Rigorous derivation of two-scale and effective damage models based on microstructure evolution

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Abstract

The reasons for the sudden complete failure of a material specimen stressed by external forces are often long-termed, gradual microscopic material changes. The formation of microscopic cracks or voids, for instance, are included in such material changes. In comparison to the expansion of the specimen, the number and the increasingly small size of the microscopic material changes lead to a high level of detail which makes the accurate simulation of such damage processes simply impossible. Therefore, reliance is placed on so-called effective models when performing a simulation. These effective models sufficiently mirror the macroscopic behavior of the specimen without, however, considering each microscopic material change.

The dissertation at hand deals with the rigorous derivation of such effective models used to describe damage processes. For different rate-independent damage processes in linear elastic material these effective models are derived as the asymptotic limit of microscopic models. The starting point is represented by a unidirectional microstructure evolution model which is based on a family of ordered admissible microstructures. Each microstructure of that family possesses the same intrinsic length scale $\varepsilon > 0$. To derive an effective model, the limit passage $\varepsilon \to 0$ is performed with the help of techniques of the two-scale convergence. For this purpose, a microstructure-regularizing term, which can be understood as a discrete gradient for piecewise constant functions, is needed to identify the limit model. The microstructure of the effective model is given pointwisely by a so-called unit cell problem which separates the microscopic scale from the macroscopic scale.

Based on these homogenization results for unidirectional microstructure evolution models, an effective model for a brutal damage process is provided. Here, the microstructure consists of only two phases, namely undamaged material which comprises inclusions of damaged material with various sizes and shapes. The size of the inclusions is scaled by $\varepsilon > 0$ and the unidirectional microstructure evolution prevents that, for fixed $\varepsilon > 0$, the inclusions shrink for progressing time. According to the unit cell problem, the material of the limit model is then given as a mixture of damaged and undamaged material. In a specific material point of the limit model, that unit cell problem does not only define the mixture ratio but also the exact geometrical mixture distribution.

Then, as a generalization, brutal damage processes are investigated, which allow the modeling of voids instead of inclusions of damaged material. This means that these considered microscopic defects do not contain any material. Despite the lack of stiffness on the set of all voids, the applicability of the provided homogenization theory is guaranteed by appropriate continuation operators. By virtue of the unit cell problem, the effective model describes a material free from voids. Compared to the ε-models, the missing defects in the limit model are taken into account due to the lower stiffness of the effective material.

Last but not least, for an evolution process describing the growth of microscopic cracks, a macroscopic crack-free model can be derived. In correspondence to the brutal damage model for voids, the effective model describes a crack-free material. In this case, in every material point of the limit model, the unit cell problem models a cracked unit cell. This crack geometry is uniquely determined and varies from point to point.

Zusammenfassung

Die Ursachen für das plötzliche totale Versagen einer durch äußere Kräfte beanspruchten Materialprobe sind häufig langfristige, schleichende mikroskopische Materialveränderungen. Zu solchen Materialveränderungen gehören zum Beispiel die Bildung von mikroskopischen Rissen oder Hohlräumen. Die Anzahl und die verschwindend geringe Größe der mikroskopischen Materialveränderungen im Vergleich zur Ausdehnung der Probe führen zu einem immensen Detailgrad, der die realitätsgetreue numerische Simulation solcher Schädigungsprozesse schlicht unmöglich macht. Daher verlässt man sich bei der Simulation auf sogenannte effektive Modelle. Diese effektiven Modelle spiegeln das makroskopische Verhalten der Probe hinreichend gut wider, ohne allerdings jede mikroskopische Materialveränderung zu berücksichtigen.

Die vorliegende Dissertation beschäftigt sich mit der rigorosen Herleitung solcher effektiven Modelle zur Beschreibung von Schädigungsprozessen. Diese effektiven Modelle werden für verschiedene raten-unabhängige Schädigungsmodelle linear elastischer Materialien hergeleitet. Den Ausgangspunkt stellt dabei ein unidirektionales Mikrostrukturevolutionsmodell dar, dessen Fundament eine Familie geordneter zulässiger Mikrostrukturen bildet. Jede Mikrostruktur dieser Familie besitzt die gleiche intrinsische Längenskala $\varepsilon > 0$. Zur Herleitung eines effektiven Modells wird der Grenzübergang $\varepsilon \to 0$ mittels Techniken aus der Theorie der Zwei-Skalen-Konvergenz durchgeführt. Um das Grenzmodell zu identifizieren, bedarf es eines mikrostruktur-regularisierenden Terms, welcher als diskreter Gradient für stückweise konstante Funktionen aufgefasst werden kann. Die Mikrostruktur des effektiven Modells ist punktweise durch ein sogenanntes Einheitszellenproblem gegeben, welches die Mikro- von der Makroskala trennt.

Ausgehend vom Homogenisierungsresultat für die unidirektionale Mikrostrukturevolution wird ein effektives Modell für einen Zwei-Phasen-Schädungsprozess hergeleitet. In diesem Fall setzt sich die Mikrostruktur aus lediglich zwei Phasen zusammen, und zwar aus ungeschädigtem Material, welches Inklusionen geschädigten Materials verschiedener Form und Größe enthält. Die Größe der Inklusionen wird mit $\varepsilon > 0$ skaliert und die unidirektionale Mikrostrukturevolution verhindert, dass bei fixiertem $\varepsilon > 0$ die Inklusionen für fortlaufende Zeit schrumpfen. Das Material des Grenzmodells ist dann in jedem Punkt als Mischung von ungeschädigtem und geschädigtem Material durch das Einheitszellenproblem gegeben. Dabei liefert das Einheitszellenproblem nicht nur das Mischungsverhältnis sondern auch die genaue geometrische Mischungsverteilung, die dem effektiven Material des jeweiligen Materialpunktes zugrunde liegt.

Als eine Verallgemeinerung werden anschließend Zwei-Phasen-Schädigungsprozesse betrachtet, die statt (der Inklusionen aus) geschädigtem Material echte Defekte zulassen, d.h. echte mikroskopische Hohlräume, die keinerlei Material enthalten. Trotz der nicht vorhandenen Steifigkeit auf der Menge aller Hohlräume wird mittels geeigneter Fortsetzungsoperatoren die Anwendbarkeit der Homogenisierungstheorie sichergestellt. Vermöge des Einheitszellenproblems beschreibt das Grenzmodell ein defektfreies Material. Den im Vergleich zu den ε-Modelle fehlenden Defekten wird durch die geringere Steifigkeit des effektiven Materials Rechnung getragen.

Abschließend wird ein effektives Modell für die Evolution mikroskopischer Risse hergeleitet. In Analogie zum Zwei-Phasen-Schädigungsmodell für Defekte beschreibt das Grenzmodell ein rissfreies Material, welches in jedem Punkt durch das Einheitszellenproblem definiert ist. Dieses Einheitszellenproblem modelliert in jedem Punkt des effektiven Modells eine gerissene Einheitszelle, wobei die Rissgeometrie eindeutig bestimmt ist und von Punkt zu Punkt variiert.

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Contents

Notation and conventions

General notation

- $\mathbb{N}$:= set of natural numbers
- $\mathbb{Z}$:= set of integers
- $\mathbb{R}$:= set of real numbers
- $\mathbb{R}^{d\times d}_{\mathrm{sym}} := \{\xi \in \mathbb{R}^{d\times d} \,|\, \xi = \xi^T\}$ for $d \in \mathbb{N}$
- $\mathbf{0} := (\underbrace{0, 0, \dots, 0}_{k\text{-components}})$, where $k \in \mathbb{N}$ depends on and follows from the context
- $\mathbf{1} := (\underbrace{1, 1, \dots, 1}_{k\text{-components}})$, where $k \in \mathbb{N}$ depends on and follows from the context
- $\mathbb{O}$:= zero element of the set $\mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}}; \mathbb{R}^{d\times d}_{\mathrm{sym}})$. The latter contains all linear, symmetric, and continuous mappings $\mathbb{C} : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$.

Functional analysis

For $d \in \mathbb{N}$ let $\mathcal{O}$ denote an element of the Lebesgue-σ-algebra $\mathcal{L}_{\mathrm{Leb}}(\mathbb{R}^d)$. Moreover, let $\Gamma \subset \partial\mathcal{O}$ with $\mu_{d-1}(\Gamma) > 0$, where $\mu_{d-1} : \mathcal{L}_{\mathrm{Leb}}(\mathbb{R}^{d-1}) \to [0, \infty]$ denotes the Lebesgue measure. Finally, let $p \in [1, \infty)$.

- $\mathbb{1}_{\mathcal{O}} : \mathbb{R}^d \to \{0, 1\}$, where $\mathbb{1}_{\mathcal{O}}(x) = 1$ if $x \in \mathcal{O}$ and $\mathbb{1}_{\mathcal{O}}(x) = 0$ if $x \in \mathbb{R}^d \backslash \mathcal{O}$
- $\mathrm{L}^p(\mathcal{O}) := \Big\{ v : \mathcal{O} \to \mathbb{R} \,\Big|\, v \text{ is measurable and } \int_{\mathcal{O}} |v(x)|^p \mathrm{d}x < \infty \Big\}$
- $\mathrm{L}^\infty(\mathcal{O}) := \Big\{ v : \mathcal{O} \to \mathbb{R} \,\Big|\, v \text{ is measurable, } \exists\, C > 0 \text{ such that } |v| \le C \text{ a.e. on } \mathcal{O} \Big\}$
- $\mathcal{M}(\mathcal{O}) := \mathrm{L}^\infty(\mathcal{O}; \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}}; \mathbb{R}^{d\times d}_{\mathrm{sym}}))$
- $\mathrm{K}_{\varepsilon\Lambda}(\mathcal{O})$:= functions being piecewise constant w.r.t. the lattice $\varepsilon\Lambda$; see (2.5)
- $\mathrm{W}^{1,p}(\mathcal{O}) := \Big\{ v \in \mathrm{L}^p(\mathcal{O}) \,\Big|\, \text{the weak derivative } \nabla v \text{ is an element of } \mathrm{L}^p(\mathcal{O})^d \Big\}$
- $\mathrm{H}^1(\mathcal{O}) := \mathrm{W}^{1,2}(\mathcal{O})$
- $\mathrm{H}^1_\Gamma(\mathcal{O}) := \Big\{ v \in \mathrm{H}^1(\mathcal{O}) \,\Big|\, \text{the trace } v|_\Gamma \text{ satisfies } v|_\Gamma = 0 \Big\}$
- $\mathrm{BV}_{\mathcal{D}}$:= functions of bounded total dissipation; see Definition 5.3
- $(\cdot)^{\mathrm{ex}} : \begin{cases} \mathrm{L}^p(\mathcal{O}) \to \mathrm{L}^p(\mathbb{R}^d), \\ \quad v \mapsto v^{\mathrm{ex}}, \end{cases}$ where $v^{\mathrm{ex}} := \begin{cases} v & \text{on } \mathcal{O}, \\ 0 & \text{otherwise.} \end{cases}$
- $\langle \cdot, \cdot \rangle : \mathcal{Q}^* \times \mathcal{Q} \to \mathbb{R}$ dual pairing of the Banach space $\mathcal{Q}$ and its dual space $\mathcal{Q}^*$ (Note that the specific choice of $\mathcal{Q}$ depends on the context.)

Definition of sets

In the following for $d \in \mathbb{N}$ the symbols A and A' denote subsets of $\mathbb{R}^d$.

- $B_R(x) := \Big\{ x' \in \mathbb{R}^d \,\Big|\, |x - x'|_d < R \Big\}$ for $x \in \mathbb{R}^d$ and $R > 0$
- $\mathrm{int}(A) := \Big\{ x \in A \,\Big|\, \exists\, R > 0 \text{ such that } B_R(x) \subset A \Big\}$
- $\mathrm{dist}(A, A') := \inf \Big\{ |x - x'|_d \,\Big|\, x \in A \text{ and } x' \in A' \Big\}$
- $\mathrm{cl}(A) := \Big\{ x \in \mathbb{R}^d \,\Big|\, \mathrm{dist}(x, A) = 0 \Big\}$
- $\partial A := \mathrm{cl}(A) \backslash \mathrm{int}(A)$
- $\mathrm{neigh}_\Delta(A) := \Big\{ x \in \mathbb{R}^d \,\Big|\, \exists\, x' \in A \text{ such that } x \in \mathrm{cl}(B_\Delta(x')) \Big\}$ for $\Delta > 0$
- $\mathrm{Im}(T) := \Big\{ x \in \mathbb{R}^k \,\Big|\, \exists\, x' \in A \text{ such that } T(x') = x \Big\}$ for $k \in \mathbb{N}$ and $T : A \to \mathbb{R}^k$

Conventions

Usage of the term *damage variable*

In the present work, damage of a body represented by a set $\Omega \in \mathcal{L}_{\mathrm{Leb}}(\mathbb{R}^d)$ initiating at time 0 and progressing until a given time $T > 0$ is captured by a vector valued variable $z : [0, T] \times \Omega \to \mathbb{R}^m$; the so-called *damage variable*. Since it is convenient to introduce most of the terms involving the evolution models for a variable $z : \Omega \to \mathbb{R}^m$ (also referred to as the damage variable) being independent of $t \in [0, T]$, there will be some inconsistency in the use of the term *damage variable*.

The relation $v \leq w$ for $v, w : \Omega \to \mathbb{R}^m$

For $v = (v_1, v_2, \dots, v_m)^T, w = (w_1, w_2, \dots, w_m)^T : \Omega \to \mathbb{R}^m$ the relation $v \leq w$ is defined as follows:

$$v \leq w \quad \overset{\text{def}}{\Longleftrightarrow} \quad v_j(x) \leq w_j(x) \quad \text{for every } x \in \Omega \text{ and all } j \in \{1, 2, \dots, m\}.$$

In the case $v, w \in \mathrm{L}^1(\Omega)^m$ the term *for every* is replaced by *for almost every*.

Neglecting the restriction of functions to shorten notation

For $\mathcal{O}, \mathcal{O}' \in \mathcal{L}_{\mathrm{Leb}}(\mathbb{R}^d)$ with $\mathcal{O}' \subset \mathcal{O}$ and $v \in \mathrm{L}^1(\mathcal{O})$ the term $\int_{\mathcal{O}'} v(x) \mathrm{d}x$ has to be understood as $\int_{\mathcal{O}'} v'(x) \mathrm{d}x$, where $v' := v|_{\mathcal{O}'} \in \mathrm{L}^1(\mathcal{O}')$. For $w \in \mathrm{L}^p(\mathcal{O})$ with $p \in [1, \infty]$, this "implies" in particular the notation

$$\big\| w \big\|_{\mathrm{L}^p(\mathcal{O}')} \overset{\text{not.}}{:=} \big\| w|_{\mathcal{O}'} \big\|_{\mathrm{L}^p(\mathcal{O}')}.$$

1 Introduction

Damage, in the sense of a decreasing durability or functionality, appears in almost every object of our everyday life. In most of the cases its beginning is not noticeable, but in the worst case scenario damage might result in catastrophic failure. Depending on the use of a particular structure, damage progression may cause danger for life or economic losses. Hence, there is a great interest in understanding such processes. The aim of scientific investigations is the reliable prediction of damage progression in a given structure . Due to such predictions, the design, functionality, or material of a specific structure can be optimized to increase its lifetime or to decrease the amount of used resources for its construction.

Depending on the material of a structure and its scope of use, damage can be the result of various effects, e.g., external loadings applied on the structure, temperature changes, chemical reactions, or phase separation. In general, damage results from a combination of multiple of these effects, which indicates the complexity of modeling damage realistically. Many damage processes have in common that they start by initiating microscopic cracks or voids. This occurs on very small scales compared to the expansion of the structure under consideration. For example, in concrete or rocks typical defects have a size of $1-10cm$, whereas in metals, alloys, or ceramics the microscopic defect size varies between $1\mu m$ and $10\mu m$. The growth of these defects, with respect to time, causes a decrease of the durability of the considered structure which might result in its total failure after a certain time. Thus, describing such a process realistically means that all involved effects need to be modeled on all appearing scales. In general, this is much too complex to be able to derive reasonable results by an appropriate effort. Therefore, most models in the theory of damage focus on partial aspects of a damage process.

To indicate the complexity of the damage theory, let us list some of the available results in the literature. The foundation of nowadays continuum damage mechanics has been built in [32, 33, 34] and [65], where the authors model the creep fracture of metals with the help of an internal variable. In [22], the idea of modeling microscopic interactions by including the gradient of this internal variable is presented. In the context of modeling crack propagation, [25] introduces the *Griffith criterion* to decide whether or not a crack propagates under given forces.

In the framework of continuum damage mechanics the dependence of macroscopic models on the internal damage variable is phenomenologically motivated, in most of the cases. Such models are considered in the following references: Concerning the existence of solutions, [55, 61] present results for rate-independent damage processes of nonlinear elastic materials in the small strain setting. In general, these solutions are not continuous with respect to time. For this reason, a novel formulation for such rate-independent damage models is introduced in [44], which provides a better description of the behavior of solutions at jumps. At jump points of a solution, this formulation determines the interplay of viscous and rate-independent effects. In the complete damage case [5, 53] yield existence of solutions for elastic and viscoelastic materials and small strains, whereas [58] is devoted to the finite-strain setting.

In [28], a phasefield model of Cahn–Hilliard type is coupled with a rate-dependent damage model and its main result provides existence of solutions in the small strain setting. This result is improved in [29, 30] by allowing also the elastic energy to depend on the phasefield variable and by enabling the consideration of logarithmic chemical energy densities.

In the context of modeling the propagation of a single crack in a bulk, [38, 40] depict the derivation of Griffith formulas for linear elastic materials. Moreover, [41] presents a rigorously derived Griffith formula for geometrical nonlinear elastic models in the quasistatic setting. Based on energy minimization, the authors of [15] provide an existence result for the quasistatic crack propagation in brittle materials. Considering the rate-independent evolution of such a crack, [42] compares solutions based on a local energy release rate criterion with those based on such a global stability condition. It turns out that the solutions based on the global stability condition tend to jump earlier than those based on the local energy release rate criterion.

As already mentioned above, in most of the cases the dependence of macroscopic damage or crack models on the internal variable is phenomenologically motivated. To improve such relations, the asymptotic behavior of such microscopic relations is investigated rigorously. In the case of static periodically distributed cracks and defects, explicit formulas for the effective bulk and surface energy densities are derived in [19]. In [66], these formulas are supplemented by homogenization results for different scalings of the microscopic surface energy and the succeeding work [67] improves these results by incorporating the non-interpenetration constraint. Among other things, the very recent paper [12] presents a homogenization result for static periodically distributed cracks satisfying the non-interpenetration constraint with the help of the *unfolding technique*. In view of rigorously motivated macroscopic models, the series [20, 21, 24] establishes a model for a quasistatic brittle damage process. There, the damage evolution of a two-phase system is considered where the growing phase models damaged material. The effective material of the macroscopic model is a mixture of the two phases and is determined with the help of the so-called *G-convergence*; see [68].

Concerning numerical simulations, the works below take place in the context of macroscopic damage evolution models. In [35, 36, 37], a two-scale evolution model is treated numerically. There, the evolution of ellipsoidal inclusions of damaged material is modeled on the microscopic scale, whereas the macroscopic quantities are obtained by averaging the microscopic ones. In contrast to this approach, in [16] a macroscopic model describing crack propagation is presented. There, the constitutive relation for the evolution model is based on a homogenization result for static periodically distributed cracks.

The thesis at hand contributes to the topic of rigorously deriving effective models by investigating the asymptotic behavior of microscopic damage progression and crack propagation models. In contrast to the macroscopic descriptions used for numerical simulations mentioned above, the here presented effective constitutive relations are rigorously derived from microscopic damage evolution models. To our knowledge, in the context of evolution models for damage there are no results which are rigorously derived and provide such an explicit structure in the literature so far. Here, we exploit Γ-convergence and unfolding techniques to derive effective damage models for the growth of microscopic defects or voids as well as effective crack models in the rate-independent setting.

1.1 Outline of this thesis

In the context of presenting the outline of this thesis, we emphasize each chapter's connection to our main results and highlight the challenges faced in each chapter. A rough overview reads as follows: The Chapters 2–5 provide the notation and the tools to verify the homogenization of a unidirectional microstructure evolution model in Chapter 6. In the Chapters 7–9 this homogenization result is exploited to some extend and partly extended to derive effective models for different types of damage phenomena. In detail the outline of this thesis reads as follows:

We start by introducing the basic notation concerning the modeling of damage processes for linear elastic materials in Chapter 2. Section 2.1 is devoted to the theory of linear elastic materials, where the deformation of a body Ω is captured by the *displacement field* $u : [0,T]\times\Omega \to \mathbb{R}^d$. In Section 2.2, the family of admissible microstructures building the foundation of all homogenization results in this thesis is introduced. For modeling damage we choose the framework of *continuum damage mechanics* which is based on a so-called *damage variable* $z : [0,T]\times\Omega \to [0,1]^m$; see Section 2.4. In the case of modeling *brutal* damage the *damage set* represents a geometrical description for microstructures and is defined in Section 2.5. Based on this geometrical description, Section 2.6 introduces the notation of the microscopic damage models considered in Chapters 7–8 and stresses some technicalities occurring there.

The main part of the asymptotic analysis of this thesis is done with respect to the theory of *two-scale convergence* developed by G. Nguetseng in [63]. For this reason Chapter 3 is devoted to the notations, the definitions, and the results concerning two-scale convergence. In this paper we use the so-called *unfolding technique* introduced in [11], which is a dual formulation of the two-scale convergence theory. With the exception of Section 3.4 all presented results are well known and can be found in [62], for instance. However, Section 3.4 deals with sequences of admissible microstructures in the sense of Section 2.2. There, for a suitable chosen sequence of admissible microstructures, a two-scale limit microstructure is determined with respect to the strong two-scale convergence in L^1. Note that although the two-scale convergence theory was introduced to gain homogenization results for periodic problems, it is possible to apply this theory in our particular non-periodic case, too.

To exploit the asymptotic analysis of Section 3.4 for the microscopic models of the Chapters 6–9, we introduce a microstructure regularization in Chapter 4. The first challenge to face when regularizing any problem is the choice of the "right" regularization. On the one hand, we regularize our microscopic models to control the asymptotic behavior of the considered microstructures. On the other hand, however, one might lose interesting effects in the limit model if the regularization is too strong. Here, we are able to present a regularization theory that allows us to keep our explicit description of microstructure in the limit without losing its variety; for more details see also Section 7.4.

This microstructure regularization is based on a regularization theory concerning piecewise constant functions defined on lattices. In Section 4.1, a discrete gradient for piecewise constant functions is introduced, which possesses the following property: For a sequences of piecewise constant functions defined on finer and finer lattices, which additionally have uniformly bounded discrete gradients, there exists a Sobolev function and a subsequence such that the subsequence converges strongly to this Sobolev function in L^p. Moreover, the subsequence of discrete gradients converges weakly to the gradient of the limit Sobolev function in L^p. By introducing this discrete gradient as a regularization term to the microscopic models of the Chapters 6–9, their microstructures show the asymptotic behavior investigated in Section 3.4. Furthermore, this regularization takes microscopic interactions into account.

The construction of the discrete gradient is inspired by the regularization theory for so-called *broken Sobolev* spaces in [8]. There, the authors introduce two terms – a discrete gradient and a regularization term. The regularization term extracts those sequences of broken Sobolev functions that converge to a classical Sobolev function, whereas the discrete gradient of such sequences converges to the gradient of the classical Sobolev function. While adapting this theory with respect to our needs, we succeeded in combining the beneficial properties of the discrete gradient and the penalty term of [8] in only one term, namely, our discrete gradient for piecewise constant functions.

In preparation for formulating the models homogenization is performed for, Chapter 5 is devoted to the *energetic formulation* of *rate-independent* systems. This general theory covers a variety of physical phenomena, e.g., linearized elastoplasticity ([27, 62]), finite-strain elastoplasticity ([50, 51]), phase transformations in shape-memory alloys ([47, 59]), models in ferromagnetism ([17]), delamination problems ([45, 57]), and crack models ([42, 43]). Section 5.2 focuses on the main notations concerning the energetic formulation consisting of a stability condition (S) as well as an energy balance (E) and states the existence of solutions in the here considered case. Moreover, a sufficient condition guaranteeing the rate independence of a system modeled by the energetic formulation is discussed. Later, the variable underlying the energetic formulation will be the tuple $(u, z) : [0, T]\times\Omega \to \mathbb{R}^d\times[0, 1]^m$ consisting of the displacement field and the damage variable.

In Chapter 6, for a family of unidirectional microstructure evolution models, homogenization is performed rigorously. This homogenization result forms the basis for the rigorously derived effective damage models of the Chapters 7–9. The microscopic models of Section 6.1, which are characterized by the intrinsic length scale (denoted by $\varepsilon > 0$) of their microstructure, are set up in an energetic formulation (S^ε) and (E^ε). For fixed $\varepsilon > 0$, these microscopic models allow for a unidirectional microstructure evolution with respect to the family of admissible microstructures introduced in Section 2.2. The microstructure evolution is captured by an internal variable $z_\varepsilon : [0,T]\times\Omega \to [0,1]^m$, whereas the deformation is modeled by the displacement field $u_\varepsilon : [0,T]\times\Omega \to \mathbb{R}^d$. To provide enough regularity with respect to the microstructure, the discrete gradient of Chapter 4 is incorporated in these models. However, observe that for fixed $\varepsilon > 0$ existence of solutions is guaranteed with or without penalty term.

The homogenized models obtained by performing the limit passage $\varepsilon \to 0$ are formulated in the subsequent sections. Section 6.2 presents a two-scale limit model defined on $\Omega\times Y$, where Y denotes the so-called *unit cell* "capturing" the microscopic behavior of the sequence of microscopic solutions. In contrast to this, in Section 6.3 a one-scale model is formulated on Ω again, which is proven to be equivalent to the two-scale model in the following sense: From any solution of one of these two models a solution of the other one can be constructed. This equivalence is based on a *unit cell problem* depending on the limit internal variable $z_0 : [0,T]\times\Omega \to [0,1]^m$. Moreover, this unit cell problem yields an explicit representation for the microstructure of the one-scale limit model. Due to our regularization in the microscopic models, the effective microstructure preserves certain properties of the microscopic ones. In the context of modeling damage, this preservation will be explained in more detail.

In the Sections 6.4 and 6.5, the rigorous verification of the two-scale limit model of Section 6.2 is performed. This is done within the framework of evolutionary Γ-convergence introduced in [56], which relies on the verification of the so-called *mutual recovery condition.* The mutual recovery condition requires the construction of certain *mutual recovery sequences* ensuring that the limit of a sequence of microscopic solutions satisfies a stability condition (S^0). Here, the crucial part of constructing a mutual recovery sequence is the provision of the component which is responsible for the internal variable. In particular, this component has to be constructed such that it respects the irreversibility constraint posed on damage evolution. This is done in Section 6.4 by extending the ideas from [61] to the discrete case. In this context, the explicit structure of the discrete gradient for piecewise constant functions is heavily exploited to verify the mutual recovery condition.

Once the stability of the limit of the sequence of microscopic solutions is ensured, we perform the limit passage $\varepsilon \to 0$ for the microscopic models rigorously. This is done in Section 6.5. For this purpose, a generalized Helly selection principal, similar to that of [56], is exploited to ensure that there exists a subsequence of $(\varepsilon)_{\varepsilon>0}$ such that for any $t \in [0,T]$ the subsequence of microscopic solutions converges to the solution of the two-scale model.

After establishing the homogenization result for the unidirectional microstructure evolution in Chapter 6, the Chapters 7–9 exploit and extend this result to derive effective models for different types of damage.

Chapter 7 presents effective models for an irreversible damage evolution, where damage progression is caused by the growth of microscopic inclusions of damaged material, i.e., the inclusions have a significantly lower stiffness than the surrounding material. The growth of the inclusions is modeled by a time dependent, vector valued damage variable $z_\varepsilon : [0,T]\times\Omega \to [0,1]^m$, where its m-components enable the modeling of various anisotropic inclusions' geometries; see Subsection 7.1.2 for some examples that are captured by our approach. Moreover, each component of the vector valued damage variable might be "equipped" with its own fracture toughness. This contributes to the discussion of how to include anisotropic damage behavior into macroscopic damage models. In Remark 7.5, the importance of carefully modeling the dissipated energy is stressed.

For a chosen family of admissible microscopic inclusions' geometries, the effective models of Section 7.2 and 7.3 are obtained by the homogenization result of Chapter 6. The effective material of the limit one-scale model in Section 7.3 is given by a unit cell problem depending on the limit damage variable $z_0 : [0,T]\times\Omega \to [0,1]^m$. In this way, the effective material is a mixture of damaged and undamaged material whose actual distribution for a given time $t \in [0,T]$ and at a certain point $x \in \Omega$ is prescribed by the damage variable's value $z_0(t,x)$. Observe that any value $z_0(t,x) \in [0,1]^m$ is associated to a specific element of the family of admissible geometries chosen for the microscopic models. In this sense, the limit models are based on the same inclusions' geometries as the microscopic ones.

In Section 7.4, our effective one-scale model is compared with the damage model of [24]. There, the authors proved existence of solutions for an effective damage model based on microstructures consisting only of damaged and undamaged material. Since this is done without any microstructure regularization, in [24], arbitrary mixtures of these two phases are allowed. In contrast to our homogenization result, their effective material in every point is described by a unit cell problem for which the ratio of damaged and undamaged material is known, but where their actual geometrical distribution is not prescribed.

In Chapter 8, the results of the previous chapter, obtained for inclusions of damaged material, are extended to the case of modeling microscopic voids. For this purpose, in the microscopic model the damage variable $z_\varepsilon : [0,T]\times\Omega \to [0,1]^m$ is associated to a non-periodically perforated, time dependent domain modeling the linear elastic body under consideration; see Subsection 8.1.1. Due to this time dependence, the state space of the displacement field varies with respect to time which causes difficulties when identifying the limit state space, for instance. To solve this problem, a continuation operator is introduced which, in dependence on the damage variable associated to the perforated domain, extends every displacement field to the fixed domain Ω, again. Although the microscopic voids might vary dramatically with respect to their size and shape, for the sequence $(u_\varepsilon, z_\varepsilon)_{\varepsilon>0}$ of microscopic solutions we succeeded in constructing an extension such that $(u_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in H^1. Observe that this uniform bound holds, although for $\varepsilon \to 0$ the number of microscopic voids tends to infinity, whereas their diameter converges to zero. Moreover, this continuation theory enables us to ensure uniform coercivity on the whole set Ω for the microscopic energy functionals.

Since the microscopic models introduced in Subsection 8.1.3 deal with time dependent, non-periodically perforated domains, Subsection 8.1.2 addresses the modeling of external forces on such domains. The discussion on the potential defects' geometries that are captured by our approach is done in Subsection 8.1.4. In contrast to Chapter 7, where inclusions of weak material are allowed to emerge in areas of undamaged material, we here need to assume preexisting voids in general. That generally means that the theory of Chapter 8 does not allow for hole initiation. In other words, even the state $z_\varepsilon \equiv \mathbf{1}$ modeling undamaged material is associated to an already perforated domain, where the preexisting microscopic voids grow with respect to time. Only in a specific scalar case (with respect to the damage variable) we are able to model hole initiation; see Remark 8.3 and Example 8.11.

By modifying the homogenization theory of Chapter 6 carefully, we are able to derive an effective two-scale model which is presented in Section 8.2. This two-scale model is formulated on $\Omega \times Y$. However, due to our homogenization approach, the microscopic voids of the microscopic models are "shifted" to the second scale, i.e., for almost every $x \in \Omega$ the set $\{x\} \times Y$ contains a subset of zero stiffness whose shape and size at time $t \in [0,T]$ is uniquely described by the value $z_0(t,x)$ of the limit damage variable. Since the perforated domains considered in the microscopic models require the assumption of an additional zero Neumann boundary condition on the boundary of each void, the limit external loading is affected by the microscopic scale, too. This is different to the previous homogenization results, where the limit external loading is only affected by the macroscopic scale.

Assuming the term of the external loading depending on the microscopic scale to be zero in the effective two-scale model, we are able to formulate an equivalent one-scale model; see Section 8.3. As in the previous homogenization results this effective one-scale model is based on a unit cell problem. In dependence on the limit damage variable's value $z_0(t,x)$ at $(t,x) \in [0,T] \times \Omega$ the unit cell problem has to be solved with respect to a unit cell Y occupied by the initially chosen material, which contains a void whose size and shape is uniquely given by $z_0(t,x)$. According to our continuation theory we are able to show that this effective material possesses positive stiffness at any time $t \in [0,T]$ and in almost every point $x \in \Omega$.

Chapter 9 is devoted to the modeling of cracks. More precise, in Section 9.1 we introduce microscopic models allowing for the propagation of various microscopic cracks. For this purpose, the damage variable $z_\varepsilon(t) : \Omega \to [0,1]^m$ is associated to a set $\mathcal{C}_\varepsilon(z_\varepsilon(t))$ consisting of various small hypersurfaces modeling microscopic cracks. Since the body Ω at time $t \in [0,T]$ contains the cracks $\mathcal{C}_\varepsilon(z_\varepsilon(t))$, the displacement field $u_\varepsilon(t) : \Omega \to \mathbb{R}^d$ might have a jump on $\mathcal{C}_\varepsilon(z_\varepsilon(t))$. According to this relation, the displacement field depends on the damage variable. Similar to Chapter 8, we do not allow for crack initiation, i.e., the initial configuration is given by a body containing preexisting cracks.

By performing the limit passage $\varepsilon \to 0$, we provide an effective two-scale model which is formulated in Section 9.2. For passing ε to zero rigorously, a compactness result is presented in Subsection 9.1.1. This result enables us to identify the two-scale limit functions of sequences $(u_\varepsilon(t))_{\varepsilon>0}$ of displacement fields $u_\varepsilon(t) : \Omega \to \mathbb{R}^d$ having jumps on the non-periodic set $\mathcal{C}_\varepsilon(z_\varepsilon(t))$. In this way, the compactness result motivates the choice of the two-scale limit function space in Section 9.2. To derive the effective model of Section 9.2 rigorously, for any function of the two-scale limit function space a mutual recovery sequence $(\widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon)_{\varepsilon>0}$ needs to be constructed. The construction of $(\widetilde{z}_\varepsilon)_{\varepsilon>0}$ can be done as in the Chapters 6–8. When constructing $(\widetilde{u}_\varepsilon)_{\varepsilon>0}$, it needs to be ensured that $\widetilde{u}_\varepsilon$ only jumps on the prescribed set $\mathcal{C}_\varepsilon(\widetilde{z}_\varepsilon)$. This is done in Subsection 9.1.2.

Similar to the result of Chapter 8, the limit external loading of the two-scale effective model is affected by the microscopic scale, too. Hence, only by assuming the term of the external loading depending on the microscopic scale to be zero, we are able to present an equivalent one-scale model; see Section 9.3. According to the homogenization theory of Chapter 6, the effective material is described by a unit cell problem. Although the unit cell in $x \in \Omega$ at time $t \in [0, T]$ contains a crack whose size is related to the damage variable's value $z_0(t, x)$, the effective material of the one-scale model is free from cracks.

Finally, this thesis concludes with Chapter 10 which provides an outlook on tasks that seem to be interesting for future works.

2 Mechanical background and notation

2.1 Linear elasticity

This section introduces the notation from the continuum mechanics of solids to describe the quasistatic evolution of a linear elastic body. For this purpose, all appearing terms are assumed to be as smooth as necessary and for the derivation of the individual relations we refer to [9].

In this thesis we deal with evolution processes of solids, whose initial shape is represented by a domain $\Omega \subset \mathbb{R}^d$ – the so-called *reference configuration*. Hence, at the beginning of the evolution Ω is the subset of $\mathbb{R}^d$ which is occupied by material. The main assumptions on the set Ω, which are assumed to hold in the whole thesis, are the following:

$$\begin{aligned}&\text{The set } \Omega \subset \mathbb{R}^d \text{ is assumed to be open, connected, bounded, and}\\ &\text{has a locally Lipschitz boundary } \partial\Omega;\ \text{see Definition 2.1 below.}\end{aligned} \tag{2.1}$$

Definition 2.1 (Locally Lipschitz boundary)**.** *A bounded set $\mathcal{O} \subset \mathbb{R}^d$ has a locally Lipschitz boundary, if for each point $x \in \partial\mathcal{O}$ there exists a neighborhood N_x such that $N_x \cap \partial\Omega$ is the graph of a Lipschitz continuous function.*

In the case of a linear elastic body its material properties are described by a symmetric, positive definite tensor $\widetilde{\mathbb{C}} : \Omega \to \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}}; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ of fourth order. To describe the deformation of the material occupying the body Ω with respect to time, the so-called *displacement field* $u : [0,T]\times\Omega \to \mathbb{R}^d$ is introduced such that the vector $u(t,x) \in \mathbb{R}^d$

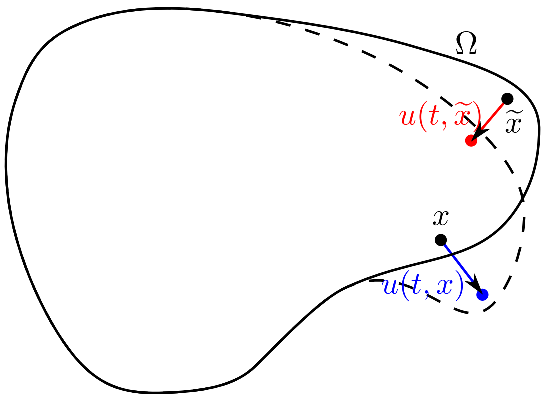


Figure 2.1: The displacement field $u : [0,T]\times\Omega \to \mathbb{R}^d$.

denotes the displacement of the reference configuration's point $x \in \Omega$ at time $t \in [0,T]$; see Figure 2.1. Thus, a local measure of the strain induced by the displacement field u is given by the *Green-St. Venant strain tensor*

$$\mathbf{E}(u) := \tfrac{1}{2}\Big(\nabla u + (\nabla u)^T + (\nabla u)^T \nabla u\Big).$$

In the *small strain setting* the displacement field's gradient is assumed to be small such that the quadratic term $(\nabla u)^T \nabla u$ is negligible. In this case the *linearized stain tensor*

$$\mathbf{e}(u) := \tfrac{1}{2}\Big(\nabla u + (\nabla u)^T\Big) \tag{2.2}$$

is a sufficiently good approximation of $\mathbf{E}(u)$. Now, we are in the position to formulate the quasi static evolution of a linear elastic body, which is fixed on a part Γ_{Dir} of its boundary $\partial\Omega$. This quasi static setting requires the external loadings $\ell_\Omega : [0,T]\times\Omega \to \mathbb{R}^d$ and $\ell_{\Gamma_{\mathrm{N}}} : [0,T]\times\Gamma_{\mathrm{N}} \to \mathbb{R}^d$, $\Gamma_{\mathrm{N}} := \partial\Omega\backslash\Gamma_{\mathrm{Dir}}$, to be slowly varying with respect to time. Thus, the linear elastic, quasi static deformation of a body Ω is described by the displacement field $u : [0,T]\times\Omega \to \mathbb{R}^d$ satisfying for all $t \in [0,T]$ the following equations of equilibrium:

$$\begin{cases} -\mathrm{div}(\widetilde{\mathbb{C}}\mathbf{e}(u(t))) = \ell_\Omega(t) & \text{on } \Omega, \\ u(t) = \mathbf{0} & \text{on } \Gamma_{\mathrm{Dir}}, \\ (\widetilde{\mathbb{C}}\mathbf{e}(u(t)))n_{\Gamma_{\mathrm{N}}} = \ell_{\Gamma_{\mathrm{N}}}(t) & \text{on } \Gamma_{\mathrm{N}}, \end{cases} \tag{2.3}$$

where $n_{\Gamma_{\mathrm{N}}} : \Gamma_{\mathrm{N}} \to \mathbb{R}^d$ denotes the outward unit normal vector on Γ_{N}.

Remark 2.2. *Observe that for the sake of shorten the notation, from now on we are always going to assume that the displacement field takes the value zero on* Γ_{Dir}. *To introduce a time dependent boundary value* $g : [0,T]\times\Gamma_{\mathrm{Dir}} \to \mathbb{R}^d$ *one might use the splitting* $\widehat{u} = u + \widehat{g}$. *Here,* $\widehat{g} : [0,T]\times\Omega \to \mathbb{R}^d$ *denotes a suitable extension of* $g : [0,T]\times\Gamma_{\mathrm{Dir}} \to \mathbb{R}^d$ *chosen as smooth as necessary. Applying this splitting to the results obtained in this thesis, they still hold true for the time dependent boundary value* $g : [0,T]\times\Gamma_{\mathrm{Dir}} \to \mathbb{R}^d$*; see also Remark* 8.8.

2.2 Microstructure

In this thesis, microstructure is understood as the heterogeneity of the material occupied body Ω. In other words, the term *microstructure* refers to the actual shape of the material properties' describing fourth order tensor $\widetilde{\mathbb{C}} \in \mathcal{M}(\Omega)$. This heterogeneity either arises from one material in different phases or from several materials appearing in different phases. In the following our assumptions on the microstructure are introduced.

In experiments, it is observed that microstructures often have an intrinsic length scale. Descriptively this length scale is related to the smallest homogeneous set of material being part of the microstructure. All models showing up such an intrinsic length scale are termed as *microscopic models* in the following. According to the huge variety of

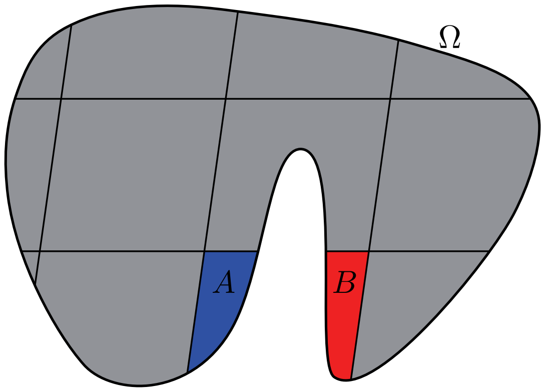


Figure 2.2: $\mathrm{K}_{\varepsilon\Lambda}(\Omega)$ prohibits its elements to take different values on A and B.

heterogeneity appearing in nature, modeling of microstructure in this general setting is hopeless and some approximation is needed. One very common kind of such an approximative microstructure is the periodic one. Here, the intrinsic length scale, denoted by $\varepsilon > 0$, is associated to the size of cells $\varepsilon(\lambda+Y)$ occupying the bounded open domain $\Omega \subset \mathbb{R}^d$, where for an arbitrary basis $\{b_1, b_2, \dots, b_d\}$ of $\mathbb{R}^d$

$$Y := \Big\{ y \in \mathbb{R}^d \,|\, y = \sum_{i=1}^{d} k_i b_i, k_i \in [0,1) \Big\}, \quad (\mu_d(Y) = 1)$$

denotes the so-called *unit cell*, which in this thesis is assumed to have volume 1. Moreover, λ is an element of the periodic lattice

$$\Lambda := \Big\{ \lambda \in \mathbb{R}^d \,|\, \lambda = \sum_{i=1}^{d} k_i b_i, k_i \in \mathbb{Z} \Big\}.$$

In the periodic case, all cells with $\varepsilon(\lambda+Y)\cap\Omega \neq \emptyset$ contain the same specific distribution of the appearing materials and their phases. That means, in this case the material tensor $\widetilde{\mathbb{C}}^{\mathrm{per}}_{\varepsilon} \in \mathcal{M}(\Omega)$ is based on a tensor $\mathbb{C}_Y \in \mathcal{M}(Y)$ given on the unit cell Y, i.e., for almost every $x \in \Omega$ it holds

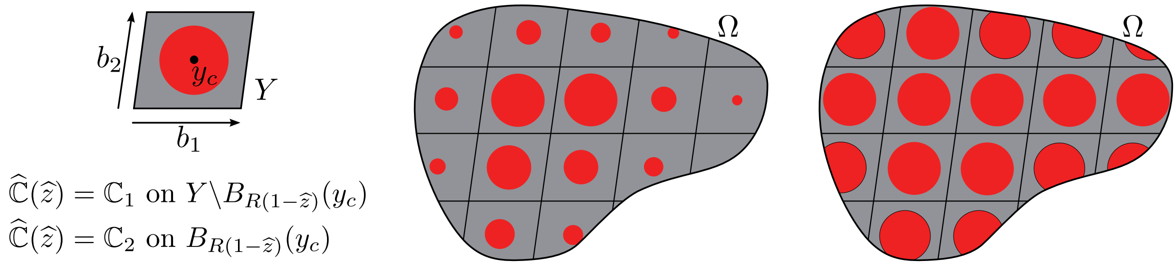
$$\widetilde{\mathbb{C}}^{\mathrm{per}}_{\varepsilon}(x) := \mathbb{C}^{\mathrm{per}}_{Y}(\tfrac{x}{\varepsilon}). \tag{2.4}$$

Here, $\mathbb{C}^{\mathrm{per}}_{Y} \in \mathcal{M}(\mathbb{R}^d)$ denotes the periodic extension of the tensor $\mathbb{C}_Y \in \mathcal{M}(Y)$. In the context of damage models we are going to consider microstructure evolution according to an internal variable; see (2.8). This evolution is considered with respect to a given family of admissible microstructures gained by generalizing the periodic ansatz in the following way: For $m \in \mathbb{N}$ let $\widehat{\mathbb{C}} : [0,1]^m \to \mathcal{M}(Y)$ be given and define the set of piecewise constant functions $\mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)$ by

$$\mathrm{K}_{\varepsilon\Lambda}(\Omega) := \{ v \in \mathrm{L}^1(\Omega) \,|\, \exists\, \widetilde{v} \in \mathrm{K}_{\varepsilon\Lambda}(\mathbb{R}^d) : \widetilde{v}|_{\Omega} = v \}, \tag{2.5}$$

where

$$\mathrm{K}_{\varepsilon\Lambda}(\mathbb{R}^d) := \{ \widetilde{v} \in \mathrm{L}^1(\mathbb{R}^d) \,|\, \forall\, \lambda \in \Lambda : \widetilde{v}|_{\varepsilon(\lambda+Y)} \equiv \mathit{const} \}.$$



$\widehat{\mathbb{C}}$ consists of two constant tensors $\mathbb{C}_1$ and $\mathbb{C}_2$

$\mathbb{C}_\varepsilon(z_\varepsilon)$ for $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1])$ with $z_\varepsilon \not\equiv const$

$\mathbb{C}_\varepsilon^{\mathrm{per}} := \mathbb{C}_\varepsilon(1)$

Figure 2.3: Comparison of a specific ($d = 2$, $m = 1$) admissible microstructures in the sense of (2.6) with the periodic case.

Then for $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)$ an element $\widetilde{\mathbb{C}}_\varepsilon = \mathbb{C}_\varepsilon(z_\varepsilon) \in \mathcal{M}(\Omega)$ of the family of admissible microstructures for almost every $x \in \Omega$ is defined by

$$\widetilde{\mathbb{C}}_\varepsilon(x) = \mathbb{C}_\varepsilon(z_\varepsilon)(x) := \Big(\widehat{\mathbb{C}}(z_\varepsilon(x))\Big)^{\mathrm{per}}(\tfrac{x}{\varepsilon}), \tag{2.6}$$

where for $\hat{z} \in [0,1]^m$ the term $(\widehat{\mathbb{C}}(\hat{z}))^{\mathrm{per}} \in \mathcal{M}(\mathbb{R}^d)$ again denotes the periodic extension of $\widehat{\mathbb{C}}(\hat{z}) \in \mathcal{M}(Y)$. Since $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)$ may vary from one cell to another, this definition allows for non-periodic coefficients. Later, these non-periodic coefficients are used to model various distributions of material defects, where the size and the shape of a particular defect in a cell $\varepsilon(\lambda+Y) \subset \Omega$ is encoded in the value $z^{\varepsilon\lambda} := z_\varepsilon|_{\varepsilon(\lambda+Y)}$.

Remark 2.3. *Note that for fixed $\varepsilon > 0$ the set $\mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)$ is finite dimensional. Hence, the strong and weak topology are the same and for $z, (z_\delta)_{\delta>0} \subset \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)$ the convergence $z_\delta \to z$ in $\mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)$ is understood as the convergence of the sequences of parameters $(z_\delta^{\varepsilon\lambda})_{\delta>0}$, with $z_\delta^{\varepsilon\lambda} :\equiv z_\delta|_{\varepsilon(\lambda+Y)\cap\Omega}$, to the parameters $z^{\varepsilon\lambda} :\equiv z|_{\varepsilon(\lambda+Y)\cap\Omega}$ for all cells with $\varepsilon(\lambda+Y)\cap\Omega \neq \emptyset$. Note that $z_\delta \to z$ in $\mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)$ is equivalent to $z_\delta \to z$ in $\mathrm{L}^q(\Omega)^m$ for any $q \in [1,\infty)$.*

2.3 Homogenization

Roughly spoken, in the classical homogenization theory one deals with a family of parameter dependent problems and the aim is the derivation of an effective problem being independent of this parameter but capturing the family's properties in a sufficiently good way. In our case the parameter denotes the intrinsic length scale of a linear elastic solid and, naturally, its size is very small compared to the size of the considered body Ω. Together with the possibly complicated shape of the microstructure this leads, for instance, to problems in the numerical investigation of such microstructures. Therefore, the derivation of an effective description being independent of the intrinsic length scale is a meaningful task.

As an example we are now considering the static deformation of a linear elastic body Ω with periodic material coefficients $\mathbb{C}^{\mathrm{per}}_{\varepsilon}(\cdot) = \mathbb{C}^{\mathrm{per}}_{Y}(\frac{\cdot}{\varepsilon}) \in \mathcal{M}(\Omega)$ (see (2.4)), i.e., we would like to apply the theory of homogenization to the family $(P(\varepsilon))_{\varepsilon>0}$ of elliptic boundary value problems $P(\varepsilon)$ given as follows:

For $\varepsilon > 0$ let $u_\varepsilon \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ denote the weak solution of

$$\begin{cases} -\mathrm{div}(\mathbb{C}^{\mathrm{per}}_{\varepsilon}\mathbf{e}(u_\varepsilon)) = \ell_\Omega & \text{on } \Omega, \\ u_\varepsilon = \mathbf{0} & \text{on } \Gamma_{\mathrm{Dir}}, \\ (\mathbb{C}^{\mathrm{per}}_{\varepsilon}\mathbf{e}(u_\varepsilon))n_{\Gamma_{\mathrm{N}}} = \ell_{\Gamma_{\mathrm{N}}} & \text{on } \Gamma_{\mathrm{N}}, \end{cases} \tag{$P(\varepsilon)$}$$

where $n_{\Gamma_{\mathrm{N}}} : \Gamma_{\mathrm{N}} \to \mathbb{R}^d$ denotes the outward unit normal vector on Γ_{N}. Regarding the classical homogenization considering periodic coefficients, a rigorous result is gained via the two-scale convergence introduced by G. Nguetseng in [63]. This result states that if $\mathbb{C}_Y \in \mathcal{M}(Y)$ (see (2.4)) is uniformly positive definite and if $u_\varepsilon \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ is the weak solution of $P(\varepsilon)$, then there exists a function $u_0 \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ such that

$$\begin{cases} u_\varepsilon \rightharpoonup u_0 & \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d, \\ \mathbb{C}^{\mathrm{per}}_{\varepsilon}\mathbf{e}(u_\varepsilon) \rightharpoonup \mathbb{C}^{\mathrm{con}}_{\mathrm{eff}}\mathbf{e}(u_0) & \text{in } \mathrm{L}^2(\Omega)^{d\times d} \end{cases} \tag{2.7}$$

and $u_0 \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ is the weak solution of $P(0)$, which is obtained from $P(\varepsilon)$ by replacing the periodic tensor $\mathbb{C}^{\mathrm{per}}_{\varepsilon}(\cdot) = \mathbb{C}^{\mathrm{per}}_{Y}(\frac{\cdot}{\varepsilon})$ by the constant tensor $\mathbb{C}^{\mathrm{con}}_{\mathrm{eff}} \in \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}}; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ defined via the so-called *unit cell problem*: For every $\xi \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$

$$\langle \mathbb{C}^{\mathrm{con}}_{\mathrm{eff}}\xi, \xi \rangle := \min\Big\{ \int_Y \langle \mathbb{C}_Y(y)(\xi + \mathbf{e}_y(v)(y)), \xi + \mathbf{e}_y(v)(y)\rangle_{d\times d} \mathrm{d}y \Big\}.$$

Here, the minimum is taken with respect to all functions $v \in \mathrm{H}^1_{\mathrm{per}}(\mathrm{cl}(Y))^d$ having zero average on Y, i.e., $\fint_Y v(y)\mathrm{d}y = 0$. In this case, homogenization results in an effective problem $P(0)$ modeling the elastic deformation of a body Ω consisting of homogeneously distributed material model by $\widetilde{\mathbb{C}}^{\mathrm{con}}_{\mathrm{eff}} \in \mathcal{M}(\Omega)$, where $\widetilde{\mathbb{C}}^{\mathrm{con}}_{\mathrm{eff}} :\equiv \mathbb{C}^{\mathrm{con}}_{\mathrm{eff}}$ on Ω. Therefore, the effective model $P(0)$ possesses no intrinsic length scale, but according to (2.7) it acts as a good approximation of $P(\varepsilon)$ for $\varepsilon > 0$ sufficiently small.

As already mentioned in Section 2.2, in this thesis we deal with microstructure evolution modeled by the time-wise behavior of an internal variable. This means that for a specific intrinsic length scale $\varepsilon > 0$ we are going to consider non-periodic microstructures modeled by $\mathbb{C}_\varepsilon(z_\varepsilon(t)) \in \mathcal{M}(\Omega)$ (see (2.6)), where $z_\varepsilon(t) \in \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)$ denotes the internal variable that may evolve in time. In this case, homogenization of a family of problems involving such microstructures is understood as the investigation of their asymptotic behavior for ε tending to zero. Here, the term *homogenization* is justified by the fact that the effective model is ε-independent and preserves the characteristic of the ε-dependent microstructure (see Section 6.2, for instance), although it will turn out that the effective material tensor is not constant and hence does not model a homogeneous material.

2.4 Modeling of damage

Mechanical models describing damage processes in solids go back to the work by L.M. Kachanov ([32, 33, 34]) and Yu.N. Rabotnov ([65]). There, in the context of creep damage of metals the authors provided the basis for the large branch of modern *continuum damage mechanics*. The basic idea for these models translated in our setting can be summarized as follows: Every point of the solid has a certain stiffness, which serves as an indicator for the damage state of the respective point. High stiffness is interpreted as the presence of a small amount of damaged material, whereas low stiffness is associated with highly progressed damage. In continuum damage mechanics, a damage variable $z : [0,T]\times\Omega \to [0,1]^m$ is introduced that on the macroscopic level represents this local damage state. Moreover, constitutive relations

$$\widetilde{\mathbb{C}}(t) = \mathbb{C}(z(t)) \tag{2.8}$$

are postulated that describe the dependence of the material constants on these damage variables and enter the equations of equilibrium; see (2.3). In the scalar case the damage variable takes values between 0 and 1, where the value 1 is related to completely undamaged material whereas the value 0 models the maximal amount of damage. Generally, the actual relation between the damage variable's value and the material depends on the specific model being under investigation; see Chapter 7 and 8, for instance.

However, in any case we are going to consider a unidirectional damage evolution, i.e., if a point $x \in \Omega$ has taken a certain amount of damage at the time $t \in [0,T]$ this damage state must not decrease for ongoing time. For this purpose, we assume the material tensor $\mathbb{C}(\cdot)$ of (2.8) to be monotone increasing with respect to the damage variable, i.e.:

$$\begin{aligned}&\text{For all } z_1, z_2 : \Omega \to [0,1]^m \text{ with } z_1 \geq z_2 \text{ and every}\\ &\xi \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \text{ it holds } \langle \mathbb{C}(z_1)\xi, \xi\rangle_{d\times d} \geq \langle \mathbb{C}(z_2)\xi, \xi\rangle_{d\times d}.\end{aligned} \tag{2.9}$$

Therefore, forcing the damage variable $z : [0,T]\times\Omega \to [0,1]^m$ to be monotone decreasing with respect to time guarantees that the damage evolution is unidirectional. This assumption on the monotonicity of the damage variable must be ensured by the evolution law which is chosen to describe its temporal behavior; see Example 2.4 below, for instance.

This approach leads to damage models of phase-field type with the damage variable as the phase field variable. The full damage evolution model consists of the macroscopic momentum balance (2.3) with $\widetilde{\mathbb{C}}(t) = \mathbb{C}(z(t))$ that is coupled with a suitable evolution law for the damage variable. As an example for the type of evolution laws we have in mind, we are now going to consider the following flow rule in the case of a scalar damage variable.

Example 2.4 (Flow rule for a scalar damage variable)**.** Let $z : [0,T]\times\Omega \to [0,1]$ denote a scalar damage variable. Then, for a positive definite tensor $\overline{\mathbb{C}} \in \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}}; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ and a given $\delta > 0$ we assume the constitutive relation (2.8) to be given by the phenomenologically motivated ansatz $\mathbb{C}(z(t)) := (z(t)+\delta)\overline{\mathbb{C}}$. For $\kappa > 0$, $\beta \geq 0$, and

$\Delta_p z(t) := \operatorname{div}(|\nabla z(t)|^{p-2}\nabla z(t))$ the evolution law modeling the time-wise behavior of the damage variable we have in mind reads as follows:

$$\begin{cases} 0 \in -\kappa+\beta\dot{z}(t) + \partial_{\mathrm{sub}} I_{(-\infty,0]}(\dot{z}(t)) & \\ \qquad + \frac{1}{2}\langle \overline{\mathbb{C}}\mathbf{e}(u(t)), \mathbf{e}(u(t))\rangle_{d\times d} - \Delta_p z(t) + \partial_{\mathrm{sub}} I_{[0,\infty)}(z(t)) & \text{on } \Omega, \\ 0 = \langle \nabla z(t), n_{\partial\Omega}\rangle_d & \text{on } \partial\Omega, \end{cases} \tag{2.10}$$

Here, $n_{\partial\Omega} : \partial\Omega \to \mathbb{R}^d$ denotes the outward unit normal vector on $\partial\Omega$ and the indicator function $I_{(-\infty,0]} : \mathbb{R} \to \{0,\infty\}$ of the interval $(-\infty,0]$ and its associated subdifferential $\partial_{\mathrm{sub}} I_{(-\infty,0]} : \mathbb{R} \to \mathcal{L}_{\mathrm{Leb}}(\mathbb{R})$ are given by:

$$I_{(-\infty,0]}(v) := \begin{cases} 0 & \text{if } v \le 0, \\ \infty & \text{if } v > 0, \end{cases} \quad \text{and} \quad \partial_{\mathrm{sub}} I_{(-\infty,0]}(v) := \begin{cases} \{0\} & \text{if } v < 0, \\ [0,\infty) & \text{if } v = 0, \\ \emptyset & \text{if } v > 0. \end{cases}$$

Moreover, it is $I_{[0,\infty)}(v) := I_{(-\infty,0]}(-v)$ and it holds $\partial_{\mathrm{sub}} I_{[0,\infty)}(v) = -\partial_{\mathrm{sub}} I_{(-\infty,0]}(-v)$. The so-called *flow rule* (2.10) is chosen such that the subdifferential $\partial_{\mathrm{sub}} I_{(-\infty,0]}(\dot{z}(t))$ accounts for the irreversibility of the damage evolution, since it forces the damage variable to be a monotonously decreasing function with respect to time. Therefore, for a suitable chosen initial condition z^0 with $z^0 \le 1$ on Ω, the damage variable $z : [0,T]\times\Omega \to [0,1]$ with $z(0) = z^0$ cannot exceed the value 1 at any time. Moreover, the second subdifferential $\partial I_{[0,\infty)}(z(t))$ entering (2.10) prevents the damage variable from taking negative values, such that altogether $z(t) \in [0,1]$ for any $t \in [0,T]$. Finally, the damage gradient $\nabla z(t)$ is incorporated to model microscopic interactions. Due to its benefits in numerical simulations, the damage gradient was used first in [22]. Also many engineering works take advantage of the damage gradient; see [4, 26], for instance.

In the following we are going to consider only the rate-independent case, i.e., we set $\beta = 0$. By introducing the energy functional $\mathcal{E} : [0,T]\times \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \times \mathrm{W}^{1,p}(\Omega;[0,1]) \to \mathbb{R}$ and the dissipation potential $\mathcal{R} : \mathrm{W}^{1,p}(\Omega;[0,1]) \to [0,\infty]$ via

$$\mathcal{E}(t,u,z) := \tfrac{1}{2}(z+\delta)\langle \overline{\mathbb{C}}\mathbf{e}(u), \mathbf{e}(u)\rangle_{d\times d} + \tfrac{1}{p}|\nabla z|^p + I_{[0,\infty)}(z) - \langle \ell(t), u\rangle$$

and

$$\mathcal{R}(v) := \begin{cases} \displaystyle\int_\Omega \kappa|v(x)|\mathrm{d}x & \text{if } v \le 0, \\ \infty & \text{otherwise} \end{cases}$$

the weak formulation of the coupled system (2.3) and (2.10) is given by

$$\begin{cases} 0 = D_u\mathcal{E}(t,u(t),z(t)) & \text{in } (\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*, \\ 0 \in D_z\mathcal{E}(t,u(t),z(t)) + \partial_{\mathrm{sub}}\mathcal{R}(\dot{z}(t)) & \text{in } (\mathrm{W}^{1,p}(\Omega;[0,1]))^*. \end{cases} \tag{2.11}$$

In specific cases this subdifferential formulation of a rate-independent process is equivalent to the so-called *energetic formulation* which is the framework rate-independent evolution processes of this thesis are modeled in; see Chapter 5. In general, this energetic formulation represents a proper generalization of the subdifferential formulation (2.11); see [47, 52, 60]. For taking various types of microstructures into account, the here considered constitutive relation of the damage variable and the material tensor will be replaced by more complicated ones in the following.

2.5 Damage set for brutal damage models

Brutal damage models are widely used in the modeling brittle solids like concrete; see [20, 21, 35, 36, 37], for instance. Here, the term *brutal* emphasizes the occurrence of a sharp interface between damaged and undamaged material. For simplicity, the damaged as well as the undamaged material are assumed to consist of only one phase, i.e., for a given damage variable $z : \Omega \to [0,1]^m$ the material tensor $\mathbb{C}^{\mathrm{D}}(z) \in \mathcal{M}(\Omega)$ is a piecewise constant function on Ω, taking either the value $\mathbb{C}_{\text{strong}} \in \mathrm{Lin}_{\text{sym}}(\mathbb{R}^{d\times d}_{\text{sym}}; \mathbb{R}^{d\times d}_{\text{sym}})$ or $\mathbb{C}_{\text{weak}} \in \mathrm{Lin}_{\text{sym}}(\mathbb{R}^{d\times d}_{\text{sym}}; \mathbb{R}^{d\times d}_{\text{sym}})$. These tensors are chosen such that for all $\xi \in \mathbb{R}^{d\times d}_{\text{sym}}$ it holds

$$0 \le \langle \mathbb{C}_{\text{weak}}\xi, \xi\rangle_{d\times d} \le \langle \mathbb{C}_{\text{strong}}\xi, \xi\rangle_{d\times d}.$$

As indicated by the subscript *strong*, the material associated to the tensor $\mathbb{C}_{\text{strong}}$ represents undamaged material, whereas $\mathbb{C}_{\text{weak}}$ models damaged material. According to this assumption the subset of damaged material $\Omega^{\mathrm{D}}(z) \subset \Omega$ is given by

$$\Omega^{\mathrm{D}}(z) := \{x \in \Omega \,|\, \widetilde{\mathbb{C}}(z)(x) = \mathbb{C}_{\text{weak}}\} \tag{2.12}$$

and is referred to as the *damage set* in the following. By writing $\mathbb{C}^{\mathrm{D}}(z) \in \mathcal{M}(\Omega)$ as

$$\mathbb{C}^{\mathrm{D}}(z) \ = \mathbb{1}_{\Omega\setminus\Omega^{\mathrm{D}}(z)}\mathbb{C}_{\text{strong}} + \mathbb{1}_{\Omega^{\mathrm{D}}(z)}\mathbb{C}_{\text{weak}} \tag{2.13}$$

the fact that the damage evolution is unidirectional (see (2.9)) results in the following monotonicity constraint on the damage set:

$$\text{For all } z_1, z_2 : \Omega \to [0,1]^m \text{ with } z_1 \ge z_2 \text{ it holds } \Omega^{\mathrm{D}}(z_1) \subset \Omega^{\mathrm{D}}(z_2). \tag{2.14}$$

Remark 2.5 (Modeling of voids, i.e., $\mathbb{C}_{\text{weak}} \equiv \mathbb{O}$)**.** *As already mentioned in Section* 2.4 *for a given damage variable* $z : \Omega \to [0,1]^m$ *the linear elastic deformation of the body* Ω *with respect to the material tensor* $\mathbb{C}^{\mathrm{D}}(z) \in \mathcal{M}(\Omega)$ *is considered; see* (2.13)*. Since Chapter* 8 *examines damage processes which model the growth of microscopic voids, we are going to assume* $\mathbb{C}_{\text{weak}} \equiv \mathbb{O}$*. However, in this case the damage set* $\Omega^{\mathrm{D}}(z)$ *(see* (2.12)*) contains no material such that actually we are interested in the linear elastic deformation of the body* $\Omega\setminus\Omega^{\mathrm{D}}(z)$ *with respect to the material tensor* $\mathbb{C}_{\text{strong}} \in \mathcal{M}(\Omega\setminus\Omega^{\mathrm{D}}(z))$*. Hence, it is reasonable to assume that for all* $z : \Omega \to [0,1]^m$ *the set* $\Omega\setminus\Omega^{\mathrm{D}}(z)$ *is connected. Since the linear elastic deformation of such a body* $\Omega\setminus\Omega^{\mathrm{D}}(z)$ *is modeled by the boundary value problem* (2.3)*, we need to ensure that for all damage variables* $z : \Omega \to [0,1]^m$ *it holds* $\mathrm{cl}(\Omega^{\mathrm{D}}(z)) \cap \partial\Omega = \emptyset$*. Otherwise the assumptions on the boundary value of the displacement field are not well defined.*

2.6 Microscopic damage models

This section is in preparation for the homogenization results presented in the Chapters 7 and 8. There, homogenization is performed for a family of microscopic brutal damage models whose underlying microstructures are similar to the non-periodic ones introduced in (2.6). However, to take the technicalities mentioned in Remark 2.5 into account, we have to modify the microstructure defining tensors of Section 2.2. For this purpose, we introduce the subsets

$$\Lambda_\varepsilon^- := \{\lambda \in \Lambda : \varepsilon(\lambda + \mathrm{cl}(Y)) \subset \Omega\} \qquad \text{and} \qquad \Lambda_\varepsilon^+ := \{\lambda \in \Lambda : \varepsilon(\lambda + Y) \cap \Omega \neq \emptyset\} \tag{2.15}$$

of Λ. Thus, the cells intersecting Ω are given by $\varepsilon(\lambda+Y)$ for $\lambda \in \Lambda_\varepsilon^+$, whereas cells $\varepsilon(\lambda+Y)$ with $\lambda \in \Lambda_\varepsilon^-$ are completely contained in Ω. The sets Ω_ε^+ and Ω_ε^- unifying all these cells are defined via

$$\Omega_\varepsilon^\pm := \bigcup_{\lambda \in \Lambda_\varepsilon^\pm} \varepsilon(\lambda + Y). \tag{2.16}$$

Note that assumption (2.1) implies

$$\lim_{\varepsilon \to 0} \Big(\mu_d(\Omega_\varepsilon^+ \backslash \Omega) + \mu_d(\Omega \backslash \Omega_\varepsilon^-) \Big) = 0. \tag{2.17}$$

In particular this condition is crucial when introducing the two-scale convergence with the help of the so-called *periodic unfolding operator* (see [62] Section 2).

Like in Section 2.2 an admissible microstructure is based on a given tensor valued mapping $\widehat{\mathbb{C}}^{\mathrm{D}} : [0,1]^m \to \mathrm{L}^\infty(Y; \{\mathbb{C}_{\mathrm{strong}}, \mathbb{C}_{\mathrm{weak}}\})$. Here, we already incorporated the fact that we are interested in brutal damage models, i.e., $\widehat{\mathbb{C}}^{\mathrm{D}}(\hat{z})$ for $\hat{z} \in [0,1]^m$ takes either the value $\mathbb{C}_{\mathrm{strong}}$ or $\mathbb{C}_{\mathrm{weak}}$. Similar to (2.6) for a given damage variable $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)$ the microstructure determining tensor $\mathbb{C}_\varepsilon^{\mathrm{D}}(z_\varepsilon)$ is defined by

$$\mathbb{C}_\varepsilon^{\mathrm{D}}(z_\varepsilon)(x) := \begin{cases} \Big(\widehat{\mathbb{C}}^{\mathrm{D}}(z_\varepsilon(x))\Big)(\frac{x}{\varepsilon}) & \text{if } x \in \Omega_\varepsilon^-, \\ \mathbb{C}_{\mathrm{strong}} & \text{if } x \in \Omega \backslash \Omega_\varepsilon^-. \end{cases} \tag{2.18}$$

In comparison to (2.6) this description only differs on the set $\Omega \backslash \Omega_\varepsilon^-$. However, as we will see in Remark 2.7 below, this slight modification ensures that the boundary value assumptions on the microscopic displacement fields are well defined independently of the choice of the tensor $\mathbb{C}_{\mathrm{weak}}$.

With the help of the damage set introduced in Section 2.5 we now replace the analytical description (2.18) of admissible microstructures by a geometrical one. This geometrical description simplifies the illustration and might be the more natural way of understanding microstructure. For this purpose, let $\mathcal{L}_{\mathrm{Leb}}(Y)$ denote the Lebesgue-σ-algebra of the set Y and introduce the set valued function $L : [0,1]^m \to \mathcal{L}_{\mathrm{Leb}}(Y)$ for $\hat{z} \in [0,1]^m$ by $L(\hat{z}) := \{y \in Y \,|\, \widehat{\mathbb{C}}^{\mathrm{D}}(\hat{z})(y) = \mathbb{C}_{\mathrm{weak}}\}$ such that $\widehat{\mathbb{C}}^{\mathrm{D}} : [0,1]^m \to \mathrm{L}^\infty(Y; \{\mathbb{C}_{\mathrm{strong}}, \mathbb{C}_{\mathrm{weak}}\})$ for almost every $y \in Y$ reads as follows:

$$\widehat{\mathbb{C}}^{\mathrm{D}}(\hat{z})(y) = \mathbb{1}_{Y \backslash L(\hat{z})}(y)\mathbb{C}_{\mathrm{strong}} + \mathbb{1}_{L(\hat{z})}(y)\mathbb{C}_{\mathrm{weak}}. \tag{2.19}$$

Moreover, following (2.12) for $z^{\varepsilon\lambda} :\equiv z_\varepsilon|_{\varepsilon(\lambda+Y)}$, where $\lambda \in \Lambda_\varepsilon^-$, we find

$$\Omega_\varepsilon^{\mathrm{D}}(z_\varepsilon) = \bigcup_{\lambda\in\Lambda_\varepsilon^-} \varepsilon\Big(\lambda + L(z^{\varepsilon\lambda})\Big). \tag{2.20}$$

Example 2.6. For gaining a microstructure consisting of balls of different size, the mapping $L : [0,1] \to \mathcal{L}_{\mathrm{Leb}}(Y)$ is defined by $L(\widehat{z}) := B_{r(\widehat{z})}(y_c)$, where $y_c \in \mathrm{int}(Y)$ is the center of the ball $B_{r(\widehat{z})}(y_c)$ with radius $r(\widehat{z})$. Moreover, $R > 0$ needs to be chosen such that $B_R(y_c) \subset Y$. If $r : [0,1] \to [0,R]$ is a non-increasing function, then condition (2.21) below is satisfied, i.e., $r(\widehat{z}) := (1-\widehat{z})R$, for instance. Thus, for a suitable $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1])$ the damage set $\Omega_\varepsilon^{\mathrm{D}}(z_\varepsilon)$ is given by the union of the red balls of the center illustration in Figure 2.2.

According to the identity (2.20) all admissible geometries of damage sets are determined by the image of the function $L : [0,1]^m \to \mathcal{L}_{\mathrm{Leb}}(Y)$. Combining this description with the monotonicity condition (2.14) shows that $L : [0,1]^m \to \mathcal{L}_{\mathrm{Leb}}(Y)$ has to be a non-increasing function in the sense of set inclusions, i.e.:

$$\text{For all } \widehat{z}_1 < \widehat{z}_2 \in [0,1]^m \text{ the set inclusion } L(\widehat{z}_2) \subset L(\widehat{z}_1) \text{ holds.} \tag{2.21}$$

Since the given tensor valued mapping $\widehat{\mathbb{C}}^{\mathrm{D}} : [0,1]^m \to \mathrm{L}^\infty(Y;\{\mathbb{C}_{\mathrm{strong}},\mathbb{C}_{\mathrm{weak}}\})$ is uniquely characterized by the set valued function $L : [0,1]^m \to \mathcal{L}_{\mathrm{Leb}}(Y)$, for $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ the tensor $\mathbb{C}_\varepsilon^{\mathrm{D}}(z_\varepsilon) \in \mathcal{M}(\Omega)$ describing an admissible microstructure reads as follows:

$$\mathbb{C}_\varepsilon^{\mathrm{D}}(z_\varepsilon) := \mathbb{1}_{\Omega\setminus\Omega_\varepsilon^{\mathrm{D}}(z_\varepsilon)}\mathbb{C}_{\mathrm{strong}} + \mathbb{1}_{\Omega_\varepsilon^{\mathrm{D}}(z_\varepsilon)}\mathbb{C}_{\mathrm{weak}}$$

Remark 2.7. (a) *According to* (2.20) *for a given damage variable* $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ *the damage set* $\Omega_\varepsilon^{\mathrm{D}}(z_\varepsilon)$ *only depends on* $z_\varepsilon|_{\varepsilon(\lambda+Y)}$ *for* $\lambda \in \Lambda_\varepsilon^-$. *In other words, two damage variables being elements of* $\mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ *and differing only on* $\Omega\setminus\Omega_\varepsilon^-$ *lead to the same damage set and the subset* $\Omega\setminus\Omega_\varepsilon^-$ *of* Ω *never contains any damaged material for every* $\varepsilon > 0$ *and any damage variable.*

Analogously the tensor $\mathbb{C}_\varepsilon^{\mathrm{D}}(z_\varepsilon)$ *takes the constant value* $\mathbb{C}_{\mathrm{strong}}$ *on the set* $\Omega\setminus\Omega_\varepsilon^-$ *independently of* $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$, *such that two functions* $z_1, z_2 \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ *satisfying* $z_1|_{\Omega_\varepsilon^-} = z_2|_{\Omega_\varepsilon^-}$ *yield the same microstructure.*

(b) *In the case of modeling voids, i.e.,* $\mathbb{C}_{\mathrm{weak}} \equiv \mathbb{O}$, *Remark* 2.5 *asks for*

$$\mathrm{dist}(L(0),\partial Y) > 0$$

to guarantee that $\Omega\setminus\Omega_\varepsilon^{\mathrm{D}}(z_\varepsilon)$ *for all* $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ *is connected. By recalling* $\Omega_\varepsilon^{\mathrm{D}}(z_\varepsilon) \subset \Omega_\varepsilon^-$, *this assumption obviously implies* $\mathrm{cl}(\Omega_\varepsilon^{\mathrm{D}}(z_\varepsilon)) \cap \partial\Omega = \emptyset$ *for any damage variable* $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$. *Therefore, the elliptic boundary value problem* (2.3) *with* $\widetilde{\mathbb{C}} := \mathbb{C}_\varepsilon^{\mathrm{D}}(z_\varepsilon)$ *is well defined for all* $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ *and independently of the choice of* $\mathbb{C}_{\mathrm{weak}}$.

3 Two-scale convergence

All homogenization results presented in this thesis rely on the two-scale convergence developed by Nguetseng in [63]. For this purpose, this chapter introduces everything needed concerning the notation and the theory of folding/unfolding and two-scale convergence but does not claim completeness. For further details we recommend to [2, 11, 13].

3.1 Folding and periodic unfolding operator

This section provides the basic definitions and notations needed to introduce two-scale convergence. Two-scale convergence is linked to a suitable two-scale embedding of the one-scale space $\mathrm{L}^p(\Omega)$ into the two-scale space $\mathrm{L}^p(\mathbb{R}^d \times Y)$. Such an embedding is called periodic unfolding operator. Roughly spoken, by unfolding a one-scale function its "macroscopic behavior" is shifted to one scale, whereas its "microscopic behavior" is shifted to a second scale. This decomposition is based on the periodic lattice Λ, the unit cell Y introduced in Section 2.2, and the following mappings:

$$[\cdot]_\Lambda : \mathbb{R}^d \to \Lambda, \quad \{\cdot\}_Y : \mathbb{R}^d \to Y, \quad \text{and} \quad x = [x]_\Lambda + \{x\}_Y \quad \text{for all } x \in \mathbb{R}^d.$$

For $\lambda \in \Lambda$ and $x \in \lambda + Y \subset \mathbb{R}^d$ it holds $[x]_\Lambda = \lambda$ and $\{x\}_Y \in Y$ is determined by $\{x\}_Y = x - [x]_\Lambda$. Moreover, for $\varepsilon > 0$ and $x \in \mathbb{R}^d$ we have the following decomposition:

$$x = \mathcal{N}_\varepsilon(x) + \varepsilon \mathcal{V}_\varepsilon(x), \quad \text{with } \mathcal{N}_\varepsilon(x) = \varepsilon \left[\frac{x}{\varepsilon}\right]_\Lambda \text{ and } \mathcal{V}_\varepsilon(x) = \left\{\frac{x}{\varepsilon}\right\}_Y, \tag{3.1}$$

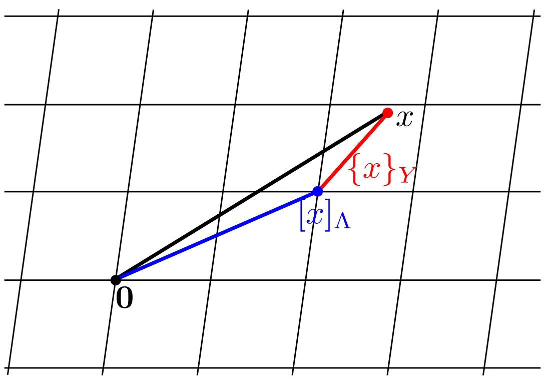


Figure 3.1: The mappings $[\cdot]_\Lambda : \mathbb{R}^d \to \Lambda$ and $\{\cdot\}_Y : \mathbb{R}^d \to Y$.

where $\mathcal{N}_\varepsilon(x) = \varepsilon\lambda$ denotes the vertex $\varepsilon\lambda \in \varepsilon\Lambda$ of the cell $\varepsilon(\lambda+Y) = \mathcal{N}_\varepsilon(x)+\varepsilon Y$ that contains x, and $\mathcal{V}_\varepsilon(x)$ is the microscopic part of x in Y. Following the lines in [69], for $\mathcal{Y} := \mathbb{R}^d/_\Lambda$ denoting the periodicity cell we introduce the mappings $\mathcal{J}_\varepsilon$ and $\mathcal{K}_\varepsilon$ as follows:

$$\mathcal{J}_\varepsilon : \begin{cases} \mathbb{R}^d & \to & \mathbb{R}^d\times\mathcal{Y}, \\ x & \mapsto & (\mathcal{N}_\varepsilon(x), \mathcal{V}_\varepsilon(x)), \end{cases} \qquad \mathcal{K}_\varepsilon : \begin{cases} \mathbb{R}^d\times\mathcal{Y} & \to & \mathbb{R}^d, \\ (x,y) & \mapsto & \mathcal{N}_\varepsilon(x) + \varepsilon y, \end{cases}$$

where in the last sum $y \in \mathcal{Y}$ is identified with $y \in Y \subset \mathbb{R}^d$. Now we are in the position to define the periodic unfolding operator which was introduced first in [11].

Definition 3.1 (Unfolding operator (see [11])). *Let $\varepsilon > 0$ and $p \in [1,\infty]$. Then the periodic unfolding operator $\mathcal{T}_\varepsilon$ is defined via*

$$\mathcal{T}_\varepsilon : \mathrm{L}^p(\Omega) \to \mathrm{L}^p(\mathbb{R}^d\times Y);\ v \mapsto v^{\mathrm{ex}} \circ \mathcal{K}_\varepsilon.$$

For all $p,q,r \in [1,\infty]$ with $\frac{1}{p}+\frac{1}{q} = \frac{1}{r}$ the periodic unfolding operator fulfills the following product rule:

$$\text{If } v_1 \in \mathrm{L}^p(\Omega) \text{ and } v_2 \in \mathrm{L}^q(\Omega), \text{ then } \mathcal{T}_\varepsilon(v_1v_2) = (\mathcal{T}_\varepsilon v_1)(\mathcal{T}_\varepsilon v_2) \in \mathrm{L}^r(\mathbb{R}^d\times Y).$$

Observe that for any function $v \in \mathrm{L}^p(\Omega)$ the support of $\mathcal{T}_\varepsilon v \in \mathrm{L}^p(\mathbb{R}^d\times Y)$ is contained in the closure of the set $[\Omega\times Y]_\varepsilon := \mathcal{K}_\varepsilon^{-1}(\Omega) = \{(x,y)|\mathcal{K}_\varepsilon(x,y) \in \Omega\}$ which is no subset of $\Omega\times Y$. Before introducing the two-scale convergence based on the unfolding operator, a folding operator is defined, which turns out to be the adjoint of the unfolding operator. For details see [62].

Definition 3.2 (Folding operator (see [62])). *Let $\varepsilon > 0$, let $p \in [1,\infty)$, and let the projection $\mathcal{P}_\varepsilon : \mathrm{L}^p(\mathbb{R}^d\times Y) \to \mathrm{K}_{\varepsilon\Lambda}(\mathbb{R}^d;\mathrm{L}^p(Y))$ to piecewise constant functions in the x-component be given by*

$$\mathcal{P}_\varepsilon V(x,y) := \fint_{\mathcal{N}_\varepsilon(x)+\varepsilon Y} V(\widehat{x},y)\mathrm{d}\widehat{x},$$

where $\fint_A g(a)\mathrm{d}a := \frac{1}{\mu_d(A)}\int_A g(a)\mathrm{d}a$ denotes the average of a function g over the set A with $\mu_d(A) > 0$. Then the folding operator $\mathcal{F}_\varepsilon^{(p)}$ is defined via:

$$\mathcal{F}_\varepsilon^{(p)} : \mathrm{L}^p(\mathbb{R}^d\times Y) \to \mathrm{L}^p(\Omega);\ V \mapsto \Big[\Big(\mathcal{P}_\varepsilon(\mathbb{1}_{[\Omega\times Y]_\varepsilon}V)\Big) \circ \mathcal{J}_\varepsilon\Big]\Big|_\Omega.$$

Note that due to the product $\mathbb{1}_{[\Omega\times Y]_\varepsilon}V \in \mathrm{L}^p(\mathbb{R}^d\times Y)$, the folded function $\mathcal{F}_\varepsilon^{(p)}V \in \mathrm{L}^p(\Omega)$ only depends on $V|_{[\Omega\times Y]_\varepsilon}$, which fits to the observation that for $v \in \mathrm{L}^p(\Omega)$ the support of the unfolded function $\mathcal{T}_\varepsilon v \in \mathrm{L}^p(\mathbb{R}^d\times Y)$ is contained in $[\Omega\times Y]_\varepsilon$. Moreover, the properties $\mathcal{P}_\varepsilon\mathbb{1}_{[\Omega\times Y]_\varepsilon} = \mathbb{1}_{[\Omega\times Y]_\varepsilon}$ and $\mathrm{supp}(\mathbb{1}_{[\Omega\times Y]_\varepsilon} \circ \mathcal{J}_\varepsilon) = \mathrm{cl}(\Omega)$ ensure that the support of the function in the square brackets is contained in $\mathrm{cl}(\Omega)$. The following proposition lists basic properties of the periodic unfolding operator and the folding operator which can be proven by decomposing $\mathbb{R}^d$ into $\cup_{\lambda\in\Lambda}\varepsilon(\lambda+Y)$.

Proposition 3.3 (Properties of the unfolding and folding operator (see [62])). *Let $\varepsilon > 0$ and $p \in (1,\infty)$. Then the periodic unfolding operator $\mathcal{T}_\varepsilon : \mathrm{L}^p(\Omega) \to \mathrm{L}^p(\mathbb{R}^d\times Y)$ and the folding operator $\mathcal{F}_\varepsilon^{(p)} : \mathrm{L}^p(\mathbb{R}^d\times Y) \to \mathrm{L}^p(\Omega)$ have the following properties:*

(a) $\|\mathcal{T}_\varepsilon v\|_{\mathrm{L}^p(\mathbb{R}^d\times Y)} = \|v\|_{\mathrm{L}^p(\Omega)}$ *and* $\mathrm{supp}(\mathcal{T}_\varepsilon v) \subset \mathrm{cl}([\Omega\times Y]_\varepsilon)$ *for all* $v \in \mathrm{L}^p(\Omega)$.

(b) $\|\mathcal{F}_\varepsilon^{(p)} V\|_{\mathrm{L}^p(\Omega)} \leq \|V\|_{\mathrm{L}^p(\mathbb{R}^d\times Y)}$ *for all* $V \in \mathrm{L}^p(\mathbb{R}^d\times Y)$.

(c) *Let* $p' := \frac{p}{p-1}$. *Then* $\mathcal{F}_\varepsilon^{(p')}$ *is the adjoint of* $\mathcal{T}_\varepsilon$, *i.e.,* $\mathcal{F}_\varepsilon^{(p')} = (\mathcal{T}_\varepsilon)'$.

(d) $\mathcal{F}_\varepsilon^{(p)} \circ \mathcal{T}_\varepsilon = \mathrm{id}_{\mathrm{L}^p(\Omega)}$ *and* $(\mathcal{T}_\varepsilon \circ \mathcal{F}_\varepsilon^{(p)})^2 = \mathcal{T}_\varepsilon \circ \mathcal{F}_\varepsilon^{(p)} = \mathbb{1}_{[\Omega\times Y]_\varepsilon}\mathcal{P}_\varepsilon$.

3.2 Strong and weak two-scale convergence

Following the lines in [62], we will now use the periodic unfolding operator to introduce the kind of two-scale convergence, which is used here; the strong and weak two-scale convergence, respectively. Moreover, we state the main results, i.e., Proposition 3.5 and Corollary 3.6, concerning the two-scale convergence needed in the following.

Definition 3.4 (Strong and weak two-scale convergence (see [62]))**.** *Let* $p \in [1,\infty)$.

(a) *A sequence* $(v_\varepsilon)_{\varepsilon>0} \subset \mathrm{L}^p(\Omega)$ *converges strongly two-scale to a function* $V \in \mathrm{L}^p(\Omega\times Y)$ *in* $\mathrm{L}^p(\Omega\times Y)$ (*notation:* $v_\varepsilon \overset{s}{\to} V$ *in* $\mathrm{L}^p(\Omega\times Y)$), *if* $\mathcal{T}_\varepsilon v_\varepsilon \to V^{\mathrm{ex}}$ *in* $\mathrm{L}^p(\mathbb{R}^d\times Y)$.

(b) *A sequence* $(v_\varepsilon)_{\varepsilon>0} \subset \mathrm{L}^p(\Omega)$ *converges weakly two-scale to a function* $V \in \mathrm{L}^p(\Omega\times Y)$ *in* $\mathrm{L}^p(\Omega\times Y)$ (*notation:* $v_\varepsilon \overset{w}{\rightharpoonup} V$ *in* $\mathrm{L}^p(\Omega\times Y)$), *if* $\mathcal{T}_\varepsilon v_\varepsilon \rightharpoonup V^{\mathrm{ex}}$ *in* $\mathrm{L}^p(\mathbb{R}^d\times Y)$.

For all $\varepsilon > 0$ the support of the function $\mathcal{T}_\varepsilon v_\varepsilon \in \mathrm{L}^p(\mathbb{R}^d\times Y)$ is contained in the set $\mathrm{cl}([\Omega\times Y]_\varepsilon) \subset \mathrm{cl}(\Omega_\varepsilon^+)\times Y$; see (2.16). This inclusion results in the fact that the support of a possible accumulation point $V \in \mathrm{L}^p(\mathbb{R}^d\times Y)$ of the sequence $(\mathcal{T}_\varepsilon v_\varepsilon)_{\varepsilon>0}$ has to be in $\mathrm{cl}(\Omega)\times Y$, since $\mu_d(\Omega_\varepsilon^+\backslash\Omega) \to 0$. According to $\mu_d(\partial\Omega) = 0$ the function spaces $\mathrm{L}^p(\Omega\times Y)$ and $\mathrm{L}^p(\mathrm{cl}(\Omega)\times Y)$ coincide and hence every accumulation point of $(\mathcal{T}_\varepsilon v_\varepsilon)_{\varepsilon>0}$ can be uniquely identified with an element of $\mathrm{L}^p(\Omega\times Y)$. However, be aware of the necessity of determining two-scale convergence with respect to the space $\mathrm{L}^p(\mathbb{R}^d\times Y)$ and not with respect to $\mathrm{L}^p(\Omega\times Y)$. We refer to [62], where in Example 2.3 it is shown that convergence in $\mathrm{L}^p(\Omega\times Y)$ is not sufficient.

