}$-predictable processes with properties analogous to those of C, P, Γ and B.

Our exponential moments assumption on the mean-variance tradeoff now takes the following form.

Assumption 5.2.4. *We assume that $\langle \lambda \cdot M \rangle_T = \int_0^T \|B_t \lambda_t\|^2 dA_t$ has a finite exponential moment of some order $c_\lambda > 0$.*

Remark 5.2.5. By He et al. [1992] Theorem 5.25 the Assumption 5.2.4 implies that $\langle \lambda^o \cdot M \rangle_T = \int_0^T \|B_t \lambda_t^o\|^2 dA_t$ has a finite exponential moment of order $c_\lambda > 0$.

The following structural assumption is then needed for the derivation of the corresponding utility maximization BSDE.

Assumption 5.2.6.

(i) $Ker(B) \subset Ker(\breve{B})$, μ^A-a.e.

(ii) The process $\|\breve{B}B^\dagger\|^2$ is bounded by a constant $c^\dagger \geq 1$. Equivalently, the eigenvalues of $(\breve{B}B^\dagger)^\intercal \breve{B}B^\dagger$ are uniformly bounded by $c^\dagger \geq 1$.

Remark 5.2.7. We mention that item (i) of the above assumption is motivated by $\langle (M^i)^o \rangle \leq \langle M^i \rangle$ for $i = 1, \ldots, d$ and hence is satisfied when $d = 1$ or, more generally, when M and M^o are made of orthogonal components, i.e. when C and $\breve{C}$ are diagonal. Using the fact that $B^\dagger B$ is the projector onto $(\mathrm{Ker}(B))^\perp$ the above item (i) is equivalent to the condition that

$$\breve{B}B^\dagger B = \breve{B}, \ \mu^A\text{-a.e.} \tag{5.2.2}$$

We also emphasize that the Assumption 5.2.6 is *automatically satisfied* if we consider the portfolio choice problem under the assumption that the filtration $(\mathcal{F}_t^R)_{t\in[0,T]}$ is part of the investor's information flow, which is given by $(\mathcal{G}_t)_{t\in[0,T]}$. In this case $R^o \equiv R$, $M^o \equiv \widetilde{M}$, $\langle M^o \rangle \equiv \langle \widetilde{M} \rangle \equiv \langle M \rangle$ and thus $\breve{B} \equiv B$. Hence (5.2.2) holds and since $BB^\dagger$ is the projector onto $\mathrm{Im}(B)$ we find that the above item (ii) is satisfied with $c^\dagger := 1$. In such a situation the focus is on utility maximization under closed constraints satisfying a boundedness condition as described in Assumption 5.2.8. The assumption that the agent can observe the evolution of the stock returns clearly is a natural condition.

Let us now work towards the investor's portfolio selection problem. As in Chapter 4 we assume that their trading strategies must belong to some constraints sets $\mathcal{K}$ which we introduce presently. In contrast to the previous chapter, in order to solve the utility maximization problem, we are going to rely purely on BSDE theory in the spririt of Hu et al. [2005] and do not refer to duality theory. As a consequence we do not assume that the trading constraints are conic. Their closedness together with a suitable boundedness property will be sufficient. Given the inclusion of partial information this is the second focus of study in the present chapter.

Assumption 5.2.8. *There exists a $(\mathcal{G}_t)_{t\in[0,T]}$-predictably measurable closed multivalued mapping $(t,\omega)\mapsto\mathcal{K}(t,\omega)\subset\mathbb{R}^d$ such that $0\in\mathcal{K}(t,\omega)$ for all (t,ω) and such that $\int_0^T \sup_{u\in\mathcal{K}_t}\|B_t u\|^2\,dA_t$ has finite exponential moments of all orders.*

Remark 5.2.9. Note that together with Assumption 5.2.6 this implies that the random variable $\int_0^T \sup_{u\in\mathcal{K}_t}\|\breve{B}_t u\|^2\,dA_t$ has finite exponential moments of all orders. Here and in what follows, a major reference for closed multivalued mappings and their measurability is Rockafellar [1976]. In particular, from Rockafellar [1976] Proposition 2.C and Theorem 2.K we see that both $(t,\omega)\mapsto\sup_{u\in\mathcal{K}_t}\|B_t u\|^2$ and $(t,\omega)\mapsto\sup_{u\in\mathcal{K}_t}\|\breve{B}_t u\|^2$ are $(\mathcal{G}_t)_{t\in[0,T]}$-predictable processes. However, there is some degeneracy related to closedness that we have to address before moving further. Namely, for an arbitrary closed set $\mathcal{K}$ the image $B\mathcal{K}$ under a linear mapping B *need not* be closed, see Pataki [2007] for a discussion of sufficient and necessary conditions when $\mathcal{K}$ is a convex cone. In Chapter 3 we use that $\mathcal{K}$ is polyhedral there, a condition that is not present here. We thus have to make use of the *closure operator* cl in the sequel. However, observing that

$$\sup_{u\in\mathcal{K}}\|Bu\|^2=\sup_{w\in B\mathcal{K}}\|w\|^2=\sup_{w\in\,\mathrm{cl}(B\mathcal{K})}\|w\|^2,$$

we find that incorporating the closure operator does not affect the above assumptions.

Remark 5.2.10. We mention that the use of the closure operator is superfluous if $(t,\omega)\mapsto\mathcal{K}(t,\omega)$ is a mapping of *compact* sets. In this case $(t,\omega)\mapsto B_t(\omega)\mathcal{K}(t,\omega)$ is a multivalued mapping of closed bounded sets. In particular, $(t,\omega)\mapsto\sup_{u\in\mathcal{K}(t,\omega)}\|B_t(\omega)u\|^2$ is well-defined and Assumption 5.2.8 is satisfied if it is a bounded process, e.g. if $\mathcal{K}(t,\omega)\equiv\mathcal{K}$ is a fixed compact set and the process $\max(\Gamma^{ii})_{i=1,\dots,d}$ of the maximal eigenvalue of C is bounded. This setting can serve us as the major example of appropriate constraints. Moreover, there is a certain tradeoff between imposing exponential moments assumptions on λ and on $\mathcal{K}$. Weakening the conditions on either object typically requires stronger assumptions on the other, see the discussion related to the BSDE and the measure change property below. We opt for fixing the assumptions as above.

The Investment Problem

We are now ready to introduce the notion of a trading strategy.

Definition 5.2.11. A d-dimensional process ν is called *admissible* and we then write $\nu\in\mathcal{A}^{\mathcal{G}}_{\mathcal{K}}$, if

(i) It is $(\mathcal{G}_t)_{t\in[0,T]}$-predictable.

(ii) It is M-integrable, i.e.

$$\mathbb{P}\left(\int_0^T \nu_t^{\intercal}\,d\langle M\rangle_t\nu_t<+\infty\right)=1.$$

(iii) We have that $B\nu \in \mathrm{cl}(B\mathcal{K})$, μ^A-a.e.

As in the preceding chapters, an admissible process ν is interpreted as an investment strategy and its components ν^i represent the *proportion of wealth* invested in each stock S^i, $i = 1, \dots, d$, now subject to the *investment and information constraints* that are determined by $\mathcal{K}$ and $\mathcal{G}$. Then, requiring that ν be $(\mathcal{G}_t)_{t\in[0,T]}$-predictable reflects the fact that only the information encoded in the smaller filtration $(\mathcal{G}_t)_{t\in[0,T]}$ may be used for choosing a trading strategy. For some initial capital $x > 0$ and an admissible strategy ν, the associated *wealth process* $X^{x,\nu}$ then evolves as follows

$$X^{x,\nu} := x\,\mathcal{E}(\nu \cdot R) = x\,\mathcal{E}(\nu \cdot M + \nu \cdot \langle M\rangle\lambda). \tag{5.2.3}$$

Observe that under the Assumptions 5.2.6 and 5.2.8 all strategies $\nu \in \mathcal{A}^{\mathcal{G}}_{\mathcal{K}}$ are also M^o-integrable, due to item (iii) above. Similarly, in the above definition, item (iii) actually also implies item (ii). Moreover, it should be regarded as the condition that "$\nu \in \mathcal{K}$", given that strategies $\nu \in \mathrm{Ker}(B)$ do not contribute to the investor's wealth and under the assumption that the case of a closed $B\mathcal{K}$ is the natural one.

As usual, our agent has preferences modelled by a power utility function U,

$$U(x) = \frac{x^p}{p},$$

where p is now restricted to be positve, i.e. $p \in (0,1)$. They start with initial capital $x > 0$, may choose admissible strategies ν, and aim to maximize the expected utility of terminal wealth.

Contrary to the previous chapter we assume that at terminal time there applies an additional *discount* or multiplicative *bonus* D, where D is a nonnegative $\mathcal{F}_T$-measurable random variable that satisfies $\mathbb{E}[D^p|\,\mathcal{H}_T] = \mathbb{E}[D^p|\,\mathcal{G}_T] =: \breve{D}^p > 0$, $\mathbb{P}$-a.s. and is such that $|\log(\breve{D})|$ has all exponential moments. As an example we may think of D as arising from a stochastic tax rate whose level depends on the state of the economy at terminal time T and hence is beyond the information which is available to the investor. Alternatively, we may think of a trader at an investment bank whose reward is a certain percentage amount of the wealth acquired for their company. Another interpretation is that if $\mathbb{E}[D^p] = 1$, then D can be considered as yielding a change of measure. The portfolio selection problem is then under the investor's subjective views determined by D, see (5.2.4).

In conclusion, we are led to considering the following optimization problem, where without loss of generality $x = 1$,

$$u(1) := \sup_{\nu\in\mathcal{A}^{\mathcal{G}}_{\mathcal{K}}} \mathbb{E}\Big[U\Big(DX^\nu_T\Big)\Big] = \sup_{\nu\in\mathcal{A}^{\mathcal{G}}_{\mathcal{K}}} \mathbb{E}\Big[D^pU\Big(X^\nu_T\Big)\Big]. \tag{5.2.4}$$

The dynamic formulation of this portfolio choice problem in the filtration $(\mathcal{G}_t)_{t\in[0,T]}$ amounts to

$$u_t(1) := \operatorname*{ess\,sup}_{\nu\in\mathcal{A}^{\mathcal{G}}_{\mathcal{K}}} \mathbb{E}\Big[U\Big(DX^\nu_{t,T}\Big)\Big|\,\mathcal{G}_t\Big], \tag{5.2.5}$$

where $X^\nu = \mathcal{E}(\nu \cdot R)$. The main idea is now to transform (5.2.5) into an equivalent problem with full information, i.e. into a problem of the form

$$u_t^{\mathcal{G}}(1) := \operatorname*{ess\,sup}_{\nu \in \mathcal{A}_{\mathcal{K}}^{\mathcal{G}}} \mathbb{E}\Big[U\Big(\breve{D}\breve{X}_{t,T}^{\nu}\breve{\kappa}_{t,T}^{\nu}\Big)\Big|\,\mathcal{G}_t\Big], \tag{5.2.6}$$

where $\breve{X}^\nu = \mathcal{E}(\nu \cdot R^o)$ and $\breve{\kappa}^\nu$ is a $(\mathcal{G}_t)_{t\in[0,T]}$-predictable process that represents a suitable adjustment. In the full information case $\breve{\kappa}^\nu \equiv 1$, of course. Actually, this also holds when $(\mathcal{F}_t^R)_{t\in[0,T]}$ is part of $(\mathcal{G}_t)_{t\in[0,T]}$. In general, we have

$$\breve{\kappa}^\nu := \exp\Bigg(\frac{p-1}{2}\int_0^{\cdot}\Big(\|B_t\nu_t\|^2 - \|\breve{B}_t\nu_t\|^2\Big)\,dA_t\Bigg). \tag{5.2.7}$$