Exploiting Proposition 3.3(c), an equivalent description of the weak two-scale convergence for $p \in (1,\infty)$ and $p' := \frac{p}{p-1}$ reads as follows:

$$v_\varepsilon \overset{w}{\rightharpoonup} V \text{ in } \mathrm{L}^p(\Omega\times Y) \quad \Leftrightarrow \quad \int_\Omega v_\varepsilon \mathcal{F}_\varepsilon^{(p')} W \,\mathrm{d}x \overset{\varepsilon\to 0}{\longrightarrow} \int_{\Omega\times Y} VW \,\mathrm{d}y\mathrm{d}x \quad \forall W \in \mathrm{L}^{p'}(\mathbb{R}^d\times Y).$$

Since the one-scale function $\mathcal{F}_\varepsilon^{(p')} W$ only depends on $W|_{[\Omega\times Y]_\varepsilon}$ it is sufficient to check this convergence only for all $W \in \mathrm{L}^{p'}(\mathrm{cl}(\Omega_{\varepsilon_0}^+)\times Y)$, where $\varepsilon_0 > 0$ is chosen arbitrarily but fixed.

Observe that according to Definition 3.4 for any two-scale convergent sequence all properties known for the L^p-topology are transmitted to the sequence of unfolded functions. Proposition 2.4 in [62] yields a detailed summary of these properties. For the convenience of the reader we state here only those properties used in the following.

Proposition 3.5 (Properties of strong and weak-two-scale convergence (see [62])). *Let $p \in (1,\infty)$ and set $p' := \frac{p}{p-1}$. If $V_0 \in \mathrm{L}^p(\Omega\times Y)$, $W_0 \in \mathrm{L}^{p'}(\Omega\times Y)$, and $M_0 \in \mathrm{L}^1(\Omega\times Y)$ are given, then for $(v_\varepsilon)_{\varepsilon>0} \subset \mathrm{L}^p(\Omega)$ and $(w_\varepsilon)_{\varepsilon>0} \subset \mathrm{L}^{p'}(\Omega)$ the following conditions hold.*

(a) *If $v_\varepsilon \xrightharpoonup{w} V_0$ in $\mathrm{L}^p(\Omega\times Y)$ and if $w_\varepsilon \xrightarrow{s} W_0$ in $\mathrm{L}^{p'}(\Omega\times Y)$, then for $\varepsilon \to 0$ it holds $\langle v_\varepsilon, w_\varepsilon\rangle_{\mathrm{L}^2(\Omega)} \to \langle V_0, W_0\rangle_{\mathrm{L}^2(\Omega\times Y)}$.*

(b) *If $v_\varepsilon \to v_0$ in $\mathrm{L}^p(\Omega)$, then $v_\varepsilon \xrightarrow{s} Ev_0$ in $\mathrm{L}^p(\Omega\times Y)$, where $E : \mathrm{L}^p(\Omega) \to \mathrm{L}^p(\Omega\times Y)$ for $v \in \mathrm{L}^p(\Omega)$ and almost every $(x,y) \in \Omega\times Y$ is defined via $Ev(x,y) := v(x)$.*

(c) *If $v_\varepsilon \xrightarrow{s} V_0$ in $\mathrm{L}^p(\Omega\times Y)$ and if $(m_\varepsilon)_{\varepsilon>0}$ is a bounded sequence of $\mathrm{L}^\infty(\Omega)$ such that $\mathcal{T}_\varepsilon m_\varepsilon(x,y) \to M_0(x,y)$ for almost every $(x,y) \in \Omega\times Y$, then $m_\varepsilon v_\varepsilon \xrightarrow{s} M_0V_0$ in $\mathrm{L}^p(\Omega\times Y)$.*

(d) *For all $V \in \mathrm{L}^p(\Omega\times Y)$ there exists a sequence $(v_\varepsilon)_{\varepsilon>0} \subset \mathrm{L}^p(\Omega)$ such that $v_\varepsilon \xrightarrow{s} V$ in $\mathrm{L}^p(\Omega\times Y)$; $v_\varepsilon := \mathcal{F}_\varepsilon(V^{\mathrm{ex}})$, for instance.*

The following corollary extends Proposition 3.5(c) to a special case appearing when applying the two-scale theory to the microscopic models of Chapter 6, 7, and 8.

Corollary 3.6. *For $p \in (1,\infty)$ let $(v_\varepsilon)_{\varepsilon>0} \subset \mathrm{L}^p(\Omega)$ and $V_0 \in \mathrm{L}^p(\Omega\times Y)$ be given such that $v_\varepsilon \xrightarrow{s} V_0$ in $\mathrm{L}^p(\Omega\times Y)$. If $(m_\varepsilon)_{\varepsilon>0}$ is a bounded sequence in $\mathrm{L}^\infty(\Omega)$ satisfying $m_\varepsilon \xrightarrow{s} M_0$ in $\mathrm{L}^1(\Omega\times Y)$ for some function $M_0 \in \mathrm{L}^1(\Omega\times Y)$, then $m_\varepsilon v_\varepsilon \xrightarrow{s} M_0V_0$ in $\mathrm{L}^p(\Omega\times Y)$.*

Proof. Note that due to the assumptions there exists a subsequence of $(\varepsilon)_{\varepsilon>0}$ for which Proposition 3.5(c) yields the desired result. Now, the validity for the whole sequence $(\varepsilon)_{\varepsilon>0}$ is proven by the following contradiction argument. Assume that the whole sequence $(m_\varepsilon v_\varepsilon)_{\varepsilon>0}$ does not converge strongly two-scale to M_0V_0 in $\mathrm{L}^p(\Omega\times Y)$. Then there exists a constant $C > 0$ and a subsequence $(\varepsilon')_{\varepsilon'>0}$ of $(\varepsilon)_{\varepsilon>0}$ such that for all $\varepsilon' > 0$ it holds $\|\mathcal{T}_{\varepsilon'}(m_{\varepsilon'}v_{\varepsilon'}) - (M_0V_0)^{\mathrm{ex}}\|_{\mathrm{L}^p(\mathbb{R}^d\times Y)} \geq C$. However, there exists a further subsequence $(\varepsilon'')_{\varepsilon''>0}$ of $(\varepsilon')_{\varepsilon'>0}$ fulfilling all assumptions of Proposition 3.5(c). Hence, $m_{\varepsilon''}v_{\varepsilon''} \xrightarrow{s} M_0V_0$ in $\mathrm{L}^p(\Omega\times Y)$ which is a contradiction to the fact that for all $\varepsilon'' > 0$ it holds $\|\mathcal{T}_{\varepsilon''}(m_{\varepsilon''}v_{\varepsilon''}) - (M_0V_0)^{\mathrm{ex}}\|_{\mathrm{L}^p(\mathbb{R}^d\times Y)} \geq C$. □

In Chapter 6 we are going to prove Γ-convergence results with respect to the weak two-scale topology. There, the following integral identity for $v \in \mathrm{L}^1(\Omega)$, which is proven by decomposing $\mathbb{R}^d$ into cells $\cup_{\lambda\in\Lambda}\varepsilon(\lambda+Y)$, will be central.

$$\int_\Omega v(x)\mathrm{d}x = \int_{\mathbb{R}^d\times Y} \mathcal{T}_\varepsilon v(x,y)\mathrm{d}y\mathrm{d}x. \tag{3.2}$$

3.3 Two-scale convergence in Sobolev spaces

This section is dedicated to two-scale convergence results for bounded sequences of Sobolev functions. In particular, for $p \in (1,\infty)$ and $\mathcal{Y} := \mathbb{R}^d/_\Lambda$ the function space

$$\mathrm{W}^{1,p}_{\mathrm{av}}(\mathcal{Y}) = \left\{ v \in \mathrm{W}^{1,p}_{\mathrm{per}}(\mathrm{cl}(Y)) \,\middle|\, \int_Y v(y)\mathrm{d}y = 0 \right\}$$

is crucial for describing two-scale convergence of sequences of gradients. Thus, the function space $\mathrm{L}^p(\Omega;\mathrm{W}^{1,p}_{\mathrm{av}}(\mathcal{Y}))$ is introduced, which is the space of two-scale functions $V \in \mathrm{L}^p(\Omega\times Y) = \mathrm{L}^p(\Omega;\mathrm{L}^p(Y))$, having the same traces on opposite faces of Y and satisfying (i) $\nabla_y V \in \mathrm{L}^p(\Omega\times Y)^d$ in the sense of distributions and (ii) $⨍_Y V(x,y)\mathrm{d}y = 0$ for almost every $x \in \Omega$. This two-scale function space is equipped with the norm $\|V\|_{\mathrm{L}^p(\Omega;\mathrm{W}^{1,p}_{\mathrm{av}}(\mathcal{Y}))} := \|\nabla_y V\|_{\mathrm{L}^p(\Omega\times Y)^d}$. The following compactness result will be exploited for sequences of displacement fields.

Proposition 3.7 (Compactness result)**.** *For $p \in (1,\infty)$ let $(v_\varepsilon)_{\varepsilon>0}$ be a bounded sequence in $\mathrm{W}^{1,p}(\Omega)$. Then there exists a subsequence of $(\varepsilon)_{\varepsilon>0}$ (not relabeled) and functions $v_0 \in \mathrm{W}^{1,p}(\Omega)$ and $V_1 \in \mathrm{L}^p(\Omega;\mathrm{W}^{1,p}_{\mathrm{av}}(\mathcal{Y}))$ such that:*

$$\begin{aligned} v_\varepsilon &\rightharpoonup v_0 && \textit{in } \mathrm{W}^{1,p}(\Omega),\\ v_\varepsilon &\xrightarrow{s} Ev_0 && \textit{in } \mathrm{L}^p(\Omega\times Y),\\ \nabla v_\varepsilon &\xrightharpoonup{w} \nabla_x Ev_0 + \nabla_y V_1 && \textit{in } \mathrm{L}^p(\Omega\times Y)^d, \end{aligned}$$

where $Ev_0(x,y) := v_0(x)$ for almost every $(x,y) \in \Omega\times Y$.

For a proof we refer to Theorem 3.1.4 in [64]. The following density result enables us to construct the so-called *mutual recovery sequence* for the displacement fields considered in Chapter 6.

Proposition 3.8 (Density result)**.** *For $p \in (1,\infty)$ and $p' := \frac{p}{p-1}$ let $w_0 \in \mathrm{W}^{1,p}_0(\Omega)$ and $W_1 \in \mathrm{L}^p(\Omega;\mathrm{W}^{1,p}_{\mathrm{av}}(\mathcal{Y}))$ be given. Moreover, let $Ew_0 \in \mathrm{W}^{1,p}(\Omega\times Y)$ for almost every $(x,y) \in \Omega\times Y$ be defined via $Ew_0(x,y) := w_0(x)$. If $w_\varepsilon \in \mathrm{W}^{1,p}_0(\Omega)$ for $\varepsilon > 0$ denotes the solution of the elliptic problem*

$$\int_\Omega (w_\varepsilon - \mathcal{F}^{(p)}_\varepsilon(Ew_0)^{\mathrm{ex}})\varphi + \langle \nabla w_\varepsilon - \mathcal{F}^{(p)}_\varepsilon(\nabla_x Ew_0 + \nabla_y W_1)^{\mathrm{ex}}, \nabla\varphi\rangle_d \mathrm{d}x = 0 \quad \forall\, \varphi \in \mathrm{W}^{1,p'}_0(\Omega),$$

then the sequence of solutions $(w_\varepsilon)_{\varepsilon>0} \subset \mathrm{W}^{1,p}_0(\Omega)$ fulfills

$$\begin{aligned} w_\varepsilon &\rightharpoonup w_0 && \textit{in } \mathrm{W}^{1,p}_0(\Omega),\\ w_\varepsilon &\xrightarrow{s} Ew_0 && \textit{in } \mathrm{L}^p(\Omega\times Y),\\ \nabla w_\varepsilon &\xrightarrow{s} \nabla_x Ew_0 + \nabla_y W_1 && \textit{in } \mathrm{L}^p(\Omega\times Y)^d. \end{aligned}$$

For a proof we refer to Proposition 2.10 in [62].

3.4 Two-scale limit identification of sequences of non-periodic coefficients

In preparation for the limit passage of the microscopic models of Section 6.1 to the effective models introduced in Section 6.2 and 6.3, we are now going to investigate the asymptotic behavior of sequences of admissible microstructures in the sense of Section 2.2; see

(2.6). In detail, for $p \in (1,\infty)$ we identify the two-scale limit of $(\mathbb{C}_\varepsilon(z_\varepsilon))_{\varepsilon>0}$ when considering a sequence $(z_\varepsilon)_{\varepsilon>0}$ satisfying $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ (see (2.5)) and $z_\varepsilon \to z_0$ in $\mathrm{L}^p(\Omega)^m$ for some function $z_0 \in \mathrm{L}^p(\Omega;[0,1]^m)$. These are reasonable assumptions in the context of the models considered in Section 6.1. For a given $\widehat{\mathbb{C}} \in \mathrm{L}^\infty([0,1]^m;\mathcal{M}(Y))$ and a function $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ we recall that the tensor $\mathbb{C}_\varepsilon(z_\varepsilon) \in \mathcal{M}(\Omega)$ for $x \in \Omega$ is defined by

$$\mathbb{C}_\varepsilon(z_\varepsilon)(x) := \widehat{\mathbb{C}}(z_\varepsilon(x))(\{\tfrac{x}{\varepsilon}\}_Y). \tag{3.3}$$

Theorem 3.9 (Two-scale limit of $(\mathbb{C}_\varepsilon(z_\varepsilon))_{\varepsilon>0}$)**.** *Assume that for $\widehat{\mathbb{C}} : \mathrm{L}^\infty([0,1]^m;\mathcal{M}(Y))$ and any measurable function $z : \mathbb{R}^d \to [0,1]^m$ the mapping*

$$\widehat{\mathbb{C}}(z(\cdot))(\cdot) : \mathbb{R}^d \times Y \to \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}};\mathbb{R}^{d\times d}_{\mathrm{sym}}))$$

is measurable on $\mathbb{R}^d \times Y$. Moreover, let $\widehat{\mathbb{C}} : [0,1]^m \to \mathcal{M}(Y)$ be continuous with respect to the strong L^1-topology, i.e., for any sequence $(\widehat{z}_\delta)_{\delta>0} \subset [0,1]^m$ satisfying $\lim_{\delta\to 0} \widehat{z}_\delta = \widehat{z}$ in $\mathbb{R}^m$ for some $\widehat{z} \in [0,1]^m$ we have

$$\lim_{\delta\to 0} \|\widehat{\mathbb{C}}(\widehat{z}_\delta) - \widehat{\mathbb{C}}(\widehat{z})\|_{\mathrm{L}^1(Y;\mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}};\mathbb{R}^{d\times d}_{\mathrm{sym}}))} = 0.$$

If $(z_\varepsilon)_{\varepsilon>0}$ denotes a sequence of functions satisfying $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ (see (2.5)) and $z_\varepsilon \to z_0$ in $\mathrm{L}^1(\Omega)^m$ for some function $z_0 \in \mathrm{L}^1(\Omega;[0,1])^m$, then

$$\mathbb{C}_\varepsilon(z_\varepsilon) \xrightarrow{s} \mathbb{C}_0(z_0) \quad \text{in } \mathrm{L}^1(\Omega\times Y;\mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}};\mathbb{R}^{d\times d}_{\mathrm{sym}})),$$

where $\mathbb{C}_\varepsilon(z_\varepsilon)$ is defined by (3.3) and $\mathbb{C}_0(z_0)$ for almost every $(x,y) \in \Omega\times Y$ is given by

$$\mathbb{C}_0(z_0)(x,y) := \widehat{\mathbb{C}}(z_0(x))(y). \tag{3.4}$$

Remark 3.10. *Observe that for $C_{\widehat{\mathbb{C}}} := \|\widehat{\mathbb{C}}\|_{\mathrm{L}^\infty([0,1]^m;\mathcal{M}(Y))}$, for any $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$, and some $z_0 \in \mathrm{L}^1(\Omega)^m$ we have $\|\mathbb{C}_\varepsilon(z_\varepsilon)\|_{\mathcal{M}(\Omega)} \le C_{\widehat{\mathbb{C}}}$ and $\|\mathbb{C}_0(z_0)\|_{\mathcal{M}(\Omega\times Y)} \le C_{\widehat{\mathbb{C}}}$ according to the definitions of the tensors $\mathbb{C}_\varepsilon(z_\varepsilon)$ and $\mathbb{C}_0(z_0)$; see (3.3) and (3.4).*

Proof. Let the sequence $(z_\varepsilon)_{\varepsilon>0}$ be given such that $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ and $z_\varepsilon \to z_0$ in $\mathrm{L}^1(\Omega)^m$ for some function $z_0 \in \mathrm{L}^1(\Omega;[0,1])^m$. We start by rewriting the two-scale function $\mathcal{T}_\varepsilon\mathbb{C}_\varepsilon(z_\varepsilon) \in \mathcal{M}(\mathbb{R}^d\times Y)$ to gain a preferably simple description to work with.

The case $x \in \mathbb{R}^d\backslash\mathrm{cl}(\Omega)$: For fixed $x \in \mathbb{R}^d\backslash\mathrm{cl}(\Omega)$ due to (2.17) there exists $\varepsilon_0 > 0$ such that $x \in \mathbb{R}^d\backslash\Omega_\varepsilon^+$ for all $\varepsilon \in (0,\varepsilon_0)$. Hence, $\mathcal{T}_\varepsilon\mathbb{C}_\varepsilon(z_\varepsilon)(x,\cdot) \equiv 0$ on Y for all $\varepsilon \in (0,\varepsilon_0)$ (see Definition 3.1). Moreover, the extension $\mathbb{C}_0^{\mathrm{ex}}(z_0)$ trivially fulfills $\mathbb{C}_0^{\mathrm{ex}}(z_0)(x,\cdot) \equiv 0$ for all $x \in \mathbb{R}^d\backslash\mathrm{cl}(\Omega)$ by definition. Altogether this shows for all $x \in \mathbb{R}^d\backslash\mathrm{cl}(\Omega)$

$$\mathcal{T}_\varepsilon\mathbb{C}_\varepsilon(z_\varepsilon)(x,\cdot) \to \mathbb{C}_0^{\mathrm{ex}}(z_0)(x,\cdot) \quad \text{in } \mathrm{L}^1(Y;\mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}};\mathbb{R}^{d\times d}_{\mathrm{sym}})). \tag{3.5}$$

The case $x \in \Omega$: Let $x \in \Omega$ be fixed. Since Ω is assumed to be open due to (2.17) there exists $\varepsilon_0 > 0$ such that $x \in \Omega_\varepsilon^-$ for all $\varepsilon \in (0,\varepsilon_0)$. Note that for $(x,y) \in \Omega_\varepsilon^-\times Y$ we have (i) $z_\varepsilon(x) = z_\varepsilon(\mathcal{N}_\varepsilon(x))$, (ii) $\mathcal{N}_\varepsilon(\mathcal{N}_\varepsilon(x)+\varepsilon y) = \mathcal{N}_\varepsilon(x)$, and (iii) $\{\frac{\mathcal{N}_\varepsilon(x)+\varepsilon y}{\varepsilon}\}_Y = y$. Keeping these observations in mind when applying $\mathcal{T}_\varepsilon$ to the tensor $\mathbb{C}_\varepsilon(z_\varepsilon)$ given by (3.3) results in

$$\mathcal{T}_\varepsilon \mathbb{C}_\varepsilon(z_\varepsilon)(x,y) = \widehat{\mathbb{C}}(z_\varepsilon(x))(y) \quad \text{for all } (x,y) \in \Omega_\varepsilon^- \times Y. \tag{3.6}$$

According to $z_\varepsilon \to z_0$ in $\mathrm{L}^1(\Omega)^m$ there exists a subsequence $(\varepsilon')_{\varepsilon'>0}$ of $(\varepsilon)_{\varepsilon>0}$ such that

$$z_{\varepsilon'}(x) \to z_0(x) \quad \text{for almost every } x \in \Omega. \tag{3.7}$$

Now, the continuity of $\widehat{\mathbb{C}}$ together with condition (3.7) enables us to pass to the limit in relation (3.6) (at least for the subsequence $(\varepsilon')_{\varepsilon'>0}$ of $(\varepsilon)_{\varepsilon>0}$), i.e., for almost every $x \in \Omega$ we have

$$\mathcal{T}_{\varepsilon'} \mathbb{C}_{\varepsilon'}(z_{\varepsilon'})(x,\cdot) \to \widehat{\mathbb{C}}(z_0(x))(\cdot) \overset{(3.4)}{=} \mathbb{C}_0(z_0)(x,\cdot) \quad \text{in } \mathrm{L}^1(Y;\mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}};\mathbb{R}^{d\times d}_{\mathrm{sym}})). \tag{3.8}$$

Define $f_{\varepsilon'} : \mathbb{R}^d \to [0,\infty)$ by $f_{\varepsilon'}(x) := \|\mathcal{T}_{\varepsilon'} \mathbb{C}_{\varepsilon'}(z_{\varepsilon'})(x,\cdot) - \mathbb{C}_0^{\mathrm{ex}}(z_0)(x,\cdot)\|_{\mathrm{L}^1(Y;\mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}};\mathbb{R}^{d\times d}_{\mathrm{sym}}))}$. Then, by combining (3.5) and (3.8) and exploiting $\mu_d(\partial\Omega) = 0$ (see (2.1)) we finally showed

$$f_{\varepsilon'} \to 0 \quad \text{almost every in } \mathbb{R}^d.$$

Note that due to $\widehat{\mathbb{C}} \in \mathrm{L}^\infty([0,1]^m;\mathcal{M}(Y))$ the sequence $(f_{\varepsilon'})_{\varepsilon'>0}$ is uniformly bounded (see Remark 3.10) and that the support of $f_{\varepsilon'} : \mathbb{R}^d \to [0,\infty)$ is contained in $\Omega_{\varepsilon_0}^+$ for all $\varepsilon' \in (0,\varepsilon_0)$. Hence, the theorem of dominated convergence yields

$$\lim_{\varepsilon'\to 0} \|f_{\varepsilon'}\|_{\mathrm{L}^1(\mathbb{R}^d)} = \lim_{\varepsilon'\to 0} \int_{\mathbb{R}^d} |f_{\varepsilon'}(x)| \mathrm{d}x = 0,$$

which proves

$$\mathbb{C}_{\varepsilon'}(z_{\varepsilon'}) \xrightarrow{s} \mathbb{C}_0(z_0) \quad \text{in } \mathrm{L}^1(\Omega\times Y;\mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}};\mathbb{R}^{d\times d}_{\mathrm{sym}})).$$

By an analog contradiction argument as applied in the proof of Corollary 3.6 we are able to show that this convergence holds for the whole sequence $(\varepsilon)_{\varepsilon>0}$ and the proof is finished. $\square$

4 Discrete gradients of piecewise constant functions

In this chapter a discrete gradient for piecewise constant function is introduced and its essential properties are stated. This discrete gradient for functions being piecewise constant on a given lattice is constructed in such a way that only an overall constant function has gradient zero. Furthermore, an in some sense bounded sequence of those piecewise constant functions (where the spacing of the lattices tends to zero) converges to a limit belonging to a Sobolev space $\mathrm{W}^{1,p}$; $p \in (1, \infty)$. Roughly spoken, we want to introduce a penalty term, extracting those sequences of BV-functions that converge strongly in L^p to a Sobolev function, such that the discrete gradient of these sequences converge weakly in L^p to the gradient of this Sobolev function. Note that although the piecewise constant functions in the Chapters 7, 8, and 9 will play the role of a damage variable in the sense of Section 2.6, the here presented calculus first of all stands on its own concerning the notation and, probably more important, it is not restricted to damage models in its application.

4.1 Definition and motivation

The construction of the discrete gradient is inspired by the so-called *lifting operator* R^{BO} introduced in [8]. There, the authors present a regularization term for so-called *broken Sobolev functions* defined on partitions of Ω consisting of disjoint, polyhedral elements. For such a decomposition of Ω the broken Sobolev space contains all functions $v \in \mathrm{L}^1(\Omega)$, whose restriction to any of these polyhedral elements is a classical Sobolev function. Then, for a sequence $(v_\varepsilon)_{\varepsilon>0}$ of broken Sobolev functions defined on finer and finer polyhedral partitions with a uniformly bounded regularization term, there exists a classical Sobolev function $v_0 \in \mathrm{W}^{1,p}(\Omega)$ and a subsequence $(\varepsilon')_{\varepsilon'>0}$ of $(\varepsilon)_{\varepsilon>0}$ such that $v_{\varepsilon'} \to v_0$ in $\mathrm{L}^p(\Omega)^d$. Moreover, the associated sequence $(\mathrm{R}^{\mathrm{BO}} v_{\varepsilon'})_{\varepsilon'>0}$ of lifted functions converges weakly to ∇v_0 in $\mathrm{L}^p(\Omega)^d$.

The reasons why this regularization approach is not in agreement with our requirements are the following: First of all, the lifting operator of [8] does not serve as a penalty term in the sense that a somehow bounded sequence of lifted broken Sobolev functions has a limit which is a classical Sobolev function. For this reason in [8] a penalty term being independent of the lifting operator is introduced. Furthermore, the lifting operator applied to a non-constant function might be zero as well; see Example 4.3 and 4.4. To combine the beneficial properties of the lifting operator and the penalty term of [8] we are now going to construct a discrete gradient for piecewise constant functions.

To avoid difficulties with cells $\varepsilon(\lambda+Y)$ intersecting the given set Ω but being not completely contained in it ($\varepsilon(\lambda+Y) \not\subset \Omega$) the discrete gradient of a function $z \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ is going to be defined on the set Ω_ε^+; see (2.16). For the subsets Λ_ε^+ and Λ_ε^- of Λ given by (2.15), an continuation operator $V_\varepsilon : \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m \to \mathrm{K}_{\varepsilon\Lambda}(\Omega_\varepsilon^+)^m$ extending a piecewise constant function $z \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ for every $\lambda \in \Lambda_\varepsilon^+ \setminus \Lambda_\varepsilon^-$ on $\varepsilon(\lambda+Y)\setminus\Omega$ constantly by the (constant) value of z on $\varepsilon(\lambda+Y) \cap \Omega$ is introduced as follows:

$$\begin{aligned}&\text{For } z \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m \text{ the function } V_\varepsilon z \in \mathrm{K}_{\varepsilon\Lambda}(\Omega_\varepsilon^+)^m \text{ for every}\\ &\lambda \in \Lambda_\varepsilon^+ \text{ and } z^{\varepsilon\lambda} :\equiv z|_{\varepsilon(\lambda+Y)\cap\Omega} \text{ is defined via } V_\varepsilon z|_{\varepsilon(\lambda+Y)} :\equiv z^{\varepsilon\lambda}.\end{aligned} \tag{4.1}$$

We now define discrete gradients for such functions. Roughly spoken, the discrete gradient of a function $z \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ on a cell $\varepsilon(\lambda+Y) \subset \Omega_\varepsilon^+$ is given by the sum of all finite differences of the values $z|_{\varepsilon(\lambda+Y)}$ and $z|_{\varepsilon(\lambda\pm b_i+Y)}$, where $z|_{\varepsilon(\lambda\pm b_i+Y)}$ for $i = 1, 2, \ldots, d$ denotes the value of z on the "neighboring" cell $\varepsilon(\lambda\pm b_i+Y)$. Hence, for fixed $i \in \{1, 2, \ldots, d\}$ two finite differences are considered on $\varepsilon(\lambda+Y)$ forcing the discrete gradient to be a piecewise constant function with respect to the finer lattice $\frac{\varepsilon}{2}\Lambda$. Observe that $\varepsilon\Lambda \subset \frac{\varepsilon}{2}\Lambda$ by the definition of Λ. The precise definition of the discrete gradient is given in the following definition.

Definition 4.1 (Discrete gradient). *For $V_\varepsilon : \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m \to \mathrm{K}_{\varepsilon\Lambda}(\Omega_\varepsilon^+)^m$ given by* (4.1) *let $R_{\frac{\varepsilon}{2}} : \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m \to \mathrm{K}_{\frac{\varepsilon}{2}\Lambda}(\Omega_\varepsilon^+)^{m\times d}$ be defined via $R_{\frac{\varepsilon}{2}} z := \sum_{i=1}^d \widetilde{R}_{\frac{\varepsilon}{2}}^{(i)}(V_\varepsilon z)$, where for $i = 1, 2, \ldots, d$ the mapping $\widetilde{R}_{\frac{\varepsilon}{2}}^{(i)} : \mathrm{K}_{\varepsilon\Lambda}(\Omega_\varepsilon^+)^m \to \mathrm{K}_{\frac{\varepsilon}{2}\Lambda}(\Omega_\varepsilon^+)^{m\times d}$ for $\overline{z} \in \mathrm{K}_{\varepsilon\Lambda}(\Omega_\varepsilon^+)^m$ reads as follows:*

$$\widetilde{R}_{\frac{\varepsilon}{2}}^{(i)}(\overline{z})(x) := \begin{cases} \frac{1}{\varepsilon|b_i|}\Big(\overline{z}(x+\frac{\varepsilon}{2}b_i) - \overline{z}(x-\frac{\varepsilon}{2}b_i)\Big) \otimes n_i & \text{if } x+\frac{\varepsilon}{2}b_i \in \Omega_\varepsilon^+ \text{ and } x-\frac{\varepsilon}{2}b_i \in \Omega_\varepsilon^+, \\ \mathbf{0} & \text{otherwise,} \end{cases}$$

with $n_i \in \mathbb{R}^d$ given by

$$n_i \in \{b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_d\}^\perp, \quad |n_i|_d = 1, \quad \text{and} \quad \langle n_i, b_i\rangle_d > 0. \tag{4.2}$$

Then the function $R_{\frac{\varepsilon}{2}} z \in \mathrm{K}_{\frac{\varepsilon}{2}\Lambda}(\Omega_\varepsilon^+)^{m\times d}$ is called discrete gradient of $z \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$.

Remark 4.2. *Note that for $z \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ and $\overline{z} := V_\varepsilon z \in \mathrm{K}_{\varepsilon\Lambda}(\Omega_\varepsilon^+)^m$ according to the relation* (4.2) *the discrete gradient $R_{\frac{\varepsilon}{2}} : \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m \to \mathrm{K}_{\frac{\varepsilon}{2}\Lambda}(\Omega_\varepsilon^+)^{m\times d}$ for the given basis $\{b_1, b_2, \ldots, b_d\}$ fulfills $(R_{\frac{\varepsilon}{2}} z)b_i = (\widetilde{R}_{\frac{\varepsilon}{2}}^{(i)}(\overline{z}))b_i \in \mathbb{R}^m$ for any $i \in \{1, 2, \ldots, d\}$.*

$$\Big(\widetilde{R}_{\frac{\varepsilon}{2}}^{(i)}(\overline{z})(x)\Big)b_i = \begin{cases} \frac{1}{\varepsilon|b_i|}\Big(\overline{z}(x+\frac{\varepsilon}{2}b_i) - \overline{z}(x-\frac{\varepsilon}{2}b_i)\Big) & \text{if } x+\frac{\varepsilon}{2}b_i \in \Omega_\varepsilon^+ \text{ and } x-\frac{\varepsilon}{2}b_i \in \Omega_\varepsilon^+, \\ \mathbf{0} & \text{otherwise.} \end{cases} \tag{4.3}$$

This property is exploited several times when proving the compactness result Theorem 4.5 *and the approximation result Theorem* 4.9 *below.*

To point out the differences between the discrete gradient $R_{\frac{\varepsilon}{2}} : \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m \to \mathrm{K}_{\frac{\varepsilon}{2}\Lambda}(\Omega_\varepsilon^+)^{m\times d}$ defined in Definition 4.1 and the lifting operator of by A. Buffa and C. Ortner in [8] we introduce the following notation: Let the so-called *broken Sobolev space* be denoted by $\mathrm{W}^{1,p}_{\varepsilon\Lambda}(\Omega) := \{v \in \mathrm{L}^1(\Omega) \,|\, v|_{\varepsilon(\lambda+Y)\cap\Omega} \in \mathrm{W}^{1,p}(\varepsilon(\lambda+Y)\cap\Omega) \quad \text{for all } \lambda \in \Lambda\}$ and let $S^j_{\varepsilon\Lambda}(\Omega)$ be the set of all piecewise polynomial functions (in the same sense as $\mathrm{K}_{\varepsilon\Lambda}(\Omega)$) with a degree less than or equal to $j \in \mathbb{N}$. Then for $\Gamma^\varepsilon_{\text{int}} := \Omega \cap \bigcup_{\lambda\in\Lambda} \varepsilon(\lambda+\partial Y)$ the lifting operator $\mathrm{R}^{\mathrm{BO}}_{\varepsilon,j} : \mathrm{W}^{1,p}_{\varepsilon\Lambda}(\Omega)^m \to S^j_{\varepsilon\Lambda}(\Omega)^{m\times d}$ introduced in [8] is defined by the relation

$$\int_\Omega \left\langle \mathrm{R}^{\mathrm{BO}}_{\varepsilon,j} v(x), \phi(x)\right\rangle_{m\times d} \mathrm{d}x = -\int_{\Gamma^\varepsilon_{\text{int}}} \left\langle [\![v(s)]\!], \{\{\phi(s)\}\}\right\rangle_{m\times d} \mathrm{d}s \quad \forall\, \phi \in S^j_{\varepsilon\Lambda}(\Omega)^{m\times d}, \tag{4.4}$$

with

$$[\![v(s)]\!] := v^+(s)\otimes n^+ + v^-(s)\otimes n^- \quad \text{and} \quad \{\{\phi(s)\}\} := \tfrac{1}{2}(\phi^+(s)+\phi^-(s)), \tag{4.5}$$

where $v^\pm$, $\phi^\pm$ are the traces of v, ϕ, and $n^\pm$ denotes the respective outward normal of length 1. Observe that $\mathrm{K}_{\varepsilon\Lambda}(\mathbb{R}^d)^m \subset \mathrm{W}^{1,p}_{\varepsilon\Lambda}(\mathbb{R}^d)^m$. Then in the case $j=0$ the relation (4.4) leads to the following explicit expression of the lifting operator for piecewise constant functions:

$$\mathrm{R}^{\mathrm{BO}}_{\varepsilon,0} : \begin{cases} \mathrm{K}_{\varepsilon\Lambda}(\mathbb{R}^d)^m \to \mathrm{K}_{\varepsilon\Lambda}(\mathbb{R}^d)^{m\times d}, \\ z \mapsto \displaystyle\sum_{i=1}^d \tfrac{1}{2\varepsilon|b_i|}\left\{z(\cdot+\varepsilon b_i) - z(\cdot-\varepsilon b_i)\right\}\otimes n_i. \end{cases} \tag{4.6}$$

Here, we replaced Ω by $\mathbb{R}^d$ such that we do not have to care about what is happening in cells $\varepsilon(\lambda+Y)$ intersecting the boundary $\partial\Omega$. Observe that for $z \in \mathrm{K}_{\varepsilon\Lambda}(\mathbb{R}^d)^m$ the function $\mathrm{R}^{\mathrm{BO}}_{\varepsilon,0} z$ is piecewise constant with respect to the lattice $\varepsilon\Lambda$, while $R_{\frac{\varepsilon}{2}} z$ is piecewise constant on the finer lattice $\frac{\varepsilon}{2}\Lambda$. According to (4.6), the value of the discrete gradient $(\mathrm{R}^{\mathrm{BO}}_{\varepsilon,0} z(x))_{k,l}$ ($k \in \{1,\ldots,m\}$ and $l \in \{1,\ldots,d\}$) for $i = 1,2,\ldots,d$ is defined by the values of the function v in the "next" ($z(x{+}\varepsilon b_i)$) and in the "previous" ($z(x{-}\varepsilon b_i)$) cell, but is independent of the value of the "actual" cell ($z(x)$). With the help of the description (4.6) the following examples show that the lifting operator $\mathrm{R}^{\mathrm{BO}}_{\varepsilon,0}$ is missing desirable properties of a discrete gradient for piecewise constant functions.

Example 4.3. Considering a periodic and piecewise constant function $z \in \mathrm{K}_{\varepsilon\Lambda}(\mathbb{R}^d)$ satisfying $z(x{+}\varepsilon b_i) = z(x{-}\varepsilon b_i)$ and $z(x) \neq z(x{+}\varepsilon b_i)$ for every $i \in \{1,\ldots,d\}$ and $x \in \mathbb{R}^d$ we obtain $\mathrm{R}^{\mathrm{BO}}_{\varepsilon,0} z \equiv \mathbf{0}$ although $z \not\equiv \mathit{const}$.

Example 4.4. For $d = 1$, $x \in \mathbb{R}$, and $k \in \mathbb{Z}$ let the sequence $(z_\varepsilon)_{\varepsilon>0}$ of piecewise constant functions $z_\varepsilon \in \mathrm{K}_{\varepsilon^p\Lambda}(\mathbb{R})$ be defined via

$$z_\varepsilon(x) := \begin{cases} 2 & \text{if } x \in \varepsilon^p[2k, 2k+1), \\ -2 & \text{if } x \in -\varepsilon^p[(2|k|+1), 2|k|), \\ 0 & \text{if } x \in \varepsilon^p[(2|k|+1), 2|k|). \end{cases} \tag{4.7}$$

Then z_ε is periodic on $[-\infty, 0)$ as well as $[0,\infty]$ and the sequence $(z_\varepsilon)_{\varepsilon>0}$ converges weakly in $\mathrm{L}^p_{\text{loc}}(\mathbb{R})$ to the Heaviside function H defined by $H(x) = 1$ if $x \geq 0$ and $H(x) = 0$ if

$x < 0$. However, observe that H does not belong to $\mathrm{W}^{1,p}_{\mathrm{loc}}(\mathbb{R})$. According to the definition of the lifting operator we have $|\mathrm{R}^{\mathrm{BO}}_{\varepsilon,0} z_\varepsilon(x)| = \frac{1}{\varepsilon}$ for $x \in [0, \varepsilon^p)$ and $\mathrm{R}^{\mathrm{BO}}_{\varepsilon,0} z_\varepsilon \equiv 0$ otherwise. This gives $\|\mathrm{R}^{\mathrm{BO}}_{\varepsilon,0} z_\varepsilon\|_{\mathrm{L}^p(\mathbb{R})} = 1$ for all $\varepsilon > 0$ which shows that this lifting operator is not the right penalty term in the sense mentioned in the beginning of this section. There is another comment on that in the following section, when explaining the strategy of the proof of Theorem 4.5.

As opposed to this, the discrete gradient defined in Definition 4.1 evaluated for z_ε from (4.7) gives us $|R_{\frac{\varepsilon}{2}} z_\varepsilon(x)| = \frac{4}{\varepsilon}$ for $x < \frac{\varepsilon^p}{2}$ and $|R_{\frac{\varepsilon}{2}} z_\varepsilon(x)| = \frac{2}{\varepsilon}$ otherwise, which leads to $\|R_{\frac{\varepsilon}{2}} z_\varepsilon\|^p_{\mathrm{L}^p(\Omega)} \geq \mu_d(\Omega)(\frac{2}{\varepsilon})^p$ for any bounded subset Ω of $\mathbb{R}$. This shows that this term along $(z_\varepsilon)_{\varepsilon>0}$ is unbounded which correlates with the fact that this sequence does not have a limit belonging to $\mathrm{W}^{1,p}_{\mathrm{loc}}(\mathbb{R})$. This indicates that the L^p-norm of the discrete gradient defined in Definition 4.1 is suitable as a penalty term filtering out sequences of piecewise constant functions converging to elements of $\mathrm{W}^{1,p}(\Omega)^m$. This is stated in the following section.

4.2 Compactness result for piecewise constant functions

The following compactness result states that the L^p-norm of the discrete gradient $R_{\frac{\varepsilon}{2}}$ serves as a penalty term extracting those sequences of piecewise constant functions converging strongly in L^p to a Sobolev function such that the sequence of discrete gradients converges weakly to the gradient of the limit function.

Theorem 4.5 (Compactness result)**.** *Let $R_{\frac{\varepsilon}{2}} : \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m \to \mathrm{K}_{\frac{\varepsilon}{2}\Lambda}(\Omega^+_\varepsilon)^{m\times d}$ be given by Definition 4.1. Then for $p \in (1,\infty)$ and every sequence $(z_\varepsilon)_{\varepsilon>0}$ of functions satisfying $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ for all $\varepsilon > 0$ and*

$$\sup_{\varepsilon>0} \Big(\|z_\varepsilon\|_{\mathrm{L}^p(\Omega)^m} + \|R_{\frac{\varepsilon}{2}} z_\varepsilon\|_{\mathrm{L}^p(\Omega^+_\varepsilon)^{m\times d}} \Big) \leq C < \infty \tag{4.8}$$

there exists a function $z_0 \in \mathrm{W}^{1,p}(\Omega)^m$ and a subsequence of $(\varepsilon)_{\varepsilon>0}$ (not relabeled) with

$$z_\varepsilon \to z_0 \text{ in } \mathrm{L}^q(\Omega)^m \quad \text{and} \quad R_{\frac{\varepsilon}{2}} z_\varepsilon|_\Omega \rightharpoonup \nabla z_0 \text{ in } \mathrm{L}^p(\Omega)^{m\times d},$$

where $1 \leq q < p^*$ and p^* denotes the Sobolev conjugate of p.

Our Theorem 4.5 is a modification of Theorem 5.2 from [8]. There, condition (4.8) is formulated with a penalty term $\int_{\Gamma^\varepsilon_{\mathrm{int}}} \varepsilon^{1-p} |[\![z_\varepsilon(s)]\!]|^p_{m\times d} \mathrm{d}s$ instead of our regularization term $\|R_{\frac{\varepsilon}{2}} z_\varepsilon\|_{\mathrm{L}^p(\Omega^+_\varepsilon)^{m\times d}}$. The authors of [8] end up with a similar convergence result with respect to their discrete gradient $\mathrm{R}^{\mathrm{BO}}_{\varepsilon,0}$. However, due to their procedure a regularized (ε-dependent) model based on functionals depending on piecewise constant functions has to contain two ingredients to arrive at a limit model described by functionals depending solely on Sobolev functions. First, the penalty term $\int_{\Gamma^\varepsilon_{\mathrm{int}}} \varepsilon^{1-p} |[\![z_\varepsilon(s)]\!]|^p_{m\times d} \mathrm{d}s$ forcing the sequence $(z_\varepsilon)_{\varepsilon>0}$ of piecewise constant functions to converge to a Sobolev function, and second, the lifted function $\mathrm{R}^{\mathrm{BO}}_{\varepsilon,0} z_\varepsilon$ to find a gradient in the limit. Thereby a further issue arises, namely, the identification and interpretation of the penalty term after passing to the limit. Clearly, due to our replacement this problem is solved. Since the proof of our Theorem 4.5 is based on that of Theorem 5.2 from [8] we need the estimate of Lemma 4.6 and the identity of Lemma 4.7 to adapt the proof from [8].

Lemma 4.6. *For $z \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ let Dz be the measure representing its distributional derivative and let $|Dz|(\Omega)$ denote the total variation of Dz. Then, there exist constants $C_1 > 0$ and $C_2 > 0$ such that for any $p \in [1, \infty)$ and for every $\varepsilon > 0$ it holds*

$$|Dz|(\Omega) \le C_1 \bigg(\int_{\Gamma^\varepsilon_{\mathrm{int}}} \varepsilon^{1-p} |[\![z(s)]\!]|^p_{m\times d} \mathrm{d}s \bigg)^{\frac{1}{p}} \le C_1 C_2 \|R_{\frac{\varepsilon}{2}} z\|_{\mathrm{L}^p(\Omega^+_\varepsilon)^{m\times d}},$$

where $\Gamma^\varepsilon_{\mathrm{int}} := \Omega \cap \bigcup_{\lambda\in\Lambda} \varepsilon(\lambda{+}\partial Y)$ and $[\![\cdot]\!]$ is defined in (4.5).

Proof. The proof of the first inequality is a straight forward generalization of Theorem 3.26 from [48] to the case of $p \neq 2$ and can be found in [8, Lemma 2] as a brief sketch, for example.

The second inequality results from the special structure of the discrete gradient. For a better understanding the calculations are split up such that the left hand side of every estimate is the same (starting point) and the only changes are on the right hand side. First of all (4.9) below is valid since every face of the cell $\varepsilon(\lambda{+}Y)$ is taken twice when summing up on the right hand side:

$$\int_{\Gamma^\varepsilon_{\mathrm{int}}} \varepsilon^{1-p} |[\![z(s)]\!]|^p_{m\times d} \mathrm{d}s = \tfrac{1}{2} \sum_{\lambda\in\Lambda^+_\varepsilon} \int_{\varepsilon(\lambda+\partial Y)} \varepsilon^{1-p} |[\![z^{\mathrm{ex}}(s)]\!]|^p_{m\times d} \mathbb{1}_\Omega(s) \mathrm{d}s \tag{4.9}$$

Since the integrand of the right hand side contains the characteristic function $\mathbb{1}_\Omega$, the function $z \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ can be replaced by any extension $\overline{z} \in \mathrm{L}^p(\Omega^+_\varepsilon)$ satisfying $\overline{z}|_\Omega = z$. We choose $\overline{z} := (V_\varepsilon z) \in \mathrm{K}_{\varepsilon\Lambda}(\Omega^+_\varepsilon)^m$; see (4.1). Since $R_{\frac{\varepsilon}{2}} z$ is a piecewise constant function with respect to the lattice $\frac{\varepsilon}{2}\Lambda$ we now artificially insert this finer lattice to the right hand side of (4.9) by decomposing every cell $\varepsilon(\lambda{+}Y) \subset \Omega^+_\varepsilon$ into 2^d equal parts in the following way: For fixed $\lambda \in \Lambda^+_\varepsilon$ there are 2^d elements $\lambda_1, \lambda_2, \ldots, \lambda_{2^d}$ of Λ such that

$$\varepsilon(\lambda{+}Y) = \bigcup_{j=1}^{2^d} \tfrac{\varepsilon}{2}(\lambda_j{+}Y). \tag{4.10}$$

Actually, on the right hand side of (4.11) below the domain of integration has increased in comparison to the right hand side of (4.9). However, since $\overline{z} \in \mathrm{K}_{\varepsilon\Lambda}(\Omega^+_\varepsilon)^m$ is constant on every cell $\varepsilon(\lambda{+}Y) \subset \Omega^+_\varepsilon$ we have $[\![\overline{z}(s)]\!] = \mathbf{0}$ for $s \in \frac{\varepsilon}{2}(\lambda_j{+}\partial Y) \backslash \varepsilon(\lambda{+}\partial Y)$ and every $j = 1, 2, \ldots, 2^d$. That is why the following equality is valid, since only zeros are added:

$$\int_{\Gamma^\varepsilon_{\mathrm{int}}} \varepsilon^{1-p} |[\![z(s)]\!]|^p_{m\times d} \mathrm{d}s = \tfrac{1}{2} \sum_{\lambda\in\Lambda^+_\varepsilon} \sum_{j=1}^{2^d} \varepsilon^{1-p} \int_{\frac{\varepsilon}{2}(\lambda_j+\partial Y)} |[\![\overline{z}(s)]\!]|^p_{m\times d} \mathbb{1}_\Omega(s) \mathrm{d}s. \tag{4.11}$$

To keep the following calculations as clear as possible, without loss of generality we are now going to assume $|b_i| = 1$ for all $i \in \{1, 2, \ldots, d\}$. Roughly spoken, we now first of all increase the domain of integration on the right hand side of (4.11) by replacing $\mathbb{1}_\Omega$ by $\mathbb{1}_{\mathrm{cl}(\Omega_\varepsilon^+)}$. Then we calculate the integral by splitting $\frac{\varepsilon}{2}(\lambda_j+\partial Y)$ into its $2d$ faces of $\frac{\varepsilon}{2}(\lambda_j+Y)$, afterwards. For $s \in \partial\Omega_\varepsilon^+$ the jump term $[\![\overline{z}(s)]\!]$ is not well-defined since $\mathrm{supp}(\overline{z}) \subset \mathrm{cl}(\Omega_\varepsilon^+)$. That is why we set $[\![\overline{z}(\cdot)]\!] := \mathbf{0}$ on $\partial\Omega_\varepsilon^+$. Since the integrand is constant on every face, integrating results in the product of this constant value and $(\frac{\varepsilon}{2})^{d-1}$, which is just the volume of one face. Moreover, the jump term of $\overline{z}$ is replaced by its definition (see (4.5)), where $z^+ = \overline{z}(\frac{\varepsilon}{2}\lambda_j)$, $z^- = \overline{z}(\frac{\varepsilon}{2}(\lambda_j+b_i))$, and $n^+ = -n^- = n_i$ are used for one face of $\frac{\varepsilon}{2}(\lambda_j+Y)$ as well as $z^+ = \overline{z}(\frac{\varepsilon}{2}\lambda_j)$, $z^- = \overline{z}(\frac{\varepsilon}{2}(\lambda_j-b_i))$, and $n^+ = -n^- = -n_i$ for the opposite one. Applying all these changes to the right hand side of (4.11) yields

$$\begin{aligned}\int_{\Gamma_{\mathrm{int}}^\varepsilon} \varepsilon^{1-p} |[\![z(s)]\!]|_{m\times d}^p \mathrm{d}s \leq \tfrac{1}{2} \sum_{\substack{j=1,\\ \lambda\in\Lambda_\varepsilon^+}}^{2^d} \varepsilon^{1-p} \sum_{i=1}^{d} (\tfrac{\varepsilon}{2})^{d-1} \Big| \Big[\overline{z}\Big(\tfrac{\varepsilon}{2}\lambda_j\Big) - \overline{z}\Big(\tfrac{\varepsilon}{2}(\lambda_j+b_i)\Big)\Big] \otimes n_i \Big|_{m\times d}^p \delta_{i,j}^{(\lambda)} \\ + (\tfrac{\varepsilon}{2})^{d-1} \Big| \Big[\overline{z}\Big(\tfrac{\varepsilon}{2}(\lambda_j-b_i)\Big) - \overline{z}\Big(\tfrac{\varepsilon}{2}\lambda_j\Big)\Big] \otimes n_i \Big|_{m\times d}^p \widetilde{\delta}_{i,j}^{(\lambda)},\end{aligned} \tag{4.12}$$

where

$$\delta_{i,j}^{(\lambda)} := \begin{cases} 0 & \text{if } \frac{\varepsilon}{2}(\lambda_j+b_i) \notin \Omega_\varepsilon^+, \\ 1 & \text{otherwise,} \end{cases} \qquad \widetilde{\delta}_{i,j}^{(\lambda)} := \begin{cases} 0 & \text{if } \frac{\varepsilon}{2}(\lambda_j-b_i) \notin \Omega_\varepsilon^+, \\ 1 & \text{otherwise.} \end{cases}$$

As already mentioned, in line (4.11) there are added a lot of zeros in comparison to (4.9), which results in the following: For fixed $\lambda \in \Lambda_\varepsilon^+$ and λ_j chosen as in (4.10) we have $\frac{\varepsilon}{2}\lambda_j \in \varepsilon(\lambda+Y)$ for all $j = 1, \ldots, 2^d$. Moreover, for any $i \in \{1, \ldots, d\}$ and all $j \in \{1, \ldots, 2^d\}$ we either we have $\frac{\varepsilon}{2}(\lambda_j+b_i) \in \varepsilon(\lambda+Y)$ or $\frac{\varepsilon}{2}(\lambda_j-b_i) \in \varepsilon(\lambda+Y)$, which either gives us $\overline{z}(\frac{\varepsilon}{2}(\lambda_j+b_i)) = \overline{z}(\frac{\varepsilon}{2}\lambda_j)$ or results in $\overline{z}(\frac{\varepsilon}{2}(\lambda_j-b_i)) = \overline{z}(\frac{\varepsilon}{2}\lambda_j)$. Thus, always one of the terms on the right hand side of (4.12) is zero and the other can be replaced in the following way ($\widehat{\delta}_{i,j}^{(\lambda)} := \delta_{i,j}^{(\lambda)}\widetilde{\delta}_{i,j}^{(\lambda)}$):

$$\int_{\Gamma_{\mathrm{int}}^\varepsilon} \varepsilon^{1-p} |[\![z(s)]\!]|_{m\times d}^p \mathrm{d}s \leq \sum_{\substack{j=1,\\ \lambda\in\Lambda_\varepsilon^+}}^{2^d} \frac{\varepsilon^d}{2^d} \sum_{i=1}^{d} \Big| \frac{1}{\varepsilon} \Big[\overline{z}\Big(\tfrac{\varepsilon}{2}(\lambda_j-b_i)\Big) - \overline{z}\Big(\tfrac{\varepsilon}{2}(\lambda_j+b_i)\Big)\Big] \otimes n_i \Big|_{m\times d}^p \widehat{\delta}_{i,j}^{(\lambda)}. \tag{4.13}$$

The next step is interchanging the sum $\sum_{i=1}^d$ and the matrix norm $|\cdot|_{m\times d}$ on the right hand side of (4.13). Therefore, we set $f_\varepsilon^{\lambda_j}(b_i) := \frac{1}{\varepsilon}[\overline{z}(\frac{\varepsilon}{2}(\lambda_j-b_i)) - \overline{z}(\frac{\varepsilon}{2}(\lambda_j+b_i))]$ to shorten notation and observe that for all $i, k = 1, \ldots, d$ according to the relation (4.2) of the vectors n_i and b_k we have $(f_\varepsilon^{\lambda_j}(b_i) \otimes n_i)b_k = f_\varepsilon^{\lambda_j}(b_i)\delta_{ik}$ (see also Remark 4.2). The interchange is based on the following trivial calculation:

$$\begin{aligned}\sum_{i,k=1}^{d} \Big| \Big(f_\varepsilon^{\lambda_j}(b_i) \otimes n_i\Big) b_k \Big|_m^p &= \sum_{i,k=1}^{d} \Big| f_\varepsilon^{\lambda_j}(b_i)\delta_{ik} \Big|_m^p = \sum_{k=1}^{d} \Big| f_\varepsilon^{\lambda_j}(b_k) \Big|_m^p \\ &= \sum_{k=1}^{d} \Big| \sum_{i=1}^{d} (f_\varepsilon^{\lambda_j}(b_i)\delta_{ik}) \Big|_m^p = \sum_{k=1}^{d} \Big| \Big(\sum_{i=1}^{d} (f_\varepsilon^{\lambda_j}(b_i) \otimes n_i) \Big) b_k \Big|_m^p\end{aligned} \tag{4.14}$$

For $A \in \mathbb{R}^{m\times d}$ and the basis $\{b_1,\dots,b_d\}$ of $\mathbb{R}^d$ let $|\cdot|_{\{b_1,\dots,b_d\}}$ denote the matrix norm defined by $|A|^p_{\{b_1,\dots,b_d\}} := \sum_{k=1}^d |Ab_k|^p_m$. Then the interchange is performed as follows:

$$\begin{aligned}\sum_{i=1}^d \Big|f_\varepsilon^{\lambda_j}(b_i)\otimes n_i\Big|^p_{m\times d} &\le C\sum_{i=1}^d \Big|f_\varepsilon^{\lambda_j}(b_i)\otimes n_i\Big|^p_{\{b_1,\dots,b_d\}} = C\sum_{i,k=1}^d \Big|\Big(f_\varepsilon^{\lambda_j}(b_i)\otimes n_i\Big)b_k\Big|^p_m\\ &\overset{(4.14)}{=} C\sum_{k=1}^d\Big|\Big(\sum_{i=1}^d (f_\varepsilon^{\lambda_j}(b_i)\otimes n_i)\Big)b_k\Big|^p_m = C\Big|\sum_{i=1}^d (f_\varepsilon^{\lambda_j}(b_i)\otimes n_i)\Big|^p_{\{b_1,\dots,b_d\}}\\ &\le \widehat{C}C\Big|\sum_{i=1}^d f_\varepsilon^{\lambda_j}(b_i)\otimes n_i\Big|^p_{m\times d},\end{aligned}$$

where we exploited the norm equivalence in dimension md two times. For $C_2^p := \widehat{C}C$ this estimate turns the right hand side of (4.13) into

$$\int_{\Gamma^\varepsilon_{\text{int}}} \varepsilon^{1-p}|[\![z(s)]\!]|^p_{m\times d}\mathrm{d}s \le C_2^p \sum_{\substack{j=1,\\ \lambda\in\Lambda^+_\varepsilon}}^{2^d} \frac{\varepsilon^d}{2^d}\Big|\sum_{i=1}^d \frac{1}{\varepsilon}\Big[\overline{z}\Big(\tfrac{\varepsilon}{2}(\lambda_j - b_i)\Big) - \overline{z}\Big(\tfrac{\varepsilon}{2}(\lambda_j + b_i)\Big)\Big]\otimes n_i\Big|^p_{m\times d}\widehat{\delta}^{(\lambda)}_{i,j}.$$

Replacing $\frac{\varepsilon^d}{2^d}$ by the integral over $\frac{\varepsilon}{2}(\lambda_j + Y)$ we finally end up with

$$\begin{aligned}&\int_{\Gamma^\varepsilon_{\text{int}}} \varepsilon^{1-p}|[\![z(s)]\!]|^p_{m\times d}\mathrm{d}s\\ &\qquad\le C_2^p \sum_{\substack{j=1,\\ \lambda\in\Lambda^+_\varepsilon}}^{2^d}\int_{\frac{\varepsilon}{2}(\lambda_j+Y)}\Big|\sum_{i=1}^d \frac{1}{\varepsilon}\Big[\overline{z}\Big(\tfrac{\varepsilon}{2}(\lambda_j - b_i)\Big) - \overline{z}\Big(\tfrac{\varepsilon}{2}(\lambda_j + b_i)\Big)\Big]\otimes n_i\Big|^p_{m\times d}\widehat{\delta}^{(\lambda)}_{i,j}\mathrm{d}x\\ &\qquad= C_2^p \sum_{\substack{j=1,\\ \lambda\in\Lambda^+_\varepsilon}}^{2^d}\int_{\frac{\varepsilon}{2}(\lambda_j+Y)}\Big|\sum_{i=1}^d \widehat{\delta}^{(\lambda)}_{i,j}\frac{1}{\varepsilon}\Big[\overline{z}\Big(x - \tfrac{\varepsilon}{2}b_i\Big) - \overline{z}\Big(x + \tfrac{\varepsilon}{2}b_i\Big)\Big]\otimes n_i\Big|^p_{m\times d}\mathrm{d}x\\ &\qquad= C_2^p\Big\|\sum_{i=1}^d \widetilde{R}^{(i)}_{\frac{\varepsilon}{2}}(\overline{z})\Big\|^p_{\mathrm{L}^p(\Omega^+_\varepsilon)^{m\times d}},\end{aligned}$$

where we used $\overline{z}(\cdot \pm \frac{\varepsilon}{2}b_i) \equiv \overline{z}(\frac{\varepsilon}{2}\lambda_j \pm \frac{\varepsilon}{2}b_i)$ on $\frac{\varepsilon}{2}(\lambda_j + Y) \subset \Omega^+_\varepsilon$, which is valid for all functions belonging to $\mathrm{K}_{\varepsilon\Lambda}(\Omega^+_\varepsilon)^m$ due to their special structure. Replacing $\overline{z}$ by $V_\varepsilon z$ concludes the proof. $\square$

Since for $z \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m \subset \mathrm{W}^{1,p}_{\varepsilon\Lambda}(\Omega)$ the proof of compactness Theorem 5.2 in [8] relies on the definition of $\mathrm{R}^{\mathrm{BO}}_{\varepsilon,0}z \in \mathrm{K}_{\frac{\varepsilon}{2}\Lambda}(\Omega)^{m\times d}$ by the identity (4.4) the next lemma states that the discrete gradient $R_{\frac{\varepsilon}{2}}z \in \mathrm{K}_{\frac{\varepsilon}{2}\Lambda}(\Omega^+_\varepsilon)^{m\times d}$ of z fulfills a similar relation.

Lemma 4.7. *Let* $R_{\frac{\varepsilon}{2}} : \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m \to \mathrm{K}_{\frac{\varepsilon}{2}\Lambda}(\Omega^+_\varepsilon)^{m\times d}$ *be given by Definition* 4.1. *Then for* $\varepsilon > 0$, *for all* $z \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$, *and every* $\varphi \in \mathrm{K}_{\varepsilon\Lambda}(\Omega^-_\varepsilon)^{m\times d}$ *it holds*

$$\int_\Omega \Big\langle R_{\frac{\varepsilon}{2}}z(x), \varphi^{\text{ex}}(x)\Big\rangle_{m\times d}\mathrm{d}x = -\int_{\Gamma^\varepsilon_{\text{int}}}\Big\langle [\![z(s)]\!], \{\{\varphi^{\text{ex}}(s)\}\}\Big\rangle_{m\times d}\mathrm{d}s, \tag{4.15}$$

where $\Gamma^\varepsilon_{\text{int}} := \Omega \cap \bigcup_{\lambda\in\Lambda}\varepsilon(\lambda + \partial Y)$ *and* $[\![\cdot]\!]$ *as well as* $\{\{\cdot\}\}$ *are defined in* (4.5).

Remark 4.8. *Observe that obviously* $\mathrm{R}^{\mathrm{BO}}_{\varepsilon,0} z \in \mathrm{K}_{\frac{\varepsilon}{2}\Lambda}(\Omega)^{m\times d}$ *satisfies relation* (4.15) *for all test-functions* $\varphi \in \mathrm{K}_{\varepsilon\Lambda}(\Omega^-_\varepsilon)^{m\times d}$ *and hence this relation does not uniquely define the discrete gradient* $R_{\frac{\varepsilon}{2}} : \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m \to \mathrm{K}_{\frac{\varepsilon}{2}\Lambda}(\Omega^+_\varepsilon)^{m\times d}$ *in contrast to the lifting operator* $\mathrm{R}^{\mathrm{BO}}_{\varepsilon,0} : \mathrm{W}^{1,p}_{\varepsilon\Lambda}(\Omega)^m \to S^0_{\varepsilon\Lambda}(\Omega)^{m\times d}$ *defined by* (4.4).

Proof. We start with rearranging the right hand side of (4.15). Since we are only testing with functions $\varphi \in \mathrm{K}_{\varepsilon\Lambda}(\Omega^-_\varepsilon)^{m\times d}$, analogously to the proof of Lemma 4.6 the function $z \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ can be replaced by the extension $\overline{z} := (V_\varepsilon z) \in \mathrm{K}_{\varepsilon\Lambda}(\Omega^+_\varepsilon)^m$.

Let $\lambda \in \Lambda$ and $s \in \varepsilon(\lambda{+}\partial Y)$. Then $\{\{\varphi^{\mathrm{ex}}(s)\}\} \neq \mathbf{0}$ implies $s \in \Gamma^\varepsilon_{\mathrm{int}}$, which is why the domain of integration can be increased to $\cup_{\lambda\in\Lambda}\varepsilon(\lambda{+}\partial Y)$. Therefore, $\overline{z} \in \mathrm{K}_{\varepsilon\Lambda}(\Omega^+_\varepsilon)^m$ needs to be replaced by its extension $\overline{z}^{\mathrm{ex}} \in \mathrm{K}_{\varepsilon\Lambda}(\mathbb{R}^d)^m$ extending it with $\mathbf{0}$ to $\mathbb{R}^d$. Note that according to $\{\{\varphi^{\mathrm{ex}}(s)\}\} \equiv \mathbf{0}$ for $s \in \partial\Omega^+_\varepsilon$ the additional jump $[\![\overline{z}^{\mathrm{ex}}(s)]\!] \neq \mathbf{0}$ does not play any role in the following calculations. On the right hand side of (4.16) below, every face of a cell $\varepsilon(\lambda{+}Y)$ is taken twice when summing up which is why this is an equality:

$$\int_{\Gamma^\varepsilon_{\mathrm{int}}} \Big\langle [\![z(s)]\!], \{\{\varphi^{\mathrm{ex}}(s)\}\} \Big\rangle_{m\times d} \mathrm{d}s = \tfrac{1}{2} \sum_{\lambda\in\Lambda} \int_{\varepsilon(\lambda+\partial Y)} \Big\langle [\![\overline{z}^{\mathrm{ex}}(s)]\!], \{\{\varphi^{\mathrm{ex}}(s)\}\} \Big\rangle_{m\times d} \mathrm{d}s. \tag{4.16}$$

Analog to the proof of Lemma 4.6 we calculate the integral which gives the factor ε^{d-1}. Note that we again assume $|b_i| = 1$ for all $i \in \{1, 2, \dots, d\}$. Furthermore, for fixed $\lambda \in \Lambda$ the jump term of $\overline{z}^{\mathrm{ex}}$ and the mean value term of φ^{ex} are replaced by $\Big(\overline{z}^{\mathrm{ex}}(\varepsilon\lambda) - \overline{z}^{\mathrm{ex}}(\varepsilon(\lambda{+}b_i))\Big) \otimes n_i$ and $\frac{1}{2}\Big(\varphi^{\mathrm{ex}}(\varepsilon\lambda) + \varphi^{\mathrm{ex}}(\varepsilon(\lambda{+}b_i))\Big)$ for one face of $\varepsilon(\lambda{+}Y)$ and by $\Big(\overline{z}^{\mathrm{ex}}(\varepsilon\lambda) - \overline{z}^{\mathrm{ex}}(\varepsilon(\lambda{-}b_i))\Big) \otimes (-n_i)$ and $\frac{1}{2}\Big(\varphi^{\mathrm{ex}}(\varepsilon\lambda) + \varphi^{\mathrm{ex}}(\varepsilon(\lambda{-}b_i))\Big)$ for the opposite one:

$$\begin{aligned}
&\int_{\Gamma^\varepsilon_{\mathrm{int}}} \Big\langle [\![z(s)]\!], \{\{\varphi^{\mathrm{ex}}(s)\}\} \Big\rangle_{m\times d} \mathrm{d}s \\
&= \tfrac{1}{2} \sum_{\lambda\in\Lambda} \varepsilon^{d-1} \sum_{i=1}^{d} \Big[\Big\langle \Big(\overline{z}^{\mathrm{ex}}(\varepsilon\lambda) {-} \overline{z}^{\mathrm{ex}}(\varepsilon(\lambda{+}b_i))\Big) {\otimes} n_i, \tfrac{1}{2}\Big(\varphi^{\mathrm{ex}}(\varepsilon\lambda) {+} \varphi^{\mathrm{ex}}(\varepsilon(\lambda{+}b_i))\Big) \Big\rangle_{m\times d} \\
&\qquad\qquad + \Big\langle \Big(\overline{z}^{\mathrm{ex}}(\varepsilon(\lambda{-}b_i)) {-} \overline{z}^{\mathrm{ex}}(\varepsilon\lambda)\Big) {\otimes} n_i, \tfrac{1}{2}\Big(\varphi^{\mathrm{ex}}(\varepsilon(\lambda{-}b_i)) {+} \varphi^{\mathrm{ex}}(\varepsilon\lambda)\Big) \Big\rangle_{m\times d} \Big]. \qquad (4.17)
\end{aligned}$$

Now, the sums are interchanged and the translation $\lambda^* = \lambda{-}b_i$ is applied to line (4.17) for every $i = 1, \dots, d$, such that we end up with

$$\begin{aligned}
&\int_{\Gamma^\varepsilon_{\mathrm{int}}} \Big\langle [\![z(s)]\!], \{\{\varphi^{\mathrm{ex}}(s)\}\} \Big\rangle_{m\times d} \mathrm{d}s \\
&= \tfrac{\varepsilon^{d-1}}{2} \sum_{i=1}^{d} \sum_{\lambda\in\Lambda} \Big\langle \Big(\overline{z}^{\mathrm{ex}}(\varepsilon\lambda) - \overline{z}^{\mathrm{ex}}(\varepsilon(\lambda{+}b_i))\Big) \otimes n_i, \varphi^{\mathrm{ex}}(\varepsilon\lambda) + \varphi^{\mathrm{ex}}(\varepsilon(\lambda{+}b_i)) \Big\rangle_{m\times d}. \qquad (4.18)
\end{aligned}$$

For rearranging the left hand side of (4.15) we introduce $Y_{b_i} = \{y \in Y \,|\, y{-}\frac{1}{2}b_i \in Y\}$ ($Y = [0,1)^d \Rightarrow Y_{e_1} = [\frac{1}{2}, 1){\times}[0,1)^{d-1}$) and $f^{(i)}_\varepsilon(x) := \frac{1}{\varepsilon}\Big(\overline{z}(x{+}\frac{\varepsilon}{2}b_i) - \overline{z}(x{-}\frac{\varepsilon}{2}b_i)\Big) \otimes n_i$ for $\overline{z} := V_\varepsilon(z) \in \mathrm{K}_{\varepsilon\Lambda}(\Omega^+_\varepsilon)^m$ to shorten notation. Then due to Definition 4.1 it holds

$$\int_\Omega \Big\langle R_{\frac{\varepsilon}{2}} z(x), \varphi^{\mathrm{ex}}(x) \Big\rangle_{m\times d} \mathrm{d}x = \sum_{\lambda\in\Lambda^-_\varepsilon} \int_{\varepsilon(\lambda+Y)} \sum_{i=1}^{d} \Big\langle f^{(i)}_\varepsilon(x), \varphi(\varepsilon\lambda) \Big\rangle_{m\times d} \mathrm{d}x, \tag{4.19}$$

where for $\lambda \in \Lambda_\varepsilon^-$ we already used $\varphi \equiv \varphi(\varepsilon\lambda)$ on $\varepsilon(\lambda{+}Y)$. Observing that

$$f_\varepsilon^{(i)}(x) = \begin{cases} \frac{1}{\varepsilon}\Big(\overline{z}(\varepsilon(\lambda{+}b_i)) - \overline{z}(\varepsilon\lambda)\Big) \otimes n_i & \text{if } x \in \varepsilon(\lambda{+}Y_{b_i}), \\ \frac{1}{\varepsilon}\Big(\overline{z}(\varepsilon\lambda) - \overline{z}(\varepsilon(\lambda{-}b_i))\Big) \otimes n_i & \text{if } x \in \varepsilon(\lambda{+}Y\backslash Y_{b_i}) \end{cases}$$

we are able to reformulate the right hand side of (4.19) by interchanging integration and summation in the following way:

$$\begin{aligned} &\int_\Omega \Big\langle R_{\frac{\varepsilon}{2}} z(x), \varphi^{\mathrm{ex}}(x)\Big\rangle_{m\times d} \mathrm{d}x \\ &\quad = \sum_{\lambda\in\Lambda_\varepsilon^-}\sum_{i=1}^d \bigg(\int_{\varepsilon(\lambda+Y_{b_i})} \Big\langle f_\varepsilon^{(i)}(x), \varphi(\varepsilon\lambda)\Big\rangle_{m\times d}\mathrm{d}x + \int_{\varepsilon(\lambda+Y\backslash Y_{b_i})} \Big\langle f_\varepsilon^{(i)}(x), \varphi(\varepsilon\lambda)\Big\rangle_{m\times d}\mathrm{d}x\bigg) \\ &\quad = \sum_{\lambda\in\Lambda_\varepsilon^-}\sum_{i=1}^d \Big[\tfrac{1}{2}\varepsilon^d \Big\langle \tfrac{1}{\varepsilon}\Big(\overline{z}(\varepsilon(\lambda{+}b_i)) - \overline{z}(\varepsilon\lambda)\Big)\otimes n_i, \varphi(\varepsilon\lambda)\Big\rangle_{m\times d} \qquad (4.20a) \\ &\qquad\qquad + \tfrac{1}{2}\varepsilon^d \Big\langle \tfrac{1}{\varepsilon}\Big(\overline{z}(\varepsilon\lambda) - \overline{z}(\varepsilon(\lambda{-}b_i))\Big)\otimes n_i, \varphi(\varepsilon\lambda)\Big\rangle_{m\times d}\Big]. \qquad (4.20b) \end{aligned}$$

Here, we already used that the function $f_\varepsilon^{(i)}$ is constant on the domain of integration. Since $\varphi^{\mathrm{ex}}(\varepsilon\lambda) = \mathbf{0}$ for all $\lambda \in \Lambda\backslash\Lambda_\varepsilon^-$, the first sum in (4.20) can be replaced by the sum of $\lambda \in \Lambda$. Afterwards, again the sums are interchanged and the translation $\lambda^* = \lambda{-}b_i$ is applied to line (4.20b) for every $i = 1, \dots, d$, such that we end up with

$$\begin{aligned} &\int_\Omega \Big\langle R_{\frac{\varepsilon}{2}} z(x), \varphi^{\mathrm{ex}}(x)\Big\rangle_{m\times d}\mathrm{d}x \\ &= \tfrac{\varepsilon^{d-1}}{2}\sum_{i=1}^d\sum_{\lambda\in\Lambda} \Big\langle\Big(\overline{z}^{\mathrm{ex}}(\varepsilon(\lambda{+}b_i)) - \overline{z}^{\mathrm{ex}}(\varepsilon\lambda)\Big)\otimes n_i, \varphi^{\mathrm{ex}}(\varepsilon\lambda) + \varphi^{\mathrm{ex}}(\varepsilon(\lambda{+}b_i))\Big\rangle_{m\times d}. \qquad (4.21) \end{aligned}$$

Comparing (4.21) and (4.18) we find that (4.15) is valid. □

Now we are in the position to prove Theorem 4.5.

Proof of Theorem 4.5. Here, we mainly follow the steps of the proof of Theorem 5.2 of [8] and explain the main differences. As already mentioned in [8] the distributional derivative Dv of a broken Sobolev function $v \in \mathrm{W}^{1,p}_{\varepsilon\Lambda}(\Omega)^m$ is given by

$$\langle Dv, \psi\rangle = \int_\Omega \langle\nabla v, \psi\rangle_{m\times d}\mathrm{d}x - \int_{\Gamma^\varepsilon_{\mathrm{int}}} \langle [\![v]\!], \psi\rangle_{m\times d}\mathrm{d}s \qquad \forall\,\psi \in \mathrm{C}_c^\infty(\Omega)^{m\times d}. \qquad (4.22)$$

This identity can be seen by using integration by parts on each cell $\varepsilon(\lambda{+}Y)$.

Now, let $(z_\varepsilon)_{\varepsilon>0}$ be given satisfying $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ for all $\varepsilon > 0$ and condition (4.8) of Theorem 4.5. Since L^p is reflexive ($p \in (1,\infty)$), there exists a subsequence and limit elements $z_0 \in \mathrm{L}^p(\Omega)^m$, $Z_0 \in \mathrm{L}^p(\Omega)^{m\times d}$ such that $z_{\varepsilon'} \rightharpoonup z_0$ in $\mathrm{L}^p(\Omega)^m$ and $R_{\frac{\varepsilon'}{2}} z_{\varepsilon'}|_\Omega \rightharpoonup Z_0$ in $\mathrm{L}^p(\Omega)^{m\times d}$ for some subsequence $(\varepsilon')_{\varepsilon'>0}$ of $(\varepsilon)_{\varepsilon>0}$. The aim is to show that $z_0 \in \mathrm{W}^{1,p}(\Omega)^m$

with $Dz_0 = Z_0$. Using (4.22) for $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ we find with $\psi \in \mathrm{C}_c^\infty(\Omega)^{m\times d}$ arbitrary but fixed

$$\langle Dz_\varepsilon, \psi\rangle = -\int_{\Gamma^\varepsilon_{\mathrm{int}}} \langle [\![z_\varepsilon]\!], \psi\rangle_{m\times d}\mathrm{d}s. \tag{4.23}$$

Choosing $\varepsilon_0 > 0$ so small such that $\mathrm{supp}(\psi) \subset \mathrm{cl}(\Omega^-_{\varepsilon_0})$ we are able to find a sequence $(\varphi_\varepsilon)_{(0<\varepsilon<\varepsilon_0)}$ with $\varphi_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega^-_\varepsilon)^{m\times d}$ such that $\|\psi - \varphi^{\mathrm{ex}}_\varepsilon\|_{\mathrm{L}^\infty(\Omega)^{m\times d}} \to 0$ for $\varepsilon \to 0$. By adding and subtracting $\varphi^{\mathrm{ex}}_\varepsilon$ we find with (4.23)

$$\begin{aligned}
&\langle Dz_\varepsilon, \psi\rangle \\
&= -\int_{\Gamma^\varepsilon_{\mathrm{int}}} \langle [\![z_\varepsilon]\!], \{\{\psi - \varphi^{\mathrm{ex}}_\varepsilon\}\}\rangle_{m\times d}\mathrm{d}s - \int_{\Gamma^\varepsilon_{\mathrm{int}}} \langle [\![z_\varepsilon]\!], \{\{\varphi^{\mathrm{ex}}_\varepsilon\}\}\rangle_{m\times d}\mathrm{d}s \\
&= -\int_{\Gamma^\varepsilon_{\mathrm{int}}} \langle [\![z_\varepsilon]\!], \{\{\psi - \varphi^{\mathrm{ex}}_\varepsilon\}\}\rangle_{m\times d}\mathrm{d}s + \int_\Omega \langle R_{\frac{\varepsilon}{2}} z_\varepsilon, \varphi^{\mathrm{ex}}_\varepsilon\rangle_{m\times d}\mathrm{d}x \\
&= -\int_{\Gamma^\varepsilon_{\mathrm{int}}} \langle [\![z_\varepsilon]\!], \{\{\psi - \varphi^{\mathrm{ex}}_\varepsilon\}\}\rangle_{m\times d}\mathrm{d}s + \int_\Omega \langle R_{\frac{\varepsilon}{2}} z_\varepsilon, \varphi^{\mathrm{ex}}_\varepsilon - \psi\rangle_{m\times d}\mathrm{d}x + \int_\Omega \langle R_{\frac{\varepsilon}{2}} z_\varepsilon, \psi\rangle_{m\times d}\mathrm{d}x, \quad (4.24)
\end{aligned}$$

where we applied Lemma 4.7 in the third line. As we will see below, the first two terms of (4.24) are bounded by $C\|\psi - \varphi^{\mathrm{ex}}_\varepsilon\|_{\mathrm{L}^\infty(\Omega)^{m\times d}}$ and hence tend to 0 as $\varepsilon \to 0$. Therefore, since $R_{\frac{\varepsilon'}{2}} z_{\varepsilon'}|_\Omega \rightharpoonup Z_0$ in $\mathrm{L}^p(\Omega)^{m\times d}$, we end up with

$$\lim_{\varepsilon'\to 0}\langle Dz_{\varepsilon'}, \psi\rangle = \int_\Omega \langle Z_0, \psi\rangle_{m\times d}\mathrm{d}s \qquad \forall\, \psi \in \mathrm{C}_c^\infty(\Omega)^{m\times d}. \tag{4.25}$$

To show the boundedness of the first two terms of (4.24) we use Hölder's inequality to conclude with Lemma 4.6

$$\begin{aligned}
\left| -\int_{\Gamma^\varepsilon_{\mathrm{int}}} \langle [\![z_\varepsilon]\!], \{\{\psi - \varphi^{\mathrm{ex}}_\varepsilon\}\}\rangle_{m\times d}\mathrm{d}s\right| &\le \|[\![z_\varepsilon]\!]\|_{\mathrm{L}^p(\Gamma^\varepsilon_{\mathrm{int}})^{m\times d}} \|\{\{\psi - \varphi^{\mathrm{ex}}_\varepsilon\}\}\|_{\mathrm{L}^{p'}(\Gamma^\varepsilon_{\mathrm{int}})^{m\times d}} \\
&\le \varepsilon^{\frac{p-1}{p}} \|R_{\frac{\varepsilon}{2}} z_\varepsilon\|_{\mathrm{L}^p(\Omega^+_\varepsilon)^{m\times d}} \|\psi - \varphi^{\mathrm{ex}}_\varepsilon\|_{\mathrm{L}^\infty(\Omega)^{m\times d}} \mu_{d-1}(\Gamma^\varepsilon_{\mathrm{int}})^{\frac{1}{p'}} \\
&\le \|R_{\frac{\varepsilon}{2}} z_\varepsilon\|_{\mathrm{L}^p(\Omega^+_\varepsilon)^{m\times d}} \|\psi - \varphi^{\mathrm{ex}}_\varepsilon\|_{\mathrm{L}^\infty(\Omega)^{m\times d}} (\mu_d(\Omega^+_\varepsilon)d)^{\frac{1}{p'}}
\end{aligned}$$

and

$$\begin{aligned}
\left|\int_\Omega \langle R_{\frac{\varepsilon}{2}} z_\varepsilon, \varphi^{\mathrm{ex}}_\varepsilon - \psi\rangle_{m\times d}\mathrm{d}x\right| &\le \|R_{\frac{\varepsilon}{2}} z_\varepsilon\|_{\mathrm{L}^p(\Omega^+_\varepsilon)^{m\times d}} \|\varphi^{\mathrm{ex}}_\varepsilon - \psi\|_{\mathrm{L}^{p'}(\Omega)^{m\times d}} \\
&\le \|R_{\frac{\varepsilon}{2}} z_\varepsilon\|_{\mathrm{L}^p(\Omega^+_\varepsilon)^{m\times d}} \|\varphi^{\mathrm{ex}}_\varepsilon - \psi\|_{\mathrm{L}^\infty(\Omega)^{m\times d}} \mu_d(\Omega)^{\frac{1}{p'}}.
\end{aligned}$$

Here, we already used $\mu_{d-1}(\Gamma^\varepsilon_{\mathrm{int}}) \le \mu_d(\Omega)d\varepsilon^{-1}$, which is valid since $\mu_{d-1}(\Gamma^\varepsilon_{\mathrm{int}})$ is bounded by the product of the number of cells contained in Ω^+_ε, which is $\mu_d(\Omega^+_\varepsilon)\varepsilon^{-d}$, and the volume of the part of $\Gamma^\varepsilon_{\mathrm{int}}$ contained in one cell, which is $d\varepsilon^{d-1}$. Thus, the assumed uniform bound of the term $\|R_{\frac{\varepsilon}{2}} z_\varepsilon\|_{\mathrm{L}^p(\Omega^+_\varepsilon)^{m\times d}}$ yields the result.

On the other hand using the definition of the distributional derivative of the function $z_{\varepsilon'} \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ together with $z_{\varepsilon'} \rightharpoonup z_0$ in $\mathrm{L}^p(\Omega)^m$, we have for all $\psi \in \mathrm{C}_c^\infty(\Omega)^{m\times d}$

$$\lim_{\varepsilon'\to 0}\langle Dz_{\varepsilon'}, \psi\rangle = \lim_{\varepsilon'\to 0} -\int_\Omega \langle z_{\varepsilon'}, \mathrm{div}\psi\rangle_m \mathrm{d}x = -\int_\Omega \langle z_0, \mathrm{div}\psi\rangle_m\mathrm{d}x. \tag{4.26}$$

Combining (4.25) and (4.26) we obtain

$$\int_\Omega \langle Z_0, \psi\rangle_{m\times d}\mathrm{d}x = -\int_\Omega \langle z_0, \mathrm{div}\psi\rangle_m \mathrm{d}x \qquad \forall\, \psi \in \mathrm{C}_c^\infty(\Omega)^{m\times d},$$

which gives us $z_0 \in \mathrm{W}^{1,p}(\Omega)^m$ and $Dz_0 = Z_0$.

Finally, we use the fact that $z_{\varepsilon'} \overset{*}{\rightharpoonup} z_0$ in $\mathrm{BV}(\Omega)^m$ implies $z_{\varepsilon'} \to z_0$ in $\mathrm{L}^1(\Omega)^m$ in order to conclude $z_{\varepsilon'} \to z_0$ in $\mathrm{L}^q(\Omega)^m$ for every $q \in [1, p^*)$. Thereby we use the following interpolation inequality obtained by Hölder's inequality for every $\zeta \in (0,1)$:

$$\|z_\varepsilon - z_0\|_{\mathrm{L}^q(\Omega)^m} \leq \|z_\varepsilon - z_0\|^{1-\zeta}_{\mathrm{L}^{p^*}(\Omega)^m} \|z_\varepsilon - z_0\|^{\zeta}_{\mathrm{L}^1(\Omega)^m},$$

and the term $\|z_\varepsilon - z_0\|_{\mathrm{L}^{p^*}(\Omega)^m}$ is bounded due to the following Sobolev-Poincare inequality proved in Theorem 4.1 of [8] and Lemma 4.6:

$$\|z_\varepsilon\|_{\mathrm{L}^{p^*}(\Omega)^m} \leq C_S\Big(\|z_\varepsilon\|_{\mathrm{L}^1(\Omega)^m} + \Big(\int_{\Gamma^\varepsilon_{\mathrm{int}}} \varepsilon^{1-p} |[\![z_\varepsilon(s)]\!]|^p_{m\times d}\mathrm{d}s\Big)^{\frac{1}{p}}\Big).$$

Thus, the proof of Theorem 4.5 is concluded. □

4.3 Recovery sequence of piecewise constant functions for the Sobolev space $\mathrm{W}^{1,p}$

In this section for an arbitrary Sobolev function a strongly in L^p converging sequence of piecewise constant functions with a lattice spacing tending to zero is constructed. This construction is done such that the associated sequence of discrete gradients converges strongly in L^p to the gradient of the Sobolev function.

Theorem 4.9 (Approximation result). *Let $R_{\frac{\varepsilon}{2}} : \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m \to \mathrm{K}_{\frac{\varepsilon}{2}\Lambda}(\Omega^+_\varepsilon)^{m\times d}$ be given by Definition 4.1. Then for every function $z_0 \in \mathrm{W}^{1,p}(\Omega)^m$ there exists a sequence $(z_\varepsilon)_{\varepsilon>0}$ satisfying $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ and*

$$\lim_{\varepsilon\to 0}\Big(\|z_0 - z_\varepsilon\|_{\mathrm{L}^p(\Omega)^m} + \|(\nabla z_0)^{\mathrm{ex}} - R_{\frac{\varepsilon}{2}} z_\varepsilon\|_{\mathrm{L}^p(\Omega^+_\varepsilon)^{m\times d}}\Big) = 0. \tag{4.27}$$

Remark 4.10. *Observe that Theorem 4.9 implies $R_{\frac{\varepsilon}{2}} z_\varepsilon|_\Omega \to \nabla z_0$ in $\mathrm{L}^p(\Omega)^{m\times d}$ due to the trivial inequality $\|\nabla z_0 - R_{\frac{\varepsilon}{2}} z_\varepsilon\|_{\mathrm{L}^p(\Omega)^{m\times d}} \leq \|(\nabla z_0)^{\mathrm{ex}} - R_{\frac{\varepsilon}{2}} z_\varepsilon\|_{\mathrm{L}^p(\Omega^+_\varepsilon)^{m\times d}}$.*

To construct a sequence of piecewise constant functions fulfilling Theorem 4.9 a projector to piecewise constant functions is introduced.

Definition 4.11 (Projector to piecewise constant functions). *Let $\varepsilon > 0$ and $p \in [1, \infty)$. The projector $P_\varepsilon : \mathrm{L}^p(\mathbb{R}^d) \to \mathrm{K}_{\varepsilon\Lambda}(\mathbb{R}^d)$ to piecewise constant functions is defined via*

$$P_\varepsilon v(x) := \fint_{\mathcal{N}_\varepsilon(x)+\varepsilon Y} v(\widehat{x})\mathrm{d}\widehat{x},$$

where $\fint_A g(a)\mathrm{d}a := \frac{1}{\mu_d(A)}\int_A g(a)\mathrm{d}a$ denotes the average of the function g over the set A with $\mu_d(A) > 0$ and $\mathcal{N}_\varepsilon : \mathbb{R}^d \to \varepsilon\Lambda$ is defined by (3.1).

Remark 4.12. *It holds* $P_\varepsilon^2 = P_\varepsilon$ *and for all* $v \in \mathrm{L}^p(\Omega)$ *we have*

(a) $\|P_\varepsilon v^{\mathrm{ex}}\|_{\mathrm{L}^p(\varepsilon(\lambda+Y))} \le \|v^{\mathrm{ex}}\|_{\mathrm{L}^p(\varepsilon(\lambda+Y))}$ *for all* $\lambda \in \Lambda$.

(b) $(P_\varepsilon v^{\mathrm{ex}})|_\Omega \to v$ *in* $\mathrm{L}^p(\Omega)$ *for* $\varepsilon \to 0$.

Proof of Theorem 4.9. Choose $\Delta > 0$ arbitrary but fixed. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have $\Omega_\varepsilon^+ \subset \mathrm{neigh}_\Delta(\Omega)$. Moreover, for given $z_0 \in \mathrm{W}^{1,p}(\Omega)^m$ there exists an extension $\overline{z}_0 \in \mathrm{W}_0^{1,p}(\mathrm{neigh}_\Delta(\Omega))^m$ with $\overline{z}_0|_\Omega = z_0$ according to Theorem A 6.12 in [3]. For $\varepsilon \in (0, \varepsilon_0)$ we define $z_\varepsilon := (P_\varepsilon \overline{z}_0^{\mathrm{ex}})|_\Omega \in \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m$ and prove that the sequence $(z_\varepsilon)_{\varepsilon\in(0,\varepsilon_0)}$ satisfies (4.27). Note that here the application of P_ε has to be understood component-wise.

1. For proving $z_\varepsilon \to z_0$ in $\mathrm{L}^p(\Omega)^m$ we start by decomposing Ω into Ω_ε^- and $\Omega\backslash\Omega_\varepsilon^-$, which allows us to exploit $(P_\varepsilon \overline{z}_0^{\mathrm{ex}})|_{\Omega_\varepsilon^-} = (P_\varepsilon z_0^{\mathrm{ex}})|_{\Omega_\varepsilon^-}$, since $\overline{z}_0|_{\Omega_\varepsilon^-} = z_0|_{\Omega_\varepsilon^-}$ by definition. Afterwards we increase the domain of integration and apply the triangle inequality. Then again the domain of integration is increased and at last condition (a) of Remark 4.12 is used:

$$\begin{aligned}
\|z_0 - P_\varepsilon \overline{z}_0^{\mathrm{ex}}\|^p_{\mathrm{L}^p(\Omega)^m} &= \|z_0 - P_\varepsilon z_0^{\mathrm{ex}}\|^p_{\mathrm{L}^p(\Omega_\varepsilon^-)^m} + \|z_0 - P_\varepsilon \overline{z}_0^{\mathrm{ex}}\|^p_{\mathrm{L}^p(\Omega\backslash\Omega_\varepsilon^-)^m} \\
&\le \|z_0 - P_\varepsilon z_0^{\mathrm{ex}}\|^p_{\mathrm{L}^p(\Omega)^m} + 2^{p-1}\|z_0\|^p_{\mathrm{L}^p(\Omega\backslash\Omega_\varepsilon^-)^m} + 2^{p-1}\|P_\varepsilon \overline{z}_0^{\mathrm{ex}}\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+\backslash\Omega_\varepsilon^-)^m} \\
&\le \|z_0 - P_\varepsilon z_0^{\mathrm{ex}}\|^p_{\mathrm{L}^p(\Omega)^m} + 2^{p-1}\|z_0\|^p_{\mathrm{L}^p(\Omega\backslash\Omega_\varepsilon^-)^m} + 2^{p-1}\|\overline{z}_0\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+\backslash\Omega_\varepsilon^-)^m}.
\end{aligned}$$

According to condition (b) of Remark 4.12 the first term converges to zero for $\varepsilon \to 0$. Since $0 \le \mu_d(\Omega\backslash\Omega_\varepsilon^-) \le \mu_d(\Omega_\varepsilon^+\backslash\Omega_\varepsilon^-) \to 0$ due to (2.17) the last two terms disappear for $\varepsilon \to 0$ and $z_\varepsilon \to z_0$ in $\mathrm{L}^p(\Omega)^m$ is verified.

2. Since $\{b_1, b_2, \dots, b_d\}$ is a basis of $\mathbb{R}^d$, proving $\lim_{\varepsilon\to0} \|(\nabla z_0)^{\mathrm{ex}} b_i - (R_{\frac{\varepsilon}{2}} z_\varepsilon) b_i\|_{\mathrm{L}^p(\Omega_\varepsilon^+)^m} = 0$ for every $i \in \{1, \dots, d\}$ implies the desired result. Thereto, let $i \in \{1, \dots, d\}$ be fixed. In the following calculations we start by adding and subtracting $(P_\varepsilon(\nabla\overline{z}_0)^{\mathrm{ex}})b_i$ to apply the triangle inequality.

$$\begin{aligned}
&\|(\nabla z_0)^{\mathrm{ex}} b_i - (R_{\frac{\varepsilon}{2}} z_\varepsilon) b_i\|_{\mathrm{L}^p(\Omega_\varepsilon^+)^m} \\
&\quad \le \|(\nabla z_0)^{\mathrm{ex}} b_i - (P_\varepsilon(\nabla\overline{z}_0)^{\mathrm{ex}}) b_i\|_{\mathrm{L}^p(\Omega_\varepsilon^+)^m} + \|(P_\varepsilon(\nabla\overline{z}_0)^{\mathrm{ex}}) b_i - (R_{\frac{\varepsilon}{2}} z_\varepsilon) b_i\|_{\mathrm{L}^p(\Omega_\varepsilon^+)^m}
\end{aligned}$$

Then analogously to step 1 the first term tends to zero when $\varepsilon \to 0$. It remains to prove that the second term converges to zero as well. As mentioned in Remark 4.2 we have $(R_{\frac{\varepsilon}{2}} z_\varepsilon) b_i = (\widetilde{R}^{(i)}_{\frac{\varepsilon}{2}}(V_\varepsilon z_\varepsilon)) b_i$ on Ω_ε^+. Hence, by plugging the identity $V_\varepsilon z_\varepsilon = [P_\varepsilon \overline{z}_0^{\mathrm{ex}}]|_{\Omega_\varepsilon^+}$ into (4.3) the second term can be transformed in the following way:

$$\begin{aligned}
&\|(P_\varepsilon(\nabla\overline{z}_0)^{\mathrm{ex}}) b_i - (R_{\frac{\varepsilon}{2}} z_\varepsilon) b_i\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^m} \\
&\quad = \|(P_\varepsilon(\nabla\overline{z}_0)^{\mathrm{ex}}) b_i - (\widetilde{R}^{(i)}_{\frac{\varepsilon}{2}}([P_\varepsilon \overline{z}_0^{\mathrm{ex}}]|_{\Omega_\varepsilon^+})) b_i\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^m} \\
&\quad = \|(P_\varepsilon(\nabla\overline{z}_0)^{\mathrm{ex}}) b_i - \tfrac{1}{\varepsilon}\Big(P_\varepsilon \overline{z}_0^{\mathrm{ex}}(\cdot + \tfrac{\varepsilon}{2} b_i) - P_\varepsilon \overline{z}_0^{\mathrm{ex}}(\cdot - \tfrac{\varepsilon}{2} b_i)\Big)\|^p_{\mathrm{L}^p(A_\varepsilon)^m} && (4.28a) \\
&\qquad + \|(P_\varepsilon(\nabla\overline{z}_0)^{\mathrm{ex}}) b_i\|^p_{\mathrm{L}^p(B_\varepsilon)^m}, && (4.28b)
\end{aligned}$$

where, $A_\varepsilon := \{x \in \Omega_\varepsilon^+ \,|\, (x+\frac{\varepsilon}{2}b_i) \in \Omega_\varepsilon^+ \text{ and } (x-\frac{\varepsilon}{2}b_i) \in \Omega_\varepsilon^+\}$ and $B_\varepsilon := \Omega_\varepsilon^+ \backslash A_\varepsilon$ for fixed $i \in \{1,\dots,d\}$. Since $B_\varepsilon \subset \Omega_\varepsilon^+ \backslash \Omega_\varepsilon^-$, the term in line (4.28b) is bounded. Moreover,

$$\|(P_\varepsilon(\nabla \overline{z}_0)^{\mathrm{ex}})b_i\|_{\mathrm{L}^p(\Omega_\varepsilon^+\backslash\Omega_\varepsilon^-)^m} \le \|(\nabla \overline{z}_0)b_i\|_{\mathrm{L}^p(\Omega_\varepsilon^+\backslash\Omega_\varepsilon^-)^m} \xrightarrow{\varepsilon\to 0} 0,$$

where again condition (a) of Remark 4.12 and $\mu_d(\Omega_\varepsilon^+\backslash\Omega_\varepsilon^-) \to 0$ for $\varepsilon \to 0$ is used. Hence, it remains to prove that the term of line (4.28a) converges to zero. Therefore, this term can be estimated by (i) increasing the domain of integration from A_ε to Ω_ε^+, (ii) exploiting condition (a) of Remark 4.12, and (iii) replacing $\frac{1}{\varepsilon}[\overline{z}_0(x+\frac{\varepsilon}{2}b_i) - \overline{z}_0(x-\frac{\varepsilon}{2}b_i)]$ by $\frac{1}{2}\int_{-1}^1 \nabla\overline{z}_0(x+\frac{\varepsilon}{2}b_i t)b_i \mathrm{d}t$ in the following way:

$$\begin{aligned}
&\|(P_\varepsilon(\nabla \overline{z}_0)^{\mathrm{ex}})b_i - \tfrac{1}{\varepsilon}\Big(P_\varepsilon\overline{z}_0^{\mathrm{ex}}(\cdot+\tfrac{\varepsilon}{2}b_i) - P_\varepsilon\overline{z}_0^{\mathrm{ex}}(\cdot-\tfrac{\varepsilon}{2}b_i)\Big)\|_{\mathrm{L}^p(A_\varepsilon)^m} \\
&\qquad \le \|(P_\varepsilon(\nabla \overline{z}_0)^{\mathrm{ex}})b_i - \tfrac{1}{\varepsilon}\Big(P_\varepsilon\overline{z}_0^{\mathrm{ex}}(\cdot+\tfrac{\varepsilon}{2}b_i) - P_\varepsilon\overline{z}_0^{\mathrm{ex}}(\cdot-\tfrac{\varepsilon}{2}b_i)\Big)\|_{\mathrm{L}^p(\Omega_\varepsilon^+)^m} \\
&\qquad \le \|(\nabla \overline{z}_0)b_i - \tfrac{1}{\varepsilon}\Big(\overline{z}_0(\cdot+\tfrac{\varepsilon}{2}b_i) - \overline{z}_0(\cdot-\tfrac{\varepsilon}{2}b_i)\Big)\|_{\mathrm{L}^p(\Omega_\varepsilon^+)^m} \\
&\qquad = \|(\nabla \overline{z}_0)b_i - \tfrac{1}{2}\textstyle\int_{-1}^1\Big(\nabla\overline{z}_0(\cdot+\tfrac{\varepsilon}{2}b_i t)\Big)b_i \mathrm{d}t\|_{\mathrm{L}^p(\Omega_\varepsilon^+)^m}.
\end{aligned} \tag{4.29}$$

Observe that this estimate holds for all parameter $\varepsilon \in (0,\varepsilon_0)$ small enough such that from $x \in \Omega_\varepsilon^+$ it follows $x+\frac{\varepsilon}{2}b_i \in \mathrm{neigh}_\Delta(\Omega)$ and $x-\frac{\varepsilon}{2}b_i \in \mathrm{neigh}_\Delta(\Omega)$. By assuming the function $\overline{z}_0$ to be an element of the space $\mathrm{C}_c^\infty(\mathrm{neigh}_\Delta(\Omega))^m$, it is easy to prove that the term in line (4.29) converges to zero. Then, by density, this also holds for $\overline{z}_0 \in \mathrm{W}_0^{1,p}(\mathrm{neigh}_\Delta(\Omega))^m$ which proves that the term in line (4.28a) converges to zero. This overall shows $\lim_{\varepsilon\to 0}\|(\nabla z_0)^{\mathrm{ex}}b_i - (R_{\frac{\varepsilon}{2}}z_\varepsilon)b_i\|_{\mathrm{L}^p(\Omega_\varepsilon^+)^m} = 0$ for every $i \in \{1,\dots,d\}$ and the proof is concluded. $\square$

5 Rate-independent systems and their energetic formulation

In this chapter the energetic formulation for rate-independent systems, introduced in [47, 59], is presented. Rate-independence means that the system's reaction on external loadings is independent of their velocities. The energetic formulation is a functional based description of rate-independent systems and in contrast to subdiffferential formulations or a description by variational inequalities it does not ask for differentiability of the underlying functionals and allows for solutions which are not continuous. Moreover, this theoretical framework is suitable for the description of various physical effects (for instance, plasticity, shape memory effects, or ferroelectric effects) and there already exists a wide theoretical basis; see [54, 59, 60], for instance. Moreover, in the case of parameter dependent Problems, methods of the calculus of variations (in particular Γ-convergence techniques) can by applied; see [56]. In the following sections this theoretical basis is presented and a sufficient criterion guarantying the system's rate independence is discussed.

5.1 Definition of rate-independent systems

In many physical and mechanical systems the interesting time scales are much longer than the intrinsic time scales. Asymptotically, these systems lead to rate-independent limit problems. For $\mathcal{Q}$ being a Banach space and denoting the state space with associated dual space $\mathcal{Q}^*$ the determining attribute of a rate-independent system is the missing of an own dynamic, which means that it only responds to changes of the external loading $\ell \in \mathrm{C}^1([t_1, t_2]; \mathcal{Q}^*)$ and the initial value $q_0 \in \mathcal{Q}$. In fact, introducing the system via an input-output-operator

$$\mathcal{H}_{[t_1,t_2]} : \mathcal{Q}\times\mathrm{C}^1([t_1, t_2]; \mathcal{Q}^*) \to \mathcal{B}([t_1, t_2]; \mathcal{Q}), (q_0, \ell) \mapsto q,$$

rate-independent systems are characterized by condition (5.1), below. Here, $\mathcal{B}([t_1, t_2]; \mathcal{Q})$ denotes the space of all measurable and bounded functions $\widetilde{q} : [t_1, t_2] \to \mathcal{Q}$, which are defined everywhere on $[t_1, t_2]$.

Definition 5.1 (Rate-independent system)**.** *For $\ell \in \mathrm{C}^1([t_1, t_2]; \mathcal{Q}^*)$ and $t_1 < t_2$ a system described by $\mathcal{H}_{[t_1,t_2]}$ is called rate-independent if for all $t_1^* < t_2^*$ and every strictly monotone re-parametrization $\theta \in \mathrm{C}^1([t_1^*, t_2^*]; [t_1, t_2])$ with $\theta(t_1^*) = t_1$ and $\theta(t_2^*) = t_2$ the* *following relation is satisfied:*

$$\mathcal{H}_{[t_1^*,t_2^*]}(q_0,\ell\circ\theta) = \Big[\mathcal{H}_{[t_1,t_2]}(q_0,\ell)\Big]\circ\theta. \tag{5.1}$$

5.2 The energetic formulation and an abstract existence result

The energetic formulation represents an evolution law for rate-independent systems which is based on the energy functional and dissipation distance introduced in the following. For the applications we have in mind, the state space $\mathcal{Q}$ is the product space of two weakly closed subspaces $\mathcal{U}$ and $\mathcal{Z}$ of reflexive Banach spaces. The amount of the system's total energy produced by components of $\mathcal{U}$ is elastic and non-dissipative, whereas the dissipative amount of energy in the system is related to the internal variable being an element of $\mathcal{Z}$.

The system's stored energy is modeled by an energy functional $\mathcal{E}:[0,T]\times\mathcal{Q}\to\mathbb{R}_\infty$ ($\mathbb{R}_\infty:=\mathbb{R}\cup\{+\infty\}$) depending on the (process) time trough the time dependent loading $\ell\in\mathrm{C}^1([0,T];\mathcal{U}^*)$. To model the dissipated energy of the system, a dissipation potential $\mathcal{R}:\mathcal{Z}\times\mathcal{Z}\to[0,\infty]$ that typically depends only on the internal variable and its velocity is introduced. The main assumptions on this dissipation potential are given by the following condition:

$$\begin{aligned}&\text{For all } z\in\mathcal{Z} \text{ the functional } \mathcal{R}(z,\cdot):\mathcal{Z}\to[0,\infty] \text{ is convex, lower}\\ &\text{semi-continuous, positive 1-homogeneous, and fulfills } \mathcal{R}(z,0)=0.\end{aligned} \tag{5.2}$$

Note that the positive homogeneity of degree 1 is the crucial condition to ensure the rate independence of the system given by the energetic formulation defined below. Assuming $\mathcal{R}:\mathcal{Z}\times\mathcal{Z}\to[0,\infty]$ to satisfy condition (5.1), the dissipated energy of the system is modeled by the dissipation distance $\mathcal{D}:\mathcal{Z}\times\mathcal{Z}\to[0,\infty]$ defined by

$$\mathcal{D}(z_1,z_2) := \inf\Big\{\int_0^1 \mathcal{R}(\hat z(s),\dot{\hat z}(s))\mathrm{d}s \;|\; \hat z\in \mathrm{W}^{1,1}_{z_1,z_2}([0,1];\mathcal{Z})\Big\}, \tag{5.3}$$

where

$$\mathrm{W}^{1,1}_{z_1,z_2}([t_1,t_2];\mathcal{Z}) := \{\hat z\in\mathrm{W}^{1,1}([t_1,t_2];\mathcal{Z}) \text{ such that } \hat z(t_1)=z_1 \text{ and } \hat z(t_2)=z_2\}. \tag{5.4}$$

Remark 5.2. *In the case of a state independent dissipation potential $\mathcal{R}:\mathcal{Z}\to[0,\infty]$ fulfilling condition* (5.2) *the dissipation distance $\mathcal{D}:\mathcal{Z}\times\mathcal{Z}\to[0,\infty]$ defined by* (5.3) *for all $z_1,z_2\in\mathcal{Z}$ satisfies $\mathcal{D}(z_1,z_2)\geq\mathcal{R}(z_2-z_1)$ according to Jensen's inequality. The opposite inequality results by integrating the dissipation potential along the explicitly given curve $\hat z\in\mathrm{W}^{1,1}_{z_1,z_2}([0,1];\mathcal{Z})$ defined by $\hat z(s):=\frac12(z_2-z_1)s^2+\frac12(z_2-z_1)s+z_1$. Obviously, it holds $\dot{\hat z}(s)=(z_2-z_1)s+\frac12(z_2-z_1)$ and*

$$\mathcal{D}(z_1,z_2)\leq\int_0^1\mathcal{R}(\dot{\hat z}(s))\mathrm{d}s \overset{s=t-\frac12}{=} \int_{\frac12}^{\frac32}\mathcal{R}((z_2-z_1)t)\mathrm{d}t = \mathcal{R}(z_2-z_1)\int_{\frac12}^{\frac32}t\mathrm{d}t = \mathcal{R}(z_2-z_1),$$

where in the second last equality the 1*-homogeneity of the potential is exploited. All together this shows that in this case the dissipation distance* $\mathcal{D} : \mathcal{Z}\times\mathcal{Z} \to [0,\infty]$ *is given by* $\mathcal{D}(z_1, z_2) = \mathcal{R}(z_2-z_1)$.

Based on $\mathcal{E} : [0,T]\times\mathcal{Q} \to \mathbb{R}_\infty$ and $\mathcal{D} : \mathcal{Z}\times\mathcal{Z} \to [0,\infty]$ we are interested in the so-called *energetic solution* of the *energetic formulation* (S) and (E); see [54].

Definition 5.3 (Energetic solution and the function space $\mathrm{BV}_\mathcal{D}([0,T];\mathcal{Z})$)**.** *A process* $(u,z) : [0,T] \to \mathcal{Q}$ *is called energetic solution of the system* $(\mathcal{Q},\mathcal{E},\mathcal{D})$ *to the initial condition* $(u^0, z^0) \in \mathcal{Q}$*, if* $(u(0), z(0)) = (u^0, z^0)$*, if* $\partial_t\mathcal{E}(\cdot, u(\cdot), z(\cdot)) \in \mathrm{L}^1((0,T))$*, if* $\mathcal{E}(t, u(t), z(t)) < \infty$*, and if the stability condition* (S) *and the energy balance* (E) *are satisfied for all* $t \in [0,T]$*.*

(S) $\mathcal{E}(t, u(t), z(t)) \le \mathcal{E}(t, \widetilde{u}, \widetilde{z}) + \mathcal{D}(z(t), \widetilde{z})$ *for all* $(\widetilde{u}, \widetilde{z}) \in \mathcal{Q}$

(E) $\mathcal{E}(t, u(t), z(t)) + \mathrm{Diss}_\mathcal{D}(z; [0,t]) = \mathcal{E}(0, u^0, z^0) + \int_0^t \partial_s\mathcal{E}(s, u(s), z(s))\mathrm{d}s$

Here, $\mathrm{Diss}_\mathcal{D}(z, [0,t]) := \sup\sum_{j=1}^N \mathcal{D}(z(t_{j-1}), z(t_j))$*, where for* $N \in \mathbb{N}$ *the supremum is taken with respect to all finite partitions* $\pi_N := \{0 = t_0 < t_1 < \ldots < t_N = t\}$ *of the interval* $[0,t]$*. Thus, we set*

$$\mathrm{BV}_\mathcal{D}([0,T];\mathcal{Z}) := \Big\{z : [0,T] \to \mathcal{Z} \,\Big|\, \mathrm{Diss}_\mathcal{D}(z, [0,T]) < \infty\Big\}.$$

Observe that by this definition of the energetic solution $(u,z) : [0,T] \to \mathcal{Q}$ the initial values $(u^0, z^0) = (u(0), z(0)) \in \mathcal{Q}$ have to be chosen such that they satisfy the stability condition (S) at $t = 0$. Introducing the set of stable states $\mathcal{S}(\widetilde{t})$ at time $\widetilde{t} \in [0,T]$ via

$$\mathcal{S}(\widetilde{t}) := \{(u,z) \in \mathcal{Q} \text{ satisfying (S) for } t = \widetilde{t} \text{ and } \mathcal{E}(\widetilde{t}, u, z) < \infty\} \tag{5.5}$$

the stability condition (S) is equivalently written as $(u(t), z(t)) \in \mathcal{S}(t)$ for all $t \in [0,T]$. For $E \in \mathbb{R}$ let

$$\mathrm{Sub}_E(t) := \{(u,z) \in \mathcal{Q} \,|\, \mathcal{E}(t,u,z) \le E\} \tag{5.6}$$

denote the energy sublevel. Assuming $\mathcal{E} : [0,T]\times\mathcal{Q} \to \mathbb{R}_\infty$ and $\mathcal{D} : \mathcal{Z}\times\mathcal{Z} \to [0,\infty]$ to satisfy the following four conditions, guarantees the existence of an energetic solution as stated in Theorem 5.5 below ([54] Theorem 3.4).

Compactness of the energy sublevels:

$$\text{For all } t \in [0,T] \text{ and every } E \in \mathbb{R} \text{ the set } \mathrm{Sub}_E(t) \text{ is weakly compact.} \tag{5.7}$$

Uniform control of the power: $\exists\, c_0, c_1 > 0\, \forall\, (\widetilde{t}, u, z) \in [0,T]\times\mathcal{Q}$ with $\mathcal{E}(\widetilde{t}, u, z){<}\infty$:

$$\mathcal{E}(\cdot, u, z) \in \mathrm{C}^1([0,T]) \text{ and } |\partial_t\mathcal{E}(t,u,z)| \le c_1(c_0 + \mathcal{E}(t,u,z)) \text{ for all } t \in [0,T]. \tag{5.8}$$

Quasi-distance:

$$\begin{aligned} &\forall\, z_1, z_2, z_3 \in \mathcal{Z} : \mathcal{D}(z_1, z_2) = 0 \quad \Leftrightarrow \quad z_1{=}z_2 \\ &\text{and} \qquad\quad \mathcal{D}(z_1, z_3) \le \mathcal{D}(z_1, z_2) + \mathcal{D}(z_2, z_3). \end{aligned} \tag{5.9}$$

Semi-continuity:

$$\mathcal{D} : \mathcal{Z}\times\mathcal{Z} \to [0,\infty] \text{ is weakly lower semi-continuous.} \tag{5.10}$$

Remark 5.4. *Note that these assumptions are solely on the energy functional and the dissipation distance. However, to guarantee the rate independence of the system* $(\mathcal{Q},\mathcal{E},\mathcal{D})$ *presented by the energetic formulation* (S) *and* (E) *we always assume the dissipation distance* $\mathcal{D} : \mathcal{Z}\times\mathcal{Z} \to [0,\infty]$ *to be given such that condition* (5.1) *is satisfied; see also Proposition* 5.6 *below.*

Theorem 5.5 (Abstract existence result; see Theorem 3.4 of [54])**.** *Assume that* $\mathcal{U}$ *and* $\mathcal{Z}$ *are weakly closed subsets of reflexive Banach spaces and set* $\mathcal{Q} := \mathcal{U}\times\mathcal{Z}$*. Let an energy functional* $\mathcal{E} : [0,T]\times\mathcal{Q} \to \mathbb{R}_\infty$ *and a dissipation distance* $\mathcal{D} : \mathcal{Z}\times\mathcal{Z} \to [0,\infty]$ *be given, such that* $(\mathcal{Q},\mathcal{E},\mathcal{D})$ *satisfies the conditions* (5.7)–(5.10)*. Furthermore, let the following compatibility conditions hold: For every sequence* $(t_k, u_k, z_k)_{k\in\mathbb{N}}$ *with* $(u_k, z_k) \in \mathcal{S}(t_k)$ *(see* (5.5)*),* $t_k \to t$ *in* $\mathbb{R}$ *and* $(u_k, z_k) \rightharpoonup (u, z)$ *in* $\mathcal{Q}$ *we have*

$$\partial_t\mathcal{E}(t_k, u_k, z_k) \to \partial_t\mathcal{E}(t, u, z), \tag{5.11}$$

$$(u, z) \in \mathcal{S}(t). \tag{5.12}$$

Then for each $(u_0, z_0) \in \mathcal{S}(0)$ *there exists an energetic solution* $(u, z) : [0,T] \to \mathcal{Q}$ *for* $(\mathcal{Q},\mathcal{E},\mathcal{D})$ *fulfilling* $(u(0), z(0)) = (u_0, z_0)$*.*

For a proof we refer to [54]; see Theorem 3.4. By presuming the existence of an energetic solution of the energetic formulation (S) and (E) the following proposition states the rate independence of the system $(\mathcal{Q},\mathcal{E},\mathcal{D})$. In addition, $\mathcal{D} : \mathcal{Z}\times\mathcal{Z} \to [0,\infty]$ satisfies a triangle inequality.

Proposition 5.6. *Assume that the dissipation potential* $\mathcal{R} : \mathcal{Z}\times\mathcal{Z} \to [0,\infty]$ *fulfills condition* (5.2) *and let the dissipation distance* $\mathcal{D} : \mathcal{Z}\times\mathcal{Z} \to [0,\infty]$ *be defined by* (5.3)*. If for* $W : \mathcal{Q} \to \mathbb{R}_\infty$ *and* $\ell \in \mathrm{C}^1([0,T];\mathcal{U}^*)$ *the energy functional* $\mathcal{E} : [0,T]\times\mathcal{Q} \to \mathbb{R}_\infty$ *is given via*

$$\mathcal{E}(t, u, z) := W(u, z) - \langle \ell(t), u\rangle, \tag{5.13}$$

then the energetic formulation (S) *and* (E) *models a rate-independent system* $(\mathcal{Q},\mathcal{E},\mathcal{D})$ *and for all* $z_1, z_2, z_3 \in \mathcal{Z}$ *the dissipation distance satisfies the following conditions:*

$$\mathcal{D}(z_1, z_1) = 0 \qquad \textit{and} \qquad \mathcal{D}(z_1, z_3) \le \mathcal{D}(z_1, z_2) + \mathcal{D}(z_2, z_3). \tag{5.14}$$

Proof. 1. We start by proving condition (5.14). Let $\mathcal{R} : \mathcal{Z}\times\mathcal{Z} \to [0,\infty]$ fulfill condition (5.2) and let $\mathcal{D} : \mathcal{Z}\times\mathcal{Z} \to [0,\infty]$ be defined by (5.3). Furthermore, for arbitrary $z_1, z_2, z_3 \in \mathcal{Z}$ let two functions $\overline{z} \in \mathrm{W}^{1,1}_{z_1,z_2}([0,1];\mathcal{Z})$ and $\overline{z}' \in \mathrm{W}^{1,1}_{z_2,z_3}([0,1];\mathcal{Z})$ be given and introduce $\widehat{z} \in \mathrm{W}^{1,1}_{z_1,z_3}([0,1];\mathcal{Z})$ by

$$\widehat{z} := \begin{cases} \overline{\overline{z}} & \text{on } [0, \frac{1}{2}], \\ \overline{\overline{z}}' & \text{on } (\frac{1}{2}, 1], \end{cases} \tag{5.15}$$

where the functions $\overline{\overline{z}} \in \mathrm{W}^{1,1}_{z_1,z_2}([0,\frac{1}{2}];\mathcal{Z})$ and $\overline{\overline{z}}' \in \mathrm{W}^{1,1}_{z_2,z_3}([\frac{1}{2},1];\mathcal{Z})$ are defined by the relations $\overline{\overline{z}} := \overline{z}(2\cdot)$ and $\overline{\overline{z}}' := \overline{z}'(2\cdot -1)$. Following the calculation below, the triangle inequality follows by taking the infimum with respect to all tuple of functions $(\overline{z},\overline{z}')$ of the space $\mathrm{W}^{1,1}_{z_1,z_2}([0,1];\mathcal{Z})\times\mathrm{W}^{1,1}_{z_2,z_3}([0,1];\mathcal{Z})$. We start with the integral transformations $s=2t$ and $s=2t-1$ for the first and the second integrand, respectively. The factors $\frac{1}{2}$ in the second arguments of the integrands are due to the chain rule. Afterwards, the positive homogeneity of degree 1 of the dissipation potential $\mathcal{R}:\mathcal{Z}\times\mathcal{Z}\to[0,\infty]$ (see (5.1)) is exploited and finally the notation (5.15) is used.

$$\begin{aligned}\int_0^1 \mathcal{R}(\overline{z}(s),\dot{\overline{z}}(s))\mathrm{d}s+\int_0^1 \mathcal{R}(\overline{z}'(s),\dot{\overline{z}}'(s))\mathrm{d}s &= \int_0^{\frac{1}{2}} \mathcal{R}(\overline{\overline{z}}(t),\dot{\overline{\overline{z}}}(t)\tfrac{1}{2})2\mathrm{d}t+\int_{\frac{1}{2}}^1 \mathcal{R}(\overline{\overline{z}}'(t),\dot{\overline{\overline{z}}}'(t)\tfrac{1}{2})2\mathrm{d}t\\ &= \int_0^{\frac{1}{2}} \mathcal{R}(\overline{\overline{z}}(t),\dot{\overline{\overline{z}}}(t))\mathrm{d}t+\int_{\frac{1}{2}}^1 \mathcal{R}(\overline{\overline{z}}'(t),\dot{\overline{\overline{z}}}'(t))\mathrm{d}t\\ &= \int_0^1 \mathcal{R}(\widehat{z}(t),\dot{\widehat{z}}(t))\mathrm{d}t \qquad (5.16)\end{aligned}$$

Taking the infimum with respect to all $(\overline{z},\overline{z}')\in\mathrm{W}^{1,1}_{z_1,z_2}([0,1];\mathcal{Z})\times\mathrm{W}^{1,1}_{z_2,z_3}([0,1];\mathcal{Z})$ results in

$$\mathcal{D}(z_1,z_2)+\mathcal{D}(z_2,z_3)\geq\mathcal{D}(z_1,z_3),$$

since on the right hand side of the identity (5.16) the infimum is taken with respect to all functions $\widehat{z}\in\mathrm{W}^{1,1}_{z_1,z_3}([0,1];\mathcal{Z})$, which additionally satisfy $\widehat{z}|_{[0,\frac{1}{2}]}\in\mathrm{W}^{1,1}_{z_1,z_2}([0,\frac{1}{2}];\mathcal{Z})$ and $\widehat{z}|_{[\frac{1}{2},1]}\in\mathrm{W}^{1,1}_{z_2,z_3}([\frac{1}{2},1];\mathcal{Z})$; see (5.15). To show $\mathcal{D}(z_1,z_1)=0$, observe that in the case of $z_2=z_1$ the function $\widehat{z}\equiv z_1\in\mathrm{W}^{1,1}_{z_1,z_1}([0,1];\mathcal{Z})$ is a minimizer of the right hand side of (5.3). This observation together with the assumption $\mathcal{R}(z_1,0)=0$ proves the desired result.

2. To prove the rate independence of $(\mathcal{Q},\mathcal{E},\mathcal{Z})$, let $T^*>0$, $\theta\in\mathrm{C}^1([0,T^*];[0,T])$ and an energetic solution $(u,z):[0,T]\to\mathcal{Q}$ be given. By setting $\underline{u}:=u\circ\theta$, $\underline{z}:=z\circ\theta$, and $\underline{\ell}:=\ell\circ\theta$ we have to verify that $(\underline{u},\underline{z}):[0,T^*]\to\mathcal{Q}$ is an energetic solution of the system $(\mathcal{Q},\underline{\mathcal{E}},\mathcal{D})$, where

$$\underline{\mathcal{E}}(t,u,z):=W(u,z)-\langle\underline{\ell}(t),u\rangle. \qquad (5.17)$$

This is done by showing that every term of the energetic formulation for the energetic solution $(u,z):[0,T]\to\mathcal{Q}$ can be replaced by the associated term for the function $(\underline{u},\underline{z}):[0,T^*]\to\mathcal{Q}$. Hence, for an arbitrary but fixed $t\in[0,T]$ and $t^*:=\theta^{-1}(t)$ it is sufficient to prove the following three equalities:

$$\mathcal{E}(t,u(t),z(t))=\underline{\mathcal{E}}(t^*,\underline{u}(t^*),\underline{z}(t^*)), \qquad (5.18)$$

$$\int_0^t\langle\dot{\ell}(s),u(s)\rangle\mathrm{d}s=\int_0^{t^*}\langle\dot{\underline{\ell}}(s),\underline{u}(s)\rangle\mathrm{d}s, \qquad (5.19)$$

$$\mathrm{Diss}_{\mathcal{D}}(z;[0,t])=\mathrm{Diss}_{\mathcal{D}}(\underline{z};[0,t^*]). \qquad (5.20)$$

Note that $\mathcal{D}(z(t),\widetilde{z})=\mathcal{D}(\underline{z}(t^*),\widetilde{z})$ for all $\widetilde{z}\in\mathcal{Z}$ since $z(t)=\underline{z}(t^*)$ due to the definition.

3. By definition, we have $u(t)=\underline{u}(t^*)$, $z(t)=\underline{z}(t^*)$, and $\ell(t)=\underline{\ell}(t^*)$. Hence, the equality (5.18) is an easy consequence of comparing (5.13) and (5.17).

4. According to the chain rule we obtain $\dot{\underline{\ell}} = (\dot{\ell} \circ \theta)\dot{\theta}$. Keeping this in mind while applying the transformation $s = \theta|_{[0,t^*]}(\underline{s})$ to the left hand side of (5.19) results in

$$\int_0^t \langle \dot{\ell}(s), u(s) \rangle \mathrm{d}s = \int_0^{t^*} \langle \dot{\underline{\ell}}(\underline{s})[\dot{\theta}|_{[0,t^*]}(\underline{s})]^{-1}, \underline{u}(\underline{s}) \rangle \dot{\theta}|_{[0,t^*]}(\underline{s})\mathrm{d}\underline{s} = \int_0^{t^*} \langle \dot{\underline{\ell}}(\underline{s}), \underline{u}(\underline{s}) \rangle \mathrm{d}\underline{s}. \quad (5.21)$$

5. To prove (5.20) for $N \in \mathbb{N}$ let $\pi_N := \{0 = t_0 < t_1 < \ldots < t_N = t\}$ be an arbitrary partition of $[0,t]$. Furthermore, for the set Z_N of arbitrary but fixed values $z_0, z_1, \ldots, z_N \in \mathcal{Z}$ let $\mathrm{W}^{1,1}_{\pi_N, Z_N}([0,t];\mathcal{Z})$ denote the subset of all functions $\widehat{z} \in \mathrm{W}^{1,1}([0,t];\mathcal{Z})$ satisfying $\widehat{z}|_{[t_{j-1},t_j]} \in \mathrm{W}^{1,1}_{z_{j-1},z_j}([t_{j-1},t_j];\mathcal{Z})$ for every $j = 1,2,\ldots,N$. In the following, as an intermediate step for $\underline{\pi}_N^* := \{\underline{t}_j^* = \theta^{-1}(t_j) \,|\, j = 0,1,\ldots,N\}$ we show

$$A(\pi_N, Z_N) := \inf \int_0^t \mathcal{R}(\widehat{z}(s), \dot{\widehat{z}}(s))\mathrm{d}s = \inf \int_0^{t^*} \mathcal{R}(\widehat{z}'(\underline{s}), \dot{\widehat{z}}'(\underline{s}))\mathrm{d}\underline{s} =: B(\underline{\pi}_N^*, Z_N),$$

where the infimum on the left hand side is taken with respect to all functions $\widehat{z}$ of $\mathrm{W}^{1,1}_{\pi_N,Z_N}([0,t];\mathcal{Z})$, whereas the infimum on the right hand side is taken with respect to all functions $\widehat{z}' \in \mathrm{W}^{1,1}_{\underline{\pi}_N^*,Z_N}([0,t^*];\mathcal{Z})$. For an arbitrary but fixed $\widehat{z} \in \mathrm{W}^{1,1}_{\pi_N,Z_N}([0,t];\mathcal{Z})$ let $\underline{\widehat{z}} := \widehat{z} \circ \theta|_{[0,t^*]} \in \mathrm{W}^{1,1}_{\underline{\pi}_N^*,Z_N}([0,t^*];\mathcal{Z})$. Then, by exploiting the integral transformation $s = \theta|_{[0,t^*]}(\underline{s})$ and the positive homogeneity of degree 1 of the dissipation potential analogously to (5.21) we have

$$\int_0^t \mathcal{R}(\widehat{z}(s), \dot{\widehat{z}}(s))\mathrm{d}s = \int_0^{t^*} \mathcal{R}(\underline{\widehat{z}}(\underline{s}), \dot{\underline{\widehat{z}}}(\underline{s}))\mathrm{d}\underline{s}.$$

Therefore, $A(\pi_N, Z_N) \geq B(\underline{\pi}_N^*, Z_N)$ by taking the infimum with respect to all functions $\widehat{z} \in \mathrm{W}^{1,1}_{\pi_N,Z_N}([0,t];\mathcal{Z})$. Starting with an arbitrary but fixed $\widehat{z}' \in \mathrm{W}^{1,1}_{\underline{\pi}_N^*,Z_N}([0,t^*];\mathcal{Z})$ an analog treatment yields the opposite inequality $A(\pi_N, Z_N) \leq B(\underline{\pi}_N^*, Z_N)$.

6. Let $(u,z) : [0,T] \to \mathcal{Q}$ be an energetic solution of $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ and for $N \in \mathbb{N}$ let $\pi_N := \{0 = t_0 < t_1 < \ldots < t_N = t\}$. Then, for $Z_N(\pi_N) := \{z(t_j) \,|\, j = 0,1,\ldots,N\}$ by the same integral transformation argument as used in (5.16) we find

$$\sum_{j=1}^N \mathcal{D}(z(t_{j-1}), z(t_j)) = \inf \int_0^t \mathcal{R}(\widehat{z}(s), \dot{\widehat{z}}(s))\mathrm{d}s = A(\pi_N, Z_N(\pi_N)),$$

where the infimum is taken with respect to all functions $\widehat{z} \in \mathrm{W}^{1,1}_{\pi_N,Z_N(\pi_N)}([0,t];\mathcal{Z})$. Referring to the definition of the total dissipation, to prove (5.20) we have to verify

$$A := \sup_{\pi_N} A(\pi_N, Z_N(\pi_N)) = \sup_{\pi_N^*} B(\pi_N^*, Z_N^*(\pi_N^*)) =: B,$$

where $\pi_N^* := \{0 = t_0^* < t_1^* < \ldots < t_N^* = t^*\}$ and $Z_N^*(\pi_N^*) := \{\underline{z}(t_j^*) \,|\, j = 0,1,\ldots,N\}$ for $\underline{z} := z \circ \theta$. Let $(\pi_N)_{N \in \mathbb{N}}$ be a maximizing sequence of $\sup_{\pi_N} A(\pi_N, Z_N(\pi_N))$. For fixed $N \in \mathbb{N}$ let $\underline{\pi}_N^*$ be defined as in step 5. Then $Z_N(\pi_N) = Z_N^*(\underline{\pi}_N^*)$ and according to step 5 we have

$$A = \lim_{N\to\infty} A(\pi_N, Z_N(\pi_N)) \overset{\text{step 5}}{=} \lim_{N\to\infty} B(\underline{\pi}_N^*, Z_N^*(\underline{\pi}_N^*)) \leq B.$$

Starting with a maximizing sequence $(\pi_N^*)_{N \in \mathbb{N}}$ of $\sup_{\pi_N^*} B(\pi_N^*, Z_N^*(\pi_N^*))$ by an analog argument we find the opposite inequality and the proof is concluded. $\square$

6 Homogenization of unidirectional microstructure evolution models

This chapter is about the homogenization of rate-independent microstructure evolution models via the evolutionary Γ-convergence method introduced in [56]. This homogenization result is in preparation for rigorously establishing effective damage and crack models, which is done in the Chapters 7–9. For this reason, we here focus on the homogenization theory and for the comparison of our results with the already existing (damage) theory we refer to the less abstract setting of the following chapters. All models introduced in this chapter are set up in the energetic formulation (see Section 5.2) modeling a linear elastic body with microstructure evolution due to an internal variable. Here, these energetic formulations are based on energy functionals and dissipation distances satisfying the abstract conditions (5.7)–(5.10) of Chapter 5. Note that (although not mentioned) all considered dissipation distances below are assumed to be given such that the associated energetic formulation defines a rate-independent process; see Definition 5.1.

The microstructures of the microscopic models considered in the following section are assumed to be admissible in sense of Section 2.2; see (2.6). These non-periodic microstructures are characterized via a given tensor valued mapping $\widehat{\mathbb{C}} \in \mathrm{L}^\infty([0,1]^m; \mathcal{M}(Y))$ and it turns out that the microstructures of the microscopic and the effective models are based on this tensor. Hence, in this chapter $\widehat{\mathbb{C}} \in \mathrm{L}^\infty([0,1]^m; \mathcal{M}(Y))$ is assumed to be given and the crucial assumptions on this tensor valued mapping are the following:

For every measurable function $z : \mathbb{R}^d \to [0,1]^m$ the mapping

$$\widehat{\mathbb{C}}(z(\cdot))(\cdot) : \begin{cases} \mathbb{R}^d \times Y \to \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}}; \mathbb{R}^{d\times d}_{\mathrm{sym}}) \\ (x,y) \mapsto \widehat{\mathbb{C}}(z(x))(y) \end{cases} \quad \text{is measurable on } \mathbb{R}^d \times Y. \tag{6.1}$$

Moreover, the mapping $\widehat{\mathbb{C}} : [0,1]^m \to \mathcal{M}(Y)$ is continuous with respect to the strong L^1-topology, i.e., for every sequence $(\widehat{z}_\delta)_{\delta>0} \subset [0,1]^m$ satisfying $\lim_{\delta\to 0} \widehat{z}_\delta = \widehat{z}$ for $\widehat{z} \in [0,1]^m$ we have

$$\lim_{\delta\to 0} \|\widehat{\mathbb{C}}(\widehat{z}_\delta) - \widehat{\mathbb{C}}(\widehat{z})\|_{\mathrm{L}^1(Y;\mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}};\mathbb{R}^{d\times d}_{\mathrm{sym}}))} = 0. \tag{6.2}$$

Finally, there exists a constant $\alpha > 0$ such that for all $y \in Y$ and every $\widehat{z} \in [0,1]^m$

$$\alpha|\xi|^2_{d\times d} \le \langle \widehat{\mathbb{C}}(\widehat{z})(y)\xi, \xi\rangle_{d\times d} \quad \text{for all } \xi \in \mathbb{R}^{d\times d}_{\mathrm{sym}}. \tag{6.3}$$

Observe that α is assumed to be independent of $y \in Y$ and $\widehat{z} \in [0,1]^m$, i.e., the inequality (6.3) has to hold uniformly with respect to $y \in Y$ and $\widehat{z} \in [0,1]^m$.

6.1 Microscopic model

This section introduces the abstract microscopic microstructure evolution models homogenization is performed for in the following. Letting the set Ω denote the domain occupied by linear elastic material its elastic deformation is captured by a displacement field $u \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$. The assumptions on $\Omega \subset \mathbb{R}^d$ are that of condition (2.1) and $\Gamma_{\mathrm{Dir}} \subset \partial\Omega$ is closed with $\mu_{d-1}(\Gamma_{\mathrm{Dir}}) > 0$. Additionally to the displacement field evolution, microstructure changes are considered due to an internal variable $z_\varepsilon : [0,T] \to \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$. Here, the parameter $\varepsilon > 0$ models the intrinsic length scale of the considered microstructure as described in Section 2.2. Hence, the state space $\mathcal{Q}_\varepsilon(\Omega)$ is the following product space:

$$\mathcal{Q}_\varepsilon(\Omega) := \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \times \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m).$$

For a given internal variable $z_\varepsilon : [0,T] \to \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ and the given tensor valued mapping $\widehat{\mathbb{C}} \in \mathrm{L}^\infty([0,1]^m;\mathcal{M}(Y))$ the actual damage state of the material is captured by the material tensor $\mathbb{C}_\varepsilon(z_\varepsilon(t)) \in \mathcal{M}(\Omega)$ defined via

$$\mathbb{C}_\varepsilon(z_\varepsilon(t))(x) := \widehat{\mathbb{C}}(z_\varepsilon(t,x))(\{\tfrac{x}{\varepsilon}\}_Y). \tag{6.4}$$

Note that the measurability of this superposition is assured by assumption (6.1). Exploiting the uniform estimate (6.3), Korn's inequality yields the existence of a constant $C_{\mathbf{e}} > 0$ such that for all $(u, z_\varepsilon) \in \mathcal{Q}_\varepsilon(\Omega)$ the following coercivity condition holds:

$$C_{\mathbf{e}} \|u\|^2_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d} \leq \tfrac{1}{2}\langle \mathbb{C}_\varepsilon(z_\varepsilon)\mathbf{e}(u), \mathbf{e}(u)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}}. \tag{6.5}$$

The continuity result here below is essential for proving existence of solutions.

Lemma 6.1. *Let $\widehat{\mathbb{C}} \in \mathrm{L}^\infty([0,1]^m;\mathcal{M}(Y))$ satisfy the conditions* (6.1) *and* (6.2). *Then the tensor valued mapping $\mathbb{C}_\varepsilon : \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m) \to \mathcal{M}(\Omega)$ defined by* (6.4) *is continuous with respect to the strong L^1-topology, i.e., for every sequence $(z_\delta)_{\delta>0} \subset \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ satisfying $z_\delta \to z$ in $\mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ with some $z \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ we have*

$$\lim_{\delta\to 0} \|\mathbb{C}_\varepsilon(z_\delta) - \mathbb{C}_\varepsilon(z)\|_{\mathrm{L}^1(\Omega;\mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}};\mathbb{R}^{d\times d}_{\mathrm{sym}}))} = 0.$$

Remark 6.2. *We recall that for fixed $\varepsilon > 0$ the space $\mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ is finite dimensional.*

Proof. Since $z_\delta \to z$ in $\mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ by assumption, for all $\lambda \in \Lambda^+_\varepsilon$ we have that $z^{\varepsilon\lambda}_\delta :\equiv z_\delta|_{\varepsilon(\lambda+Y)\cap\Omega} \to z^{\varepsilon\lambda} :\equiv z|_{\varepsilon(\lambda+Y)\cap\Omega}$ in $\mathbb{R}^m$; see Remark 2.3. Hence,

$$\lim_{\delta\to 0} \sup_{\lambda\in\Lambda^+_\varepsilon} \|\widehat{\mathbb{C}}(z^{\varepsilon\lambda}_\delta) - \widehat{\mathbb{C}}(z^{\varepsilon\lambda})\|_{\mathrm{L}^1(Y;\mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}};\mathbb{R}^{d\times d}_{\mathrm{sym}}))} = 0$$

according to (6.2). Combining this condition with the calculations below proves the desired result. In the second line here below, $\{\frac{x}{\varepsilon}\}_Y = \frac{x}{\varepsilon} - \lambda$ for $x \in \varepsilon(\lambda+Y)$ is exploited and in line three the transformation $y = \frac{x}{\varepsilon} - \lambda$ of the integral is performed:

$$
\begin{aligned}
&\|\mathbb{C}_\varepsilon(z_\delta) - \mathbb{C}_\varepsilon(z)\|_{\mathrm{L}^1(\Omega;\mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}};\mathbb{R}^{d\times d}_{\mathrm{sym}}))} \\
&\qquad = \sum_{\lambda\in\Lambda_\varepsilon^+} \int_{\varepsilon(\lambda+Y)\cap\Omega} |\mathbb{C}_\varepsilon(z_\delta)(x) - \mathbb{C}_\varepsilon(z)(x)| \mathrm{d}x \\
&\qquad \le \sum_{\lambda\in\Lambda_\varepsilon^+} \int_{\varepsilon(\lambda+Y)} |\widehat{\mathbb{C}}(z_\delta^{\varepsilon\lambda})(\tfrac{x}{\varepsilon} - \lambda) - \widehat{\mathbb{C}}(z^{\varepsilon\lambda})(\tfrac{x}{\varepsilon} - \lambda)| \mathrm{d}x \\
&\qquad = \sum_{\lambda\in\Lambda_\varepsilon^+} \varepsilon^d \int_Y |\widehat{\mathbb{C}}(z_\delta^{\varepsilon\lambda})(y) - \widehat{\mathbb{C}}(z^{\varepsilon\lambda})(y)| \mathrm{d}y \\
&\qquad \le \mu_d(\Omega_\varepsilon^+) \sup_{\lambda\in\Lambda_\varepsilon^+} \|\widehat{\mathbb{C}}(z_\delta^{\varepsilon\lambda}) - \widehat{\mathbb{C}}(z^{\varepsilon\lambda})\|_{\mathrm{L}^1(Y;\mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}};\mathbb{R}^{d\times d}_{\mathrm{sym}}))}.
\end{aligned}
$$

Applying the limit $\delta \to 0$ to this estimate concludes the proof. □

To introduce the energetic formulation of the microscopic model we once choose the parameter $p \in (1, \infty)$ and keep it fixed for rest of this chapter, i.e., in the following, any p refers to this choice. One ingredient of the energetic formulation is the energy functional $\mathcal{E}_\varepsilon : [0,T]\times\mathcal{Q}_\varepsilon(\Omega) \to \mathbb{R}$ based on the material tensor $\mathbb{C}_\varepsilon : \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m) \to \mathcal{M}(\Omega)$.

$$
\mathcal{E}_\varepsilon(t, u, z_\varepsilon) := \tfrac{1}{2}\langle \mathbb{C}_\varepsilon(z_\varepsilon)\mathbf{e}(u), \mathbf{e}(u)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} + \|R_{\frac{\varepsilon}{2}} z_\varepsilon\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}} - \langle \ell(t), u\rangle \tag{6.6}
$$

Here, $\ell \in \mathrm{C}^1([0,T]; (\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$ models the external loading consisting of volume and surface forces.

Remark 6.3. *Here, $\|R_{\frac{\varepsilon}{2}} z_\varepsilon\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}}$ is a regularization term yielding better convergence properties when looking for an effective limit model. In fact, due to this regularization the effective microstructures in Section* 6.2 *and* 6.3 *are uniquely described by the limit internal variable. Moreover, certain aspects of the microstructure described by $\mathbb{C}_\varepsilon(z_\varepsilon)$ are preserved in the effective models; for more details see Remark* 6.6*. For a homogenization result without any microstructure regularization we refer to the papers* [20, 21, 24]*. The comparison between our effective microstructure and the more general but less explicit one of* [24] *is done in Section* 7.4*.*

Note that this regularization term is neither necessary nor problematic for proving existence of solutions of the microscopic model, i.e., for $\varepsilon > 0$ fixed. Descriptively, it can be seen as a local interaction of a material point and its neighborhood. Therefore, the type of interaction depends on the physical effect described by the internal variable. In the case of modeling damage (*Chapter* 7–9) *this penalty term prefers the progression of damage nearby already damaged material and penalizes its appearance in areas of intact material.*

The dissipation of the rate-independent system is modeled by the dissipation distance $\mathcal{D}_\varepsilon : \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)\times\mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m) \to [0,\infty]$ given by

$$
\mathcal{D}_\varepsilon(z_1, z_2) := \begin{cases} \widetilde{\mathcal{D}}_\varepsilon(z_1, z_2) & \text{if } z_1 \ge z_2, \\ \infty & \text{otherwise,} \end{cases} \tag{6.7}
$$

where $\widetilde{\mathcal{D}}_\varepsilon : \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)\times\mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m) \to [0,\infty)$ is continuous and $(\widetilde{\mathcal{D}}_\varepsilon)_{\varepsilon>0}$ is assumed to converge continuously to $\widetilde{\mathcal{D}}_0 : \mathrm{L}^p(\Omega;[0,1]^m)\times\mathrm{L}^p(\Omega;[0,1]^m) \to [0,\infty)$, i.e.:

$$\begin{aligned}&\text{For all } (z_\varepsilon,\widetilde{z}_\varepsilon)_{\varepsilon>0} \text{ and } z_0,\, \widetilde{z}_0 \in \mathrm{L}^p(\Omega;[0,1]^m) \text{ satisfying } z_\varepsilon,\, \widetilde{z}_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m),\\ &z_\varepsilon \to z_0 \text{ in } \mathrm{L}^1(\Omega)^m \text{ and } \widetilde{z}_\varepsilon \to \widetilde{z}_0 \text{ in } \mathrm{L}^1(\Omega)^m \text{ it holds } \lim_{\varepsilon\to 0} \widetilde{\mathcal{D}}_\varepsilon(z_\varepsilon,\widetilde{z}_\varepsilon) = \widetilde{\mathcal{D}}_0(z_0,\widetilde{z}_0).\end{aligned} \tag{6.8}$$

Remark 6.4. *Since the elements of the sequence $(z_\varepsilon,\widetilde{z}_\varepsilon)_{\varepsilon>0}$ of condition* (6.8) *are uniformly bounded by* 1 *on Ω, the assumed convergences actually hold for all $q \in [1,\infty)$.*

Additionally, $\mathcal{D}_\varepsilon : \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)\times\mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m) \to [0,\infty]$ has to be chosen such that

$$\begin{aligned}&\forall\, z_1,z_2,z_3 \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m) : \mathcal{D}_\varepsilon(z_1,z_2) = 0 \quad \Leftrightarrow \quad z_1 = z_2\\ &\text{and} \qquad\qquad\qquad\qquad \mathcal{D}_\varepsilon(z_1,z_3) \le \mathcal{D}_\varepsilon(z_1,z_2) + \mathcal{D}_\varepsilon(z_2,z_3).\end{aligned} \tag{6.9}$$

For $t \in [0,T]$ and $z_\varepsilon : [0,T] \to \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ the total dissipation $\mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon;[0,t])$ reads as follows:

$$\mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon;[0,t]) := \sup\Big\{\sum_{j=1}^{N} \mathcal{D}_\varepsilon(z_\varepsilon(t_{j-1}), z_\varepsilon(t_j))\Big\},$$

where for $N \in \mathbb{N}$ the supremum is taken with respect to all finite partitions $\pi_N := \{0 = t_0 < t_1 < \ldots < t_N = t\}$ of the interval $[0,t]$. Note that according to the definition of $\mathcal{D}_\varepsilon : \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)\times\mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m) \to [0,\infty]$ the total dissipation is finite if and only if $z_\varepsilon : [0,T] \to \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ is a (component-wise) non-increasing function.

For given initial values $(u^0_\varepsilon, z^0_\varepsilon) \in \mathcal{Q}_\varepsilon(\Omega)$ the rate-independent microstructure evolution is modeled by the ε-dependent energetic formulation (S^ε) and (E^ε), where $\varepsilon > 0$ denotes the intrinsic length-scale of the microstructure and where the energy functional and the dissipation distance are chosen as in (6.6) and (6.7), respectively.

Stability condition (S^ε) **and energy balance** (E^ε) **for all** $t \in [0,T]$**:**

$$\begin{aligned}&\mathcal{E}_\varepsilon(t,u_\varepsilon(t),z_\varepsilon(t)) \le \mathcal{E}_\varepsilon(t,\widetilde{u},\widetilde{z}) + \mathcal{D}_\varepsilon(z_\varepsilon(t),\widetilde{z}) \quad \text{for all } (\widetilde{u},\widetilde{z}) \in \mathcal{Q}_\varepsilon(\Omega)\\ &\mathcal{E}_\varepsilon(t,u_\varepsilon(t),z_\varepsilon(t)) + \mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon;[0,t]) = \mathcal{E}_\varepsilon(0,u^0_\varepsilon,z^0_\varepsilon) - \int_0^t \langle \dot{\ell}(s), u_\varepsilon(s)\rangle \mathrm{d}s\end{aligned}$$

Since $\ell \in \mathrm{C}^1([0,T];(\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$, for any energetic solution $(u_\varepsilon,z_\varepsilon) : [0,T] \to \mathcal{Q}_\varepsilon(\Omega)$ the right hand side of the energy balance (E^ε) is finite for all $t \in [0,T]$. Therefore, $z_\varepsilon : [0,T] \to \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ is a (component-wise) non-increasing function. Following the procedure of Section 5.2 for $\widetilde{t} \in [0,T]$ by $\mathcal{S}_\varepsilon(\widetilde{t})$ the set of stable states is denoted, i.e.,

$$\mathcal{S}_\varepsilon(\widetilde{t}) := \{(u_\varepsilon,z_\varepsilon) \in \mathcal{Q}_\varepsilon(\Omega) \text{ satisfying } (\mathrm{S}^\varepsilon) \text{ for } t = \widetilde{t} \text{ and } \mathcal{E}_\varepsilon(\widetilde{t},u_\varepsilon,z_\varepsilon) < \infty\}.$$

For the energetic formulation (S^ε) and (E^ε) the abstract Theorem 5.5 guarantees the existence of a solution.

Proposition 6.5 (Existence of solutions)**.** *Assume that* (6.1), (6.2)*, and* (6.5) *hold. For* $\ell \in \mathrm{C}^1([0,T];(\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$ *let the energy functional* $\mathcal{E}_\varepsilon : [0,T]\times\mathcal{Q}_\varepsilon(\Omega) \to \mathbb{R}$ *be defined via* (6.6)*. Moreover, for a continuous* $\widetilde{\mathcal{D}}_\varepsilon : \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)\times\mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m) \to [0,\infty)$ *let the dissipation distance* $\mathcal{D}_\varepsilon : \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)\times\mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m) \to [0,\infty]$ *be defined by* (6.7) *and satisfy* (6.9)*.*

Then for all $(u^0_\varepsilon, z^0_\varepsilon) \in \mathcal{S}_\varepsilon(0)$*, there exists an energetic solution* $(u_\varepsilon, z_\varepsilon) : [0,T] \to \mathcal{Q}_\varepsilon(\Omega)$ *of the rate-independent system* $(\mathcal{Q}_\varepsilon(\Omega), \mathcal{E}_\varepsilon, \mathcal{D}_\varepsilon)$ *satisfying* $(u_\varepsilon(0), z_\varepsilon(0)) = (u^0_\varepsilon, z^0_\varepsilon)$ *and*

$$
\begin{aligned}
u_\varepsilon &\in \mathrm{L}^\infty([0,T];\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d),\\
z_\varepsilon &\in \mathrm{L}^\infty([0,T];\mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)) \cap \mathrm{BV}_{\mathcal{D}_\varepsilon}([0,T];\mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)).
\end{aligned}
$$

Proof. We have to check the conditions (5.7)–(5.12).

(5.7): Due to (6.5) we have for every $u \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$, $z \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$, and $t \in [0,T]$:

$$
\begin{aligned}
C_{\mathbf{e}}\|u\|^2_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d} &\overset{(6.5)}{\leq} \tfrac{1}{2}\langle \mathbb{C}_\varepsilon(z)\mathbf{e}(u), \mathbf{e}(u)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}}\\
&\leq \mathcal{E}_\varepsilon(t,u,z) + \langle \ell(t), u\rangle\\
&\leq \mathcal{E}_\varepsilon(t,u,z) + C_\ell\|u\|_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d},
\end{aligned}
\tag{6.10}
$$

where $C_\ell := \|\ell\|_{\mathrm{C}^1([0,T];(\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)} < \infty$. (Note that since $\varepsilon > 0$ is fixed in the whole proof, here the index ε for $z \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ is neglected.) By artificially introducing the product $\|u\|_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d}\cdot 1$ to the right hand side of (6.10), the application of the scaled version of Young's inequality yields

$$
C\|u\|^2_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d} \leq \mathcal{E}_\varepsilon(t,u,z) + \widehat{C} \tag{6.11}
$$

for some constants $C, \widehat{C} > 0$ and a suitable chosen scaling parameter. For a constant $E \in \mathbb{R}$ and a sequence $(u_\delta, z_\delta)_{\delta>0} \subset \mathcal{Q}_\varepsilon(\Omega)$ belonging to the sublevel set $\mathrm{Sub}_E(t)$ (see (5.6)) estimate (6.11) yields a uniform upper bound for the sequence $(\|u_\delta\|^2_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d})_{\delta>0}$. Hence, due to the reflexivity of $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ there exists $u_0 \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ and a subsequence of $(u_\delta)_{\delta>0}$ converging weakly to u_0 in $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$.

Moreover, $(z_\delta)_{\delta>0} \subset \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ is uniformly bounded by assumption. Hence, there exists a function $z_0 \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ and a subsequence of $(z_\delta)_{\delta>0}$ converging to z_0 in $\mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$; see Remark 2.3. Applying Lemma 6.1 yields the strong convergence of $(\mathbb{C}_\varepsilon(z_\delta))_{\delta>0}$ to $\mathbb{C}_\varepsilon(z_0)$ in $\mathrm{L}^1(\Omega;\mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}};\mathbb{R}^{d\times d}_{\mathrm{sym}}))$ for this subsequence of $(\delta)_{\delta>0}$. By possibly choosing a further subsequence $(\delta')_{\delta'>0}$ of $(\delta)_{\delta>0}$ we have $\mathbf{e}(u_{\delta'}) \rightharpoonup \mathbf{e}(u_0)$ in $\mathrm{L}^2(\Omega)^{d\times d}$ and $\mathbb{C}_\varepsilon(z_{\delta'}) \to \mathbb{C}_\varepsilon(z_0)$ in $\mathrm{L}^1(\Omega;\mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}};\mathbb{R}^{d\times d}_{\mathrm{sym}}))$. For $x \in \Omega$, $\widehat{z} \in [0,1]^m$, and $\xi \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ set $f(x,\widehat{z},\xi) := \langle \mathbb{C}_\varepsilon(\widehat{z})\xi, \xi\rangle_{d\times d}$. Then all assumptions of the lower semicontinuity Theorem 3.23 of [14] are fulfilled and it follows

$$
\liminf_{\delta'\to 0} \int_\Omega \langle \mathbb{C}_\varepsilon(z_{\delta'})\mathbf{e}(u_{\delta'}), \mathbf{e}(u_{\delta'})\rangle_{d\times d}\mathrm{d}x \geq \int_\Omega \langle \mathbb{C}_\varepsilon(z_0)\mathbf{e}(u_0), \mathbf{e}(u_0)\rangle_{d\times d}\mathrm{d}x. \tag{6.12}
$$

According to Remark 2.3 it holds $z_{\delta'} \to z_0$ in $\mathrm{L}^q(\Omega)^m$ for every $q \in [1,\infty)$. Combining this convergence with the continuity of the mapping $R_{\frac{\varepsilon}{2}} : \mathrm{K}_{\varepsilon\Lambda}(\Omega)^m \to \mathrm{K}_{\frac{\varepsilon}{2}\Lambda}(\Omega_\varepsilon^+)^{m\times d}$ ($\varepsilon > 0$ fixed) with respect to the strong L^p-topology results in

$$\lim_{\delta'\to 0} \|R_{\frac{\varepsilon}{2}} z_{\delta'}\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}} = \|R_{\frac{\varepsilon}{2}} z_0\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}}. \tag{6.13}$$

Trivially, $\langle \ell(t), u_{\delta'}\rangle \to \langle \ell(t), u_0\rangle$ is valid, since $u_{\delta'} \rightharpoonup u_0$ in $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ which together with (6.12) and (6.13) yields

$$E \geq \liminf_{\delta'\to 0} \mathcal{E}_\varepsilon(t, u_{\delta'}, z_{\delta'}) \geq \mathcal{E}_\varepsilon(t, u_0, z_0),$$

such that the compactness of the energy sublevel sets is proven.

(5.8): Since $\ell \in \mathrm{C}^1([0,T]; (\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$, we have

$$|\partial_t \mathcal{E}_\varepsilon(t,u,z)| = |\langle \dot{\ell}(t), u\rangle| \leq C_\ell \|u\|_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d} \leq \tfrac{C_\ell}{2}(\|u\|^2_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d} + 1).$$

Combining this estimate with inequality (6.11) gives $|\partial_t \mathcal{E}_\varepsilon(t,u,z)| \leq c_1(c_0 + \mathcal{E}_\varepsilon(t,u,z))$ for some constants $c_0, c_1 > 0$ such that the uniform control of the power is shown.

(5.9): See assumption (6.9).

(5.10): According to Remark 2.3 the weak and strong topology of $\mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ are the same. Hence, condition (5.10) holds, since $\mathcal{D}_\varepsilon : \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m) \times \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m) \to [0,\infty]$ is lower continuous by definition.

(5.11): Since $\partial_t \mathcal{E}_\varepsilon(t,u,z) = -\langle \dot{\ell}(t), u\rangle$, this condition is trivially satisfied.

(5.12): For $u \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$, $z \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$, and $t \in [0,T]$ considering $(t_\delta, u_\delta, z_\delta)_{\delta>0}$ with $(u_\delta, z_\delta) \in \mathcal{S}_\varepsilon(t_\delta)$, $t_\delta \to t$, $u_\delta \rightharpoonup u$ in $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$, and $z_\delta \to z$ in $\mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ we have to check that $(u,z) \in \mathcal{S}_\varepsilon(t)$. For an arbitrary function $\widetilde{z} \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ with $\widetilde{z} \leq z$ let $\widetilde{z}_\delta := \min\{\widetilde{z}, z_\delta\}$. Then $\widetilde{z}_\delta \to \widetilde{z}$ in $\mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ and $\infty > \widetilde{\mathcal{D}}_\varepsilon(z_\delta, \widetilde{z}_\delta) \to \widetilde{\mathcal{D}}_\varepsilon(z, \widetilde{z})$ for $\delta \to 0$ due to the assumed continuity of $\widetilde{\mathcal{D}}_\varepsilon : \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m) \times \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m) \to [0,\infty)$. Since $\langle \ell(t_\delta), u_\delta\rangle \to \langle \ell(t), u\rangle$ analogously to the first step of this proof, for all functions $(\widetilde{u}, \widetilde{z}) \in \mathcal{Q}_\varepsilon(\Omega)$ with $\widetilde{z} \leq z$ we find

$$\begin{aligned}
\mathcal{E}_\varepsilon(t,u,z) &\overset{\text{step 1}}{\leq} \liminf_{\delta\to 0} \mathcal{E}_\varepsilon(t_\delta, u_\delta, z_\delta) \\
&\leq \lim_{\delta\to 0} \Big(\mathcal{E}_\varepsilon(t_\delta, \widetilde{u}, \widetilde{z}_\delta) + \widetilde{\mathcal{D}}_\varepsilon(z_\delta, \widetilde{z}_\delta) \Big) \\
&= \mathcal{E}_\varepsilon(t, \widetilde{u}, \widetilde{z}_0) + \widetilde{\mathcal{D}}_\varepsilon(z, \widetilde{z}).
\end{aligned}$$

The second inequality is due to the stability of the sequence $(u_\delta, z_\delta)_{\delta>0}$ and in the last line Lemma 6.1 was exploited. Note that in the case $\widetilde{z} \not\leq z$, due to $\mathcal{D}_\varepsilon(z, \widetilde{z}) = \infty$, the stability condition (S) is trivially satisfied.

Finally, letting $(u_\varepsilon, z_\varepsilon) : [0,T] \to \mathcal{Q}_\varepsilon(\Omega)$ be a solution of (S^ε) and (E^ε) its time regularity needs to be proven. Since (E^ε) is fulfilled for all $t \in [0,T]$ and since its right hand side is finite we have $\mathcal{E}_\varepsilon(t, u_\varepsilon(t), z_\varepsilon(t)) < \infty$ and $\mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon; [0,T]) < \infty$. Hence, we already have $z_\varepsilon \in \mathrm{BV}_{\mathcal{D}_\varepsilon}([0,T];\mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m))$. The proof of Theorem 5.5 is based on a generalized version of Helly's selection principal implying the piecewise continuity of $z_\varepsilon : [0,T] \to \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$. Therefore, it holds $z_\varepsilon \in \mathrm{L}^\infty([0,T],\mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m))$. The estimate $\mathcal{E}_\varepsilon(t,u_\varepsilon(t),z_\varepsilon(t)) < \infty$ immediately yields $u_\varepsilon \in \mathrm{L}^\infty([0,T],\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)$ according to inequality (6.11). Since the proof of the abstract existence result (Theorem 5.5) is based on interpolants constructed for a time discretization and since all these interpolants of the energetic solution are measurable in time, the displacement field is measurable, too. This concludes the proof. □

6.2 Two-scale effective model

By investigating the asymptotic behavior of the microscopic models (S^ε) and (E^ε), the two-scale model ($\mathbf{S}^0$) and ($\mathbf{E}^0$) introduced below will turn out to be their rigorous limit for $\varepsilon \to 0$. For an internal variable $z_0 \in \mathrm{W}^{1,p}(\Omega;[0,1]^m)$ and the given tensor valued mapping $\widehat{\mathbb{C}} \in \mathrm{L}^\infty([0,1]^m;\mathcal{M}(Y))$, the microstructure is described by the two-scale elasticity tensor $\mathbb{C}_0(z_0) \in \mathcal{M}(\Omega{\times}Y)$ for almost every $(x,y) \in \Omega{\times}Y$ defined via

$$\mathbb{C}_0(z_0)(x,y) := \widehat{\mathbb{C}}(z_0(x))(y). \tag{6.14}$$

Note that the measurability of this superposition is ensured by assumption (6.1). Since it is the strong two-scale limit of a sequence of microscopic tensors of the previous section (see Theorem 3.9), this tensor is actually the natural candidate.

Remark 6.6. *The constitutive relation* (6.14) *states that in almost every point x of Ω there is a unit cell $\{x\}{\times}Y$ containing the microstructure modeled by the mapping $\widehat{\mathbb{C}}(z_0(x)) \in \mathcal{M}(\{x\}{\times}Y)$. For a more descriptive illustration of this limit microstructure, see the less abstract setting of Section* 7.2. *Observe that the effective microstructure modeled by $\mathbb{C}_0(z_0) \in \mathcal{M}(\Omega{\times}Y)$ for the given mapping $\widehat{\mathbb{C}} \in \mathrm{L}^\infty([0,1]^m;\mathcal{M}(Y))$ is uniquely described by the limit internal variable $z_0 \in \mathrm{W}^{1,p}(\Omega;[0,1]^m)$. Without the microstructure regularization in the microscopic models this would not be the case; see Section* 7.4.

The fact that the energetic formulation (S^ε) and (E^ε) is solely based on functionals makes this approach well adapted to the theory of Γ-convergence when looking for an effective limit model. Let $\mathcal{Y} := \mathbb{R}^d/_\Lambda$ denote the periodicity cell. For a given $t \in [0,T]$ we are going to apply the theory of Γ-convergence to the sequence $(\mathcal{E}_\varepsilon(t,\cdot,\cdot))_{\varepsilon>0}$ of microscopic energy functionals. For this purpose, we choose the two-scale topology for the displacement field component u_ε (see Proposition 3.7) and the topology implied by Theorem 4.5 for the internal variable z_ε. Hence, the limit function space $\mathbf{Q}_0$ has the following structure:

$$\mathbf{Q}_0 := \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d{\times}\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d{\times}\mathrm{W}^{1,p}(\Omega;[0,1]^m).$$

For $(u_0,U_1) \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d{\times}\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ we define $\widetilde{\mathbf{e}}(u_0,U_1) := \mathbf{e}_x(u_0) + \mathbf{e}_y(U_1)$, where $\mathbf{e}_x(u_0)$ and $\mathbf{e}_y(U_1)$ denote the linearized strain tensors with respect to the x and y-variable; see (2.2). Then the stored energy of the system is modeled by the two-scale energy functional $\mathbf{E}_0 : [0,T]\times\mathbf{Q}_0 \to \mathbb{R}$ defined via

$$\mathbf{E}_0(t,u_0,U_1,z_0) := \tfrac{1}{2}\langle\mathbb{C}_0(z_0)\widetilde{\mathbf{e}}(u_0,U_1),\widetilde{\mathbf{e}}(u_0,U_1)\rangle_{\mathrm{L}^2(\Omega\times Y)^{d\times d}} + \|\nabla z_0\|^p_{\mathrm{L}^p(\Omega)^{m\times d}} - \langle\ell(t),u_0\rangle.$$

The dissipated energy of the effective models is based on the given, continuous mapping $\widetilde{\mathcal{D}}_0 : \mathrm{L}^p(\Omega;[0,1]^m)\times\mathrm{L}^p(\Omega;[0,1]^m) \to [0,\infty)$, see (6.8). Thus, the limit dissipation distance $\mathbf{D}_0 : \mathrm{W}^{1,p}(\Omega;[0,1]^m)\times\mathrm{W}^{1,p}(\Omega;[0,1]^m) \to [0,\infty]$ is given by

$$\mathbf{D}_0(z_1,z_2) := \begin{cases} \widetilde{\mathcal{D}}_0(z_1,z_2) & \text{if } z_1 \geq z_2, \\ \infty & \text{otherwise} \end{cases}$$

and for $t \in [0,T]$ and a function $z_0 : [0,T] \to \mathrm{W}^{1,p}(\Omega;[0,1]^m)$ the total dissipation $\mathrm{Diss}_{\mathbf{D}_0}(z_0;[0,t])$ is defined by

$$\mathrm{Diss}_{\mathbf{D}_0}(z_0;[0,t]) := \sup\Big\{\sum_{j=1}^{N}\mathbf{D}_0(z_0(t_{j-1}),z_0(t_j))\Big\},$$

where for $N \in \mathbb{N}$ the supremum is taken with respect to all finite partitions $\pi_N := \{0 = t_0 < t_1 < \ldots < t_N = t\}$ of the interval $[0,t]$. Finally, for given initial values $(u_0^0,U_1^0,z_0^0) \in \mathbf{Q}_0$ the rate-independent microstructure evolution model is given by the energetic formulation $(\mathbf{S}^0)$ and $(\mathbf{E}^0)$:

Stability condition $(\mathbf{S}^0)$ and energy balance $(\mathbf{E}^0)$ for all $t \in [0,T]$:

$$\mathbf{E}_0(t,u_0(t),U_1(t),z_0(t)) \leq \mathbf{E}_0(t,\widetilde{u},\widetilde{U},\widetilde{z}) + \mathbf{D}_0(z_0(t),\widetilde{z}) \quad \text{for all } (\widetilde{u},\widetilde{U},\widetilde{z}) \in \mathbf{Q}_0$$
$$\mathbf{E}_0(t,u_0(t),U_1(t),z_0(t)) + \mathrm{Diss}_{\mathbf{D}_0}(z_0;[0,t]) = \mathbf{E}_0(0,u_0^0,U_1^0,z_0^0) - \int_0^t\langle\dot{\ell}(s),u_0(s)\rangle\mathrm{d}s$$

For $\widetilde{t} \in [0,T]$ we denote by $\mathbf{S}_0(\widetilde{t})$ the set of stable states, i.e.,

$$\mathbf{S}_0(\widetilde{t}) := \{(u_0,U_1,z_0) \in \mathbf{Q}_0 \text{ satisfying } (\mathbf{S}^0) \text{ for } t = \widetilde{t} \text{ and } \mathbf{E}_0(\widetilde{t},u_0,U_1,z_0) < \infty\}.$$

Remark 6.7. *The existence of a solution of the two-scale model is proven via the convergence result in Section* 6.5*, where for $\varepsilon \to 0$ the convergence of a subsequence of energetic solutions $(u_\varepsilon,z_\varepsilon) : [0,T] \to \mathcal{Q}_\varepsilon(\Omega)$ of the microscopic models* (S^ε) *and* (E^ε) *to a function $(u_0,U_1,z_0) : [0,T] \to \mathbf{Q}_0$ satisfying the two-scale energetic formulation* $(\mathbf{S}^0)$ *and* $(\mathbf{E}^0)$ *is shown.*

6.3 One-scale effective model

In this section we formulate a one-scale model which is equivalent to the two-scale model of Section 6.2 in the following sense: From any solution of one of those systems a solution of the other model can be constructed.

Let the state space $\mathcal{Q}_0(\Omega)$ be given by

$$\mathcal{Q}_0(\Omega) := \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \times \mathrm{W}^{1,p}(\Omega;[0,1]^m).$$

The energy functional is based on a mapping $\mathbb{C}_{\mathrm{eff}} : \mathrm{W}^{1,p}(\Omega;[0,1]^m) \to \mathcal{M}(\Omega)$. The precise structure of this microstructure describing function is motivated by the following observation: Let $t \in [0,T]$ and $(u_0, U_1, z_0) \in \mathbf{S}_0(t)$ be given. Then, by assuming the testfunctions $(\widetilde{u}, \widetilde{z})$ to take the values (u_0, z_0) in the stability condition ($\mathbf{S}^0$) we find that U_1 is the unique solution of

$$\min\{I_0(z_0, u_0, U) \,|\, U \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d\}, \tag{6.15}$$

where

$$I_0(z_0, u_0, U) := \int_{\Omega\times Y} \langle \widehat{\mathbb{C}}(z_0(x))(y)(\widetilde{\mathbf{e}}(u_0, U)(x,y)), \widetilde{\mathbf{e}}(u_0, U)(x,y)\rangle_{d\times d} \mathrm{d}y\mathrm{d}x.$$

This motivates the introduction of a tensor $\widetilde{\mathbb{C}}_{\mathrm{eff}} \in \mathrm{L}^\infty([0,1]^m; \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}}; \mathbb{R}^{d\times d}_{\mathrm{sym}}))$ given by the following unit cell problem: For $\xi \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, $\widehat{z} \in [0,1]^m$, and the given mapping $\widehat{\mathbb{C}} \in \mathrm{L}^\infty([0,1]^m; \mathcal{M}(Y))$ let

$$C_{\mathrm{eff}}(\widehat{z}, \xi) := \min_{v \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d} I(\widehat{z}, \xi, v), \tag{6.16a}$$

$$I(\widehat{z}, \xi, v) := \int_Y \Big\langle \widehat{\mathbb{C}}(\widehat{z})(y)\Big(\xi + \mathbf{e}_y(v)(y)\Big), \xi + \mathbf{e}_y(v)(y) \Big\rangle_{d\times d} \mathrm{d}y. \tag{6.16b}$$

Then $C_{\mathrm{eff}}(\widehat{z}, \cdot)$ defines the tensor $\widetilde{\mathbb{C}}_{\mathrm{eff}}(\widehat{z})$ as stated in the following proposition.

Proposition 6.8. *Let condition* (6.3) *hold and let* $C_{\mathrm{eff}} : [0,1]^m \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}$ *be defined by* (6.16). *Then for* $\widehat{z} \in [0,1]^m$ *there exists a unique solution of* (6.16a). *Moreover, for all* $\widehat{z} \in [0,1]^m$ *there exists* $\widetilde{\mathbb{C}}_{\mathrm{eff}}(\widehat{z}) \in \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}}, \mathbb{R}^{d\times d}_{\mathrm{sym}})$ *such that*

$$\forall\, \xi \in \mathbb{R}^{d\times d}_{\mathrm{sym}} : \qquad C_{\mathrm{eff}}(\widehat{z}, \xi) = \langle \widetilde{\mathbb{C}}_{\mathrm{eff}}(\widehat{z})\xi, \xi\rangle_{d\times d}.$$

Proof. For given $\xi \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and $\widehat{z} \in [0,1]^m$ the functional $I(\widehat{z}, \xi, \cdot) : \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d \to \mathbb{R}$ is continuous and strictly convex due to (6.3). Hence, there exists a unique minimizer of (6.16a) fulfilling the Euler-Lagrange equation

$$D_v(I(\widehat{z}, \xi, v))[\widetilde{v}] = 0 \qquad \forall\, \widetilde{v} \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d$$

and according to the Lemma of Lax–Milgram

$$\mathcal{L}_{\hat{z}}(\xi) := \mathrm{Argmin}\{I(\widehat{z}, \xi, v) \,|\, v \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d\} \tag{6.17}$$

defines a linear solution operator $\mathcal{L}_{\hat{z}} : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})$. For $i,j,k,l \in \{1,\dots,d\}$ let $\delta_{ij,kl}$ denote the Kronecker delta. Then for $e_{ij} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ given by $(e_{ij})_{kl} := \frac{1}{2}(\delta_{ij,kl} + \delta_{ji,kl})$ we define

$$\widetilde{\mathbb{C}}_{\mathrm{eff}_{ijkl}}(\widehat{z}) := \int_Y \Big\langle \widehat{\mathbb{C}}(\widehat{z})(y)\Big(e_{ij} + \mathbf{e}_y(\mathcal{L}_{\hat{z}}(e_{ij}))(y)\Big), e_{kl} + \mathbf{e}_y(\mathcal{L}_{\hat{z}}(e_{kl}))(y) \Big\rangle_{d\times d} \mathrm{d}y.$$

First of all we have $\widetilde{\mathbb{C}}_{\text{eff}}(\widehat{z}) \in \text{Lin}_{\text{sym}}(\mathbb{R}^{d\times d}_{\text{sym}}, \mathbb{R}^{d\times d}_{\text{sym}})$ and by

$$\begin{aligned}\langle \widetilde{\mathbb{C}}_{\text{eff}}(\widehat{z})\xi, \xi\rangle_{d\times d} &= \sum_{i,j=1}^{d} \langle \xi, e_{ij}\rangle_{d\times d} \sum_{k,l=1}^{d} \langle \xi, e_{kl}\rangle_{d\times d} \langle \widetilde{\mathbb{C}}_{\text{eff}_{ijkl}}(\widehat{z}) e_{ij}, e_{kl}\rangle_{d\times d} \\ &= \int_Y \langle \widehat{\mathbb{C}}(\widehat{z})(y)(\xi + \mathbf{e}_y(\mathcal{L}_{\hat{z}}(\xi))(y)), \xi + \mathbf{e}_y(\mathcal{L}_{\hat{z}}(\xi))(y)\rangle_{d\times d} \mathrm{d}y\end{aligned}$$

we find $C_{\text{eff}}(\widehat{z}, \xi) = \langle \widetilde{\mathbb{C}}_{\text{eff}}(\widehat{z})\xi, \xi\rangle_{d\times d}$, which concludes the proof. □

Proposition 6.8 enables us to introduce $\mathbb{C}_{\text{eff}} : \mathrm{W}^{1,p}(\Omega; [0,1]^m) \to \mathcal{M}(\Omega)$ describing the microstructure evolution in the energetic formulation below. For $\xi \in \mathbb{R}^{d\times d}_{\text{sym}}$, $x \in \Omega$, $z_0 \in \mathrm{W}^{1,p}(\Omega; [0,1]^m)$, and the given $\widehat{\mathbb{C}} \in \mathrm{L}^\infty([0,1]^m; \mathcal{M}(Y))$ the unit cell problem

$$\langle \mathbb{C}_{\text{eff}}(z_0)(x)\xi, \xi\rangle_{d\times d} := \min_{v \in \mathrm{H}^1_{\text{av}}(\mathcal{Y})^d} \int_Y \langle \widehat{\mathbb{C}}(z_0(x))(y)(\xi + \mathbf{e}_y(v)(y)), \xi + \mathbf{e}_y(v)(y)\rangle_{d\times d} \mathrm{d}y \tag{6.18}$$

is well defined. Thus, the one-scale model is based on the one-scale energy functional $\mathcal{E}_0 : [0,T] \times \mathcal{Q}_0(\Omega) \to \mathbb{R}$ defined in the following way:

$$\mathcal{E}_0(t, u_0, z_0) := \tfrac{1}{2} \langle \mathbb{C}_{\text{eff}}(z_0)\mathbf{e}(u_0), \mathbf{e}(u_0)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} + \|\nabla z_0\|^p_{\mathrm{L}^p(\Omega)^{m\times d}} - \langle \ell(t), u_0\rangle$$

For the same given, continuous mapping $\widetilde{\mathcal{D}}_0 : \mathrm{L}^p(\Omega; [0,1]^m) \times \mathrm{L}^p(\Omega; [0,1]^m) \to [0,\infty)$ as considered in the previous sections (see (6.8)) the dissipated energy of the system is based on the limit dissipation distance $\mathcal{D}_0 : \mathrm{W}^{1,p}(\Omega; [0,1]^m) \times \mathrm{W}^{1,p}(\Omega; [0,1]^m) \to [0,\infty]$, which reads as follows:

$$\mathcal{D}_0(z_1, z_2) := \begin{cases} \widetilde{\mathcal{D}}_0(z_1, z_2) & \text{if } z_1 \geq z_2, \\ \infty & \text{otherwise.} \end{cases}$$

Observe that in contrast to the limit energy functional, due to the regularization with respect to the damage variable there is no second scale appearing in the limit dissipation distance. Therefore, (here and in all following limit models) the dissipation distances of the two-scale and one-scale limit models coincide, i.e., it holds $\mathcal{D}_0(z_1, z_2) = \mathbf{D}_0(z_1, z_2)$ for all $z_1, z_2 \in \mathrm{W}^{1,p}(\Omega; [0,1]^m)$ by definition. For $z_0 : [0,T] \to \mathrm{W}^{1,p}(\Omega; [0,1]^m)$ the total dissipation $\text{Diss}_{\mathcal{D}_0}(z_0; [0,t])$ until the time $t \in [0,T]$ is defined by

$$\text{Diss}_{\mathcal{D}_0}(z_0; [0,t]) := \sup \Big\{ \sum_{j=1}^{N} \mathcal{D}_0(z_0(t_{j-1}), z_0(t_j)) \Big\},$$

where for $N \in \mathbb{N}$ the supremum is taken with respect to all finite partitions π_N, with $\pi_N := \{0 = t_0 < t_1 < \ldots < t_N = t\}$, of the interval $[0,t]$. Finally, for given initial values $(u_0^0, z_0^0) \in \mathcal{Q}_0(\Omega)$ the energetic formulation (S^0) and (E^0) of the one-scale rate-independent system $(\mathcal{Q}_0(\Omega), \mathcal{E}_0, \mathcal{D}_0)$ reads as follows:

Stability condition (S^0) **and energy balance** (E^0) **for all** $t \in [0,T]$**:**

$$\mathcal{E}_0(t,u_0(t),z_0(t)) \le \mathcal{E}_0(t,\widetilde{u},\widetilde{z}) + \mathcal{D}_0(z_0(t),\widetilde{z}) \quad \text{for all } (\widetilde{u},\widetilde{z}) \in \mathcal{Q}_0(\Omega)$$
$$\mathcal{E}_0(t,u_0(t),z_0(t)) + \mathrm{Diss}_{\mathcal{D}_0}(z_0;[0,t]) = \mathcal{E}_0(0,u_0^0,z_0^0) - \int_0^t \langle \dot{\ell}(s), u_0(s)\rangle \mathrm{d}s$$

Furthermore, for $\widetilde{t} \in [0,T]$ we define the set of stable states $\mathcal{S}_0(\widetilde{t})$ via

$$\mathcal{S}_0(\widetilde{t}) := \{(u_0,z_0) \in \mathcal{Q}_0(\Omega) \text{ satisfying } (\mathrm{S}^0) \text{ for } t=\widetilde{t} \text{ and } \mathcal{E}_0(\widetilde{t},u_0,z_0) < \infty\}.$$

The following theorem states the equivalence of the two-scale model introduced in Section 6.2 and the here considered one-scale model. For the definition of the in following appearing two-scale terms we refer to Section 6.2.

Theorem 6.9 (Equivalence of the two-scale and one-scale model)**.** *Assume that condition* (6.3) *holds. For* $\widehat{z} \in [0,1]^m$ *let* $\mathcal{L}_{\widehat{z}} : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d$ *denote the linear operator defined by* (6.17)*. Furthermore, let* $(u_0,U_1) \in \mathrm{L}^\infty([0,T];\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \times \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d)$ *and let* $z_0 \in \mathrm{L}^\infty([0,T];\mathrm{W}^{1,p}(\Omega;[0,1]^m)) \cap \mathrm{BV}_{\mathcal{D}_0}([0,T];\mathrm{W}^{1,p}(\Omega;[0,1]^m))$*. Then for initial values* $(u_0^0,U_1^0,z_0^0) \in \mathbf{S}_0(0)$ *the following two statements are equivalent:*

(a) *The function* $(u_0,U_1,z_0) : [0,T] \to \mathbf{Q}_0$ *with* $(u_0(0),U_1(0),z_0(0)) = (u_0^0,U_1^0,z_0^0)$ *is a solution of* $(\mathbf{S}^0)$ *and* $(\mathbf{E}^0)$*.*

(b) *The function* $(u_0,z_0) : [0,T] \to \mathcal{Q}_0(\Omega)$ *with* $(u_0(0),z_0(0)) = (u_0^0,z_0^0)$ *is a solution of* (S^0) *and* (E^0)*, and* $U_1(t) := \mathcal{L}_{z_0(t,\cdot)}(\mathbf{e}_x(u_0(t))(\cdot))$ *for all* $t \in [0,T]$*.*

The statement of Theorem 6.9 is a direct consequence of Proposition 6.10 and Corollary 6.11 below.