Remark 5.2.12. A well established fact in dynamic programming is that $(u_t(1))_{t\in[0,T]}$ and $\big(u_t^{\mathcal{G}}(1)\big)_{t\in[0,T]}$ are $(\mathcal{G}_t)_{t\in[0,T]}$-supermartingales, hence admit modifications that are càdlàg. In the sequel we thus assume that $(u_t(1))_{t\in[0,T]}$ and $\big(u_t^{\mathcal{G}}(1)\big)_{t\in[0,T]}$ have been chosen in their càdlàg modification. For the proof of the supermartingale property observe that for $t \in [0,T]$ the family $\big(\mathbb{E}[U(DX_{t,T}^{\nu})|\,\mathcal{G}_t]\big)_{\nu\in\mathcal{A}_{\mathcal{K}}^{\mathcal{G}}}$ is stable by supremum, i.e. for all $\nu_1,\nu_2 \in \mathcal{A}_{\mathcal{K}}^{\mathcal{G}}$ we obtain from a calculation that

$$\mathbb{E}\big[U\big(DX_{t,T}^{\nu_1}\big)\big|\,\mathcal{G}_t\big] \vee \mathbb{E}\big[U\big(DX_{t,T}^{\nu_2}\big)\big|\,\mathcal{G}_t\big] = \mathbb{E}\big[U\big(DX_{t,T}^{\nu}\big)\big|\,\mathcal{G}_t\big],\ \mathbb{P}\text{-a.s.}$$

for $\nu := \nu_1\mathbf{1}_E + \nu_2\mathbf{1}_{E^c} \in \mathcal{A}_{\mathcal{K}}^{\mathcal{G}}$ where

$$E := \Big\{\mathbb{E}\big[U\big(DX_{t,T}^{\nu_1}\big)\big|\,\mathcal{G}_t\big] > \mathbb{E}\big[U\big(DX_{t,T}^{\nu_2}\big)\big|\,\mathcal{G}_t\big]\Big\} \in \mathcal{G}_t.$$

We thus can deduce an increasing limit property of the mentioned family. More explicitly, there is a sequence $(\nu_n)_{n\in\mathbb{N}}$ in $\mathcal{A}_{\mathcal{K}}^{\mathcal{G}}$ such that $\mathbb{P}$-a.s.

$$u_t(1) = \lim_{n\to+\infty} \uparrow \mathbb{E}\big[U\big(DX_{t,T}^{\nu_n}\big)\big|\,\mathcal{G}_t\big].$$

For $s \le t$ we then deduce from the monotone convergence theorem that $\mathbb{P}$-a.s.

$$\begin{aligned}
\mathbb{E}\big[u_t(1)\big|\,\mathcal{G}_s\big] &= \lim_{n\to+\infty} \uparrow \mathbb{E}\Big[\mathbb{E}\big[U\big(DX_{t,T}^{\nu_n}\big)\big|\,\mathcal{G}_t\big]\Big|\mathcal{G}_s\Big] \le \operatorname*{ess\,sup}_{\nu\in\mathcal{A}_{\mathcal{K}}^{\mathcal{G}}} \mathbb{E}\big[U\big(DX_{t,T}^{\nu}\big)\big|\,\mathcal{G}_s\big] \\
&= \operatorname*{ess\,sup}_{\nu\in\mathcal{A}_{\mathcal{K}}^{\mathcal{G}},\,\nu\equiv 0 \text{ on } (s,t]} \mathbb{E}\big[U\big(DX_{s,T}^{\nu}\big)\big|\,\mathcal{G}_s\big] \le u_s(1),
\end{aligned}$$

which shows that $(u_t(1))_{t\in[0,T]}$ indeed is a supermartingale for the filtration $(\mathcal{G}_t)_{t\in[0,T]}$ and the proof for $\big(u_t^{\mathcal{G}}(1)\big)_{t\in[0,T]}$ follows the same pattern.

The BSDE in the Smaller Filtration and the Existence Result

As already announced we proceed in the spirit of Hu et al. [2005] and Mania and Santacroce [2010]. We solve the portfolio choice problem (5.2.6) by finding a solution to a specific BSDE, performing a verification argument and finally constructing an optimal strategy where in the last step we refer to measurable selection. The BSDE in question is defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{G}_t)_{0\le t\le T}, \mathbb{P})$ and takes the following form,

$$d\Psi_t = Z_t^\intercal \, dM_t^o + dN_t - F(t, Z_t)\, dA_t - \frac{1}{2}\, d\langle N\rangle_t, \quad \Psi_T = \log\big(U(\breve{D})\big), \tag{5.2.8}$$

where N is a continuous $(\mathcal{G}_t)_{0\le t\le T}$-local martingale that is orthogonal to M^o and the driver F is given by

$$F(\cdot, z) = \frac{p(p-1)}{2}\operatorname{dist}^2\left(\frac{B\lambda^o + (B^\dagger)^\intercal \breve{C} z}{1-p}, B\mathcal{K}\right) + \frac{p(1-p)}{2}\left\|\frac{B\lambda^o + (B^\dagger)^\intercal \breve{C} z}{1-p}\right\|^2 + \frac{1}{2}\|\breve{B}z\|^2.$$

Here, dist denotes the distance function on $\mathbb{R}^d$ from a non-empty set.

We argue that the driver F is $(\mathcal{G}_t)_{0\le t\le T}$-predictable for fixed $z \in \mathbb{R}^d$. We first observe that

$$\operatorname{dist}\left(\frac{B\lambda^o + (B^\dagger)^\intercal \breve{C} z}{1-p}, B\mathcal{K}\right) = \operatorname{dist}\left(\frac{B\lambda^o + (B^\dagger)^\intercal \breve{C} z}{1-p}, \operatorname{cl}(B\mathcal{K})\right).$$

Using Rockafellar [1976] Corollary 1.P and Assumption 5.2.8 the multivalued mapping $(t,\omega) \mapsto \operatorname{cl}(B_t(\omega)\mathcal{K}(t,\omega))$ is $(\mathcal{G}_t)_{t\in[0,T]}$-predictably measurable. The $(\mathcal{G}_t)_{t\in[0,T]}$-predictability of F then follows from Rockafellar [1976] Theorem 2.K.

We are now ready to state the first main result of this chapter restricting ourselves to the question of existence.

Theorem 5.2.13. *Let the Assumptions 5.2.2, 5.2.4, 5.2.6 and 5.2.8 hold. Then there exists a solution* $(\hat{\Psi}, \hat{Z}, \hat{N})$ *to the BSDE* (5.2.8) *on* $(\Omega, \mathcal{F}, (\mathcal{G}_t)_{0\le t\le T}, \mathbb{P})$*, i.e. to the BSDE*

$$d\Psi_t = Z_t^\intercal \, dM_t^o + dN_t - F(t, Z_t)\, dA_t - \frac{1}{2}\, d\langle N\rangle_t, \quad \Psi_T = \log\big(U(\breve{D})\big),$$

with driver

$$F(\cdot, z) = -\frac{p(1-p)}{2}\operatorname{dist}^2\left(\frac{B\lambda^o + (B^\dagger)^\intercal \breve{C} z}{1-p}, B\mathcal{K}\right) - \frac{q}{2}\left\|B\lambda^o + (B^\dagger)^\intercal \breve{C} z\right\|^2 + \frac{1}{2}\left\|\breve{B}z\right\|^2 \tag{5.2.9}$$

where $q = \frac{p}{p-1}$ *and where* $\mathbb{E}[\exp(\delta^* \hat{\Psi}^*)] < +\infty$ *for some* $\delta^* > 2$*. The dynamic value process* $\big(u_t(1)\big)_{t\in[0,T]}$ *from* (5.2.5) *is then given by*

$$u(1) \equiv \exp(\hat{\Psi})$$

and, after fixing an appropriate measurable selector (which indeed exists), we find that

$$\hat{\nu} := B^{\dagger}\Pi_{\mathrm{cl}(B\mathcal{K})}\left(\frac{B\lambda^o + (B^{\dagger})^{\intercal}\check{C}\hat{Z}}{1-p}\right) \tag{5.2.10}$$

defines an optimal strategy for the portfolio choice problem (5.2.4)*, i.e.*

$$u(1) = \mathbb{E}\Big[U\Big(DX_T^{\hat{\nu}}\Big)\Big],$$

where $\hat{\nu} \in \mathcal{A}_{\mathcal{K}}^{\mathcal{G}}$ *and* Π *stands for the nearest point operator onto the indicated set.*

We recall that in the present framework, for a $(\mathcal{G}_t)_{t\in[0,T]}$-predictably measurable multivalued function $G : [0,T] \times \Omega \to 2^{\mathbb{R}^d}$, a *measurable selector* is a $(\mathcal{G}_t)_{t\in[0,T]}$-predictable process $g : [0,T] \times \Omega \to \mathbb{R}^d$ such that $g(t,\omega) \in G(t,\omega)$ for all $(t,\omega) \in [0,T] \times \Omega$.

5.3 Information Constrained Utility Maximization

In this section we provide the proof of Theorem 5.2.13. We first transform the original problem (5.2.5) to the auxiliary optimization problem (5.2.6), then provide a solution to the BSDE (5.2.8) and finally use the martingale optimality paradigm to verify that this solution gives us the optimizers.

5.3.1 The Auxiliary Utility Maximization Problem

This subsection concerns the derivation of the auxiliary optimization problem (5.2.6), which can be interpreted as a problem for which $(\mathcal{G}_t)_{t\in[0,T]}$ encodes full information. We show that it is equivalent to the original portfolio choice problem under the present constraints and partial information. We begin with a preparatory lemma.

Lemma 5.3.1. *Let the Assumptions 5.2.2 and 5.2.4 hold and let* ν *be a* $(\mathcal{G}_t)_{t\in[0,T]}$*-predictable process which is* M*- and* M^o*-integrable. Then we have* $\nu \cdot M^o \equiv (\nu \cdot \widetilde{M})^o$ *and* $\mathbb{E}[\mathcal{E}(\nu \cdot \widetilde{M})_t | \, \mathcal{G}_t] = \mathcal{E}((\nu \cdot \widetilde{M})^o)_t$ *for all* $t \in [0,T]$*,* $\mathbb{P}$*-a.s. As a consequence,* $\mathbb{P}$*-a.s. for all* $t \in [0,T]$*,*

$$\mathbb{E}[\mathcal{E}(\nu \cdot \widetilde{M})_t | \, \mathcal{G}_t] = \mathcal{E}(\nu \cdot M^o)_t.$$

Proof. We observe that $\nu\cdot\widetilde{M}$ and $\nu\cdot M^o$ exist and that the Assumption 5.2.2 is satisfied for $\nu \cdot \widetilde{M}$ and for each of the components of $\widetilde{M}$ in place of M. We thus may refer to Mania and Santacroce [2010] Lemma 1 for the second statement and to Mania et al. [2008] Proposition 2.2 to show that for any $(\mathcal{G}_t)_{t\in[0,T]}$-local martingale $\overline{M}$ and all $i = 1,\ldots,d$, $\langle \widetilde{M}^i, \overline{M}\rangle \equiv \langle (M^i)^o, \overline{M}\rangle$. Together with Jacod and Shiryaev [2003] Theorem III.4.5 we then deduce

$$\langle (\nu\cdot\widetilde{M})^o,\overline{M}\rangle_t = \int_0^t \frac{d\langle \nu\cdot\widetilde{M},\overline{M}\rangle_s}{d\langle\overline{M}\rangle_s}\,d\langle\overline{M}\rangle_s = \int_0^t \frac{\sum_{i=1}^d \nu^i\,\frac{d\langle\widetilde{M}^i,\overline{M}\rangle_s}{dA_s}\,dA_s}{d\langle\overline{M}\rangle_s}\,d\langle\overline{M}\rangle_s$$
$$= \int_0^t \frac{\sum_{i=1}^d \nu^i\,\frac{d\langle (M^i)^o,\overline{M}\rangle_s}{dA_s}\,dA_s}{d\langle\overline{M}\rangle_s}\,d\langle\overline{M}\rangle_s = \langle \nu\cdot M^o,\overline{M}\rangle_t,$$

where we use $\frac{d(\cdot)}{d(\cdot)}$ to denote the respective Radon-Nikodým derivatives. Setting $\overline{M} := (\nu\cdot\widetilde{M})^o - \nu\cdot M^o$ then gives the claim and the final assertion is immediate. $\square$

If in the above lemma we replace ν by $\nu\mathbf{1}_{[0,s]}$ for a fixed $s\in[0,T]$, then the claim can be stated as $\mathbb{E}[\mathcal{E}(\nu\cdot\widetilde{M})_{s,t}|\,\mathcal{G}_t] = \mathcal{E}(\nu\cdot M^o)_{s,t}$. We use this to derive the auxiliary dynamic formulation (5.2.6) of the portfolio choice problem.