Proposition 6.10. *Assume that* (6.3) *holds. For* $\widehat{z} \in [0,1]^m$ *let* $\mathcal{L}_{\widehat{z}} : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d$ *denote the linear operator defined by* (6.17)*. Then for* $t \in [0,T]$*, for* $u_0 \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$*, and for* $z_0 \in \mathrm{W}^{1,p}(\Omega;[0,1]^m)$ *the following statements are equivalent:*

(a) U_1 *is the unique solution of* (6.15)*.*

(b) $U_1 = \mathcal{L}_{z_0(\cdot)}(\mathbf{e}_x(u_0)(\cdot))$*.*

(c) $\mathbf{E}_0(t,u_0,U_1,z_0) = \mathcal{E}_0(t,u_0,z_0)$*.*

Proof. For the given $(u_0,z_0) \in \mathcal{Q}_0(\Omega)$ the function $U_1 \in \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ is the unique minimizer of (6.15), if and only if the following inequality holds for all testfunctions $\widetilde{U} \in \mathrm{L}^2(\Omega,\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$.

$$I_0(z_0,u_0,\widetilde{U}) \ge I_0(z_0,u_0,U_1), \tag{6.19}$$

where $I_0(z_0,u_0,\cdot) : \mathrm{L}^2(\Omega,\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d \to \mathbb{R}$ is the continuous functional given by

$$I_0(z_0,u_0,U) := \int_{\Omega\times Y} \langle \widehat{\mathbb{C}}(z_0(x))(y)(\widetilde{\mathbf{e}}(u_0,U)(x,y)), \widetilde{\mathbf{e}}(u_0,U)(x,y)\rangle_{d\times d} \mathrm{d}y\mathrm{d}x.$$

(b)⇒(a): With $\mathcal{L}_{\hat{z}} : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d$ defined by (6.17) we show that the function $\overline{U}_1 \in \mathrm{L}^2(\Omega, \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$, which is defined by $\overline{U}_1(x,y) := \mathcal{L}_{z_0(x)}(\mathbf{e}_x(u_0)(x))(y)$ for almost every $(x,y) \in \Omega{\times}Y$, is also a solution of (6.19). For almost every $x \in \Omega$ the function $\overline{U}_1(x,\cdot) \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d$ satisfies the following inequality for all $\widetilde{v} \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d$ by definition; see (6.16b) and (6.17).

$$I(z_0(x), \mathbf{e}_x(u_0)(x), \widetilde{v}) \geq I(z_0(x), \mathbf{e}_x(u_0)(x), \overline{U}_1(x,\cdot)). \tag{6.20}$$

For arbitrary but fixed $(f_i, v_i) \in \mathrm{L}^2(\Omega){\times}\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})$, $i = 1, 2, \ldots, d$, and almost every $x \in \Omega$ we now choose the following specific testfunction $\widetilde{v} := \sum_{i=1}^d f_i(x) v_i e_i \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d$. Here, the k-th component $(e_i)_k$ of the vector $e_i \in \mathbb{R}^d$ is defined by $(e_i)_k := \delta_{ik}$, where δ_{ik} for $i,k \in \{1,2,\ldots,d\}$ denotes the Kronecker delta. By integrating (6.20) over Ω for $\widehat{U} := (f_1 v_1, \ldots, f_d v_d)^T$ we obtain

$$I_0(z_0, u_0, \widehat{U}) \geq I_0(z_0, u_0, \overline{U}_1). \tag{6.21}$$

By choosing suitable linear combinations of such testfunctions, we find that (6.21) holds for any function $\widehat{U}$ of the linear span of $\{(f_1 v_1, \ldots, f_d v_d)^T \,|\, (f_i, v_i) \in \mathrm{L}^2(\Omega){\times}\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}), i = 1, \ldots, d\}$. Observe that by basic density properties for Bochner spaces the linear span of $\{(f_1 v_1, \ldots, f_d v_d)^T \,|\, (f_i, v_i) \in \mathrm{L}^2(\Omega){\times}\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}), i = 1, \ldots, d\}$ is dense in $\mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$, which combined with the continuity of $I_0(z_0, u_0, \cdot) : \mathrm{L}^2(\Omega, \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d \to \mathbb{R}$ results in the fact that (6.21) holds for any function $\widehat{U} \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$. Hence, the function $\overline{U}_1 \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ solves (6.19) for the given function $(u_0, z_0) \in \mathcal{Q}_0(\Omega)$.

(a)⇒(b): For given $(u_0, z_0) \in \mathcal{Q}_0(\Omega)$ let $U_1 \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^n$ be the unique solution of (6.19). As already proven in the first step, $\overline{U}_1(x,y) := \mathcal{L}_{z_0(x)}(\nabla_x u_0(x))(y)$ is also a solution of (6.19). According to the uniqueness of the minimizer this results in $U_1 = \overline{U}_1$.

(a),(b)⇔(c): Following the trivial transformations below for given $(u_0, z_0) \in \mathcal{Q}_0(\Omega)$ we find

$$U_1 = \mathcal{L}_{z_0(\cdot)}(\mathbf{e}_x(u_0)(\cdot)) \quad \Leftrightarrow \quad \mathbf{E}_0(t, u_0, U_1, z_0) = \mathcal{E}_0(t, u_0, z_0).$$

Indeed, using the definitions of $\mathbb{C}_{\mathrm{eff}}(z_0) \in \mathcal{M}(\Omega)$ and $\mathcal{L}_{z_0(x)} : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d$ we have:

$$\begin{aligned}
\mathcal{E}_0(t, u_0, z_0) &= \tfrac{1}{2}\langle \mathbb{C}_{\mathrm{eff}}(z_0)\mathbf{e}_x(u_0), \mathbf{e}_x(u_0)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} + \|\nabla z_0\|^p_{\mathrm{L}^p(\Omega)^{m\times d}} - \langle \ell(t), u_0\rangle && \text{(6.22a)}\\
&= \tfrac{1}{2}\int_\Omega I\Big(z_0(x), \mathbf{e}_x(u_0)(x), \mathcal{L}_{z_0(x)}(\mathbf{e}_x(u_0)(x))\Big)\mathrm{d}x + \|\nabla z_0\|^p_{\mathrm{L}^p(\Omega)^{m\times d}} - \langle \ell(t), u_0\rangle \\
&= \tfrac{1}{2} I_0\Big(z_0, u_0, \mathcal{L}_{z_0(\cdot)}(\mathbf{e}_x(u_0)(\cdot))\Big) + \|\nabla z_0\|^p_{\mathrm{L}^p(\Omega)^{m\times d}} - \langle \ell(t), u_0\rangle && \text{(6.22b)}\\
&= \mathbf{E}_0(t, u_0, U_1, z_0) && \text{(6.22c)}
\end{aligned}$$

In the case of "⇐" line (6.22c) is equal to line (6.22a) by assumption and the identity $U_1 = \mathcal{L}_{z_0(\cdot)}(\mathbf{e}_x(u_0)(\cdot))$ follows by comparing line (6.22b) and (6.22c). On the other hand in the case of "⇒" in line (6.22c) $U_1 = \mathcal{L}_{z_0(\cdot)}(\mathbf{e}_x(u_0)(\cdot))$ was exploited.

Note that in the case of $U_1 = \mathcal{L}_{z_0(\cdot)}(\mathbf{e}_x(u_0)(\cdot))$ the function U_1 is the unique minimizer of (6.15) such that there is no function $\widehat{U}_1 \neq U_1$fulfilling $\mathbf{E}_0(t, u_0, \widehat{U}_1, z_0) = \mathcal{E}_0(t, u_0, z_0)$. □

Corollary 6.11. *Assume that* (6.3) *holds. For* $\widehat{z} \in [0,1]^m$ *let* $\mathcal{L}_{\hat{z}} : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d$ *denote the linear operator defined by* (6.17)*. Then for* $t \in [0,T]$ *the following statements are equivalent:*

(a) $(u_0, U_1, z_0) \in \mathbf{S}_0(t)$.

(b) $U_1 = \mathcal{L}_{z_0(\cdot)}(\mathbf{e}_x(u_0)(\cdot))$ *and* $(u_0, z_0) \in \mathcal{S}_0(t)$.

Proof. (a)⇒(b): As already mentioned in the beginning of this section the condition $(u_0, U_1, z_0) \in \mathbf{S}_0(t)$ implies that $U_1 \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ is the minimizer of (6.15). Hence, we have $U_1 = \mathcal{L}_{z_0(\cdot)}(\mathbf{e}_x(u_0)(\cdot))$ according to Proposition 6.10. Moreover, the following inequality holds, by taking the minimum over all $\widetilde{U} \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ on the right hand side of the stability condition ($\mathbf{S}^0$).

$$\mathbf{E}_0(t, u_0, U_1, z_0) \le \min_{\widetilde{U}} \mathbf{E}_0(t, \widetilde{u}, \widetilde{U}, \widetilde{z}) + \mathbf{D}_0(z_0, \widetilde{z}) \qquad \forall\, (\widetilde{u}, \widetilde{z}) \in \mathcal{Q}_0(\Omega).$$

However, for the first two terms of this inequality Proposition 6.10 can be exploited such that we end up with

$$\mathcal{E}_0(t, u_0, z_0) \le \mathcal{E}_0(t, \widetilde{u}, \widetilde{z}) + \mathcal{D}_0(z_0, \widetilde{z}) \qquad \forall\, (\widetilde{u}, \widetilde{z}) \in \mathcal{Q}_0(\Omega).$$

(b)⇐(a): Due to Proposition 6.10 we have

$$\mathbf{E}_0(t, u_0, U_1, z_0) = \mathcal{E}_0(t, u_0, z_0) \le \mathcal{E}_0(t, \widetilde{u}, \widetilde{z}) + \mathcal{D}_0(z_0, \widetilde{z}) \qquad \forall\, (\widetilde{u}, \widetilde{z}) \in \mathcal{Q}_0(\Omega).$$

Moreover, it holds $\mathcal{E}_0(t, \widetilde{u}, \widetilde{z}) \le \mathbf{E}_0(t, \widetilde{u}, \widetilde{U}, \widetilde{z})$ for all $\widetilde{U} \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ since there is equality for the unique minimizer $\widetilde{U}_1 = \mathcal{L}_{\widetilde{z}(\cdot)}(\mathbf{e}_x(\widetilde{u})(\cdot))$. This estimate finally gives

$$\mathbf{E}_0(t, u_0, U_1, z_0) \le \mathbf{E}_0(t, \widetilde{u}, \widetilde{U}, \widetilde{z}) + \mathbf{D}_0(z_0, \widetilde{z}) \qquad \forall\, (\widetilde{u}, \widetilde{U}, \widetilde{z}) \in \mathbf{Q}_0,$$

which implies $(u_0, U_1, z_0) \in \mathbf{S}_0(t)$ and Corollary 6.11 is proven. □

6.4 Mutual recovery sequence

This section is in preparation for proving the convergence of the microscopic models introduced in Section 6.1 to the effective models of Section 6.2 and 6.3. For this purpose, we are going to apply the evolutionary Γ-convergence method which is presented in [56] in an abstract setting. There, the authors pointed out that the crucial issue in performing the limit passage is to guarantee the stability of the limit when starting with a *stable sequence*. Hence, one of the main concerns of [56] is the provision of various sufficient conditions ensuring this stability. Therefore, the existence of a so-called *mutual recovery sequence* is requested and we are going to focus on one suitable definition and refer to [56] for the general theory.

The state spaces and functionals underlying the following definitions and theorems are those introduced in the Sections 6.1 and 6.2. Summarizing, this section contains the proof that there are subsequences of solutions of the microscopic models (S^ε) and (E^ε) which converge to a function satisfying the two-scale stability condition ($\mathbf{S}^0$) for all $t \in [0,T]$. We start with the following definitions:

Definition 6.12 (Stable sequence with respect to $t \in [0,T]$)**.** *Let the discrete gradient $R_{\frac{\varepsilon}{2}} : \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m) \to \mathrm{K}_{\frac{\varepsilon}{2}\Lambda}(\Omega_\varepsilon^+)^{m\times d}$ be given by Definition* 4.1. *Then a sequence $(u_\varepsilon, z_\varepsilon)_{\varepsilon>0}$ satisfying $(u_\varepsilon, z_\varepsilon) \in \mathcal{Q}_\varepsilon(\Omega)$ for every $\varepsilon > 0$ is called stable sequence with respect to the time $t \in [0,T]$ if the conditions* (a) *and* (b) *hold:*

(a) *There exists a function $(u_0, U_1, z_0) \in \mathbf{Q}_0$ such that:*

$$
\begin{aligned}
u_\varepsilon &\rightharpoonup u_0 && \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d, & z_\varepsilon &\to z_0 && \text{in } \mathrm{L}^p(\Omega)^m,\\
u_\varepsilon &\xrightarrow{s} Eu_0 && \text{in } \mathrm{L}^2(\Omega\times Y)^d, & R_{\frac{\varepsilon}{2}} z_\varepsilon|_\Omega &\rightharpoonup \nabla z_0 && \text{in } \mathrm{L}^p(\Omega)^{m\times d},\\
\nabla u_\varepsilon &\xrightharpoonup{w} \nabla_x E u_0 + \nabla_y U_1 && \text{in } \mathrm{L}^2(\Omega\times Y)^{d\times d}.
\end{aligned}
$$

(b) *$(u_\varepsilon, z_\varepsilon) \in \mathcal{S}_\varepsilon(t)$ for every $\varepsilon > 0$.*

Definition 6.13 (Mutual recovery condition and mutual recovery sequence)**.** *A sequences of functionals $(\mathcal{E}_\varepsilon, \mathcal{D}_\varepsilon)_{\varepsilon>0}$ fulfills the mutual recovery condition, if for every function $(\widetilde{u}_0, \widetilde{U}_1, \widetilde{z}_0) \in \mathbf{Q}_0$ and for every stable sequence $(u_\varepsilon, z_\varepsilon)_{\varepsilon>0}$ with respect to $t \in [0,T]$ the following holds:*

There exists a sequence $(\widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon)_{\varepsilon>0}$ with $(\widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon) \in \mathcal{Q}_\varepsilon(\Omega)$ for all $\varepsilon > 0$ such that

$$
\limsup_{\varepsilon\to 0} \mathcal{D}_\varepsilon(z_\varepsilon, \widetilde{z}_\varepsilon) \le \mathbf{D}_0(z_0, \widetilde{z}_0) \tag{6.23}
$$

as well as

$$
\limsup_{\varepsilon\to 0} \Big(\mathcal{E}_\varepsilon(t, \widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon) - \mathcal{E}_\varepsilon(t, u_\varepsilon, z_\varepsilon)\Big) \le \mathbf{E}_0(t, \widetilde{u}_0, \widetilde{U}_1, \widetilde{z}_0) - \mathbf{E}_0(t, u_0, U_1, z_0). \tag{6.24}
$$

Such a sequence $(\widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon)_{\varepsilon>0}$ is called mutual recovery sequence.

Remark 6.14. *Observe that Definition* 6.13 *does not ask the mutual recovery sequence $(\widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon)_{\varepsilon>0}$ to converge to $(\widetilde{u}_0, \widetilde{U}_1, \widetilde{z}_0) \in \mathbf{Q}_0$ in any sense.*

Theorem 6.15 (Mutual recovery sequence)**.** *Assume that the conditions* (6.1)*,* (6.2)*, and* (6.5) *hold. For $\varepsilon > 0$ and $\ell \in \mathrm{C}^1([0,T]; (\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$ let $\mathcal{E}_\varepsilon : [0,T]\times\mathcal{Q}_\varepsilon(\Omega) \to \mathbb{R}$ be defined via* (6.6) *and let $\mathcal{D}_\varepsilon : \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)\times\mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m) \to [0,\infty]$ be defined by* (6.7) *fulfilling the conditions* (6.8) *and* (6.9)*. Furthermore, let $\mathbf{E}_0 : [0,T]\times\mathbf{Q}_0 \to \mathbb{R}$ and $\mathbf{D}_0 : \mathrm{W}^{1,p}(\Omega;[0,1]^m)\times\mathrm{W}^{1,p}(\Omega;[0,1]^m) \to [0,\infty]$ be given as introduced in Section* 6.2*. If $(u_\varepsilon, z_\varepsilon)_{\varepsilon>0}$ is a stable sequence with respect to $t \in [0,T]$ with limit $(u_0, U_1, z_0) \in \mathbf{Q}_0$, then:*

(a) *For every $(\widetilde{u}_0, \widetilde{U}_1, \widetilde{z}_0) \in \mathbf{Q}_0$ there exists a mutual recovery sequence $(\widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon)_{\varepsilon>0}$.*

(b) *$(u_0, U_1, z_0) \in \mathbf{S}_0(t)$.*

The construction of the u-component of the mutual recovery sequence is based on the two-scale density result concerning Sobolev functions stated in Proposition 3.8. Starting with a given stable sequence $(u_\varepsilon, z_\varepsilon)_{\varepsilon>0}$ the z-component $\widetilde{z}_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ is explicitly constructed out of $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ in the proof of the following theorem.

Theorem 6.16 (z-component of the mutual recovery sequence)**.** *Let the discrete gradient $R_{\frac{\varepsilon}{2}} : \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m) \to \mathrm{K}_{\frac{\varepsilon}{2}\Lambda}(\Omega_\varepsilon^+)^{m\times d}$ be given by Definition 4.1 and let $(u_\varepsilon, z_\varepsilon)_{\varepsilon>0}$ be a stable sequence with respect to $t \in [0,T]$ with limit $(u_0, U_1, z_0) \in \mathbf{Q}_0$.*

Then for every $\widetilde{z}_0 \in \mathrm{W}^{1,p}(\Omega;[0,1])^m$ with $\widetilde{z}_0 \le z_0$ there exists a sequence $(\widetilde{z}_\varepsilon)_{\varepsilon>0}$ satisfying $\widetilde{z}_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$, $\widetilde{z}_\varepsilon \le z_\varepsilon$, $\widetilde{z}_\varepsilon \to \widetilde{z}_0$ in $\mathrm{L}^p(\Omega)^m$, $R_{\frac{\varepsilon}{2}}\widetilde{z}_\varepsilon|_\Omega \rightharpoonup \nabla\widetilde{z}_0$ in $\mathrm{L}^p(\Omega)^{m\times d}$, and

$$\limsup_{\varepsilon\to 0}\Big(\|R_{\frac{\varepsilon}{2}}\widetilde{z}_\varepsilon\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}} - \|R_{\frac{\varepsilon}{2}}z_\varepsilon\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}}\Big) \le \|\nabla\widetilde{z}_0\|^p_{\mathrm{L}^p(\Omega)^{m\times d}} - \|\nabla z_0\|^p_{\mathrm{L}^p(\Omega)^{m\times d}}. \tag{6.25}$$

The construction of the z-component of the mutual recovery sequence is based on that done in [61]. There, the authors constructed a mutual recovery sequence for scalar Sobolev functions. Here, the main steps of the proof stay the same but due to the discrete setting on the ε-level and the vectorial case some new technicalities come into play.

Proof. 1. Let $z_0, \widetilde{z}_0 \in \mathrm{W}^{1,p}(\Omega;[0,1]^m)$ and $(z_\varepsilon)_{\varepsilon>0}$ be given as assumed in Theorem 6.16. Choose $\Delta > 0$ arbitrary but fixed. Then there exists $\varepsilon_0 > 0$ such that $\Omega_\varepsilon^+ \subset \mathrm{neigh}_\Delta(\Omega)$ for all $\varepsilon \in (0,\varepsilon_0)$. Moreover, there exists an extension $\overline{z}_0 \in \mathrm{W}^{1,p}_0(\mathrm{neigh}_\Delta(\Omega);[0,1]^m)$ of $\widetilde{z}_0 \in \mathrm{W}^{1,p}(\Omega;[0,1]^m)$ satisfying $\overline{z}_0|_\Omega = \widetilde{z}_0$ according to Theorem A 6.12 in [3]. Let $P_\varepsilon : \mathrm{L}^p(\mathbb{R}^d) \to \mathrm{K}_{\varepsilon\Lambda}(\mathbb{R}^d)$ denote the projector to piecewise constant functions introduced in Definition 4.11. Then $\overline{z}_\varepsilon := (P_\varepsilon(\overline{z}_0^{\mathrm{ex}}))|_\Omega$ satisfies

$$\lim_{\varepsilon\to 0}\Big(\|\widetilde{z}_0 - \overline{z}_\varepsilon\|_{\mathrm{L}^p(\Omega)^m} + \|(\nabla\widetilde{z}_0)^{\mathrm{ex}} - R_{\frac{\varepsilon}{2}}\overline{z}_\varepsilon\|_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}}\Big) = 0, \tag{6.26}$$

as shown in the proof of Theorem 4.9. Observe that the application of the projector P_ε to the function $\overline{z}_0^{\mathrm{ex}} \in \mathrm{L}^p(\mathbb{R}^d)^m$ has to be understood component-wise. Following the proof in [61] we introduce the function $\widetilde{z}_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$, decomposed for every component $\widetilde{z}_\varepsilon^{(j)}$, $j \in \{1,2,\dots,m\}$, in the following way:

$$\widetilde{z}_\varepsilon^{(j)}(x) := \begin{cases} \max\{0, \overline{z}_\varepsilon^{(j)}(x) - \delta_\varepsilon^{(j)}\} & \text{if } x \in A_\varepsilon^{(j)} := \Omega_\varepsilon^- \backslash B_\varepsilon^{(j)}, \\ z_\varepsilon^{(j)}(x) & \text{if } x \in B_\varepsilon^{(j)} \cup (\Omega\backslash\Omega_\varepsilon^-), \end{cases}$$

where $B_\varepsilon^{(j)} := \{x \in \Omega_\varepsilon^- : z_\varepsilon^{(j)}(x) < \max\{0, \overline{z}_\varepsilon^{(j)}(x) - \delta_\varepsilon^{(j)}\}\}$. For $j \in \{1,2,\dots,m\}$ the positive constant δ_ε^j will later be chosen in such a way that $\delta_\varepsilon^j \to 0$ for $\varepsilon \to 0$. This definition immediately results in $0 \le \widetilde{z}_\varepsilon \le z_\varepsilon$.

2. Now, we prove that $\widetilde{z}_\varepsilon \to \widetilde{z}_0$ in $\mathrm{L}^p(\Omega)^m$. Since $\widetilde{z}_\varepsilon \to \widetilde{z}_0$ in $\mathrm{L}^p(\Omega)^m$ is equivalent to $\widetilde{z}_\varepsilon^{(j)} \to \widetilde{z}_0^{(j)}$ in $\mathrm{L}^p(\Omega)$ for every $j \in \{1,2,\dots,m\}$ we will restrict ourselves to the case $m = 1$. Hence, let $A_\varepsilon := A_\varepsilon^{(1)}$, $B_\varepsilon := B_\varepsilon^{(1)}$, and $\delta_\varepsilon := \delta_\varepsilon^{(1)}$ to shorten notation. According to $|z_\varepsilon(x) - \widetilde{z}_0(x)| \le 1$, especially on B_ε, we find

$$\|\widetilde{z}_\varepsilon - \widetilde{z}_0\|^p_{\mathrm{L}^p(\Omega)} \le \|\max\{0, \overline{z}_\varepsilon - \delta_\varepsilon\} - \widetilde{z}_0\|^p_{\mathrm{L}^p(A_\varepsilon)} + \mu_d(B_\varepsilon). \tag{6.27}$$

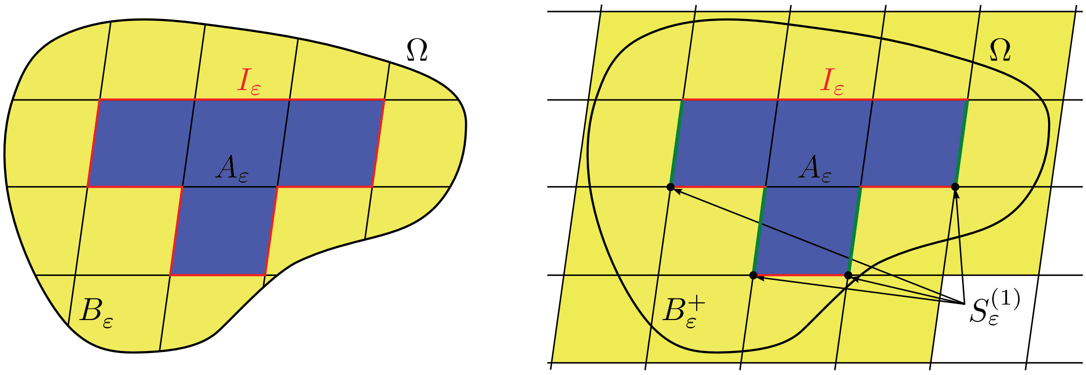


Figure 6.1: Decomposition of Ω into the subsets A_ε and B_ε.

By increasing the domain of integration from A_ε to Ω, adding zero $(-\overline{z}_\varepsilon+\overline{z}_\varepsilon)$ and applying the triangle inequality, the first term of (6.27) is bounded by the expression $2^{p-1}\|\max\{0,\overline{z}_\varepsilon-\delta_\varepsilon\}-\overline{z}_\varepsilon\|^p_{\mathrm{L}^p(\Omega)}+2^{p-1}\|\overline{z}_\varepsilon-\widetilde{z}_0\|^p_{\mathrm{L}^p(\Omega)}$. Hence, due to (6.26) the right hand side of (6.27) converges to zero if the sequence $(\delta_\varepsilon)_{\varepsilon>0}$ can be chosen such that $\delta_\varepsilon\to 0$ and $\mu_d(B_\varepsilon)\to 0$.

3. Choice of $\delta_\varepsilon>0$ such that $\delta_\varepsilon\to 0$ and $\mu_d(B_\varepsilon)\to 0$: As before let $m=1$. Since $\widetilde{z}_0=\overline{z}_0$ on Ω_ε^- by definition the identity $\overline{z}_\varepsilon=P_\varepsilon\widetilde{z}_0^{\mathrm{ex}}$ on Ω_ε^- holds. Combining this identity with the assumption $\widetilde{z}_0\le z_0$ results in $\overline{z}_\varepsilon\le P_\varepsilon z_0^{\mathrm{ex}}$ on Ω_ε^-. Due to this estimate

$$B_\varepsilon\subset\Big\{x\in\Omega_\varepsilon^-\;\Big|\;z_\varepsilon(x)<\max\{0,P_\varepsilon z_0^{\mathrm{ex}}(x)-\delta_\varepsilon\}\Big\}\subset\Big\{x\in\Omega_\varepsilon^-\;\Big|\;\delta_\varepsilon<|P_\varepsilon z_0^{\mathrm{ex}}(x)-z_\varepsilon(x)|\Big\}=:\widehat{B}_\varepsilon$$

such that Markov's inequality (**M**) can be exploited in the following way:

$$\mu_d(B_\varepsilon)\le\mu_d\big(\widehat{B}_\varepsilon\big)\overset{(\mathbf{M})}{\le}\tfrac{1}{\delta_\varepsilon^p}\int_{\Omega_\varepsilon^-}|P_\varepsilon z_0^{\mathrm{ex}}(x)-z_\varepsilon(x)|^p\mathrm{d}x.$$

By choosing $\delta_\varepsilon^p:=\|P_\varepsilon z_0^{\mathrm{ex}}-z_\varepsilon\|_{\mathrm{L}^p(\Omega_\varepsilon^-)}\le\|P_\varepsilon z_0^{\mathrm{ex}}-z_0\|_{\mathrm{L}^p(\Omega)}+\|z_0-z_\varepsilon\|_{\mathrm{L}^p(\Omega)}$, for instance, the assumed convergence $z_\varepsilon\to z_0$ in $\mathrm{L}^p(\Omega)$ yields $\delta_\varepsilon\to 0$ and $\mu_d(B_\varepsilon)\to 0$ as $\varepsilon\to 0$. As already mentioned in [61], $\delta_\varepsilon>0$ is necessary to apply Markov's inequality. However, in the case of $\delta_\varepsilon=0$ the assumed convergence $z_\varepsilon\to z_0$ in $\mathrm{L}^p(\Omega)$ implies $(P_\varepsilon z_0^{\mathrm{ex}})|_\Omega-z_\varepsilon\to 0$ in $\mathrm{L}^p(\Omega)$ such that $\lim_{\varepsilon\to 0}\mu_d(\widehat{B}_\varepsilon)=0$ results immediately.

4. To show: $\limsup_{\varepsilon\to 0}\Big(\|R_{\frac{\varepsilon}{2}}\widetilde{z}_\varepsilon\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^d}-\|R_{\frac{\varepsilon}{2}}z_\varepsilon\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^d}\Big)\le\|\nabla\widetilde{z}_0\|^p_{\mathrm{L}^p(\Omega)^d}-\|\nabla z_0\|^p_{\mathrm{L}^p(\Omega)^d}$:

Roughly spoken, the fact $\mu_d(B_\varepsilon)\to 0$ for $\varepsilon\to 0$ means that in the case of a sequence of Sobolev functions $(z_\varepsilon\in\mathrm{W}^{1,p}(\Omega))$ it is sufficient to prove (6.25) for A_ε instead of Ω_ε^+ on the left hand side. However, since we are interested in the case of piecewise constant functions we have to pay some special attention to the region around the interface $I_\varepsilon=\partial A_\varepsilon\cap\partial B_\varepsilon^+$, where $B_\varepsilon^+:=B_\varepsilon\cup(\Omega_\varepsilon^+\backslash\Omega_\varepsilon^-)$. Note that due to the definition of A_ε and B_ε there are disjoint subsets $\Lambda_{A_\varepsilon},\Lambda_{B_\varepsilon}\subset\Lambda_\varepsilon^-$ such that $A_\varepsilon=\bigcup_{\lambda\in\Lambda_{A_\varepsilon}}\varepsilon(\lambda+Y)$ and $B_\varepsilon=\bigcup_{\lambda\in\Lambda_{B_\varepsilon}}\varepsilon(\lambda+Y)$. Hence, for $\Lambda_{B_\varepsilon^+}:=\Lambda_{B_\varepsilon}\cup(\Lambda_\varepsilon^+\backslash\Lambda_\varepsilon^-)$ we have $B_\varepsilon^+=\bigcup_{\lambda\in\Lambda_{B_\varepsilon^+}}\varepsilon(\lambda+Y)$.

For $i\in\{1,2,\ldots,d\}$ let $n_i\in\mathbb{R}^d$ be given by condition (4.2) and let $F_{n_i}(\varepsilon\lambda)$ denote the face of $\varepsilon(\lambda+Y)$ orthogonal to $n_i\in\mathbb{R}^d$ which is contained in $\varepsilon(\lambda+Y)$. Then, the

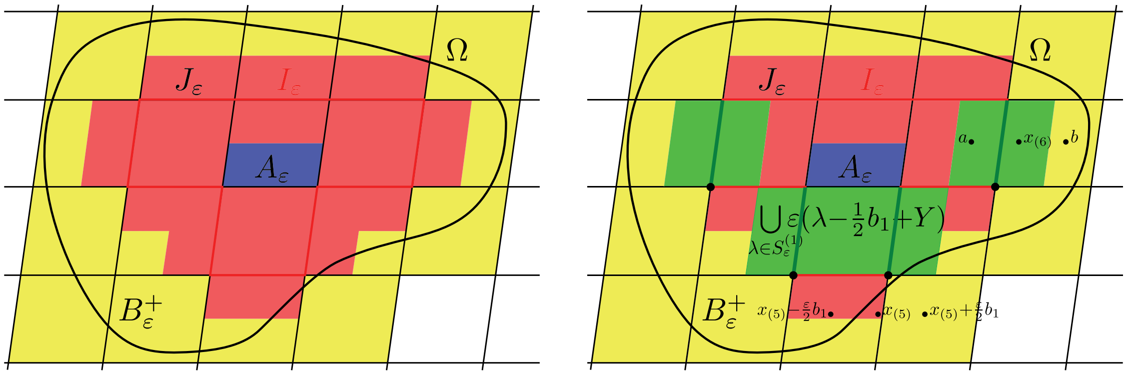


Figure 6.2: Here, $x_{(5)}$ and $x_{(6)}$ denote points considered in step 5 and 6, respectively.

interface I_ε can be uniquely represented by $I_\varepsilon = \bigcup_{i=1}^d \bigcup_{\lambda\in S_\varepsilon^{(i)}} \mathrm{cl}(F_{n_i}(\varepsilon\lambda))$, where $S_\varepsilon^{(i)} \subset \Lambda$ is a suitable finite subset and $\bigcup_{\lambda\in S_\varepsilon^{(i)}} \mathrm{cl}(F_{n_i}(\varepsilon\lambda))$ are all faces of the interface I_ε that are orthogonal to $n_i \in \mathbb{R}^d$. Observe that $|S_\varepsilon^{(i)}| \le |\Lambda_{B_\varepsilon^+}|$ since the number of faces in $S_\varepsilon^{(i)}$ is bounded by the number of all cells contained in B_ε^+.

Taking the union of all cells

$$J_\varepsilon := \bigcup_{i=1}^d \bigcup_{\lambda\in S_\varepsilon^{(i)}} \varepsilon(\lambda-\tfrac{1}{2}b_i+Y)$$

containing the face $F_{n_i}(\varepsilon\lambda)$ in the middle (see Figure 6.2) we have $I_\varepsilon \subset \mathrm{cl}(J_\varepsilon)$ and

$$\mu_d(J_\varepsilon) \le \sum_{i=1}^d \sum_{\lambda\in S_\varepsilon^{(i)}} \varepsilon^d = \sum_{i=1}^d |S_\varepsilon^{(i)}|\varepsilon^d \le \sum_{i=1}^d |\Lambda_{B_\varepsilon^+}|\varepsilon^d = d\mu_d(B_\varepsilon^+). \tag{6.28}$$

The set J_ε has been constructed in such a way that $x \in A_\varepsilon \backslash J_\varepsilon$ implies $x+\frac{\varepsilon}{2}b_i \in A_\varepsilon$ and $x-\frac{\varepsilon}{2}b_i \in A_\varepsilon$ and the analog statement is valid on $B_\varepsilon^+ \backslash J_\varepsilon$. Hence, by exploiting the structure of $R_{\frac{\varepsilon}{2}} : \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m) \to \mathrm{K}_{\frac{\varepsilon}{2}\Lambda}(\Omega_\varepsilon^+)^{m\times d}$ given by Definition 4.1 we have

$$R_{\frac{\varepsilon}{2}}\widetilde{z}_\varepsilon = \begin{cases} R_{\frac{\varepsilon}{2}}(\max\{0,\overline{z}_\varepsilon-\delta_\varepsilon\}) & \text{in } A_\varepsilon\backslash J_\varepsilon, \\ R_{\frac{\varepsilon}{2}}\widetilde{z}_\varepsilon & \text{in } J_\varepsilon, \\ R_{\frac{\varepsilon}{2}}z_\varepsilon & \text{in } B_\varepsilon^+\backslash J_\varepsilon. \end{cases} \tag{6.29}$$

Keeping (6.25) in mind, we want to estimate $|R_{\frac{\varepsilon}{2}}\widetilde{z}_\varepsilon|^p$ from above by terms depending only on z_ε and $\overline{z}_\varepsilon$. Due to (6.29) we only have to care about the case $x \in J_\varepsilon$. Therefore, we consider every component $\big(R_{\frac{\varepsilon}{2}}\widetilde{z}_\varepsilon(x)\big)b_i$ separately.

5. The case $x \in J_\varepsilon \backslash \bigcup_{\lambda\in S_\varepsilon^{(i)}} \varepsilon(\lambda-\frac{1}{2}b_i+Y)$ for $i\in\{1,\dots,d\}$ fixed:

In this case either $x+\frac{\varepsilon}{2}b_i \in A_\varepsilon$ and $x-\frac{\varepsilon}{2}b_i \in A_\varepsilon$ or $x+\frac{\varepsilon}{2}b_i \in B_\varepsilon^+$ and $x-\frac{\varepsilon}{2}b_i \in B_\varepsilon^+$. Combining this result with the definition of the function $\widetilde{z}_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ and the structure of the discrete gradient yields the desired estimate

$$\Big|\Big(R_{\frac{\varepsilon}{2}}\widetilde{z}_\varepsilon(x)\Big)b_i\Big| \le \max\Big\{\Big|\Big(R_{\frac{\varepsilon}{2}}(\max\{0,\overline{z}_\varepsilon(x)-\delta_\varepsilon\})\Big)b_i\Big|, \Big|\Big(R_{\frac{\varepsilon}{2}}z_\varepsilon(x)\Big)b_i\Big|\Big\}. \tag{6.30}$$

6. The case $x \in \bigcup_{\lambda \in S_\varepsilon^{(i)}} \varepsilon(\lambda - \frac{1}{2}b_i + Y)$ for $i \in \{1, \dots, d\}$ fixed:

In this case either $x + \frac{\varepsilon}{2}b_i \in A_\varepsilon$ and $x - \frac{\varepsilon}{2}b_i \in B_\varepsilon^+$ or $x + \frac{\varepsilon}{2}b_i \in B_\varepsilon^+$ and $x - \frac{\varepsilon}{2}b_i \in A_\varepsilon$ according to the definition of $S_\varepsilon^{(i)}$. Without loss of generality set $a := x + \frac{\varepsilon}{2}b_i \in A_\varepsilon$ and $b := x - \frac{\varepsilon}{2}b_i \in B_\varepsilon^+$. Then due to the definitions of A_ε and B_ε^+ we have

$$1 \geq z_\varepsilon(a) \geq \widetilde{z}_\varepsilon(a) = \max\{0, \overline{z}_\varepsilon(a) - \delta_\varepsilon\} \geq 0, \tag{6.31a}$$

$$1 \geq \max\{0, \overline{z}_\varepsilon(b) - \delta_\varepsilon\} > \widetilde{z}_\varepsilon(b) = z_\varepsilon(b) \geq 0. \tag{6.31b}$$

Since $b \in B_\varepsilon^+ \backslash B_\varepsilon = \Omega_\varepsilon^+ \backslash \Omega_\varepsilon^-$ is possible, in relation (6.31b) and in the following table every function has to be understood as its extension with respect to the continuation operator $V_\varepsilon : \mathrm{K}_{\varepsilon\Lambda}(\Omega) \to \mathrm{K}_{\varepsilon\Lambda}(\Omega_\varepsilon^+)$ given by (4.1). Keeping this remark in mind the following estimates are valid.

	if $\widetilde{z}_\varepsilon(a) \geq \widetilde{z}_\varepsilon(b)$	if $\widetilde{z}_\varepsilon(a) < \widetilde{z}_\varepsilon(b)$
$\lvert\widetilde{z}_\varepsilon(a) - \widetilde{z}_\varepsilon(b)\rvert$	$= \widetilde{z}_\varepsilon(a) - \widetilde{z}_\varepsilon(b)$ $\overset{(6.31a)}{\leq} z_\varepsilon(a) - \widetilde{z}_\varepsilon(b)$ $\overset{(6.31b)}{=} z_\varepsilon(a) - z_\varepsilon(b)$	$= \widetilde{z}_\varepsilon(b) - \widetilde{z}_\varepsilon(a)$ $\overset{(6.31b)}{<} \max\{0, \overline{z}_\varepsilon(b) - \delta_\varepsilon\} - \widetilde{z}_\varepsilon(a)$ $\overset{(6.31a)}{=} \max\{0, \overline{z}_\varepsilon(b) - \delta_\varepsilon\} - \max\{0, \overline{z}_\varepsilon(a) - \delta_\varepsilon\}$

Hence, we also find

$$\left|\left(R_{\frac{\varepsilon}{2}} \widetilde{z}_\varepsilon(x)\right) b_i\right| \leq \max\left\{\left|\left(R_{\frac{\varepsilon}{2}}(\max\{0, \overline{z}_\varepsilon(x) - \delta_\varepsilon\})\right) b_i\right|, \left|\left(R_{\frac{\varepsilon}{2}} z_\varepsilon(x)\right) b_i\right|\right\}, \tag{6.32}$$

for all $x \in \bigcup_{\lambda \in S_\varepsilon^{(i)}} \varepsilon(\lambda - \frac{1}{2}b_i + Y)$.

7. Summary of the case $x \in J_\varepsilon$: Combining (6.30) and (6.32) these inequalities hold for every $x \in J_\varepsilon$, which finally results in

$$|R_{\frac{\varepsilon}{2}} \widetilde{z}_\varepsilon|^p \leq \begin{cases} |R_{\frac{\varepsilon}{2}} \overline{z}_\varepsilon|^p & \text{in } A_\varepsilon \backslash J_\varepsilon, \\ |R_{\frac{\varepsilon}{2}} \overline{z}_\varepsilon|^p + |R_{\frac{\varepsilon}{2}} z_\varepsilon|^p & \text{in } J_\varepsilon, \\ |R_{\frac{\varepsilon}{2}} z_\varepsilon|^p & \text{in } B_\varepsilon^+ \backslash J_\varepsilon, \end{cases} \tag{6.33}$$

by recalling (6.29), since $|\max\{C_1, C_2\}|^p \leq |C_1|^p + |C_2|^p$ and since

$$|R_{\frac{\varepsilon}{2}} \max\{0, \overline{z}_\varepsilon(x) - \delta_\varepsilon\}| \leq |R_{\frac{\varepsilon}{2}}(\overline{z}_\varepsilon(x) - \delta_\varepsilon)| = |R_{\frac{\varepsilon}{2}} \overline{z}_\varepsilon(x)|.$$

Exploiting (6.33) we conclude in the case $m = 1$ that

$$\begin{aligned} &\limsup_{\varepsilon \to 0} \left(\|R_{\frac{\varepsilon}{2}} \widetilde{z}_\varepsilon\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^d} - \|R_{\frac{\varepsilon}{2}} z_\varepsilon\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^d}\right) \\ &\quad \leq \limsup_{\varepsilon \to 0} \left(\int_{A_\varepsilon \backslash J_\varepsilon} |R_{\frac{\varepsilon}{2}} \overline{z}_\varepsilon(x)|^p - |R_{\frac{\varepsilon}{2}} z_\varepsilon(x)|^p \mathrm{d}x \right. \\ &\qquad + \int_{B_\varepsilon^+ \backslash J_\varepsilon} |R_{\frac{\varepsilon}{2}} z_\varepsilon(x)|^p - |R_{\frac{\varepsilon}{2}} z_\varepsilon(x)|^p \mathrm{d}x \\ &\qquad \left. + \int_{J_\varepsilon} |R_{\frac{\varepsilon}{2}} \overline{z}_\varepsilon(x)|^p + |R_{\frac{\varepsilon}{2}} z_\varepsilon(x)|^p - |R_{\frac{\varepsilon}{2}} z_\varepsilon(x)|^p \mathrm{d}x \right) \end{aligned}$$

$$
\begin{aligned}
&= \limsup_{\varepsilon\to 0}\left(\int_{A_\varepsilon\cup J_\varepsilon}|R_{\frac{\varepsilon}{2}}\overline{z}_\varepsilon(x)|^p\mathrm{d}x - \int_{A_\varepsilon\setminus J_\varepsilon}|R_{\frac{\varepsilon}{2}}z_\varepsilon(x)|^p\mathrm{d}x\right)\\
&\le \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^+}|R_{\frac{\varepsilon}{2}}\overline{z}_\varepsilon(x)|^p\mathrm{d}x - \liminf_{\varepsilon\to 0}\int_\Omega|\mathbb{1}_{A_\varepsilon\setminus J_\varepsilon}(x)R_{\frac{\varepsilon}{2}}z_\varepsilon(x)|^p\mathrm{d}x\\
&= \|\nabla\widetilde{z}_0\|^p_{\mathrm{L}^p(\Omega)^d} - \|\nabla z_0\|^p_{\mathrm{L}^p(\Omega)^d},
\end{aligned}
$$

where in the second last line the first term converges to $\|\nabla\widetilde{z}_0\|^p_{\mathrm{L}^p(\Omega)^d}$ according to (6.26). Moreover, weak lower semi-continuity of the norm together with the weak convergence $\mathbb{1}_{A_\varepsilon\setminus J_\varepsilon}R_{\frac{\varepsilon}{2}}z_\varepsilon \rightharpoonup \nabla z_0$ in $\mathrm{L}^p(\Omega)^d$ is exploited for the second one. Note that due to estimate (6.28) we have $\mathbb{1}_{A_\varepsilon\setminus J_\varepsilon} \to \mathbb{1}_\Omega$ in $\mathrm{L}^q(\Omega)$ for every $q\in[1,\infty)$, since $\lim_{\varepsilon\to 0}\mu_d(B_\varepsilon)=0$ implies $\lim_{\varepsilon\to 0}\mu_d(B_\varepsilon^+)=0$.

8. The general case $m>1$: Up to now, in the case $m>1$ it holds ($j\in\{1,2,\dots,m\}$)

$$
\limsup_{\varepsilon\to 0}\left(\|R_{\frac{\varepsilon}{2}}\widetilde{z}_\varepsilon^{(j)}\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^d} - \|R_{\frac{\varepsilon}{2}}v_\varepsilon^{(j)}\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^d}\right) \le \|\nabla\widetilde{z}_0^{(j)}\|^p_{\mathrm{L}^p(\Omega)^d} - \|\nabla v_0^{(j)}\|^p_{\mathrm{L}^p(\Omega)^d}
$$

for every component $\widetilde{z}_\varepsilon^{(j)}, v_\varepsilon^{(j)}, \widetilde{z}_0^{(j)}, v_0^{(j)}$ of the functions $\widetilde{z}_\varepsilon, z_\varepsilon\in\mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ and $\widetilde{z}_0, z_0\in\mathrm{W}^{1,p}(\Omega;[0,1])^m$. Summing up these inequalities for all $j=1,2,\dots,m$ we finally have

$$
\limsup_{\varepsilon\to 0}\left(\|R_{\frac{\varepsilon}{2}}\widetilde{z}_\varepsilon\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}} - \|R_{\frac{\varepsilon}{2}}z_\varepsilon\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}}\right) \le \|\nabla\widetilde{z}_0\|^p_{\mathrm{L}^p(\Omega)^{m\times d}} - \|\nabla z_0\|^p_{\mathrm{L}^p(\Omega)^{m\times d}}.
$$

9. $R_{\frac{\varepsilon}{2}}\widetilde{z}_\varepsilon|_\Omega \rightharpoonup \nabla\widetilde{z}_0$ in $\mathrm{L}^p(\Omega)^{m\times d}$: According to step 8 Theorem 4.5 can be applied for the sequence $(\widetilde{z}_\varepsilon)_{\varepsilon>0}$. Moreover, due to step 2 the limit-function of Theorem 4.5 is identified as $\widetilde{z}_0\in\mathrm{W}^{1,p}(\Omega;[0,1]^m)$ which altogether yields $R_{\frac{\varepsilon}{2}}\widetilde{z}_\varepsilon|_\Omega \rightharpoonup \nabla\widetilde{z}_0$ in $\mathrm{L}^p(\Omega)^{m\times d}$ for a subsequence (not relabeled). □

Now, Theorem 6.16 enables us to construct the mutual recovery sequence $(\widetilde{u}_\varepsilon,\widetilde{z}_\varepsilon)_{\varepsilon>0}$.

Proof of Theorem 6.15. Part (a): Let $(u_\varepsilon,z_\varepsilon)_{\varepsilon>0}$ be a the stable sequence with respect to $t\in[0,T]$ converging to the limit $(u_0,U_1,z_0)\in\mathbf{Q}_0$; see Definition 6.12. Then, for a given function $(\widetilde{u}_0,\widetilde{U}_1,\widetilde{z}_0)\in\mathbf{Q}_0$ we start by constructing the mutual recovery sequence $(\widetilde{u}_\varepsilon,\widetilde{z}_\varepsilon)_{\varepsilon>0}$.

1. First, the z-component $\widetilde{z}_\varepsilon\in\mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ is constructed and (6.23) is verified. Observe that in the case of $\mathbf{D}_0(z_0,\widetilde{z}_0)=\infty$, the lim sup-inequality (6.23) is trivially fulfilled for the sequence $(\widetilde{z}_\varepsilon)_{\varepsilon>0}$ constructed in the proof of Theorem 4.9. Hence, without loss of generality we assume $\widetilde{z}_0\le z_0$ from now on. According to Theorem 6.16 there exists a sequence $(\widetilde{z}_\varepsilon)_{\varepsilon>0}$ satisfying $\widetilde{z}_\varepsilon\in\mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$, $\widetilde{z}_\varepsilon\le z_\varepsilon$, $\widetilde{z}_\varepsilon\to\widetilde{z}_0$ in $\mathrm{L}^p(\Omega)^m$, $R_{\frac{\varepsilon}{2}}\widetilde{z}_\varepsilon|_\Omega\rightharpoonup\nabla\widetilde{z}_0$ in $\mathrm{L}^p(\Omega)^{m\times d}$, and

$$
\limsup_{\varepsilon\to 0}\left(\|R_{\frac{\varepsilon}{2}}\widetilde{z}_\varepsilon\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}} - \|R_{\frac{\varepsilon}{2}}z_\varepsilon\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}}\right) \le \|\nabla\widetilde{z}_0\|^p_{\mathrm{L}^p(\Omega)^{m\times d}} - \|\nabla z_0\|^p_{\mathrm{L}^p(\Omega)^{m\times d}}.
$$

Recalling assumption (6.8) results in $\lim_{\varepsilon\to 0}\mathcal{D}_\varepsilon(z_\varepsilon,\widetilde{z}_\varepsilon)=\mathbf{D}_0(z_0,\widetilde{z}_0)$ and (6.23) is shown.

2. Now the u-component $\widetilde{u}_\varepsilon\in\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ is constructed. Adopting the notation of Proposition 3.8 let $w_\varepsilon\in\mathrm{H}^1_0(\Omega)^d$ be the solution of the elliptic problem stated there with

$w_0 :\equiv \mathbf{0} \in \mathrm{H}^1_0(\Omega)^d$ and $W_1 := \widetilde{U}_1 \in \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$. Then according to Proposition 3.8 we have $w_\varepsilon \rightharpoonup \mathbf{0}$ in $\mathrm{H}^1_0(\Omega)^d$, $w_\varepsilon \xrightarrow{s} \mathbf{0}$ in $\mathrm{L}^2(\Omega{\times}Y)^d$, and $\nabla w_\varepsilon \xrightarrow{s} \nabla_y \widetilde{U}_1$ in $\mathrm{L}^2(\Omega{\times}Y)^{d\times d}$. Thus, the u-component of the mutual recovery sequence is defined via

$$\widetilde{u}_\varepsilon := \widetilde{u}_0 + w_\varepsilon.$$

Using property (b) of Proposition 3.5 and the convergence results for $(w_\varepsilon)_{\varepsilon>0}$ we find

$$\begin{aligned} \widetilde{u}_\varepsilon &\rightharpoonup \widetilde{u}_0 && \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d,\\ \widetilde{u}_\varepsilon &\xrightarrow{s} E\widetilde{u}_0 && \text{in } \mathrm{L}^2(\Omega{\times}Y)^d,\\ \nabla\widetilde{u}_\varepsilon &\xrightarrow{s} \nabla_x E\widetilde{u}_0 + \nabla_y \widetilde{U}_1 && \text{in } \mathrm{L}^2(\Omega{\times}Y)^{d\times d}. \end{aligned}$$

3. Now we are in the position to prove the lim sup-inequality stated in (6.24). According to the assumption and step 2 we have $u_\varepsilon \rightharpoonup u_0$ in $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ and $\widetilde{u}_\varepsilon \rightharpoonup \widetilde{u}_0$ in $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ which implies

$$\lim_{\varepsilon\to 0}\Big(\langle \ell(t), u_\varepsilon\rangle - \langle \ell(t), \widetilde{u}_\varepsilon\rangle\Big) = \langle \ell(t), u_0\rangle - \langle \ell(t), \widetilde{u}_0\rangle.$$

4. Next we prove that

$$\begin{aligned} &\limsup_{\varepsilon\to 0}\Big(\langle \mathbb{C}_\varepsilon(\widetilde{z}_\varepsilon)\mathbf{e}(\widetilde{u}_\varepsilon), \mathbf{e}(\widetilde{u}_\varepsilon)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} - \langle \mathbb{C}_\varepsilon(z_\varepsilon)\mathbf{e}(u_\varepsilon), \mathbf{e}(u_\varepsilon)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}}\Big)\\ &\le \langle \mathbb{C}_0(\widetilde{z}_0)\widetilde{\mathbf{e}}(\widetilde{u}_0,\widetilde{U}_1), \widetilde{\mathbf{e}}(\widetilde{u}_0,\widetilde{U}_1)\rangle_{\mathrm{L}^2(\Omega\times Y)^{d\times d}} - \langle \mathbb{C}_0(z_0)\widetilde{\mathbf{e}}(u_0,U_1), \widetilde{\mathbf{e}}(u_0,U_1)\rangle_{\mathrm{L}^2(\Omega\times Y)^{d\times d}}. \end{aligned} \tag{6.34}$$

Combining this with the convergence results of step 1 and 3 implies the lim sup-inequality (6.24). To show relation (6.34) we are going to prove

$$\lim_{\varepsilon\to 0}\langle \mathbb{C}_\varepsilon(\widetilde{z}_\varepsilon)\mathbf{e}(\widetilde{u}_\varepsilon), \mathbf{e}(\widetilde{u}_\varepsilon)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} = \langle \mathbb{C}_0(\widetilde{z}_0)\widetilde{\mathbf{e}}(\widetilde{u}_0,\widetilde{U}_1), \widetilde{\mathbf{e}}(\widetilde{u}_0,\widetilde{U}_1)\rangle_{\mathrm{L}^2(\Omega\times Y)^{d\times d}} \tag{6.35}$$

and

$$\liminf_{\varepsilon\to 0}\langle \mathbb{C}_\varepsilon(z_\varepsilon)\mathbf{e}(u_\varepsilon), \mathbf{e}(u_\varepsilon)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} \ge \langle \mathbb{C}_0(z_0)\widetilde{\mathbf{e}}(u_0,U_1), \widetilde{\mathbf{e}}(u_0,U_1)\rangle_{\mathrm{L}^2(\Omega\times Y)^{d\times d}}. \tag{6.36}$$

Ad (6.35): Since $\widetilde{z}_\varepsilon \to \widetilde{z}_0$ in $\mathrm{L}^p(\Omega)^m$ according to Theorem 3.9 we have $\mathbb{C}_\varepsilon(\widetilde{z}_\varepsilon) \xrightarrow{s} \mathbb{C}_0(\widetilde{z}_0)$ in $\mathrm{L}^1(\Omega{\times}Y;\mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}};\mathbb{R}^{d\times d}_{\mathrm{sym}}))$. Adopting the notation of Corollary 3.6 let $m_\varepsilon := \mathbb{C}_\varepsilon(\widetilde{z}_\varepsilon)$, $M_0 := \mathbb{C}_0(\widetilde{z}_0)$, and $v_\varepsilon := \mathbf{e}(\widetilde{u}_\varepsilon)$, $V_0 := \widetilde{\mathbf{e}}(\widetilde{u}_0,\widetilde{U}_1)$. Then Corollary 3.6 together with the convergence results for $(\widetilde{u}_\varepsilon)_{\varepsilon>0}$ give $w_\varepsilon := \mathbb{C}_\varepsilon(\widetilde{z}_\varepsilon)\mathbf{e}(\widetilde{u}_\varepsilon) \xrightarrow{s} \mathbb{C}_0(\widetilde{z}_0)\widetilde{\mathbf{e}}(\widetilde{u}_0,\widetilde{U}_1) =: W_0$ in $\mathrm{L}^2(\Omega{\times}Y)^{d\times d}$. With this, Proposition 3.5(a) yields (6.35).

Ad (6.36): We start with the following integral identity valid according to identity (3.2) and the product rule for the unfolding operator $\mathcal{T}_\varepsilon$:

$$\langle \mathbb{C}_\varepsilon(z_\varepsilon)\mathbf{e}(u_\varepsilon), \mathbf{e}(u_\varepsilon)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} = \langle \mathcal{T}_\varepsilon\mathbb{C}_\varepsilon(z_\varepsilon)\mathcal{T}_\varepsilon\mathbf{e}(u_\varepsilon), \mathcal{T}_\varepsilon\mathbf{e}(u_\varepsilon)\rangle_{\mathrm{L}^2(\mathbb{R}^d\times Y)^{d\times d}}. \tag{6.37}$$

Since $z_\varepsilon \to z_0$ in $\mathrm{L}^p(\Omega)^m$ according to Theorem 3.9 we have $\mathcal{T}_\varepsilon\mathbb{C}_\varepsilon(z_\varepsilon) \to \mathbb{C}_0^{\mathrm{ex}}(z_0)$ in $\mathrm{L}^1(\mathbb{R}^d{\times}Y;\mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}};\mathbb{R}^{d\times d}_{\mathrm{sym}}))$. Moreover, due to the definition of two-scale convergence it holds $\mathcal{T}_\varepsilon \mathbf{e}(u_\varepsilon) \rightharpoonup \widetilde{\mathbf{e}}^{\mathrm{ex}}(u_0, U_1)$ in $\mathrm{L}^2(\mathbb{R}^d{\times}Y)^{d\times d}$, which enables us to apply Theorem 3.23 of [14] which yields the following inequality:

$$\liminf_{\varepsilon'\to 0} \langle \mathcal{T}_\varepsilon \mathbb{C}_\varepsilon(z_\varepsilon)\mathcal{T}_\varepsilon \mathbf{e}(u_\varepsilon), \mathcal{T}_\varepsilon \mathbf{e}(u_\varepsilon)\rangle_{\mathrm{L}^2(\mathbb{R}^d\times Y)^{d^2}} \geq \langle \mathbb{C}_0^{\mathrm{ex}}(z_0)\widetilde{\mathbf{e}}^{\mathrm{ex}}(u_0,U_1), \widetilde{\mathbf{e}}^{\mathrm{ex}}(u_0,U_1)\rangle_{\mathrm{L}^2(\mathbb{R}^d\times Y)^{d^2}}.$$

Taking into account that $\mathrm{supp}(\mathbb{C}_0^{\mathrm{ex}}(z_0)) \subset \Omega{\times}Y$ this inequality together with (6.37) gives (6.36) and the proof of point (a) in Theorem 6.15 is done.

Part (b) is a consequence of point (a): Let $(u_\varepsilon, z_\varepsilon)_{\varepsilon>0}$ be a stable sequence with respect to $t \in [0,T]$ converging to the limit $(u_0, U_1, z_0) \in \mathbf{Q}_0$; see Definition 6.12. Then, for an arbitrary function $(\widetilde{u}_0, \widetilde{U}_1, \widetilde{z}_0) \in \mathbf{Q}_0$ with $\widetilde{z}_0 \leq z_0$ choose $(\widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon)_{\varepsilon>0}$ as constructed in the steps 1 and 2. Note that in the case $\widetilde{z}_0 \not\leq z_0$ according to $\mathbf{D}_0(\widetilde{z}_0, z_0) = \infty$ the stability condition ($\mathbf{S}^0$) is trivially fulfilled. Due to the stability of $(u_\varepsilon, z_\varepsilon) \in \mathcal{Q}_\varepsilon(\Omega)$ at time $t \in [0,T]$ we have

$$0 \leq \mathcal{E}_\varepsilon(t, \widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon) + \mathcal{D}_\varepsilon(z_\varepsilon, \widetilde{z}_\varepsilon) - \mathcal{E}_\varepsilon(t, u_\varepsilon, z_\varepsilon).$$

Applying the limsup with respect to the sequence $(\varepsilon)_{\varepsilon>0}$ to the right hand side according to (6.23) and (6.24) results in

$$0 \leq \mathbf{E}_0(t, \widetilde{u}_0, \widetilde{U}_1, \widetilde{z}_0) + \mathbf{D}_0(z_0, \widetilde{z}_0) - \mathbf{E}_0(t, u_0, U_1, z_0),$$

which is nothing else than the stability condition ($\mathbf{S}^0$) of $(u_0, U_1, z_0) \in \mathbf{Q}_0$ at time $t \in [0,T]$ for the arbitrarily chosen test-function $(\widetilde{u}_0, \widetilde{U}_1, \widetilde{z}_0) \in \mathbf{Q}_0$. □

6.5 Convergence result

This section provides the main result of this chapter, saying that the models of the Sections 6.2 and 6.3 are the limit of the microscopic models introduced in Section 6.1. However, before that we show that $\mathbf{E}_0 : [0,T]{\times}\mathbf{Q}_0 \to \mathbb{R}$ is the Γ-limit of the sequence $(\mathcal{E}_\varepsilon)_{\varepsilon>0}$ of functionals $\mathcal{E}_\varepsilon : [0,T]{\times}\mathcal{Q}_\varepsilon(\Omega) \to \mathbb{R}$ with respect to our special topology.

Theorem 6.17 (Mosco convergence of $(\mathcal{E}_\varepsilon)_{\varepsilon>0}$ to $\mathbf{E}_0$)**.** *Let $(u_\varepsilon, z_\varepsilon)_{\varepsilon>0}$ be a sequence satisfying $(u_\varepsilon, z_\varepsilon) \in \mathcal{Q}_\varepsilon(\Omega)$ for all $\varepsilon > 0$ and*

$$\begin{array}{llll} u_\varepsilon \rightharpoonup u_0 & \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d, & z_\varepsilon \to z_0 & \text{in } \mathrm{L}^p(\Omega)^m, \\ u_\varepsilon \xrightarrow{s} Eu_0 & \text{in } \mathrm{L}^2(\Omega{\times}Y)^d, & R_{\frac{\varepsilon}{2}} z_\varepsilon|_\Omega \rightharpoonup \nabla z_0 & \text{in } \mathrm{L}^p(\Omega)^{m\times d}, \\ \nabla u_\varepsilon \xrightharpoonup{w} \nabla_x E u_0 + \nabla_y U_1 & \text{in } \mathrm{L}^2(\Omega{\times}Y)^{d\times d}. & & \end{array}$$

Then for every $t \in [0,T]$ it holds $\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t, u_\varepsilon, z_\varepsilon) \geq \mathbf{E}_0(t, u_0, U_1, z_0)$*. Moreover, for every function $(\widetilde{u}_0, \widetilde{U}_1, \widetilde{z}_0) \in \mathbf{Q}_0$ there exists a sequence $(\widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon)_{\varepsilon>0}$ with $(\widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon) \in \mathcal{Q}_\varepsilon(\Omega)$ for every $\varepsilon > 0$, with*

$$\begin{array}{llll} \widetilde{u}_\varepsilon \rightharpoonup \widetilde{u}_0 & \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d, & \widetilde{z}_\varepsilon \to \widetilde{z}_0 & \text{in } \mathrm{L}^p(\Omega)^m, \\ \widetilde{u}_\varepsilon \xrightarrow{s} E\widetilde{u}_0 & \text{in } \mathrm{L}^2(\Omega{\times}Y)^d, & R_{\frac{\varepsilon}{2}} \widetilde{z}_\varepsilon|_\Omega \to \nabla \widetilde{z}_0 & \text{in } \mathrm{L}^p(\Omega)^{m\times d}, \\ \nabla \widetilde{u}_\varepsilon \xrightarrow{s} \nabla_x E\widetilde{u}_0 + \nabla_y \widetilde{U}_1 & \text{in } \mathrm{L}^2(\Omega{\times}Y)^{d\times d}, & & \end{array}$$

and with $\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t, \widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon) = \mathbf{E}_0(t, \widetilde{u}_0, \widetilde{U}_1, \widetilde{z}_0)$.

Observe that here the term *Mosco* refers to the strong two-scale convergence of the recovery sequence's u-component $(\widetilde{u}_\varepsilon, \nabla\widetilde{u}_\varepsilon)_{\varepsilon>0}$, and not to the weak convergence of $(\widetilde{u}_\varepsilon)_{\varepsilon>0}$ in $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$.

Proof. Ad lim inf-inequality: Due to the assumptions of Theorem 6.17 we already have $\lim_{\varepsilon\to 0}\langle \ell(t), u_\varepsilon\rangle = \langle \ell(t), u_0\rangle$ and $\liminf_{\varepsilon\to 0}\|R_{\frac{\varepsilon}{2}} z_\varepsilon\|_{\mathrm{L}^p(\Omega)^{m\times d}} \geq \|\nabla z_0\|_{\mathrm{L}^p(\Omega)^{m\times d}}$. Moreover, Theorem 3.9 states $\mathcal{T}_\varepsilon\mathbb{C}_\varepsilon(z_\varepsilon) \to \mathbb{C}_0^{\mathrm{ex}}(z_0)$ in $\mathrm{L}^1(\mathbb{R}^d\times Y; \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}}; \mathbb{R}^{d\times d}_{\mathrm{sym}}))$. Thus, we are in the position to apply Theorem 3.23 of [14] which yields the following inequality:

$$\liminf_{\varepsilon\to 0}\langle \mathcal{T}_\varepsilon\mathbb{C}_\varepsilon(z_\varepsilon)\mathcal{T}_\varepsilon\mathbf{e}(u_\varepsilon), \mathcal{T}_\varepsilon\mathbf{e}(u_\varepsilon)\rangle_{\mathrm{L}^2(\mathbb{R}^d\times Y)^{d^2}} \geq \langle \mathbb{C}_0^{\mathrm{ex}}(z_0)\widetilde{\mathbf{e}}^{\mathrm{ex}}(u_0, U_1), \widetilde{\mathbf{e}}^{\mathrm{ex}}(u_0, U_1)\rangle_{\mathrm{L}^2(\mathbb{R}^d\times Y)^{d^2}}.$$

Altogether we proved $\liminf_{\varepsilon\to 0}\mathcal{E}_\varepsilon(t, u_\varepsilon, z_\varepsilon) \geq \mathbf{E}_0(t, u_0, U_1, z_0)$ for every $t \in [0,T]$, by taking the integral identity (3.2) and $\mathrm{supp}(\mathbb{C}_0^{\mathrm{ex}}(z_0)) \subset \Omega\times Y$ into account.

Ad lim(sup)-(in)equality: For a given $(\widetilde{u}_0, \widetilde{U}_1, \widetilde{z}_0) \in \mathbf{Q}_0$ choosing $\widetilde{u}_\varepsilon \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ as in step 2 of the proof of Theorem 6.15 yields the stated convergence results for the sequence $(\widetilde{u}_\varepsilon)_{\varepsilon>0}$.

According to Theorem 4.9 for $\widetilde{z}_0 \in \mathrm{W}^{1,p}(\Omega; [0,1]^m)$ there exists a sequence $(\widetilde{z}_\varepsilon)_{\varepsilon>0}$ such that $\widetilde{z}_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)$, $\widetilde{z}_\varepsilon \to \widetilde{z}_0$ in $\mathrm{L}^p(\Omega)^m$, and $R_{\frac{\varepsilon}{2}}\widetilde{z}_\varepsilon|_\Omega \to \nabla\widetilde{z}_0$ in $\mathrm{L}^p(\Omega)^{m\times d}$. Moreover, condition (4.27) implies

$$\lim_{\varepsilon\to 0}\|R_{\frac{\varepsilon}{2}}\widetilde{z}_\varepsilon\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}} = \|\nabla\widetilde{z}_0\|^p_{\mathrm{L}^p(\Omega)^{m\times d}}. \tag{6.38}$$

Finally, Theorem 3.9 yields $\mathbb{C}_\varepsilon(\widetilde{z}_\varepsilon) \xrightarrow{s} \mathbb{C}_0(\widetilde{z}_0)$ in $\mathrm{L}^1(\Omega\times Y; \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}}; \mathbb{R}^{d\times d}_{\mathrm{sym}}))$. By adopting the notation of Corollary 3.6, with $m_\varepsilon := \mathbb{C}_\varepsilon(\widetilde{z}_\varepsilon)$, $M_0 := \mathbb{C}_0(\widetilde{z}_0)$, $v_\varepsilon := \mathbf{e}(\widetilde{u}_\varepsilon)$, and $V_0 := \widetilde{\mathbf{e}}(\widetilde{u}_0, \widetilde{U}_1)$ we have $w_\varepsilon := \mathbb{C}_\varepsilon(\widetilde{z}_\varepsilon)\mathbf{e}(\widetilde{u}_\varepsilon) \xrightarrow{s} \mathbb{C}_0(\widetilde{z}_0)\widetilde{\mathbf{e}}(\widetilde{u}_0, \widetilde{U}_1) =: W_0$ in $\mathrm{L}^2(\Omega\times Y)$. Additionally exploiting Proposition 3.5(a) results in

$$\lim_{\varepsilon\to 0}\langle \mathbb{C}_\varepsilon(\widetilde{z}_\varepsilon)\mathbf{e}(\widetilde{u}_\varepsilon), \mathbf{e}(\widetilde{u}_\varepsilon)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} = \langle \mathbb{C}_0(\widetilde{z}_0)\widetilde{\mathbf{e}}(\widetilde{u}_0, \widetilde{U}_1), \widetilde{\mathbf{e}}(\widetilde{u}_0, \widetilde{U}_1)\rangle_{\mathrm{L}^2(\Omega\times Y)^{d\times d}}. \tag{6.39}$$

Combining (6.38), (6.39), and $\lim_{\varepsilon\to 0}\langle \ell(t), \widetilde{u}_\varepsilon\rangle = \langle \ell(t), \widetilde{u}_0\rangle$ concludes the proof. $\square$

Now we are in the position to state the final result of this section, saying that the sequence of solutions of the microscopic models (S^ε) and (E^ε) introduced in Section 6.1 converges to a solution of the effective two-scale model ($\mathbf{S}^0$) and ($\mathbf{E}^0$) introduced in Section 6.2.

Theorem 6.18 (Convergence result ensuring the existence of solutions to $(\mathbf{S}^0)\&(\mathbf{E}^0)$)**.** *Assume that* (6.1), (6.2), *and* (6.5) *hold. For* $\varepsilon > 0$ *and* $\ell \in \mathrm{C}^1([0,T]; (\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$ *let the energy functional* $\mathcal{E}_\varepsilon : [0,T]\times\mathcal{Q}_\varepsilon(\Omega) \to \mathbb{R}$ *be defined via* (6.6) *and let the dissipation distance* $\mathcal{D}_\varepsilon : \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)\times\mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m) \to [0,\infty]$ *be defined by* (6.7) *such that* (6.8) *and* (6.9) *hold. Moreover, let the limit energy functional* $\mathbf{E}_0 : [0,T]\times\mathbf{Q}_0 \to \mathbb{R}$ *and the limit dissipation distance* $\mathbf{D}_0 : \mathrm{W}^{1,p}(\Omega; [0,1]^m)\times\mathrm{W}^{1,p}(\Omega; [0,1]^m) \to [0,\infty]$ *be given as introduced in Section* 6.2. *If for every* $\varepsilon > 0$ *the function* $(u_\varepsilon, z_\varepsilon) : [0,T] \to \mathcal{Q}_\varepsilon(\Omega)$ *is* *an energetic solution of* (S^ε) *and* (E^ε) *with* $(u_\varepsilon(0), z_\varepsilon(0)) = (u_\varepsilon^0, z_\varepsilon^0)$ *and if there exists a triple* $(u_0^0, U_1^0, z_0^0) \in \mathbf{Q}_0$ *such that the initial values satisfy*

$$\begin{aligned} u_\varepsilon^0 &\rightharpoonup u_0^0 && \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d, & z_\varepsilon^0 &\to z_0^0 && \text{in } \mathrm{L}^p(\Omega)^m, \\ u_\varepsilon^0 &\xrightarrow{s} Eu_0^0 && \text{in } \mathrm{L}^2(\Omega\times Y)^d, & R_{\frac{\varepsilon}{2}} z_\varepsilon^0|_\Omega &\to \nabla z_0^0 && \text{in } \mathrm{L}^p(\Omega)^{m\times d}, \\ \nabla u_\varepsilon^0 &\xrightarrow{s} \nabla_x E u_0^0 + \nabla_y U_1^0 && \text{in } \mathrm{L}^2(\Omega\times Y)^{d\times d}, \end{aligned}$$

then there exists a function $(u_0, U_1, z_0) : [0,T] \to \mathbf{Q}_0$ *with*

$$\begin{aligned} &(u_0, U_1) \in \mathrm{L}^\infty([0,T]; \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \times \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d), \\ &\qquad z_0 \in \mathrm{L}^\infty([0,T]; \mathrm{W}^{1,p}(\Omega;[0,1]^m)) \cap \mathrm{BV}_{\mathbf{D}_0}([0,T]; \mathrm{W}^{1,p}(\Omega;[0,1]^m)) \end{aligned}$$

and a subsequence of $(\varepsilon)_{\varepsilon>0}$ *(not relabeled) satisfying for all* $t \in [0,T]$

$$\begin{aligned} u_\varepsilon(t) &\rightharpoonup u_0(t) && \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d, & z_\varepsilon(t) &\to z_0(t) && \text{in } \mathrm{L}^p(\Omega)^m, \\ u_\varepsilon(t) &\xrightarrow{s} Eu_0(t) && \text{in } \mathrm{L}^2(\Omega\times Y)^d, & R_{\frac{\varepsilon}{2}}(z_\varepsilon(t))|_\Omega &\to \nabla z_0(t) && \text{in } \mathrm{L}^p(\Omega)^{m\times d}, \\ \nabla u_\varepsilon(t) &\xrightarrow{s} \nabla_x E u_0(t) + \nabla_y U_1(t) && \text{in } \mathrm{L}^2(\Omega\times Y)^{d\times d}. \end{aligned}$$

Furthermore, $(u_0, U_1, z_0) : [0,T] \to \mathbf{Q}_0$ *is an energetic solution to* $(\mathbf{S}^0)$ *and* $(\mathbf{E}^0)$ *with* $(u_0(0), U_1(0), z_0(0)) = (u_0^0, U_1^0, z_0^0) \in \mathbf{S}_0(0)$*. Additionally, for all* $t \in [0,T]$ *it holds*

$$\begin{aligned} &\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t, u_\varepsilon(t), z_\varepsilon(t)) = \mathbf{E}_0(t, u_0(t), U_1(t), z_0(t)), \\ &\lim_{\varepsilon\to 0} \mathrm{Diss}_{\mathcal{D}\varepsilon}(z_\varepsilon; [0,t]) = \mathrm{Diss}_{\mathbf{D}_0}(z_0; [0,t]). \end{aligned}$$

Proof. 1. Let $(u_\varepsilon, z_\varepsilon) : [0,T] \to \mathcal{Q}_\varepsilon(\Omega)$ be an energetic solution of (S^ε) and (E^ε) with $(u_\varepsilon(0), z_\varepsilon(0)) = (u_\varepsilon^0, z_\varepsilon^0)$. We start by proving a priori estimates. Due to (6.5), for $C_\ell := \|\ell\|_{\mathrm{C}^1([0,T];(\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)} < \infty$ inequality (6.40) below is obtained and is further estimated by exploiting the non-negativity of $\mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon; [0,t])$ in the energy balance (E^ε).

$$\begin{aligned} C_{\mathbf{e}} \|u_\varepsilon(t)\|^2_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d} &\le \mathcal{E}_\varepsilon(t, u_\varepsilon(t), z_\varepsilon(t)) + C_\ell \|u_\varepsilon(t)\|_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d} \\ &\overset{(\mathrm{E}^\varepsilon)}{\le} \mathcal{E}_\varepsilon(0, u_\varepsilon^0, z_\varepsilon^0) - \int_0^t \langle \dot{\ell}(s), u_\varepsilon(s)\rangle \mathrm{d}s + C_\ell \|u_\varepsilon(t)\|_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d} \end{aligned} \tag{6.40}$$

According to the assumptions on $(u_\varepsilon^0, z_\varepsilon^0)_{\varepsilon>0}$ there exists a constant $C_0 > 0$ such that $\mathcal{E}_\varepsilon(0, u_\varepsilon^0, z_\varepsilon^0) \le C_0$ for all $\varepsilon > 0$. Applying the scaled version of Young's estimate to the product $C_\ell \|u_\varepsilon(t)\|_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d}$ on the right hand side of (6.40) and taking the supremum with respect to $t \in [0,T]$ on both sides afterwards, yields the uniform estimate

$$\sup_{t\in[0,T]} \|u_\varepsilon(t)\|_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d} \le c, \tag{6.41}$$

where $c > 0$ only depends on $C_0 > 0$, $T > 0$, and $\ell \in \mathrm{C}^1([0,T]; (\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$. This estimate implies that the energy balance's right hand side is uniformly bounded which results in a uniform bound for the total dissipation $\mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon;[0,t])$ on its left hand side. Hence, $z_\varepsilon:[0,T]\to \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ is a (component-wise) non-increasing function. Estimating $\|R_{\frac{\varepsilon}{2}}(z_\varepsilon(t))\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}}$ in the same way as in (6.40) gives

$$\sup_{\varepsilon>0}\sup_{t\in[0,T]}\|R_{\frac{\varepsilon}{2}}(z_\varepsilon(t))\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}}\leq C_0+cC_\ell(T+1),$$

where we already exploited (6.41). Moreover, $\|z_\varepsilon(t)\|^p_{\mathrm{L}^p(\Omega)^m}\leq m\mu_d(\Omega)$ for every $\varepsilon>0$ and all $t\in[0,T]$ since $0\leq z_\varepsilon(t)\leq 1$ by definition. Combining all estimates results in the following uniform bound of the solution $(u_\varepsilon,z_\varepsilon):[0,T]\to\mathcal{Q}_\varepsilon(\Omega)$: There exists a constant $C>0$ depending only on $C_0>0$, $T>0$, and $\ell\in\mathrm{C}^1([0,T];(\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$ such that for all $\varepsilon>0$ it holds:

$$\sup_{\varepsilon>0}\sup_{t\in[0,T]}\Big(\|u_\varepsilon(t)\|_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d}+\|z_\varepsilon(t)\|^p_{\mathrm{L}^p(\Omega)^m}+\|R_{\frac{\varepsilon}{2}}(z_\varepsilon(t))\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}}\Big)\leq C. \tag{6.42}$$

2. Now we are going to construct a function $z_0:[0,T]\to\mathrm{W}^{1,p}(\Omega;[0,1]^m)$ and choose a subsequence $(\widetilde{\varepsilon})_{\widetilde{\varepsilon}>0}$ of $(\varepsilon)_{\varepsilon>0}$ such that for any $t\in[0,T]$ the sequence $(z_{\widetilde{\varepsilon}}(t))_{\widetilde{\varepsilon}>0}$ converges to $z_0(t)$ with respect to the strong L^1-topology. Similarly to the proceeding in [49, Section 3], we start by constructing the function $z_0:[0,T]\to\mathrm{W}^{1,p}(\Omega;[0,1]^m)$. This construction is based on the limit of the sequence $(F_\varepsilon)_{\varepsilon>0}$ of functions $F_\varepsilon:[0,T]\to\mathbb{R}$ defined via

$$F_\varepsilon(t):=\|z_\varepsilon(t)\|_{\mathrm{L}^1_1(\Omega)^m}, \tag{6.43}$$

where the subscript *1* denotes that the space $\mathrm{L}^1(\Omega)^m$ for $v\in\mathrm{L}^1(\Omega)^m$ is equipped with the norm $\|v\|_{\mathrm{L}^1_1(\Omega)^m}:=\sum_{j=1}^m\|v_j\|_{\mathrm{L}^1(\Omega)}$. As already mentioned in step 1, $F_\varepsilon:[0,T]\to\mathbb{R}$ is monotonously decreasing and uniformly bounded by $m\mu_d(\Omega)$. Therefore, the Helly selection principle is applicable saying that there exists a monotonously decreasing function $F_0\in\mathrm{BV}([0,T];\mathbb{R})$ and a subsequence $(\varepsilon')_{\varepsilon'>0}$ of $(\varepsilon)_{\varepsilon>0}$ such that for all $t\in[0,T]$ it holds

$$F_{\varepsilon'}(t)\overset{\varepsilon'\to 0}{\longrightarrow}F_0(t). \tag{6.44}$$

Let $J_0\subset[0,T]$ be the jump set of F_0, which is at most countable since $F_0\in\mathrm{BV}([0,T];\mathbb{R})$ is monotone. Furthermore, let $K_T\subset[0,T]\backslash J_0$ be a dense and countable subset and choose $(t_n)_{n\in\mathbb{N}}$ such that $(t_n)_{n\in\mathbb{N}}=K_T\cup J_0$. For arbitrary but fixed $n\in\mathbb{N}$ according to the uniform bound (6.42) the assumptions of Theorem 4.5 and Theorem 3.9 for the sequence $(z_{\varepsilon'}(t_n))_{\varepsilon'>0}$ are satisfied. Hence, there exists a function $z_0^{(t_n)}\in\mathrm{W}^{1,p}(\Omega;[0,1]^m)$ and a subsequence $(\varepsilon'')_{\varepsilon''>0}$ of $(\varepsilon')_{\varepsilon'>0}$ satisfying for $\varepsilon\to 0$

$$z_{\varepsilon''}(t_n)\to z_0^{(t_n)}\qquad \text{in } \mathrm{L}^p(\Omega)^m, \tag{6.45a}$$

$$R_{\frac{\varepsilon''}{2}}(z_{\varepsilon''}(t_n))|_\Omega\rightharpoonup\nabla z_0^{(t_n)}\qquad \text{in } \mathrm{L}^p(\Omega)^{m\times d}, \tag{6.45b}$$

$$\mathbb{C}_{\varepsilon''}(z_{\varepsilon''}(t_n))\overset{s}{\to}\mathbb{C}_0(z_0^{(t_n)})\quad \text{in } \mathrm{L}^1(\Omega\times Y;\mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}};\mathbb{R}^{d\times d}_{\mathrm{sym}})). \tag{6.45c}$$

Let $(z_0^{(t_n)})_{n\in\mathbb{N}}\subset\mathrm{W}^{1,p}(\Omega;[0,1]^m)$ denote the set of all limit functions. Since $(t_n)_{n\in\mathbb{N}}$ is a countable set, by a diagonalization argument we are able to construct a (possibly different but not relabeled) subsequence $(\varepsilon'')_{\varepsilon''>0}$ of $(\varepsilon')_{\varepsilon'>0}$ satisfying (6.45) for all $n\in\mathbb{N}$.

Due to (6.45a) for all $n \in \mathbb{N}$ we have $F_{\varepsilon''}(t_n) = \|z_{\varepsilon''}(t_n)\|_{\mathrm{L}^1_1(\Omega)^m} \xrightarrow{\varepsilon''\to 0} \|z_0^{(t_n)}\|_{\mathrm{L}^1_1(\Omega)^m}$ which results in $F_0(t_n) = \|z_0^{(t_n)}\|_{\mathrm{L}^1_1(\Omega)^m}$ by keeping (6.44) in mind. Moreover, the monotonicity of $z_{\varepsilon''} : [0,T] \to \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ together with (6.45a) results in $z_0^{(t_l)} \leq z_0^{(t_k)}$ for all $t_k < t_l \in K_T$. According to this relation of $z_0^{(t_k)}$ and $z_0^{(t_l)}$ for $t_k < t_l \in K_T$ we find

$$C_m\|z_0^{(t_k)} - z_0^{(t_l)}\|_{\mathrm{L}^1(\Omega)^m} \leq \|z_0^{(t_k)} - z_0^{(t_l)}\|_{\mathrm{L}^1_1(\Omega)^m} = \|z_0^{(t_k)}\|_{\mathrm{L}^1_1(\Omega)^m} - \|z_0^{(t_l)}\|_{\mathrm{L}^1_1(\Omega)^m} = F_0(t_k) - F_0(t_l)$$

which due to the continuity of F_0 on $[0,T]\backslash J_0 \supset K_T$ converges to 0 for $t_k \nearrow t_l$ or $t_l \searrow t_k$. Here, $C_m > 0$ is the constant resulting from the utilization of the norm equivalence in dimension m. Hence, the function $\zeta_0 : K_T \to \mathrm{W}^{1,p}(\Omega;[0,1]^m)$ for all $t_k \in K_T$ defined by $\zeta_0(t_k) := z_0^{(t_k)}$ is continuous with respect to $\|\cdot\|_{\mathrm{L}^1(\Omega)^m}$. This function enables us to construct the limit function $z_0 : [0,T] \to \mathrm{L}^1(\Omega)^m$ in the following way:

(a) $z_0(t_n) = z_0^{(t_n)}$ for all $n \in \mathbb{N}$,

(b) $z_0|_{[0,T]\backslash J_0}$ is the continuous extension of ζ_0 with respect to $\|\cdot\|_{\mathrm{L}^1(\Omega)^m}$.

Observe that according to $J_0 \subset (t_n)_{n\in\mathbb{N}}$ and the density of $K_T \subset [0,T]\backslash J_0$ the function $z_0 : [0,T] \to \mathrm{L}^1(\Omega)^m$ is defined everywhere on $[0,T]$.

3. Now we show that the sequence $(z_{\varepsilon''}(t))_{\varepsilon''>0}$ for all $t \in [0,T]$ converges to the function $z_0(t)$ in the sense of (6.45). Since the monotonicity of $z_{\varepsilon''} : [0,T] \to \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ has to be understood as $z_{\varepsilon''}(\widetilde{t}) \leq z_{\varepsilon''}(t)$ (component-wise) for all $t < \widetilde{t} \in [0,T]$ it holds

$$\|z_{\varepsilon''}(t) - z_{\varepsilon''}(\widetilde{t})\|_{\mathrm{L}^1_1(\Omega)^m} = \|z_{\varepsilon''}(t)\|_{\mathrm{L}^1_1(\Omega)^m} - \|z_{\varepsilon''}(\widetilde{t})\|_{\mathrm{L}^1_1(\Omega)^m} \overset{(6.43)}{=} F_{\varepsilon''}(t) - F_{\varepsilon''}(\widetilde{t}).$$

Exploiting this relation in the following calculation yields $z_{\varepsilon''}(t) \to z_0(t)$ in $\mathrm{L}^1(\Omega)^m$. For $t \in [0,T]\backslash(t_n)_{n\in\mathbb{N}} \subset [0,T]\backslash J_0$ we choose $t_m \in K_T$ such that $t < t_m$. Then

$$\begin{aligned}
\lim_{\varepsilon''\to 0} \|z_{\varepsilon''}(t) - z_0(t)\|_{\mathrm{L}^1(\Omega)^m} &\leq \lim_{\varepsilon''\to 0} \Big(\|z_{\varepsilon''}(t) - z_{\varepsilon''}(t_m)\|_{\mathrm{L}^1(\Omega)^m} + \|z_{\varepsilon''}(t_m) - z_0(t_m)\|_{\mathrm{L}^1(\Omega)^m} \Big) \\
&\quad + \|z_0(t_m) - z_0(t)\|_{\mathrm{L}^1(\Omega)^m} \\
&\overset{(6.45a)}{\leq} \lim_{\varepsilon''\to 0} C_m^{-1}\Big(F_{\varepsilon''}(t) - F_{\varepsilon''}(t_m)\Big) + \|z_0(t_m) - z_0(t)\|_{\mathrm{L}^1(\Omega)^m} \\
&\overset{(6.44)}{=} C_m^{-1}\Big(F_0(t) - F_0(t_m)\Big) + \|z_0(t_m) - z_0(t)\|_{\mathrm{L}^1(\Omega)^m}.
\end{aligned} \tag{6.46}$$

Since F_0 and z_0 are continuous on $[0,T]\backslash J_0$, $t_m \in K_T$ with $t < t_m$ can be chosen such that (6.46) gets arbitrarily small, which proves $z_{\varepsilon''}(t) \to z_0(t)$ in $\mathrm{L}^1(\Omega)^m$ for every $t \in [0,T]$.

On the other hand, according to estimate (6.42) we are able to apply Theorem 4.5 and Theorem 3.9 again such that for arbitrary but fixed $t \in [0,T]\backslash(t_n)_{n\in\mathbb{N}}$ there exists a function $z^{(t)} \in \mathrm{W}^{1,p}(\Omega;[0,1]^m)$ and a subsequence $(\varepsilon''')_{\varepsilon'''>0}$ of $(\varepsilon'')_{\varepsilon''>0}$ satisfying

$$z_{\varepsilon'''}(t) \to z^{(t)} \quad \text{in } \mathrm{L}^p(\Omega)^m, \tag{6.47a}$$

$$R_{\frac{\varepsilon'''}{2}}(z_{\varepsilon'''}(t))|_\Omega \rightharpoonup \nabla z^{(t)} \quad \text{in } \mathrm{L}^p(\Omega)^{m\times d}, \tag{6.47b}$$

$$\mathbb{C}_{\varepsilon'''}(z_{\varepsilon'''}(t)) \overset{s}{\to} \mathbb{C}_0(z^{(t)}) \quad \text{in } \mathrm{L}^1(\Omega\times Y;\mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}};\mathbb{R}^{d\times d}_{\mathrm{sym}})). \tag{6.47c}$$

Since $t \in [0,T] \backslash (t_n)_{n\in\mathbb{N}}$ was chosen arbitrarily and we already proved $z_{\varepsilon''}(t) \to z_0(t)$ in $\mathrm{L}^1(\Omega)^m$ for all $t \in [0,T]$, this convergence result first of all gives $z_0(t) \in \mathrm{W}^{1,p}(\Omega;[0,1]^m)$ for every $t \in [0,T]$. Observe that the validity of this statement for all $t \in (t_n)_{n\in\mathbb{N}}$ is already guaranteed by (6.45). Secondly, with $z^{(t)} = z_0(t)$ the convergence result (6.47) is valid for all converging subsequences of $(\varepsilon'')_{\varepsilon''>0}$ such that we conclude that (6.47) holds for the whole sequence $(\varepsilon'')_{\varepsilon''>0}$.

Recapitulating all results proven in step 2 and 3 there exists a piecewise continuous, monotone function $z_0 \in \mathrm{L}^\infty([0,T];\mathrm{W}^{1,p}(\Omega;[0,1]^m))$ and a subsequence of $(\varepsilon)_{\varepsilon>0}$ (not relabeled) such that the following is valid for all $t \in [0,T]$ if $\varepsilon \to 0$:

$$z_\varepsilon(t) \to z_0(t) \qquad \text{in } \mathrm{L}^p(\Omega)^m, \tag{6.48a}$$

$$R_{\frac{\varepsilon}{2}}(z_\varepsilon(t))|_\Omega \rightharpoonup \nabla z_0(t) \qquad \text{in } \mathrm{L}^p(\Omega)^{m\times d}, \tag{6.48b}$$

$$\mathbb{C}_\varepsilon(z_\varepsilon(t)) \xrightarrow{s} \mathbb{C}_0(z_0(t)) \quad \text{in } \mathrm{L}^1(\Omega\times Y;\mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}};\mathbb{R}^{d\times d}_{\mathrm{sym}})). \tag{6.48c}$$

4. Now for every $t \in [0,T]$ we prove the displacement field's convergence for the same subsequence constructed in step 2 and 3. For this purpose, let $u_0 : [0,T] \to \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ and $U_1 : [0,T] \to \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ be uniquely defined by

$$u_0(t) \in \mathrm{Argmin}\{\mathcal{E}_0(t,u,z_0(t)) \,|\, u \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d\}, \tag{6.49a}$$

$$U_1(t) := \mathcal{L}_{z_0(t)}(\mathbf{e}_x(u_0(t))) \qquad \text{(see Theorem 6.9)}, \tag{6.49b}$$

where $z_0 : [0,T] \to \mathrm{W}^{1,p}(\Omega)$ is the function defined in step 2.