Proposition 5.3.2. *Let the Assumptions 5.2.2, 5.2.4 and 5.2.8 hold. Then the dynamic optimization problems* (5.2.5) *and* (5.2.6) *are equivalent, i.e.*

$$u_t(1) = \operatorname*{ess\,sup}_{\nu\in\mathcal{A}^{\mathcal{G}}_{\mathcal{K}}} \mathbb{E}\Big[U\Big(DX^\nu_{t,T}\Big)\Big|\,\mathcal{G}_t\Big] = \operatorname*{ess\,sup}_{\nu\in\mathcal{A}^{\mathcal{G}}_{\mathcal{K}}} \mathbb{E}\Big[U\Big(\breve{D}\breve{X}^\nu_{t,T}\breve{\kappa}^\nu_{t,T}\Big)\Big|\,\mathcal{G}_t\Big] = u^{\mathcal{G}}_t(1),$$

where $\breve{\kappa}^\nu$ is defined in (5.2.7).

Proof. Using the definitions, the condition on D, the above lemma and the preceding observation we obtain that for $\nu\in\mathcal{A}^{\mathcal{G}}_{\mathcal{K}}$,

$$\begin{aligned}
&\mathbb{E}\Big[U\Big(DX^\nu_{t,T}\Big)\Big|\,\mathcal{G}_t\Big]\\
&= \frac{1}{p}\,\mathbb{E}\Bigg[\mathbb{E}[D^p|\,\mathcal{H}_T]\exp\Bigg(p(\nu\cdot\widetilde{M})_{t,T} + p\int_t^T \nu_s^\intercal\,d\langle M\rangle_s\lambda^o_s - \frac{p}{2}\int_t^T \|B_s\nu_s\|^2\,dA_s\Bigg)\Bigg|\,\mathcal{G}_t\Bigg]\\
&= \frac{1}{p}\,\mathbb{E}\Bigg[\breve{D}^p\,\mathbb{E}\Big[\mathcal{E}(p\nu\cdot\widetilde{M})_{t,T}\Big|\,\mathcal{G}_T\Big]\exp\Bigg(p\int_t^T \nu_s^\intercal\,d\langle M\rangle_s\lambda^o_s + \frac{p^2-p}{2}\int_t^T \|B_s\nu_s\|^2\,dA_s\Bigg)\Bigg|\,\mathcal{G}_t\Bigg]\\
&= \frac{1}{p}\,\mathbb{E}\Bigg[\breve{D}^p\,\mathcal{E}(p\nu\cdot M^o)_{t,T}\exp\Bigg(p\int_t^T \nu_s^\intercal\,d\langle M\rangle_s\lambda^o_s + \frac{p^2-p}{2}\int_t^T \|B_s\nu_s\|^2\,dA_s\Bigg)\Bigg|\,\mathcal{G}_t\Bigg]\\
&= \mathbb{E}\Bigg[U\Big(\breve{D}\breve{X}^\nu_{t,T}\Big)\exp\Bigg(\frac{p^2-p}{2}\int_t^T\Big(\|B_s\nu_s\|^2-\|\breve{B}_s\nu_s\|^2\Big)\,dA_s\Bigg)\Bigg|\,\mathcal{G}_t\Bigg]\\
&= \mathbb{E}\Bigg[U\Big(\breve{D}\breve{X}^\nu_{t,T}\,\breve{\kappa}_{t,T}\Big)\Bigg|\,\mathcal{G}_t\Bigg],
\end{aligned}$$

which shows the desired identity. $\square$

Remark 5.3.3. Observe that the existence of the auxiliary object $(\mathcal{H}_t)_{t\in[0,T]}$ is used in the first equality of the above calculation. As an alternative, we could ask for the

$(\mathcal{G}_t)_{t\in[0,T]}$-predictability of *both* $\langle M\rangle$ *and* λ. Then the introduction of $(\mathcal{H}_t)_{t\in[0,T]}$ is not necessary and we can restrict ourselves to using $(\mathcal{G}_t)_{t\in[0,T]}$ in all the other structural conditions (e.g. on D), see also Remark 5.2.1.

5.3.2 A Solution to the BSDE

Before turning our attention to solving the above portfolio selection problem by using the martingale optimality paradigm we collect together the crucial properties of the driver F from (5.2.9).

Lemma 5.3.4. *Let the Assumption 5.2.6 (ii) hold. Then we have $\mathbb{P}$-a.s.*

(i) For all $t\in[0,T]$ the driver $F(t,\cdot)$ is continuous and convex (in z).

(ii) For all $\varepsilon_0>0$ and all t,z,

$$|F(t,z)|\leq p\varepsilon_0\|B_t\lambda^o_t\|^2+\frac{p(1-p+1/\varepsilon_0)}{2}\sup_{u\in\mathcal{K}_t}\|B_tu\|^2+\frac{\gamma(\varepsilon_0)}{2}\|\breve{B}_tz\|^2, \tag{5.3.1}$$

where $\gamma(\varepsilon_0):=1+2pc^\dagger\varepsilon_0$.

(iii) For all $t\in[0,T]$ the function $z\mapsto F(t,z)$ is locally Lipschitz continuous, i.e. for all t,z_1,z_2

$$|F(t,z_1)-F(t,z_2)|\leq\Big(2|q|\cdot\|B_t\lambda^o_t\|+\frac{1+2|q|}{2}\Big(\|\breve{B}_tz_1\|+\|\breve{B}_tz_2\|\Big)\Big)\cdot\|\breve{B}_t(z_1-z_2)\|.$$

Proof. As in the proof of Proposition 4.6.3 we obtain the representation

$$F(t,z)=\frac{1}{2}\|\breve{B}_tz\|^2-\frac{p(1-p)}{2}\inf_{u\in B_t\mathcal{K}_t}\Big(\|u\|^2-\frac{2}{1-p}\langle B_t\lambda^o_t+(B^\dagger)^\intercal\breve{C}z,u\rangle\Big)$$

where the infimum defines a concave function in z from which item (i) follows together with the continuity of the distance function. Item (ii) is then derived by an application of the generalized Young inequality to the last expression upon using Assumption 5.2.6 (ii). The last item follows from the Lipschitz continuity of the distance function using Assumption 5.2.6 (ii) again. □

Remark 5.3.5. We remark that Lemma 5.3.4 (ii) gives us a family of growth estimates which are parameterized by $\varepsilon_0>0$. In particular, we are able to choose an appropriate ε_0^* for which specific conditions that we find to be necessary hold.

Consistently to the notation in Chapter 3 we now set

$$|\alpha(\varepsilon_0)|_1:=p\varepsilon_0\int_0^T\|B_t\lambda^o_t\|^2\,dA_t+\frac{p(1-p+1/\varepsilon_0)}{2}\int_0^T\sup_{u\in\mathcal{K}_t}\|B_tu\|^2\,dA_t$$

and observe that there is a tradeoff between the conditions that have to be imposed on the first and the second summand as ε_0 varies. As we will see below an important argument in performing the verification hinges on proving that a solution to the BSDE provides a measure change in the spirit of Section 3.7. In particular, we need that $\gamma(\varepsilon_0) < 2$, so that we restrict ourselves to $\varepsilon_0 < \frac{1}{2pc^\dagger}$ in the sequel.

Proposition 5.3.6. *Let the Assumptions 5.2.2 (i) and (iv), 5.2.4, 5.2.6 (ii) and 5.2.8 hold. If $\varepsilon_0^* > 0$ is small enough, i.e. $\varepsilon_0^* < \frac{c_\lambda \wedge (1/c^\dagger)}{2p}$, then*

(i) The random variable $|\log(U(\breve{D}))| + \big|\alpha(\varepsilon_0^*)\big|_1$ has an exponential moment of some order $\delta^* > 2$, where δ^* depends on p, c_λ, $c^\dagger$ and ε_0^*.

(ii) The BSDE (5.2.8) has a solution $(\hat{\Psi}, \hat{Z}, \hat{N})$ such that $\mathbb{E}\big[\exp(\delta^* \hat{\Psi}^*)\big] < +\infty$, where δ^* is from item (i).

(iii) For all $\nu \in \mathcal{A}_{\mathcal{K}}^{\mathcal{G}}$ the stochastic exponential $\mathcal{E}\big((\hat{Z} + p\nu)\cdot M^o + \hat{N}\big)$ is a true martingale on $[0,T]$ for the filtration $(\mathcal{G}_t)_{t\in[0,T]}$.

Proof. With regards to item (i) let $\varepsilon_0^* > 0$ be some number that is small enough to satisfy $\varepsilon_0^* < \frac{c_\lambda \wedge (1/c^\dagger)}{2p}$. Take $\beta > 1$ such that $1 < \beta^2 < \frac{c_\lambda}{2p\varepsilon_0^*}$, $\delta^* := \frac{c_\lambda}{p\varepsilon_0^* \beta^2} > 2$ and $\varrho := \frac{\beta}{\beta - 1} > 1$. Then by the Hölder inequality we obtain

$$\begin{aligned}
\mathbb{E}\Big[\exp\big(\delta^*\big(\big|\log(U(\breve{D}))\big| + |\alpha(\varepsilon_0^*)|_1\big)\big)\Big] &\le \frac{1}{p^{\delta^*}}\mathbb{E}\Big[\exp\big(p\varrho\delta^* |\log(\breve{D})|\big)\Big]^{1/\varrho} \\
&\quad \cdot \mathbb{E}\left[\exp\left(p\varepsilon_0^*\beta^2\delta^* \int_0^T \left\|B_t\lambda_t^o\right\|^2 dA_t\right)\right]^{1/\beta^2} \\
&\quad \cdot \mathbb{E}\left[\exp\left(\frac{p\varrho\beta\delta^*(1 - p + 1/\varepsilon_0^*)}{2}\int_0^T \sup_{u\in\mathcal{K}_t}\|B_t u\|^2\, dA_t\right)\right]^{1/(\beta\varrho)} < +\infty,
\end{aligned}$$

since $p\varepsilon_0^*\beta^2\delta^* = c_\lambda$.

Item (ii) then follows from Corollary 3.4.3 (i) since $\delta^* > 2 > \gamma(\varepsilon_0^*)$. We fix a suitable ε_0^* and δ^*.