On the other hand for fixed $t \in [0,T]$ we have $(u_\varepsilon(t), z_\varepsilon(t)) \in \mathcal{S}_\varepsilon(t)$ by assumption. Due to (6.42) and Proposition 3.7 there exist $(u_0^{(t)}, U_1^{(t)}) \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \times \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ and a subsequence $(\varepsilon')_{\varepsilon'>0}$ of the sequence $(\varepsilon)_{\varepsilon>0}$ considered in (6.48) such that

$$\begin{aligned} u_{\varepsilon'}(t) &\rightharpoonup u_0^{(t)} && \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d, \\ u_{\varepsilon'}(t) &\xrightarrow{s} Eu_0^{(t)} && \text{in } \mathrm{L}^2(\Omega\times Y)^d, \\ \nabla u_{\varepsilon'}(t) &\xrightharpoonup{w} \nabla_x Eu_0^{(t)} + \nabla_y U_1^{(t)} && \text{in } \mathrm{L}^2(\Omega\times Y)^{d\times d}. \end{aligned}$$

Thus, we verified the applicability of Theorem 6.15 which states $(u_0^{(t)}, U_1^{(t)}, z_0(t)) \in \mathbf{S}_0(t)$. Following Corollary 6.11 this is equivalent to

$$U_1^{(t)} = \mathcal{L}_{z_0(t,\cdot)}(\mathbf{e}_x(u_0^{(t)})(\cdot)) \quad \text{and} \quad (u_0^{(t)}, z_0(t)) \in \mathcal{S}_0(t). \tag{6.50a}$$

By choosing $\widetilde{z} = z_0(t)$ in the stability condition (S^0) we find

$$u_0^{(t)} \in \mathrm{Argmin}\{\mathcal{E}_0(t,u,z_0(t)) \,|\, u \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d\}. \tag{6.50b}$$

Comparing (6.49) and (6.50) we obtain $(u_0^{(t)}, U_1^{(t)}) = (u_0(t), U_1(t))$. This identification shows

$$u_\varepsilon(t) \rightharpoonup u_0(t) \qquad \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d, \tag{6.51a}$$

$$u_\varepsilon(t) \xrightarrow{s} Eu_0(t) \qquad \text{in } \mathrm{L}^2(\Omega\times Y)^d, \tag{6.51b}$$

$$\nabla u_\varepsilon(t) \xrightharpoonup{w} \nabla_x Eu_0(t) + \nabla_y U_1(t) \quad \text{in } \mathrm{L}^2(\Omega\times Y)^{d\times d}, \tag{6.51c}$$

where the validity for the whole sequence $(\varepsilon)_{\varepsilon>0}$ considered in (6.48) is proven via a contradiction argument similar to that applied in the proof of Corollary 3.6.

Note that in this step we already proved $(u_0(t), U_1(t), z_0(t)) \in \mathbf{S}_0(t)$ for all $t \in [0,T]$, which includes $(u_0^0, U_1^0, z_0^0) \in \mathbf{S}_0(0)$. Since the pointwise limit of a sequence of measurable functions is measurable again, according to the uniform estimate (6.42) we have $(u_0, U_1) \in \mathrm{L}^\infty([0,T]; \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \times \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d)$.

5. For proving that $(u_0, U_1, z_0) : [0,T] \to \mathbf{Q}_0$ satisfies the energy balance $(\mathbf{E}^0)$ we pass in (E^ε) to the limit $\varepsilon \to 0$. We start with the right hand side. Due to the uniform bound (6.42) we have $|\langle \dot{\ell}(s), u_\varepsilon(s)\rangle| \le C_\ell C$ for every $\varepsilon > 0$ and all $s \in [0,T]$ such that

$$\lim_{\varepsilon\to 0} \int_0^t \langle \dot{\ell}(s), u_\varepsilon(s)\rangle \mathrm{d}s = \int_0^t \lim_{\varepsilon\to 0} \langle \dot{\ell}(s), u_\varepsilon(s)\rangle \mathrm{d}s = \int_0^t \langle \dot{\ell}(s), u_0(s)\rangle \mathrm{d}s$$

by applying the theorem of dominated convergence and making use of $u_\varepsilon(s) \rightharpoonup u_0(s)$ in $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ for all $s \in [0,t]$.

For $m_\varepsilon := \mathbb{C}_\varepsilon(z_\varepsilon^0)$, $M_0 := \mathbb{C}_0(z_0^0)$, $v_\varepsilon := \mathbf{e}(u_\varepsilon^0)$, and $V_0 := \widetilde{\mathbf{e}}(u_0^0, U_1^0)$ applying Corollary 3.6 gives $w_\varepsilon := \mathbb{C}_\varepsilon(z_\varepsilon^0)\mathbf{e}(u_\varepsilon^0) \xrightarrow{s} \mathbb{C}_0(z_0^0)\widetilde{\mathbf{e}}(u_0^0, U_1^0) =: W_0$ in $\mathrm{L}^2(\Omega\times Y)^{d\times d}$ due to the assumptions for the sequences of initial values $(z_\varepsilon^0)_{\varepsilon>0}$ and $(u_\varepsilon^0)_{\varepsilon>0}$. Thus, Proposition 3.5(a) yields

$$\lim_{\varepsilon\to 0} \langle \mathbb{C}_\varepsilon(z_\varepsilon^0)\mathbf{e}(u_\varepsilon^0), \mathbf{e}(u_\varepsilon^0)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} = \langle \mathbb{C}_0(z_0^0)\widetilde{\mathbf{e}}(u_0^0, U_1^0), \widetilde{\mathbf{e}}(u_0^0, U_1^0)\rangle_{\mathrm{L}^2(\Omega\times Y)^{d\times d}},$$

which finally results in $\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t, u_\varepsilon^0, z_\varepsilon^0) = \mathbf{E}_0(t, u_0^0, U_1^0, z_0^0)$.

6. Left hand side of (E^ε): According to the convergence results (6.48) and (6.51) all assumptions of Theorem 6.17 are fulfilled, such that for all $t \in [0,T]$ we have

$$\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t, u_\varepsilon(t), z_\varepsilon(t)) \ge \mathbf{E}_0(t, u_0(t), U_1(t), z_0(t)). \tag{6.52}$$

For $N \in \mathbb{N}$ let $\pi_N := \{0 = t_0 < t_1 < \ldots < t_N = t\}$ be an arbitrary partition of the interval $[0,t]$. Then, by exploiting (i) the definition of $\mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon; [0,t])$, (ii) the assumption (6.8), and (iii) the convergence result (6.48a) the following estimate holds:

$$\liminf_{\varepsilon\to 0} \mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon; [0,t]) \ge \lim_{\varepsilon\to 0} \sum_{j=1}^N \mathcal{D}_\varepsilon(z_\varepsilon(t_{j-1}), z_\varepsilon(t_j)) = \sum_{j=1}^N \mathbf{D}_0(z_0(t_{j-1}), z_0(t_j)).$$

By taking the supremum with respect to all finite partition π_N of the interval $[0,t]$ on the right hand side this inequality yields

$$\liminf_{\varepsilon\to 0} \mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon; [0,t]) \ge \mathrm{Diss}_{\mathbf{D}_0}(z_0; [0,t]). \tag{6.53}$$

Since $\mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon; [0,t])$ is uniformly bounded with respect to $\varepsilon > 0$ and $t \in [0,T]$, relation (6.53) implies $z_0 \in \mathrm{BV}_{\mathbf{D}_0}([0,T]; \mathrm{W}^{1,p}(\Omega; [0,1]^m))$. Adding (6.52) and (6.53) and combing this with the convergence results of step 5 for all $t \in [0,T]$ we have

$$(\mathbf{E}^0_{\mathrm{l}}) \le \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t, u_\varepsilon(t), z_\varepsilon(t)) + \liminf_{\varepsilon\to 0} \mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon; [0,t]) \le \lim_{\varepsilon\to 0} (\mathrm{E}^\varepsilon_{\mathrm{r}}) \overset{\text{step5}}{=} (\mathbf{E}^0_{\mathrm{r}}), \tag{6.54}$$

where the index l and r denote the left and right hand side of the respective energy balance. Due to the stability $(u_0(t), U_1(t), z_0(t)) \in \mathcal{S}_0(t)$ proved in step 4 we immediately obtain the opposite inequality $(\mathbf{E}^0_{\mathrm{l}}) \geq (\mathbf{E}^0_{\mathrm{r}})$ according to Proposition 2.4 of [56], such that finally $(u_0, U_1, z_0) : [0,T] \to \mathbf{Q}_0$ satisfies for all $t \in [0,T]$ the energy balance

$$\mathbf{E}_0(t, u_0(t), U_1(t), z_0(t)) + \mathrm{Diss}_{\mathbf{D}_0}(z_0; [0,t]) = \mathbf{E}_0(t, u_0(0), U_1(0), z_0(0)) - \int_0^t \langle \dot{\ell}(s), u_0(s)\rangle \mathrm{d}s.$$

Due to the validity of the energy balance $(\mathbf{E}^0)$ actually all inequalities in (6.54) are equalities. This implies that (6.52) and (6.53) also have to be equalities and that their limits exist. Hence, it holds

$$\begin{aligned} &\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t, u_\varepsilon(t), z_\varepsilon(t)) = \mathbf{E}_0(t, u_0(t), U_1(t), z_0(t)), \\ &\lim_{\varepsilon\to 0} \mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon; [0,t]) = \mathrm{Diss}_{\mathbf{D}_0}(z_0; [0,t]). \end{aligned} \tag{6.55}$$

7. So far we verified that $(u_0(t), U_1(t), z_0(t)) \in \mathbf{Q}_0$ is a solution of $(\mathbf{S}^0)$ and $(\mathbf{E}^0)$ and it remains to prove the strong convergence properties. Since weak convergence combined with norm convergence implies strong convergence (see [3] Exercise 6.6) for the z-component this proof is done by showing $\lim_{\varepsilon\to 0} \|R_{\frac{\varepsilon}{2}}(z_\varepsilon(t))\|^p_{\mathrm{L}^p(\Omega)^{m\times d}} = \|\nabla z_0(t)\|^p_{\mathrm{L}^p(\Omega)^{m\times d}}$. Therefore, let $a_\varepsilon(t) := \langle \mathbb{C}_\varepsilon(z_\varepsilon(t))\mathbf{e}(u_\varepsilon(t)), \mathbf{e}(u_\varepsilon(t))\rangle_{\mathrm{L}^2(\Omega)^{d\times d}}$ and $b_\varepsilon(t) := \|R_{\frac{\varepsilon}{2}}(z_\varepsilon(t))\|^p_{\mathrm{L}^p(\Omega^+_\varepsilon)^{m\times d}}$ and start by recalling

$$\liminf_{\varepsilon\to 0} a_\varepsilon(t) \geq \langle \mathbb{C}_0(z_0(t))\widetilde{\mathbf{e}}(u_0(t), U_1(t)), \widetilde{\mathbf{e}}(u_0(t), U_1(t))\rangle_{\mathrm{L}^2(\Omega\times Y)^{d\times d}} =: a(t) \tag{6.56a}$$

analogously to (6.36) and

$$\liminf_{\varepsilon\to 0} b_\varepsilon(t) \geq \liminf_{\varepsilon\to 0} \|R_{\frac{\varepsilon}{2}}(z_\varepsilon(t))\|^p_{\mathrm{L}^p(\Omega)^{m\times d}} \geq \|\nabla z_0(t)\|^p_{\mathrm{L}^p(\Omega)^{m\times d}} =: b(t), \tag{6.56b}$$

since $R_{\frac{\varepsilon}{2}}(z_\varepsilon(t))|_\Omega \rightharpoonup \nabla z_0(t)$ in $\mathrm{L}^p(\Omega)^{m\times d}$ was proven in (6.48b). These lim inf-inequalities together with condition (6.55) yield

$$a(t)+b(t) \leq \liminf_{\varepsilon\to 0} a_\varepsilon(t) + \liminf_{\varepsilon\to 0} b_\varepsilon(t) \leq \lim_{\varepsilon\to 0}(a_\varepsilon(t)+b_\varepsilon(t)) \overset{(6.55)}{=} a(t)+b(t),$$

stating that the relations in (6.56a) and (6.56b) actually are all equalities. By assuming the opposite, finally the existence of the limits $\lim_{\varepsilon\to 0} a_\varepsilon(t) = a(t)$ and $\lim_{\varepsilon\to 0} b_\varepsilon(t) = b(t)$ is shown, which together with (6.56b) results in the desired convergence result; namely $\lim_{\varepsilon\to 0} \|R_{\frac{\varepsilon}{2}}(z_\varepsilon(t))\|^p_{\mathrm{L}^p(\Omega)^{m\times d}} = \|\nabla z_0(t)\|^p_{\mathrm{L}^p(\Omega)^{m\times d}}$.

8. To prove $\nabla u_\varepsilon(t) \overset{s}{\to} \nabla_x E u_0(t) + \nabla_y U_1(t)$ in $\mathrm{L}^2(\Omega\times Y)^{d\times d}$, choose $\widetilde{u}_\varepsilon(t) := u_0(t) + v_\varepsilon(t)$, where $v_\varepsilon(t) \in \mathrm{H}^1_0(\Omega)^d$ is the solution of the elliptic problem stated in Proposition 3.8 with $v_0(t) :\equiv \mathbf{0} \in \mathrm{H}^1_0(\Omega)^d$ and $V_1(t) := U_1(t) \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$. Thus, the proof is based on estimating the term $\|\nabla(u_\varepsilon(t) - \widetilde{u}_\varepsilon(t))\|^2_{\mathrm{L}^2(\Omega)^{d\times d}}$ by assumption (6.5) in the following

way:

$$\begin{aligned}
&C_{\mathbf{e}}\|\nabla(u_\varepsilon(t)-\widetilde{u}_\varepsilon(t))\|^2_{\mathrm{L}^2(\Omega)^{d\times d}} \\
&\qquad \le \tfrac{1}{2}\langle \mathbb{C}_\varepsilon(z_\varepsilon(t))\mathbf{e}(u_\varepsilon(t)-\widetilde{u}_\varepsilon(t)), \mathbf{e}(u_\varepsilon(t)-\widetilde{u}_\varepsilon(t))\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} \\
&\qquad = \tfrac{1}{2}\langle \mathbb{C}_\varepsilon(z_\varepsilon(t))\mathbf{e}(u_\varepsilon(t)), \mathbf{e}(u_\varepsilon(t))\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} - \tfrac{1}{2}\langle \mathbb{C}_\varepsilon(z_\varepsilon(t))\mathbf{e}(\widetilde{u}_\varepsilon(t)), \mathbf{e}(\widetilde{u}_\varepsilon(t))\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} \\
&\qquad\quad + \langle \mathbb{C}_\varepsilon(z_\varepsilon(t))\mathbf{e}(\widetilde{u}_\varepsilon(t)), \mathbf{e}(\widetilde{u}_\varepsilon(t)-u_\varepsilon(t))\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} \\
&\qquad = \mathcal{E}_\varepsilon(t,u_\varepsilon(t),z_\varepsilon(t)) - \mathcal{E}_\varepsilon(t,\widetilde{u}_\varepsilon(t),z_\varepsilon(t)) + \langle \ell(t), u_\varepsilon(t)-\widetilde{u}_\varepsilon(t)\rangle && (6.57a)\\
&\qquad\quad + \langle \mathbb{C}_\varepsilon(z_\varepsilon(t))\mathbf{e}(\widetilde{u}_\varepsilon(t)), \mathbf{e}(\widetilde{u}_\varepsilon(t)-u_\varepsilon(t))\rangle_{\mathrm{L}^2(\Omega)^{d\times d}}. && (6.57b)
\end{aligned}$$

Now we show that $\lim_{\varepsilon\to 0}\|\nabla(u_\varepsilon(t)-\widetilde{u}_\varepsilon(t))\|^2_{\mathrm{L}^2(\Omega)^{d\times d}} = 0$. Observe that

$$\begin{aligned}
\widetilde{u}_\varepsilon(t) &\rightharpoonup u_0(t) && \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d, && (6.58a)\\
\widetilde{u}_\varepsilon(t) &\overset{s}{\to} Eu_0(t) && \text{in } \mathrm{L}^2(\Omega\times Y)^d, && (6.58b)\\
\nabla\widetilde{u}_\varepsilon(t) &\overset{s}{\to} \nabla_x Eu_0(t)+\nabla_y U_1(t) && \text{in } \mathrm{L}^2(\Omega\times Y)^{d\times d}, && (6.58c)
\end{aligned}$$

due to Proposition 3.8. Then analogously to step 4 of the proof of Theorem 6.15 (see (6.35)) we have

$$\langle \mathbb{C}_\varepsilon(z_\varepsilon(t))\mathbf{e}(\widetilde{u}_\varepsilon(t)), \mathbf{e}(\widetilde{u}_\varepsilon(t))\rangle_{\mathrm{L}^2(\Omega)^{d^2}} \xrightarrow{\varepsilon\to 0} \langle \mathbb{C}_0(z_0(t))\widetilde{\mathbf{e}}(u_0(t),U_1(t)), \widetilde{\mathbf{e}}(u_0(t),U_1(t))\rangle_{\mathrm{L}^2(\Omega\times Y)^{d^2}}.$$

Note that here the function $(\widetilde{u}_0,\widetilde{U}_1,\widetilde{z}_0)$ of the proof of Theorem 6.15 (see (6.35)) is replaced by $(u_0(t),U_1(t),z_0(t))$. Moreover, due to convergence result (6.58a) it holds $\lim_{\varepsilon\to 0}\langle \ell(t),\widetilde{u}_\varepsilon(t)\rangle = \langle \ell(t),u_0(t)\rangle$, and $\lim_{\varepsilon\to 0}\|R_{\frac{\varepsilon}{2}}(z_\varepsilon(t))\|_{\mathrm{L}^p(\Omega)^{m\times d}} = \|\nabla z_0(t)\|_{\mathrm{L}^p(\Omega)^{m\times d}}$ was proved in step 7. Taking all together results in

$$\lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon(t,\widetilde{u}_\varepsilon(t),z_\varepsilon(t)) = \mathbf{E}_0(t,u_0(t),U_1(t),z_0(t)),$$

which proves that the first two terms of line (6.57a) in the limit ($\varepsilon\to 0$) sum up to zero; see (6.55). Trivially, according to $u_\varepsilon(t)\rightharpoonup u_0(t)$ in $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ and (6.58a) the last term of line (6.57a) converges to zero, too. Since $\mathbf{e}(\widetilde{u}_\varepsilon(t)-u_\varepsilon(t)) \overset{w}{\rightharpoonup} \mathbf{0}$ in $\mathrm{L}^2(\Omega\times Y)^{d\times d}$ according to (6.51c) and (6.58c), and $\mathbb{C}_\varepsilon(z_\varepsilon(t))\mathbf{e}(\widetilde{u}_\varepsilon(t)) \overset{s}{\to} \mathbb{C}_0(z_0(t))\widetilde{\mathbf{e}}(u_0(t),U_1(t))$ in $\mathrm{L}^2(\Omega\times Y)^{d\times d}$ analogously to step 4 of the proof of Theorem 6.15, also the L^2-scalar product term of line (6.57b) converges to zero.

Now we conclude the proof by the following estimate, where we start by adding zero to apply the triangle inequality afterwards.

$$\begin{aligned}
&\|\mathcal{T}_\varepsilon(\nabla u_\varepsilon(t)) - (\nabla_x Eu_0(t)+\nabla_y U_1(t))^{\mathrm{ex}}\|_{\mathrm{L}^2(\mathbb{R}^d\times Y)^{d^2}} \\
&\quad \le \|\mathcal{T}_\varepsilon(\nabla(u_\varepsilon(t)-\widetilde{u}_\varepsilon(t)))\|_{\mathrm{L}^2(\mathbb{R}^d\times Y)^{d^2}} + \|\mathcal{T}_\varepsilon(\nabla\widetilde{u}_\varepsilon(t)) - (\nabla_x Eu_0(t)+\nabla_y U_1(t))^{\mathrm{ex}}\|_{\mathrm{L}^2(\mathbb{R}^d\times Y)^{d^2}}.
\end{aligned}$$

Since $\lim_{\varepsilon\to 0}\|\nabla(u_\varepsilon(t)-\widetilde{u}_\varepsilon(t))\|^2_{\mathrm{L}^2(\Omega)^{d\times d}} = 0$, by exploiting the norm preservation of the unfolding operator the first term converges to zero. Furthermore, according to (6.58c) also the last term converges to zero, which concludes the proof. □

7 Effective damage models based on unidirectional evolution of microscopic inclusions of weak material

Here and in the following chapters the homogenization result of Chapter 6 is exploited to derive effective models for different types of damage models. In this context the asymptotic behavior of families of brutal microscopic models are investigated whose underlying microstructures are those of Section 2.6. That means, that the analytical point of view of Chapter 6, i.e., the microstructure description via $\widehat{\mathbb{C}} \in \mathrm{L}^\infty([0,1]^m; \mathcal{M}(Y))$, is replaced by the geometrical ansatz described in Section 2.6. This procedure includes, that the assumptions guaranteeing existence of solutions and enabling the derivation of effective models are made on the damage set, which is based on a given set valued function $L : [0,1]^m \to \mathcal{L}_{\mathrm{Leb}}(Y)$. The advantage of this ansatz is the internal variable's monotonicity inherited to the stiffness of the considered body; see also Section 2.6. To apply the homogenization theory presented in Chapter 6 to this geometrical description of microstructure, we have to guarantee the validity of the crucial assumptions (6.1)–(6.3). This is assured by the following conditions (7.1) and (7.2) as we will see below. As already mentioned in Section 2.5 our brutal damage models are based on two positive definite tensors $\mathbb{C}_{\mathrm{strong}}, \mathbb{C}_{\mathrm{weak}} \in \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}}; \mathbb{R}^{d\times d}_{\mathrm{sym}})$, i.e., there exists a positive constant α such that

$$\text{for all } \xi \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \text{ it holds} \qquad \alpha|\xi|^2_{d\times d} \leq \langle \mathbb{C}_{\mathrm{weak}}\xi, \xi\rangle_{d\times d} \leq \langle \mathbb{C}_{\mathrm{strong}}\xi, \xi\rangle_{d\times d}. \tag{7.1}$$

The assumptions on the microstructure determining mapping $L : [0,1]^m \to \mathcal{L}_{\mathrm{Leb}}(Y)$ are the following:

- $L : [0,1]^m \to \mathcal{L}_{\mathrm{Leb}}(Y)$ is a non-increasing function; see (2.21). (7.2a)
- For all $\widehat{z} \in [0,1]^m$ with $\widehat{z} \neq \mathbf{1}$ it holds $\mu_d(L(\widehat{z})) > 0$. (7.2b)
- For all $\widehat{z} \in [0,1]^m$ the set $L(\widehat{z})$ is a closed subset of $\mathrm{cl}(Y)$. (7.2c)

For any given $\widehat{z} \in [0,1]^m$ and every $(\widehat{z}_\delta)_{\delta>0} \subset [0,1]^m$ satisfying $\widehat{z}_\delta \to \widehat{z}$ in $\mathbb{R}^m$ it holds

- $\mu_d(L(\widehat{z})\backslash L(\widehat{z}_\delta)) + \mu_d(L(\widehat{z}_\delta)\backslash L(\widehat{z})) \to 0$ for $\delta \to 0$ and (7.2d)
- $\forall \Delta > 0\, \exists\, \delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ it holds $L(\widehat{z}_\delta) \subset \mathrm{neigh}_\Delta(L(\widehat{z}))$. (7.2e)

We will discuss concrete examples for $L : [0,1]^m \to \mathcal{L}_{\mathrm{Leb}}(Y)$ in Subsection 7.1.2.

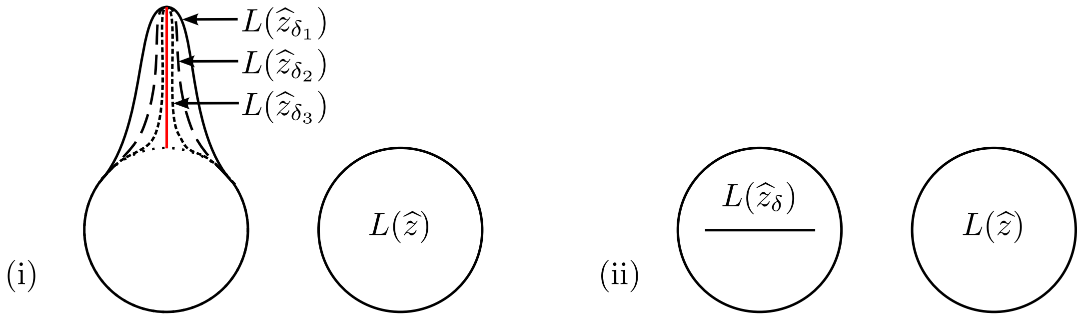


Figure 7.1: Examples for the mapping $L : [0,1]^m \to \mathcal{L}_{\text{Leb}}(Y)$ fulfilling condition (7.2d) but violating (7.2e). (i) Due to the red illustrated subset the indicated sequence violates (7.2e). (ii) Any element of the sequence $(L(\widehat{z}_\delta))_{\delta>0}$ is given by the illustrated ball from which a lower dimensional subset is cut off.

Starting with the given set valued function $L : [0,1]^m \to \mathcal{L}_{\text{Leb}}(Y)$ the tensor valued mapping $\widehat{\mathbb{C}}^{\text{In}} : [0,1]^m \to \mathrm{L}^\infty(Y;\{\mathbb{C}_{\text{strong}},\mathbb{C}_{\text{weak}}\})$ building the foundation of Chapter 6, for all $\widehat{z} \in [0,1]^m$ and every $y \in Y$ is given by (2.19) in the following way:

$$\widehat{\mathbb{C}}^{\text{In}}(\widehat{z})(y) := \mathbb{1}_{Y\setminus L(\widehat{z})}(y)\mathbb{C}_{\text{strong}} + \mathbb{1}_{L(\widehat{z})}(y)\mathbb{C}_{\text{weak}}. \tag{7.3}$$

Here, the superscript *In* refers to the modeling of the inclusions of weak material in a bulk of much stronger material. By assuming the relations (7.1) and (7.2) to hold, for $\widehat{\mathbb{C}}^{\text{In}} : [0,1]^m \to \mathrm{L}^\infty(Y;\{\mathbb{C}_{\text{strong}},\mathbb{C}_{\text{weak}}\})$ defined by (7.3) the conditions (6.1)–(6.3) will be verified in the following.

According to (7.1) for $\widehat{C}_{\mathbf{e}} := \frac{\alpha}{2}$ the tensor $\widehat{\mathbb{C}}^{\text{In}} : [0,1]^m \to \mathrm{L}^\infty(Y;\{\mathbb{C}_{\text{strong}},\mathbb{C}_{\text{weak}}\})$ fulfills inequality (6.3). The monotonicity constraint (7.2a) is to ensure that the damage evolution is unidirectional and has to be understood as stated in (2.21). Since microstructure changes of measure zero do have no effect on the stored energy of the systems considered in the following, asking for (7.2b) prevents the damage sets from being sets of measure zero. Due to the integral description of the models below, depending on the choice of the dissipation distance, violating (7.2b) allows for damage progression without dissipating energy; see Remark 7.5. Alternatively, one could renounce the condition (7.2b) and take into account microstructure changes of measure zero in the dissipation distance, but this is just a question of modeling and does not add any mathematical difficulties. Observe that since $\mu_d(L(\mathbf{1})) = 0$ is allowed, there is no need of a small amount of weak material to initiate damage progression. That means, if we start with a body completely occupied by strong material, occurrence of weak material takes place once the evolution causes a decrease of the damage variable.

The continuity assumption (7.2d) implies that $\widehat{\mathbb{C}}^{\text{In}} : [0,1]^m \to \mathrm{L}^\infty(Y;\{\mathbb{C}_{\text{strong}},\mathbb{C}_{\text{weak}}\})$ is continuous with respect to the strong L^1-topology, which means that it satisfies (6.2). Hence, condition (7.2d) forces changes of the damage set to be continuous with respect to changes of its volume.

The remaining assumptions (7.2c) and (7.2e) together with (7.2a) ensure the validity of condition (6.1), which is stated in the lemma below. Condition (7.2e) represents some kind of uniformity assumption on the boundary $L(\widehat{z})$ with respect to changes of the parameters $\widehat{z} \in [0,1]^m$; see Figure 7.1 for two examples. According to Chapter 6, condition (6.1) assures the measurability of the energy densities introduced in the following. This measurability is noteworthy, since due to Remark 7.2 none of the here considered energy densities are Carathéodory functions.

Lemma 7.1. *Let $L : [0,1]^m \to \mathcal{L}_{\mathrm{Leb}}(Y)$ satisfy the conditions (7.2a), (7.2c), and (7.2e). Moreover, let $\widehat{\mathbb{C}}^{\mathrm{In}} : [0,1]^m \to \mathrm{L}^\infty(Y;\{\mathbb{C}_{\mathrm{strong}},\mathbb{C}_{\mathrm{weak}}\})$ be defined by (7.3). Then, for any measurable function $z : \mathbb{R}^d \to [0,1]^m$ the mapping*

$$\widehat{\mathbb{C}}^{\mathrm{In}}(z(\cdot))(\cdot) : \begin{cases} \mathbb{R}^d{\times}Y \to \{\mathbb{C}_{\mathrm{strong}},\mathbb{C}_{\mathrm{weak}}\} \\ \quad (x,y) \mapsto \widehat{\mathbb{C}}^{\mathrm{In}}(z(x))(y) \end{cases} \quad \text{is measurable on } \mathbb{R}^d{\times}Y, \tag{7.4}$$

i.e., condition (6.1) of Chapter 6 is fulfilled.

Remark 7.2. *Let the mapping $f : Y{\times}\mathbb{R}^{d\times d}_{\mathrm{sym}}{\times}\mathbb{R}^m \to \mathbb{R}$ be defined by*

$$f(y,\xi,\widehat{z}) := \langle \widehat{\mathbb{C}}^{\mathrm{In}}(\widehat{z})(y)\xi,\xi\rangle_{d\times d} = \begin{cases} \langle \mathbb{C}_{\mathrm{strong}}\xi,\xi\rangle_{d\times d} & \text{if } y \in Y\backslash L(\widehat{z}), \\ \langle \mathbb{C}_{\mathrm{weak}}\xi,\xi\rangle_{d\times d} & \text{if } y \in L(\widehat{z}). \end{cases}$$

Then for fixed $y \in Y$ the mapping $f(y,\cdot,\cdot) : \mathbb{R}^{d\times d}_{\mathrm{sym}}{\times}\mathbb{R}^m \to \mathbb{R}$ is not continuous on $\mathbb{R}^{d\times d}_{\mathrm{sym}}{\times}\mathbb{R}^m$ as for fixed $\xi \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ it only takes the values $\langle \mathbb{C}_{\mathrm{strong}}\xi,\xi\rangle_{d\times d}$ and $\langle \mathbb{C}_{\mathrm{weak}}\xi,\xi\rangle_{d\times d}$. Hence, $f : Y{\times}\mathbb{R}^{d\times d}_{\mathrm{sym}}{\times}\mathbb{R}^m \to \mathbb{R}$ does not satisfy the Carathéodory condition. However, as follows from the previous lemma, for every measurable function $z : \mathbb{R}^d \to [0,1]^m$ the mapping $\widehat{f}_z : \mathbb{R}^d{\times}Y{\times}\mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}$ with $\widehat{f}_z(x,y,\xi) := \langle \widehat{\mathbb{C}}^{\mathrm{In}}(z(x))(y)\xi,\xi\rangle_{d\times d}$ is a Carathéodory function, since the mapping $\xi \mapsto \widehat{f}_z(x,y,\xi)$ for all $(x,y) \in \mathbb{R}^d{\times}Y$ is continuous and since $(x,y) \mapsto \widehat{f}_z(x,y,\xi)$ for any $\xi \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ is measurable.

Proof. To verify condition (7.4) let $z : \mathbb{R}^d \to [0,1]^m$ be an arbitrary but fixed measurable function. Due to the definition of $\widehat{\mathbb{C}}^{\mathrm{In}} : [0,1]^m \to \mathrm{L}^\infty(Y;\{\mathbb{C}_{\mathrm{strong}},\mathbb{C}_{\mathrm{weak}}\})$ given by (7.3), the mapping $\widehat{\mathbb{C}}^{\mathrm{In}}(z(\cdot))(\cdot) : \mathbb{R}^d{\times}Y \to \{\mathbb{C}_{\mathrm{strong}},\mathbb{C}_{\mathrm{weak}}\}$ is constant on the two sets $M(z) := \bigcup_{x\in\mathbb{R}^d}\{x\}{\times}L(z(x))$ and $(\mathbb{R}^d{\times}Y)\backslash M(z)$. Hence, (7.4) is proven by showing that $M(z)$ is a measurable subset of $\mathbb{R}^d{\times}Y$.

For this purpose, we choose a countable sequence $(z_\delta)_{(\delta>0)}$ of simple functions approximating the measurable mapping $z : \mathbb{R}^d \to [0,1]^m$ from below, i.e., $z_\delta(x) \nearrow z(x)$ (componentwise) for all $x \in \mathbb{R}^d$. Here, the term *simple function* means, that there is a finite number of disjoint, measurable sets $A_1^\delta, A_2^\delta, \ldots, A_{n_\delta}^\delta \subset \mathbb{R}^d$ and constant vectors $z_1^\delta, z_2^\delta, \ldots, z_{n_\delta}^\delta \in [0,1]^m$ such that $\bigcup_{k=1}^{n_\delta} A_k = \mathbb{R}^d$ and $z_\delta = \sum_{k=1}^{n_\delta} \mathbb{1}_{A_k^\delta} z_k^\delta$. Thus, we now consider the sequence $(M(z_\delta))_{\delta>0}$ of $M(z)$ approximating sets. For $\delta > 0$ the measurability of $M(z_\delta)$ is a consequence of the fact, that it can be written as a finite union of measurable sets in the following way:

$$M(z_\delta) = \bigcup_{k=1}^{n_\delta}\Big(\bigcup_{x\in A_k^\delta}\{x\}{\times}L(z_\delta(x))\Big) = \bigcup_{k=1}^{n_\delta}\Big(A_k^\delta{\times}L(z_k^\delta)\Big).$$

Note that for fixed $\delta > 0$ the measurability of the set $L(z_k^\delta)$ for all $k \in \{1, 2, \dots, n_\delta\}$ is ensured by assumption (7.2c). Due to the relation $z_\delta \le z$ on $\mathbb{R}^d$ and condition (7.2a) we have $M(z) \subset M(z_\delta)$ for every $\delta > 0$ by definition. Moreover, $\bigcap_{\delta>0} M(z_\delta) \subset M(z)$ is shown by the following contradiction argument:

Let $(x^*, y^*) \in \bigcap_{\delta>0} M(z_\delta)$ but $(x^*, y^*) \notin M(z)$. Then for all $\delta > 0$

$$y^* \in L(z_\delta(x^*)) \tag{7.5}$$

but

$$\operatorname{dist}(y^*, L(z(x^*))) =: 2\Delta > 0 \tag{7.6}$$

since $L(z(x^*))$ was assumed to be closed; see (7.2c). Condition (7.6) implies

$$y^* \notin \operatorname{neigh}_\Delta(L(z(x^*))). \tag{7.7}$$

Since $z_\delta(x^*) \to z(x^*)$ by assumption, there exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ it holds

$$y^* \overset{(7.5)}{\in} L(z_\delta(x^*)) \overset{(7.2e)}{\subset} \operatorname{neigh}_\Delta(L(z(x^*)))$$

which is a contradiction to (7.7).

All together we proved $M(z) = \bigcap_{\delta>0} M(z_\delta)$. Since $M(z)$ can be written as the countable intersection of measurable sets, this shows its measurability and hence condition (7.4) is verified. $\square$

7.1 Inclusions of weak material causing damage progression

In this section for fixed $\varepsilon > 0$ we start by introducing various microscopic damage models fitting into the homogenization theory of Chapter 6. These different models are given by the energetic formulation based on an energy functional and a dissipation distance. The microstructure is modeled by a function $L : [0,1]^m \to \mathcal{L}_{\text{Leb}}(Y)$ introduced in (7.2). Before giving explicit examples for $L : [0,1]^m \to \mathcal{L}_{\text{Leb}}(Y)$, we formulate the abstract setting. For a given damage variable $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)$ the microstructure for almost every $x \in \Omega$ is modeled by the tensor

$$\mathbb{C}_\varepsilon^{\text{In}}(z_\varepsilon)(x) := \mathbb{1}_{\Omega \setminus \Omega_\varepsilon^{\text{D}}(z_\varepsilon)}(x)\mathbb{C}_{\text{strong}} + \mathbb{1}_{\Omega_\varepsilon^{\text{D}}(z_\varepsilon)}(x)\mathbb{C}_{\text{weak}}, \tag{7.8}$$

where the damage set $\Omega_\varepsilon^{\text{D}}(z_\varepsilon)$ for $\lambda \in \Lambda_\varepsilon^-$ (see (2.15)) and $z^{\varepsilon\lambda} :\equiv z_\varepsilon|_{\varepsilon(\lambda+Y)}$ according to (2.20) reads as follows:

$$\Omega_\varepsilon^{\text{D}}(z_\varepsilon) := \bigcup_{\lambda \in \Lambda_\varepsilon^-} \varepsilon(\lambda + L(z^{\varepsilon\lambda})). \tag{7.9}$$

Note that due to assumption (2.1) Korn's inequality is applicable for the domain Ω which together with (7.1) for all $u \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ results in

$$\tfrac{\alpha}{2} C_{\mathrm{Korn}} \|u\|^2_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d} \leq \tfrac{\alpha}{2} \|\mathbf{e}(u)\|^2_{\mathrm{L}^2(\Omega)^{d\times d}} \leq \tfrac{1}{2} \langle \mathbb{C}^{\mathrm{In}}_\varepsilon(z_\varepsilon)\mathbf{e}(u), \mathbf{e}(u)\rangle_{d\times d}. \tag{7.10}$$

Therefore, the coercivity condition (6.5) for the microscopic models considered in this section is fulfilled.

7.1.1 Energy functional and dissipation potential

Analog to Section 6.1, the state space $\mathcal{Q}^{\mathrm{In}}_\varepsilon(\Omega)$ is given by

$$\mathcal{Q}^{\mathrm{In}}_\varepsilon(\Omega) := \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \times \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m).$$

Again we once choose $p \in (1,\infty)$ and keep it fixed for the rest of this chapter. Then the stored energy of the system is modeled by the energy functional $\mathcal{E}^{\mathrm{In}}_\varepsilon : [0,T] \times \mathcal{Q}^{\mathrm{In}}_\varepsilon(\Omega) \to \mathbb{R}$ defined via

$$\mathcal{E}^{\mathrm{In}}_\varepsilon(t, u, z_\varepsilon) := \tfrac{1}{2} \langle \mathbb{C}^{\mathrm{In}}_\varepsilon(z_\varepsilon)\mathbf{e}(u), \mathbf{e}(u)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} + \|R_{\frac{\varepsilon}{2}} z_\varepsilon\|^p_{\mathrm{L}^p(\Omega^+_\varepsilon)^{m\times d}} - \langle \ell(t), u\rangle, \tag{7.11}$$

where $\ell \in \mathrm{C}^1([0,T]; (\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$ and $\mathbf{e}(u)$ denotes the linearized strain tensor. Observe that the energy density's measurability is ensured by Lemma 7.1, and according to Remark 7.2 it does not satisfy the Carathéodory condition.

Referring to Section 5.2 the modeling of the dissipated energy of the system is based on a dissipation potential. The choice of this potential specifies the relation of (i) the amount of dissipated energy of the system from one time step to another and (ii) the behavior of the damage variable along this time interval. Including the examples considered in the Subsections 7.1.2 and 7.1.4 below, all investigated potentials fulfill condition (5.2) and hence ensure the rate independence of the respective system; see Proposition 5.6. To show the applicability of the homogenization theory of Chapter 6, the associated dissipation distances are calculated such that condition (6.8) can be verified in the respective case.

Let an arbitrary mapping $L : [0,1]^m \to \mathcal{L}_{\mathrm{Leb}}(Y)$ satisfying (7.1) and (7.2) be given. In this general case the amount of dissipated energy of the system from one time step to another is assumed to be proportional to the change of the damage variable in this time interval but with possibly different proportionality factors for every component. This behavior is modeled by a dissipation potential $\mathcal{R}^{\mathrm{In}}_\varepsilon : \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m) \to [0,\infty]$ depending only on the damage variable's velocity. Choose $q' \in (1,\infty)$ and keep it fixed for the rest of this chapter, i.e., in the following every q' refers to this choice. Then, for a given sequence $(\kappa^{\mathrm{In}}_\varepsilon)_{\varepsilon>0} \subset \mathrm{L}^{q'}(\Omega; [0,\infty)^m)$ satisfying $\kappa^{\mathrm{In}}_\varepsilon \rightharpoonup \kappa^{\mathrm{In}}_0$ in $\mathrm{L}^{q'}(\Omega)^m$ for some function $\kappa^{\mathrm{In}}_0 \in \mathrm{L}^{q'}(\Omega; [0,\infty)^m)$ the dissipation potential associated to $\varepsilon > 0$ reads as follows:

$$\mathcal{R}^{\mathrm{In}}_\varepsilon(v_\varepsilon) := \begin{cases} \displaystyle\int_{\Omega^-_\varepsilon} |\langle \kappa^{\mathrm{In}}_\varepsilon(x), v_\varepsilon(x)\rangle_m| \mathrm{d}x & \text{if } \mathbf{0} \geq v_\varepsilon \text{ on } \Omega, \\ \infty & \text{otherwise.} \end{cases}$$

Hence, according to Remark 5.2, $\mathcal{D}_\varepsilon^{\mathrm{In}} : \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)\times\mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m) \to [0,\infty]$ is given by

$$\mathcal{D}_\varepsilon^{\mathrm{In}}(z_1,z_2) := \begin{cases} \int_{\Omega_\varepsilon^-} |\langle \kappa_\varepsilon^{\mathrm{In}}(x), z_2(x) - z_1(x)\rangle_m| \mathrm{d}x & \text{if } z_1 \geq z_2 \text{ on } \Omega, \\ \infty & \text{otherwise.} \end{cases} \tag{7.12}$$

Note that here and in the following the dissipation distance depends only on the arguments' values on the set Ω_ε^-. This relation is due the fact that the system's dissipated energy depends on the evolution of inclusions of weak material, which according to (7.9) is restricted to the set Ω_ε^-. Observe that a vector valued damage variable might be used to model anisotropic inclusions of weak material; see Example 7.4 and 7.6, for instance. On the other hand, the problem underlying the modeling ansatz may show an anisotropic behavior with respect to changes of the inclusions geometry. In this case, this anisotropic response to changes of the inclusions geometry can be modeled by the vector valued fracture toughness $\kappa_\varepsilon^{\mathrm{In}} \in \mathrm{L}^{q'}(\Omega;[0,\infty)^m)$. Thus, in addition to the possible anisotropy entering the model due to the inclusions geometries, the system's reaction on a single component of the damage variable can be modeled individually.

To indicate the variety of damage caused microstructures captured by this energetic approach, we are now going to consider several dissipation potentials in dependence of different explicit choices of the microstructure determining function $L : [0,1]^m \to \mathcal{L}_{\mathrm{Leb}}(Y)$ fulfilling the conditions (7.1) and (7.2). There the focus is placed on associating the system's dissipated energy to the changes of the inclusions' geometries.

7.1.2 Potential inclusions' geometries

Example 7.3 (Spherical inclusions of weak material)**.** In this case the damage progression is assumed to cause spherical inclusions of weak material emerging in the center of cells $\varepsilon(\lambda{+}Y) \subset \Omega$. To simplify notation for this example, for a given basis $\{b_1, b_2, \ldots, b_d\}$ of $\mathbb{R}^d$ the unit cell is redefined in the following way:

$$Y := \Big\{ y \in \mathbb{R}^d \,\Big|\, y = \sum_{i=1}^d k_i b_i, k_i \in [-\tfrac{1}{2}, \tfrac{1}{2}) \Big\}.$$

Let the maximal radius $R > 0$ be chosen such that $\mathrm{cl}(B_R(\mathbf{0})) \subset Y$. Then, for $m = 1$ and $\widehat{z} \in [0,1]$ the microstructure determining function $L : [0,1] \to \mathcal{L}_{\mathrm{Leb}}(Y)$ is defined by

$$L(\widehat{z}) := (1{-}\widehat{z})\mathrm{cl}(B_R(\mathbf{0})) := \begin{cases} \{y \in Y \,|\, \frac{1}{(1-\widehat{z})} y \in \mathrm{cl}(B_R(\mathbf{0}))\} & \text{if } \widehat{z} \in [0,1), \\ \emptyset & \text{if } \widehat{z} = 1. \end{cases}$$

Thus, for a suitable damage variable $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1])$ the distribution of weak material is illustrated in the center of Figure 2.2.

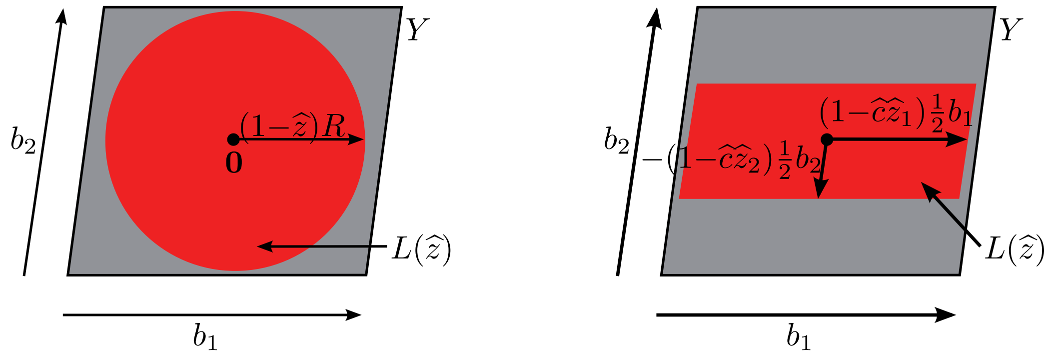


Figure 7.2: Left: Spherical inclusions of weak material. Right: Quadrangular inclusions of weak material.

For $d \geq 2$ we consider two different dissipation potentials associating the dissipated energy to the change of the volume or surface of the damage set.

$$\mathcal{R}^{\mathrm{vol}}_{\varepsilon}(z_\varepsilon, v_\varepsilon) := \begin{cases} -\,d\,\kappa^{\mathrm{vol}}(d)\displaystyle\int_{\Omega_\varepsilon^-}(1-z_\varepsilon(x))^{d-1}v_\varepsilon(x)\mathrm{d}x & \text{if } 0 \geq v_\varepsilon, \\ \infty & \text{otherwise,} \end{cases}$$
$$\mathcal{R}^{\mathrm{sur}}_{\varepsilon}(z_\varepsilon, v_\varepsilon) := \begin{cases} -\,(d-1)\kappa^{\mathrm{sur}}(d)\displaystyle\int_{\Omega_\varepsilon^-}(1-z_\varepsilon(x))^{d-2}v_\varepsilon(x)\mathrm{d}x & \text{if } 0 \geq v_\varepsilon, \\ \infty & \text{otherwise.} \end{cases}$$

Here, $\kappa^{\mathrm{vol}}(d) := \pi^{\frac{d}{2}}R^d(\Gamma(\frac{d}{2}+1))^{-1}$ denotes the volume of the d-dimensional ball with radius R and $\kappa^{\mathrm{sur}}(d) := 2\pi^{\frac{d}{2}}R^{d-1}(\Gamma(\frac{d}{2}))^{-1}$ its surface area. Observe that as an illustration these constants are chosen such that the proportionality factor of dissipated energy and the change of the volume or surface of the damage set is exactly 1. Without any difficulties one also might consider x-dependent factors scaling the dissipated energy in dependence on the damage variable's value instead of the constants $\kappa^{\mathrm{vol}}(d)$ and $\kappa^{\mathrm{sur}}(d)$.

The associated dissipation distances are determined by the minimizing problem (5.3). To receive an explicit description for these distances, for arbitrary $z_1 \geq z_2 \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1])$ and some arbitrary chosen function $\overline{z} \in \mathrm{W}^{1,1}_{z_1,z_2}([0,1];\mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]))$ (see (5.4)) we calculate the value $\int_0^1 \mathcal{R}^{\mathrm{vol}}_\varepsilon(\overline{z}(s),\dot{\overline{z}}(s))\mathrm{d}s$. Note that this function has to be non-increasing to be a possible minimizer of (5.3). Integrating by parts with respect to s yields

$$-d\int_0^1\int_{\Omega_\varepsilon^-}(1-\overline{z}(s,x))^{d-1}\dot{\overline{z}}(s,x)\mathrm{d}x\mathrm{d}s = \int_{\Omega_\varepsilon^-}(1-\overline{z}(1,x))^d - (1-\overline{z}(0,x))^d\mathrm{d}x.$$

Hence, the term $\int_0^1 \mathcal{R}^{\mathrm{vol}}_\varepsilon(\overline{z}(s),\dot{\overline{z}}(s))\mathrm{d}s$ is independent of the path described by the function $\overline{z}:[0,1]\to\mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1])$ and depends only on the boundary values z_1 and z_2. Since an analog argument is valid for $\mathcal{R}^{\mathrm{sur}}_\varepsilon$, the dissipation distances defined by the minimizing problem (5.3) read as follows:

$$\mathcal{D}^{\mathrm{vol}}_\varepsilon(z_1,z_2) := \begin{cases} \kappa^{\mathrm{vol}}(d)\displaystyle\int_{\Omega_\varepsilon^-}(1-z_2(x))^d-(1-z_1(x))^d\mathrm{d}x & \text{if } z_1\geq z_2,\\ \infty & \text{otherwise,}\end{cases}$$
$$\mathcal{D}^{\mathrm{sur}}_\varepsilon(z_1,z_2) := \begin{cases} \kappa^{\mathrm{sur}}(d)\displaystyle\int_{\Omega_\varepsilon^-}(1-z_2(x))^{d-1}-(1-z_1(x))^{d-1}\mathrm{d}x & \text{if } z_1\geq z_2,\\ \infty & \text{otherwise.}\end{cases}$$

Decomposing Ω_ε^- into small cells $\varepsilon(\lambda+Y)$, $\lambda\in\Lambda_\varepsilon^-$, and exploiting the definition of the damage set (see (7.9)) the dissipation distances are equivalently written as

$$\mathcal{D}^{\mathrm{vol}}_\varepsilon(z_1,z_2) \;=\begin{cases} \mu_d(\Omega^{\mathrm{D}}_\varepsilon(z_2))-\mu_d(\Omega^{\mathrm{D}}_\varepsilon(z_1)) & \text{if } z_1\geq z_2,\\ \infty & \text{otherwise,}\end{cases}$$
$$\mathcal{D}^{\mathrm{sur}}_\varepsilon(z_1,z_2) \;=\begin{cases} \mu_{d-1}(\partial\Omega^{\mathrm{D}}_\varepsilon(z_2))-\mu_{d-1}(\partial\Omega^{\mathrm{D}}_\varepsilon(z_1)) & \text{if } z_1\geq z_2,\\ \infty & \text{otherwise,}\end{cases}$$

showing explicitly the desired relation of dissipated energy and changes of the damage set.

Example 7.4 (Quadrangular inclusions of weak material)**.** For the sake of keeping the notation as simple as possible, we set $d=2$ in this case, but there are no problems when generalizing this example to some arbitrary dimension. In the following the unit cell Y is defined as in Example 7.3. Furthermore, the amount of dissipated energy from one time step to another is assumed to be equal to the volume of strong material undergoing the damage process (and hence transforming into weak material) during this time interval.

We start with an example of quadrangular inclusions of weak material. For this purpose, for $m=2$ and $\widehat{z}\in[0,1]^2$ we set

$$Y(\widehat{z}) := \Big\{y\in\mathbb{R}^2\,\Big|\,y=\sum_{i=1}^{2}\widehat{z}_ik_ib_i,\,k_i\in[-\tfrac{1}{2},\tfrac{1}{2})\Big\}. \tag{7.13}$$

As already mentioned in the beginning of this chapter, assuming condition (7.2b) to hold prevents the damage set from being a set of measure zero. Since we are going to associate the dissipated energy to the volume change of the damage set, microstructure changes of measure zero will not be captured by the dissipation distance. On the other hand, by assuming the dissipated energy to be proportional to the surface change of the damage set one might renounce condition (7.2b).

To ensure the validity of condition (7.2b) we choose a constant $\widehat{c}\in(0,1)$. Thus, the mapping $L:[0,1]^2\to\mathcal{L}_{\mathrm{Leb}}(Y)$ for $\widehat{z}\in[0,1]^2$ is defined via $L(\widehat{z}):=\mathrm{cl}(Y(\mathbf{1}-\widehat{c}\widehat{z}))$. Since $\widehat{c}<1$, every cell $\varepsilon(\lambda+Y)\subset\Omega$ contains at least a small amount of weak material at any time, even for a constant damage variable taking the value $\mathbf{1}$. Note that the smaller the difference $1-\widehat{c}>0$ the smaller the amount of initially appearing weak material in the "undamaged" system. Here, the term *undamaged* refers to the internal variable and does not mean that there is no weak material.

Remark 7.5 (Necessity of condition (7.2b))**.** *Let* $\widehat{L}(\widehat{z}) := \mathrm{cl}(Y(\mathbf{1}-\widehat{z}))$ *and choose* $z_\varepsilon^{(1)} :\equiv (\widehat{z}_1, 1)^T \in [0,1]^2$ *for some arbitrary* $\widehat{z}_1 \in [0,1)$*. Since the set* $\Omega_\varepsilon^{\mathrm{D}}(z_\varepsilon^{(1)})$ *is* $(d-1)$*-dimensional, there is no difference between the values* $\mathcal{E}_\varepsilon^{\mathrm{In}}(t,u,\mathbf{1})$ *and* $\mathcal{E}_\varepsilon^{\mathrm{In}}(t,u,z_\varepsilon^{(1)})$ *of stored energy. Therefore, in the case of associating the dissipated energy to the volume change of the damage set, there neither energy is dissipated when switching from the "undamaged" state* $(u,\mathbf{1})$ *to the "damaged" state* $(u,z_\varepsilon^{(1)})$ *nor does their stored energy change.*

On the other hand, starting from $(u,\mathbf{1})$ *or* (u,z_ε) *and considering damage progression in* b_2*-direction yields dramatically different results. Choose* $z_\varepsilon^{(2)} :\equiv (1,\widehat{z}_2)^T \in [0,1]^2$ *and* $z_\varepsilon^{(1,2)} :\equiv (\widehat{z}_1,\widehat{z}_2)^T \in [0,1]^2$ *for some arbitrary* $\widehat{z}_2 \in [0,1)$*. Then* $\mathcal{E}_\varepsilon^{\mathrm{In}}(t,u,\mathbf{1}) = \mathcal{E}_\varepsilon^{\mathrm{In}}(t,u,z_\varepsilon^{(2)})$ *but* $\mathcal{E}_\varepsilon^{\mathrm{In}}(t,u,z_\varepsilon^{(1)}) \neq \mathcal{E}_\varepsilon^{\mathrm{In}}(t,u,z_\varepsilon^{(1,2)})$*. Moreover, there is no energy dissipated when switching from* $(u,\mathbf{1})$ *to* $(u,z_\varepsilon^{(2)})$*, but the energy dissipated when switching from* $(u,z_\varepsilon^{(1)})$ *to* $(u,z_\varepsilon^{(1,2)})$ *is proportional to* $\mu_d(\Omega_\varepsilon^{\mathrm{D}}(z_\varepsilon^{(1,2)})) > 0$*. This investigation indicates the necessity of condition* (7.2b) *if the relation of dissipated energy and changes of the damage variable is not chosen carefully.*

Letting $\gamma \in (0,\pi)$ denote the angle between the two basis vectors b_1 and b_2 (see (7.13)) and setting $\kappa := \sin(\gamma)|b_1|_2|b_2|_2$, the dissipation potential leading to a proportional relation of dissipated energy and volume changes of the damage set is defined by

$$\mathcal{R}_\varepsilon^{\mathrm{qua}}(z,v) := \begin{cases} -\kappa \displaystyle\int_{\Omega_\varepsilon^-} (1-\widehat{c}z_1(x))\widehat{c}v_2(x) + (1-\widehat{c}z_2(x))\widehat{c}v_1(x)\mathrm{d}x & \text{if } \mathbf{0} \geq v, \\ \infty & \text{otherwise.} \end{cases}$$

Again, the choice of the constant $\kappa > 0$ is motivated by keeping the following as simple as possible and could be easily replaced by a non-negative function, for instance. Proceeding analogously to Example 7.3, the associated dissipation distance reads as follows:

$$\mathcal{D}_\varepsilon^{\mathrm{qua}}(z^{(1)},z^{(2)}) := \begin{cases} \kappa \displaystyle\int_{\Omega_\varepsilon^-} \prod_{i=0}^{1}(1-\widehat{c}z_{1+i}^{(2)}(x)) - \prod_{i=0}^{1}(1-\widehat{c}z_{1+i}^{(1)}(x))\mathrm{d}x & \text{if } z^{(1)} \geq z^{(2)}, \\ \infty & \text{otherwise.} \end{cases}$$

Again, by decomposing Ω_ε^- into small cells $\varepsilon(\lambda+Y)$, $\lambda \in \Lambda_\varepsilon^-$, and exploiting the definition of the damage set (see (7.9)) the following description hold:

$$\mathcal{D}_\varepsilon^{\mathrm{quad}}(z^{(1)},z^{(2)}) = \begin{cases} \mu_d(\Omega_\varepsilon^{\mathrm{D}}(z^{(2)})) - \mu_d(\Omega_\varepsilon^{\mathrm{D}}(z^{(1)})) & \text{if } z^{(1)} \geq z^{(2)}, \\ \infty & \text{otherwise.} \end{cases}$$

Obviously, this example can be easily generalized by allowing the quadrangular inclusions to grow independently in all four directions. Therefore, the number of damage variable's components has to be doubled and the potential needs to be adapted.

Example 7.6 (Anisotropic inclusions of weak material)**.** In contrast to Example 7.4 we here allow for more general anisotropic inclusions of weak material. Again we set

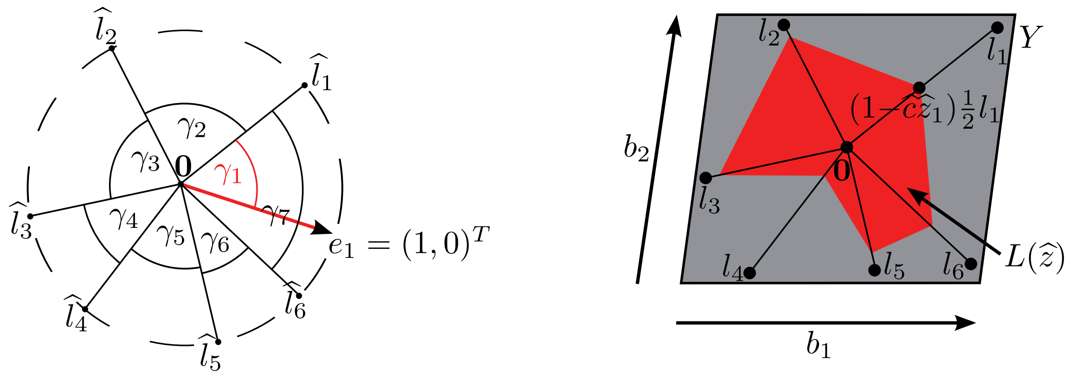


Figure 7.3: Left: Construction of $\widehat{l}_1, \widehat{l}_2, \dots, \widehat{l}_6 \in \mathbb{R}^d$ for given angels $\gamma_1 \in [0,\pi)$ and $\gamma_2, \gamma_3, \dots, \gamma_7 \in (0,\pi)$. Right: Anisotropic inclusions of weak material.

$d = 2$ and let the unit cell Y be defined as in Example 7.3. Analog to Example 7.4, the dissipated energy is assumed to be proportional to the volumetric change of the damage set. To construct the anisotropic inclusions, let $m > 2$ and for $j \in \{2, 3, \dots, m, m+1\}$ choose m angles $\gamma_j \in (0,\pi)$ with $\sum_{j=2}^{m+1} \gamma_j = 2\pi$. Then, for $\gamma_1 \in [0,\pi)$ let $\widehat{l}_1 \in \mathbb{R}^2$ with $|\widehat{l}_1|_2 = 1$ be one of the two vectors satisfying $\cos(\gamma_1) = \langle \widehat{l}_1, e_1 \rangle_2$, where $e_1 := (1,0)^T$. Thus, the vectors $\widehat{l}_2, \widehat{l}_3, \dots, \widehat{l}_m \in Y$ of length 1 for $j = 2, 3, \dots, m$ are iteratively given by

$$\cos\Big(\sum_{k=1}^{j} \gamma_k\Big) = \langle \widehat{l}_j, e_1 \rangle_2 \qquad \text{and} \qquad \cos(\gamma_j) = \langle \widehat{l}_j, \widehat{l}_{j-1} \rangle_2.$$

For $a_1, a_2, a_3 \in \mathbb{R}^2$ let triangle$[a_1, a_2, a_3]$ denote the closed triangle generated by the vertices a_1, a_2 and a_3. For $j = 1, 2, \dots, m$ choose c_j such that $l_j := c_j \widehat{l}_j$ is an element of $\mathrm{cl}(Y)$. Then, for $\widehat{c} \in (0,1)$ we set

$$L(\widehat{z}) := \bigcup_{j=1}^{m} \mathrm{triangle}[(1-\widehat{c}\widehat{z}_j)l_j, (1-\widehat{c}\widehat{z}_{j+1})l_{j+1}, \mathbf{0}],$$

where here and in the following $\widehat{z}_{m+1} := \widehat{z}_1$ and $l_{m+1} := l_1$. Again the here appearing constant $\widehat{c} \in (0,1)$ ensures assumption (7.2b). For $\kappa_j := \frac{1}{2}\sin(\gamma_{j+1})|l_j|_2|l_{j+1}|_2$ the dissipation potential modeling the desired behavior reads as follows:

$$\mathcal{R}_\varepsilon^{\mathrm{tria}}(z,v) := \begin{cases} -\sum_{j=1}^{m} \kappa_j \int_{\Omega_\varepsilon^-} (1-\widehat{c}z_j(x))\widehat{c}v_{j+1}(x) + (1-\widehat{c}z_{j+1}(x))\widehat{c}v_j(x)\mathrm{d}x & \text{if } \mathbf{0} \geq v, \\ \infty & \text{otherwise.} \end{cases}$$

An analog argument as used for Example 7.3 yields the explicit formula

$$\mathcal{D}_\varepsilon^{\mathrm{tria}}(z^{(1)}, z^{(2)}) := \begin{cases} \sum_{j=1}^{m} \kappa_j \int_{\Omega_\varepsilon^-} \prod_{i=0}^{1}(1-\widehat{c}z_{j+i}^{(1)}(x)) - \prod_{i=0}^{1}(1-\widehat{c}z_{j+i}^{(2)}(x))\mathrm{d}x & \text{if } z^{(1)} \geq z^{(2)}, \\ \infty & \text{otherwise} \end{cases}$$

for the dissipation distance, which can be reformulated in the following way:

$$\mathcal{D}_\varepsilon^{\mathrm{tria}}(z^{(1)}, z^{(2)}) = \begin{cases} \mu_d(\Omega_\varepsilon^{\mathrm{D}}(z^{(2)})) - \mu_d(\Omega_\varepsilon^{\mathrm{D}}(z^{(1)})) & \text{if } z^{(1)} \geq z^{(2)}, \\ \infty & \text{otherwise.} \end{cases}$$

Remark 7.7. (a) *Observe that from the application point of view the proceeding in Example* 7.3, 7.4, *and* 7.6 *rather would be conversely. Most likely one would like to start with the relation of the dissipated energy to the increase of the damage set. In this case, the existence of an underlying dissipation potential fulfilling condition* (5.2) *needs to be checked afterwards; see also Subsection* 7.1.4.

(b) *By considering sequences* $(z_\varepsilon)_{\varepsilon>0}$ *and* $(\widetilde{z}_\varepsilon)_{\varepsilon>0}$ *of functions* $z_\varepsilon, \widetilde{z}_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)$ *satisfying condition* (6.8), *the derivation of the limit dissipation distance in Example* 7.3, 7.4, *and* 7.6 *is straight forward. Hence, in the following* (*including the limit models of Section* 7.2 *and* 7.3) *everything is written exemplarily for the case of a general mapping* $L : [0,1]^m \to \mathcal{L}_{\mathrm{Leb}}(Y)$ *satisfying* (7.1) *and* (7.2), *where the dissipation distance is given by relation* (7.12).

7.1.3 The microscopic model and existence of solutions

For given initial values $(u_\varepsilon^0, z_\varepsilon^0) \in \mathcal{Q}_\varepsilon^{\mathrm{In}}(\Omega)$ the rate-independent damage evolution is modeled by the ε-dependent energetic formulation $(\mathrm{S}_{\mathrm{In}}^\varepsilon)$ and $(\mathrm{E}_{\mathrm{In}}^\varepsilon)$, where $\varepsilon > 0$ scales the size of the damage structure.

Stability condition $(\mathrm{S}_{\mathrm{In}}^\varepsilon)$ **and energy balance** $(\mathrm{E}_{\mathrm{In}}^\varepsilon)$ **for all** $t \in [0,T]$**:**

$$\mathcal{E}_\varepsilon^{\mathrm{In}}(t, u_\varepsilon(t), z_\varepsilon(t)) \leq \mathcal{E}_\varepsilon^{\mathrm{In}}(t, \widetilde{u}, \widetilde{z}) + \mathcal{D}_\varepsilon^{\mathrm{In}}(z_\varepsilon(t), \widetilde{z}) \quad \text{for all } (\widetilde{u}, \widetilde{z}) \in \mathcal{Q}_\varepsilon^{\mathrm{In}}(\Omega)$$

$$\mathcal{E}_\varepsilon^{\mathrm{In}}(t, u_\varepsilon(t), z_\varepsilon(t)) + \mathrm{Diss}_{\mathcal{D}_\varepsilon^{\mathrm{In}}}(z_\varepsilon; [0,t]) = \mathcal{E}_\varepsilon^{\mathrm{In}}(0, u_\varepsilon^0, z_\varepsilon^0) - \int_0^t \langle \dot{\ell}(s), u_\varepsilon(s) \rangle \mathrm{d}s$$

Here, $\mathrm{Diss}_{\mathcal{D}_\varepsilon^{\mathrm{In}}}(z_\varepsilon; [0,t]) := \sup \sum_{j=1}^N \mathcal{D}_\varepsilon^{\mathrm{In}}(z_\varepsilon(t_{j-1}), z_\varepsilon(t_j))$, where for $N \in \mathbb{N}$ the supremum is taken with respect to all finite partitions $\pi_N := \{0 = t_0 < t_1 < \ldots < t_N = t\}$ of the interval $[0,T]$. Following Section 5.2 for $\widetilde{t} \in [0,T]$ by $\mathcal{S}_\varepsilon^{\mathrm{In}}(\widetilde{t})$ the set of stable states is denoted.

$$\mathcal{S}_\varepsilon^{\mathrm{In}}(\widetilde{t}) := \{(u_\varepsilon, z_\varepsilon) \in \mathcal{Q}_\varepsilon^{\mathrm{In}}(\Omega) \text{ satisfying } (\mathrm{S}_{\mathrm{In}}^\varepsilon) \text{ for } t = \widetilde{t} \text{ and } \mathcal{E}_\varepsilon^{\mathrm{In}}(\widetilde{t}, u_\varepsilon, z_\varepsilon) < \infty\}.$$

The following corollary, ensuring the existence of solutions for $(\mathrm{S}_{\mathrm{In}}^\varepsilon)$ and $(\mathrm{E}_{\mathrm{In}}^\varepsilon)$, is a direct consequence of Proposition 6.5.

Corollary 7.8 (Existence of solutions)**.** *Assume that the conditions* (7.1) *and* (7.2) *hold. For* $\ell \in \mathrm{C}^1([0,T]; (\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$ *let* $\mathcal{E}_\varepsilon^{\mathrm{In}} : [0,T] \times \mathcal{Q}_\varepsilon^{\mathrm{In}}(\Omega) \to \mathbb{R}$ *be defined via* (7.11). *Moreover, for* $\kappa_\varepsilon^{\mathrm{In}} \in \mathrm{L}^{q'}(\Omega; [0,\infty)^m)$ *let* $\mathcal{D}_\varepsilon^{\mathrm{In}} : \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m) \times \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m) \to [0,\infty]$ *be given by relation* (7.12).

Then for $(u_\varepsilon^0, z_\varepsilon^0) \in \mathcal{S}_\varepsilon^{\mathrm{In}}(0)$*, there exists an energetic solution* $(u_\varepsilon, z_\varepsilon) : [0,T] \to \mathcal{Q}_\varepsilon^{\mathrm{In}}(\Omega)$ *of the rate-independent system* $(\mathcal{Q}_\varepsilon^{\mathrm{In}}(\Omega), \mathcal{E}_\varepsilon^{\mathrm{In}}, \mathcal{D}_\varepsilon^{\mathrm{In}})$ *satisfying* $(u_\varepsilon(0), z_\varepsilon(0)) = (u_\varepsilon^0, z_\varepsilon^0)$ *and*

$$\begin{aligned} &u_\varepsilon \in \mathrm{L}^\infty([0,T], \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d) \\ &z_\varepsilon \in \mathrm{L}^\infty([0,T], \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)) \cap \mathrm{BV}_{\mathcal{D}_\varepsilon^{\mathrm{In}}}([0,T], \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)). \end{aligned}$$

Proof. To apply Proposition 6.5 we need to ensure the validity of the conditions (6.1), (6.2), (6.5), and (6.9). Due to Lemma 7.1 condition (6.1) holds, and as already sketched in the beginning of this chapter, assumption (7.2d) implies (6.2). Korn's inequality together with assumption (7.1) result in the coercivity condition (6.5); see (7.10). Finally, condition (6.9) trivially holds for all dissipation distances considered in this section. □

7.1.4 Rate independence with respect to another internal variable

Similar to the Examples 7.3, 7.4, and 7.6 this subsection is about ensuring the rate independence of a specific model given by the energetic formulation. As before, the rate independence of an energetic formulation is guaranteed by assuming the underlying dissipation potential to satisfy condition (5.2). However, in contrast to Subsection 7.1.2, here the starting point is the dissipation distance, modeling the amount of dissipated energy needed to switch from one damage state to another. Proceeding in this way seems natural from the modeling point of view. Only after determining the relation of the dissipated energy to the microstructural changes, the internal variable is chosen. This choice is preferably done such that the dissipation distance's structure becomes rather simple. Thus, in best case scenario the underlying dissipation potential is immediately evident. In this case the difficulty of finding the underlying potential is replaced by showing the necessary convergence results for the new internal variable.

Example 7.9. In the following we are going to present an example generalizing Example 7.3. Let $m = 1$ and choose Y as in Example 7.3, i.e.,

$$Y := \Big\{ y \in \mathbb{R}^d \,\Big|\, y = \sum_{i=1}^d k_i b_i, k_i \in [-\tfrac{1}{2}, \tfrac{1}{2}) \Big\}.$$

Let $L : [0,1] \to \mathcal{L}_{\mathrm{Leb}}(Y)$ be given by $L(\widehat{z}) := (1-\widehat{z})D$, where $D \subset Y$ is a closed set which is starshaped with respect to the center of the cell Y and satisfies $1 \geq \mu_d(D) > 0$. Again, we want the dissipated energy to be proportional to the volume change of the damage set $\Omega_\varepsilon^{\mathrm{D}}(z_\varepsilon)$. Hence, the natural candidate for the internal variable is the characteristic function $\mathbb{1}_{\Omega \setminus \Omega_\varepsilon^{\mathrm{D}}(z_\varepsilon)}$, instead of z_ε. Then, the internal variable's function space is

$$\mathbb{X}^D_{\varepsilon\Lambda}(\Omega) := \{ \chi \in \mathrm{L}^\infty(\Omega; \{0,1\}) \,|\, \exists\, z \in \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]) : \chi = \mathbb{1}_{\Omega \setminus \Omega_\varepsilon^{\mathrm{D}}(z)} \}.$$

Observe that for two functions $z_1, z_2 \in \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1])$ satisfying $\mathbb{1}_{\Omega \setminus \Omega_\varepsilon^{\mathrm{D}}(z_1)} = \mathbb{1}_{\Omega \setminus \Omega_\varepsilon^{\mathrm{D}}(z_2)}$ we obtain $z_1|_{\Omega_\varepsilon^-} = z_2|_{\Omega_\varepsilon^-}$ but $z_1 \neq z_2$, in general; see also Remark 2.7. Thus, the dissipation distance $\mathcal{D}_\varepsilon^{\mathrm{Ex}} : \mathbb{X}_{\varepsilon\Lambda}^D(\Omega) \times \mathbb{X}_{\varepsilon\Lambda}^D(\Omega) \to [0,\infty]$ creating proportionality of the dissipated energy and the increase of the damage set is defined via

$$\mathcal{D}_\varepsilon^{\mathrm{Ex}}(\chi_1,\chi_2) := \begin{cases} \int_\Omega \kappa_\varepsilon^{\mathrm{Ex}}(x)|\chi_2(x)-\chi_1(x)|\mathrm{d}x & \text{if } \chi_1 \geq \chi_2, \\ \infty & \text{otherwise,} \end{cases}$$

where $(\kappa_\varepsilon^{\mathrm{Ex}})_{\varepsilon>0} \subset \mathrm{L}^1(\Omega;[0,\infty))$ is assumed to be given such that $\kappa_\varepsilon^{\mathrm{Ex}} \to \kappa_0^{\mathrm{Ex}}$ in $\mathrm{L}^1(\Omega)$ for some $\kappa_0^{\mathrm{Ex}} \in \mathrm{L}^1(\Omega;[0,\infty))$. The underlying dissipation potential $\mathcal{R}_\varepsilon^{\mathrm{Ex}} : \mathbb{X}_{\varepsilon\Lambda}^D(\Omega) \to [0,\infty]$ is obviously given by (see also Remark 5.2)

$$\mathcal{R}_\varepsilon^{\mathrm{Ex}}(v_\varepsilon) := \begin{cases} \int_\Omega \kappa_\varepsilon^{\mathrm{Ex}}(x)|v_\varepsilon(x)|\mathrm{d}x & \text{if } 0 \geq v_\varepsilon, \\ \infty & \text{otherwise.} \end{cases}$$

Since the damage variable is scalar, for a monotone decreasing $\chi_\varepsilon : [0,T] \to \mathbb{X}_{\varepsilon\Lambda}^D(\Omega)$ the total dissipation simplifies to the following expression (see Definition 5.3):

$$\mathrm{Diss}_{\mathcal{D}_\varepsilon^{\mathrm{Ex}}}(\chi_\varepsilon;[0,t]) = \int_\Omega \kappa_\varepsilon^{\mathrm{Ex}}(x)|\chi_\varepsilon(0,x)-\chi_\varepsilon(t,x)|\mathrm{d}x. \tag{7.14}$$

For modeling the stored energy of the system we want to stay with the functional defined by (7.11), replacing only the previous internal variable $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1])$ by $\chi_\varepsilon = \mathbb{1}_{\Omega\setminus\Omega_\varepsilon^{\mathrm{D}}(z_\varepsilon)} \in \mathbb{X}_{\varepsilon\Lambda}^D(\Omega)$. For this purpose, we introduce $Q_\varepsilon : \mathbb{X}_{\varepsilon\Lambda}^D(\Omega) \to \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1])$ by

$$Q_\varepsilon(\chi_\varepsilon) := \max\{z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]) \,|\, \chi_\varepsilon = \mathbb{1}_{\Omega\setminus\Omega_\varepsilon^{\mathrm{D}}(z_\varepsilon)}\} \tag{7.15}$$

identifying a characteristic function $\chi_\varepsilon \in \mathbb{X}_{\varepsilon\Lambda}^D(\Omega)$ with an element z_ε of the set of piecewise constant functions $\mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1])$. The opposite direction is implemented by the identification operator $N_\varepsilon : \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]) \to \mathbb{X}_{\varepsilon\Lambda}^D(\Omega)$ for $x \in \Omega_\varepsilon^-$ defined by

$$N_\varepsilon(z_\varepsilon)(x) := \mathbb{1}_{\mathcal{N}_\varepsilon(x)+\varepsilon(Y\setminus L(z_\varepsilon(x)))}(x) \tag{7.16a}$$

and

$$N_\varepsilon(z_\varepsilon)|_{\Omega\setminus\Omega_\varepsilon^-} :\equiv 1. \tag{7.16b}$$

The operators' properties are stated the following proposition, which is an immediate consequence of the operators' definition.

Proposition 7.10. *Let $Q_\varepsilon : \mathbb{X}_{\varepsilon\Lambda}^D(\Omega) \to \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1])$ and $N_\varepsilon : \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]) \to \mathbb{X}_{\varepsilon\Lambda}^D(\Omega)$ be defined by* (7.15) *and* (7.16)*, respectively. Then:*

(a) *$N_\varepsilon \circ Q_\varepsilon : \mathbb{X}_{\varepsilon\Lambda}^D(\Omega) \to \mathbb{X}_{\varepsilon\Lambda}^D(\Omega)$ is the identity.*

(b) *$Q_\varepsilon \circ N_\varepsilon : \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]) \to \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1])$ is a projection.*

(c) *$(Q_\varepsilon \circ N_\varepsilon(z))|_{\Omega_\varepsilon^-} = z|_{\Omega_\varepsilon^-}$ for all $z \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1])$.*

Remark 7.11. *Since $m = 1$ and $L(z) = (1{-}z)D$, for any given element $\chi_\varepsilon \in \mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$ the value $Q_\varepsilon(\chi_\varepsilon)(x)$ can be determined explicitly by*

$$Q_\varepsilon(\chi_\varepsilon)(x) = \begin{cases} \dfrac{1}{\mu_d(D)} \displaystyle\fint_{\mathcal{N}_\varepsilon(x)+\varepsilon Y} \chi_\varepsilon(y)\mathrm{d}y - \dfrac{1{-}\mu_d(D)}{\mu_d(D)} & \text{for all } x \in \Omega^-_\varepsilon, \\ 1 & \text{for } x \in \Omega\backslash\Omega^-_\varepsilon. \end{cases}$$

Referring to (7.11), the energy functional $\mathcal{E}^{\mathrm{Ex}}_\varepsilon : [0,T]{\times}\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d{\times}\mathbb{X}^D_{\varepsilon\Lambda}(\Omega) \to \mathbb{R}$ is defined by

$$\mathcal{E}^{\mathrm{Ex}}_\varepsilon(t,u,\chi_\varepsilon) := \tfrac{1}{2}\langle \mathbb{C}^{\mathrm{Ex}}_\varepsilon(\chi_\varepsilon)\mathbf{e}(u), \mathbf{e}(u)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} + \|R_{\frac{\varepsilon}{2}}(Q_\varepsilon(\chi_\varepsilon))\|^p_{\mathrm{L}^p(\Omega^+_\varepsilon)^d} - \langle \ell(t), u\rangle,$$

where $\mathbb{C}^{\mathrm{Ex}}_\varepsilon(\chi_\varepsilon)$ for almost every $x \in \Omega$ is given by

$$\mathbb{C}^{\mathrm{Ex}}_\varepsilon(\chi_\varepsilon)(x) := \chi_\varepsilon(x)\mathbb{C}_{\mathrm{strong}} + (1 - \chi_\varepsilon(x))\mathbb{C}_{\mathrm{weak}}. \tag{7.17}$$

Observe that $\mathbb{C}^{\mathrm{Ex}}_\varepsilon(\chi_\varepsilon) = \mathbb{C}^{\mathrm{In}}_\varepsilon(Q_\varepsilon(\chi_\varepsilon))$; see the definition in line (7.8). Finally, for given initial values $(u^0_\varepsilon, \chi^0_\varepsilon) \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d{\times}\mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$ and by exploiting the description (7.14) the systems evolution is modeled by the energetic formulation ($\mathrm{S}^\varepsilon_{\mathrm{Ex}}$) and ($\mathrm{E}^\varepsilon_{\mathrm{Ex}}$).

Stability condition ($\mathrm{S}^\varepsilon_{\mathrm{Ex}}$) and energy balance ($\mathrm{E}^\varepsilon_{\mathrm{Ex}}$) for all $t \in [0,T]$:

$$\mathcal{E}^{\mathrm{Ex}}_\varepsilon(t,u_\varepsilon(t),\chi_\varepsilon(t)) \leq \mathcal{E}^{\mathrm{Ex}}_\varepsilon(t,\widetilde{u},\widetilde{\chi}) + \mathcal{D}^{\mathrm{Ex}}_\varepsilon(\chi_\varepsilon(t),\widetilde{\chi}) \quad \text{for all } (\widetilde{u},\widetilde{\chi}) \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d{\times}\mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$$
$$\mathcal{E}^{\mathrm{Ex}}_\varepsilon(t,u_\varepsilon(t),\chi_\varepsilon(t)) + \int_\Omega \kappa^{\mathrm{Ex}}_\varepsilon|\chi_\varepsilon(0) - \chi_\varepsilon(t)|\mathrm{d}x = \mathcal{E}^{\mathrm{Ex}}_\varepsilon(0,u^0_\varepsilon,\chi^0_\varepsilon) - \int_0^t \langle \dot{\ell}(s), u_\varepsilon(s))\rangle \mathrm{d}s$$

In preparation for the limit investigation $\varepsilon \to 0$ of ($\mathrm{S}^\varepsilon_{\mathrm{Ex}}$) and ($\mathrm{E}^\varepsilon_{\mathrm{Ex}}$) we need to determine the limit tensor of $(\mathbb{C}^{\mathrm{Ex}}_\varepsilon(\chi_\varepsilon))_{\varepsilon>0}$ analogously to Theorem 3.9. Thereby, the assumptions on $(\chi_\varepsilon)_{\varepsilon>0}$ are motivated by available uniform bounds of the energy functionals.

Proposition 7.12. *Let the conditions* (7.1) *and* (7.2) *be fulfilled. Moreover, let $(\chi_\varepsilon)_{\varepsilon>0}$ be a sequence with $\chi_\varepsilon \in \mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$ and $Q_\varepsilon(\chi_\varepsilon) \to z_0$ in $\mathrm{L}^p(\Omega)$ for some $z_0 \in \mathrm{L}^p(\Omega;[0,1])$. Then*

$$\chi_\varepsilon \overset{*}{\rightharpoonup} \mu_d(D)z_0 + (1{-}\mu_d(D)) \text{ in } \mathrm{L}^\infty(\Omega),$$
$$\mathbb{C}^{\mathrm{Ex}}_\varepsilon(\chi_\varepsilon) \xrightarrow{s} \mathbb{C}^{\mathrm{In}}_0(z_0) \text{ in } \mathrm{L}^1(\Omega{\times}Y;\mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}};\mathbb{R}^{d\times d}_{\mathrm{sym}})),$$

where $\mathbb{C}^{\mathrm{Ex}}_\varepsilon(\chi_\varepsilon)$ is defined by (7.17) *and $\mathbb{C}^{\mathrm{In}}_0(z_0)$ for all $(x,y) \in \Omega{\times}Y$ is given by*

$$\mathbb{C}^{\mathrm{In}}_0(z_0)(x,y) := \widehat{\mathbb{C}}^{\mathrm{In}}(z_0(x))(y).$$

Proof. We start by proving $\chi_\varepsilon \overset{*}{\rightharpoonup} \mu_d(D)z_0 + (1{-}\mu_d(D))$ in $\mathrm{L}^\infty(\Omega)$. Thereto, let $(\chi_\varepsilon)_{\varepsilon>0}$ be given such that $\chi_\varepsilon \in \mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$ and $Q_\varepsilon(\chi_\varepsilon) \to z_0$ in $\mathrm{L}^p(\Omega)$. Moreover, let $\varepsilon_0 > 0$ be fixed, let $\varphi \in \mathrm{K}_{\varepsilon_0\Lambda}(\Omega^-_{\varepsilon_0})$, and set $\varepsilon_k := \frac{\varepsilon_0}{2^k}$. By exploiting for all $k \in \mathbb{N}$ that $\varphi \in \mathrm{K}_{\varepsilon_k\Lambda}(\Omega^-_{\varepsilon_0})$ is piecewise constant ($\varphi^{\varepsilon_k\lambda} := \varphi^{\mathrm{ex}}|_{\varepsilon_k(\lambda+Y)}$ for $\lambda \in \Lambda^-_{\varepsilon_k}$) in (7.18a) here below, using the explicit formula of Remark 7.11 in (7.18b), and taking advantage of the fact that

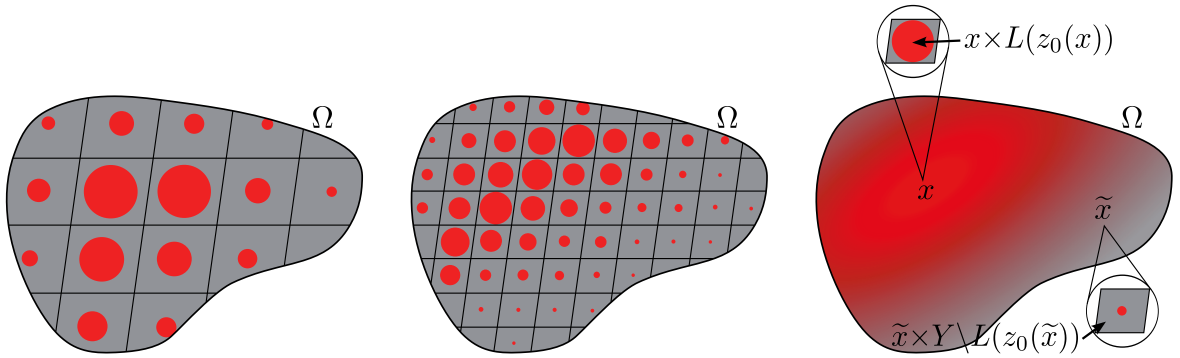


Figure 7.4: Schematic representation of the limit passage of the microscopic one-scale model of Section 7.1 to the two-scale limit model of Section 7.2, where the microscopic inclusions are assumed modeled as in Example 7.3.

$\varphi \in \mathrm{K}_{\varepsilon_k\Lambda}(\Omega^-_{\varepsilon_0})$ and $Q_{\varepsilon_k}(\chi_{\varepsilon_k}) \in \mathrm{K}_{\varepsilon_k\Lambda}(\Omega^-_{\varepsilon_0})$ are piecewise constant in (7.18c), we obtain the following result:

$$\int_\Omega \varphi^{\mathrm{ex}}(x)\chi_{\varepsilon_k}(x)\mathrm{d}x = \sum_{\lambda\in\Lambda^-_{\varepsilon_k}} \varepsilon_k^d \varphi^{\varepsilon_k\lambda}\frac{1}{\varepsilon_k^d}\int_{\varepsilon_k(\lambda+Y)} \chi_{\varepsilon_k}(x)\mathrm{d}x \tag{7.18a}$$

$$= \sum_{\lambda\in\Lambda^-_{\varepsilon_k}} \varepsilon_k^d \varphi^{\varepsilon_k\lambda}\Big(\mu_d(D)Q_{\varepsilon_k}(\chi_{\varepsilon_k})(\varepsilon_k\lambda) + (1-\mu_d(D))\Big) \tag{7.18b}$$

$$= \sum_{\lambda\in\Lambda^-_{\varepsilon_k}} \int_{\varepsilon_k(\lambda+Y)} \varphi^{\mathrm{ex}}(x)\Big(\mu_d(D)Q_{\varepsilon_k}(\chi_{\varepsilon_k})(x) + (1-\mu_d(D))\Big)\mathrm{d}x \tag{7.18c}$$

$$= \int_\Omega \varphi^{\mathrm{ex}}(x)\Big(\mu_d(D)Q_{\varepsilon_k}(\chi_{\varepsilon_k})(x) + (1-\mu_d(D))\Big)\mathrm{d}x. \tag{7.18d}$$

Combining the assumption $Q_{\varepsilon_k}(\chi_{\varepsilon_k}) \to z_0$ in $\mathrm{L}^p(\Omega)$ with (7.18d) for all $\varphi \in \mathrm{K}_{\varepsilon_0\Lambda}(\Omega^-_{\varepsilon_0})$ results in

$$\lim_{\varepsilon_k\to 0}\int_\Omega \varphi^{\mathrm{ex}}(x)\chi_{\varepsilon_k}(x)\mathrm{d}x = \int_\Omega \varphi^{\mathrm{ex}}(x)\Big(\mu_d(D)z_0(x) + (1-\mu_d(D))\Big)\mathrm{d}x.$$

Since any function $v \in \mathrm{L}^1(\Omega)$ can be approximated by a sequence of piecewise constant functions $(\varphi_{\hat\varepsilon})_{\hat\varepsilon>0}$, with $\varphi_{\hat\varepsilon} \in \mathrm{K}_{\hat\varepsilon\Lambda}(\Omega)$, and since the sequence $(Q_\varepsilon(\chi_\varepsilon))_{\varepsilon>0}$ is uniformly bounded in $\mathrm{L}^\infty(\Omega)$, we conclude $\chi_\varepsilon \overset{*}{\rightharpoonup} \mu_d(D)z_0(x) + (1-\mu_d(D))$ in $\mathrm{L}^\infty(\Omega)$.

Due to $\mathbb{C}^{\mathrm{Ex}}_\varepsilon(\chi_\varepsilon) = \mathbb{C}^{\mathrm{In}}_\varepsilon(Q_\varepsilon(\chi_\varepsilon))$, the strong two-scale convergence $\mathbb{C}^{\mathrm{Ex}}_\varepsilon(\chi_\varepsilon) \xrightarrow{s} \mathbb{C}^{\mathrm{In}}_0(z_0)$ in $\mathrm{L}^1(\Omega\times Y; \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}}; \mathbb{R}^{d\times d}_{\mathrm{sym}}))$ is an immediate consequence of Theorem 3.9. □

7.2 Two-scale effective damage model based on the growth of inclusions of weak material

In this section we formulate the two-scale effective damage model $(\mathbf{S}^0_{\mathrm{In}})$ and $(\mathbf{E}^0_{\mathrm{In}})$, which for $\varepsilon \to 0$ according to Theorem 6.18 is the limit model of the microscopic models $(\mathrm{S}^\varepsilon_{\mathrm{In}})$

and $(\mathrm{E}^{\varepsilon}_{\mathrm{In}})$. The stored energy of the system $(\mathbf{S}^0_{\mathrm{In}})$ and $(\mathbf{E}^0_{\mathrm{In}})$ is based on the tensor valued mapping $\mathbb{C}^{\mathrm{In}}_0 : \mathrm{W}^{1,p}(\Omega;[0,1]^m) \to \mathrm{L}^\infty(\Omega\times Y;\{\mathbb{C}_{\mathrm{strong}},\mathbb{C}_{\mathrm{weak}}\})$, which for almost every $(x,y)\in\Omega\times Y$ and any $z_0\in\mathrm{W}^{1,p}(\Omega;[0,1]^m)$ is defined via

$$\mathbb{C}^{\mathrm{In}}_0(z_0)(x,y) := \mathbb{1}_{Y\setminus L(z_0(x))}(y)\mathbb{C}_{\mathrm{strong}} + \mathbb{1}_{L(z_0(x))}(y)\mathbb{C}_{\mathrm{weak}}.$$

Remark 7.13. *By comparing the tensors $\mathbb{C}^{\mathrm{In}}_\varepsilon(z_\varepsilon)$ (see (7.8)) and $\mathbb{C}^{\mathrm{In}}_0(z_0)$ we observe that the microstructure is preserved by shifting it to the second scale in the following sense: Considering a damage variable $z_\varepsilon\in\mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ the damage set of a cell $\varepsilon(\lambda{+}Y)\subset\Omega$ for $z^{\varepsilon\lambda} := z_\varepsilon|_{\varepsilon(\lambda+Y)}$ is given by $\varepsilon(\lambda{+}L(z^{\varepsilon\lambda}))$, whereas in the limit ($z_\varepsilon(x)\to z_0(x)$ for almost every $x\in\Omega$) in almost every point $x\in\Omega$ there is a unit cell $\{x\}\times Y$ containing the damage set $L(z_0(x))$; see also Figure* 7.4.

Referring to Section 6.2 the limit function space $\mathbf{Q}^{\mathrm{In}}_0$ has the following structure:

$$\mathbf{Q}^{\mathrm{In}}_0 := \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d\times\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d\times\mathrm{W}^{1,p}(\Omega;[0,1]^m),$$

where $\mathcal{Y} := \mathbb{R}^d/_\Lambda$ denotes the periodicity cell. For $(u_0,U_1)\in\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d\times\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ we define $\widetilde{\mathbf{e}}(u_0,U_1) := \mathbf{e}_x(u_0)+\mathbf{e}_y(U_1)$. Thus, the stored energy of the two-scale system is modeled by the functional $\mathbf{E}^{\mathrm{In}}_0 : [0,T]\times\mathbf{Q}^{\mathrm{In}}_0\to\mathbb{R}$ defined via

$$\mathbf{E}^{\mathrm{In}}_0(t,u_0,U_1,z_0) := \tfrac12\langle\mathbb{C}^{\mathrm{In}}_0(z_0)\widetilde{\mathbf{e}}(u_0,U_1),\widetilde{\mathbf{e}}(u_0,U_1)\rangle_{\mathrm{L}^2(\Omega\times Y)^{d\times d}} + \|\nabla z_0\|^p_{\mathrm{L}^p(\Omega)^{m\times d}} - \langle\ell(t),u_0\rangle.$$

Checking condition (6.8) for the sequence of functionals $(\mathcal{D}^{\mathrm{In}}_\varepsilon)_{\varepsilon>0}$ given by (7.12) results in a limit dissipation distance $\mathbf{D}^{\mathrm{In}}_0 : \mathrm{W}^{1,p}(\Omega;[0,1]^m)\times\mathrm{W}^{1,p}(\Omega;[0,1]^m)\to[0,\infty]$ given by

$$\mathbf{D}^{\mathrm{In}}_0(z_1,z_2) := \begin{cases} \displaystyle\int_\Omega |\langle\kappa^{\mathrm{In}}_0(x), z_2(x)-z_1(x)\rangle_m|\mathrm{d}x & \text{if } z_1\geq z_2,\\ \infty & \text{otherwise,}\end{cases}$$

where $\kappa^{\mathrm{In}}_0\in\mathrm{L}^{q'}(\Omega;[0,\infty)^m)$ is the same function as chosen in the definition of the microscopic dissipation distance; see (7.12). For given initial values $(u^0_0,U^0_1,z^0_0)\in\mathbf{Q}^{\mathrm{In}}_0$ the rate-independent damage evolution is modeled by the energetic formulation $(\mathbf{S}^0_{\mathrm{In}})$ and $(\mathbf{E}^0_{\mathrm{In}})$.

Stability condition $(\mathbf{S}^0_{\mathrm{In}})$ and energy balance $(\mathbf{E}^0_{\mathrm{In}})$ for all $t\in[0,T]$:

$$\mathbf{E}^{\mathrm{In}}_0(t,u_0(t),U_1(t),z_0(t)) \leq \mathbf{E}^{\mathrm{In}}_0(t,\widetilde{u},\widetilde{U},\widetilde{z}) + \mathbf{D}^{\mathrm{In}}_0(z_0(t),\widetilde{z}) \quad \text{for all } (\widetilde{u},\widetilde{U},\widetilde{z})\in\mathbf{Q}^{\mathrm{In}}_0$$

$$\mathbf{E}^{\mathrm{In}}_0(t,u_0(t),U_1(t),z_0(t)) + \mathrm{Diss}_{\mathbf{D}^{\mathrm{In}}_0}(z_0;[0,t]) = \mathbf{E}^{\mathrm{In}}_0(0,u^0_0,U^0_1,z^0_0) - \int_0^t\langle\dot{\ell}(s),u_0(s)\rangle\mathrm{d}s$$

Here, $\mathrm{Diss}_{\mathbf{D}^{\mathrm{In}}_0}(z_0;[0,t]) := \sup\sum_{j=1}^N\mathbf{D}^{\mathrm{In}}_0(z_0(t_{j-1}),z_0(t_j))$, where for $N\in\mathbb{N}$ the supremum is taken with respect to all finite partitions $\pi_N := \{0=t_0<t_1<\ldots<t_N=t\}$ of the interval $[0,T]$. Analog to Remark 6.7, the existence of a solution of the two-scale damage model is proven by the convergence result of Section 6.5.

Corollary 7.14 (Existence of solutions). *Assume that the conditions* (7.1) *and* (7.2) *hold. Let* $\mathbf{E}_0^{\mathrm{In}} : [0,T]\times\mathbf{Q}_0^{\mathrm{In}} \to \mathbb{R}$ *and* $\mathbf{D}_0^{\mathrm{In}} : \mathrm{W}^{1,p}(\Omega;[0,1]^m)\times\mathrm{W}^{1,p}(\Omega;[0,1]^m) \to [0,\infty]$ *be defined as described above. Let* $(u_0^0, U_1^0, z_0^0) \in \mathbf{Q}_0^{\mathrm{In}}$ *be given such that it is the limit of a stable sequence* $(u_\varepsilon^0, z_\varepsilon^0)_{\varepsilon>0}$ *with respect to* $0 \in [0,T]$ *in the sense of Definition* 6.12. *If* $\nabla u_\varepsilon^0 \xrightarrow{s} \nabla_x E u_0^0 + \nabla_y U_1^0$ *in* $\mathrm{L}^2(\Omega\times Y)^{d\times d}$ *and* $R_{\frac{\varepsilon}{2}} z_\varepsilon^0|_\Omega \to \nabla z_0^0$ *in* $\mathrm{L}^p(\Omega)^{m\times d}$, *then there exists an energetic solution* $(u_0, U_1, z_0) : [0,T] \to \mathbf{Q}_0^{\mathrm{In}}$ *of the rate-independent system* $(\mathbf{Q}_0^{\mathrm{In}}, \mathbf{E}_0^{\mathrm{In}}, \mathbf{D}_0^{\mathrm{In}})$ *with initial condition* (u_0^0, U_1^0, z_0^0) *satisfying*

$$\begin{aligned}(u_0, U_1) &\in \mathrm{L}^\infty([0,T];\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d\times\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d),\\ z_0 &\in \mathrm{L}^\infty([0,T];\mathrm{W}^{1,p}(\Omega;[0,1]^m)) \cap \mathrm{BV}_{\mathbf{D}_0^{\mathrm{In}}}([0,T];\mathrm{W}^{1,p}(\Omega;[0,1]^m)).\end{aligned}$$

Proof. This statement is a direct consequence of Theorem 6.18. Therefore, we need to verify the theorem's assumptions (6.1), (6.2), (6.5), (6.8), and (6.9). As we already saw in the proof of Corollary 7.8 the conditions (7.1) and (7.2) guarantee the validity of (6.1), (6.2), (6.5), and (6.9). Due to Remark 6.4 it is sufficient to verify condition (6.8) for some arbitrary $q \in [1,\infty)$. Choose $q := \frac{q'}{q'-1}$ such that the convergence claimed in (6.8) results from the fact that the dual pairing of a weak converging sequence in $\mathrm{L}^{q'}(\Omega)^m$ and a strong converging sequence in $\mathrm{L}^q(\Omega)^m$ converges to the dual pairing of their limits. □

7.2.1 Two-scale limit energy functional and dissipation distance for Example 7.9

Referring to Example 7.9 it holds $\mathcal{E}_\varepsilon^{\mathrm{Ex}}(t, u_\varepsilon, \chi_\varepsilon) = \mathcal{E}_\varepsilon^{\mathrm{In}}(t, u_\varepsilon, Q_\varepsilon(\chi_\varepsilon))$ for every $\varepsilon > 0$, all $(u_\varepsilon, \chi_\varepsilon) \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d\times\mathbb{X}^D_{\varepsilon\Lambda}(\Omega)$, and any $t \in [0,T]$. Therefore, according to the convergence result Theorem 6.18 the limit energy functional $\mathbf{E}_0^{\mathrm{Ex}} : [0,T]\times\mathbf{Q}_0^{\mathrm{In}} \to \mathbb{R}$ is given by

$$\mathbf{E}_0^{\mathrm{Ex}}(t, u_0, U_1, z_0) := \mathbf{E}_0^{\mathrm{In}}(t, u_0, U_1, z_0)$$

for every $(u_0, U_1, z_0) \in \mathbf{Q}_0^{\mathrm{In}}$ and all $t \in [0,T]$. Adapting condition (6.8) to the "new" internal variable and exploiting the convergence result of Proposition 7.12 yields that the limit dissipation distance $\mathbf{D}_0^{\mathrm{Ex}} : \mathrm{W}^{1,p}(\Omega;[0,1])\times\mathrm{W}^{1,p}(\Omega;[0,1]) \to [0,\infty]$ for the limit function $\kappa_0^{\mathrm{Ex}} \in \mathrm{L}^1(\Omega;[0,\infty))$ is defined by

$$\mathbf{D}_0^{\mathrm{Ex}}(z_1, z_2) := \begin{cases} (1-\mu_d(D)) \displaystyle\int_\Omega \kappa_0^{\mathrm{Ex}}(x)|z_2(x) - z_1(x)|\mathrm{d}x & \text{if } z_1 \geq z_2,\\ \infty & \text{otherwise.}\end{cases}$$

7.3 One-scale effective damage model based on the growth of inclusions of weak material

For the sake of completeness the one-scale model being equivalent to the two-scale model of Section 7.2 is formulated. Let $\mathcal{Q}_0^{\mathrm{In}}(\Omega)$ denote the state space defined via

$$\mathcal{Q}_0^{\mathrm{In}}(\Omega) := \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d\times\mathrm{W}^{1,p}(\Omega;[0,1]^m).$$

For a given function $z_0 \in \mathrm{W}^{1,p}(\Omega;[0,1]^m)$ the tensor $\widehat{\mathbb{C}}^{\mathrm{In}}(z_0(x)) \in \mathrm{L}^\infty(Y;\{\mathbb{C}_{\mathrm{strong}}, \mathbb{C}_{\mathrm{weak}}\})$ for almost every $(x,y) \in \Omega{\times}Y$ is given by

$$\widehat{\mathbb{C}}^{\mathrm{In}}(z_0(x)) = \mathbb{1}_{Y\setminus L(z_0(x))}(y)\mathbb{C}_{\mathrm{strong}} + \mathbb{1}_{L(z_0(x))}(y)\mathbb{C}_{\mathrm{weak}};$$

see (7.3). Since assumption (7.1) ensures the validity of condition (6.3), Proposition 6.8 yields that for $\xi \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ the unit cell problem

$$\langle \mathbb{C}^{\mathrm{In}}_{\mathrm{eff}}(z_0)(x)\xi,\xi\rangle_{d\times d} := \min_{v\in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d} \int_Y \langle \widehat{\mathbb{C}}^{\mathrm{In}}(z_0(x))(y)(\xi{+}\mathbf{e}_y(v)(y)),\xi{+}\mathbf{e}_y(v)(y)\rangle_{d\times d}\mathrm{d}y \quad (7.19)$$

defines a mapping $\mathbb{C}^{\mathrm{In}}_{\mathrm{eff}} : \mathrm{W}^{1,p}(\Omega;[0,1]^m) \to \mathcal{M}(\Omega)$. Thus, the one-scale model is based on the one-scale energy functional $\mathcal{E}^{\mathrm{In}}_0 : [0,T]{\times}\mathcal{Q}^{\mathrm{In}}_0(\Omega) \to \mathbb{R}$ defined in the following way:

$$\mathcal{E}^{\mathrm{In}}_0(t,u_0,z_0) := \tfrac{1}{2}\langle \mathbb{C}^{\mathrm{In}}_{\mathrm{eff}}(z_0)\mathbf{e}(u_0),\mathbf{e}(u_0)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} + \|\nabla z_0\|^p_{\mathrm{L}^p(\Omega)^{m\times d}} - \langle \ell(t),u_0\rangle.$$

Furthermore, for the same function $\kappa^{\mathrm{In}}_0 \in \mathrm{L}^{q'}(\Omega;[0,\infty)^m)$ as chosen in the definition of the microscopic dissipation distance given by (7.12), the limit dissipation distance $\mathcal{D}^{\mathrm{In}}_0 : \mathrm{W}^{1,p}(\Omega;[0,1]^m){\times}\mathrm{W}^{1,p}(\Omega;[0,1]^m) \to [0,\infty]$ is given by

$$\mathcal{D}^{\mathrm{In}}_0(z_1,z_2) = \begin{cases} \int_\Omega |\langle \kappa^{\mathrm{In}}_0(x), z_2(x)-z_1(x)\rangle_m|\mathrm{d}x & \text{if } z_1 \geq z_2, \\ \infty & \text{otherwise.} \end{cases}$$

For given initial values $(u^0_0,z^0_0) \in \mathcal{Q}^{\mathrm{In}}_0(\Omega)$ the energetic formulation $(\mathrm{S}^0_{\mathrm{In}})$ and $(\mathrm{E}^0_{\mathrm{In}})$ of the rate-independent system $(\mathcal{Q}^{\mathrm{In}}_0(\Omega),\mathcal{E}^{\mathrm{In}}_0,\mathcal{D}^{\mathrm{In}}_0)$ reads as follows:

Stability condition $(\mathrm{S}^0_{\mathrm{In}})$ **and energy balance** $(\mathrm{E}^0_{\mathrm{In}})$ **for all** $t \in [0,T]$**:**

$$\mathcal{E}^{\mathrm{In}}_0(t,u_0(t),z_0(t)) \leq \mathcal{E}^{\mathrm{In}}_0(t,\widetilde{u},\widetilde{z}) + \mathcal{D}^{\mathrm{In}}_0(z_0(t),\widetilde{z}) \quad \text{for all } (\widetilde{u},\widetilde{z}) \in \mathcal{Q}^{\mathrm{In}}_0(\Omega)$$

$$\mathcal{E}^{\mathrm{In}}_0(t,u_0(t),z_0(t)) + \mathrm{Diss}_{\mathcal{D}^{\mathrm{In}}_0}(z_0;[0,t]) = \mathcal{E}^{\mathrm{In}}_0(0,u^0_0,z^0_0) - \int_0^t \langle \dot{\ell}(s),u_0(s)\rangle\mathrm{d}s$$

Here, $\mathrm{Diss}_{\mathcal{D}^{\mathrm{In}}_0}(z_0;[0,t]) := \sup\sum_{j=1}^N \mathcal{D}^{\mathrm{In}}_0(z_0(t_{j-1}),z_0(t_j))$, where for $N \in \mathbb{N}$ the supremum is taken with respect to all finite partitions $\pi_N := \{0 = t_0 < t_1 < \ldots < t_N = t\}$ of the interval $[0,T]$.

Corollary 7.15 (Existence of solutions)**.** *Assume that the conditions* (7.1) *and* (7.2) *hold. Let* $\mathcal{E}^{\mathrm{In}}_0 : [0,T]{\times}\mathcal{Q}^{\mathrm{In}}_0(\Omega) \to \mathbb{R}$ *and* $\mathcal{D}^{\mathrm{In}}_0 : \mathrm{W}^{1,p}(\Omega;[0,1]^m){\times}\mathrm{W}^{1,p}(\Omega;[0,1]^m) \to [0,\infty]$ *be defined as described above. Let* $(u^0_0,U^0_1,z^0_0) \in \mathbf{Q}^{\mathrm{In}}_0$ *be given such that it is the limit of a stable sequence* $(u^0_\varepsilon,z^0_\varepsilon)_{\varepsilon>0}$ *with respect to* $0 \in [0,T]$ *in the sense of Definition* 6.12*. If* $\nabla u^0_\varepsilon \xrightarrow{s} \nabla_x E u^0_0 + \nabla_y U^0_1$ *in* $\mathrm{L}^2(\Omega{\times}Y)^{d\times d}$ *and* $R_{\frac{\varepsilon}{2}} z^0_\varepsilon|_\Omega \to \nabla z^0_0$ *in* $\mathrm{L}^p(\Omega)^{m\times d}$*, then there exists an energetic solution* $(u_0,z_0) : [0,T] \to \mathcal{Q}^{\mathrm{In}}_0(\Omega)$ *of the rate-independent system* $(\mathcal{Q}^{\mathrm{In}}_0(\Omega),\mathcal{E}^{\mathrm{In}}_0,\mathcal{D}^{\mathrm{In}}_0)$ *with initial condition* (u^0_0,z^0_0) *satisfying*

$$u_0 \in \mathrm{L}^\infty([0,T];\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d),$$
$$z_0 \in \mathrm{L}^\infty([0,T];\mathrm{W}^{1,p}(\Omega;[0,1]^m)) \cap \mathrm{BV}_{\mathcal{D}^{\mathrm{In}}_0}([0,T];\mathrm{W}^{1,p}(\Omega;[0,1]^m)).$$

Proof. Subject to the assumptions of Corollary 7.15 there exists an energetic solution $(u_0, U_1, z_0) : [0,T] \to \mathbf{Q}_0^{\mathrm{In}}$ of the rate-independent system $(\mathbf{Q}_0^{\mathrm{In}}, \mathbf{E}_0^{\mathrm{In}}, \mathbf{D}_0^{\mathrm{In}})$ with initial condition (u_0^0, U_1^0, z_0^0); see Corollary 7.14. Since condition (7.1) implies (6.3), Theorem 6.9 states the existence of an energetic solution $(u_0, z_0) : [0,T] \to \mathcal{Q}_0^{\mathrm{In}}(\Omega)$ of the rate-independent system $(\mathcal{Q}_0^{\mathrm{In}}(\Omega), \mathcal{E}_0^{\mathrm{In}}, \mathcal{D}_0^{\mathrm{In}})$ with initial condition (u_0^0, z_0^0) and the proof is concluded. □

7.4 Discussion of the results

Summarizing the results of this chapter, we are able to provide existence of solutions for effective models modeling rate-independent damage progression caused by inclusions of weak material. These effective models allow for various inclusions' geometries and present a constitutive relation of the limit damage variable and the effective material tensor, which is uniquely described by the unit cell problem (7.19) (one scale effective model). Due to the asymptotic analysis of Chapter 6, this constitutive relation is rigorously derived from the modeling of microscopic inclusions of weak material in a bulk of undamaged material. Comparing this homogenization result with the damage model of [24] shows that there has to be some kind of microstructure regularization in the microscopic models to obtain this unique relation of the limit damage variable and the effective material tensor, as we will see below.

In [24] the authors prove existence of solutions for the so-called *energy minimization problem* which is based on the following energy functional:

$$\mathcal{E}^{\mathrm{GL}}(t, \mathbb{C}(t), u(t), z(t)) := \tfrac{1}{2}\langle \mathbb{C}(t)\mathbf{e}(u(t)), \mathbf{e}(u(t))\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} + k\|z(t)\|_{\mathrm{L}^1(\Omega)} - \langle \ell(t), u(t)\rangle.$$

Here, the first term accounts for the stored elastic energy of the system, the second term models the dissipated energy in dependence on the change of the damage variable, and the last term denotes the energy caused by the external loadings. Comparing this functional with the model of Section 7.3 reveals that this energy functional lacks any regularization with respect to the damage variable. Moreover, here the material tensor $\mathbb{C}$ occurs as a variable, i.e., there is no constitutive relation determining $\mathbb{C}(t,x)$ uniquely in dependence on the damage variable's value $z(t,x)$.

Now, the function $(\mathbb{C}, u, z) : [0,T]\times\Omega \to \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}}; \mathbb{R}^{d\times d}_{\mathrm{sym}})\times\mathbb{R}^d\times[0,1]$ is a solution of the energy minimization problem if the following three conditions are fulfilled:

1. $(\mathbb{C}, z) : [0,T]\times\Omega \to \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}}; \mathbb{R}^{d\times d}_{\mathrm{sym}})\times[0,1]$ is monotonically decreasing as a function of $t \in [0,T]$.

2. For all $t \in [0,T]$ the function $u(t) \in \mathrm{H}^1_0(\Omega)^d$ is a solution of $-\mathrm{div}(\mathbb{C}(t)\mathbf{e}(u(t))) = \ell(t)$ in Ω and the following energy balance holds

$$\mathcal{E}^{\mathrm{GL}}(t, \mathbb{C}(t), u(t), z(t)) = \mathcal{E}^{\mathrm{GL}}(0, \mathbb{C}(0), u(0), z(0)) - \int_0^t \langle \dot{\ell}(s), u(s)\rangle \mathrm{d}s.$$

3. For every $t \in [0,T]$ and for all admissible $(\widetilde{\mathbb{C}}, \widetilde{u}, \widetilde{z})$ we have

$$\mathcal{E}^{\mathrm{GL}}(t, \mathbb{C}(t), u(t), z(t)) \leq \mathcal{E}^{\mathrm{GL}}(t, \widetilde{\mathbb{C}}, \widetilde{u}, \widetilde{z})$$

and there exists a sequence $(\chi_n)_{n\in\mathbb{N}}$ of characteristic functions $\chi_n : [0,T]\times\Omega \to \{0,1\}$ which are monotonically decreasing with respect to time such that it holds

$$\begin{cases} \chi_n(t) \overset{*}{\rightharpoonup} z(t), \\ \chi_n(t)\mathbb{C}_{\mathrm{strong}} + (1-\chi_n(t))\mathbb{C}_{\mathrm{weak}} \overset{G}{\to} \mathbb{C}(t). \end{cases}$$

Here, the sets $\mathrm{supp}(\chi_n)$ and $\mathrm{supp}(1-\chi_n)$ might be interpreted as microscopic distributions of undamaged and damaged material leading to the effective energy minimization problem. Since besides condition 3 there are no further assumptions on the characteristic functions χ_n, in contrast to our microscopic models (see Section 7.1.4, for instance) this model allows for arbitrary distributions of damaged and undamaged material.

However, as a result of this generality and the lack of a microstructure regularization there is no relation determining the material tensor $\mathbb{C}(t)$ uniquely in dependence on the damage variable $z(t)$. In fact, for every $t \in [0,T]$ and any point $x \in \Omega$ the material tensor $\mathbb{C}(t,x)$ is an element of the so-called *G-closure* of the two constant tensors $\mathbb{C}_{\mathrm{strong}}$ and $\mathbb{C}_{\mathrm{weak}}$ with the volume fraction $z(t,x)$. That means that the material tensor $\mathbb{C}(t,x)$ is given by a unit cell problem similar to (7.19), where the set $Y\backslash L(z(t,x))$ of undamaged material could be any subset of Y with the volume fraction $z(t,x) \in [0,1]$, i.e., the geometry of $L(z(t,x))$ is not prescribed. This shows that if one is interested in improving the constitutive relation between the damage variable and the effective material tensor, some kind of microstructure regularization is needed.

The aim of future tasks is the exploitation of the here presented results for numerical simulations of the damage progression in complex structures. In this context the effective models provide a large degree of freedom with respect to the choice of the inclusions' geometry. Moreover, the effective models separate the microscopic and macroscopic scale. By phenomenologically motivating the macroscopic quantities, similar models possessing separated scales have been investigated in the engineering community already; see [35, 36, 37], for instance. There, the authors provide numerical results for a two-scale damage model allowing the evolution of microscopic ellipsoidal inclusions of weak material. However, in contrast to the here presented rigorously derived effective models there the considered homogenized quantities are obtained by averaging corresponding microscopic ones. Moreover, in [35, 36, 37] no regularization with respect to the damage variable is considered which should result in a different limit model; see also the discussion on the model of [24] above. Due to the here presented rigorous derivation of the effective models presented in Section 7.2 and 7.3 it would be interesting to see if numerical simulations of these models yield better results compared to those presented in [35, 36, 37].

8 Effective damage models based on the unidirectional evolution of microscopic defects

As in the previous chapter, we are here going to investigate the asymptotic behavior of a family of brutal microscopic models. The main difference to Chapter 7 is that here damage progression does not increase the amount of damaged material but enlarges the size of preexisting defects, which are modeled by holes in the set being associated to the considered body. Analytically, this is modeled by setting the tensor modeling the weak material in Chapter 7 to zero. As already discussed in Section 2.5 (see Remark 2.5) this causes some technicalities forcing us to adapt the previous chapter's assumptions made on the microstructure determining mapping $L : [0,1]^m \to \mathcal{L}_{\mathrm{Leb}}(Y)$. Since $\mathbb{C}_{\mathrm{weak}} = \mathbb{O}$, the positive definiteness is now assumed on $\mathbb{C}_{\mathrm{strong}} \in \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}}; \mathbb{R}^{d\times d}_{\mathrm{sym}})$, only, i.e., there exists a positive constant α such that

$$\text{for all } \xi \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \text{ it holds} \qquad \alpha|\xi|^2_{d\times d} \le \langle \mathbb{C}_{\mathrm{strong}}\xi, \xi\rangle_{d\times d}. \tag{8.1}$$

The mapping $L : [0,1]^m \to \mathcal{L}_{\mathrm{Leb}}(Y)$ is adapted by the following assumptions:

- $L : [0,1]^m \to \mathcal{L}_{\mathrm{Leb}}(Y)$ is a non-increasing function; see (2.21). (8.2a)
- For all $\hat{z} \in [0,1]^m$ it holds $\mu_d(L(\hat{z})) > 0$. (8.2b)
- For all $\hat{z} \in [0,1]^m$ the set $L(\hat{z})$ is a closed subset of Y. (8.2c)
- For all $\hat{z} \in [0,1]^m$ the set $L(\hat{z})$ has a locally Lipschitz boundary $\partial L(\hat{z})$ (see Definition 2.1) and it holds $\mathrm{dist}(L(\mathbf{0}), \partial Y) > 0$. (8.2d)

For any given $\hat{z} \in [0,1]^m$ and every $(\hat{z}_\delta)_{\delta>0} \subset [0,1]^m$ satisfying $\hat{z}_\delta \to \hat{z}$ in $\mathbb{R}^m$ it holds

- $\mu_d(L(\hat{z})\backslash L(\hat{z}_\delta)) + \mu_d(L(\hat{z}_\delta)\backslash L(\hat{z})) \to 0$ for $\delta \to 0$ and (8.2e)
- $\forall \Delta > 0\, \exists\, \delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ it holds $L(\hat{z}_\delta) \subset \mathrm{neigh}_\Delta(L(\hat{z}))$. (8.2f)

There exist bi-Lipschitz transformations $(T_{\hat{z}})_{\hat{z}\in[0,1]^m}$, $T_{\hat{z}} : Y \to \mathbb{R}^d$, such that

- $\sup\limits_{\hat{z}\in[0,1]^m} \Big(\|\nabla T_{\hat{z}}\|_{\mathrm{L}^\infty(Y)^{d\times d}} + \|\nabla T_{\hat{z}}^{-1}\|_{\mathrm{L}^\infty(\mathrm{Im}(T_{\hat{z}}))^{d\times d}}\Big) =: C_T < \infty.$ (8.2g)
- For all $\hat{z} \in [0,1]^m$ it holds $\mathrm{Im}(T_{\hat{z}}|_{L(\hat{z})}) = L(\mathbf{0})$ as well as $Y \subset \mathrm{Im}(T_{\hat{z}})$. (8.2h)
- For any given $\hat{z} \in [0,1]^m$ and for every $(\hat{z}_\delta)_{\delta>0} \subset [0,1]^m$ with $\hat{z}_\delta \to \hat{z}$ in $\mathbb{R}^m$ it holds $T_{\hat{z}_\delta} \to T_{\hat{z}}$ pointwise in Y. (8.2i)

Observe that we do not assume $\mathrm{Im}(T_{\hat{z}}) = Y$, which simplifies the construction of the transformations $(T_{\hat{z}})_{\hat{z}\in[0,1]^m}$ in the specific cases considered in Subsection 8.1.4. Moreover, condition (8.2i) asks for some continuity of the family $(T_{\hat{z}})_{\hat{z}\in[0,1]^m}$ with respect to $\widehat{z} \in [0,1]^m$. However, note that normally these transformations are constructed such that (8.2i) is automatically fulfilled, as we will see in Subsection 8.1.4.

Following Chapter 7 for the given function $L : [0,1]^m \to \mathcal{L}_{\mathrm{Leb}}(Y)$ the tensor valued mapping $\widehat{\mathbb{C}}^{\mathrm{H}} : [0,1]^m \to \mathrm{L}^\infty(Y; \{\mathbb{C}_{\mathrm{strong}}, \mathbb{O}\})$ for $\widehat{z} \in [0,1]^m$ and every $y \in Y$ is introduced via

$$\widehat{\mathbb{C}}^{\mathrm{H}}(\widehat{z})(y) := \mathbb{1}_{Y\setminus L(\hat{z})}(y)\mathbb{C}_{\mathrm{strong}}. \tag{8.3}$$

Since the "material" tensor $\mathbb{C}_{\mathrm{weak}}$ is set to zero in this chapter, the superscript H refers to the modeling of holes. As in the previous chapter, (8.1) and (8.2) ensure the crucial conditions (6.1) and (6.2) of the homogenization theory presented in Chapter 6 made on the tensor valued mapping $\widehat{\mathbb{C}}^{\mathrm{H}} : [0,1]^m \to \mathrm{L}^\infty(Y; \{\mathbb{C}_{\mathrm{strong}}, \mathbb{O}\})$ to hold. Among other things the additional assumptions (8.2d), (8.2g), and (8.2h) in comparison to Chapter 7 are made to prove a uniform coercivity condition for the microscopic models introduced in Section 8.1. In preparation for this coercivity condition the existence of an extension operator is stated in Lemma 8.2 below. However, before that, we note that the additional assumption (8.2d) together with (8.2e) do not imply (8.2f). This is shown by the example illustrated in Figure 7.1(i). Moreover, we are going to show that the conditions (8.2a), (8.2e), (8.2g), and (8.2h) imply assumption (8.2b), if additionally $\mu_d(L(\mathbf{0})) > 0$ is assumed.

Proposition 8.1. *Let $L : [0,1]^m \to \mathcal{L}_{\mathrm{Leb}}(Y)$ be given, satisfying* (8.2a)*,* (8.2e)*,* (8.2g)*, and* (8.2h)*. If additionally it holds $\mu_d(L(\mathbf{0})) > 0$, then condition* (8.2b) *is fulfilled.*

Proof. The statement is proven via a contradiction argument. For this purpose, we start with some a priori estimates. Due to (8.2g) there exist positive constants C_1 and C_2 such that

$$\sup_{\hat{z}\in[0,1]^m} \|\det(\nabla T_{\hat{z}})\|_{\mathrm{L}^\infty(Y)} \le C_1 \qquad \text{and} \qquad \sup_{\hat{z}\in[0,1]^m} \|\det(\nabla T_{\hat{z}}^{-1})\|_{\mathrm{L}^\infty(\mathrm{Im}(T_{\hat{z}}))} \le C_2. \tag{8.4}$$

Moreover, since for all $\widehat{z} \in [0,1]^m$ the mapping $T_{\hat{z}} : Y \to \mathbb{R}^d$ is assumed to be a bi-Lipschitz transformation, for almost every $x \in \mathrm{Im}(T_{\hat{z}})$ we have

$$|\det(\nabla T_{\hat{z}}^{-1}(x))\det(\nabla T_{\hat{z}}(T_{\hat{z}}^{-1}(x)))| = 1.$$

Combining this equality with the a priori estimates given by (8.4) for any $\widehat{z} \in [0,1]^m$ and almost every $x \in \mathrm{Im}(T_{\hat{z}})$ results in

$$C_1|\det(\nabla T_{\hat{z}}^{-1}(x))| \ge 1. \tag{8.5}$$

Now we are going to produce a contradiction by assuming that $\mu_d(L(\mathbf{1})) = 0$. Choose $(\widehat{z}_\delta)_{\delta>0} \subset [0,1]^m$ with $\widehat{z}_\delta \nearrow \mathbf{1}$ in $\mathbb{R}^m$. Then, $\mu_d(L(\widehat{z}_\delta)) = \mu_d(L(\widehat{z}_\delta)\setminus L(\mathbf{1})) \to 0$ according to (8.2e). On the other hand, by applying the transformation $T_{\hat{z}_\delta}|_{L(\hat{z}_\delta)} : L(\widehat{z}_\delta) \to L(\mathbf{0})$ for fixed $\delta > 0$ to the integral $\int_{L(\hat{z}_\delta)} 1\mathrm{d}y$, we finally end up with

$$\mu_d(L(\widehat{z}_\delta)) = \int_{L(\hat{z}_\delta)} 1\mathrm{d}y \overset{(8.2\mathrm{h})}{=} \int_{L(\mathbf{0})} |\det(\nabla T_{\hat{z}_\delta}^{-1}(x))|\mathrm{d}x \overset{(8.5)}{\geq} \tfrac{1}{C_1}\mu_d(L(\mathbf{0})) > 0.$$

Since the left hand side converges to 0 for $\delta \to 0$, this estimate is a contradiction to $\mu_d(L(\mathbf{0})) > 0$ and hence the assumption $\mu_d(L(\mathbf{1})) = 0$ was wrong. Hence, $\mu_d(L(\mathbf{1})) > 0$ has to hold, which implies the validity of condition (8.2b) by keeping assumption (8.2a) in mind. ☐

Lemma 8.2. *Assume that $L : [0,1]^m \to \mathcal{L}_{\mathrm{Leb}}(Y)$ satisfies the conditions* (8.2d), (8.2g), *and* (8.2h). *Then for every $\widehat{z} \in [0,1]^m$ there exists a linear strong* 1*-extension operator $\mathcal{X}_{\hat{z}} : \mathrm{H}^1(Y \backslash L(\widehat{z}))^d \to \mathrm{H}^1(Y)^d$ and there exists a constant $C_{\mathcal{X}} > 0$ being independent of $\widehat{z} \in [0,1]^m$ such that for all $v_{\hat{z}} \in \mathrm{H}^1(Y\backslash L(\widehat{z}))^d$ it holds*

$$\|\mathcal{X}_{\hat{z}}(v_{\hat{z}})\|_{\mathrm{L}^2(Y)^d} \leq C_{\mathcal{X}}\|v_{\hat{z}}\|_{\mathrm{L}^2(Y\backslash L(\hat{z}))^d} \quad \textit{and} \quad \|\nabla\mathcal{X}_{\hat{z}}(v_{\hat{z}})\|_{\mathrm{L}^2(Y)^{d\times d}} \leq C_{\mathcal{X}}\|\nabla v_{\hat{z}}\|_{\mathrm{L}^2(Y\backslash L(\hat{z}))^{d\times d}}.$$

Proof. For fixed $\widehat{z} \in [0,1]^m$ the continuation operator $\mathcal{X}_{\hat{z}} : \mathrm{H}^1(Y\backslash L(\widehat{z}))^d \to \mathrm{H}^1(Y)^d$ is constructed as follows: First a given function $v_{\hat{z}} \in \mathrm{H}^1(Y\backslash L(\widehat{z}))^d$ is transformed to the domain $\mathrm{Im}(T_{\hat{z}})\backslash L(\mathbf{0})$. Then, this transformed function is extended across the hole $L(\mathbf{0})$ and afterwards it is rescaled again. This construction is based on the existence of a strong 1-extension operator $\mathcal{X}_0 : \mathrm{H}^1(Y\backslash L(\mathbf{0}))^d \to \mathrm{H}^1(Y)^d$, which is ensured by assumption (8.2d); see Theorem 5.24 in [1], for instance. Hence, there exists some constant $C_{\mathcal{X}_0} > 0$ such that for all $w \in \mathrm{H}^1(Y\backslash L(\mathbf{0}))^d$ the inequalities

$$\|\mathcal{X}_0(w)\|_{\mathrm{L}^2(Y)^d} \leq C_{\mathcal{X}_0}\|w\|_{\mathrm{L}^2(Y\backslash L(\mathbf{0}))^d}, \tag{8.6a}$$
$$\|\nabla\mathcal{X}_0(w)\|_{\mathrm{L}^2(Y)^{d\times d}} \leq C_{\mathcal{X}_0}\|\nabla w\|_{\mathrm{L}^2(Y\backslash L(\mathbf{0}))^{d\times d}} \tag{8.6b}$$

hold. Let $\widehat{z} \in [0,1]^m$ be arbitrary but fixed and observe that due to assumption (8.2h) we have $Y\backslash L(\mathbf{0}) \subset \mathrm{Im}(T_{\hat{z}})\backslash L(\mathbf{0})$. For $\widehat{w}_{\hat{z},0} \in \mathrm{H}^1(\mathrm{Im}(T_{\hat{z}})\backslash L(\mathbf{0}))^d$ the continuation operator $\mathcal{X}_0 : \mathrm{H}^1(Y\backslash L(\mathbf{0}))^d \to \mathrm{H}^1(Y)^d$ enables us to construct a strong 1-extension operator $\widehat{\mathcal{X}}_{0,\hat{z}} : \mathrm{H}^1(\mathrm{Im}(T_{\hat{z}})\backslash L(\mathbf{0}))^d \to \mathrm{H}^1(\mathrm{Im}(T_{\hat{z}}))^d$ via

$$\widehat{\mathcal{X}}_{0,\hat{z}}(\widehat{w}_{\hat{z},0}) = \begin{cases} \mathcal{X}_0(\widehat{w}_{\hat{z},0}|_{Y\backslash L(\mathbf{0})}) & \text{on } Y, \\ \widehat{w}_{\hat{z},0} & \text{on } \mathrm{Im}(T_{\hat{z}})\backslash Y. \end{cases} \tag{8.7}$$

Observe that by this definition for all functions $\widehat{w}_{\hat{z},0} \in \mathrm{H}^1(\mathrm{Im}(T_{\hat{z}})\backslash L(\mathbf{0}))^d$ it holds $\widehat{\mathcal{X}}_{0,\hat{z}}(\widehat{w}_{\hat{z},0})|_{\mathrm{Im}(T_{\hat{z}})\backslash L(\mathbf{0})} = \widehat{w}_{\hat{z},0}$ and $\|\widehat{\mathcal{X}}_{0,\hat{z}}(\widehat{w}_{\hat{z},0})\|_{\mathrm{H}^1(\mathrm{Im}(T_{\hat{z}}))^d} \leq (C_{\mathcal{X}_0}+1)\|\widehat{w}_{\hat{z},0}\|_{\mathrm{H}^1(\mathrm{Im}(T_{\hat{z}})\backslash L(\mathbf{0}))^d}$, where $C_{\mathcal{X}_0} > 0$ is the constant of (8.6b), which is independent of $\widehat{z} \in [0,1]^m$. Introducing the transformations

$$\mathbf{T}_{\hat{z}} : \begin{cases} \mathrm{H}^1(\mathrm{Im}(T_{\hat{z}}))^d \to \mathrm{H}^1(Y)^d, \\ \qquad \widetilde{w}_{\hat{z}} \mapsto \widetilde{w}_{\hat{z}} \circ T_{\hat{z}} \end{cases} \quad \text{and} \quad \mathbb{T}_{\hat{z}} : \begin{cases} \mathrm{H}^1(Y\backslash L(\widehat{z}))^d \to \mathrm{H}^1(\mathrm{Im}(T_{\hat{z}})\backslash L(\mathbf{0}))^d, \\ \qquad v_{\hat{z}} \mapsto v_{\hat{z}} \circ T_{\hat{z}}^{-1}|_{\mathrm{Im}(T_{\hat{z}})\backslash L(\mathbf{0})} \end{cases} \tag{8.8}$$

the desired operator $\mathcal{X}_{\hat{z}} : \mathrm{H}^1(Y \backslash L(\hat{z}))^d \to \mathrm{H}^1(Y)^d$ is given by

$$\mathcal{X}_{\hat{z}}(v_{\hat{z}}) := \mathbf{T}_{\hat{z}}(\widehat{\mathcal{X}_{0,\hat{z}}}(\mathbb{T}_{\hat{z}}(v_{\hat{z}}))). \tag{8.9}$$

By definition for all $v_{\hat{z}} \in \mathrm{H}^1(Y \backslash L(\hat{z}))^d$ we have $\mathcal{X}_{\hat{z}}(v_{\hat{z}})|_{Y\backslash L(\hat{z})} = v_{\hat{z}}$. To prove the inequalities stated in Lemma 8.2, observe that estimating $\|\mathcal{X}_{\hat{z}}(v_{\hat{z}})\|_{\mathrm{L}^2(Y)^d}$ and $\|\nabla\mathcal{X}_{\hat{z}}(v_{\hat{z}})\|_{\mathrm{L}^2(Y)^{d\times d}}$ is done in a similar way, which is why we focus on the more complicated latter term. For this purpose, we start by decomposing Y into the disjoint sets $\mathrm{Im}(T_{\hat{z}}^{-1}|_Y)$ and $Y\backslash \mathrm{Im}(T_{\hat{z}}^{-1}|_Y)$ such that

$$\|\nabla\mathcal{X}_{\hat{z}}(v_{\hat{z}})\|^2_{\mathrm{L}^2(Y)^{d\times d}} = \|\nabla\mathcal{X}_{\hat{z}}(v_{\hat{z}})\|^2_{\mathrm{L}^2(\mathrm{Im}(T_{\hat{z}}^{-1}|_Y))^{d\times d}} + \|\nabla\mathcal{X}_{\hat{z}}(v_{\hat{z}})\|^2_{\mathrm{L}^2(Y\backslash\mathrm{Im}(T_{\hat{z}}^{-1}|_Y))^{d\times d}}.$$

Note that $\mathrm{Im}(T_{\hat{z}}^{-1}|_Y) \supset \mathrm{Im}(T_{\hat{z}}^{-1}|_{L(\mathbf{0})}) = L(\hat{z})$ yields $\mathcal{X}_{\hat{z}}(v_{\hat{z}})|_{Y\backslash\mathrm{Im}(T_{\hat{z}}^{-1}|_Y)} = v_{\hat{z}}|_{Y\backslash\mathrm{Im}(T_{\hat{z}}^{-1}|_Y)}$, which immediately gives

$$\|\nabla\mathcal{X}_{\hat{z}}(v_{\hat{z}})\|^2_{\mathrm{L}^2(Y\backslash\mathrm{Im}(T_{\hat{z}}^{-1}|_Y))^{d\times d}} \leq \|\nabla v_{\hat{z}}\|^2_{\mathrm{L}^2(Y\backslash L(\hat{z}))^{d\times d}}.$$

Hence, it is sufficient to prove the existence of $C_{\mathcal{X}} > 1$ (being independent of $\hat{z} \in [0,1]^m$) such that for all $v_{\hat{z}} \in \mathrm{H}^1(Y\backslash L(\hat{z}))^d$ it holds

$$\|\nabla\mathcal{X}_{\hat{z}}(v_{\hat{z}})\|^2_{\mathrm{L}^2(\mathrm{Im}(T_{\hat{z}}^{-1}|_Y))^{d\times d}} \leq (C_{\mathcal{X}}^2 - 1)\|\nabla v_{\hat{z}}\|^2_{\mathrm{L}^2(Y\backslash L(\hat{z}))^{d\times d}}.$$

The proof of this inequality is performed in calculation (8.11) below. There the following inequality being valid for any $\hat{z} \in [0,1]^m$ and almost every $y \in Y$ is required:

$$C_2|\det(\nabla T_{\hat{z}}(y))| \geq 1. \tag{8.10}$$

This estimate can be proven analogously to (8.5). In calculation (8.11) below, at the beginning of every line the respectively exploited relation is indicated. Additionally, in line (8.11b) the non-negative integrand of line (8.11a) is increased by multiplying it with the left hand side of (8.10). Moreover, the chain-rule is applied in the lines (8.11a) and (8.11d). Finally, we substitute $T_{\hat{z}}(y)$ by x in line (8.11c) and do the opposite substitution in line (8.11e). Taking all these mentioned transformations into account, for an arbitrary but fixed chosen $v_{\hat{z}} \in \mathrm{H}^1(Y\backslash L(\hat{z}))^d$ it holds

$$\begin{aligned}
&\Big\|\nabla\mathcal{X}_{\hat{z}}(v_{\hat{z}})\Big\|^2_{\mathrm{L}^2(\mathrm{Im}(T_{\hat{z}}^{-1}|_Y))^{d\times d}} \\
&= \int_{\mathrm{Im}(T_{\hat{z}}^{-1}|_Y)} \Big|\nabla\Big[\widehat{\mathcal{X}_{0,\hat{z}}}(\mathbb{T}_{\hat{z}}(v_{\hat{z}}))\Big](T_{\hat{z}}(y))\nabla T_{\hat{z}}(y)\Big|^2_{d\times d}\mathrm{d}y && \text{(8.11a)} \\
&\overset{(8.2g)}{\leq} C_T^2 C_2 \int_{\mathrm{Im}(T_{\hat{z}}^{-1}|_Y)} \Big|\nabla\Big[\widehat{\mathcal{X}_{0,\hat{z}}}(\mathbb{T}_{\hat{z}}(v_{\hat{z}}))\Big](T_{\hat{z}}(y))\Big|^2_{d\times d}|\det(\nabla T_{\hat{z}}(y))|\mathrm{d}y && \text{(8.11b)} \\
&\overset{(8.7)}{=} C_T^2 C_2 \int_Y \Big|\nabla\Big[\mathcal{X}_0(\mathbb{T}_{\hat{z}}(v_{\hat{z}}))\Big](x)\Big|^2_{d\times d}\mathrm{d}x && \text{(8.11c)} \\
&\overset{(8.6b)}{\leq} C_{\mathcal{X}_0}^2 C_T^2 C_2 \int_{Y\backslash L(\mathbf{0})} \Big|\nabla\Big[\mathbb{T}_{\hat{z}}(v_{\hat{z}})\Big](x)\Big|^2_{d\times d}\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= && C_3 \int_{Y\setminus L(\mathbf{0})} \left|\nabla v_{\hat z}(T_{\hat z}^{-1}|_{\mathbb{R}^d\setminus L(\mathbf{0})}(x))\nabla T_{\hat z}^{-1}|_{\mathbb{R}^d\setminus L(\mathbf{0})}(x)\right|_{d\times d}^2 \mathrm{d}x && \text{(8.11d)}\\
&\overset{(8.2g)}{\le} && C_T^2 C_3 \int_{Y\setminus L(\mathbf{0})} \left|\nabla v_{\hat z}(T_{\hat z}^{-1}|_{\mathbb{R}^d\setminus L(\mathbf{0})}(x))\right|_{d\times d}^2 \mathrm{d}x \\
&= && C_T^2 C_3 \int_{\mathrm{Im}(T_{\hat z}^{-1}|_{Y\setminus L(\mathbf{0})})} \left|\nabla v_{\hat z}(y)\right|_{d\times d}^2 |\det(\nabla T_{\hat z}(y))| \mathrm{d}y && \text{(8.11e)}\\
&\overset{(8.4)}{\le} && C_1 C_T^2 C_3 \int_{\mathrm{Im}(T_{\hat z}^{-1}|_{Y\setminus L(\mathbf{0})})} \left|\nabla v_{\hat z}(y)\right|_{d\times d}^2 \mathrm{d}y \;=\; C_1 C_T^2 C_3 \left\|\nabla v_{\hat z}\right\|^2_{\mathrm{L}^2(\mathrm{Im}(T_{\hat z}^{-1}|_{Y\setminus L(\mathbf{0})}))^{d\times d}},
\end{aligned}
$$

where $C_3 := C_{\mathcal{X}_0}^2 C_T^2 C_2$. Recalling $\mathrm{Im}(T_{\hat z}^{-1}|_{L(\mathbf{0})}) = L(\hat z)$ and setting $C_{\mathcal{X}}^2 := C_1 C_T^2 C_3 + 1$, calculation (8.11) yields the desired estimate

$$
\|\nabla \mathcal{X}_{\hat z}(v_{\hat z})\|^2_{\mathrm{L}^2(\mathrm{Im}(T_{\hat z}^{-1}|_Y))^{d\times d}} \le (C_{\mathcal{X}}^2 - 1)\|\nabla v_{\hat z}\|^2_{\mathrm{L}^2(Y\setminus L(\hat z))^{d\times d}}.
$$

and the proof is concluded. □

Remark 8.3. *Observe that for $m = 1$ in the particular case described below, the statement of Lemma 8.2 stays valid, if one neglects the assumption (8.2g) and if one assumes the conditions (8.2b) and (8.2h) to hold only for $\hat z \in [0, 1)$. Since this setting allow for $\mu_d(L(1)) = 0$, in this particular case hole initiation can be modeled.*

Let $g : [0, 1] \to [1, \infty]$ be strictly monotone with $g(0) = 1$, $g(\hat z) < \infty$ for $\hat z \in [0, 1)$, and $\lim_{\hat z \nearrow 1} g(\hat z) = \infty$. Then, the transformations $(T_{\hat z})_{\hat z\in[0,1]}$ with $T_{\hat z}(y) = g(\hat z)y$ and $T_1(y) = y$ correspond to the family of scaled sets $(L(\hat z))_{\hat z\in[0,1]}$, which are given by $L(\hat z) := \frac{1}{g(\hat z)} L(0)$ for $\hat z \in [0, 1)$ and $L(1) := \emptyset$. Since $\mathrm{H}^1(Y\setminus L(1))^d = \mathrm{H}^1(Y)^d$, there is no need of an extension in the case $\hat z = 1$, i.e., for $v_1 \in \mathrm{H}^1(Y\setminus L(1))^d = \mathrm{H}^1(Y)^d$ we define

$$
\mathcal{X}_1(v_1) := v_1. \tag{8.12}
$$

Moreover, for any $\hat z^* \in [0, 1)$ it holds

$$
\sup_{\hat z\in[0,\hat z^*]} \left(\|\nabla T_{\hat z}\|_{\mathrm{L}^\infty(Y)^{d\times d}} + \|\nabla T_{\hat z}^{-1}\|_{\mathrm{L}^\infty(\mathrm{Im}(T_{\hat z}))^{d\times d}}\right) =: C_T(\hat z^*) < \infty, \tag{8.13}
$$

where the constant $C_T(\hat z^*)$ depends on the value $\hat z^* \in [0, 1)$. To prove Lemma 8.2 under these modified assumptions, we proceed as before. The first and only changes concern the calculation (8.11) and look like follows: For $\hat z \in [0, 1)$ we find $\nabla T_{\hat z} \equiv g(\hat z)\mathrm{Id}$. To substitute $T_{\hat z}(y)$ by x, line (8.11a) is multiplied by $1 = g^{d-2}(\hat z)g^{2-d}(\hat z)$ which together with the term $g^2(\hat z)$ coming from the chain rule ($\nabla T_{\hat z} \equiv g(\hat z)\mathrm{Id}$) leads to a factor $g^d(\hat z)g^{2-d}(\hat z) = |\det(\nabla T_{\hat z})|g^{2-d}(\hat z)$. After the integral transformation there is a factor $g^{2-d}(\hat z)$ and as before inequality (8.6b) is exploited. Applying the chain rule in line (8.11d) ($\nabla T_{\hat z}^{-1} \equiv g^{-1}(\hat z)\mathrm{Id}$) causes an additional term $g^{-2}(\hat z)$ which altogether leads to a factor $g^{-d}(\hat z)$. Substituting x by $T_{\hat z}(y)$ in line (8.11e) yields $|\det(\nabla T_{\hat z})| = g^d(\hat z)$ which cancels out the factor $g^{-d}(\hat z)$. In this way calculation (8.11) is established by exploiting only the estimate (8.6b). Therefore, in this case Lemma 8.2 holds for the constant $C_{\mathcal{X}_0} > 0$ of the inequalities (8.6), without assuming the uniform estimate (8.2g) to hold and by claiming (8.2b) and (8.2h) only for $\hat z \in [0, 1)$.

By assuming (8.1) and (8.2) to hold, for $\widehat{\mathbb{C}}^{\mathrm{H}} : [0,1]^m \to \mathrm{L}^\infty(Y; \{\mathbb{C}_{\mathrm{strong}}, \mathbb{O}\})$ we are now going to verify the crucial conditions (6.1) and (6.2) of Chapter 6. Since the statement of Lemma 7.1 is independent of the value of $\mathbb{C}_{\mathrm{weak}}$, it is applicable for the here considered $\widehat{\mathbb{C}}^{\mathrm{H}} : [0,1]^m \to \mathrm{L}^\infty(Y; \{\mathbb{C}_{\mathrm{strong}}, \mathbb{O}\})$, too. Hence, for all measurable $z : \mathbb{R}^d \to [0,1]^m$ the mapping $\widehat{\mathbb{C}}^{\mathrm{H}}(z(\cdot))(\cdot) : \mathbb{R}^d \times Y \to \{\mathbb{C}_{\mathrm{strong}}, \mathbb{O}\}$ is measurable on $\mathbb{R}^d \times Y$ and assumption (6.1) of Chapter 6 is fulfilled. Analog to Chapter 7, condition (8.2e) implies that $\widehat{\mathbb{C}}^{\mathrm{H}} : [0,1]^m \to \mathrm{L}^\infty(Y; \{\mathbb{C}_{\mathrm{strong}}, \mathbb{O}\})$ is continuous with respect to the strong L^1-topology such that (6.2) is established as well.

In contrast to the previous chapter, for an arbitrary $\widehat{z} \in [0,1]^m$ the tensor $\widehat{\mathbb{C}}^{\mathrm{H}}(\widehat{z})$ takes the value zero on the set $L(\widehat{z})$ and hence condition (6.3) obviously cannot be satisfied uniformly in $y \in Y \supset L(\widehat{z})$. Taking a closer look to the theory presented in Chapter 6, we find that assumption (6.3) was made to ensure the unique solvability of the unit cell problem being the basis of the one-scale modeled formulated in Section 6.3. For this reason we refer to Section 8.3 for a sufficient condition replacing (6.3) and enabling the formulation of an equivalent one-scale model in this particular case.

8.1 Damage progression caused by microscopic defects

In this section the microscopic models describing damage progression by increasing the size of preexisting defects, which are modeled by holes in the displacement field's reference configuration, are introduced.

8.1.1 Displacement field's reference configuration

As already mentioned in the beginning of this chapter, the main difference to Chapter 7 is that the material tensor $\mathbb{C}_{\mathrm{weak}}$ is set to zero. This banal appearing assumption leads to the fact that for a given damage variable $z_\varepsilon : [0,T] \to \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)$ the displacement field's reference configuration is given by the t- and ε-dependent set

$$\Omega \backslash \Omega_\varepsilon^{\mathrm{D}}(z_\varepsilon(t)) = \Omega \backslash \bigcup_{\lambda \in \Lambda_\varepsilon^-} \varepsilon(\lambda + L(z_{(t)}^{\varepsilon\lambda})),$$

where $z_{(t)}^{\varepsilon\lambda} :\equiv z_\varepsilon(t)|_{\varepsilon(\lambda+Y)}$ for every $\lambda \in \Lambda_\varepsilon^-$; see (2.15). Note that up to now the displacement field's reference configuration of the considered models was given by the time- and micro-scale-independent set Ω, such that there was no need to comment on this. Since for a given damage variable $z_\varepsilon : [0,T] \to \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)$ the displacement field at time $t \in [0,T]$ is technically a function defined on $\Omega \backslash \Omega_\varepsilon^{\mathrm{D}}(z_\varepsilon(t))$, its state space actually depends on the damage variable. Recalling the evolutionary models of Chapter 6 and 7, both the displacement field and the damage variable, are unknowns. Hence, assuming the damage variable $z_\varepsilon : [0,T] \to \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)$ to be a priori known for the considered time interval $[0,T]$ is not realistic.

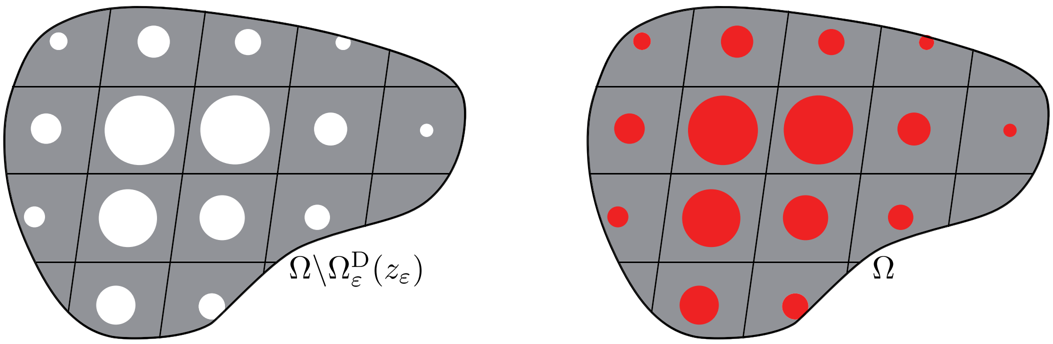


Figure 8.1: Left: Perforated domain $\Omega\backslash\Omega_\varepsilon^{\mathrm{D}}(z_\varepsilon)$ of positive stiffness depending on z_ε. Right: Artificial body described by the z_ε-independent domain Ω, but containing the red subsets of zero stiffness.

To overcome this inconsistency of the displacement field's state space we are going to investigate the damage progression of an artificial body whose reference configuration is given by the t- and ε-independent set Ω. Here, *artificial* refers to the fact that this body allows for "material" with zero stiffness. Therefore, for $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ the "material" distribution of this artificial body for almost every $x \in \Omega$ is given by

$$\mathbb{C}_\varepsilon^{\mathrm{H}}(z_\varepsilon)(x) := \mathbb{1}_{\Omega\backslash\Omega_\varepsilon^{\mathrm{D}}(z_\varepsilon)}(x)\mathbb{C}_{\mathrm{strong}}.$$

The introduction of this artificial body now allows us to choose $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ as the displacement field's state space, again. However, since for $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ there is zero stiffness on the set $\Omega_\varepsilon^{\mathrm{D}}(z_\varepsilon)$, there is no physical sense of the displacement field's value on that part of Ω. For this reason, for $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ the displacement field $u \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ is assumed to satisfy the constraint, that $u|_{\Omega_\varepsilon^{\mathrm{D}}(z_\varepsilon)}$ is uniquely described by $u|_{\Omega\backslash\Omega_\varepsilon^{\mathrm{D}}(z_\varepsilon)}$. In other words, the evolution only affects the displacement field on the set of positive stiffness and its physically senseless values are defined by the constraint described above. Observe that the displacement field's physically senseless values are cut off by the material tensor anyway. Modeling this constraint is enabled by the following corollary stating the existence of an extension operator that is uniformly bounded with respect to $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ and $\varepsilon > 0$.

Theorem 8.4. *Let $L : [0,1]^m \to \mathcal{L}_{\mathrm{Leb}}(Y)$ fulfill* (8.2d)*. Moreover, assume that there exist bi-Lipschitz transformations $(T_{\hat z})_{\hat z\in[0,1]^m}$, $T_{\hat z} : Y \to \mathbb{R}^d$, satisfying* (8.2g) *and* (8.2h)*.*

Then for all $\varepsilon > 0$ and any $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ there exists a strong 1*-extension operator $\mathcal{X}_{\varepsilon,z_\varepsilon} : \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega\backslash\Omega_\varepsilon^{\mathrm{D}}(z_\varepsilon))^d \to \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$, i.e., for all $u_{z_\varepsilon} \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega\backslash\Omega_\varepsilon^{\mathrm{D}}(z_\varepsilon))^d$ it holds*

$$\|\mathcal{X}_{\varepsilon,z_\varepsilon}u_{z_\varepsilon}\|_{\mathrm{L}^2(\Omega)^d} \le C_{\mathcal{X}}\|u_{z_\varepsilon}\|_{\mathrm{L}^2(\Omega\backslash\Omega_\varepsilon^{\mathrm{D}}(z_\varepsilon))^d}, \tag{8.14a}$$
$$\|\nabla(\mathcal{X}_{\varepsilon,z_\varepsilon}u_{z_\varepsilon})\|_{\mathrm{L}^2(\Omega)^{d\times d}} \le C_{\mathcal{X}}\|\nabla u_{z_\varepsilon}\|_{\mathrm{L}^2(\Omega\backslash\Omega_\varepsilon^{\mathrm{D}}(z_\varepsilon))^{d\times d}}, \tag{8.14b}$$

where the constant $C_{\mathcal{X}} > 0$ is the same as in Lemma 8.2*. Therefore, $C_{\mathcal{X}} > 0$ is independent of $\varepsilon > 0$ and $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$.*

Remark 8.5. *Later, for a given damage variable $z_\varepsilon : [0,T] \to \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ the stored energy at time $t \in [0,T]$ of the evolution model of Subsection 8.1.3 below is only finite if the displacement field $u(t) \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ satisfies*

$$u(t) = \mathcal{X}_{\varepsilon,z_\varepsilon(t)}(u(t)|_{\Omega\setminus\Omega^{\mathrm{D}}_\varepsilon(z_\varepsilon(t))}). \tag{8.15}$$

Therefore, all displacement fields violating this constraint are not physically plausible. In this way, for all $u(t) \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ fulfilling (8.15) the function $u(t)|_{\Omega^{\mathrm{D}}_\varepsilon(z_\varepsilon(t))}$ is uniquely defined by $u(t)|_{\Omega\setminus\Omega^{\mathrm{D}}_\varepsilon(z_\varepsilon(t))}$.

The construction of $\mathcal{X}_{\varepsilon,z_\varepsilon} : \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega\setminus\Omega^{\mathrm{D}}_\varepsilon(z_\varepsilon))^d \to \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ being performed in the following proof relies on the separated estimates for the function's and the gradient's L^2-norm stated in Lemma 8.2. Thus, we are able to verify the ε and z_ε independence of $C_\mathcal{X}$, which is crucial for applying the homogenization theory of Chapter 6. Moreover, by replacing ∇ by $\mathbf{e} := \frac{1}{2}(\nabla+\nabla^T)$ in the proof of Lemma 8.2 and Theorem 8.4, estimate (8.14b) holds true for the symmetric part of the gradient, too. In this case the proofs are exactly the same.

Proof of Theorem 8.4. Let $\varepsilon > 0$ and $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ be given. Then the continuation operator $\mathcal{X}_{\varepsilon,z_\varepsilon} : \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega\setminus\Omega^{\mathrm{D}}_\varepsilon(z_\varepsilon))^d \to \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ is constructed with the help of the following two scaling operators

$$\widehat{T}_{\varepsilon\lambda} : \begin{cases} \varepsilon(\lambda+Y) \to \mathcal{Y}, \\ \qquad x \mapsto \mathcal{V}_\varepsilon(x) \quad \text{(see (3.1))} \end{cases} \quad \text{and} \quad \widehat{T}^{-1}_{\varepsilon\lambda} : \begin{cases} \mathcal{Y} \to \varepsilon(\lambda+Y), \\ y \mapsto \varepsilon(\lambda+y). \end{cases}$$

Introducing the operators

$$\widehat{\mathbf{T}}_{\varepsilon\lambda} : \begin{cases} \mathrm{H}^1(Y)^d \to \mathrm{H}^1(\varepsilon(\lambda+Y))^d, \\ \qquad \widehat{v} \mapsto \widehat{v} \circ \widehat{T}_{\varepsilon\lambda} \end{cases} \quad \text{and} \quad \widehat{\mathbb{T}}^{\hat{z}}_{\varepsilon\lambda} : \begin{cases} \mathrm{H}^1(\varepsilon(\lambda+Y\setminus L(\widehat{z})))^d \to \mathrm{H}^1(Y\setminus L(\widehat{z}))^d, \\ \qquad u \mapsto u \circ \widehat{T}^{-1}_{\varepsilon\lambda}|_{Y\setminus L(\hat{z})} \end{cases}$$

for $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ the strong 1-extension operator $\mathcal{X}_{\varepsilon,z_\varepsilon} : \mathrm{H}^1(\Omega\setminus\Omega^{\mathrm{D}}_\varepsilon(z_\varepsilon))^d \to \mathrm{H}^1(\Omega)^d$ is given by

$$\mathcal{X}_{\varepsilon,z_\varepsilon}(u)(x) := \widehat{\mathbf{T}}_{\mathcal{N}_\varepsilon(x)}(\mathcal{X}_{z_\varepsilon(x)}(\widehat{\mathbb{T}}^{z_\varepsilon(x)}_{\mathcal{N}_\varepsilon(x)}(u(x)))), \tag{8.16}$$

where for $\widehat{z} \in [0,1]^m$ the strong 1-extension operator $\mathcal{X}_{\hat{z}} : \mathrm{H}^1(Y\setminus L(\widehat{z}))^d \to \mathrm{H}^1(Y)^d$ is that of Lemma 8.2. By decomposing Ω into small cells $\varepsilon(\lambda+Y)\cap\Omega \neq \emptyset$, with this definition the inequalities (8.14) are proven in the same way as performed in the proof of Lemma 8.2. While estimating $\|\mathcal{X}_{\varepsilon,z_\varepsilon}(u)\|_{\mathrm{L}^2(\Omega)^d}$ and $\|\nabla\mathcal{X}_{\varepsilon,z_\varepsilon}(u)\|_{\mathrm{L}^2(\Omega)^{d\times d}}$ analogously to (8.11), according to $\nabla\widehat{T}_{\varepsilon\lambda} = \varepsilon^{-1}\mathrm{Id}$ and $\nabla\widehat{T}^{-1}_{\varepsilon\lambda} = \varepsilon\mathrm{Id}$ in this case no estimates have to be applied in the whole calculation (except of that stated in Lemma 8.2); see also Remark 8.3. Since the chain rule is applied twice in calculation (8.11), the appearing terms $(\varepsilon^{-1}\mathrm{Id})$ and $(\varepsilon\mathrm{Id})$ cancel out. To substitute $\widehat{T}_{\varepsilon\lambda}(x)$ by y one inserts the factor $1 = (\frac{\varepsilon}{\varepsilon})^d = \varepsilon^d|\det(\nabla\widehat{T}_{\varepsilon\lambda})|$. The remaining factor after performing this transformation then cancels out by applying the opposite substitution. Therefore, the inequalities of Lemma 8.2 are the only estimates coming into play, explaining that (8.14) holds for the same constant $C_\mathcal{X}$. $\square$

Remark 8.6. *As already stated in Proposition* 8.1 *the assumptions* (8.2a), (8.2d), (8.2g), *and* (8.2h) *prohibit hole initiation in the microscopic damage model. That means for all* $\varepsilon > 0$ *and any time* $t \in [0, T]$ *in every cell* $\varepsilon(\lambda+Y) \subset \Omega$ *there is a hole containing the set* $\varepsilon(\lambda+L(\mathbf{1}))$ *at least. Hence, the body associated to the undamaged state with respect to the damage variable* ($z_\varepsilon \equiv \mathbf{1}$ *on* Ω) *is already perforated by periodically distributed holes* $\varepsilon L(\mathbf{1})$*. However, observe that at least in the case described in Remark* 8.3 *hole initiation is allowed, i.e., in this particular case* ($m = 1$) *Theorem* 8.4 *is valid without assuming the uniform estimate* (8.2g) *to hold and by claiming* (8.2b) *and* (8.2h) *only for* $\widehat{z} \in [0, 1)$.

8.1.2 External loading

This subsection addresses to the choice of the external loading for a specific microscopic model. For this purpose, let $\varepsilon > 0$ be chosen arbitrarily but fixed. Thinking of the artificial body defined by the reference configuration Ω and the tensor valued mapping $\mathbb{C}^{\mathrm{H}}_\varepsilon : \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0, 1]^m) \to \mathrm{L}^\infty(\Omega; \{\mathbb{C}_{\text{strong}}, \mathbb{O}\})$, one might model the external loading by a function $\ell \in \mathrm{C}^1([0, T]; (\mathrm{H}^1_{\Gamma_{\text{Dir}}}(\Omega)^d)^*)$, as it is done in the Chapters 6 and 7. Since the displacement field's state space is given by $\mathrm{H}^1_{\Gamma_{\text{Dir}}}(\Omega)^d$, this choice would cause no mathematical issues. On the other hand, such an external loading is also applied to regions of Ω with zero stiffness, which in general is not a physically plausible behavior. However, due to the constraint (8.15), for a given damage variable $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0, 1]^m)$ and $\ell \in \mathrm{C}^1([0, T]; (\mathrm{H}^1_{\Gamma_{\text{Dir}}}(\Omega)^d)^*)$ for all $t \in [0, T]$ we find

$$\langle \ell(t), u\rangle := \Big\langle \ell(t), \mathcal{X}_{\varepsilon,z_\varepsilon}\Big(u|_{\Omega\backslash\Omega^{\mathrm{D}}_\varepsilon(z_\varepsilon)}\Big)\Big\rangle = \Big\langle \mathcal{X}^*_{\varepsilon,z_\varepsilon}(\ell(t)), u|_{\Omega\backslash\Omega^{\mathrm{D}}_\varepsilon(z_\varepsilon)}\Big\rangle_{\Omega\backslash\Omega^{\mathrm{D}}_\varepsilon(z_\varepsilon)}.$$

Here, at any time $t \in [0, T]$ the term $\mathcal{X}^*_{\varepsilon,z_\varepsilon}(\ell(t)) \in (\mathrm{H}^1(\Omega\backslash\Omega^{\mathrm{D}}_\varepsilon(z_\varepsilon))^d)^*$ models a physically reasonable external loading for the body associated to the set $\Omega\backslash\Omega^{\mathrm{D}}_\varepsilon(z_\varepsilon)$ of positive stiffness, where $\mathcal{X}^*_{\varepsilon,z_\varepsilon} : (\mathrm{H}^1(\Omega)^d)^* \to (\mathrm{H}^1(\Omega\backslash\Omega^{\mathrm{D}}_\varepsilon(z_\varepsilon))^d)^*$ denotes the adjoint operator of $\mathcal{X}_{\varepsilon,z_\varepsilon} : \mathrm{H}^1(\Omega\backslash\Omega^{\mathrm{D}}_\varepsilon(z_\varepsilon))^d \to \mathrm{H}^1(\Omega)^d$; see Theorem 8.4. The disadvantage of modeling the external loading in this way is that $\mathcal{X}^*_{\varepsilon,z_\varepsilon} : (\mathrm{H}^1(\Omega)^d)^* \to (\mathrm{H}^1(\Omega\backslash\Omega^{\mathrm{D}}_\varepsilon(z_\varepsilon))^d)^*$ is not explicitly given. At time $t \in [0, T]$ for $\ell \in \mathrm{C}^1([0, T]; (\mathrm{H}^1_{\Gamma_{\text{Dir}}}(\Omega)^d)^*)$ and a given $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0, 1]^m)$ the actual force on the body associated to the set $\Omega\backslash\Omega^{\mathrm{D}}_\varepsilon(z_\varepsilon)$ of positive stiffness is defined by the non-explicit term $\mathcal{X}^*_{\varepsilon,z_\varepsilon}(\ell(t))$.

Another way of introducing an external loading $\ell^{\ell_0,\ell_1}_{z_\varepsilon} \in \mathrm{C}^1([0, T]; (\mathrm{H}^1_{\Gamma_{\text{Dir}}}(\Omega)^d)^*)$ for the body associated to the set $\Omega\backslash\Omega^{\mathrm{D}}_\varepsilon(z_\varepsilon)$ of positive stiffness is the following: Let the function $(\ell_0, \ell_1) \in \mathrm{C}^1([0, T]; \mathrm{L}^2(\Omega)^d\times\mathrm{L}^2(\Omega)^{d\times d})$ be given. Then, for a given $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0, 1]^m)$ the external loading $\ell^{\ell_0,\ell_1}_{z_\varepsilon} \in \mathrm{C}^1([0, T]; (\mathrm{H}^1_{\Gamma_{\text{Dir}}}(\Omega)^d)^*)$ on the body associated to $\Omega\backslash\Omega^{\mathrm{D}}_\varepsilon(z_\varepsilon)$ for all $u \in \mathrm{H}^1_{\Gamma_{\text{Dir}}}(\Omega)^d$ and every $t \in [0, T]$ is defined by

$$\langle \ell^{\ell_0,\ell_1}_{z_\varepsilon}(t), u\rangle := \langle \mathbb{1}_{\Omega\backslash\Omega^{\mathrm{D}}_\varepsilon(z_\varepsilon)}\ell_0(t), u\rangle_{\mathrm{L}^2(\Omega)^d} + \langle \mathbb{1}_{\Omega\backslash\Omega^{\mathrm{D}}_\varepsilon(z_\varepsilon)}\ell_1(t), \nabla u\rangle_{\mathrm{L}^2(\Omega)^{d\times d}}. \tag{8.17}$$

The advantage of introducing the external loading in this way is its explicit structure, which enables us to investigate the external loading's asymptotic behavior for $\varepsilon \to 0$;

see Lemma 8.7 below. For this reason, from now we use (8.17) to model the external loading for the microscopic models introduced in the following subsection.

Lemma 8.7. *Let $u_0 \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$, $(u_\varepsilon)_{\varepsilon>0} \subset \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$, $(z_\varepsilon)_{\varepsilon>0}$ with $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$, and $z_0 \in \mathrm{W}^{1,p}(\Omega;[0,1]^m)$ be given. Moreover, for $(\ell_0,\ell_1) \in \mathrm{C}^1([0,T];\mathrm{L}^2(\Omega)^d{\times}\mathrm{L}^2(\Omega)^{d\times d})$ let $\ell^{\ell_0,\ell_1}_{z_\varepsilon} \in \mathrm{C}^1([0,T];(\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$ be defined by (8.17). If $u_\varepsilon \xrightarrow{s} Eu_0$ in $\mathrm{L}^2(\Omega{\times}Y)^d$, if $\nabla u_\varepsilon \xrightarrow{w} \nabla_x Eu_0 + \nabla_y U_1$ in $\mathrm{L}^2(\Omega{\times}Y)^{d\times d}$, and if $z_\varepsilon \to z_0$ in $\mathrm{L}^p(\Omega)^m$, then*

$$\lim_{\varepsilon\to 0}\langle \ell^{\ell_0,\ell_1}_{z_\varepsilon}(t), u_\varepsilon\rangle = \langle \ell^{\ell_0,\ell_1}_{z_0}(t), (u_0,U_1)\rangle,$$

where $\ell^{\ell_0,\ell_1}_{z_0} \in \mathrm{C}^1([0,T];(\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d{\times}\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d)^*)$ for all $t\in[0,T]$ is given by

$$\langle \ell^{\ell_0,\ell_1}_{z_0}(t), (u_0,U_1)\rangle := \langle h(z_0)\ell_0(t), u_0\rangle_{\mathrm{L}^2(\Omega)^d} + \langle H(z_0)\ell_1(t), \nabla_x Eu_0 + \nabla_y U_1\rangle_{\mathrm{L}^2(\Omega\times Y)^{d\times d}} \quad (8.18)$$

for $(h(z_0), H(z_0)) \in \mathrm{L}^1(\Omega){\times}\mathrm{L}^1(\Omega{\times}Y)$, which for almost every $(x,y)\in\Omega{\times}Y$ are defined by $h(z_0)(x) := \int_Y \mathbb{1}_{Y\setminus L(z_0(x))}(\widetilde{y})\mathrm{d}\widetilde{y}$ and $H(z_0)(x,y) := \mathbb{1}_{Y\setminus L(z_0(x))}(y)$.

Proof. Let $u_0 \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$, let $(u_\varepsilon)_{\varepsilon>0} \subset \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$, let $(z_\varepsilon)_{\varepsilon>0}$ with $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$, and let $z_0 \in \mathrm{W}^{1,p}(\Omega;[0,1]^m)$ be given, such that $u_\varepsilon \xrightarrow{s} Eu_0$ in $\mathrm{L}^2(\Omega{\times}Y)^d$, such that $\nabla u_\varepsilon \xrightarrow{w} \nabla_x Eu_0 + \nabla_y U_1$ in $\mathrm{L}^2(\Omega{\times}Y)^{d\times d}$, and such that $z_\varepsilon \to z_0$ in $\mathrm{L}^p(\Omega)^m$. Applying Theorem 3.9 to the sequence $(\mathbb{C}^{\mathrm{H}}_\varepsilon(z_\varepsilon))_{\varepsilon>0}$ implies $\mathbb{1}_{\Omega\setminus\Omega^{\mathrm{D}}_\varepsilon(z_\varepsilon)} \xrightarrow{s} H(z_0)$ in $\mathrm{L}^1(\Omega{\times}Y)$; see (8.19) and (8.30). Thus, Lemma 8.7 is proven by the following calculation.

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\langle \ell^{\ell_0,\ell_1}_{z_\varepsilon}(t), u_\varepsilon\rangle \\
&\overset{(8.17)}{=} \lim_{\varepsilon\to 0}\Big(\langle \mathbb{1}_{\Omega\setminus\Omega^{\mathrm{D}}_\varepsilon(z_\varepsilon)}\ell_0(t), u_\varepsilon\rangle_{\mathrm{L}^2(\Omega)^d} + \langle \mathbb{1}_{\Omega\setminus\Omega^{\mathrm{D}}_\varepsilon(z_\varepsilon)}\ell_1(t), \nabla u_\varepsilon\rangle_{\mathrm{L}^2(\Omega)^{d\times d}}\Big) \\
&\overset{(3.2)}{=} \lim_{\varepsilon\to 0}\Big(\langle \mathcal{T}_\varepsilon(\mathbb{1}_{\Omega\setminus\Omega^{\mathrm{D}}_\varepsilon(z_\varepsilon)}\ell_0(t)), \mathcal{T}_\varepsilon u_\varepsilon\rangle_{\mathrm{L}^2(\mathbb{R}^d\times Y)^d} + \langle \mathcal{T}_\varepsilon(\mathbb{1}_{\Omega\setminus\Omega^{\mathrm{D}}_\varepsilon(z_\varepsilon)}\ell_1(t)), \mathcal{T}_\varepsilon \nabla u_\varepsilon\rangle_{\mathrm{L}^2(\mathbb{R}^d\times Y)^{d\times d}}\Big) \\
&\overset{\text{Cor. 3.6}}{=} \langle h(z_0)\ell_0(t), u_0\rangle_{\mathrm{L}^2(\Omega)^d} + \langle H(z_0)\ell_1(t), \nabla_x Eu_0 + \nabla_y U_1\rangle_{\mathrm{L}^2(\Omega\times Y)^{d\times d}},
\end{aligned}$$

where in the last line we already exploited that the first two-scale limit of the second last line is constant with respect to $y\in Y$. □

Remark 8.8. *If one needs to model time dependent Dirichlet data on Γ_{Dir}, following Remark 2.2 u_ε is replaced by $u_\varepsilon + \widehat{g}(t)$, where $\widehat{g} : [0,T]{\times}\Omega \to \mathbb{R}^d$ describes the desired boundary value on Γ_{Dir}. Among other things, the additional term*

$$\langle \mathbb{1}_{\Omega\setminus\Omega^{\mathrm{D}}_\varepsilon(z_\varepsilon)}\ell_0(t), \widehat{g}(t)\rangle_{\mathrm{L}^2(\Omega)^d} + \langle \mathbb{1}_{\Omega\setminus\Omega^{\mathrm{D}}_\varepsilon(z_\varepsilon)}\ell_1(t), \nabla\widehat{g}(t)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}}$$

enters the microscopic energy functional in line (8.20) below. Performing the limit passage $\varepsilon\to 0$, the term $\langle h(z_0)\ell_0(t), \widehat{g}(t)\rangle_{\mathrm{L}^2(\Omega)^d} + \langle h(z_0)\ell_1(t), \nabla\widehat{g}(t)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}}$ results from the time dependent Dirichlet data in the limit. According to the perforated domains considered in the microscopic models, here the scaling factor $h(z_0)$ depending on the actual damage state shows up.

8.1.3 Energy functional, dissipation distance, the microscopic model, and existence of solutions

Due to the introduction of the artificial body in Subsection 8.1.1 the state space $\mathcal{Q}_\varepsilon^{\mathrm{H}}(\Omega)$ can be chosen as

$$\mathcal{Q}_\varepsilon^{\mathrm{H}}(\Omega) := \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \times \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m).$$

Let $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$. For the microstructure being modeled by the tensor

$$\mathbb{C}_\varepsilon^{\mathrm{H}}(z_\varepsilon) := \mathbb{1}_{\Omega\setminus\Omega_\varepsilon^{\mathrm{D}}(z_\varepsilon)}\mathbb{C}_{\mathrm{strong}}, \tag{8.19}$$

the energy functional $\mathcal{E}_\varepsilon^{\mathrm{H}} : [0,T]\times\mathcal{Q}_\varepsilon^{\mathrm{H}}(\Omega) \to \mathbb{R}_\infty := \mathbb{R}\cup\{\infty\}$ is defined as follows: First, we once choose $p \in (1,\infty)$ and keep it fixed for the rest of this chapter. Then, the functional $\widetilde{\mathcal{E}}_\varepsilon^{\mathrm{H}} : [0,T]\times\mathcal{Q}_\varepsilon^{\mathrm{H}}(\Omega) \to \mathbb{R}$ is introduced by

$$\widetilde{\mathcal{E}}_\varepsilon^{\mathrm{H}}(t,u,z_\varepsilon) := \tfrac{1}{2}\langle \mathbb{C}_\varepsilon^{\mathrm{H}}(z_\varepsilon)\mathbf{e}(u),\mathbf{e}(u)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} + \|R_{\frac{\varepsilon}{2}} z_\varepsilon\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}} - \langle \ell_{z_\varepsilon}^{\ell_0,\ell_1}(t),u\rangle,$$

where the external loading $\ell_{z_\varepsilon}^{\ell_0,\ell_1} \in \mathrm{C}^1([0,T];(\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$ for a given tuple of functions $(\ell_0,\ell_1) \in \mathrm{C}^1([0,T];\mathrm{L}^2(\Omega)^d\times\mathrm{L}^2(\Omega)^{d\times d})$ is defined by (8.17). Thus, we are in the position to define the energy functional $\mathcal{E}_\varepsilon^{\mathrm{H}} : [0,T]\times\mathcal{Q}_\varepsilon^{\mathrm{H}}(\Omega) \to \mathbb{R}_\infty$ via

$$\mathcal{E}_\varepsilon^{\mathrm{H}}(t,u,z_\varepsilon) := \begin{cases} \widetilde{\mathcal{E}}_\varepsilon^{\mathrm{H}}(t,u,z_\varepsilon) & \text{if } u = \mathcal{X}_{\varepsilon,z_\varepsilon}(u|_{\Omega\setminus\Omega_\varepsilon^{\mathrm{D}}(z_\varepsilon)}) \text{ (see (8.15))},\\ \infty & \text{otherwise.}\end{cases} \tag{8.20}$$

Observe that setting the energy's value to infinity for all displacement fields violating the constraint (8.15) excludes physically non-plausible values; see also Remark 8.5. To apply the homogenization theory of Chapter 6, the coercivity condition (6.5) needs to be established. By exploiting Korn's inequality, applying Theorem 8.4 afterwards (see also Remark 8.5), and keeping assumption (8.1) in mind, the following estimate holds for all $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ and every $u \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ with $u = \mathcal{X}_{\varepsilon,z_\varepsilon}(u|_{\Omega\setminus\Omega_\varepsilon^{\mathrm{D}}(z_\varepsilon)})$.

$$\begin{aligned}\tfrac{\alpha}{2}C_{\mathcal{X}}^{-2}C_{\mathrm{Korn}}\|u\|^2_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d} \le \tfrac{\alpha}{2}C_{\mathcal{X}}^{-2}\|\mathbf{e}(u)\|^2_{\mathrm{L}^2(\Omega)^{d\times d}} &\le \tfrac{\alpha}{2}\|\mathbf{e}(u)\|^2_{\mathrm{L}^2(\Omega\setminus\Omega_\varepsilon^{\mathrm{D}}(z_\varepsilon))^{d\times d}}\\ &\le \tfrac{1}{2}\langle\mathbb{C}_\varepsilon^{\mathrm{H}}(z_\varepsilon)\mathbf{e}(u),\mathbf{e}(u)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}}\end{aligned} \tag{8.21}$$

Therefore, the energy functional is coercive by definition; see (8.20). As already commented on in Subsection 7.1.1 the dissipation potential (and thus the dissipation distance) specifies the relation of dissipated energy and changes of the damage variable. Therefore, its choice largely depends on the behavior one wants to model. Here, we consider the prototypical choice of Subsection 7.1.1; see (7.12). For this purpose, we choose $q' \in (1,\infty)$ and keep it fixed for the rest of this chapter. Then, for a given sequence $(\kappa_\varepsilon^{\mathrm{H}})_{\varepsilon>0} \subset \mathrm{L}^{q'}(\Omega;[0,\infty)^m)$ satisfying $\kappa_\varepsilon^{\mathrm{H}} \rightharpoonup \kappa_0^{\mathrm{H}}$ in $\mathrm{L}^{q'}(\Omega)^m$ for some function $\kappa_0^{\mathrm{H}} \in \mathrm{L}^{q'}(\Omega;[0,\infty)^m)$ the dissipation distance associated to $\varepsilon > 0$ reads as follows:

$$\mathcal{D}_\varepsilon^{\mathrm{H}}(z_1,z_2) = \begin{cases} \displaystyle\int_{\Omega_\varepsilon^-} |\langle\kappa_\varepsilon^{\mathrm{H}}(x), z_2(x)-z_1(x)\rangle_m|\mathrm{d}x & \text{if } z_1 \ge z_2,\\ \infty & \text{otherwise.}\end{cases} \tag{8.22}$$

For given initial values $(u_\varepsilon^0, z_\varepsilon^0) \in \mathcal{Q}_\varepsilon^{\mathrm{H}}(\Omega)$ satisfying the constraint $u_\varepsilon^0 = \mathcal{X}_{\varepsilon,z_\varepsilon^0}(u_\varepsilon^0|_{\Omega\setminus\Omega_\varepsilon^{\mathrm{D}}(z_\varepsilon^0)})$ for all $\varepsilon > 0$, the rate-independent damage evolution is modeled by the ε-dependent energetic formulation $(\mathrm{S}_{\mathrm{H}}^\varepsilon)$ and $(\mathrm{E}_{\mathrm{H}}^\varepsilon)$, where $\varepsilon > 0$ scales the size of the appearing holes.

Stability condition $(\mathrm{S}_{\mathrm{H}}^\varepsilon)$ and energy balance $(\mathrm{E}_{\mathrm{H}}^\varepsilon)$ for all $t \in [0,T]$:

$$\mathcal{E}_\varepsilon^{\mathrm{H}}(t, u_\varepsilon(t), z_\varepsilon(t)) \leq \mathcal{E}_\varepsilon^{\mathrm{H}}(t, \widetilde{u}, \widetilde{z}) + \mathcal{D}_\varepsilon^{\mathrm{H}}(z_\varepsilon(t), \widetilde{z}) \quad \text{for all } (\widetilde{u}, \widetilde{z}) \in \mathcal{Q}_\varepsilon^{\mathrm{H}}(\Omega)$$

$$\mathcal{E}_\varepsilon^{\mathrm{H}}(t, u_\varepsilon(t), z_\varepsilon(t)) + \mathrm{Diss}_{\mathcal{D}_\varepsilon^{\mathrm{H}}}(z_\varepsilon; [0,t]) = \mathcal{E}_\varepsilon^{\mathrm{H}}(0, u_\varepsilon^0, z_\varepsilon^0) - \int_0^t \langle \dot{\ell}_{z_\varepsilon(s)}^{\ell_0,\ell_1}(s), u_\varepsilon(s)\rangle \mathrm{d}s$$

Here, $\mathrm{Diss}_{\mathcal{D}_\varepsilon^{\mathrm{H}}}(z_\varepsilon; [0,t]) := \sup \sum_{j=1}^N \mathcal{D}_\varepsilon^{\mathrm{H}}(z_\varepsilon(t_{j-1}), z_\varepsilon(t_j))$, where for $N \in \mathbb{N}$ the supremum is taken with respect to all finite partitions $\pi_N := \{0 = t_0 < t_1 < \ldots < t_N = t\}$ of the interval $[0,T]$. For $\widetilde{t} \in [0,T]$

$$\mathcal{S}_\varepsilon^{\mathrm{H}}(\widetilde{t}) := \{(u_\varepsilon, z_\varepsilon) \in \mathcal{Q}_\varepsilon^{\mathrm{H}}(\Omega) \text{ satisfying } (\mathrm{S}_{\mathrm{H}}^\varepsilon) \text{ for } t = \widetilde{t} \text{ and } \mathcal{E}_\varepsilon^{\mathrm{H}}(\widetilde{t}, u_\varepsilon, z_\varepsilon) < \infty\}$$

denotes the set of stable states. The following corollary states the existence of solutions for $(\mathrm{S}_{\mathrm{H}}^\varepsilon)$ and $(\mathrm{E}_{\mathrm{H}}^\varepsilon)$ and is proven analogously to Proposition 6.5.