With regards to item (iii) we need some more preparation. We assume that $\varepsilon_0 > 0$ is an arbitrary number that is small enough to satisfy $\varepsilon_0 < \frac{1}{2(1+pc^\dagger)}$. Then define $\tilde{\varrho}(\varepsilon_0) := 1 - \varepsilon_0$ and

$$\varepsilon(\varepsilon_0) := \frac{1}{2}\min\left(\varepsilon_0, \frac{1}{p}, \frac{2\tilde{\varrho}(\varepsilon_0) - \gamma(\varepsilon_0)}{2p\tilde{\varrho}(\varepsilon_0)}\right),$$

where $\gamma(\varepsilon_0)$ is as in Lemma 5.3.4 (ii). Finally, set

$$\varrho_0(\varepsilon_0) := \frac{\tilde{\varrho}(\varepsilon_0)(1 - p\varepsilon(\varepsilon_0))}{2\tilde{\varrho}(\varepsilon_0)(1 - p\varepsilon(\varepsilon_0)) - \gamma(\varepsilon_0)}.$$

Then $1 > \tilde{\varrho}(\varepsilon_0) > \gamma(\varepsilon_0)/2 > 1/2$, $\varepsilon(\varepsilon_0) > 0$, $\varrho_0(\varepsilon_0) > 0$ and $\lim_{\varepsilon_0\downarrow 0}\varrho_0(\varepsilon_0) = 1+$ (which means that the limit is from above). The first three of these assertions are derived easily. We also observe that $\lim_{\varepsilon_0\downarrow 0}\varepsilon(\varepsilon_0) = 0$ and that $\varepsilon(\varepsilon_0) = \varepsilon_0$ for ε_0 sufficiently small. Since $\varepsilon_0 < \frac{1}{2(1+pc^\dagger)} < 1 + 4c^\dagger + 2/p$ we also have that $\varrho_0(\varepsilon_0) > 1$ for ε_0 sufficiently small from which the last claim follows. In particular, there exists an $\varepsilon_0^\star \in \left(0, \frac{1}{2(1+pc^\dagger)} \wedge \frac{\delta^*-2}{\delta^*} \wedge \varepsilon_0^*\right)$ such that $1 < \varrho_0(\varepsilon_0^\star) < \frac{\delta^*}{2}$. We now proceed similarly to the proof of Theorem 3.7.2 and consider for $\tilde{\varrho} := 1 - \varepsilon_0^\star$, $\eta > 0$, and $\nu \in \mathcal{A}_{\mathcal{K}}^{\mathcal{G}}$,

$$\log G_\eta(t) := \tilde{\varrho}\eta \left[((\hat{Z} + p\nu) \cdot M^o)_t + \hat{N}_t\right] + \tilde{\varrho}^2 \left(\frac{1}{2} - \eta\right) \left\langle (\hat{Z} + p\nu) \cdot M^o + \hat{N} \right\rangle_t$$

to get from the BSDE (5.2.8) and the growth condition in (5.3.1),

$$\begin{aligned}
\log G_\eta(t) &= \tilde{\varrho}\eta\left(\hat{\Psi}_t - \hat{\Psi}_0 + \int_0^t F(s, \hat{Z}_s)\, dA_s + \frac{1}{2}\, d\langle \hat{N} \rangle_t\right) \\
&\quad + p\,(\nu \cdot M^o)_t + \tilde{\varrho}^2\left(\frac{1}{2} - \eta\right)\langle (\hat{Z} + p\nu) \cdot M^o + \hat{N} \rangle_t \\
&\le \tilde{\varrho}\eta(\hat{\Psi}^* + |\hat{\Psi}_0|) + \tilde{\varrho}\eta\left|\alpha(\varepsilon_0^\star)\right|_1 + \tilde{\varrho}\eta\left(\frac{\gamma}{2} + \frac{\tilde{\varrho}}{\eta}\left(\frac{1}{2} - \eta\right)\right)\langle \hat{Z} \cdot M^o + \hat{N} \rangle_t \\
&\quad + p\,(\nu \cdot M^o)_t + p^2\tilde{\varrho}^2\left(\frac{1}{2} - \eta\right)\langle \nu \cdot M^o \rangle_t + 2p\tilde{\varrho}^2\left(\frac{1}{2} - \eta\right)\langle \hat{Z} \cdot M^o, \nu \cdot M^o \rangle_t \\
&\le \tilde{\varrho}\eta(\hat{\Psi}^* + |\hat{\Psi}_0|) + \tilde{\varrho}\eta\left|\alpha(\varepsilon_0^\star)\right|_1 + \tilde{\varrho}\eta\left(\frac{\gamma}{2} + \frac{\tilde{\varrho}}{\eta}\left(\frac{1}{2} - \eta\right)\right)\langle \hat{N} \rangle_t \\
&\quad + \tilde{\varrho}\eta\left(\frac{\gamma}{2} + \frac{\tilde{\varrho}}{\eta}\left(\frac{1}{2} - \eta\right) + \frac{p\varepsilon\tilde{\varrho}}{\eta}\left|\frac{1}{2} - \eta\right|\right)\langle \hat{Z} \cdot M^o \rangle_t \\
&\quad + p\,(\nu \cdot M^o)_t + p\tilde{\varrho}^2\left(p\left(\frac{1}{2} - \eta\right) + \frac{1}{\varepsilon}\left|\frac{1}{2} - \eta\right|\right)\langle \nu \cdot M^o \rangle_t,
\end{aligned}$$

where $\varepsilon > 0$ is arbitrary, by the Young inequality. In what follows we choose $\varepsilon > 0$ to satisfy

$$\varepsilon = \frac{1}{2}\min\left(\varepsilon_0^\star, \frac{1}{p}, \frac{2\tilde{\varrho} - \gamma}{2p\tilde{\varrho}}\right) = \varepsilon(\varepsilon_0^\star)$$

where $\gamma = \gamma(\varepsilon_0^\star)$. Noting that for $\eta > 1/2$,

$$\frac{\gamma}{2} + \frac{\tilde{\varrho}}{\eta}\left(\frac{1}{2} - \eta\right) + \frac{p\varepsilon\tilde{\varrho}}{\eta}\left|\frac{1}{2} - \eta\right| \le 0 \iff \eta \ge \frac{\tilde{\varrho}(1 - p\varepsilon)}{2\tilde{\varrho}(1 - p\varepsilon) - \gamma} = \varrho_0(\varepsilon_0^\star) =: \varrho_0^\star,$$

where indeed $\varrho_0^\star > 1 > 1/2$, we have that $\mathbb{P}$-a.s. for all $t \in [0, T]$,

$$\begin{aligned}
G_\eta(t) \le{}& \exp\left(\tilde{\varrho}\eta(\hat{\Psi}^* + |\hat{\Psi}_0|)\right)\exp\left(\tilde{\varrho}\eta\left|\alpha(\varepsilon_0^\star)\right|_1\right) \\
&\cdot \exp\left(p(\nu \cdot M^o)^*\right)\exp\left(\frac{p\tilde{\varrho}^2}{\varepsilon}\left|\frac{1}{2} - \eta\right|\langle \nu \cdot M^o \rangle_T\right),
\end{aligned} \tag{5.3.2}$$

for all $\eta \ge \varrho_0^\star$. Setting $\eta^* := \delta^*/2 > \varrho_0^\star > 1$ and using the Hölder inequality twice, firstly with exponent $\beta := 1/\tilde{\varrho} > 1$ and secondly with exponent $\beta := 2$, we conclude from the exponential moments conditions on $\hat{\Psi}^*$, $\left|\alpha(\varepsilon_0^\star)\right|_1$ and $\langle \nu \cdot M^o \rangle_T$ that

$$\sup_{\substack{\tau \text{ stopping time} \\ \text{valued in [0,T]}}} \mathbb{E}\big[G_{\eta^*}(\tau)\big] < +\infty. \tag{5.3.3}$$

It now follows from Lemma 3.7.1 that the stochastic exponential $\mathcal{E}\big(\tilde{\varrho}\eta\big[(\hat{Z}+p\nu)\cdot M^o+\hat{N}\big]\big)$ is a true martingale for all $\nu \in \mathcal{A}_{\mathcal{K}}^{\mathcal{G}}$ and all $\eta \in \Big(1, \frac{\delta^*}{2}\Big)$, in particular for $\eta := \frac{1}{\tilde{\varrho}} = \frac{1}{1-\varepsilon_0^\star} < \delta^*/2$. □

5.3.3 The Martingale Optimality Principle

We are now ready to prove Theorem 5.2.13 by using the martingale optimality principle. Let $(\hat{\Psi}, \hat{Z}, \hat{N})$ be the solution to the BSDE (5.2.8) obtained in the previous proposition. For arbitrary $\nu \in \mathcal{A}_{\mathcal{K}}^{\mathcal{G}}$ we then set

$$J^\nu := U\Big(\breve{X}^{1,\nu}\breve{\kappa}\Big)$$

and deduce from Itô's formula that

$$\begin{aligned}
d(J_t^\nu \exp(\hat{\Psi}_t)) &= J_t^\nu \exp(\hat{\Psi}_t)\Bigg[(\hat{Z}_t + p\nu_t)\, dM_t^o + d\hat{N}_t \\
&+ \frac{p(1-p)}{2}\Bigg(\mathrm{dist}^2\Big(\frac{B_t\lambda_t^o + (B_t^\dagger)^\intercal \breve{C}_t \hat{Z}_t}{1-p}, B_t\mathcal{K}_t\Big) - \Bigg\|\frac{B_t\lambda_t^o + (B_t^\dagger)^\intercal \breve{C}_t \hat{Z}_t}{1-p} - B_t\nu_t\Bigg\|^2 dA_t\Bigg)\Bigg] \\
&= J_t^\nu \exp(\hat{\Psi}_t)\Big((\hat{Z}_t + p\nu_t)\, dM_t^o + d\hat{N}_t + \frac{p(1-p)}{2}\, \upsilon(t, \nu_t, \hat{Z}_t)\, dA_t\Big),
\end{aligned} \tag{5.3.4}$$

where for $\nu \in \mathcal{A}_{\mathcal{K}}^{\mathcal{G}}$ and $z \in \mathbb{R}^d$,

$$\begin{aligned}
\upsilon(t, \nu_t, z) : &= \mathrm{dist}^2\Big(\frac{B_t\lambda_t^o + (B_t^\dagger)^\intercal \breve{C}_t z}{1-p}, B_t\mathcal{K}_t\Big) - \Bigg\|\frac{B_t\lambda_t^o + (B_t^\dagger)^\intercal \breve{C}_t z}{1-p} - B_t\nu_t\Bigg\|^2 \\
&= \mathrm{dist}^2\Big(\frac{B_t\lambda_t^o + (B_t^\dagger)^\intercal \breve{C}_t z}{1-p}, \mathrm{cl}(B_t\mathcal{K}_t)\Big) - \Bigg\|\frac{B_t\lambda_t^o + (B_t^\dagger)^\intercal \breve{C}_t z}{1-p} - B_t\nu_t\Bigg\|^2 \le 0.
\end{aligned}$$

Note that Assumption 5.2.6 (i) is used in the application of Itô's formula in (5.3.4) via the formula (5.2.2). Namely, there occur both $B\nu$ and $\breve{B}\nu$ in (5.2.7). However, the BSDE (5.2.8) is a BSDE for the local martingale M^o, hence necessarily involves $\breve{B}\hat{Z}$. In order to derive υ, which is given in the above convenient form, in the multidimensional case considered here we hence require a consistency relation between B and $\breve{B}$, the Assumption 5.2.6 (i). From (5.3.4) we derive that

$$J^\nu \exp(\hat{\Psi}) = \frac{1}{p}\, \exp(\hat{\Psi}_0)\, \mathcal{E}\Big((\hat{Z} + p\nu)\cdot M^o + \hat{N}\Big) \exp\Big(\frac{p(1-p)}{2}\int_0^{\cdot} \upsilon(t, \nu_t, \hat{Z}_t)\, dA_t\Big),$$

which according to Proposition 5.3.6 (iii) is the product of a uniformly integrable martingale for the filtration $(\mathcal{G}_t)_{t\in[0,T]}$ and a positive nonincreasing $(\mathcal{G}_t)_{t\in[0,T]}$-predictable process, hence a $(\mathcal{G}_t)_{t\in[0,T]}$-supermartingale, see Jacod and Shiryaev [2003] Theorem II.8.21. It follows that for $t \in [0,T]$,

$$J_t^\nu \exp(\hat{\Psi}_t) \geq \mathbb{E}\Big[J_T^\nu \exp(\hat{\Psi}_T)\Big|\,\mathcal{G}_t\Big], \quad \mathbb{P}\text{-a.s.}$$

from which

$$\exp(\hat{\Psi}_t) \geq \mathbb{E}\Big[U\Big(\breve{D}\breve{X}^\nu_{t,T}\breve{\kappa}^\nu_{t,T}\Big)\Big|\,\mathcal{G}_t\Big], \quad \mathbb{P}\text{-a.s.}$$