Corollary 8.9 (Existence of solutions)**.** *Assume that the conditions* (8.1) *and* (8.2) *hold. For* $(\ell_0, \ell_1) \in \mathrm{C}^1([0,T]; \mathrm{L}^2(\Omega)^d \times \mathrm{L}^2(\Omega)^{d\times d})$ *let* $\ell_{z_\varepsilon}^{\ell_0,\ell_1} \in \mathrm{C}^1([0,T]; (\mathrm{H}_{\Gamma_{\mathrm{Dir}}}^1(\Omega)^d)^*)$ *be defined by* (8.17)*. Moreover, let* $\mathcal{E}_\varepsilon^{\mathrm{H}} : [0,T]\times\mathcal{Q}_\varepsilon^{\mathrm{H}}(\Omega) \to \mathbb{R}_\infty$ *be defined via* (8.20) *and for* $\kappa_\varepsilon^{\mathrm{H}} \in \mathrm{L}^{q'}(\Omega; [0,\infty)^m)$ *let* $\mathcal{D}_\varepsilon^{\mathrm{H}} : \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)\times\mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m) \to [0,\infty]$ *be given by* (8.22)*.*

Then for $(u_\varepsilon^0, z_\varepsilon^0) \in \mathcal{S}_\varepsilon^{\mathrm{H}}(0)$*, there exists an energetic solution* $(u_\varepsilon, z_\varepsilon) : [0,T] \to \mathcal{Q}_\varepsilon^{\mathrm{H}}(\Omega)$ *of the rate-independent system* $(\mathcal{Q}_\varepsilon^{\mathrm{H}}(\Omega), \mathcal{E}_\varepsilon^{\mathrm{H}}, \mathcal{D}_\varepsilon^{\mathrm{H}})$ *satisfying* $(u_\varepsilon(0), z_\varepsilon(0)) = (u_\varepsilon^0, z_\varepsilon^0)$ *and*

$$\begin{aligned} &u_\varepsilon \in \mathrm{L}^\infty([0,T], \mathrm{H}_{\Gamma_{\mathrm{Dir}}}^1(\Omega)^d), \\ &z_\varepsilon \in \mathrm{L}^\infty([0,T], \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)) \cap \mathrm{BV}_{\mathcal{D}_\varepsilon^{\mathrm{H}}}([0,T], \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)). \end{aligned}$$

Proof. Referring to the proof of Corollary 7.8 the assumptions (6.1), (6.2), (6.5), and (6.9) of Proposition 6.5 are fulfilled. Therefore, the proof of Corollary 8.9 is completely analog to that of Proposition 6.5. The only point that remains to be shown is that the energy sublevel sets are weakly compact, i.e.: For $u_0 \in \mathrm{H}_{\Gamma_{\mathrm{Dir}}}^1(\Omega)^d$, for $z_0 \in \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)$, and for a sequence $(u_\delta, z_\delta)_{\delta>0}$ in $\mathcal{Q}_\varepsilon^{\mathrm{H}}(\Omega)$ belonging to the sublevel set $\mathrm{Sub}_E(t)$ (see (5.6)) with $u_\delta \rightharpoonup u_0$ in $\mathrm{H}_{\Gamma_{\mathrm{Dir}}}^1(\Omega)^d$ and $z_\delta \to z_0$ in $\mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)$ we need to verify $u_0 = \mathcal{X}_{\varepsilon,z_0}(u_0|_{\Omega\setminus\Omega_\varepsilon^{\mathrm{D}}(z_0)})$. Otherwise, $\mathcal{E}_\varepsilon^{\mathrm{H}}(t, u_0, z_0) = \infty$ by definition and hence $(u_0, z_0) \notin \mathrm{Sub}_E(t)$.

To show $u_0 = \mathcal{X}_{\varepsilon,z_0}(u_0|_{\Omega\setminus\Omega_\varepsilon^{\mathrm{D}}(z_0)})$ on $\Omega_\varepsilon^{\mathrm{D}}(z_0) \subset \Omega_\varepsilon^-$, observe that for all $\delta > 0$ we have $u_\delta = \mathcal{X}_{\varepsilon,z_\delta}(u_\delta|_{\Omega\setminus\Omega_\varepsilon^{\mathrm{D}}(z_\delta)})$ due to the assumption $(u_\delta, z_\delta) \in \mathrm{Sub}_E(t)$. By estimating the difference of u_0 and $\mathcal{X}_{\varepsilon,z_0}(u_0|_{\Omega\setminus\Omega_\varepsilon^{\mathrm{D}}(z_0)})$ on Ω_ε^- with the help of the triangle inequality we obtain

$$\begin{aligned} \big\|u_0 - \mathcal{X}_{\varepsilon,z_0}(u_0|_{\Omega\setminus\Omega_\varepsilon^{\mathrm{D}}(z_0)})\big\|_{\mathrm{L}^2(\Omega_\varepsilon^-)^d} &\leq \big\|u_0 - u_\delta\big\|_{\mathrm{L}^2(\Omega)^d} + \Big\|\mathcal{X}_{\varepsilon,z_\delta}\Big((u_\delta - u_0)|_{\Omega\setminus\Omega_\varepsilon^{\mathrm{D}}(z_\delta)}\Big)\Big\|_{\mathrm{L}^2(\Omega)^d} + R_{z_\delta} \\ &\leq \big\|u_0 - u_\delta\big\|_{\mathrm{L}^2(\Omega)^d} + C_{\mathcal{X}}\big\|u_\delta - u_0\big\|_{\mathrm{L}^2(\Omega\setminus\Omega_\varepsilon^{\mathrm{D}}(z_\delta))^d} + R_{z_\delta}. \end{aligned}$$

Hence, it is sufficient to show that $R_{z_\delta} := \|\mathcal{X}_{\varepsilon,z_\delta}(u_0|_{\Omega\backslash\Omega_\varepsilon^{\mathrm{D}}(z_\delta)}) - \mathcal{X}_{\varepsilon,z_0}(u_0|_{\Omega\backslash\Omega_\varepsilon^{\mathrm{D}}(z_0)})\|_{\mathrm{L}^2(\Omega_\varepsilon^-)^d}$ converges to zero for $\delta \to 0$. Recalling the construction of the strong 1-extension operator $\mathcal{X}_{\varepsilon,z_\varepsilon} : \mathrm{H}^1(\Omega\backslash\Omega_\varepsilon^{\mathrm{D}}(z_\varepsilon))^d \to \mathrm{H}^1(\Omega)^d$ (see (8.16)), by decomposing Ω_ε^- into small cells $\varepsilon(\lambda+Y)$, for fixed $\lambda \in \Lambda_\varepsilon^-$ we have to show

$$\mathcal{X}_{\hat{z}_\delta^{(\lambda)}}(v^{(\lambda)}|_{Y\backslash L(\hat{z}_\delta^{(\lambda)})}) \to \mathcal{X}_{\hat{z}_0^{(\lambda)}}(v^{(\lambda)}|_{Y\backslash L(\hat{z}_0^{(\lambda)})}) \qquad \text{in } \mathrm{L}^2(Y)^d, \tag{8.23}$$

where $\widehat{z}_\delta^{(\lambda)} :\equiv z_\delta|_{\varepsilon(\lambda+Y)}$, $\widehat{z}_0^{(\lambda)} :\equiv z_0|_{\varepsilon(\lambda+Y)}$, and $v^{(\lambda)} := u_0(\varepsilon(\lambda+\cdot))$. Observe that for any $\widehat{z} \in [0,1]^m$ it holds $v^{(\lambda)}|_{Y\backslash L(\hat{z})} = u_0|_{\varepsilon(\lambda+Y\backslash L(\hat{z}))}(\varepsilon(\lambda+\cdot))$. Since for all $\widehat{z} \in [0,1]^m$ and every $v \in \mathrm{H}^1(Y)^d$ it holds $\mathcal{X}_{\hat{z}}(v|_{Y\backslash L(\hat{z})}) = \mathbf{T}_{\hat{z}}(\widehat{\mathcal{X}}_{0,\hat{z}}(\mathbb{T}_{\hat{z}}(v|_{Y\backslash L(\hat{z})})))$ (see (8.9)) the convergence in (8.23) results, if for $\widehat{z}_0 \in [0,1]$, for $(\widehat{z}_\delta)_{\delta>0} \subset [0,1]^m$ with $\widehat{z}_\delta \to \widehat{z}_0$ in $\mathbb{R}^m$, and for all $w \in \mathrm{H}^1(\mathbb{R}^d)^d$ the following convergences hold true.

$$\Big(\mathbb{T}_{\hat{z}_\delta}(v|_{Y\backslash L(\hat{z}_\delta)})\Big)^{\mathrm{ex}} \overset{\delta\to 0}{\longrightarrow} \Big(\mathbb{T}_{\hat{z}_0}(v|_{Y\backslash L(\hat{z}_0)})\Big)^{\mathrm{ex}} \qquad \text{in } \mathrm{L}^2(\mathbb{R}^d)^d, \tag{8.24a}$$

$$\Big(\widehat{\mathcal{X}}_{0,\hat{z}_\delta}(w|_{\mathrm{Im}(T_{\hat{z}_\delta})\backslash L(0)})\Big)^{\mathrm{ex}} \overset{\delta\to 0}{\longrightarrow} \Big(\widehat{\mathcal{X}}_{0,\hat{z}_0}(w|_{\mathrm{Im}(T_{\hat{z}_0})\backslash L(0)})\Big)^{\mathrm{ex}} \qquad \text{in } \mathrm{L}^2(\mathbb{R}^d)^d, \tag{8.24b}$$

$$\mathbf{T}_{\hat{z}_\delta}(w|_{\mathrm{Im}(T_{\hat{z}_\delta})}) \overset{\delta\to 0}{\longrightarrow} \mathbf{T}_{\hat{z}_0}(w|_{\mathrm{Im}(T_{\hat{z}_0})}) \qquad \text{in } \mathrm{L}^2(Y)^d. \tag{8.24c}$$

Observe that the convergences in (8.24a) and (8.24c) are proven with similar arguments and that (8.24b) is trivially fulfilled for the strong L^2-topology; see (8.7). Therefore, we focus on proving (8.24c). For this purpose, for $w \in \mathrm{H}^1(\mathbb{R}^d)^d$ choose $(w_n)_{n\in\mathbb{N}} \subset \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^d)^d$ such that $w_n \to w$ in $\mathrm{H}^1(\mathbb{R}^d)^d$. Then, according to assumption (8.2i) for any fixed $n \in \mathbb{N}$ it holds $\mathbf{T}_{\hat{z}_\delta}(w_n|_{\mathrm{Im}(T_{\hat{z}_\delta})}) \to \mathbf{T}_{\hat{z}_0}(w_n|_{\mathrm{Im}(T_{\hat{z}_0})})$ pointwise in Y for $\delta \to 0$. Hence, the theorem of dominated convergence yields for every fixed $n \in \mathbb{N}$

$$\lim_{\delta\to 0}\Big\|\mathbf{T}_{\hat{z}_\delta}(w_n|_{\mathrm{Im}(T_{\hat{z}_\delta})}) - \mathbf{T}_{\hat{z}_0}(w_n|_{\mathrm{Im}(T_{\hat{z}_0})})\Big\|_{\mathrm{L}^2(Y)^d} = 0. \tag{8.25}$$

For an arbitrary but fixed $\Delta > 0$ we now choose $n_\Delta \in \mathbb{N}$ such that

$$\|w - w_{n_\Delta}\|_{\mathrm{L}^2(\mathbb{R}^d)^d} \leq \tfrac{\Delta}{3\sqrt{C_2}}. \tag{8.26}$$

According to the convergence result of line (8.25) there exits $\delta_{n_\Delta} > 0$ such that for all $\delta \in (0, \delta_{n_\Delta})$ it holds $\|\mathbf{T}_{\hat{z}_\delta}(w_{n_\Delta}|_{\mathrm{Im}(T_{\hat{z}_\delta})}) - \mathbf{T}_{\hat{z}_0}(w_{n_\Delta}|_{\mathrm{Im}(T_{\hat{z}_0})})\|_{\mathrm{L}^2(Y)^d} \leq \frac{\Delta}{3}$. Keeping this estimate in mind, by triangle inequality we find

$$\begin{aligned}&\Big\|\mathbf{T}_{\hat{z}_\delta}(w|_{\mathrm{Im}(T_{\hat{z}_\delta})}) - \mathbf{T}_{\hat{z}_0}(w|_{\mathrm{Im}(T_{\hat{z}_0})})\Big\|_{\mathrm{L}^2(Y)^d} \\ &\qquad \leq \Big\|\mathbf{T}_{\hat{z}_\delta}\Big((w - w_{n_\Delta})|_{\mathrm{Im}(T_{\hat{z}_\delta})}\Big)\Big\|_{\mathrm{L}^2(Y)^d} + \tfrac{\Delta}{3} + \Big\|\mathbf{T}_{\hat{z}_0}\Big((w_{n_\Delta} - w)|_{\mathrm{Im}(T_{\hat{z}_0})}\Big)\Big\|_{\mathrm{L}^2(Y)^d}.\end{aligned} \tag{8.27}$$

Now, we are going to show that the first and the last term of (8.27) can be estimated by $\frac{\Delta}{3}$. These estimates are verified by the following calculation, where the non-negative integrand of the second line is increased by the left hand side of (8.10). Finally, in line (8.28b) the integral transformation with respect to the substitution $T_{\hat{z}_\delta}(y) = x$ is performed.

$$
\begin{aligned}
&\Big\|\mathbf{T}_{\hat{z}_\delta}\Big((w-w_{n_\Delta})|_{\mathrm{Im}(T_{\hat{z}_\delta})}\Big)\Big\|^2_{\mathrm{L}^2(Y)^d}\\
&\quad= \int_Y \Big|w|_{\mathrm{Im}(T_{\hat{z}_\delta})}(T_{\hat{z}_\delta}(y)) - w_{n_\Delta}|_{\mathrm{Im}(T_{\hat{z}_\delta})}(T_{\hat{z}_\delta}(y))\Big|^2_d \mathrm{d}y\\
&\quad\overset{(8.10)}{\leq} C_2\int_Y \Big|w|_{\mathrm{Im}(T_{\hat{z}_\delta})}(T_{\hat{z}_\delta}(y)) - w_{n_\Delta}|_{\mathrm{Im}(T_{\hat{z}_\delta})}(T_{\hat{z}_\delta}(y))\Big|^2_d |\det(\nabla T_{\hat{z}_\delta}(y))|\mathrm{d}y && (8.28a)\\
&\quad= C_2\int_{\mathrm{Im}(T_{\hat{z}_\delta})} \Big|w|_{\mathrm{Im}(T_{\hat{z}_\delta})}(x) - w_{n_\Delta}|_{\mathrm{Im}(T_{\hat{z}_\delta})}(x)\Big|^2_d \mathrm{d}x && (8.28b)\\
&\quad\leq C_2\Big\|w-w_{n_\Delta}\Big\|^2_{\mathrm{L}^2(\mathbb{R}^d)^d} \overset{(8.26)}{\leq} \left(\tfrac{\Delta}{3}\right)^2 && (8.28c)
\end{aligned}
$$

Observe that the same calculation holds true for the last term of line (8.27) such that for the arbitrary chosen $\Delta > 0$ we showed

$$\Big\|\mathbf{T}_{\hat{z}_\delta}(w|_{\mathrm{Im}(T_{\hat{z}_\delta})}) - \mathbf{T}_{\hat{z}_0}(w|_{\mathrm{Im}(T_{\hat{z}_0})})\Big\|_{\mathrm{L}^2(Y)^d} \leq \Delta$$

by combining (8.27) and (8.28c). Thus, the convergences of (8.24) are verified and altogether, by showing $u_0 = \mathcal{X}_{\varepsilon,z_0}(u_0|_{\Omega\setminus\Omega^{\mathrm{D}}_\varepsilon(z_0)})$, we proved the weak compactness of the energy sublevel sets. □

Remark 8.10. *Note that the statement of Corollary* 8.9 *also holds true in the case of modeling hole initiation as described in Remark* 8.3, *i.e., condition* (8.2g) *does not hold and the assumptions* (8.2b), (8.2h), *and* (8.2i) *are only claimed for* $\widehat{z} \in [0,1)$. *Since Remark* 8.6 *states that the crucial coercivity condition* (8.21) *is valid in this case too, there is no problem in this regard. By keeping the following observations in mind, the proof of the weak compactness of the energy sublevels is analog to that of the proof of Corollary* 8.9. *When estimating*

$$\Big\|u_0 - \mathcal{X}_{\varepsilon,z_0}(u_0|_{\Omega\setminus\Omega^{\mathrm{D}}_\varepsilon(z_0)})\Big\|^2_{\mathrm{L}^2(\Omega^-_\varepsilon)^d} = \sum_{\lambda\in\Lambda^-_\varepsilon}\Big\|u_0 - \mathcal{X}_{\varepsilon,z_0}(u_0|_{\Omega\setminus\Omega^{\mathrm{D}}_\varepsilon(z_0)})\Big\|^2_{\mathrm{L}^2(\varepsilon(\lambda+Y))^d},$$

we find that for $z_0^{(\lambda)} = 1$, *where* $z_0^{(\lambda)} :\equiv z_0|_{\varepsilon(\lambda+Y)}$ *for* $\lambda \in \Lambda^-_\varepsilon$, *we already have that* $\|u_0 - \mathcal{X}_{\varepsilon,z_0}(u_0|_{\Omega\setminus\Omega^{\mathrm{D}}_\varepsilon(z_0)})\|^2_{\mathrm{L}^2(\varepsilon(\lambda+Y))^d} = 0$ *according to* (8.12). *For this reason without loss of generality we assume that* $z_0^{(\lambda)} \leq C < 1$ *for all* $\lambda \in \Lambda^-_\varepsilon$. *Then, we proceed as in the proof of Corollary* 8.9, *i.e., for* $\widehat{z}_0 \leq C < 1$, *for* $(\widehat{z}_\delta)_{\delta>0} \subset [0,1]$ *with* $\widehat{z}_\delta \to \widehat{z}_0$ *in* $\mathbb{R}$, *for* $v \in \mathrm{H}^1(Y)^d$, *and for* $w \in \mathrm{H}^1(\mathbb{R}^d)^d$ *we need to verify* (8.24). *This verification is done in the same manner as before, but observe that in line* (8.28a) *we exploited that estimate* (8.10) *holds for all* $\widehat{z} \in [0,1]$, *which in this case is not available up to now. According to Remark* 8.3 *for* $\widehat{z}^* \in (C,1)$ *the uniform estimate* (8.13), *implying the desired estimate* (8.10) *for all* $\widehat{z} \in [0,\widehat{z}^*]$, *is available. Due to the convergence* $\widehat{z}_\delta \to \widehat{z}_0$ *in* $\mathbb{R}$ *there exists* $\delta_0 > 0$ *such that for all* $\delta \in (0,\delta_0)$ *it holds* $\widehat{z}_\delta \in [0,\widehat{z}^*]$, *which ensures that for* $\delta \in (0,\delta_0)$ *we can apply estimate* (8.10) *in line* (8.28a).

8.1.4 Verification of this chapter's additional assumptions made on $L : [0,1]^m \to \mathcal{L}_{\mathrm{Leb}}(Y)$ for specific choices

In this subsection we are going to verify the in comparison to Chapter 7 additional assumptions made on the microstructure determining function $L : [0,1]^m \to \mathcal{L}_{\mathrm{Leb}}(Y)$ for the specific choices made in Subsection 7.1.2. For this purpose, in the respective case we start by modifying the mapping $L : [0,1]^m \to \mathcal{L}_{\mathrm{Leb}}(Y)$ to ensure that assumption (8.2d) holds. Afterwards, the transformations $(T_{\hat{z}})_{\hat{z}\in[0,1]^m}$ are constructed such that condition (8.2h) is fulfilled and finally assumption (8.2g) is verified for this construction. As one can easily see, all constructed transformations $(T_{\hat{z}})_{\hat{z}\in[0,1]^m}$ automatically satisfy condition (8.2i).

Example 8.11 (Spherical holes)**.** Coming back to Example 7.3 we now assume the damage progression to cause the growth of spherical holes emerging in the center of cells $\varepsilon(\lambda+Y) \subset \Omega$. For this purpose, we adopt the notation of Example 7.3, i.e.,

$$Y := \left\{ y \in \mathbb{R}^d \,\middle|\, y = \sum_{i=1}^{d} k_i b_i, k_i \in [-\tfrac{1}{2}, \tfrac{1}{2}) \right\}$$

and for $m = 1$ and $\hat{z} \in [0,1]$ the mapping $L : [0,1] \to \mathcal{L}_{\mathrm{Leb}}(Y)$ is defined by

$$L(\hat{z}) := (1-\hat{z})\mathrm{cl}(B_R(\mathbf{0})) := \begin{cases} \{y \in Y \mid \frac{1}{(1-\hat{z})} y \in \mathrm{cl}(B_R(\mathbf{0}))\} & \text{if } \hat{z} \in [0,1), \\ \emptyset & \text{if } \hat{z} = 1. \end{cases}$$

According to (8.2c) the set $L(0) = \mathrm{cl}(B_R(\mathbf{0}))$ is assumed to be contained in the non-closed set Y with the same center. Hence, condition (8.2d) is trivially fulfilled. The simplest choice of the bi-Lipschitz transformations $(T_{\hat{z}})_{\hat{z}\in[0,1]}$ is the following: For $\hat{z} \in [0,1]$ let $T_{\hat{z}} : Y \to \mathbb{R}^d$ be defined by

$$T_{\hat{z}}(y) := \begin{cases} \frac{1}{(1-\hat{z})} y & \text{if } \hat{z} \in [0,1), \\ y & \text{if } \hat{z} = 1. \end{cases}$$

Recalling Remark 8.3, for bi-Lipschitz transformations $(T_{\hat{z}})_{\hat{z}\in[0,1]}$ chosen like this, we are able to neglect condition (8.2g) and the assumptions (8.2b) and (8.2h) only have to hold for $\hat{z} \in [0,1)$. Therefore, these modified assumptions (8.2) hold for these specific choices of $L : [0,1] \to \mathcal{L}_{\mathrm{Leb}}(Y)$ and $(T_{\hat{z}})_{\hat{z}\in[0,1]}$.

Example 8.12 (Quadrangular holes)**.** This example is about damage progression causing quadrangular holes, which is similar to the modeling of quadrangular inclusions of weak material discussed in Subsection 7.1.2. Recalling the notation of Example 7.4 for $d = m = 2$ and $\hat{z} \in [0,1]^2$ we set

$$Y(\hat{z}) := \left\{ y \in \mathbb{R}^2 \,\middle|\, y = \sum_{i=1}^{2} \hat{z}_i k_i b_i, k_i \in [-\tfrac{1}{2}, \tfrac{1}{2}) \right\}$$

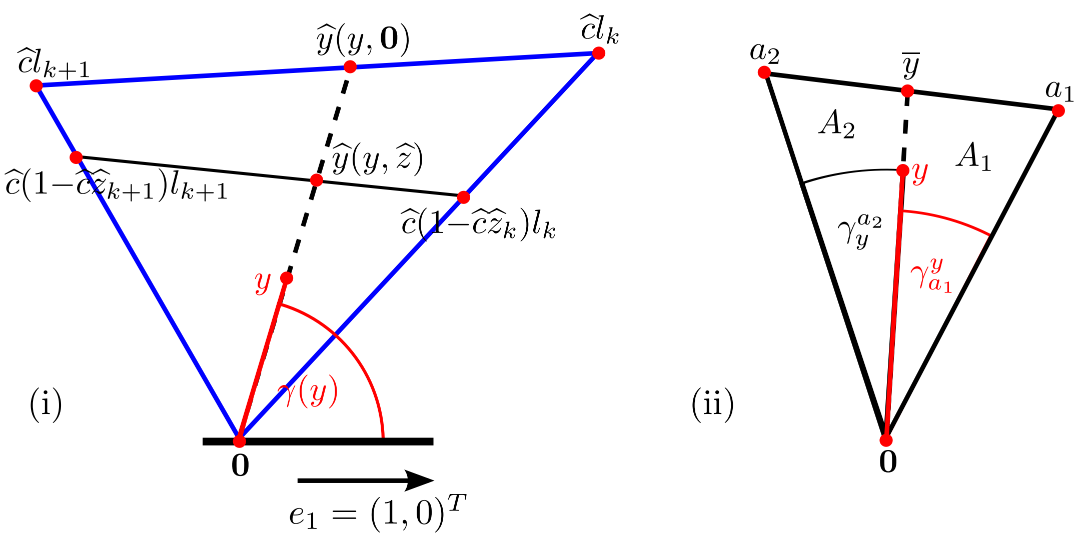


Figure 8.2: Notations for Example 8.13

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and the unit cell Y is defined as in Example 8.11. To ensure the validity of assumption (8.2d) the mapping $L : [0,1]^2 \to \mathcal{L}_{\mathrm{Leb}}(Y)$ needs to be slightly modified in contrast to Example 7.4. For this purpose, let $\widehat{c} \in (0,1)$ and set $L(\widehat{z}) := \mathrm{cl}(Y(\widehat{c}(\mathbf{1}-\widehat{c}\widehat{z})))$. To ensure that condition (8.2h) holds, the bi-Lipschitz transformations need to scale the component y_1 of $y \in Y$ with respect to the value of the $\widehat{z}_1$-component of $\widehat{z} \in [0,1]^2$ (analog for the y_2-component). One possible choice for $(T_{\hat{z}})_{\hat{z}\in[0,1]^2}$ fulfilling (8.2h) is the following: For $\widehat{z} \in [0,1]^2$ let $T_{\hat{z}} : Y \to \mathbb{R}^2$ be defined via

$$T_{\hat{z}}(y) := \left(\tfrac{1}{(1-\hat{c}\hat{z}_1)} y_1, \tfrac{1}{(1-\hat{c}\hat{z}_2)} y_2\right)^T.$$

Calculating

$$\nabla T_{\hat{z}}(y) \ = \mathrm{diag}_{2\times 2}\left(\tfrac{1}{(1-\hat{c}\hat{z}_1)}, \tfrac{1}{(1-\hat{c}\hat{z}_2)}\right)$$

we find $\|\nabla T_{\hat{z}}\|_{\mathrm{L}^\infty(Y)^{2\times 2}} \leq \frac{1}{(1-\hat{c})}$ and $\|\nabla T_{\hat{z}}^{-1}\|_{\mathrm{L}^\infty(\mathrm{Im}(T_{\hat{z}}))^{2\times 2}} \leq 1$ which establishes assumption (8.2g).

Example 8.13 (Anisotropic holes)**.** Here, we enable the model to generate anisotropic defects in the sense of Example 7.6. For this purpose, let $d = 2$, $m > 2$, and for $j \in \{2, 3, \dots, m, m{+}1\}$ choose $\gamma_j \in (0, \pi)$ such that $\sum_{j=2}^{m+1} \gamma_j = 2\pi$. Without loss of generality set $\gamma_1 := 0$ and construct the elements $l_1, l_2, \cdots, l_m \in \mathrm{cl}(Y)$ as in Example 7.6.

Then, for $\widehat{c} \in (0,1)$ we are going to consider the set valued mapping $L : [0,1]^m \to \mathcal{L}_{\text{Leb}}(Y)$ defined by

$$L(\widehat{z}) := \bigcup_{j=1}^{m} \text{triangle}[\widehat{c}(1-\widehat{c}\widehat{z}_j)l_j, \widehat{c}(1-\widehat{c}\widehat{z}_{j+1})l_{j+1}, \mathbf{0}].$$

By inserting the factor $\widehat{c} \in (0,1)$, this mapping obviously fulfills condition (8.2d). To verify the conditions (8.2g) and (8.2h) the bi-Lipschitz transformations $(T_{\hat{z}})_{\hat{z}\in[0,1]^m}$ are constructed as follows: For $\widetilde{y} \in \mathbb{R}^2$ and $\widehat{z} \in [0,1]^m$ let $\widehat{y}(\widetilde{y},\widehat{z}) \in \mathbb{R}^2$ denote the point indicated in Figure 8.2(i). Thus, for $\widehat{z} \in [0,1]^m$ the function $T_{\hat{z}} : Y \to \mathbb{R}^2$ maps $y \in Y$ onto a point which is given by the product of the point $\widehat{y}(y,\mathbf{0})$ and the scaling factor $\frac{|y|_2}{|\hat{y}(y,\hat{z})|_2}$, i.e.,

$$T_{\hat{z}}(y) := \begin{cases} \dfrac{|y|_2}{|\widehat{y}(y,\widehat{z})|_2}\widehat{y}(y,\mathbf{0}) = \dfrac{|\widehat{y}(y,\mathbf{0})|_2}{|\widehat{y}(y,\widehat{z})|_2}y & \text{if } \sum\limits_{j=1}^{k}\gamma_j \leq \gamma(y) < \sum\limits_{j=1}^{k+1}\gamma_j, \\ \mathbf{0} & \text{if } y = \mathbf{0}, \end{cases}$$

where the angle $\gamma(y) \in [0,2\pi)$ for $y = (y_1,y_2)^T \neq \mathbf{0}$ is defined by

$$\gamma(y) := \begin{cases} \arctan(\frac{y_2}{y_1}) & \text{if } y_1 > 0 \text{ and } y_2 \geq 0, \\ \frac{\pi}{2} & \text{if } y_1 = 0 \text{ and } y_2 > 0, \\ \arctan(\frac{y_2}{y_1}) + \pi & \text{if } y_1 < 0, \\ \frac{3\pi}{2} & \text{if } y_1 = 0 \text{ and } y_2 < 0, \\ \arctan(\frac{y_2}{y_1}) + 2\pi & \text{if } y_1 > 0 \text{ and } y_2 < 0. \end{cases}$$

Therefore, for $\widehat{z} \in [0,1]^m$ the mapping $T_{\hat{z}}^{-1} : \mathbb{R}^2 \to Y$ is given by

$$T_{\hat{z}}^{-1}(\widetilde{y}) := \begin{cases} \dfrac{|\widehat{y}(\widetilde{y},\widehat{z})|_2}{|\widehat{y}(\widetilde{y},\mathbf{0})|_2}\widetilde{y} & \text{if } \sum\limits_{j=1}^{k}\gamma_j \leq \gamma(\widetilde{y}) < \sum\limits_{j=1}^{k+1}\gamma_j, \\ \mathbf{0} & \text{if } \widetilde{y} = \mathbf{0}. \end{cases}$$

Due to the similar structure of $T_{\hat{z}} : Y \to \mathbb{R}^2$ and $T_{\hat{z}}^{-1} : \mathbb{R}^2 \to Y$, the verification of the estimate $\sup_{\hat{z}\in[0,1]^m} \|\nabla T_{\hat{z}}^{-1}\|_{\mathrm{L}^\infty(\mathrm{Im}(T_{\hat{z}}))^{2\times 2}} < \infty$ can be done analogously to the proof of $\sup_{\hat{z}\in[0,1]^m} \|\nabla T_{\hat{z}}\|_{\mathrm{L}^\infty(Y)^{2\times 2}} < \infty$ below.

In preparation for specifying the term $T_{\hat{z}}(y)$ in dependence of $y = (y_1,y_2)^T \in Y$ and $\widehat{z} = (\widehat{z}_1,\widehat{z}_2,\ldots,\widehat{z}_m)^T \in [0,1]^m$, we calculate the area $A := \mu_2(\text{triangle}[a_1,a_2,\mathbf{0}])$ (see Figure 8.2(ii)) via

$$A = |a_1|_2|a_2|_2\sin(\gamma_{a_1}^{y}+\gamma_{y}^{a_2}) = A_1+A_2 = |a_1|_2|\overline{y}|_2\sin(\gamma_{a_1}^{y}) + |\overline{y}|_2|a_2|_2\sin(\gamma_{y}^{a_2}),$$

which results in the following description for the value $|\overline{y}|_2$:

$$|\overline{y}|_2 = \frac{|a_1|_2|a_2|_2\sin(\gamma_{a_1}^{y}+\gamma_{y}^{a_2})}{|a_1|_2\sin(\gamma_{a_1}^{y}) + |a_2|_2\sin(\gamma_{y}^{a_2})}. \tag{8.29}$$

For an arbitrary but fixed $\mathbf{0} \neq y \in Y$ there exists a unique $k \in \{1, 2, \dots, m\}$ such that $\sum_{j=1}^{k} \gamma_j \leq \gamma(y) < \sum_{j=1}^{k+1} \gamma_j$. Exploiting (8.29) in some simple but lengthy calculation yields

$$T_{\hat{z}}(y) = \frac{(1-\widehat{c}\widehat{z}_k)|l_k|_2 \sin\Big(\gamma(y) - \sum\limits_{j=1}^{k} \gamma_j\Big) + (1-\widehat{c}\widehat{z}_{k+1})|l_{k+1}|_2 \sin\Big(\sum\limits_{j=1}^{k+1} \gamma_j - \gamma(y)\Big)}{(1-\widehat{c}\widehat{z}_k)(1-\widehat{c}\widehat{z}_{k+1})\Big[|l_k|_2 \sin\Big(\gamma(y) - \sum\limits_{j=1}^{k} \gamma_j\Big) + |l_{k+1}|_2 \sin\Big(\sum\limits_{j=1}^{k+1} \gamma_j - \gamma(y)\Big)\Big]} y$$
$$=: f(\gamma(y), \widehat{z}) y.$$

To verify condition (8.2g) for $y = (y_1, y_2)^T \neq \mathbf{0}$ we calculate

$$\nabla T_{\hat{z}}(y) = \begin{pmatrix} \partial_{y_1} f(\gamma(y), \widehat{z}) y_1 + f(\gamma(y), \widehat{z}) & \partial_{y_2} f(\gamma(y), \widehat{z}) y_1 \\ \partial_{y_2} f(\gamma(y), \widehat{z}) y_1 & \partial_{y_2} f(\gamma(y), \widehat{z}) y_2 + f(\gamma(y), \widehat{z}) \end{pmatrix}$$

To prove $\sup_{\hat{z} \in [0,1]^m} \|\nabla T_{\hat{z}}\|_{\mathrm{L}^\infty(Y)^{2\times 2}} < \infty$ we apply the quotient rule on the explicit expression of $f(\gamma(y), \widehat{z})$. Therefore, for the numerator $f_1(\gamma(y), \widehat{z})$ of $f(\gamma(y), \widehat{z}))$ and its denominator $f_2(\gamma(y), \widehat{z})$ it is sufficient to show that for $i, n, l \in \{1, 2\}$ there exist constants $C_i, C_{i,l,n}, C_0 > 0$ such that $|f_i(\gamma(y), \widehat{z})| \leq C_i$, $|\partial_{y_i} f_l(\gamma(y), \widehat{z}) y_n| \leq C_{i,l,n}$, as well as $|f_2(\gamma(y), \widehat{z})| \geq C_0$ for all $\mathbf{0} \neq y \in Y$ and every $\widehat{z} \in [0, 1]^m$. Trivially, for $i \in \{1, 2\}$, for all $\mathbf{0} \neq y \in Y$ and every $\widehat{z} \in [0, 1]^m$ it holds

$$|f_i(\gamma(y), \widehat{z})| \leq \max_{k \in \{1,2,\dots,m\}} (|l_k|_2 + |l_{k+1}|_2).$$

To show $|\partial_{y_i} f_l(\gamma(y), \widehat{z}) y_n| \leq C_{i,l,n}$ for $i, l, n \in \{1, 2\}$, for all $\mathbf{0} \neq y \in Y$ and every $\widehat{z} \in [0, 1]^m$ it is sufficient to prove that

$$|y_i \partial_{y_l}(\sin(\gamma(y) + C))| = |y_i \cos(\gamma(y) + C) \partial_{y_l} \gamma(y)| \leq |y_i \partial_{y_l} \gamma(y)|$$

($C \in \mathbb{R}$) is bounded. For this purpose, we calculate the weak derivative $\partial_{y_l} \gamma$, which finally yields the following uniform bounds:

$$|y_i \partial_{y_l} \gamma(y)| = \begin{cases} \left| \frac{y_1 y_2}{y_1^2 + y_2^2} \right| \leq 2 & \text{if } i = l = 1 \text{ or } i = l = 2, \\ \left| \frac{y_1^2}{y_1^2 + y_2^2} \right| \leq 1 & \text{if } i = 1 \text{ or } l = 2, \\ \left| \frac{y_2^2}{y_1^2 + y_2^2} \right| \leq 1 & \text{if } i = 2 \text{ or } l = 1. \end{cases}$$

It remains to be shown that there exists a constant $C_0 > 0$ such that $|f_2(\gamma(y), \widehat{z})| \geq C_0$ for all $\mathbf{0} \neq y \in Y$ and every $\widehat{z} \in [0, 1]^m$. Let $\gamma_{\max} := \max_{k \in \{1,2,\dots,m\}} \gamma_k$. Then

$$|f_2(\gamma(y), \widehat{z})| \geq (1-\widehat{c})^2 \min_{k \in \{1,2,\dots,m\}} |l_k|_2 \min_{\tilde{\gamma} \in [0, \gamma_{\max}]} \Big(\sin(\widetilde{\gamma}) + \sin(\gamma_{\max} - \widetilde{\gamma})\Big),$$

where $\min_{\tilde{\gamma} \in [0, \gamma_{\max}]} (\sin(\widetilde{\gamma}) + \sin(\gamma_{\max} - \widetilde{\gamma})) > 0$ since $0 < \gamma_{\max} < \pi$. Thus, the desired estimates are established and hence it holds $\sup_{\hat{z} \in [0,1]^m} \|\nabla T_{\hat{z}}\|_{\mathrm{L}^\infty(Y)^{2\times 2}} < \infty$.

8.2 Two-scale effective damage model based on unidirectional defects' evolution

In this section we formulate the two-scale effective damage model ($\mathbf{S}_{\mathrm{H}}^{0}$) and ($\mathbf{E}_{\mathrm{H}}^{0}$) which will turn out to be the limit model of ($\mathrm{S}_{\mathrm{H}}^{\varepsilon}$) and ($\mathrm{E}_{\mathrm{H}}^{\varepsilon}$) for $\varepsilon \to 0$. Here, damage is described by the two-scale elasticity tensor $\mathbb{C}_0^{\mathrm{H}} : \mathrm{W}^{1,p}(\Omega;[0,1]^m) \to \mathrm{L}^\infty(\Omega\times Y;\{\mathbb{C}_{\mathrm{strong}},\mathbb{O}\})$ which for almost every $(x,y)\in\Omega\times Y$ and $z_0\in\mathrm{W}^{1,p}(\Omega;[0,1]^m)$ is defined via

$$\mathbb{C}_0^{\mathrm{H}}(z_0)(x,y) := \mathbb{1}_{Y\setminus L(z_0(x))}(y)\mathbb{C}_{\mathrm{strong}}. \tag{8.30}$$

As in Section 7.2 this two-scale tensor is motivated by the strong two-scale limit of a sequence $\mathbb{C}_\varepsilon^{\mathrm{H}}(z_\varepsilon)$ of microscopic tensors; see Theorem 3.9. For $\mathcal{Y} := \mathbb{R}/_\Lambda$ denoting the periodicity cell the limit function space $\mathbf{Q}_0^{\mathrm{H}}$ has the following structure:

$$\mathbf{Q}_0^{\mathrm{H}} := \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d\times\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d\times\mathrm{W}^{1,p}(\Omega;[0,1]^m).$$

Recalling Lemma 8.7, concerning the asymptotic behavior of the external forces applied to the microscopic models of Subsection 8.1.3, the external loading for the two-scale limit model is introduced as follows: For $(\ell_0,\ell_1)\in\mathrm{C}^1([0,T];\mathrm{L}^2(\Omega)^d\times\mathrm{L}^2(\Omega)^{d\times d})$ and $z_0\in\mathrm{W}^{1,p}(\Omega;[0,1]^m)$ the external loading $\ell_{z_0}^{\ell_0,\ell_1}\in\mathrm{C}^1([0,T];(\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d\times\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d)^*)$ for all $t\in[0,T]$ and $(u_0,U_1)\in\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d\times\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ is defined by

$$\langle\ell_{z_0}^{\ell_0,\ell_1}(t),(u_0,U_1)\rangle:=\langle h(z_0)\ell_0(t),u_0\rangle_{\mathrm{L}^2(\Omega)^d}+\langle H(z_0)\ell_1(t),\nabla_x Eu_0+\nabla_y U_1\rangle_{\mathrm{L}^2(\Omega\times Y)^{d\times d}}. \tag{8.31}$$

Here, $(h(z_0),H(z_0))\in\mathrm{L}^1(\Omega)\times\mathrm{L}^1(\Omega\times Y)$ for almost every $(x,y)\in\Omega\times Y$ are defined by $h(z_0)(x) := \int_Y \mathbb{1}_{Y\setminus L(z_0(x))}(\widetilde{y})\mathrm{d}\widetilde{y} = \mu_d(Y\setminus L(z_0(x)))$ and $H(z_0)(x,y) := \mathbb{1}_{Y\setminus L(z_0(x))}(y)$. Observe that the measurbility of the function $h(z_0):\Omega\to[0,1]$ is ensured by Funbini's Theorem; see [3, Theorem A 4.10]. For $(u_0,U_1)\in\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d\times\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ we set $\widetilde{\mathbf{e}}(u_0,U_1) := \mathbf{e}_x(u_0)+\mathbf{e}_y(U_1)$. Thus, the energy functional $\mathbf{E}_0^{\mathrm{H}} : [0,T]\times\mathbf{Q}_0^{\mathrm{H}}\to\mathbb{R}$ is given by

$$\begin{aligned}&\mathbf{E}_0^{\mathrm{H}}(t,u_0,U_1,z_0)\\&\quad:= \tfrac{1}{2}\langle\mathbb{C}_0^{\mathrm{H}}(z_0)\widetilde{\mathbf{e}}(u_0,U_1),\widetilde{\mathbf{e}}(u_0,U_1)\rangle_{\mathrm{L}^2(\Omega\times Y)^{d\times d}} + \|\nabla z_0\|^p_{\mathrm{L}^p(\Omega)^{m\times d}} - \langle\ell_{z_0}^{\ell_0,\ell_1}(t),(u_0,U_1)\rangle.\end{aligned} \tag{8.32}$$

Remark 8.14. *Observe that this two-scale limit functional is motivated by investigating the asymptotic behavior of the microscopic energy functionals $(\mathcal{E}_\varepsilon^{\mathrm{H}})_{\varepsilon>0}$ with respect to $(z_\varepsilon)_{\varepsilon>0}$ with $z_\varepsilon\in\mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ and $(u_\varepsilon)_{\varepsilon>0}\subset\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ with $u_\varepsilon = \mathcal{X}_{\varepsilon,z_\varepsilon}(u_\varepsilon|_{\Omega\setminus\Omega_\varepsilon^{\mathrm{D}}(z_\varepsilon)})$, which converge in the following sense: Let $(u_0,U_1,z_0)\in\mathbf{Q}_0^{\mathrm{H}}$ be given such that*

$$\begin{array}{llll} u_\varepsilon \rightharpoonup u_0 & \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d, & z_\varepsilon\to z_0 & \text{in } \mathrm{L}^p(\Omega)^m,\\ u_\varepsilon \xrightarrow{s} Eu_0 & \text{in } \mathrm{L}^2(\Omega\times Y)^d, & R_{\frac{\varepsilon}{2}}z_\varepsilon|_\Omega \rightharpoonup \nabla z_0 & \text{in } \mathrm{L}^p(\Omega)^{m\times d},\\ \nabla u_\varepsilon \xrightharpoonup{w} \nabla_x Eu_0+\nabla_y U_1 & \text{in } \mathrm{L}^2(\Omega\times Y)^{d\times d}. & & \end{array}$$

By improving the assumptions on the transformations $(T_{\hat{z}})_{\hat{z}\in[0,1]^m}$ one might force the limit function $U_1\in\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ to satisfy

$$U_1(x,\cdot) = \mathcal{X}_{z_0(x)}(U_1(x,\cdot)|_{Y\setminus L(z_0(x))}) \quad \textit{for almost every } x\in\Omega. \tag{8.33}$$

Here, $\mathcal{X}_{\hat{z}} : \mathrm{H}^1(Y\backslash L(\widehat{z}))^d \to \mathrm{H}^1(Y)^d$ $(\widehat{z} \in [0,1]^m)$ *denotes the strong* 1*-extension operator of Lemma* 8.2*. In this case the limit energy functional* $\overline{\mathbf{E}}_0^{\mathrm{H}} : [0,T]\times\mathbf{Q}_0^{\mathrm{H}} \to \mathbb{R}_\infty$ *would be defined by*

$$\overline{\mathbf{E}}_0^{\mathrm{H}}(t,u_0,U_1,z_0) := \begin{cases} \mathbf{E}_0^{\mathrm{H}}(t,u_0,U_1,z_0) & \textit{if } U_1 \textit{ satisfies } (8.33), \\ \infty & \textit{otherwise.} \end{cases} \tag{8.34}$$

This would have the advantage that for $M(z_0) := \bigcup_{x\in\Omega}\{x\}\times L(z_0(x))$ *the physically senseless part* $U_1|_{M(z_0)}$ *of the two-scale displacement field is uniquely described by the part* $U_1|_{(\Omega\times Y)\backslash M(z_0)}$ *given on the set of positive stiffness. Therefore, the energy functional* $\overline{\mathbf{E}}_0^{\mathrm{H}} : [0,T]\times\mathbf{Q}_0^{\mathrm{H}} \to \mathbb{R}_\infty$ *is coercive, which is not the case for the functional defined in line* (8.32)*. However, since these physically senseless values are cut off by the material tensor* $\mathbb{C}_0^{\mathrm{H}}(z_0)$ *and the external loading* $\ell_{z_0}^{\ell_0,\ell_1}$ *anyway, we do not continue to pursue this ansatz in the following.*

Since there were no changes in the microscopic dissipation distances in comparison to those of Chapter 7, for the limit function $\kappa_0^{\mathrm{H}} \in \mathrm{L}^{q'}(\Omega;[0,\infty))^m$ (see (8.22)) the limit dissipation distance $\mathbf{D}_0^{\mathrm{H}} : \mathrm{W}^{1,p}(\Omega;[0,1]^m)\times\mathrm{W}^{1,p}(\Omega;[0,1]^m) \to [0,\infty]$ of the sequence $(\mathcal{D}_\varepsilon^{\mathrm{H}})_{\varepsilon>0}$ is defined via (see Section 7.2)

$$\mathbf{D}_0^{\mathrm{H}}(z_1,z_2) := \begin{cases} \displaystyle\int_\Omega |\langle \kappa_0^{\mathrm{H}}(x), z_2(x)-z_1(x)\rangle_m|\mathrm{d}x & \text{if } z_1 \geq z_2, \\ \infty & \text{otherwise.} \end{cases} \tag{8.35}$$

For given initial values $(u_0^0, U_1^0, z_0^0) \in \mathbf{Q}_0^{\mathrm{H}}$ the rate-independent damage evolution is modeled by the energetic formulation $(\mathbf{S}_{\mathrm{H}}^0)$ and $(\mathbf{E}_{\mathrm{H}}^0)$.

Stability condition $(\mathbf{S}_{\mathrm{H}}^0)$ and energy balance $(\mathbf{E}_{\mathrm{H}}^0)$ for all $t \in [0,T]$:

$$\mathbf{E}_0^{\mathrm{H}}(t,u_0(t),U_1(t),z_0(t)) \leq \mathbf{E}_0^{\mathrm{H}}(t,\widetilde{u},\widetilde{U},\widetilde{z}) + \mathbf{D}_0^{\mathrm{H}}(z_0(t),\widetilde{z}) \quad \text{for all } (\widetilde{u},\widetilde{U},\widetilde{z}) \in \mathbf{Q}_0^{\mathrm{H}}$$

$$\mathbf{E}_0^{\mathrm{H}}(t,u_0(t),U_1(t),z_0(t)) + \mathrm{Diss}_{\mathbf{D}_0^{\mathrm{H}}}(z_0;[0,t]) = \mathbf{E}_0^{\mathrm{H}}(0,u_0^0,U_1^0,z_0^0) - \int_0^t \langle \dot{\ell}_{z_0(s)}^{\ell_0,\ell_1}(s), (u_0(s),U_1(s))\rangle \mathrm{d}s$$

Here, $\mathrm{Diss}_{\mathbf{D}_0^{\mathrm{H}}}(z_0;[0,t]) := \sup \sum_{j=1}^N \mathbf{D}_0^{\mathrm{H}}(z_0(t_{j-1}),z_0(t_j))$, where for $N \in \mathbb{N}$ the supremum is taken with respect to all finite partitions $\pi_N := \{0 = t_0 < t_1 < \ldots < t_N = t\}$ of the interval $[0,T]$. The following existence result extends Theorem 6.18 to the situation with voids.

Theorem 8.15 (Existence of solutions)**.** *Assume that the conditions* (8.1) *and* (8.2) *hold. Let* $\mathbf{E}_0^{\mathrm{H}} : [0,T]\times\mathbf{Q}_0^{\mathrm{H}} \to \mathbb{R}$ *and* $\mathbf{D}_0^{\mathrm{H}} : \mathrm{W}^{1,p}(\Omega;[0,1]^m)\times\mathrm{W}^{1,p}(\Omega;[0,1]^m) \to [0,\infty]$ *be defined as described above. Moreover, let* $(u_0^0,U_1^0,z_0^0) \in \mathbf{Q}_0^{\mathrm{H}}$ *be given such that it is the limit of a stable sequence* $(u_\varepsilon^0, z_\varepsilon^0)_{\varepsilon>0}$ *with respect to* $0 \in [0,T]$ *in the sense of Definition* 6.12*. If* $\nabla u_\varepsilon^0 \stackrel{s}{\to} \nabla_x E u_0^0 + \nabla_y U_1^0$ *in* $\mathrm{L}^2(\Omega\times Y)^{d\times d}$ *and* $R_{\frac{\varepsilon}{2}} z_\varepsilon^0|_\Omega \to \nabla z_0^0$ *in*

$\mathrm{L}^p(\Omega)^{m\times d}$, *then there exists an energetic solution* $(u_0, U_1, z_0) : [0,T] \to \mathbf{Q}_0^{\mathrm{H}}$ *of the rate-independent system* $(\mathbf{Q}_0^{\mathrm{H}}, \mathbf{E}_0^{\mathrm{H}}, \mathbf{D}_0^{\mathrm{H}})$ *with initial condition* (u_0^0, U_1^0, z_0^0) *satisfying*

$$\begin{aligned}(u_0, U_1) &\in \mathrm{L}^\infty([0,T]; \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \times \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d),\\ z_0 &\in \mathrm{L}^\infty([0,T]; \mathrm{W}^{1,p}(\Omega;[0,1]^m)) \cap \mathrm{BV}_{\mathbf{D}_0^{\mathrm{H}}}([0,T]; \mathrm{W}^{1,p}(\Omega;[0,1]^m)).\end{aligned}$$

Remark 8.16. *For* $(u_0, U_1, z_0) : [0,T] \to \mathbf{Q}_0^{\mathrm{H}}$ *being an energetic solution to* $(\mathbf{S}_{\mathrm{H}}^0)$ *and* $(\mathbf{E}_{\mathrm{H}}^0)$ *let* $M^c(z_0(t)) := (\Omega\times Y)\backslash M(z_0(t))$ *denote the set of positive stiffness at* $t \in [0,T]$ $(M(z_0(t)) := \bigcup_{x\in\Omega}\{x\}\times L(z_0(t,x)))$. *Then due to the zero stiffness on* $M(z_0(t))$ *any function* $(u_0, \widehat{U}_1, z_0) : [0,T] \to \mathbf{Q}_0^{\mathrm{H}}$ *with* $\widehat{U}_1(t)|_{M^c(z_0(t))} = U_1(t)|_{M^c(z_0(t))}$ *for all* $t \in [0,T]$ *is an energetic solution of* $(\mathbf{S}_{\mathrm{H}}^0)$ *and* $(\mathbf{E}_{\mathrm{H}}^0)$*, too.*

Proof. To proceed analogously to the proof of Theorem 6.18, observe that all assumptions of Theorem 6.18 are satisfied in this case (ignoring the fact that here the microscopic energy functionals are allowed to take the value ∞); see the proof of Corollary 7.14. Since the proof of Theorem 6.18 exploits the results of Theorem 6.15 and Theorem 6.17, they have to be modified, too. These adaptations are done in Corollary 8.17 and Corollary 8.18 below. Afterwards, by following the proof of Theorem 6.18, establishing Theorem 8.15 is straight forward. □

Corollary 8.17 (Mutual recovery sequence)**.** *Assume that* (8.1) *and* (8.2) *hold. For* $(\ell_0, \ell_1) \in \mathrm{C}^1([0,T]; \mathrm{L}^2(\Omega)^d \times \mathrm{L}^2(\Omega)^{d\times d})$, $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$, *and* $z_0 \in \mathrm{W}^{1,p}(\Omega;[0,1]^m)$ *let* $\ell_{z_\varepsilon}^{\ell_0,\ell_1} \in \mathrm{C}^1([0,T]; (\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$ *and* $\ell_{z_0}^{\ell_0,\ell_1} \in \mathrm{C}^1([0,T]; (\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$ *be defined by* (8.17) *and* (8.31)*, respectively. Moreover, let* $\mathcal{E}_\varepsilon^{\mathrm{H}} : [0,T]\times\mathcal{Q}_\varepsilon^{\mathrm{H}}(\Omega) \to \mathbb{R}_\infty$ *be defined via* (8.20) *and let* $\mathcal{D}_\varepsilon^{\mathrm{H}} : \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)\times\mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m) \to [0,\infty]$ *be defined by* (8.22)*. Finally, let* $\mathbf{E}_0^{\mathrm{H}} : [0,T]\times\mathbf{Q}_0^{\mathrm{H}} \to \mathbb{R}$ *and* $\mathbf{D}_0^{\mathrm{H}} : \mathrm{W}^{1,p}(\Omega;[0,1]^m)\times\mathrm{W}^{1,p}(\Omega;[0,1]^m) \to [0,\infty]$ *be given as described above. If* $(u_\varepsilon, z_\varepsilon)_{\varepsilon>0}$ *is a stable sequence with respect to* $\widetilde{t} \in [0,T]$ *(see Definition* 6.12*) with limit* $(u_0, U_1, z_0) \in \mathbf{Q}_0^{\mathrm{H}}$*, then:*

(a) *For every* $(\widetilde{u}_0, \widetilde{U}_1, \widetilde{z}_0) \in \mathbf{Q}_0^{\mathrm{H}}$ *there exists a mutual recovery sequence* $(\widetilde{\widetilde{u}}_\varepsilon, \widetilde{z}_\varepsilon)_{\varepsilon>0}$*; see Definition* 6.13*.*

(b) (u_0, U_1, z_0) *satisfies the stability condition* $(\mathbf{S}_{\mathrm{H}}^0)$ *for* $t = \widetilde{t}$*.*

Proof. Part (a): Let $(u_\varepsilon, z_\varepsilon)_{\varepsilon>0}$ be a the stable sequence with respect to $t \in [0,T]$ converging to the limit $(u_0, U_1, z_0) \in \mathbf{Q}_0^{\mathrm{H}}$; see Definition 6.12. Then, for a given function $(\widetilde{u}_0, \widetilde{U}_1, \widetilde{z}_0) \in \mathbf{Q}_0$ we start by constructing the mutual recovery sequence $(\widetilde{\widetilde{u}}_\varepsilon, \widetilde{z}_\varepsilon)_{\varepsilon>0}$.

1. This first step is completely analog to the proof of Theorem 6.15. Without loss of generality let $\widetilde{z}_0 \le z_0$. Then, according to Theorem 6.16 there exists a sequence $(\widetilde{z}_\varepsilon)_{\varepsilon>0}$ satisfying $\widetilde{z}_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$, $\widetilde{z}_\varepsilon \le z_\varepsilon$, $\widetilde{z}_\varepsilon \to \widetilde{z}_0$ in $\mathrm{L}^p(\Omega)^m$, $R_{\frac{\varepsilon}{2}}\widetilde{z}_\varepsilon|_\Omega \rightharpoonup \nabla\widetilde{z}_0$ in $\mathrm{L}^p(\Omega)^{m\times d}$,

$$\limsup_{\varepsilon\to 0}\Big(\|R_{\frac{\varepsilon}{2}}\widetilde{z}_\varepsilon\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}} - \|R_{\frac{\varepsilon}{2}}z_\varepsilon\|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}}\Big) \le \|\nabla\widetilde{z}_0\|^p_{\mathrm{L}^p(\Omega)^{m\times d}} - \|\nabla z_0\|^p_{\mathrm{L}^p(\Omega)^{m\times d}}, \quad (8.36)$$

and $\lim_{\varepsilon\to} \mathcal{D}_\varepsilon^{\mathrm{H}}(z_\varepsilon, \widetilde{z}_\varepsilon) = \mathbf{D}_0^{\mathrm{H}}(z_0, \widetilde{z}_0)$.

2. Now, the u-component $\widetilde{\widetilde{u}}_\varepsilon \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ of the mutual recovery sequence is constructed. For the sequence $(\widetilde{z}_\varepsilon)_{\varepsilon>0}$ introduced in step 1 the sequence $(\widetilde{\widetilde{u}}_\varepsilon)_{\varepsilon>0} \subset \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ is defined by

$$\widetilde{\widetilde{u}}_\varepsilon := \mathcal{X}_{\varepsilon,\widetilde{z}_\varepsilon}(\widetilde{u}_\varepsilon|_{\Omega\backslash\Omega^{\mathrm{D}}_\varepsilon(\widetilde{z}_\varepsilon)}), \tag{8.37}$$

where $(\widetilde{u}_\varepsilon)_{\varepsilon>0} \subset \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ is the sequence constructed in the second step of the proof of Theorem 6.15 and satisfies

$$\begin{aligned} \widetilde{u}_\varepsilon &\rightharpoonup \widetilde{u}_0 && \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d, \\ \widetilde{u}_\varepsilon &\xrightarrow{s} E\widetilde{u}_0 && \text{in } \mathrm{L}^2(\Omega\times Y)^d, \\ \nabla\widetilde{u}_\varepsilon &\xrightarrow{s} \nabla_x E\widetilde{u}_0 + \nabla_y \widetilde{U}_1 && \text{in } \mathrm{L}^2(\Omega\times Y)^{d\times d}. \end{aligned}$$

Note that we do not claim, that this convergence result also holds for the sequence $(\widetilde{\widetilde{u}}_\varepsilon)_{\varepsilon>0}$. In fact, whenever the asymptotic behavior of a term involving the function $\widetilde{\widetilde{u}}_\varepsilon$ is investigated in the following, then by definition this term actually depends on the product $\mathbb{1}_{\Omega\backslash\Omega^{\mathrm{D}}_\varepsilon(\widetilde{z}_\varepsilon)}\widetilde{\widetilde{u}}_\varepsilon$. Since $\widetilde{\widetilde{u}}_\varepsilon$ and $\widetilde{u}_\varepsilon$ coincide on $\mathrm{supp}(\mathbb{1}_{\Omega\backslash\Omega^{\mathrm{D}}_\varepsilon(\widetilde{z}_\varepsilon)}) = \Omega\backslash\Omega^{\mathrm{D}}_\varepsilon(\widetilde{z}_\varepsilon)$, thus the convergence results for the sequence $(\widetilde{u}_\varepsilon)_{\varepsilon>0}$ can be exploited. The adaptation (8.37) is only made to ensures $\mathcal{E}^{\mathrm{H}}_\varepsilon(\widetilde{t}, \widetilde{\widetilde{u}}_\varepsilon, \widetilde{z}_\varepsilon) < \infty$ for all $\varepsilon > 0$, which, for instance, is necessary to prove the lim sup-inequality (6.24). In fact, every sequence $(\widetilde{u}_\varepsilon)_{\varepsilon>0}$ appearing in the proofs of Theorem 6.17 and Theorem 6.18 has to be replaced by $(\widetilde{\widetilde{u}}_\varepsilon)_{\varepsilon>0}$ to prove the modified results stated in Theorem 8.15 and Corollary 8.18.

3. According to Lemma 8.7, for the given $\widetilde{t} \in [0,T]$ we have

$$\lim_{\varepsilon\to 0}\Big(\langle \ell^{\ell_0,\ell_1}_{z_\varepsilon}(\widetilde{t}), u_\varepsilon\rangle - \langle \ell^{\ell_0,\ell_1}_{\widetilde{z}_\varepsilon}(\widetilde{t}), \widetilde{\widetilde{u}}_\varepsilon\rangle\Big) = \langle \ell^{\ell_0,\ell_1}_{z_0}(\widetilde{t}), (u_0,U_1)\rangle - \langle \ell^{\ell_0,\ell_1}_{\widetilde{z}_0}(\widetilde{t}), (\widetilde{u}_0,\widetilde{U}_1)\rangle, \tag{8.38}$$

where we already exploited, that $\widetilde{\widetilde{u}}_\varepsilon$ and $\widetilde{u}_\varepsilon$ coincide on $\mathrm{supp}(\mathbb{1}_{\Omega\backslash\Omega^{\mathrm{D}}_\varepsilon(\widetilde{z}_\varepsilon)}) = \Omega\backslash\Omega^{\mathrm{D}}_\varepsilon(\widetilde{z}_\varepsilon)$ which implies $\langle \ell^{\ell_0,\ell_1}_{\widetilde{z}_\varepsilon}(\widetilde{t}), \widetilde{\widetilde{u}}_\varepsilon\rangle = \langle \ell^{\ell_0,\ell_1}_{\widetilde{z}_\varepsilon}(\widetilde{t}), \widetilde{u}_\varepsilon\rangle$ for all $\varepsilon > 0$; see (8.17).

4. Analog to the fourth step of the proof of Theorem 6.15, we have

$$\begin{aligned} &\limsup_{\varepsilon\to 0}\Big(\langle \mathbb{C}^{\mathrm{H}}_\varepsilon(\widetilde{z}_\varepsilon)\mathbf{e}(\widetilde{u}_\varepsilon), \mathbf{e}(\widetilde{u}_\varepsilon)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} - \langle \mathbb{C}^{\mathrm{H}}_\varepsilon(z_\varepsilon)\mathbf{e}(u_\varepsilon), \mathbf{e}(u_\varepsilon)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}}\Big) \qquad (8.39) \\ &\quad \le \langle \mathbb{C}^{\mathrm{H}}_0(\widetilde{z}_0)\widetilde{\mathbf{e}}(\widetilde{u}_0,\widetilde{U}_1), \widetilde{\mathbf{e}}(\widetilde{u}_0,\widetilde{U}_1)\rangle_{\mathrm{L}^2(\Omega\times Y)^{d\times d}} - \langle \mathbb{C}^{\mathrm{H}}_0(z_0)\widetilde{\mathbf{e}}(u_0,U_1), \widetilde{\mathbf{e}}(u_0,U_1)\rangle_{\mathrm{L}^2(\Omega\times Y)^{d\times d}}. \end{aligned}$$

Since the functions $\widetilde{\widetilde{u}}_\varepsilon$ and $\widetilde{u}_\varepsilon$ coincide on the set $\mathrm{supp}(\mathbb{C}^{\mathrm{H}}_\varepsilon(\widetilde{z}_\varepsilon)) = \Omega\backslash\Omega^{\mathrm{D}}_\varepsilon(\widetilde{z}_\varepsilon)$ it holds

$$\mathcal{E}^{\mathrm{H}}_\varepsilon(\widetilde{t}, \widetilde{\widetilde{u}}_\varepsilon, \widetilde{z}_\varepsilon) = \tfrac{1}{2}\langle \mathbb{C}^{\mathrm{H}}_\varepsilon(\widetilde{z}_\varepsilon)\mathbf{e}(\widetilde{u}_\varepsilon), \mathbf{e}(\widetilde{u}_\varepsilon)\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} + \|R_{\frac{\varepsilon}{2}}\widetilde{z}_\varepsilon\|^p_{\mathrm{L}^p(\Omega^+_\varepsilon)^{m\times d}} - \langle \ell^{\ell_0,\ell_1}_{\widetilde{z}_\varepsilon}(\widetilde{t}), \widetilde{u}_\varepsilon\rangle,$$

although $\mathcal{E}^{\mathrm{H}}_\varepsilon(\widetilde{t}, \widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon)$ might be infinite due to the energy functional's definition; see (8.20). Combining (8.36), (8.38), and (8.39) we finally end up with

$$\limsup_{\varepsilon\to 0}\Big(\mathcal{E}^{\mathrm{H}}_\varepsilon(\widetilde{t}, \widetilde{\widetilde{u}}_\varepsilon, \widetilde{z}_\varepsilon) - \mathcal{E}^{\mathrm{H}}_\varepsilon(\widetilde{t}, u_\varepsilon, z_\varepsilon)\Big) \le \mathbf{E}^{\mathrm{H}}_0(\widetilde{t}, \widetilde{u}_0, \widetilde{U}_1, \widetilde{z}_0) - \mathbf{E}^{\mathrm{H}}_0(\widetilde{t}, u_0, U_1, z_0)$$

i.e., $(\widetilde{\widetilde{u}}_\varepsilon, \widetilde{z}_\varepsilon)_{\varepsilon>0}$ is a mutual recovery sequence; see Definition 6.13

The proof of part (b) is exactly the same as in the proof of Theorem 6.15. □

Corollary 8.18 ("Mosco" convergence of $(\mathcal{E}^{\mathrm{H}}_\varepsilon)_{\varepsilon>0}$ to $\mathbf{E}^{\mathrm{H}}_0$). *Let $(u_\varepsilon, z_\varepsilon)_{\varepsilon>0}$ be a sequence satisfying $(u_\varepsilon, z_\varepsilon) \in \mathcal{Q}^{\mathrm{H}}_\varepsilon(\Omega)$ for all $\varepsilon > 0$ and*

$$\begin{array}{rlrl} u_\varepsilon \rightharpoonup u_0 & \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d, & z_\varepsilon \to z_0 & \text{in } \mathrm{L}^p(\Omega)^m, \\ u_\varepsilon \xrightarrow{s} Eu_0 & \text{in } \mathrm{L}^2(\Omega\times Y)^d, & R_{\frac{\varepsilon}{2}} z_\varepsilon|_\Omega \rightharpoonup \nabla z_0 & \text{in } \mathrm{L}^p(\Omega)^{m\times d}, \\ \nabla u_\varepsilon \xrightharpoonup{w} \nabla_x Eu_0 + \nabla_y U_1 & \text{in } \mathrm{L}^2(\Omega\times Y)^{d\times d}. & & \end{array}$$

Then for every $t \in [0,T]$ it holds $\liminf_{\varepsilon\to 0} \mathcal{E}^{\mathrm{H}}_\varepsilon(t, u_\varepsilon, z_\varepsilon) \geq \mathbf{E}^{\mathrm{H}}_0(t, u_0, U_1, z_0)$.

Moreover, for every function $(\widetilde{u}_0, \widetilde{U}_1, \widetilde{z}_0) \in \mathbf{Q}^{\mathrm{H}}_0$ there exists a sequence $(\widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon)_{\varepsilon>0}$ with $(\widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon) \in \mathcal{Q}^{\mathrm{H}}_\varepsilon(\Omega)$ for every $\varepsilon > 0$ such that

$$\begin{array}{rlrl} \widetilde{u}_\varepsilon \rightharpoonup \widetilde{u}_0 & \text{in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d, & \widetilde{z}_\varepsilon \to \widetilde{z}_0 & \text{in } \mathrm{L}^p(\Omega)^m, \\ \widetilde{u}_\varepsilon \xrightarrow{s} E\widetilde{u}_0 & \text{in } \mathrm{L}^2(\Omega\times Y)^d, & R_{\frac{\varepsilon}{2}} \widetilde{z}_\varepsilon|_\Omega \to \nabla \widetilde{z}_0 & \text{in } \mathrm{L}^p(\Omega)^{m\times d}, \\ \nabla \widetilde{u}_\varepsilon \xrightarrow{s} \nabla_x E\widetilde{u}_0 + \nabla_y \widetilde{U}_1 & \text{in } \mathrm{L}^2(\Omega\times Y)^{d\times d}, & & \end{array}$$

and $\lim_{\varepsilon\to 0} \mathcal{E}^{\mathrm{H}}_\varepsilon(t, \widetilde{\widetilde{u}}_\varepsilon, \widetilde{z}_\varepsilon) = \mathbf{E}^{\mathrm{H}}_0(t, \widetilde{u}_0, \widetilde{U}_1, \widetilde{z}_0)$, where $\widetilde{\widetilde{u}}_\varepsilon := \mathcal{X}_{\varepsilon,\tilde{z}_\varepsilon}(\widetilde{u}_\varepsilon|_{\Omega\setminus\Omega^{\mathrm{D}}_\varepsilon(\tilde{z}_\varepsilon)})$.

Proof. The proof is completely analog to that of Theorem 6.17. However, observe that to guarantee that $\mathcal{E}_\varepsilon(t, \widetilde{\widetilde{u}}_\varepsilon, \widetilde{z}_\varepsilon)$ is finite for all $\varepsilon > 0$, the recovery sequence needs to be redefined ($\widetilde{\widetilde{u}}_\varepsilon := \mathcal{X}_{\varepsilon,\tilde{z}_\varepsilon}(\widetilde{u}_\varepsilon|_{\Omega\setminus\Omega^{\mathrm{D}}_\varepsilon(\tilde{z}_\varepsilon)})$) on the set $\Omega^{\mathrm{D}}_\varepsilon(\widetilde{z}_\varepsilon)$ of zero stiffness. Again, we do not claim that the sequence $(\widetilde{\widetilde{u}}_\varepsilon)_{\varepsilon>0} \subset \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ converges in any sense to the functions $(\widetilde{u}_0, \widetilde{U}_1) \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \times \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$. That is why, roughly spoken, Corollary 8.18 states Mosco convergence only on the set of positive stiffness, which is sufficient to prove Theorem 8.15 analogously to Theorem 6.18. $\square$

8.3 One-scale effective damage model based on unidirectional defects' evolution

As in the Chapters 6–7 we are able to formulate a one-scale model which is equivalent to that of Section 8.2. However, in contrast to these previous results this equivalence does only hold by making some further assumptions. Let the state space $\mathcal{Q}^{\mathrm{H}}_0(\Omega)$ be given by

$$\mathcal{Q}^{\mathrm{H}}_0(\Omega) := \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \times \mathrm{W}^{1,p}(\Omega; [0,1]^m).$$

Recalling the motivation of Section 6.3, the equivalence of the models in Section 6.2 and 6.3 is based on the fact that for given stable state $(u_0, U_1, z_0) \in \mathbf{S}_0(t)$ the function $U_1 \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ is the unique solution of (6.15). This minimizing property of the function $U_1 \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ results from the stability condition ($\mathbf{S}^0$) by choosing $(\widetilde{u}, \widetilde{z}) = (u_0, z_0)$. However, since in the two-scale model of Section 8.2 the function $U_1 \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ enters the term $\langle \ell^{\ell_0,\ell_1}_{z_0}(t), (u_0, U_1)\rangle$ in the two-scale energy $\mathbf{E}^{\mathrm{H}}_0(t, u_0, U_1, z_0)$, we need to restrict ourselves to the following external loadings:

For given $\ell_0 \in \mathrm{C}^1([0,T];\mathrm{L}^2(\Omega)^d)$ and $z_0 \in \mathrm{W}^{1,p}(\Omega;[0,1]^m)$ let the external loading $\ell_{z_0}^{\ell_0} \in \mathrm{C}^1([0,T];(\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$ for all $t \in [0,T]$ and every $u_0 \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ be defined by

$$\langle \ell_{z_0}^{\ell_0}(t), u_0\rangle := \langle h(z_0)\ell_0(t), u_0\rangle_{\mathrm{L}^2(\Omega)^d}. \tag{8.40}$$

Here, $h(z_0) \in \mathrm{L}^1(\Omega)$ for almost every $x \in \Omega$ is defined by $h(z_0)(x) := \fint_Y \mathbb{1}_{Y\setminus L(z_0(x))}(y)\mathrm{d}y$.

Remark 8.19. *Observe that for any function* $(u_0, U_1) \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \times \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ *it holds* $\langle \ell_{z_0}^{\ell_0}(t), u_0\rangle = \langle \ell_{z_0}^{\ell_0,\mathbf{0}}(t), (u_0, U_1)\rangle$*; see* (8.31)*. Therefore, in the case* $\ell_1 \equiv \mathbf{0}$ *the external loading* $\ell_{z_0}^{\ell_0,\mathbf{0}}$ *for the two-scale model of Section* 8.2 *can be understood as an element of* $\mathrm{C}^1([0,T];(\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$.

Analog to Section 6.3, the energy functional is based on $\mathbb{C}^{\mathrm{H}}_{\mathrm{eff}} : \mathrm{W}^{1,p}(\Omega;[0,1]^m) \to \mathcal{M}(\Omega)$, which for $\xi \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, for almost every $x \in \Omega$, and for $z_0 \in \mathrm{W}^{1,p}(\Omega;[0,1]^m)$ is defined via the unit cell problem

$$\langle \mathbb{C}^{\mathrm{H}}_{\mathrm{eff}}(z_0)(x)\xi, \xi\rangle_{d\times d} := \min_{v\in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d} I^{\mathrm{H}}(z_0(x), \xi, v). \tag{8.41}$$

Here, for $\widehat{z} \in [0,1]^m$ the functional $I^{\mathrm{H}}(\widehat{z}, \xi, \cdot) : \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d \to \mathbb{R}_\infty$ is introduced via

$$I^{\mathrm{H}}(\widehat{z}, \xi, v) := \begin{cases} \widetilde{I}^{\mathrm{H}}(\widehat{z}, \xi, v) & \text{if } v = \mathcal{X}_{\hat{z}}(v|_{Y\setminus L(\hat{z})}), \\ \infty & \text{otherwise,} \end{cases}$$

where the continuous functional $\widetilde{I}^{\mathrm{H}}(\widehat{z}, \xi, \cdot) : \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d \to \mathbb{R}$ for the given tensor valued mapping $\widehat{\mathbb{C}}^{\mathrm{H}} : [0,1]^m \to \mathrm{L}^\infty(Y;\{\mathbb{C}_{\mathrm{strong}}, \mathbb{O}\})$ is given by

$$\widetilde{I}^{\mathrm{H}}(\widehat{z}, \xi, v) := \int_Y \Big\langle \widehat{\mathbb{C}}^{\mathrm{H}}(\widehat{z})(y)\Big(\xi + \mathbf{e}_y(v)(y)\Big), \xi + \mathbf{e}_y(v)(y)\Big\rangle_{d\times d}\mathrm{d}y.$$

Since $I^{\mathrm{H}}(\widehat{z}, \xi, v)$ is only finite if $v \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d$ coincides with its continuation $\mathcal{X}_{\hat{z}}(v|_{Y\setminus L(\hat{z})})$ with respect to the strong 1-extension $\mathcal{X}_{\hat{z}} : \mathrm{H}^1(Y\setminus L(\widehat{z}))^d \to \mathrm{H}^1(Y)^d$, according to Lemma 8.2 the functional $I^{\mathrm{H}}(\widehat{z}, \xi, \cdot) : \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d \to \mathbb{R}_\infty$ is coercive. Hence, showing that the minimizing problem (8.41) is well defined is done analogously to the proof of Proposition 6.8. Moreover, this coercivity implies that for almost every $x \in \Omega$ the tensor $\mathbb{C}^{\mathrm{H}}_{\mathrm{eff}}(z_0)(x) \in \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}_{\mathrm{sym}};\mathbb{R}^{d\times d}_{\mathrm{sym}})$ is positive definite independent of the function $z_0 \in \mathrm{W}^{1,p}(\Omega;[0,1]^m)$. Thus, we are able to introduce the one-scale limit energy functional $\mathcal{E}^{\mathrm{H}}_0 : [0,T]\times\mathcal{Q}^{\mathrm{H}}_0(\Omega) \to \mathbb{R}$ via

$$\mathcal{E}^{\mathrm{H}}_0(t, u_0, z_0) := \tfrac{1}{2}\langle \mathbb{C}^{\mathrm{H}}_{\mathrm{eff}}(z_0)\mathbf{e}(u_0), \mathbf{e}(u_0)\rangle_{\mathrm{L}^2(\Omega\times Y)^{d\times d}} + \|\nabla z_0\|^p_{\mathrm{L}^p(\Omega)^{m\times d}} - \langle \ell^{\ell_0}_{z_0}(t), u_0\rangle.$$

For $\kappa^{\mathrm{H}}_0 \in \mathrm{L}^{q'}(\Omega;[0,\infty)^m)$ denoting the limit function mentioned in the definition of the microscopic dissipation distance defined in line (8.22), the limit dissipation distance $\mathcal{D}^{\mathrm{H}}_0 : \mathrm{W}^{1,p}(\Omega;[0,1]^m)\times\mathrm{W}^{1,p}(\Omega;[0,1]^m) \to [0,\infty]$ reads as follows:

$$\mathcal{D}^{\mathrm{H}}_0(z_1, z_2) := \begin{cases} \displaystyle\int_\Omega |\langle \kappa^{\mathrm{H}}_0(x), z_2(x) - z_1(x)\rangle_m|\mathrm{d}x & \text{if } z_1 \geq z_2, \\ \infty & \text{otherwise.} \end{cases}$$

For given initial values $(u_0^0, z_0^0) \in \mathcal{Q}_0^{\mathrm{H}}(\Omega)$ the existence of an energetic solution of the rate-independent system $(\mathcal{Q}_0^{\mathrm{H}}(\Omega), \mathcal{E}_0^{\mathrm{H}}, \mathcal{D}_0^{\mathrm{H}})$ results from Theorem 8.15 by keeping Corollary 8.21 below in mind.

Stability condition $(\mathrm{S}_{\mathrm{H}}^0)$ **and energy balance** $(\mathrm{E}_{\mathrm{H}}^0)$ **for all** $t \in [0, T]$**:**

$$\mathcal{E}_0^{\mathrm{H}}(t, u_0(t), z_0(t)) \leq \mathcal{E}_0^{\mathrm{H}}(t, \widetilde{u}, \widetilde{z}) + \mathcal{D}_0^{\mathrm{H}}(z_0(t), \widetilde{z}) \quad \text{for all } (\widetilde{u}, \widetilde{z}) \in \mathcal{Q}_0^{\mathrm{H}}(\Omega)$$
$$\mathcal{E}_0^{\mathrm{H}}(t, u_0(t), z_0)(t)) + \mathrm{Diss}_{\mathcal{D}_0^{\mathrm{H}}}(z_0; [0, t]) = \mathcal{E}_0^{\mathrm{H}}(0, u_0^0, z_0^0) - \int_0^t \langle \dot{\ell}^{\ell_0}_{z_0(s)}(s), u_0(s) \rangle \mathrm{d}s$$

Here, $\mathrm{Diss}_{\mathcal{D}_0^{\mathrm{H}}}(z_0; [0, t]) := \sup \sum_{j=1}^{N} \mathcal{D}_0^{\mathrm{H}}(z_0(t_{j-1}), z_0(t_j))$, where for $N \in \mathbb{N}$ the supremum is taken with respect to all finite partitions $\pi_N := \{0 = t_0 < t_1 < \ldots < t_N = t\}$ of the interval $[0, T]$. The following theorem states the equivalence of the here formulated effective one-scale model and a two-scale model which is a slight modification of the model presented in Section 8.2. Afterwards, Theorem 8.15 might be combined with Corollary 8.21 to prove existence of solutions for the model considered in this section.

Theorem 8.20 (Equivalence of the two-scale and one-scale model)**.** *Assume that the conditions* (8.1) *and* (8.2) *hold. For* $\widehat{z} \in [0, 1]^m$ *let* $\mathcal{L}_{\hat{z}}^{\mathrm{H}} : \mathbb{R}_{\mathrm{sym}}^{d \times d} \to \mathrm{H}_{\mathrm{av}}^1(\mathcal{Y})^d$ *denote the linear operator, which for* $\xi \in \mathbb{R}_{\mathrm{sym}}^{d \times d}$ *is defined by*

$$\mathcal{L}_{\hat{z}}^{\mathrm{H}}(\xi) := \mathrm{Argmin}\{ I^{\mathrm{H}}(\widehat{z}, \xi, v) \,|\, v \in \mathrm{H}_{\mathrm{av}}^1(\mathcal{Y})^d \}.$$

For $\ell_0 \in \mathrm{C}^1([0, T]; (\mathrm{H}_{\Gamma_{\mathrm{Dir}}}^1(\Omega)^d)^*)$*, for* $\ell_1 :\equiv \mathbf{0}$*, and for* $\overline{z}_0 \in \mathrm{W}^{1,p}(\Omega; [0, 1]^m)$ *let the external loadings* $\ell_{\overline{z}_0}^{\ell_0, \mathbf{0}} \in \mathrm{C}^1([0, T]; (\mathrm{H}_{\Gamma_{\mathrm{Dir}}}^1(\Omega)^d)^*)$ *and* $\ell_{\overline{z}_0}^{\ell_0} \in \mathrm{C}^1([0, T]; (\mathrm{H}_{\Gamma_{\mathrm{Dir}}}^1(\Omega)^d)^*)$ *be given by* (8.31) *and* (8.40)*; see also Remark* 8.19*. Let* $(\overline{\mathbf{S}}_{\mathrm{H}}^0)$ *and* $(\overline{\mathbf{E}}_{\mathrm{H}}^0)$ *denote the energetic formulation with respect to the energy functional of* (8.34) *and the dissipation potential of* (8.35)*.*

Furthermore, let $z_0 \in \mathrm{L}^\infty([0, T]; \mathrm{W}^{1,p}(\Omega; [0, 1]^m)) \cap \mathrm{BV}_{\mathcal{D}_0^{\mathrm{H}}}([0, T]; \mathrm{W}^{1,p}(\Omega; [0, 1]^m))$ *and let* $(u_0, \overline{U}_1) \in \mathrm{L}^\infty([0, T]; \mathrm{H}_{\Gamma_{\mathrm{Dir}}}^1(\Omega)^d \times \mathrm{L}^2(\Omega; \mathrm{H}_{\mathrm{av}}^1(\mathcal{Y}))^d)$*. Then for* $(u_0^0, \overline{U}_1^0, z_0^0)$ *satisfying the stability condition* $(\overline{\mathbf{S}}_{\mathrm{H}}^0)$ *for* $t = 0$ *the following two statements are equivalent:*

(a) *The function* $(u_0, \overline{U}_1, z_0) : [0, T] \to \mathbf{Q}_0^{\mathrm{H}}$ *with* $(u_0(0), \overline{U}_1(0), z_0(0)) = (u_0^0, \overline{U}_1^0, z_0^0)$ *is a solution of* $(\overline{\mathbf{S}}_{\mathrm{H}}^0)$ *and* $(\overline{\mathbf{E}}_{\mathrm{H}}^0)$*.*

(b) *The function* $(u_0, z_0) : [0, T] \to \mathcal{Q}_0^{\mathrm{H}}(\Omega)$ *with* $(u_0(0), z_0(0)) = (u_0^0, z_0^0)$ *is a solution of* $(\mathrm{S}_{\mathrm{H}}^0)$ *and* $(\mathrm{E}_{\mathrm{H}}^0)$*, and* $\overline{U}_1(t) := \mathcal{L}_{z_0(t,\cdot)}^{\mathrm{H}}(\mathbf{e}_x(u_0(t))(\cdot))$ *for all* $t \in [0, T]$*.*

Proof. Let $\ell_0 \in \mathrm{C}^1([0, T]; (\mathrm{H}_{\Gamma_{\mathrm{Dir}}}^1(\Omega)^d)^*)$ be given. The assumption $\ell_1 :\equiv \mathbf{0}$ enables us to gain the following result by choosing $(\widetilde{u}, \widetilde{z}) = (u_0, z_0)$ for the testfunctions in the stability condition $(\overline{\mathbf{S}}_{\mathrm{H}}^0)$: For $(u_0, \overline{U}_1, z_0) \in \mathbf{Q}_0^{\mathrm{H}}$ satisfying the stability condition $(\overline{\mathbf{S}}_{\mathrm{H}}^0)$ at some time $t \in [0, T]$ the function $\overline{U}_1 \in \mathrm{L}^2(\Omega; \mathrm{H}_{\mathrm{av}}^1(\mathcal{Y}))^d$ is the unique solution of

$$\min\{ I_0^{\mathrm{H}}(z_0, u_0, U) \,|\, U \in \mathrm{L}^2(\Omega; \mathrm{H}_{\mathrm{av}}^1(\mathcal{Y}))^d \}. \tag{8.42}$$

Here, $I_0^{\mathrm{H}}(z_0, u_0, \cdot) : \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d \to \mathbb{R}_\infty$ is defined by

$$I_0^{\mathrm{H}}(z_0, u_0, U) := \begin{cases} \widetilde{I}_0^{\mathrm{H}}(z_0, u_0, U) & \text{if } U \text{ satisfies (8.43) below,} \\ \infty & \text{otherwise,} \end{cases}$$

where the continuous functional $\widetilde{I}_0^{\mathrm{H}}(z_0, u_0, \cdot) : \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d \to \mathbb{R}$ is given by

$$\widetilde{I}_0^{\mathrm{H}}(z_0, u_0, U) := \langle \mathbb{C}_0^{\mathrm{H}}(z_0)\widetilde{\mathbf{e}}(u_0, U), \widetilde{\mathbf{e}}(u_0, U)\rangle_{\mathrm{L}^2(\Omega\times Y)^{d\times d}}.$$

For $\mathcal{X}_{\widehat{z}} : \mathrm{H}^1(Y\backslash L(\widehat{z}))^d \to \mathrm{H}^1(Y)^d$ ($\widehat{z} \in [0,1]^m$) denoting the strong 1-extension operator of Lemma 8.2, condition (8.43) reads as follows:

$$U(x,\cdot) = \mathcal{X}_{z_0(x)}(U(x,\cdot)|_{Y\backslash L(z_0(x))}) \quad \text{for almost every } x \in \Omega. \tag{8.43}$$

Observe that for the given (u_0, z_0) the functional $I_0^{\mathrm{H}}(z_0, u_0, \cdot) : \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d \to \mathbb{R}_\infty$ is coercive according to Lemma 8.2.