Since this holds for all $\nu \in \mathcal{A}^{\mathcal{G}}_{\mathcal{K}}$ we obtain, $\mathbb{P}$-a.s. (see Remark 5.2.12),

$$\exp(\hat{\Psi}_t) \geq u_t^{\mathcal{G}}(1) = \operatorname*{ess\,sup}_{\nu\in\mathcal{A}^{\mathcal{G}}_{\mathcal{K}}} \mathbb{E}\Big[U\Big(\breve{D}\breve{X}^\nu_{t,T}\breve{\kappa}^\nu_{t,T}\Big)\Big|\,\mathcal{G}_t\Big]\,.$$

In particular, $\mathbb{P}$-a.s.

$$\exp(\hat{\Psi}_0) \geq u_0^{\mathcal{G}}(1) = u(1) = \sup_{\nu\in\mathcal{A}^{\mathcal{G}}_{\mathcal{K}}} \mathbb{E}\Big[U\Big(DX_T^\nu\Big)\Big]\,.$$

Finally, we define

$$\hat{\nu} := B^\dagger \Pi_{\mathrm{cl}(B\mathcal{K})}\left(\frac{B\lambda^o + (B^\dagger)^\intercal \breve{C}\hat{Z}}{1-p}\right)$$

where we fix a suitable measurable selector. Note that such a measurable selector exists thanks to Rockafellar [1976] Corollary 1.C. Thus, $\hat{\nu} \in \mathcal{A}^{\mathcal{G}}_{\mathcal{K}}$. Namely,

$$B\hat{\nu} = BB^\dagger \Pi_{\mathrm{cl}(B\mathcal{K})}\left(\frac{B\lambda^o + (B^\dagger)^\intercal \breve{C}\hat{Z}}{1-p}\right) = \Gamma^{\frac{1}{2}}\widetilde{\Gamma}^{\frac{1}{2}} \Pi_{\mathrm{cl}(B\mathcal{K})}\left(\frac{B\lambda^o + (B^\dagger)^\intercal \breve{C}\hat{Z}}{1-p}\right)$$

and, moreover,

$$v(\cdot, \hat{\nu}, \hat{Z}) \equiv 0,$$

since all elements in the images of $\Gamma^{\frac{1}{2}}\widetilde{\Gamma}^{\frac{1}{2}}B$ and $B = \Gamma^{\frac{1}{2}}P$ have the same zero components (which are closed) and for the nonzero components the closure does not affect the distance function. We conclude that $J^{\hat{\nu}}\exp(\hat{\Psi})$ is a martingale so that we obtain, $\mathbb{P}$-a.s.

$$\exp(\hat{\Psi}_t) \geq u_t^{\mathcal{G}}(1) = \operatorname*{ess\,sup}_{\nu\in\mathcal{A}^{\mathcal{G}}_{\mathcal{K}}} \mathbb{E}\Big[U\Big(\breve{D}\breve{X}^\nu_{t,T}\breve{\kappa}^\nu_{t,T}\Big)\Big|\,\mathcal{G}_t\Big] \geq \mathbb{E}\Big[U\Big(\breve{D}\breve{X}^{\hat{\nu}}_{t,T}\breve{\kappa}^{\hat{\nu}}_{t,T}\Big)\Big|\,\mathcal{G}_t\Big] = \exp(\hat{\Psi}_t)$$

and

$$\mathbb{E}\Big[U\Big(DX_T^{\hat{\nu}}\Big)\Big] = \exp(\hat{\Psi}_0) = u(1) = \sup_{\nu\in\mathcal{A}^{\mathcal{G}}_{\mathcal{K}}} \mathbb{E}\Big[U\Big(DX_T^\nu\Big)\Big]\,.$$

The continuity of $\hat{\Psi}$ and the càdlàg property of $(u_t(1))_{t\in[0,T]}$ and $(u_t^{\mathcal{G}}(1))_{t\in[0,T]}$, see Remark 5.2.12, show that $\exp(\hat{\Psi}) \equiv u^{\mathcal{G}}(1) \equiv u(1)$ holds up to indistinguishability.

Remark 5.3.7. It becomes clear again that the verification argument consists of proving that some object is a true martingale, which is a recurrent theme in the present thesis and at the heart of stochastic control theory, see condition (A) in the introduction.

5.4 Stability of the Optimization Problem

In this section we prove that the constrained utility maximization problem (5.2.4) is continuous with respect to the input parameters p, λ, $\mathcal{K}$ and D of the model, i.e. the risk-aversion parameter of the agent, the market price of risk process, the constraints sets and the discount or measure change factor. We proceed similarly to Section 4.6 under the appropriate assumption on the mean-variance tradeoff. More specifically, we impose the following condition.

Assumption 5.4.1. *We assume that $\langle \lambda \cdot M \rangle_T = \int_0^T \|B_t \lambda_t\|^2 \, dA_t$ has finite exponential moments of all orders.*

We then have

Theorem 5.4.2. *Let the Assumptions 5.2.2, 5.2.6, 5.2.8 and 5.4.1 hold. Then there exists a unique solution $(\hat{\Psi}, \hat{Z}, \hat{N})$ to the BSDE* (5.2.8) *with $\hat{\Psi} \in \mathfrak{E}$ and we have $u(1) \equiv \exp(\hat{\Psi})$ where the corresponding optimal strategy $\hat{\nu} \in \mathcal{A}^{\mathcal{G}}_{\mathcal{K}}$ is given by* (5.2.10).

Proof. For all $\varepsilon_0 > 0$ the random variable $|\log(U(\breve{D}))| + |\alpha(\varepsilon_0)|_1$ has exponential moments of all orders. The first claim then follows from the Theorems 3.2.5 and 3.2.6. Moreover, for $\varepsilon_0 > 0$ sufficiently small and all $\nu \in \mathcal{A}^{\mathcal{G}}_{\mathcal{K}}$, the stochastic exponential $\mathcal{E}\big((\hat{Z} + p\nu) \cdot M^o + \hat{N}\big)$ is a true $(\mathcal{G}_t)_{t \in [0,T]}$-martingale on $[0, T]$, which follows as in the proof of 5.3.6 (iii). The reasoning from the previous section then yields the result. $\square$

With regards to the stability analysis let

$$(\lambda^n)_{n \in \mathbb{N}},\ (p^n)_{n \in \mathbb{N}},\ (\mathcal{K}^n)_{n \in \mathbb{N}} \text{ and } (D^n)_{n \in \mathbb{N}}$$

be sequences of parameters that converge to $\lambda =: \lambda^0$, $p =: p^0$, $\mathcal{K} =: \mathcal{K}^0$ and $D =: D^0$ in a sense that will be specified shortly. Fix $n \in \mathbb{N}$. We have that each λ^n is a predictable M-integrable process so that it leads to dynamics for the asset $S = S^n$ of the form

$$dS^n_t = \mathrm{Diag}(S^n_t)\Big(dM_t + d\langle M \rangle_t \lambda^n_t\Big).$$

In addition, assume that $(\lambda^n)^o = (\lambda^n)^{\mathcal{G}} = (\lambda^n)^{\mathcal{H}}$, μ^A-a.e. for all $n \in \mathbb{N}$. Each risk aversion parameter p^n is valued in $(0, 1)$ and corresponds to a utility function

$$U^n(x) := \frac{1}{p^n} x^{p^n}, \quad x > 0.$$

Each D^n is an $\mathcal{F}_T$-measurable random variable satisfying

$$\mathbb{E}\big[(D^n)^{p^n} \,\big|\, \mathcal{H}_T\big] = \mathbb{E}\big[(D^n)^{p^n} \,\big|\, \mathcal{G}_T\big] =: (\breve{D}^n)^{p^n} > 0, \quad \mathbb{P}\text{-a.s.}$$

and each constraints set $\mathcal{K}^n$ is assumed to satisfy Assumption 5.2.8 so that we can consider the primal problem as a function of the inputs,

$$u_t^n(1) := \operatorname*{ess\,sup}_{\nu \in \mathcal{A}^{\mathcal{G}}_{\mathcal{K}^n}} \mathbb{E}\Big[U^n\Big(D^n X_T^{n,\nu}\Big)\Big], \tag{5.4.1}$$

where $X^{n,\nu}$ represents the wealth acquired from an investment in S^n so that we have

$$X^{n,\nu} = \mathcal{E}(\nu \cdot M + \nu \cdot \langle M \rangle \lambda^n). \tag{5.4.2}$$

In what follows we prove the continuity of the optimizers

$$\hat{X}^n := X^{n,\hat{\nu}^n} \text{ and } \hat{\nu}^n := \hat{\nu}(\lambda^n, p^n, \mathcal{K}^n, D^n),$$

for this problem. In view of Theorem 3.2.7 the following assumption is appropriate for our purposes.

Assumption 5.4.3. *The preferences and markets converge in the following sense,*

$$\lim_{n\to+\infty} p^n = p \quad \textit{and}$$
$$\lim_{n\to+\infty} \Big(\langle [(\lambda^n)^o - \lambda^o] \cdot M \rangle_T + |\breve{D}^n - \breve{D}|\Big) = 0$$

in $\mathbb{P}$-probability. The constraints sets converge in the sense of the closed topology

$$\operatorname*{Lim}_{n\to+\infty} \operatorname{cl}(B\mathcal{K}^n) = \operatorname{cl}(B\mathcal{K}), \quad \mu^A\text{-}a.e.$$

see Appendix 6.2 for more details. Moreover, if $\varepsilon_0 < \frac{1}{2\max(p^n, n \geq 0) c^\dagger}$, we assume that

$$\sup_{n\geq 0} \mathbb{E}\Big[e^{\varrho\left(|\log(\breve{D}^n)| + |\alpha^n(\varepsilon_0)|_1\right)}\Big] < +\infty$$

for all $\varrho > 0$ where

$$|\alpha^n(\varepsilon_0)|_1 := p^n \varepsilon_0 \int_0^T \left\|B_t (\lambda_t^n)^o\right\|^2 dA_t + \frac{p^n(1 - p^n + 1/\varepsilon_0)}{2} \int_0^T \sup_{u \in \mathcal{K}_t^n} \|B_t u\|^2 \, dA_t.$$

Remark 5.4.4. With regards to the appropriate mode of convergence of the market price of risk processes and in analogy to the conditions present in Chapter 4 we could have required that $\langle (\lambda^n - \lambda) \cdot M \rangle_T$ have all exponential moments, uniformly in n, and that $\langle (\lambda^n - \lambda) \cdot M \rangle_T \to 0$ as $n \to +\infty$ in $\mathbb{P}$-probability. These conditions imply the corresponding conditions in the above assumption as is easily checked using the universal property of the optional projection together with He et al. [1992] Theorem 5.25. In view of Proposition 6.2.2 and the invariance of the distance function under the closure operator a condition equivalent to $\operatorname{Lim}_{n\to+\infty} \operatorname{cl}(B\mathcal{K}^n) = \operatorname{cl}(B\mathcal{K})$ is that $\operatorname{dist}(\cdot, B\mathcal{K}^n) \to \operatorname{dist}(\cdot, B\mathcal{K})$ pointwise as $n \to +\infty$.

Under the Assumptions 5.2.2, 5.2.6 and 5.4.3 and for fixed $n \in \mathbb{N}_0$ we have from Theorem 5.4.2 that there exists a unique solution $(\hat{\Psi}^n, \hat{Z}^n, \hat{N}^n)$ to the BSDE (written in generic variables)

$$d\Psi_t = Z_t^\intercal \, dM_t^o + dN_t - F^n(t, Z_t)\, dA_t - \frac{1}{2}\, d\langle N\rangle_t, \quad \Psi_T = \log\big(U(\breve{D}^n)\big), \tag{5.4.3}$$

with $\hat{\Psi}^n \in \mathfrak{E}$ and where F^n is given by

$$F^n(\cdot, z) = -\frac{p^n(1-p^n)}{2}\operatorname{dist}^2\left(\frac{B(\lambda^n)^o + (B^\dagger)^\intercal \breve{C} z}{1-p^n}, B\mathcal{K}^n\right) - \frac{q^n}{2}\left\|B(\lambda^n)^o + (B^\dagger)^\intercal \breve{C} z\right\|^2 + \frac{1}{2}\|\breve{B} z\|^2.$$

Moreover, $u^n(1) \equiv \exp\big(\hat{\Psi}^n\big)$ and

$$\hat{\nu}^n = B^\dagger \Pi_{\mathrm{cl}(B\mathcal{K}^n)}\left(\frac{B(\lambda^n)^o + (B^\dagger)^\intercal \breve{C}\hat{Z}^n}{1-p^n}\right)$$

defines an optimal strategy of the portfolio choice problem for the input parameters λ^n, p^n, $\mathcal{K}^n$ and D^n. Clearly, our goal is now to use Theorem 3.2.7 for which we need that the drivers F^n converge appropriately. This is the content of the following lemma.

Lemma 5.4.5. *Let the Assumptions 5.2.6 (ii) and 5.4.3 hold. Then, setting $\hat{Z} = \hat{Z}^0$,*

$$\lim_{n\to+\infty} \mathbb{E}\left[\int_0^T \big|F^n(t,\hat{Z}_t) - F(t,\hat{Z}_t)\big|\, dA_t\right] = 0.$$

Proof. Using the definition of the drivers one can derive the following inequality

$$\begin{aligned}
&|F^n(t,\hat{Z}_t) - F(t,\hat{Z}_t)| \\
&\le \frac{p(1-p)}{2}\left|\operatorname{dist}^2\left(\frac{B_t\lambda_t^o + (B_t^\dagger)^\intercal \breve{C}_t\hat{Z}_t}{1-p}, B_t\mathcal{K}_t^n\right) - \operatorname{dist}^2\left(\frac{B_t\lambda_t^o + (B_t^\dagger)^\intercal \breve{C}_t\hat{Z}_t}{1-p}, B_t\mathcal{K}_t\right)\right| \\
&+ \frac{p^n(1-p^n)}{2}\left|\operatorname{dist}^2\left(\frac{B_t\lambda_t^o + (B_t^\dagger)^\intercal \breve{C}_t\hat{Z}_t}{1-p}, B_t\mathcal{K}_t^n\right) - \operatorname{dist}^2\left(\frac{B_t(\lambda^n)_t^o + (B_t^\dagger)^\intercal \breve{C}_t\hat{Z}_t}{1-p^n}, B_t\mathcal{K}_t^n\right)\right| \\
&+ \frac{\big|p(1-p) - p^n(1-p^n)\big|}{2}\operatorname{dist}^2\left(\frac{B_t\lambda_t^o + (B_t^\dagger)^\intercal \breve{C}_t\hat{Z}_t}{1-p}, B_t\mathcal{K}_t^n\right) \\
&+ \tfrac{|q-q^n|}{2}\big\|B_t(\lambda^n)_t^o + (B_t^\dagger)^\intercal \breve{C}_t\hat{Z}_t\big\|^2 + \frac{q}{2}\Big|\big\|B_t\lambda_t^o + (B_t^\dagger)^\intercal \breve{C}_t\hat{Z}_t\big\|^2 - \big\|B_t(\lambda^n)_t^o + (B_t^\dagger)^\intercal \breve{C}_t\hat{Z}_t\big\|^2\Big| \\
&=: G_t^n + H_t^n + I_t^n + J_t^n.
\end{aligned}$$

As in the proof of Proposition 4.6.3 we have to show that

$$\lim_{n\to+\infty} \mathbb{E}\left[\int_0^T (G_t^n + H_t^n + I_t^n + J_t^n)\, dA_t\right] = 0,$$

for which we can work term by term. For instance, by Proposition 6.2.2, $(G^n)_{n\in\mathbb{N}}$ converges to zero μ^A-a.e. and is dominated by $|q|(\|\breve{B}\hat{Z}\|^2 + \|B\lambda^o)\|^2)$. In particular, thanks to the dominated convergence theorem, we have

$$\lim_{n\to+\infty} \mathbb{E}\left[\int_0^T G_t^n\, dA_t\right] = \lim_{n\to+\infty} \int_{[0,T]\times\Omega} G^n\, d\mu^A = 0.$$

Similarly, following the same pattern as in the proof of Proposition 4.6.3, we use the local Lipschitz estimate for the distance function, Proposition 6.2.2, the boundedness of $\frac{p^n(1-p^n)}{2}$, Assumption 5.2.6 (ii) and the given integrability to finally derive all the desired individual convergences. □

We are now ready to state the BSDE stability result 3.2.7 adapted to the present framework.

Theorem 5.4.6. *Let the Assumptions 5.2.2, 5.2.6 and 5.4.3 hold and let the triple* $(\hat{\Psi}^n, \hat{Z}^n, \hat{N}^n)$ *denote the unique solution to the BSDE* (5.4.3) *with* $\hat{\Psi}^n \in \mathfrak{E}$ *then*

$$\lim_{n\to+\infty} \mathbb{E}\Big[\exp\Big(\varrho\,(\hat{\Psi}^n - \hat{\Psi})^*\Big)\Big] = 1,$$
$$\lim_{n\to+\infty} \mathbb{E}\left[\Big(\langle(\hat{Z}^n - \hat{Z})\cdot M^o\rangle_T + \langle\hat{N}^n - \hat{N}\rangle_T\Big)^{\varrho/2}\right] = 0,$$

for all $\varrho \geq 1$*, where* $(\hat{\Psi}, \hat{Z}, \hat{N})$ *denotes the unique solution triple of the BSDE* (5.2.8) *with* $\hat{\Psi} \in \mathfrak{E}$*.*

As a consequence, we immediately derive the convergence of the value processes. We have the following theorem.

Theorem 5.4.7. *Let the Assumptions 5.2.2, 5.2.6 and 5.4.3 hold and let* $(u^n(1))_{n\in\mathbb{N}}$ *be the sequence of the dynamic value functions for the constrained utility maximization problem* (5.4.1) *which we know to satisfy* $u^n(1) \equiv \hat{\Psi}^n$*. Similarly, let* $u(1)$ *be the dynamic value function for the problem* (5.2.5) *with* $u(1) \equiv \hat{\Psi} \equiv \hat{\Psi}^0$*. Here,* $\hat{\Psi}^n$*,* $n \geq 0$*, are components of the corresponding unique BSDE solutions with* $\hat{\Psi}^n \in \mathfrak{E}$*. Then for all* $\varrho > 0$*,*

$$\lim_{n\to+\infty} \mathbb{E}\left[\sup_{t\in[0,T]} \big|u_t^n(1) - u_t(1)\big|^{\varrho}\right] = 0.$$

Proof. From the proof of Theorem 3.6.1 we know that $\exp\big((\hat{\Psi}^n - \hat{\Psi})^*\big)$ converges to one in $L^{\beta\varrho}(\mathbb{P})$ for $\beta > \max(1/\varrho, 1)$. Hence,

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}\left|u_t^n(1)-u_t(1)\right|^{\varrho}\right] &= \mathbb{E}\left[\sup_{t\in[0,T]}\left|e^{\hat{\Psi}_t^n}-e^{\hat{\Psi}_t}\right|^{\varrho}\right] \\ &\leq \mathbb{E}\left[\left(e^{\sup_{t\in[0,T]}\left|\hat{\Psi}_t^n-\hat{\Psi}_t\right|}-1\right)^{\varrho}\exp(\varrho\hat{\Psi}^*)\right] \\ &\leq \mathbb{E}\left[\left(e^{\left(\hat{\Psi}_t^n-\hat{\Psi}_t\right)^*}-1\right)^{\beta\varrho}\right]^{1/\beta}\mathbb{E}\Big[\exp\big(\beta\varrho/(\beta-1)\hat{\Psi}^*\big)\Big]^{(\beta-1)/\beta} \\ &\longrightarrow 0\end{aligned}$$

as $n \to +\infty$ by the Hölder inequality and $\hat{\Psi} \in \mathfrak{E}$. □

If, in addition to the Assumption 5.2.8, the constraints sets are *convex*, then the nearest point operator is a well-defined mapping on $\mathbb{R}^d$. In particular, optimal strategies are unique in the sense that the corresponding stochastic integrals with respect to M are indistinguishable. In this case we have the following stability results.

Theorem 5.4.8. *Let the Assumptions 5.2.2, 5.2.6 and 5.4.3 hold and assume that the $\mathcal{K}^n$ are convex. Then for all $\varrho \geq 1$*

$$\lim_{n\to+\infty}\mathbb{E}\left[\langle(\hat{\nu}^n-\hat{\nu})\cdot M\rangle_T^{\varrho/2}\right]=0.$$

In particular, $(\hat{\nu}^n - \hat{\nu}) \cdot M$ converges to zero in $\mathcal{M}^2$ and hence in the semimartingale topology.

Proof. Using the definitions, it follows that

$$\lim_{n\to+\infty}\mathbb{E}\left[\langle(\hat{\nu}^n-\hat{\nu})\cdot M\rangle_T^{\varrho/2}\right]=0$$

can be derived from

$$\begin{aligned}&\left(\int_0^T\left\|\Gamma^{\frac{1}{2}}\widetilde{\Gamma}^{\frac{1}{2}}\Pi_{\mathrm{cl}\left(B_t\mathcal{K}_t^n\right)}\left(\tfrac{B_t(\lambda^n)_t^o+(B_t^\dagger)^\intercal\check{C}_t\hat{Z}_t^n}{1-p^n}\right)-\Gamma^{\frac{1}{2}}\widetilde{\Gamma}^{\frac{1}{2}}\Pi_{\mathrm{cl}\left(B_t\mathcal{K}_t\right)}\left(\tfrac{B_t\lambda_t^o+(B_t^\dagger)^\intercal\check{C}_t\hat{Z}_t}{1-p}\right)\right\|^2dA_t\right)^{\varrho/2} \\ &\quad\leq\left(\int_0^T\left\|\Pi_{\mathrm{cl}\left(B_t\mathcal{K}_t^n\right)}\left(\tfrac{B_t(\lambda^n)_t^o+(B_t^\dagger)^\intercal\check{C}_t\hat{Z}_t^n}{1-p^n}\right)-\Pi_{\mathrm{cl}\left(B_t\mathcal{K}_t\right)}\left(\tfrac{B_t\lambda_t^o+(B_t^\dagger)^\intercal\check{C}_t\hat{Z}_t}{1-p}\right)\right\|^2dA_t\right)^{\varrho/2}.\end{aligned}$$

To establish the result we proceed similarly to the proof of Lemma 5.4.5, now using Proposition 6.2.3, so that Proposition 4.2.9 (i) then yields the final assertion. □

Theorem 5.4.9. *Let the assumptions of the previous theorem hold and suppose that the conditions on the convergence of the market price of risk processes are as in Remark 5.4.4, then the sequence of processes $\hat{X}^n = X^{\hat{\nu}^n}$, $n \in \mathbb{N}$, converges to $\hat{X} = \hat{X}^{\hat{\nu}}$ in the semimartingale topology.*

Proof. We note the dynamics of the optimal wealth processes given by (5.4.2) and set

$$\Upsilon^n := \hat{\nu}^n \cdot M + \hat{\nu}^n \cdot \langle M \rangle \lambda^n,$$

for $n \in \mathbb{N}_0$. We show the convergence in $\mathcal{H}^2$ of the sequence $(\Upsilon^n)_{n\in\mathbb{N}}$ so that the result of the theorem follows from Proposition 4.2.9 (ii) since $\hat{X}^n = \mathcal{E}(\Upsilon^n)$ and $\hat{X} = \mathcal{E}(\Upsilon^0)$. Observe from Theorem 5.4.8 that $(\hat{\nu}^n - \hat{\nu}) \cdot M$ converges to zero in $\mathcal{M}^2$ so that we need only show the convergence of the finite variation parts, namely that

$$\lim_{n\to+\infty} \mathbb{E}\left[\left(\int_0^T |\, d\big(\langle \hat{\nu}^n \cdot M, \lambda^n \cdot M\rangle - \langle \hat{\nu} \cdot M, \lambda \cdot M\rangle\big)|\right)^2\right] = 0.$$

Adding and subtracting $\langle \hat{\nu} \cdot M, \lambda^n \cdot M\rangle$ and then applying the Kunita-Watanabe inequality, we see that the above holds due to Theorem 5.4.8 together with the assumed convergence of $\langle(\lambda^n - \lambda) \cdot M\rangle_T$ to zero in all $L^\varrho(\mathbb{P})$ spaces. □

6 Appendix

6.1 The Solution for Logarithmic Utility

For completeness we here describe the unconstrained continuous case when $p = 0$. The results are well-known. Our method of finding the optimizers relies on the precise description of the dual domain given in Theorem 4.4.1 and hence leads readily to the results. If it exists, the optimal $\hat{Y}$ is given by

$$\hat{Y} = y\,\mathcal{E}(-\lambda \cdot M + \hat{N}) = y\,\mathcal{E}(-\lambda \cdot M)\,\mathcal{E}(\hat{N}),$$

where $\hat{N}$ is a local martingale orthogonal to M. In the present situation we can explicitly compute $\hat{N}$. Using $\mathbb{E}[\mathcal{E}(N)_T] \leq 1$ for an arbitrary local martingale N orthogonal to M together with Jensen's inequality, we have for $Y = y\,\mathcal{E}(-\lambda \cdot M + N) \in \mathcal{Y}(y)$,

$$\begin{aligned}\mathbb{E}\Big[\widetilde{U}(Y_T)\Big] &= -\mathbb{E}[\log(Y_T)] - 1 = -\mathbb{E}[\log(y\,\mathcal{E}(-\lambda \cdot M)_T)] - \mathbb{E}[\log(\mathcal{E}(N)_T)] - 1 \\ &\geq -\mathbb{E}[\log(y\,\mathcal{E}(-\lambda \cdot M)_T)] - \log(\mathbb{E}[\mathcal{E}(N)_T]) - 1 \\ &\geq -\mathbb{E}[\log(y\,\mathcal{E}(-\lambda \cdot M)_T)] - 1.\end{aligned}$$

This shows that

$$\inf_{Y = y\,\mathcal{E}(-\lambda \cdot M + N)} \mathbb{E}\big[\widetilde{U}(Y_T)\big] \geq -\mathbb{E}[\log(y\,\mathcal{E}(-\lambda \cdot M)_T)] - 1,$$

with equality attained when $N \equiv 0$. We conclude that $\hat{Y} = y\,\mathcal{E}(-\lambda \cdot M)$ by its uniqueness. We necessarily have that $\hat{Y}_T = U'(\hat{X}_T) = 1/\hat{X}_T$ so that $\hat{X}\hat{Y} \equiv \mathbb{E}[\hat{X}_T \hat{Y}_T | \mathcal{F}_\cdot] \equiv 1$. Moreover, we see from

$$\begin{aligned}\exp\Big(\lambda \cdot M + \frac{1}{2}\,\lambda \cdot \langle M \rangle \lambda\Big) &= \frac{y}{\hat{Y}} = y\hat{X} \\ &= \exp\Big(\hat{\nu} \cdot M - \frac{1}{2}\,\hat{\nu} \cdot \langle M \rangle \hat{\nu} + \hat{\nu} \cdot \langle M \rangle \lambda\Big)\end{aligned}$$

that the relation $(\lambda - \hat{\nu}) \cdot M \equiv 0$ holds. We can therefore regard λ as the optimal strategy. In order to complete the picture, we note that

$$\hat{\Psi} = \log\left(\frac{\hat{Y}}{U'(\hat{X})}\right) = \log(\hat{X}\hat{Y}) \equiv 0,$$

so that our BSDE (1.3.1) reduces to the trivial one with solution triple $(\hat{\Psi}, \hat{Z}, \hat{N}) \equiv (\hat{\Psi}, \hat{\nu} - \lambda, \hat{N}) \equiv (0,0,0)$.

To describe the opportunity process in the logarithmic case we observe that the value function is given by

$$u(x) = \log(x) + \mathbb{E}\left[(\lambda \cdot M)_T + \frac{1}{2}\langle \lambda \cdot M\rangle_T\right] = \log(x) + \Lambda_0,$$

where we have set

$$\begin{aligned}\Lambda_t &:= \mathbb{E}\left[\int_t^T \lambda_s\, dM_s + \frac{1}{2}\lambda_s^\intercal\, d\langle M\rangle_s \lambda_s \,\middle|\, \mathcal{F}_t\right] = \mathbb{E}\Big[\log\Big(\hat{X}_T/\hat{X}_t\Big)\,\Big|\,\mathcal{F}_t\Big] \\ &= \mathbb{E}[\log(\hat{X}_T)|\,\mathcal{F}_t] - \log(\hat{X}_t) = \mathbb{E}[U(\hat{X}_T)|\,\mathcal{F}_t] - U(\hat{X}_t),\end{aligned}$$

so Λ is nothing else but the opportunity process for logarithmic utility in the spirit of Nutz [2010b], see also Mania and Tevzadze [2008]. In the present case of a continuous filtration, Λ satisfies the linear BSDE

$$d\Lambda_t = \zeta_t\, dM_t - \frac{1}{2}\lambda_t^\intercal\, d\langle M\rangle_t \lambda_t + dL_t, \quad \Lambda_T = 0,$$

where L is orthogonal to M. This BSDE also appears in Hu et al. [2005], Mania and Tevzadze [2008] and Morlais [2009] and should be regarded as the right object of study when dealing with logarithmic utility.

6.2 Set Valued Analysis

In this appendix we provide the necessary definitions from set valued analysis relevant to the present thesis. We fix a sequence $(\mathcal{J}^n)_{n\in\mathbb{N}}$ of closed nonempty subsets of $\mathbb{R}^d$ and begin with the analogue of lim inf and lim sup for sets, see Aubin and Frankowska [1990].

Definition 6.2.1. The *upper limit* of the sequence $(\mathcal{J}^n)_{n\in\mathbb{N}}$ is the subset

$$\begin{aligned}\operatorname*{Lim\,sup}_{n\to+\infty} \mathcal{J}^n &:= \left\{x \in \mathbb{R}^d \,\middle|\, \liminf_{n\to+\infty} \operatorname{dist}(x, \mathcal{J}^n) = 0\right\} \\ &= \left\{x \in \mathbb{R}^d \,\middle|\, x \text{ a cluster point of an } (x_n)_{n\in\mathbb{N}},\ x_n \in \mathcal{J}^n \text{ for all } n \in \mathbb{N}\right\},\end{aligned}$$

where dist denotes the usual distance function from a set in $\mathbb{R}^d$. Similarly, the *lower limit* of the sequence $(\mathcal{J}^n)_{n\in\mathbb{N}}$ is the subset

$$\begin{aligned}\operatorname*{Lim\,inf}_{n\to+\infty} \mathcal{J}^n &:= \left\{x \in \mathbb{R}^d \,\middle|\, \lim_{n\to+\infty} \operatorname{dist}(x, \mathcal{J}^n) = 0\right\} \\ &= \left\{x \in \mathbb{R}^d \,\middle|\, x = \lim_{n\to+\infty} x_n,\ \text{where } x_n \in \mathcal{J}^n \text{ for all } n \in \mathbb{N}\right\}.\end{aligned}$$

A set $\mathcal{J}$ is called the *closed set limit* of the sequence $(\mathcal{J}^n)_{n\in\mathbb{N}}$ if the upper and lower limit sets coincide, i.e.

$$\mathcal{J} = \operatorname*{Lim\,sup}_{n\to+\infty} \mathcal{J}^n = \operatorname*{Lim\,inf}_{n\to+\infty} \mathcal{J}^n,$$

in which case we write $\mathcal{J} = \mathrm{Lim}_{n\to+\infty} \mathcal{J}^n$.

We note that if $(\mathcal{J}^n)_{n\in\mathbb{N}}$ is a sequence of (convex) predictably measurable multivalued mappings then $\mathcal{J} = \mathrm{Lim}_{n\to+\infty} \mathcal{J}^n$ is a (convex) predictably measurable multivalued mapping as well, see Proposition 6.2.2 and Rockafellar [1976] Proposition 1A.

The following proposition shows that the above notion of set convergence implies pointwise convergence of the distance functions. Actually, it is *equivalent* to the latter. This motivates the choice of the above set convergence, which is often called Kuratowski convergence in the literature, as the appropriate notion of convergence of sets.

Proposition 6.2.2 (Beer [1987] Lemma 2.0 and Theorem 2.3)**.** *The following are equivalent:*

(i) The sequence $(\mathcal{J}^n)_{n\in\mathbb{N}}$ of closed nonempty sets converges to $\mathcal{J}$, i.e.

$$\mathcal{J} = \operatorname*{Lim}_{n\to+\infty} \mathcal{J}^n.$$

(ii) The sequence $(\mathrm{dist}(\cdot, \mathcal{J}^n))_{n\in\mathbb{N}}$ of functions converges pointwise to $\mathrm{dist}(\cdot, \mathcal{J})$.

In the case that the sets $\mathcal{J}^n$, $n \in \mathbb{N}$, are also convex, we derive from Schochetman and Smith [1992] Theorem 3.3 that the above set convergence is also equivalent to pointwise convergence of the nearest point operators. We state the claim that is important to our study.

Proposition 6.2.3 (Schochetman and Smith [1992] Theorem 3.2)**.** *Let Π stand for the nearest point operator onto the indicated (closed and convex) set. If the sequence $(\mathcal{J}^n)_{n\in\mathbb{N}}$ of closed and convex nonempty sets has a set limit denoted by $\mathcal{J}$ then the sequence $(\Pi_{\mathcal{J}^n})_{n\in\mathbb{N}}$ of mappings converges pointwise on $\mathbb{R}^d$ to $\Pi_{\mathcal{J}}$.*

We include a different and direct proof for the convenience of the reader.

Proof. Let $z \in \mathbb{R}^d$ and $x \in \mathcal{J} = \mathrm{Lim}_{n\to+\infty} \mathcal{J}^n$, i.e. there exist $x_n \in \mathcal{J}^n$ such that $x = \lim_{n\to+\infty} x_n$. We have that $\langle z - \Pi_{\mathcal{J}^n}(z), \Pi_{\mathcal{J}^n}(z) - x_n\rangle \geq 0$ due to the characterization of the nearest point operator. It follows that

$$\begin{aligned} 0 &\leq \langle z - \Pi_{\mathcal{J}^n}(z), \Pi_{\mathcal{J}^n}(z) - z\rangle + \langle z - \Pi_{\mathcal{J}^n}(z), z - x_n\rangle \\ &\leq -\|z - \Pi_{\mathcal{J}^n}(z)\|^2 + \|z - \Pi_{\mathcal{J}^n}(z)\| \cdot \|z - x_n\|, \end{aligned}$$

so that

$$\big|\|z\| - \|\Pi_{\mathcal{J}^n}(z)\|\big|^2 \leq \big(\|z\| + \|\Pi_{\mathcal{J}^n}(z)\|\big) \cdot \|z - x_n\|. \tag{6.2.1}$$

We argue that the sequence $(\Pi_{\mathcal{J}^n}(z))_{n\in\mathbb{N}}$ is bounded. If not, there would be a subsequence of it with norm tending towards $+\infty$. This would contradict (6.2.1) since, due to the convergence of $\|z - x_n\|$, the right hand side would tend to $+\infty$ in first order while the left hand side would go to $+\infty$ in second order. We therefore can choose a subsequence $(\Pi_{\mathcal{J}^{n_k}}(z))_{k\in\mathbb{N}}$ of $(\Pi_{\mathcal{J}^n}(z))_{n\in\mathbb{N}}$ which converges to some z_0. By our assumption on the set convergence, $z_0 \in \mathcal{J}$. Moreover, we have that

$$\|z_0 - z\| = \lim_{k\to+\infty} \|\Pi_{\mathcal{J}^{n_k}}(z) - z\| \leq \lim_{k\to+\infty} \|x_{n_k} - z\| = \|x - z\|,$$

i.e. $\|z_0 - z\| \leq \|x - z\|$. Since $x \in \mathcal{J}$ was arbitrary, we conclude that $\|z_0 - z\| = \min_{x\in\mathcal{J}} \|x - z\|$ from which $z_0 = \Pi_{\mathcal{J}}(z)$. In fact, the above reasoning shows that *any* cluster point of the sequence $(\Pi_{\mathcal{J}^n}(z))_{n\in\mathbb{N}}$ coincides with $z_0 = \Pi_{\mathcal{J}}(z)$, which shows that there is exactly one accumulation point of this bounded sequence, hence $\Pi_{\mathcal{J}}(z) = z_0 = \lim_{n\to+\infty} \Pi_{\mathcal{J}^n}(z)$. □

The final proposition shows that the alternative assumption given in Remark 4.2.19 and used in Kardaras [2010] also leads to the appropriate convergence of the projections.

Proposition 6.2.4. *Let the sequence $(\mathcal{J}^n)_{n\in\mathbb{N}}$ of closed and convex nonempty sets have a set limit denoted by $\mathcal{J}$ and suppose that Q is a $d \times d$ matrix such that* $\mathrm{Ker}(Q) \subseteq \mathcal{J}^n$ *for all $n \in \mathbb{N}$ and* $\mathrm{Ker}(Q) \subseteq \mathcal{J}$. *Then $Q\mathcal{J} = \mathrm{Lim}_{n\to+\infty} Q\mathcal{J}^n$.*

Proof. We must show that

$$Q\mathcal{J} \subseteq \operatorname*{Lim\,inf}_{n\to+\infty} Q\mathcal{J}^n \subseteq \operatorname*{Lim\,sup}_{n\to+\infty} Q\mathcal{J}^n \subseteq Q\mathcal{J}.$$

The first containment is an easy consequence of the definitions and we omit the details. Since one always has $\mathrm{Lim\,inf}_{n\to+\infty} Q\mathcal{J}^n \subseteq \mathrm{Lim\,sup}_{n\to+\infty} Q\mathcal{J}^n$ we need only prove the final containment.

Let $y \in \mathrm{Lim\,sup}_{n\to+\infty} Q\mathcal{J}^n$, this means we may find sequences $(y_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ for which $y_n = Qx_n$ and $x_n \in \mathcal{J}^n$ for all $n \in \mathbb{N}$ and such that $(y_{n_k})_{k\in\mathbb{N}}$ converges to y for a subsequence $(n_k)_{k\in\mathbb{N}}$. We must show that we can construct x with $x \in \mathcal{J}$ and $Qx = y$. For each $k \in \mathbb{N}$ we may decompose x_{n_k} uniquely as $x_{n_k} = x^1_{n_k} + x^2_{n_k}$ with $x^1_{n_k} \in \mathrm{Ker}(Q)$ and $x^2_{n_k} \in \mathrm{Ker}(Q)^\perp$. From the assumption $\mathrm{Ker}(Q) \subset \mathcal{J}^n$ we see that for all $\varepsilon \in (0,1)$, $\frac{-(1-\varepsilon)}{\varepsilon} x^1_{n_k} \in \mathcal{J}^{n_k}$ so that

$$(1-\varepsilon)x^2_{n_k} = \varepsilon \tfrac{-(1-\varepsilon)}{\varepsilon} x^1_{n_k} + (1-\varepsilon)x_{n_k} \in \mathcal{J}^{n_k}$$

by convexity. Since each $\mathcal{J}^{n_k}$ is also closed, letting ε tend to zero we see $x^2_{n_k} \in \mathcal{J}^{n_k}$. From the above construction it follows that $x^2_{n_k} = Q^\dagger Q x_{n_k}$, where $Q^\dagger$ is the Moore-Penrose pseudoinverse of Q. Define now the vector $x := Q^\dagger y$, then we have $x = \lim_{k\to+\infty} x^2_{n_k}$ since

$$\|x^2_{n_k} - x\| \leq \|Q^\dagger\| \cdot \|Qx^2_{n_k} - y\| = \|Q^\dagger\| \cdot \|Qx_{n_k} - y\| = \|Q^\dagger\| \cdot \|y_{n_k} - y\|,$$

where the right hand side tends to zero. We derive that $x \in \mathrm{Lim\,sup}_{n\to+\infty} \mathcal{J}^n = \mathcal{J}$ and $y = \lim_{k\to+\infty} Qx_{n_k} = \lim_{k\to+\infty} Qx^2_{n_k} = Qx$, hence $y \in Q\mathcal{J}$. □

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List of Symbols

A	predictable increasing process that encodes the quadratic variation of M
A_ϱ	Muckenhoupt inequality
α, α^n	process or random variable
$\lvert\alpha\rvert_1$	integral of α over $[0,T]$ with respect to A, $\lvert\alpha\rvert_1 := \int_0^T \alpha_t \, dA_t$
$\mathcal{A}, \mathcal{A}_\mathcal{K}, \mathcal{A}_\mathcal{K}^\mathcal{G}$	families of admissible strategies
$b, b(\overline{M}), b(\lambda \cdot M)$	critical exponent, see Definition (2.5.3)
B	square root of C, $C = B^\intercal B$, $B = \Gamma^{1/2} P$
$B^\dagger$	Moore-Penrose pseudoinverse of B
$\breve{B}$	square root of $\breve{C}$, $\breve{C} = \breve{B}^\intercal \breve{B}$, $\breve{B} = \breve{\Gamma}^{1/2} \breve{P}$
β, β^n	process or constant
$\overline{\beta}, \beta_f, \beta^*$	constants
$\langle M \rangle$	quadratic variation of M
$\langle M, N \rangle$	quadratic covariation of M and N
C	covariation matrix of M, $\langle M \rangle = C \cdot A$
$\breve{C}$	covariation matrix of M^o, $\langle M^o \rangle = \breve{C} \cdot A$
$c, c_p, \widetilde{c}_p, c_{rH,p}, c_0, c_A$	
$c_{p,\gamma}, c_{p,\gamma}, c_{\varrho,\gamma}, c_\lambda, c^\dagger$	constants
cl	closure operator
$\cdot$	stochastic integration
d	dimension
D	$\mathcal{F}_T$-measurable random variable, discount, tax rate or bonus
dist	distance function from a set
ΔN	jump part of N
E	predictable subset of $[0,T] \times \Omega$
ess inf	essential infimum
ess sup	essential supremum
$\mathfrak{E}$	space of processes whose supremum has all exponential moments
$\mathcal{E}$	stochastic exponential
F, F^n, f, g	drivers of BSDEs
$\mathcal{F}, \mathcal{F}_t, \mathcal{F}_t^R, \mathcal{F}_t^S, \mathcal{G}_t, \mathcal{H}_t$	σ-algebras (items of filtrations)
Γ	diagonal matrix of eigenvalues of C
$\widetilde{\Gamma}$	pseudoinverse of Γ
$\mathcal{K}, \mathcal{K}^n$	constraints set

$\mathcal{K}^\circ$	polar cone
$K^j, j = 1, \dots, m$	rays generating the cone $\mathcal{K}$
K_A	bound on A
k_q	critical value, see (2.5.6)
$\hat{\kappa}$	integrand in the decomposition $\hat{Y}^1 = \mathcal{E}(-\hat{\kappa} \cdot M + \hat{N})$
$\breve{\kappa}^\nu$	adjustment for partial information, see (5.2.7)
Ker	kernel of a linear map (preimage of zero)
Im	image of a linear map
inf	infimum
$\mathbf{1}$	indicator function
L	opportunity process
$\widetilde{L}$	dual opportunity process
$\mathcal{L}$	constraints set in the additive formulation
$L^0, L^0(\mathbb{P})$	space of measurable random variables
$L^1, L^1(\mathbb{P})$	space of integrable random variables
$L^\varrho, L^\varrho(\mathbb{P})$	space of ϱ-integrable random variables
L^1_+	family of nonnegative integrable random variables
L^2	space of square-integrable random variables/processes
L^∞	space of bounded random variables
λ	predictable M-integrable process, market price of risk
λ^o	optional projection of λ
$\lambda^{\mathcal{G}}, \lambda^{\mathcal{H}}$	predictable projection onto the indicated filtration
$\cdot \vee \cdot$	maximum/supremum
$\cdot \wedge \cdot$	minimum/infimum
M	fixed continuous local martingale
M^o	optional projection of M
$\overline{M}$	generic continuous (local) martingale
$\mathcal{M}^\varrho$	ϱ-integrable continuous local martingales
$\mathfrak{M}^\varrho$	predictable ϱ-integrable processes
μ^A	Doléans measure associated to M via A
N	continuous local martingale orthogonal to M
N^c	continuous part of N
$\hat{N}$	continuous local martingale orthogonal to M in the decomposition of $\hat{Y}$, $\hat{Y}^1 = \mathcal{E}(-\hat{\kappa} \cdot M + \hat{N})$
$\mathbb{N}$	the positive integers, $\mathbb{N} = \{1, 2, 3, \dots\}$
$\mathbb{N}_0$	the nonnegative integers, $\mathbb{N} = \{0, 1, 2, 3, \dots\}$
$\lVert \cdot \rVert$	Euclidean norm in $\mathbb{R}^d$
$\lVert \cdot \rVert_{\mathrm{BMO}_2}$	BMO_2 norm, see (2.2.6)
$\lVert \cdot \rVert_{L^\infty}$	L^∞ norm
ν	trading strategy, proportion of wealth
$\hat{\nu}$	optimal trading strategy
$(\Omega, \mathcal{F}, \mathbb{P})$	probability space
p	investor's relative risk avesion, $p \in (-\infty, 1)$

P	matrix of eigenvectors of C, $C = P^\intercal \Gamma P$
Π	nearest point operator
$\mathbb{P}^n, \widetilde{\mathbb{P}}$	probability measures equivalent to $\mathbb{P}$
$\mathcal{P}$	predictable σ-algebra on $[0,T] \times \Omega$
Φ	standard normal cumulative distribution function
Ψ, Ψ^n	first component of a BSDE solution triple
$\hat{\Psi}$	opportunity process, $\hat{\Psi} = \log(\hat{Y}/U'(\hat{X}))$
q	dual number to p, $q := p/(p-1)$
R	stock returns process
ϱ	generic constant
R_ϱ	reverse Hölder inequality
$\mathbb{R}$	the real numbers
S	stock price vector
$\mathcal{S}^\varrho$	continuous processes with ϱ-integrable supremum
$\mathcal{S}^\infty$	bounded continuous processes
sgn	sign function
sup	supremum
σ	stopping time
$\langle x, y \rangle$	scalar product of $x, y \in \mathbb{R}^d$
T	positive time horizon, $t \in [0,T]$
$^\intercal$	transposition
$\tau, \widetilde{\tau}, \tau_n$	stopping times
$u, u^{\mathcal{G}}$	primal value function/process
U, U^n	utility functions, mostly of power type
$\widetilde{u}$	dual value function
$\widetilde{U}$	convex dual to U, $\widetilde{U}(y) = \sup_{x>0}(U(x) - xy)$
W	Brownian motion
$\widetilde{W}$	Brownian motion under $\widetilde{\mathbb{P}}$
x	real number, mostly positive, initial value of an X
$X, X^\nu, X^{x,\nu}$	wealth processes
$\hat{X}$	optimal wealth process
ξ, ξ^n	random variable, terminal condition of a BSDE
$\mathcal{X}(x), \mathcal{X}^{add}(x)$	family of admissible wealth processes
y	(positive) real number, initial value of a dual variable Y
Y, Y^y, Y^1	dual variables
$\hat{Y}$	optimal dual variable
Y^λ	density of the minimal martingale measure, $Y^\lambda = \mathcal{E}(-\lambda \cdot M)$
$\mathcal{Y}(y), \mathcal{Y}^{add}(y)$	dual domains
Υ	generic stochastic process
Υ^*	supremum of a process Υ over $[0,T]$
Z, Z^n	control processes in BSDE solution triples
$\hat{Z}$	optimal control process in the BSDE triple involving $\hat{\Psi}$

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Selbstständigkeitserklärung

Ich versichere, dass ich die vorliegende Arbeit selbstständig und ohne unerlaubte Hilfe angefertigt habe.

Berlin, den 13. Juli 2011 Markus Severin Mocha