To prove Theorem 8.20 we proceed similar to the proof of Theorem 6.9, which consists of the proofs of Proposition 6.10 and Corollary 6.11. For all functions $\widetilde{U} \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ it holds $\widetilde{I}_0^{\mathrm{H}}(z_0, u_0, \widetilde{U}) = I_0^{\mathrm{H}}(z_0, u_0, \widetilde{U}_{z_0})$, where $\widetilde{U}_{z_0} \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ for almost every $x \in \Omega$ is defined by $\widetilde{U}_{z_0}(x,\cdot) := \mathcal{X}_{z_0(x)}(\widetilde{U}(x,\cdot)|_{Y\backslash L(z_0(x))})$. Therefore, $\overline{U}_1 \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ is the unique solution of (8.42), if and only if for all $\widetilde{U} \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ it holds

$$\widetilde{I}_0^{\mathrm{H}}(z_0, u_0, \widetilde{U}) \geq I_0^{\mathrm{H}}(z_0, u_0, \overline{U}_1), \tag{8.44}$$

which is relation (6.19) translated to the here considered case. Let $\underline{U}_1 \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ for almost every $(x,y) \in \Omega\times Y$ be defined by $\underline{U}_1(x,y) := \mathcal{L}^{\mathrm{H}}_{z_0(x)}(\mathbf{e}_x(u_0)(x))(y)$. Referring to relation (6.20) of the proof of Theorem 6.9, for almost every $x \in \Omega$ and for any $\widetilde{v} \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d$ it holds

$$\widetilde{I}^{\mathrm{H}}(z_0(x), \mathbf{e}_x(u_0)(x), \widetilde{v}) \geq I^{\mathrm{H}}(z_0(x), \mathbf{e}_x(u_0)(x), \underline{U}_1(x,\cdot)),$$

which is derived analogously to (8.44). Since $\widetilde{I}_0^{\mathrm{H}}(z_0, u_0, \cdot) : \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d \to \mathbb{R}$ and $\widetilde{I}^{\mathrm{H}}(z_0(x), u_0(x), \cdot) : \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d \to \mathbb{R}$ are continuous, the rest of the proof of Theorem 8.20 is completely analog to that of Theorem 6.9. □

Corollary 8.21 (Equivalence of the two-scale and one-scale model)**.** *Assume that the conditions* (8.1) *and* (8.2) *hold. For* $\widehat{z} \in [0,1]^m$ *let* $\mathcal{L}^{\mathrm{H}}_{\widehat{z}} : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d$ *denote the linear operator, which for* $\xi \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ *is defined by*

$$\mathcal{L}^{\mathrm{H}}_{\widehat{z}}(\xi) := \mathrm{Argmin}\{I^{\mathrm{H}}(\widehat{z}, \xi, v) \,|\, v \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})^d\}.$$

For $\ell_0 \in \mathrm{C}^1([0,T]; (\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$*, for* $\ell_1 :\equiv \mathbf{0}$*, and for* $\overline{z}_0 \in \mathrm{W}^{1,p}(\Omega;[0,1]^m)$ *let the external loadings* $\ell^{\ell_0,\mathbf{0}}_{\overline{z}_0} \in \mathrm{C}^1([0,T]; (\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$ *and* $\ell^{\ell_0}_{\overline{z}_0} \in \mathrm{C}^1([0,T]; (\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$ *be defined by* (8.31) *and* (8.40)*; see also Remark* 8.19*.*

Furthermore, let $z_0 \in \mathrm{L}^\infty([0,T]; \mathrm{W}^{1,p}(\Omega;[0,1]^m)) \cap \mathrm{BV}_{\mathcal{D}^{\mathrm{H}}_0}([0,T]; \mathrm{W}^{1,p}(\Omega;[0,1]^m))$ *and let* $(u_0, U_1) \in \mathrm{L}^\infty([0,T]; \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \times \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d)$*. Then for* (u_0^0, U_1^0, z_0^0) *satisfying the stability condition* $(\mathbf{S}^0_{\mathrm{H}})$ *for* $t = 0$ *the following two statements hold:*

(a) *If* $(u_0, U_1, z_0) : [0,T] \to \mathbf{Q}_0^{\mathrm{H}}$ *with* $(u_0(0), U_1(0), z_0(0)) = (u_0^0, U_1^0, z_0^0)$ *is a solution of* $(\mathbf{S}_{\mathrm{H}}^0)$ *and* $(\mathbf{E}_{\mathrm{H}}^0)$*, then* $(u_0, z_0) : [0,T] \to \mathcal{Q}_0^{\mathrm{H}}(\Omega)$ *with* $(u_0(0), z_0(0)) = (u_0^0, z_0^0)$ *is a solution of* $(\mathrm{S}_{\mathrm{H}}^0)$ *and* $(\mathrm{E}_{\mathrm{H}}^0)$*.*

(b) *If* $(u_0, z_0) : [0,T] \to \mathcal{Q}_0^{\mathrm{H}}(\Omega)$ *with* $(u_0(0), z_0(0)) = (u_0^0, z_0^0)$ *is a solution of* $(\mathrm{S}_{\mathrm{H}}^0)$ *and* $(\mathrm{E}_{\mathrm{H}}^0)$*, then* $(u_0, \overline{U}_1, z_0) : [0,T] \to \mathbf{Q}_0^{\mathrm{H}}$ *with* $(u_0(0), \overline{U}_1(0), z_0(0)) = (u_0^0, \overline{U}_1^0, z_0^0)$ *is a solution of* $(\mathbf{S}_{\mathrm{H}}^0)$ *and* $(\mathbf{E}_{\mathrm{H}}^0)$*, where* $\overline{U}_1(t) := \mathcal{L}_{z_0(t,\cdot)}^{\mathrm{H}}(\mathbf{e}_x(u_0(t))(\cdot))$ *for all* $t \in [0,T]$ *and* $\overline{U}_1^0 := \overline{U}_1(0)$*.*

Proof. For $\hat{z} \in [0,1]^m$ let $\mathcal{X}_{\hat{z}} : \mathrm{H}^1(Y \backslash L(\hat{z}))^d \to \mathrm{H}^1(Y)^d$ denote the strong 1-extension operator of Lemma 8.2. Moreover, let $(u_0, \overline{U}_1, z_0) \in \mathbf{Q}_0^{\mathrm{H}}$ be given such that

$$\overline{U}_1(x,\cdot) := \mathcal{X}_{z_0(x)}(U_1(x,\cdot)|_{Y \backslash L(z_0(x))}) \quad \text{for almost every } x \in \Omega.$$

Then, by definition for all $t \in [0,T]$ and every function $U_1 \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ which coincides with $\overline{U}_1$ on the set $(\Omega \times Y) \backslash M(z_0) := \bigcup_{x \in \Omega} \{x\} \times Y \backslash L(z_0(x))$ of positive stiffness, it holds

$$\mathbf{E}_0^{\mathrm{H}}(t, u_0, U_1, z_0) = \overline{\mathbf{E}}_0^{\mathrm{H}}(t, u_0, \overline{U}_1, z_0); \tag{8.45}$$

see (8.32) and (8.34).

(a) If the function $(u_0, U_1, z_0) : [0,T] \to \mathbf{Q}_0^{\mathrm{H}}$ with $(u_0(0), U_1(0), z_0(0)) = (u_0^0, U_1^0, z_0^0)$ is a solution of $(\mathbf{S}_{\mathrm{H}}^0)$ and $(\mathbf{E}_{\mathrm{H}}^0)$, then according to (8.45) the function $(u_0, \overline{U}_1, z_0) : [0,T] \to \mathbf{Q}_0^{\mathrm{H}}$ with $(u_0(0), \overline{U}_1(0), z_0(0)) = (u_0^0, \overline{U}_1^0, z_0^0)$ is a solution of $(\overline{\mathbf{S}}_{\mathrm{H}}^0)$ and $(\overline{\mathbf{E}}_{\mathrm{H}}^0)$. Here, the initial value $\overline{U}_1^0$ is given by $\overline{U}_1^0 := \overline{U}_1(0)$, where for all $t \in [0,T]$ the function $\overline{U}_1(t)$ is defined by

$$\overline{U}_1(t,x,\cdot) := \mathcal{X}_{z_0(t,x)}(U_1(t,x,\cdot)|_{Y \backslash L(z_0(t,x))}) \quad \text{for almost every } x \in \Omega.$$

Thus, Theorem 8.20 yields the desired result.

(b) For all $t \in [0,T]$ set $\overline{U}_1(t) := \mathcal{L}_{z_0(t,\cdot)}^{\mathrm{H}}(\mathbf{e}_x(u_0(t))(\cdot))$. Furthermore, $\overline{U}_1^0 := \overline{U}_1(0)$. Then, according to Theorem 8.20 the function $(u_0, \overline{U}_1, z_0) : [0,T] \to \mathbf{Q}_0^{\mathrm{H}}$ satisfying $(u_0(0), \overline{U}_1(0), z_0(0)) = (u_0^0, \overline{U}_1^0, z_0^0)$ is a solution of $(\overline{\mathbf{S}}_{\mathrm{H}}^0)$ and $(\overline{\mathbf{E}}_{\mathrm{H}}^0)$. However, due to (8.45) for all $t \in [0,T]$ it holds $\mathbf{E}_0^{\mathrm{H}}(t, u_0(t), \overline{U}_1(t), z_0(t)) = \overline{\mathbf{E}}_0^{\mathrm{H}}((t, u_0(t), \overline{U}_1(t), z_0(t))$, such that $(u_0, \overline{U}_1, z_0) : [0,T] \to \mathbf{Q}_0^{\mathrm{H}}$ also is a solution of $(\mathbf{S}_{\mathrm{H}}^0)$ and $(\mathbf{E}_{\mathrm{H}}^0)$. $\square$

8.4 Discussion of the results

This chapter presents rigorously derived effective models for a linear elastic body which is damaged by the growth of preexisting microscopic voids. Due to the underlying non-periodically perforated domains this verification is mathematically challenging and requires a specific extension theory, which suits the admissible microstructures considered in this thesis.

The changes with respect to the results of Chapter 7 concerning the external loadings are the following: Since at time $t \in [0,T]$ the external loadings of the microscopic models are only applied to the set $\Omega\backslash\Omega_\varepsilon^{\mathrm{D}}(z_\varepsilon(t)) \subset \Omega$ possessing positive stiffness, the external loadings of the effective models are scaled by factors depending on the limit damage variable $z_0(t) \in \mathrm{W}^{1,p}(\Omega; [0,1]^m)$. Moreover, there is an additional part of the boundary coming from the microscopic voids. In terms of a strong formulation this means that we impose a zero Neumann boundary condition for this additional part of the boundary. In turn, this forces the external loadings of the effective two-scale model to depend on the second scale, too. Only by assuming the two-scale external loadings to be independent of this second scale (by setting it to zero) we are able to obtain an equivalent effective one-scale model.

Due to the asymptotic analysis of Chapter 6 the microscopic voids considered in the microscopic models of Subsection 8.1.3 are "shifted" to the second scale of the effective two-scale model presented in Section 8.2. That means, for $t \in [0,T]$ and for almost every $x \in \Omega$ the tensor $\mathbb{C}_0^{\mathrm{H}}(z_0(t))(x,\cdot)$ is zero on the set $\{x\}\times L(z_0(t,x)) \subset \Omega\times Y$.

However, according to the continuation theory the material tensor $\mathbb{C}_{\mathrm{eff}}^{\mathrm{H}}(z_0(t))(x)$ is positive definite for all $t \in [0,T]$ and almost every $x \in \Omega$, independent of the value $z_0(t,x)$ of the limit damage variable. Hence, the effective one-scale model describes the damage progression in a linear elastic body represented by the set Ω, i.e., there are no voids present. Here, the voids of the microscopic models are reflected by the unit cell problem (8.41) defining the effective material tensor $\mathbb{C}_{\mathrm{eff}}^{\mathrm{H}}(z_0(t))(x)$.

Observe that for a given sequence $(z_\varepsilon)_{\varepsilon>0}$ of damage variables $z_\varepsilon : [0,T] \to \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)$ the sequence $(\Omega\backslash\Omega_\varepsilon^{\mathrm{D}}(z_\varepsilon(t)))_{\varepsilon>0}$ of perforated sets for any $t \in [0,T]$ satisfies the *condition of strong connectivity*; see [39, Subsection 1.7.1]. This is the crucial assumption of the homogenization theory for strongly perforated domains in [39]. This theory enables the derivation of effective descriptions for problems defined on sequences of non-periodically perforated domains. It is based on the introduction of a mesoscopic scale, which is small relative to the size of the whole domain but large compared to the microscopic scale. Due to this mesoscopic scale the effective quantities, like the effective material tensor in our case, are given by the limit of so-called *conductivity tensors* introduced on this mesoscopic scale; see [39, Chapter 5]. Comparing this theory to our approach, the discrete gradient appearing in the microscopic models in some sense introduces a third scale – lying between the microscopic and the macroscopic scale – to these models, too. This discrete gradient prevents the defects' shapes and sizes from varying in a uncontrollable manner. In this way we are able to perform the limit passage directly with respect to the microscopic scale. In contrast to the techniques in [39] this enables us to avoid the further consideration of a limit passage from the mesoscopic to the microscopic scale, where the identification of the limit model might be difficult.

9 Effective models for the evolution of microscopic cracks

This chapter deals with damage models describing the growth of microscopic cracks along paths known in advance. To describe these crack paths let $\mathcal{C} \subset Y$ be a hypersurface which is the graph of a Lipschitz function. The material under investigation will be modeled by the positive definite tensor $\mathbb{C}_{\text{strong}} \in \mathrm{Lin}_{\text{sym}}(\mathbb{R}^{d\times d}_{\text{sym}}; \mathbb{R}^{d\times d}_{\text{sym}})$, i.e., there exists a positive constant α such that

$$\text{for all } \xi \in \mathbb{R}^{d\times d}_{\text{sym}} \text{ it holds} \qquad \alpha|\xi|^2_{d\times d} \leq \langle \mathbb{C}_{\text{strong}}\xi, \xi\rangle_{d\times d}. \tag{9.1}$$

To model the growth of a crack along the path $\mathcal{C} \subset Y$ we assume that there exists a mapping $L^{\mathrm{C}} : [0,1]^m \to \mathcal{L}_{\text{Leb}}(\mathcal{C})$ with the following properties:

- $L^{\mathrm{C}} : [0,1]^m \to \mathcal{L}_{\text{Leb}}(\mathcal{C})$ is a non-increasing function; see (2.21). (9.2a)
- For all $\widehat{z} \in [0,1]^m$ it holds $\mu_{d-1}(L^{\mathrm{C}}(\widehat{z})) > 0$. (9.2b)
- For all $\widehat{z} \in [0,1]^m$ the set $L^{\mathrm{C}}(\widehat{z})$ is a closed subset of $\mathcal{C}$. (9.2c)
- For all $\widehat{z} \in [0,1]^m$ the relative boundary $\partial_{\text{rel}}L^{\mathrm{C}}(\widehat{z})$ of the set $L^{\mathrm{C}}(\widehat{z})$ is a locally Lipschitz domain (see Def. 2.1) and it holds $3\eta^* := \mathrm{dist}(L^{\mathrm{C}}(\mathbf{0}), \partial Y) > 0$. (9.2d)

For any given $\widehat{z} \in [0,1]^m$ and every $(\widehat{z}_\delta)_{\delta>0} \subset [0,1]^m$ satisfying $\widehat{z}_\delta \to \widehat{z}$ in $\mathbb{R}^m$ it holds

- $\mu_{d-1}(L^{\mathrm{C}}(\widehat{z})\backslash L^{\mathrm{C}}(\widehat{z}_\delta)) + \mu_{d-1}(L^{\mathrm{C}}(\widehat{z}_\delta)\backslash L^{\mathrm{C}}(\widehat{z})) \to 0$ for $\delta \to 0$ and (9.2e)
- $\forall \Delta > 0\, \exists\, \delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ it holds $L^{\mathrm{C}}(\widehat{z}_\delta) \subset \mathrm{neigh}_\Delta(L^{\mathrm{C}}(\widehat{z}))$. (9.2f)

Note that for any $\Delta > 0$ it holds $\mu_d(\mathrm{neigh}_\Delta(L^{\mathrm{C}}(\widehat{z}))) > 0$ according to assumption (9.2b). There exist bi-Lipschitz transformations $(T^{\mathrm{C}}_{\hat{z}})_{\hat{z}\in[0,1]^m}$, $T^{\mathrm{C}}_{\hat{z}} : Y \to Y$, such that

- $\sup_{\hat{z}\in[0,1]^m} \Big(\|\nabla T^{\mathrm{C}}_{\hat{z}}\|_{\mathrm{L}^\infty(Y)^{d\times d}} + \|\nabla (T^{\mathrm{C}}_{\hat{z}})^{-1}\|_{\mathrm{L}^\infty(Y)^{d\times d}} \Big) =: C_T < \infty.$ (9.2g)
- For all $\widehat{z} \in [0,1]^m$ it holds $\mathrm{Im}(T^{\mathrm{C}}_{\hat{z}}|_{L^{\mathrm{C}}(\hat{z})}) = L^{\mathrm{C}}(\mathbf{0})$ and $T^{\mathrm{C}}_{\hat{z}}|_{\mathrm{neigh}_{\eta^*}(\partial Y)\cap Y} = \mathrm{id}$. (9.2h)
- For any $\widehat{z} \in [0,1]^m$ and for every $(\widehat{z}_\delta)_{\delta>0} \subset [0,1]^m$ with $\widehat{z}_\delta \to \widehat{z}$ in $\mathbb{R}^m$ it holds $\|T^{\mathrm{C}}_{\hat{z}_\delta} - T^{\mathrm{C}}_{\hat{z}}\|_{\mathrm{W}^{1,\infty}(Y)^d} \to 0$ and $\|(T^{\mathrm{C}}_{\hat{z}_\delta})^{-1} - (T^{\mathrm{C}}_{\hat{z}})^{-1}\|_{\mathrm{W}^{1,\infty}(Y)^d} \to 0$. (9.2i)

Let $L^\eta : [0,1]^m \to \mathcal{L}_{\text{Leb}}(Y)$ for all $\widehat{z} \in [0,1]^m$ be defined by $L^\eta(\widehat{z}) := \mathrm{neigh}_\eta(L^{\mathrm{C}}(\widehat{z}))$. Then, we assume that the following condition is satisfied.

$$\text{For all } \eta \in (0,\eta^*) \text{ and every } \widehat{z} \in [0,1]^m \text{ it holds } \mathrm{Im}(T^{\mathrm{C}}_{\hat{z}}|_{L^\eta(\hat{z})}) = L^\eta(\mathbf{0}). \tag{9.3}$$

Observe that for $\widehat{z} \in [0,1]^m$ due to (9.2h) and (9.3) the value of $T^{\mathrm{C}}_{\widehat{z}} : Y \to Y$ is prescribed on the set $(\mathrm{neigh}_{\eta^*}(\partial Y) \cap Y) \cup L^{\eta^*}(\widehat{z})$, which according to assumption (9.2d) satisfies $\mathrm{dist}(\mathrm{neigh}_{\eta^*}(\partial Y), L^{\eta^*}(\widehat{z})) = \eta^* > 0$.

Assuming (9.1), (9.2), and (9.3) to hold, for any $\eta \in (0, \eta^*)$ the crucial conditions (8.1) and (8.2) of Chapter 8 are fulfilled for $L^\eta : [0,1]^m \to \mathcal{L}_{\mathrm{Leb}}(Y)$. Therefore, the here considered crack models might be understood as the limits which are formally derived by letting η tend to zero in the respective damage model of Chapter 8 based on the mapping $L^\eta : [0,1]^m \to \mathcal{L}_{\mathrm{Leb}}(Y)$. For $\widehat{z} \in [0,1]^m$ the function space $\mathrm{H}^1(Y \backslash L^\eta(\widehat{z}))^d$ builds the foundation for the displacement fields considered in Chapter 8. Hence, by formally letting η tend to zero, we have to deal with the function space $\mathrm{H}^1(Y \backslash L^{\mathrm{C}}(\widehat{z}))^d$ allowing any function to have a jump on the set $L^{\mathrm{C}}(\widehat{z})$.

9.1 Damage progression caused by microscopic cracks

Similar to the damage set considered in Section 2.6, the set of all microscopic cracks $\mathcal{C}_\varepsilon(z_\varepsilon) \subset \Omega$ associated to a given damage variable $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)$ is defined via

$$\mathcal{C}_\varepsilon(z_\varepsilon) := \bigcup_{\lambda \in \Lambda_\varepsilon^-} \varepsilon(\lambda + L^{\mathrm{C}}(z^{\varepsilon\lambda})),$$

where $z^{\varepsilon\lambda} := z_\varepsilon|_{\varepsilon(\lambda+Y)}$ for all $\lambda \in \Lambda_\varepsilon^-$; see (2.15). Since the investigated body Ω for a given damage variable is assumed to contain the microscopic cracks modeled by the set $\mathcal{C}_\varepsilon(z_\varepsilon)$, the displacement field u_ε is allowed to jump across this set, i.e., u_ε is assumed to be an element of the space $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega \backslash \mathcal{C}_\varepsilon(z_\varepsilon))^d$. To keep the state space $\mathcal{Q}^{\mathrm{C}}_\varepsilon(\Omega)$ independent of the damage variable $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)$, we set

$$\mathcal{Q}^{\mathrm{C}}_\varepsilon(\Omega) := \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega \backslash \mathcal{C}_\varepsilon(\mathbf{0}))^d \times \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)$$

and incorporate the restriction of the displacement field u_ε to only jump on the subset $\mathcal{C}_\varepsilon(z_\varepsilon) \subset \mathcal{C}_\varepsilon(\mathbf{0})$ in the energy functional. For $(\ell_0, \ell_1) \in \mathrm{C}^1([0,T]; \mathrm{L}^2(\Omega)^d \times \mathrm{L}^2(\Omega)^{d\times d})$ the external loading $\ell^{\ell_0,\ell_1}_{\mathrm{C},\varepsilon} \in \mathrm{C}^1([0,T]; (\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega \backslash \mathcal{C}_\varepsilon(\mathbf{0}))^d)^*)$ for all $t \in [0,T]$ is defined by

$$\langle \ell^{\ell_0,\ell_1}_{\mathrm{C},\varepsilon}(t), u_\varepsilon \rangle := \langle \ell_0(t), u_\varepsilon \rangle_{\mathrm{L}^2(\Omega \backslash \mathcal{C}_\varepsilon(\mathbf{0}))^d} + \langle \ell_1(t), \nabla u_\varepsilon \rangle_{\mathrm{L}^2(\Omega \backslash \mathcal{C}_\varepsilon(\mathbf{0}))^{d\times d}}. \tag{9.4}$$

To introduce the energy functional $\mathcal{E}^{\mathrm{C}}_\varepsilon : [0,T] \times \mathcal{Q}^{\mathrm{C}}_\varepsilon(\Omega) \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ we start by choosing once $p \in (1,\infty)$ and keep it fix for the rest of this chapter. Moreover, let $\widetilde{\mathcal{E}}^{\mathrm{C}}_\varepsilon : [0,T] \times \mathcal{Q}^{\mathrm{C}}_\varepsilon(\Omega) \to \mathbb{R}$ be given by

$$\widetilde{\mathcal{E}}^{\mathrm{C}}_\varepsilon(t, u_\varepsilon, z_\varepsilon) := \tfrac{1}{2} \langle \mathbb{C}_{\mathrm{strong}} \mathbf{e}(u_\varepsilon), \mathbf{e}(u_\varepsilon) \rangle_{\mathrm{L}^2(\Omega \backslash \mathcal{C}_\varepsilon(\mathbf{0}))^{d\times d}} + \| R_{\frac{\varepsilon}{2}} z_\varepsilon \|^p_{\mathrm{L}^p(\Omega_\varepsilon^+)^{m\times d}} - \langle \ell^{\ell_0,\ell_1}_{\mathrm{C},\varepsilon}(t), u_\varepsilon \rangle.$$

Thus, the energy functional $\mathcal{E}^{\mathrm{C}}_\varepsilon : [0,T] \times \mathcal{Q}^{\mathrm{C}}_\varepsilon(\Omega) \to \mathbb{R}_\infty$ is defined by

$$\mathcal{E}^{\mathrm{C}}_\varepsilon(t, u_\varepsilon, z_\varepsilon) = \begin{cases} \widetilde{\mathcal{E}}^{\mathrm{C}}_\varepsilon(t, u_\varepsilon, z_\varepsilon) & \text{if } u_\varepsilon \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega \backslash \mathcal{C}_\varepsilon(z_\varepsilon))^d, \\ \infty & \text{otherwise.} \end{cases} \tag{9.5}$$

Proposition 9.1 (Uniform Korn inequality for a periodic crack distribution (see [12])). *For all $v_\varepsilon \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega\backslash\mathcal{C}_\varepsilon(\mathbf{0}))^d$ there exists a constant $C_{\mathrm{Korn}} > 0$ independent of $\varepsilon > 0$ such that*

$$C_{\mathrm{Korn}}\|v_\varepsilon\|^2_{\mathrm{H}^1(\Omega\backslash\mathcal{C}_\varepsilon(\mathbf{0})^d} \leq \|\mathbf{e}(v_\varepsilon)\|^2_{\mathrm{L}^2(\Omega\backslash\mathcal{C}_\varepsilon(\mathbf{0}))^{d\times d}}.$$

For a proof of this statement we refer to Proposition 4.1 of [12]. Therefore, for any function $u_\varepsilon \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega\backslash\mathcal{C}_\varepsilon(z_\varepsilon))^d$ the following uniform coercivity condition holds:

$$\begin{aligned}\tfrac{\alpha}{2}C_{\mathrm{Korn}}\|u_\varepsilon\|^2_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega\backslash\mathcal{C}_\varepsilon(z_\varepsilon))^d} = \tfrac{\alpha}{2}C_{\mathrm{Korn}}\|u_\varepsilon\|^2_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega\backslash\mathcal{C}_\varepsilon(\mathbf{0}))^d} &\leq \tfrac{\alpha}{2}\|\mathbf{e}(u_\varepsilon)\|^2_{\mathrm{L}^2(\Omega\backslash\mathcal{C}_\varepsilon(\mathbf{0}))^{d\times d}} \\ &\leq \tfrac{1}{2}\langle\mathbb{C}_{\mathrm{strong}}\mathbf{e}(u_\varepsilon), \mathbf{e}(u_\varepsilon)\rangle_{\mathrm{L}^2(\Omega\backslash\mathcal{C}_\varepsilon(\mathbf{0}))^{d\times d}}. \end{aligned} \tag{9.6}$$

Let $q' \in (1, \infty)$ be given and fixed for the rest of this chapter. Moreover, assume that there exists a sequence $(\kappa^{\mathrm{C}}_\varepsilon)_{\varepsilon>0} \subset \mathrm{L}^{q'}(\Omega; [0,\infty)^m)$ and a function $\kappa^{\mathrm{C}}_0 \in \mathrm{L}^{q'}(\Omega; [0,\infty)^m)$ such that it holds $\kappa^{\mathrm{C}}_\varepsilon \rightharpoonup \kappa^{\mathrm{C}}_0$ in $\mathrm{L}^{q'}(\Omega)^m$. Then, the microscopic dissipation distance $\mathcal{D}^{\mathrm{C}}_\varepsilon : \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)\times\mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m) \to [0,\infty]$ reads as follows:

$$\mathcal{D}^{\mathrm{C}}_\varepsilon(z_1, z_2) = \begin{cases} \displaystyle\int_{\Omega^-_\varepsilon} |\langle\kappa^{\mathrm{C}}_\varepsilon(x), z_2(x) - z_1(x)\rangle_m|\mathrm{d}x & \text{if } z_1 \geq z_2, \\ \infty & \text{otherwise.}\end{cases} \tag{9.7}$$

For given initial values $z^0_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)$ and $u^0_\varepsilon \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega\backslash\mathcal{C}_\varepsilon(z^0_\varepsilon))^d$ the growth of microscopic cracks is modeled by the ε-dependent energetic formulation ($\mathrm{S}^\varepsilon_{\mathrm{C}}$) and ($\mathrm{E}^\varepsilon_{\mathrm{C}}$), where $\varepsilon > 0$ scales the length of the appearing cracks.

Stability condition ($\mathrm{S}^\varepsilon_{\mathrm{C}}$) **and energy balance** ($\mathrm{E}^\varepsilon_{\mathrm{C}}$) **for all** $t \in [0,T]$**:**

$$\mathcal{E}^{\mathrm{C}}_\varepsilon(t, u_\varepsilon(t), z_\varepsilon(t)) \leq \mathcal{E}^{\mathrm{C}}_\varepsilon(t, \widetilde{u}, \widetilde{z}) + \mathcal{D}^{\mathrm{C}}_\varepsilon(z_\varepsilon(t), \widetilde{z}) \quad \text{for all } (\widetilde{u}, \widetilde{z}) \in \mathcal{Q}^{\mathrm{C}}_\varepsilon(\Omega)$$

$$\mathcal{E}^{\mathrm{C}}_\varepsilon(t, u_\varepsilon(t), z_\varepsilon(t)) + \mathrm{Diss}_{\mathcal{D}^{\mathrm{C}}_\varepsilon}(z_\varepsilon; [0,t]) = \mathcal{E}^{\mathrm{C}}_\varepsilon(0, u^0_\varepsilon, z^0_\varepsilon) - \int_0^t \langle\dot{\ell}^{\ell_0,\ell_1}_{\mathrm{C},\varepsilon}(s), u_\varepsilon(s)\rangle\mathrm{d}s$$

Here, $\mathrm{Diss}_{\mathcal{D}^{\mathrm{C}}_\varepsilon}(z_\varepsilon; [0,t]) := \sup\sum_{j=1}^N \mathcal{D}^{\mathrm{C}}_\varepsilon(z_\varepsilon(t_{j-1}), z_\varepsilon(t_j))$, where for $N \in \mathbb{N}$ the supremum is taken with respect to all finite partitions $\pi_N := \{0 = t_0 < t_1 < \ldots < t_N = t\}$ of the interval $[0,T]$. For $\widetilde{t} \in [0,T]$

$$\mathcal{S}^{\mathrm{C}}_\varepsilon(\widetilde{t}) := \{(u_\varepsilon, z_\varepsilon) \in \mathcal{Q}^{\mathrm{C}}_\varepsilon(\Omega) \text{ satisfying } (\mathrm{S}^\varepsilon_{\mathrm{C}}) \text{ for } t = \widetilde{t} \text{ and } \mathcal{E}^{\mathrm{C}}_\varepsilon(\widetilde{t}, u_\varepsilon, z_\varepsilon) < \infty\}$$

denotes the set of stable states. Note that by assuming $u^0_\varepsilon \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega\backslash\mathcal{C}_\varepsilon(z^0_\varepsilon))^d$ for the initial values $(u^0_\varepsilon, z^0_\varepsilon) \in \mathcal{Q}^{\mathrm{C}}_\varepsilon(\Omega)$ the right hand side of the energy balance ($\mathrm{E}^\varepsilon_{\mathrm{C}}$) is finite. The following corollary states existence of solutions for the here presented microscopic models.

Corollary 9.2 (Existence of solutions). *Assume that the conditions* (9.1) *and* (9.2) *hold. For $(\ell_0, \ell_1) \in \mathrm{C}^1([0,T]; \mathrm{L}^2(\Omega)^d\times\mathrm{L}^2(\Omega)^{d\times d})$ let $\ell^{\ell_0,\ell_1}_{\mathrm{C},\varepsilon} \in \mathrm{C}^1([0,T]; (\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega\backslash\mathcal{C}_\varepsilon(\mathbf{0}))^d)^*)$ be defined by* (9.4)*. Moreover, let $\mathcal{E}^{\mathrm{C}}_\varepsilon : [0,T]\times\mathcal{Q}^{\mathrm{C}}_\varepsilon(\Omega) \to \mathbb{R}_\infty$ be defined via* (9.5) *and for $\kappa^{\mathrm{C}}_\varepsilon \in \mathrm{L}^{q'}(\Omega; [0,\infty)^m)$ let $\mathcal{D}^{\mathrm{C}}_\varepsilon : \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)\times\mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m) \to [0,\infty]$ be given by* (9.7).

Then for $(u_\varepsilon^0, z_\varepsilon^0) \in \mathcal{S}_\varepsilon^{\mathrm{C}}(0)$*, there exists an energetic solution* $(u_\varepsilon, z_\varepsilon) : [0,T] \to \mathcal{Q}_\varepsilon^{\mathrm{C}}(\Omega)$ *of the rate-independent system* $(\mathcal{Q}_\varepsilon^{\mathrm{C}}(\Omega), \mathcal{E}_\varepsilon^{\mathrm{C}}, \mathcal{D}_\varepsilon^{\mathrm{C}})$ *satisfying* $(u_\varepsilon(0), z_\varepsilon(0)) = (u_\varepsilon^0, z_\varepsilon^0)$ *and*

$$\begin{aligned} &u_\varepsilon \in \mathrm{L}^\infty([0,T], \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega\backslash\mathcal{C}_\varepsilon(\mathbf{0}))^d), \\ &z_\varepsilon \in \mathrm{L}^\infty([0,T], \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)) \cap \mathrm{BV}_{\mathcal{D}_\varepsilon^{\mathrm{C}}}([0,T], \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)). \end{aligned}$$

Proof. Analog to the proof of Corollary 8.9, the assumptions (6.1), (6.2), (6.5), and (6.9) of Proposition 6.5 are fulfilled. Again, the only point that remains to be shown is the proof of the weak compactness of the energy sublevel sets. In detail, for a limit function $(u_0, z_0) \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega\backslash\mathcal{C}_\varepsilon(\mathbf{0}))^d \times \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ of a sequence $(u_\delta, z_\delta)_{\delta>0}$ in $\mathcal{Q}_\varepsilon^{\mathrm{H}}(\Omega)$ belonging to $\mathrm{Sub}_E(t)$ (see (5.6)) we have to show $u_0 \in \mathrm{Sub}_E(t)$, i.e., for the functions $u_\delta \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega\backslash\mathcal{C}_\varepsilon(z_\delta))^d \subset \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega\backslash\mathcal{C}_\varepsilon(\mathbf{0}))^d$ we have

$$u_\delta \to u_0 \quad \text{in } \mathrm{L}^2(\Omega\backslash\mathcal{C}_\varepsilon(\mathbf{0}))^d \tag{9.8a}$$

$$\nabla u_\delta \rightharpoonup \nabla u_0 \quad \text{in } \mathrm{L}^2(\Omega\backslash\mathcal{C}_\varepsilon(\mathbf{0}))^{d\times d}, \tag{9.8b}$$

and we need to verify $u_0 \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega\backslash\mathcal{C}_\varepsilon(z_0))^d$. For this purpose, let $\varphi \in \mathrm{C}_{\mathrm{c}}^\infty(\Omega\backslash\mathcal{C}_\varepsilon(z_0))^{d\times d}$ be chosen arbitrarily but fixed. Then $\mathrm{dist}(\mathcal{C}_\varepsilon(z_0), \mathrm{supp}(\varphi)) > 0$. Hence, according to assumption (9.2f) by choosing $\delta_0 > 0$ sufficiently small it holds $\mathcal{C}_\varepsilon(z_\delta) \cap \mathrm{supp}(\varphi) = \emptyset$ for all $\delta \in (0, \delta_0)$. This implies

$$\varphi \in \mathrm{C}_{\mathrm{c}}^\infty(\Omega\backslash\mathcal{C}_\varepsilon(z_\delta))^{d\times d} \quad \text{for all } \delta \in (0,\delta_0). \tag{9.9}$$

Thus, the following calculation is justified.

$$\begin{aligned} \int_\Omega \langle \nabla u_0(x), \varphi(x)\rangle_{d\times d} \mathrm{d}x &\overset{(9.8b)}{=} \lim_{\delta\to 0} \int_\Omega \langle \nabla u_\delta(x), \varphi(x)\rangle_{d\times d} \mathrm{d}x \\ &\overset{(9.9)}{=} -\lim_{\delta\to 0} \int_\Omega \langle u_\delta(x), \mathrm{div}(\varphi(x))\rangle_d \mathrm{d}x \\ &\overset{(9.8a)}{=} -\int_\Omega \langle u_0(x), \mathrm{div}(\varphi(x))\rangle_d \mathrm{d}x \end{aligned}$$

Note that in the second equality additionally $u_\delta \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega\backslash\mathcal{C}_\varepsilon(z_\delta))^d$ is exploited for all $\delta \in (0, \delta_0)$. Since the testfunction $\varphi \in \mathrm{C}_{\mathrm{c}}^\infty(\Omega\backslash\mathcal{C}_\varepsilon(z_0))^{d\times d}$ was chosen arbitrarily, this calculation shows $u_0 \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega\backslash\mathcal{C}_\varepsilon(z_0))^d$. □

9.1.1 Compactness result for sequences of microscopic displacement fields

To derive effective models by investigating the asymptotic behavior of the microscopic models presented here, we are going to proceed similar to Chapter 6. There, the two-scale limit of a sequences of displacement fields belonging to the microscopic models is identified by Proposition 3.7. According to coercivity condition (9.6), for a sequence $(z_\varepsilon)_{\varepsilon>0}$ of damage variables with $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ and $\sup_{\varepsilon>0} \|R_{\frac{\varepsilon}{2}} z_\varepsilon\|_{\mathrm{L}^p(\Omega_\varepsilon^+)^{d\times d}} < \infty$, here we will have to deal with sequences $(u_\varepsilon)_{\varepsilon>0}$ of displacement fields with $u_\varepsilon \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega\backslash\mathcal{C}_\varepsilon(z_\varepsilon))^d$ and $\sup_{\varepsilon>0}\|u_\varepsilon\|_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega\backslash\mathcal{C}_\varepsilon(z_\varepsilon))^d} < \infty$. For this reason the compactness with respect to the two-scale topology of such sequences is discussed in the following theorem.

Theorem 9.3 (Compactness result)**.** *Assume that the conditions* (9.1), (9.2), *and* (9.3) *hold. Let* $(u_\varepsilon, z_\varepsilon)_{\varepsilon>0}$ *be a sequence with* $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1])$ *and* $u_\varepsilon \in \mathrm{H}^1(\Omega\backslash\mathcal{C}_\varepsilon(z_\varepsilon))^d$ *such that there exists a function* $z_0 \in \mathrm{L}^p(\Omega;[0,1]^m)$ *satisfying* $z_\varepsilon \to z_0$ *in* $\mathrm{L}^p(\Omega)^m$*. Moreover, assume that there exists a constant* $C > 0$ *independent of* $\varepsilon > 0$ *and* z_ε *such that*

$$\sup_{\varepsilon>0}\|u_\varepsilon\|_{\mathrm{H}^1(\Omega\backslash\mathcal{C}_\varepsilon(z_\varepsilon))^d} \leq C.$$

Then there exists a subsequence of $(\varepsilon)_{\varepsilon>0}$ *(not relabeled) and functions* $u_0 \in \mathrm{H}^1(\Omega)^d$ *and* $U_1 \in \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(\mathbf{0})))^d$ *such that*

$$\textit{for almost every } x \in \Omega \textit{ it holds } U_1(x,\cdot) \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(z_0(x)))^d \textit{ and} \tag{9.10}$$

$$u_\varepsilon \xrightarrow{s} Eu_0 \qquad \textit{in } \mathrm{L}^2(\Omega{\times}Y)^d, \tag{9.11a}$$

$$\nabla u_\varepsilon \xrightharpoonup{w} \nabla_x Eu_0 + \nabla_y U_1 \quad \textit{in } \mathrm{L}^2(\Omega{\times}Y)^{d\times d}. \tag{9.11b}$$

The proof of this theorem is an easy consequence of the following lemma.

Lemma 9.4. *Assume that* (9.1), (9.2), *and* (9.3) *hold. For* $z_0 \in \mathrm{L}^p(\Omega;[0,1]^m)$ *let* $(u_\varepsilon, z_\varepsilon)_{\varepsilon>0}$ *denote a sequence with* $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ *and* $u_\varepsilon \in \mathrm{H}^1(\Omega\backslash\mathcal{C}_\varepsilon(z_\varepsilon))^d$ *such that* $z_\varepsilon \to z_0$ *in* $\mathrm{L}^p(\Omega)^m$*. Moreover, assume that there exists a constant* $C > 0$ *independent of* $\varepsilon > 0$ *and* z_ε *such that*

$$\sup_{\varepsilon>0}\|u_\varepsilon\|_{\mathrm{H}^1(\Omega\backslash\mathcal{C}_\varepsilon(z_\varepsilon))^d} \leq C.$$

For any $\eta \in (0,\eta^*)$ *and for* $\widehat{z} \in [0,1]^m$ *set* $L^\eta(\widehat{z}) := \mathrm{neigh}_\eta(L^{\mathrm{C}}(\widehat{z}))$ *and*

$$\Omega^{\mathrm{D}}_{\varepsilon,\eta}(z_\varepsilon) := \bigcup_{\lambda\in\Lambda^-_\varepsilon} \varepsilon(\lambda + L^\eta(z^{\varepsilon\lambda})),$$

where $z^{\varepsilon\lambda} := z_\varepsilon|_{\varepsilon(\lambda+Y)}$ *for all* $\lambda \in \Lambda^-_\varepsilon$*; see* (2.15)*. Let* $\mathcal{X}^\eta_{\varepsilon,z_\varepsilon} : \mathrm{H}^1(\Omega\backslash\Omega^{\mathrm{D}}_{\varepsilon,\eta}(z_\varepsilon))^d \to \mathrm{H}^1(\Omega)^d$ *denote the continuation operator of Theorem 8.4. If* $u^\eta_\varepsilon := \mathcal{X}^\eta_{\varepsilon,z_\varepsilon}(u_\varepsilon|_{\Omega\backslash\Omega^{\mathrm{D}}_{\varepsilon,\eta}(z_\varepsilon)})$*, then there exists a subsequence of* $(\varepsilon)_{\varepsilon>0}$ *(not relabeled) and* $(u_0, U^\eta_1) \in \mathrm{H}^1(\Omega)^d{\times}\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$*, where* u_0 *is independent of* $\eta \in (0,\eta^*)$*, such that for* $\varepsilon \to 0$ *it holds*

$$u^\eta_\varepsilon \rightharpoonup u_0 \qquad \textit{in } \mathrm{H}^1(\Omega)^d, \tag{9.12a}$$

$$u^\eta_\varepsilon \xrightarrow{s} Eu_0 \qquad \textit{in } \mathrm{L}^2(\Omega{\times}Y)^d, \tag{9.12b}$$

$$\nabla u^\eta_\varepsilon \xrightharpoonup{w} \nabla_x Eu_0 + \nabla_y U^\eta_1 \quad \textit{in } \mathrm{L}^2(\Omega{\times}Y)^{d\times d}. \tag{9.12c}$$

Here, the characteristic function $H^\eta(z_0) \in \mathrm{L}^\infty(\Omega{\times}Y)$ *for almost every* $(x,y) \in \Omega{\times}Y$ *is defined by* $H^\eta(z_0)(x,y) := \mathbb{1}_{Y\backslash L^\eta(z_0(x))}(y)$*. Furthermore, there exists a* η*-independent function* $U_1 \in \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(\mathbf{0})))^d$ *which satisfies* (9.10) *and*

$$\nabla_y U^\eta_1|_{\mathrm{supp}(H^\eta(z_0))} = \nabla_y U_1|_{\mathrm{supp}(H^\eta(z_0))} \quad \textit{almost everywhere.} \tag{9.13}$$

Proof. 1. According to Theorem 8.4 it holds

$$\sup_{\varepsilon>0} \|u_\varepsilon^\eta\|_{\mathrm{H}^1(\Omega)^d} \leq \sup_{\varepsilon>0} C_{\mathcal{X}}^\eta \|u_\varepsilon\|_{\mathrm{H}^1(\Omega\backslash\Omega^{\mathrm{D}}_{\varepsilon,\eta}(z_\varepsilon))^d} \leq C_{\mathcal{X}}^\eta C.$$

Therefore, due to Proposition 3.7 for fixed $\eta \in (0,\eta^*)$ there exists a subsequence of $(\varepsilon)_{\varepsilon>0}$ (not relabeled) and a function $(u_0^\eta, U_1^\eta) \in \mathrm{H}^1(\Omega)^d \times \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ such that

$$\begin{aligned} u_\varepsilon^\eta &\rightharpoonup u_0^\eta && \text{in } \mathrm{H}^1(\Omega)^d, && (9.14a)\\ u_\varepsilon^\eta &\xrightarrow{s} Eu_0^\eta && \text{in } \mathrm{L}^2(\Omega\times Y)^d, && (9.14b)\\ \nabla u_\varepsilon^\eta &\xrightharpoonup{w} \nabla_x Eu_0^\eta + \nabla_y U_1^\eta && \text{in } \mathrm{L}^2(\Omega\times Y)^{d\times d}. && (9.14c)\end{aligned}$$

Now we show that the limit function $u_0^\eta \in \mathrm{H}^1(\Omega)^d$ is η-independent. For this purpose, let $0<\eta_1<\eta_2<\eta^*$ be arbitrary but fixed. Then

$$\begin{aligned} u_\varepsilon^{\eta_1} &= u_\varepsilon \quad \text{on } \Omega\backslash\Omega^{\mathrm{D}}_{\varepsilon,\eta_1}(z_\varepsilon), && (9.15a)\\ u_\varepsilon^{\eta_2} &= u_\varepsilon \quad \text{on } \Omega\backslash\Omega^{\mathrm{D}}_{\varepsilon,\eta_2}(z_\varepsilon) \subset \Omega\backslash\Omega^{\mathrm{D}}_{\varepsilon,\eta_1}(z_\varepsilon) && (9.15b)\end{aligned}$$

by definition and there exist limit functions $u_0^{\eta_1}, u_0^{\eta_2} \in \mathrm{H}^1(\Omega)^d$ and a subsequence of $(\varepsilon)_{\varepsilon>0}$ (possibly different to that mentioned above but again not relabeled) such that $u_\varepsilon^{\eta_1} \rightharpoonup u_0^{\eta_1}$ in $\mathrm{H}^1(\Omega)^d$ and $u_\varepsilon^{\eta_2} \rightharpoonup u_0^{\eta_2}$ in $\mathrm{H}^1(\Omega)^d$. To prove the η-independence of $u_0^\eta \in \mathrm{H}^1(\Omega)^d$ we show that $u_0^{\eta_1} = u_0^{\eta_2}$ almost everywhere on Ω. For this purpose, we estimate the difference $u_0^{\eta_1} - u_0^{\eta_2}$ on $\Omega\backslash\Omega^{\mathrm{D}}_{\varepsilon,\eta_2}(z_\varepsilon)$ by applying the triangle inequality after subtracting and adding the function u_ε.

$$\begin{aligned} \|u_0^{\eta_1} - u_0^{\eta_2}\|_{\mathrm{L}^2(\Omega\backslash\Omega^{\mathrm{D}}_{\varepsilon,\eta_2}(z_\varepsilon))^d} &\overset{(9.15)}{\leq} \|u_0^{\eta_1} - u_\varepsilon^{\eta_1}\|_{\mathrm{L}^2(\Omega\backslash\Omega^{\mathrm{D}}_{\varepsilon,\eta_2}(z_\varepsilon))^d} + \|u_\varepsilon^{\eta_2} - u_0^{\eta_2}\|_{\mathrm{L}^2(\Omega\backslash\Omega^{\mathrm{D}}_{\varepsilon,\eta_2}(z_\varepsilon))^d}\\ &\leq \|u_0^{\eta_1} - u_\varepsilon^{\eta_1}\|_{\mathrm{L}^2(\Omega)^d} + \|u_\varepsilon^{\eta_2} - u_0^{\eta_2}\|_{\mathrm{L}^2(\Omega)^d} && (9.16)\end{aligned}$$

Due to (9.14a) the terms of the last line converge to zero for $\varepsilon \to 0$. Thus, performing the limit passage $\varepsilon \to 0$ in (9.16) with respect to the two-scale topology yields the desired result as we will see in the calculation below. According to Theorem 3.9 we have $\mathbb{1}_{\Omega\backslash\Omega^{\mathrm{D}}_{\varepsilon,\eta}(z_\varepsilon)} \xrightarrow{s} H^\eta(z_0)$ in $\mathrm{L}^1(\Omega\times Y)$. The limit of $(\mathcal{T}_\varepsilon(u_0^{\eta_1} - u_0^{\eta_2}))_{\varepsilon'>0}$ is given by Proposition 3.5(b) and finally the product converges according to Corollary 3.6.

$$\begin{aligned} 0 &= \lim_{\varepsilon\to0} \|u_0^{\eta_1} - u_0^{\eta_2}\|^2_{\mathrm{L}^2(\Omega\backslash\Omega^{\mathrm{D}}_{\varepsilon,\eta_2}(z_\varepsilon))^d}\\ &\overset{(3.2)}{=} \lim_{\varepsilon\to0} \int_{\mathbb{R}^d\times Y} \mathcal{T}_\varepsilon \mathbb{1}_{\Omega\backslash\Omega^{\mathrm{D}}_{\varepsilon,\eta_2}(z_\varepsilon)}(x,y) |\mathcal{T}_\varepsilon(u_0^{\eta_1} - u_0^{\eta_2})(x,y)|_d^2 \mathrm{d}y\mathrm{d}x\\ &= \int_{\Omega\times Y} H^{\eta_2}(z_0)(x,y) |E(u_0^{\eta_1} - u_0^{\eta_2})(x,y)|_d^2 \mathrm{d}y\mathrm{d}x\\ &\geq \int_\Omega \mu_d(Y\backslash L^{\eta^*}(0)) |u_0^{\eta_1}(x) - u_0^{\eta_2}(x)|_d^2 \mathrm{d}x \geq 0\end{aligned}$$

Independent of the choice of $\eta_2 \in (0,\eta^*)$ for almost every $x \in \Omega$ we obtain that the factor $\int_Y H^{\eta_2}(z_0)(x,y)\mathrm{d}y \geq \mu_d(Y\backslash L^{\eta^*}(0)) > 0$ is positive. Therefore, the last line of this calculation forces the limit functions $u_0^{\eta_1}, u_0^{\eta_2} \in \mathrm{H}^1(\Omega)^d$ to coincide almost everywhere on Ω.

2. Now we verify the existence of $U_1 \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(0)))^d$ satisfying condition (9.10) and fulfilling (9.13) for any $\eta \in (0, \eta^*)$. For this purpose, for $0 < \eta_1 < \eta_2 < \eta^*$ we start by showing $\nabla_y U_1^{\eta_1}|_{\mathrm{supp}(H^{\eta_2}(z_0))} = \nabla_y U_1^{\eta_2}|_{\mathrm{supp}(H^{\eta_2}(z_0))}$ almost everywhere. For an arbitrary but fixed function $V \in \mathrm{L}^2(\Omega{\times}Y)^{d\times d}$ let $(v_\varepsilon)_{\varepsilon>0} \subset \mathrm{L}^2(\Omega)^{d\times d}$ be given by $v_\varepsilon := \mathcal{F}_\varepsilon^{(2)}(V)$. Then $v_\varepsilon \overset{s}{\to} V$ in $\mathrm{L}^2(\Omega{\times}Y)^{d\times d}$ due to Proposition 3.5(d). By recalling $\mathbb{1}_{\Omega\backslash\Omega^{\mathrm{D}}_{\varepsilon,\eta}(z_\varepsilon)} \overset{s}{\to} H^\eta(z_0)$ in $\mathrm{L}^1(\Omega{\times}Y)$, Corollary 3.6 yields $\mathbb{1}_{\Omega\backslash\Omega^{\mathrm{D}}_{\varepsilon,\eta}(z_\varepsilon)} v_\varepsilon \overset{s}{\to} H^\eta(z_0)V$ in $\mathrm{L}^2(\Omega{\times}Y)^{d\times d}$. Hence, the weak convergence of (9.14c) for η_1 and η_2 results in

$$0 \overset{(9.15)}{=} \Big\langle \nabla u_\varepsilon^{\eta_1} - \nabla u_\varepsilon^{\eta_2}, \mathbb{1}_{\Omega\backslash\Omega^{\mathrm{D}}_{\varepsilon,\eta_2}(z_\varepsilon)} v_\varepsilon \Big\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} \overset{\varepsilon\to 0}{\longrightarrow} \Big\langle \nabla_y U_1^{\eta_1} - \nabla_y U_1^{\eta_2}, H^{\eta_2}(z_0)V \Big\rangle_{\mathrm{L}^2(\Omega\times Y)^{d\times d}}.$$

Therefore, for any $V \in \mathrm{L}^2(\Omega{\times}Y)^{d\times d}$ it holds $0 = \langle \nabla_y U_1^{\eta_1} - \nabla_y U_1^{\eta_2}, V\rangle_{\mathrm{L}^2(\mathrm{supp}(H^{\eta_2}(z_0)))^{d\times d}}$ which implies $\nabla_y U_1^{\eta_1}|_{\mathrm{supp}(H^{\eta_2}(z_0))} = \nabla_y U_1^{\eta_2}|_{\mathrm{supp}(H^{\eta_2}(z_0))}$ almost everywhere due to the fundamental lemma of calculus of variations. Since this holds true for any $0 < \eta_1 < \eta_2 < \eta^*$, the function $U_1 \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(0)))^d$, which satisfies the condition (9.10) for almost every $(x, y) \in \mathrm{supp}(H^0(z_0))$, can be defined by $\nabla_y U_1(x, y) := \lim_{\eta\to 0} \nabla_y U_1^\eta(x, y)$. Here, for fixed $(x, y) \in \mathrm{supp}(H^0(z_0))$ the limit $\lim_{\eta\to 0} \nabla_y U_1^\eta(x, y)$ does exist, since $(x, y) \in \mathrm{supp}(H^\eta(z_0))$ for all $\eta \in (0, \eta_0)$ and $\eta_0 > 0$ sufficiently small. Hence, ignoring the fact that all considered functions are only defined almost everywhere on $\Omega{\times}Y$, the sequence $(\nabla_y U_1^\eta(x, y))_{\eta\in(0,\eta_0)}$ is constant. □

Proof of Theorem 9.3. Observe that

$$\mathbf{L}^2_{\eta^*}(\mathbb{R}^d{\times}Y) := \{v \in \mathrm{L}^2(\mathbb{R}^d{\times}Y) \,|\, \exists\, \widehat{\eta} \in (0, \eta^*) \text{ such that } \mathrm{supp}(v) \subset \mathbb{R}^d{\times}Y\backslash L^{\widehat{\eta}}(\mathbf{0})\}$$

is a dense subset of $\mathrm{L}^2(\mathbb{R}^d{\times}Y)$. Combining Proposition 3.3(a) with the given a priori estimate $\sup_{\varepsilon>0} \|u_\varepsilon\|_{\mathrm{H}^1(\Omega\backslash\mathcal{C}_\varepsilon(z_\varepsilon))^d} \le C$ we have

$$\|\mathcal{T}_\varepsilon u_\varepsilon\|^2_{\mathrm{L}^2(\mathbb{R}^d\times Y\backslash L^{\mathrm{C}}(\mathbf{0}))^d} + \|\mathcal{T}_\varepsilon(\nabla u_\varepsilon)\|^2_{\mathrm{L}^2(\mathbb{R}^d\times Y\backslash L^{\mathrm{C}}(\mathbf{0}))^{d\times d}} = \|u_\varepsilon\|^2_{\mathrm{H}^1(\Omega\backslash\mathcal{C}_\varepsilon(z_\varepsilon))^d} \le C^2.$$

1. To verify the weak convergences in (9.11), it is sufficient to show that there exist functions $u_0 \in \mathrm{H}^1(\Omega)^d$ and $U_1 \in \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(\mathbf{0})))^d$ with $U_1(x,\cdot) \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(z_0(x)))^d$ for almost every $x \in \Omega$ such that for all $(\varphi, \Phi) \in \mathbf{L}^2_{\eta^*}(\mathbb{R}^d{\times}Y)^d{\times}\mathbf{L}^2_{\eta^*}(\mathbb{R}^d{\times}Y)^{d\times d}$ it holds

$$\langle \mathcal{T}_\varepsilon u_\varepsilon, \varphi\rangle_{\mathrm{L}^2(\mathbb{R}^d\times Y)^d} \overset{\varepsilon\to 0}{\longrightarrow} \langle (Eu_0)^{\mathrm{ex}}, \varphi\rangle_{\mathrm{L}^2(\mathbb{R}^d\times Y)^d},$$
$$\langle \mathcal{T}_\varepsilon(\nabla u_\varepsilon), \Phi\rangle_{\mathrm{L}^2(\mathbb{R}^d\times Y)^{d\times d}} \overset{\varepsilon\to 0}{\longrightarrow} \langle (\nabla_x E u_0 + \nabla_y U_1)^{\mathrm{ex}}, \Phi\rangle_{\mathrm{L}^2(\mathbb{R}^d\times Y)^{d\times d}}.$$

Note that this verification in both cases is done by the same arguments, which is why we focus on the latter relation. Let $\Phi \in \mathbf{L}^2_{\eta^*}(\mathbb{R}^d{\times}Y)^{d\times d}$ be arbitrary but fixed. By definition there exists $\eta \in (0, \eta^*)$ such that $\mathrm{supp}(\Phi) \subset \mathbb{R}^d{\times}Y\backslash L^\eta(\mathbf{0})$. Proposition 3.3 allows us to replace the left hand side in the following way:

$$\langle \mathcal{T}_\varepsilon(\nabla u_\varepsilon), \Phi\rangle_{\mathrm{L}^2(\mathbb{R}^d\times Y)^{d\times d}} = \langle \nabla u_\varepsilon, \mathcal{F}_\varepsilon \Phi\rangle_{\mathrm{L}^2(\Omega)^{d\times d}} \tag{9.17}$$

By the definition of $\mathbf{L}^2_{\eta^*}(\mathbb{R}^d{\times}Y)^{d\times d}$ we have $\operatorname{supp}(\mathcal{F}_\varepsilon\Phi) \subset \Omega\backslash\Omega^{\mathrm{D}}_{\varepsilon,\eta}(\mathbf{0})$. Furthermore, according to the assumptions Lemma 9.4 is applicable, yielding the existence of functions $(u_0, U_1^\eta) \in \mathrm{H}^1(\Omega)^d{\times}\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d$ and $U_1 \in \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(\mathbf{0})))^d$ satisfying (9.10) and (9.13) such that $u^\eta_\varepsilon := \mathcal{X}^\eta_{\varepsilon,z_\varepsilon}(u_\varepsilon|_{\Omega\backslash\Omega^{\mathrm{D}}_{\varepsilon,\eta}(z_\varepsilon)})$ converges in the sense of (9.12). Since the function u_ε by definition coincides with u^η_ε on $\Omega\backslash\Omega^{\mathrm{D}}_{\varepsilon,\eta}(z_\varepsilon) \supset \Omega\backslash\Omega^{\mathrm{D}}_{\varepsilon,\eta}(\mathbf{0})$, condition (9.17) enables us to do the following replacement to conclude the proof.

$$\begin{aligned}
\lim_{\varepsilon\to 0}\langle\mathcal{T}_\varepsilon(\nabla u_\varepsilon),\Phi\rangle_{\mathrm{L}^2(\mathbb{R}^d\times Y)^{d\times d}} &= \lim_{\varepsilon\to 0}\langle\mathcal{T}_\varepsilon(\nabla u^\eta_\varepsilon),\Phi\rangle_{\mathrm{L}^2(\mathbb{R}^d\times Y)^{d\times d}} \\
&\overset{(9.12c)}{=} \langle(\nabla_x E u_0 + \nabla_y U_1^\eta)^{\mathrm{ex}},\Phi\rangle_{\mathrm{L}^2(\mathbb{R}^d\times Y)^{d\times d}} \\
&\overset{(9.13)}{=} \langle(\nabla_x E u_0 + \nabla_y U_1)^{\mathrm{ex}},\Phi\rangle_{\mathrm{L}^2(\mathbb{R}^d\times Y)^{d\times d}}
\end{aligned}$$

Here, the last equality results from the combination of condition (9.13) with the fact that $\operatorname{supp}(H^\eta(z_0)) \supset \operatorname{supp}(\Phi)\cap\Omega{\times}Y$.

2. To show the strong convergence of line (9.11a), for some arbitrary but fixed $\eta > 0$ and $u^\eta_\varepsilon := \mathcal{X}^\eta_{\varepsilon,z_\varepsilon}(u_\varepsilon|_{\Omega\backslash\Omega^{\mathrm{D}}_{\varepsilon,\eta}(z_\varepsilon)})$ we exploit the following decomposition.

$$\|u_\varepsilon - u_0\|^2_{\mathrm{L}^2(\Omega)^d} = \|u^\eta_\varepsilon - u_0\|^2_{\mathrm{L}^2(\Omega\backslash\Omega^{\mathrm{D}}_{\varepsilon,\eta}(z_\varepsilon))^d} + \|u_\varepsilon - u_0\|^2_{\mathrm{L}^2(\Omega^{\mathrm{D}}_{\varepsilon,\eta}(z_\varepsilon))^d}$$

By increasing the domain of integration in the first term and by applying Hölder's inequality to the second one we obtain

$$\begin{aligned}
\|u_\varepsilon - u_0\|^2_{\mathrm{L}^2(\Omega)^d} &\le \|u^\eta_\varepsilon - u_0\|^2_{\mathrm{L}^2(\Omega)^d} + \Big(\mu_d(\Omega^{\mathrm{D}}_{\varepsilon,\eta}(z_\varepsilon))\Big)^{\frac{r-2}{r}}\|u_\varepsilon - u_0\|^2_{\mathrm{L}^r(\Omega^{\mathrm{D}}_{\varepsilon,\eta}(z_\varepsilon))^d} \\
&\le \|u^\eta_\varepsilon - u_0\|^2_{\mathrm{L}^2(\Omega)^d} + \Big(\mu_d(\Omega^{\mathrm{D}}_{\varepsilon,\eta}(z_\varepsilon))\Big)^{\frac{r-2}{r}}\|u_\varepsilon - u_0\|^2_{\mathrm{L}^r(\Omega)^d}
\end{aligned}$$

where for the Sobolev exponent $2^* := \frac{2d}{d-2}$ the parameter r is an element of the interval $(2, 2^*)$. Therefore, according to (9.12b) the first term converges to zero and due to the continuous embedding of $\mathrm{H}^1(\Omega\backslash\mathcal{C}_\varepsilon(z_\varepsilon))^d$ into $\mathrm{L}^r(\Omega\backslash\mathcal{C}_\varepsilon(z_\varepsilon))^d = \mathrm{L}^r(\Omega)^d$ the second term is bounded by $(\mu_d(\Omega^{\mathrm{D}}_{\varepsilon,\eta}(z_\varepsilon)))^{\frac{r-2}{r}}C^2_{\mathrm{Sob}}(C + \|u_0\|_{\mathrm{H}^1(\Omega)^d})^2$. Since for fixed $\eta > 0$ it holds $\mu_d(\Omega^{\mathrm{D}}_{\varepsilon,\eta}(z_\varepsilon)) \to 0$ for $\varepsilon \to 0$ this concludes the proof; see Proposition 3.5(b). $\square$

9.1.2 Recovery sequence for two-scale displacement fields with jumps

The construction of the displacement field component of the mutual recovery sequence of Theorem 6.15 as well as the recovery sequence of the Γ-convergence result of Theorem 6.17 rely on the density result of Proposition 3.8. As we have already seen in Chapter 8, the implicit dependence of the displacement field on the damage variable (due to the energy functional of line (9.5)) causes some technicalities when constructing these recovery sequences. However, in contrast to Chapter 8, here we have to provide a completely new construction. Referring to Section 6.5, for an arbitrary function $(\widetilde{u}_0, \widetilde{U}_1, \widetilde{z}_0) \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d{\times}\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(\mathbf{0})))^d{\times}\mathrm{W}^{1,p}(\Omega;[0,1]^m)$ satisfying for almost every $x \in \Omega$ the condition $\widetilde{U}_1(x,\cdot) \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(\widetilde{z}_0(x)))^d$, the recovery sequence

$(\widetilde{z}_\varepsilon)_{\varepsilon>0}$ for the damage component will be given by Theorem 4.9. Translated in the here considered setting, the construction of the displacement field component of the recovery sequence $(\widetilde{u}_\varepsilon)_{\varepsilon>0}$ has to be done in such a way that it holds

$$\lim_{\varepsilon\to 0}\langle \mathbb{C}_{\mathrm{strong}}\mathbf{e}(\widetilde{u}_\varepsilon), \mathbf{e}(\widetilde{u}_\varepsilon)\rangle_{\mathrm{L}^2(\Omega\backslash\mathcal{C}_\varepsilon(\widetilde{z}_\varepsilon))^{d\times d}} = \langle \mathbb{C}_{\mathrm{strong}}\widetilde{\mathbf{e}}(\widetilde{u}_0, \widetilde{U}_1), \widetilde{\mathbf{e}}(\widetilde{u}_0, \widetilde{U}_1)\rangle_{\mathrm{L}^2(\Omega\times Y\backslash L^{\mathrm{C}}(\mathbf{0}))^{d\times d}};$$

see (6.39). This means that for fixed $\varepsilon > 0$ the set on which the function $\widetilde{u}_\varepsilon$ is allowed to jump has to be contained in $\mathcal{C}_\varepsilon(\widetilde{z}_\varepsilon)$, i.e., $\widetilde{u}_\varepsilon \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega\backslash\mathcal{C}_\varepsilon(\widetilde{z}_\varepsilon))^d$. In contrast to the construction of Chapter 6, here the displacement field component of the recovery sequences depends on the already constructed damage component.

Theorem 9.5. *Let $(\widetilde{z}_\varepsilon)_{\varepsilon>0}$ with $\widetilde{z}_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ be given such that there exists $\widetilde{z}_0 \in \mathrm{L}^p(\Omega;[0,1]^m)$ with $\widetilde{z}_\varepsilon \to \widetilde{z}_0$ in $\mathrm{L}^p(\Omega)^m$. Moreover, let $\widetilde{U}_1 \in \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(\mathbf{0})))^d$ satisfy $\widetilde{U}_1(x,\cdot) \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(\widetilde{z}_0(x)))^d$ for almost every $x \in \Omega$. Then, for $\widetilde{u}_0 \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ there exists a sequence $(\widetilde{u}_\varepsilon)_{\varepsilon>0}$ with $\widetilde{u}_\varepsilon \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega\backslash\mathcal{C}_\varepsilon(\widetilde{z}_\varepsilon))^d$ such that*

$$\begin{aligned} \widetilde{u}_\varepsilon &\xrightarrow{s} E\widetilde{u}_0 && \text{in } \mathrm{L}^2(\Omega\times Y)^d,\\ \nabla\widetilde{u}_\varepsilon &\xrightarrow{s} \nabla_x E\widetilde{u}_0+\nabla_y\widetilde{U}_1 && \text{in } \mathrm{L}^2(\Omega\times Y)^{d\times d}. \end{aligned}$$

Proof. For $i = 1,2,\ldots,d$ let $[\widetilde{U}_1]_i \in \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(\mathbf{0})))$ denote the i-th component of the function $\widetilde{U}_1 \in \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(\mathbf{0})))^d$. If the function $v^{(i)}_\varepsilon \in \mathrm{H}^1_0(\Omega\backslash\mathcal{C}_\varepsilon(\widetilde{z}_\varepsilon))$ for $V^{(i)}_1 := [\widetilde{U}_1]_i \in \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(\mathbf{0})))$ denotes the unique solution of (9.18) below and if $\widetilde{v}_\varepsilon \in \mathrm{H}^1_0(\Omega\backslash\mathcal{C}_\varepsilon(\widetilde{z}_\varepsilon))^d$ is defined by $[\widetilde{v}_\varepsilon]_i := v^{(i)}_\varepsilon$, then the sequence $(\widetilde{u}_\varepsilon)_{\varepsilon>0}$ is given by $\widetilde{u}_\varepsilon := \widetilde{u}_0 + \widetilde{v}_\varepsilon \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega\backslash\mathcal{C}_\varepsilon(\widetilde{z}_\varepsilon))^d$. Thus, the statement of Theorem 9.5 is an immediate consequence of Lemma 9.7 below. $\square$

Remark 9.6. *Observe that in the case of $\widetilde{z}_\varepsilon :\equiv \mathbf{0}$ (for all $\varepsilon > 0$), this result contains the periodic case, i.e., $\widetilde{u}_\varepsilon \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega\backslash\mathcal{C}_\varepsilon(\mathbf{0}))^d$ and $\mathcal{C}_\varepsilon(\mathbf{0})$ is a periodic set. As a result this theorem is a proper generalization of the periodic case, which is treated in* [10], *for instance.*

Lemma 9.7. *Let $(\widetilde{z}_\varepsilon)_{\varepsilon>0}$ with $\widetilde{z}_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ be given such that there exists a function $\widetilde{z}_0 \in \mathrm{L}^p(\Omega;[0,1]^m)$ with $\widetilde{z}_\varepsilon \to \widetilde{z}_0$ in $\mathrm{L}^p(\Omega)^m$. Moreover, let $V_1 \in \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(\mathbf{0})))$ satisfy $V_1(x,\cdot) \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(\widetilde{z}_0(x)))$ for almost every $x \in \Omega$. If $v_\varepsilon \in \mathrm{H}^1_0(\Omega\backslash\mathcal{C}_\varepsilon(\widetilde{z}_\varepsilon))$ for $\varepsilon > 0$ denotes the solution of the elliptic problem*

$$\int_\Omega v_\varepsilon\varphi_\varepsilon + \langle\nabla v_\varepsilon - \mathcal{F}^{(2)}_\varepsilon(\nabla_y V_1)^{\mathrm{ex}}, \nabla\varphi_\varepsilon\rangle_d \mathrm{d}x = 0 \quad \forall\,\varphi_\varepsilon \in \mathrm{H}^1_0(\Omega\backslash\mathcal{C}_\varepsilon(\widetilde{z}_\varepsilon)), \tag{9.18}$$

where the folding operator $\mathcal{F}^{(2)}_\varepsilon : \mathrm{L}^2(\mathbb{R}^d\times Y)^d \to \mathrm{L}^2(\Omega)^d$ is given by Definition 3.2, *then the sequence of solutions $(v_\varepsilon)_{\varepsilon>0}$ fulfills*

$$v_\varepsilon \xrightarrow{s} \mathbf{0} \quad \text{in } \mathrm{L}^2(\Omega\times Y), \tag{9.19a}$$

$$\nabla v_\varepsilon \xrightarrow{s} \nabla_y V_1 \quad \text{in } \mathrm{L}^2(\Omega\times Y)^d. \tag{9.19b}$$

Proof. Choosing $\varphi_\varepsilon = v_\varepsilon$ in (9.18), by rearranging its terms we find

$$\|v_\varepsilon\|_{\mathrm{H}^1_0(\Omega\setminus\mathcal{C}_\varepsilon(\tilde{z}_\varepsilon))} \le \|\mathcal{F}^{(2)}_\varepsilon(\nabla_y V_1)^{\mathrm{ex}}\|_{\mathrm{L}^2(\Omega\setminus\mathcal{C}_\varepsilon(\mathbf{0}))^d} \le \|\nabla_y V_1\|_{\mathrm{L}^2(\Omega\times Y\setminus L^{\mathrm{C}}(\mathbf{0}))^d}, \tag{9.20}$$

where the last inequality is due to Proposition 3.3(b).

1. We start by investigating the case, where the sequence $(\widetilde{z}_\varepsilon)_{\varepsilon>0}$ with $\widetilde{z}_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ for any $\varepsilon > 0$ is given by $\widetilde{z}_\varepsilon :\equiv \mathbf{0}$. Hence, $\widetilde{z}_\varepsilon \to \widetilde{z}_0 :\equiv \mathbf{0}$ in $\mathrm{L}^p(\Omega)^m$. Due to [70] the linear span of $\mathrm{C}^\infty_\mathrm{c}(\Omega)\times(\mathrm{C}^\infty(\mathcal{Y}\setminus L^{\mathrm{C}}(\mathbf{0})) \cap \mathrm{H}^1(\mathcal{Y}\setminus L^{\mathrm{C}}(\mathbf{0})))$ is a dense subset of $\mathrm{L}^2(\Omega;\mathrm{H}^1(\mathcal{Y}\setminus L^{\mathrm{C}}(\mathbf{0})))$. Therefore, according to the a priori estimate (9.20) it is sufficient to prove (9.19) only for all functions $V_1 \in \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\setminus L^{\mathrm{C}}(\mathbf{0})))^d \subset \mathrm{L}^2(\Omega;\mathrm{H}^1(\mathcal{Y}\setminus L^{\mathrm{C}}(\mathbf{0})))$ that are given by finite linear combinations of functions $W = w_0 w_1$, where $w_0 \in \mathrm{C}^\infty_\mathrm{c}(\Omega)$ and $w_1 \in \mathrm{C}^\infty(\mathcal{Y}\setminus L^{\mathrm{C}}(\mathbf{0}))\cap\mathrm{H}^1(\mathcal{Y}\setminus L^{\mathrm{C}}(\mathbf{0}))$. This enables us to define $g_\varepsilon, r_\varepsilon \in \mathrm{H}^1_0(\Omega\setminus\mathcal{C}_\varepsilon(\mathbf{0}))$ for all $x\in\Omega$ by $g_\varepsilon(x) := \varepsilon V_1(x,\{\frac{x}{\varepsilon}\}_Y)$ and $r_\varepsilon := v_\varepsilon - g_\varepsilon$, where $v_\varepsilon \in \mathrm{H}^1_0(\Omega\setminus\mathcal{C}_\varepsilon(\mathbf{0}))$ is the solution of (9.18). Exploiting the definition of the unfolding operator (see Definition 3.1) and the continuity of V_1 we immediately obtain

$$g_\varepsilon \overset{s}{\to} \mathbf{0} \qquad \text{in } \mathrm{L}^2(\Omega\times Y), \tag{9.21a}$$

$$\nabla g_\varepsilon \overset{s}{\to} \nabla_y V_1 \quad \text{in } \mathrm{L}^2(\Omega\times Y)^d. \tag{9.21b}$$

Now we are going to prove that it holds

$$\|\mathcal{T}_\varepsilon r_\varepsilon\|^2_{\mathrm{L}^2(\mathbb{R}^d\times Y\setminus L^{\mathrm{C}}(\mathbf{0}))} + \|\mathcal{T}_\varepsilon(\nabla r_\varepsilon)\|^2_{\mathrm{L}^2(\mathbb{R}^d\times Y\setminus L^{\mathrm{C}}(\mathbf{0}))^d} = \|r_\varepsilon\|^2_{\mathrm{H}^1_0(\Omega\setminus\mathcal{C}_\varepsilon(\mathbf{0}))} \overset{\varepsilon\to 0}{\longrightarrow} 0,$$

which together with $v_\varepsilon = g_\varepsilon + r_\varepsilon$ and (9.21) concludes the proof of step 1. For this purpose we choose $\varphi_\varepsilon = r_\varepsilon$ as a testfunction in (9.18) and rearrange its terms in the calculation below. Moreover, we exploit Proposition 3.3(a) in the last line.

$$\begin{aligned}
&\int_\Omega r_\varepsilon^2 + \langle \nabla r_\varepsilon, \nabla r_\varepsilon\rangle_d \mathrm{d}x \\
&\qquad = \int_\Omega -g_\varepsilon r_\varepsilon + \langle \mathcal{F}^{(2)}_\varepsilon(\nabla_y V_1)^{\mathrm{ex}} - \nabla g_\varepsilon, \nabla r_\varepsilon\rangle_d \mathrm{d}x \\
&\qquad \le \|g_\varepsilon\|_{\mathrm{L}^2(\Omega)}\|r_\varepsilon\|_{\mathrm{L}^2(\Omega)} + \|\mathcal{F}^{(2)}_\varepsilon(\nabla_y V_1)^{\mathrm{ex}} - \nabla g_\varepsilon\|_{\mathrm{L}^2(\Omega)^d}\|\nabla r_\varepsilon\|_{\mathrm{L}^2(\Omega)^d} \\
&\qquad \le \Big(\|\mathcal{T}_\varepsilon g_\varepsilon\|_{\mathrm{L}^2(\mathbb{R}^d\times Y)} + \|\nabla_y V_1 - \mathcal{T}_\varepsilon(\nabla g_\varepsilon)\|_{\mathrm{L}^2(\mathbb{R}^d\times Y)^d}\Big)\|r_\varepsilon\|_{\mathrm{H}^1_0(\Omega\setminus\mathcal{C}_\varepsilon(\mathbf{0}))}
\end{aligned}$$

Observe that the left hand side of this estimate can be estimated from below by the term $\frac{1}{2}\|r_\varepsilon\|^2_{\mathrm{H}^1_0(\Omega\setminus\mathcal{C}_\varepsilon(\mathbf{0}))}$ and hence the convergence (9.21) yields the desire result.

2. To prove the general case, where $(\widetilde{z}_\varepsilon)_{\varepsilon>0}$ with $\widetilde{z}_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ is a given sequence such that there exists $\widetilde{z}_0 \in \mathrm{L}^p(\Omega;[0,1]^m)$ with $\widetilde{z}_\varepsilon \to \widetilde{z}_0$ in $\mathrm{L}^p(\Omega)^m$, we start by proving (9.19) with respect to the weak two-scale topology. For this purpose, let $\Xi_{\varepsilon,\tilde{z}_\varepsilon} : \Omega\setminus\mathcal{C}_\varepsilon(\widetilde{z}_\varepsilon) \to \Omega\setminus\mathcal{C}_\varepsilon(\mathbf{0})$ denote the transformation defined by $\Xi_{\varepsilon,\tilde{z}_\varepsilon}|_{\Omega\setminus\Omega^-_\varepsilon} := \mathrm{id}$ and $\Xi_{\varepsilon,\tilde{z}_\varepsilon}(x) := \mathcal{N}_\varepsilon(x) + \varepsilon T^{\mathrm{C}}_{\tilde{z}_\varepsilon(x)}(\{\frac{x}{\varepsilon}\}_Y)$, $x\in\Omega^-_\varepsilon$. Here, for $\widehat{z}\in[0,1]^m$ the bi-Lipschitz transformation $T^{\mathrm{C}}_{\hat{z}} : Y \to Y$ satisfies the conditions (9.2g), (9.2h), (9.2i), and (9.3). Thus, $\Xi^{-1}_{\varepsilon,\tilde{z}_\varepsilon} : \Omega\setminus\mathcal{C}_\varepsilon(\mathbf{0}) \to \Omega\setminus\mathcal{C}_\varepsilon(\widetilde{z}_\varepsilon)$ is given by $\Xi^{-1}_{\varepsilon,\tilde{z}_\varepsilon}(\overline{x}) := \mathcal{N}_\varepsilon(\overline{x}) + \varepsilon(T^{\mathrm{C}}_{\tilde{z}_\varepsilon(\overline{x})})^{-1}(\{\frac{\overline{x}}{\varepsilon}\}_Y)$, $\overline{x}\in\Omega^-_\varepsilon$, and $\Xi^{-1}_{\varepsilon,\tilde{z}_\varepsilon}|_{\Omega\setminus\Omega^-_\varepsilon} := \mathrm{id}$. By applying this transformation to (9.18) we are going to prove

(9.19) by exploiting the result of step 1. The function $v_\varepsilon \in \mathrm{H}_0^1(\Omega\backslash\mathcal{C}_\varepsilon(\tilde{z}_\varepsilon))^d$ is the solution of (9.18) if and only if $\overline{v}_\varepsilon := v_\varepsilon \circ \Xi_{\varepsilon,\tilde{z}_\varepsilon}^{-1} \in \mathrm{H}_0^1(\Omega\backslash\mathcal{C}_\varepsilon(\mathbf{0}))$ for all $\overline{\varphi}_\varepsilon \in \mathrm{H}_0^1(\Omega\backslash\mathcal{C}_\varepsilon(\mathbf{0}))$ satisfies

$$\int_\Omega |\det(\nabla\Xi_{\varepsilon,\tilde{z}_\varepsilon}^{-1})|\Big(\overline{v}_\varepsilon\overline{\varphi}_\varepsilon + \Big\langle\nabla\overline{v}_\varepsilon(\nabla\Xi_{\varepsilon,\tilde{z}_\varepsilon}^{-1})^{-1}, \nabla\overline{\varphi}_\varepsilon(\nabla\Xi_{\varepsilon,\tilde{z}_\varepsilon}^{-1})^{-1}\Big\rangle_d\Big)\mathrm{d}\overline{x} \tag{9.22}$$
$$= \int_\Omega |\det(\nabla\Xi_{\varepsilon,\tilde{z}_\varepsilon}^{-1})|\Big\langle\mathcal{F}_\varepsilon^{(2)}(\nabla_y V_1)^{\mathrm{ex}} \circ \Xi_{\varepsilon,\tilde{z}_\varepsilon}^{-1}, \nabla\overline{\varphi}_\varepsilon(\nabla\Xi_{\varepsilon,\tilde{z}_\varepsilon}^{-1})^{-1}\Big\rangle_d\mathrm{d}\overline{x}.$$

For an arbitrary $z \in \mathrm{L}^p(\Omega)^m$ let $\Xi_{0,z} : \bigcup_{x\in\Omega}\{x\}\times Y\backslash L^{\mathrm{C}}(z(x)) \to \Omega\times Y\backslash L^{\mathrm{C}}(\mathbf{0})$ denote the transformation, which for $(x,y) \in \Omega\times Y$ is defined by $\Xi_{0,z}(x,y) := (x, T_{z(x)}^{\mathrm{C}}(y))$. Thus, the inverse transformation $\Xi_{0,z}^{-1} : \Omega\times Y\backslash L^{\mathrm{C}}(\mathbf{0}) \to \bigcup_{x\in\Omega}\{x\}\times Y\backslash L^{\mathrm{C}}(z(x))$ for $(\overline{x},\overline{y}) \in \Omega\times Y$ is given by $\Xi_{0,z}^{-1}(\overline{x},\overline{y}) := (\overline{x}, (T_{z(\overline{x})}^{\mathrm{C}})^{-1}(\overline{y}))$. Now we assume that there exist functions $(\widehat{v}_0, \widehat{V}_1) \in \mathrm{H}_0^1(\Omega)\times\mathrm{L}^2(\Omega;\mathrm{H}_{\mathrm{av}}^1(\mathcal{Y}\backslash L^{\mathrm{C}}(\mathbf{0})))$ such that the following convergences hold for a subsequence of $(\varepsilon)_{\varepsilon>0}$ (not relabeled).

$$\overline{v}_\varepsilon \xrightarrow{w} \widehat{v}_0 \qquad \text{in } \mathrm{L}^2(\Omega\times Y), \tag{9.23a}$$
$$\nabla\overline{v}_\varepsilon \xrightarrow{w} \nabla_{\overline{x}}E\widehat{v}_0+\nabla_{\overline{y}}\widehat{V}_1 \qquad \text{in } \mathrm{L}^2(\Omega\times Y)^d, \tag{9.23b}$$
$$(\nabla\Xi_{\varepsilon,\tilde{z}_\varepsilon}^{-1})^{-1} \xrightarrow{s} (\nabla_{\overline{y}}[\Xi_{0,\tilde{z}_0}^{-1}]_2)^{-1} \qquad \text{in } \mathrm{L}^2(\Omega\times Y)^{d\times d}, \tag{9.23c}$$
$$\det(\nabla\Xi_{\varepsilon,\tilde{z}_\varepsilon}^{-1}) \xrightarrow{s} \det(\nabla_{\overline{y}}[\Xi_{0,\tilde{z}_0}^{-1}]_2) \qquad \text{in } \mathrm{L}^2(\Omega\times Y), \tag{9.23d}$$
$$\mathcal{F}_\varepsilon^{(2)}(\nabla_y V_1)^{\mathrm{ex}} \circ \Xi_{\varepsilon,\tilde{z}_\varepsilon}^{-1} \xrightarrow{s} \nabla_{\overline{y}}\overline{V}_1(\nabla_{\overline{y}}[\Xi_{0,\tilde{z}_0}^{-1}]_2)^{-1} \qquad \text{in } \mathrm{L}^2(\Omega\times Y)^d, \tag{9.23e}$$

Here, $\overline{V}_1 := V_1 \circ \Xi_{0,\tilde{z}_0}^{-1} \in \mathrm{L}^2(\Omega;\mathrm{H}_{\mathrm{av}}^1(\mathcal{Y}\backslash L^{\mathrm{C}}(\mathbf{0})))$ and $[(a,b)]_2 := b$ denotes the second component of the tuple (a,b). Exploiting this assumptions we are now going to perform the limit passage $\varepsilon \to 0$ in (9.22), whereas the convergences of (9.23) are justified by the steps 4–6 below. For arbitrary but fixed testfunctions $(\overline{\varphi}_0, \overline{\Phi}_1) \in \mathrm{H}_0^1(\Omega)\times\mathrm{L}^2(\Omega;\mathrm{H}_{\mathrm{av}}^1(\mathcal{Y}\backslash L^{\mathrm{C}}(\mathbf{0})))$ according to step 1 there exists a sequence $(\overline{\varphi}_\varepsilon)_{\varepsilon>0}$ with $\overline{\varphi}_\varepsilon \in \mathrm{H}_0^1(\Omega\backslash\mathcal{C}_\varepsilon(\mathbf{0}))$ such that

$$\overline{\varphi}_\varepsilon \xrightarrow{s} \overline{\varphi}_0 \qquad \text{in } \mathrm{L}^2(\Omega\times Y), \tag{9.24a}$$
$$\nabla\overline{\varphi}_\varepsilon \xrightarrow{s} \nabla_{\overline{x}}E\overline{\varphi}_0+\nabla_{\overline{y}}\overline{\Phi}_1 \qquad \text{in } \mathrm{L}^2(\Omega\times Y)^d. \tag{9.24b}$$

By exploiting the convergences (9.23) and (9.24) in relation (9.22) for all testfunctions $(\overline{\varphi}_0, \overline{\Phi}_1) \in \mathrm{H}_0^1(\Omega)\times\mathrm{L}^2(\Omega;\mathrm{H}_{\mathrm{av}}^1(\mathcal{Y}\backslash L^{\mathrm{C}}(\mathbf{0})))$ it holds

$$\int_{\Omega\times Y} |\det(\nabla_{\overline{y}}[\Xi_{0,\tilde{z}_0}^{-1}]_2)|\Big(\widehat{v}_0\overline{\varphi}_0 \tag{9.25}$$
$$+\Big\langle(\nabla_{\overline{x}}E\widehat{v}_0+\nabla_{\overline{y}}\widehat{V}_1)(\nabla_{\overline{y}}[\Xi_{0,\tilde{z}_0}^{-1}]_2)^{-1}, (\nabla_{\overline{x}}E\overline{\varphi}_0+\nabla_{\overline{y}}\overline{\Phi}_1)(\nabla_{\overline{y}}[\Xi_{0,\tilde{z}_0}^{-1}]_2)^{-1}\Big\rangle_d\Big)\mathrm{d}\overline{y}\mathrm{d}\overline{x}$$
$$= \int_{\Omega\times Y} |\det(\nabla_{\overline{y}}[\Xi_{0,\tilde{z}_0}^{-1}]_2)|\Big\langle\nabla_{\overline{y}}\overline{V}_1(\nabla_{\overline{y}}[\Xi_{0,\tilde{z}_0}^{-1}]_2)^{-1}, (\nabla_{\overline{x}}E\overline{\varphi}_0+\nabla_{\overline{y}}\overline{\Phi}_1)(\nabla_{\overline{y}}[\Xi_{0,\tilde{z}_0}^{-1}]_2)^{-1}\Big\rangle_d\mathrm{d}\overline{y}\mathrm{d}\overline{x}.$$

Note that (9.25) is the Euler-Lagrange equation of the following minimizing problem

$$\min\{J(v,V)\,|\,(v,V) \in \mathrm{H}_0^1(\Omega)\times\mathrm{L}^2(\Omega;\mathrm{H}_{\mathrm{av}}^1(\mathcal{Y}\backslash L^{\mathrm{C}}(\mathbf{0})))\}, \tag{9.26}$$

where for $\sigma_{\tilde{z}_0} := (\det(\nabla_y[\Xi_{0,\tilde{z}_0}^{-1}]_2))^{\frac{1}{2}}$ the functional $J : \mathrm{H}_0^1(\Omega)\times\mathrm{L}^2(\Omega;\mathrm{H}_{\mathrm{av}}^1(\mathcal{Y}\backslash L^{\mathrm{C}}(\mathbf{0}))) \to \mathbb{R}$ is defined by

$$J(v,V) := \tfrac{1}{2}\|\sigma_{\tilde{z}_0}v\|_{\mathrm{L}^2(\Omega)}^2 + \tfrac{1}{2}\|\sigma_{\tilde{z}_0}(\nabla_{\overline{x}}Ev+\nabla_{\overline{y}}V-\nabla_{\overline{y}}\overline{V}_1)(\nabla_{\overline{y}}[\Xi_{0,\tilde{z}_0}^{-1}]_2)^{-1}\|_{\mathrm{L}^2(\Omega\times Y)^d}^2.$$

Therefore, the function $(\widehat{v}_0, \widehat{V}_1)$ is the unique solution of (9.26). On the other hand, the function $(\mathbf{0}, \overline{V}_1)$ is obviously the minimizer of (9.26) such that we have $(\widehat{v}_0, \widehat{V}_1) = (\mathbf{0}, \overline{V}_1)$. Summarizing step 2, up to now we showed

$$\overline{v}_\varepsilon \overset{w}{\rightharpoonup} \mathbf{0} \quad \text{in } \mathrm{L}^2(\Omega{\times}Y), \tag{9.27a}$$

$$\nabla\overline{v}_\varepsilon \overset{w}{\rightharpoonup} \nabla_{\overline{y}}\overline{V}_1 \quad \text{in } \mathrm{L}^2(\Omega{\times}Y)^d. \tag{9.27b}$$

To verify (9.19) with respect to the weak two-scale topology, we exemplarily prove (9.19b). According to the uniform bound (9.20) it if sufficient (see Proposition 3.3 and Section 3.2) to show for all $\Psi \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^d{\times}Y)^d$ that it holds

$$\lim_{\varepsilon\to 0}\int_\Omega \langle \nabla v_\varepsilon, \mathcal{F}_\varepsilon^{(2)}\Psi\rangle_d \mathrm{d}x = \int_{\Omega\times Y} \langle \nabla_y V_1, \Psi\rangle_d \mathrm{d}y\mathrm{d}x. \tag{9.28}$$

For this purpose, we start by applying the transformation $\Xi^{-1}_{\varepsilon,\tilde{z}_\varepsilon} : \Omega\backslash\mathcal{C}_\varepsilon(\mathbf{0}) \to \Omega\backslash\mathcal{C}_\varepsilon(\widetilde{z}_\varepsilon)$ to the left hand side of (9.28). Afterwards, we exploit the convergence results (9.23c), (9.23d), and (9.27b) as well as that due to the continuity of Ψ, assumption (9.2i), and Proposition 3.3(b) we have $\mathcal{F}_\varepsilon^{(2)}\Psi \circ \Xi^{-1}_{\varepsilon,\tilde{z}_\varepsilon} \overset{s}{\to} \Psi \circ \Xi^{-1}_{0,\tilde{z}_0}$ in $\mathrm{L}^2(\Omega{\times}Y)^d$. Finally, we apply the transformation $\Xi_{0,\tilde{z}_0} : \bigcup_{x\in\Omega}\{x\}{\times}Y\backslash L^\mathrm{C}(\widetilde{z}_0(x)) \to \Omega{\times}Y\backslash L^\mathrm{C}(\mathbf{0})$.

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_\Omega \langle \nabla_x v_\varepsilon, \mathcal{F}_\varepsilon^{(2)}\Psi\rangle_d \mathrm{d}x &= \lim_{\varepsilon\to 0}\int_\Omega |\det(\nabla_{\overline{x}}\Xi^{-1}_{\varepsilon,\tilde{z}_\varepsilon})|\langle \nabla_{\overline{x}}\overline{v}_\varepsilon(\nabla_{\overline{x}}\Xi^{-1}_{\varepsilon,\tilde{z}_\varepsilon})^{-1}, \mathcal{F}_\varepsilon^{(2)}\Psi\circ\Xi^{-1}_{\varepsilon,\tilde{z}_\varepsilon}\rangle_d \mathrm{d}\overline{x} \\
&= \int_{\Omega\times Y} |\det(\nabla_{\overline{y}}[\Xi^{-1}_{0,\tilde{z}_0}]_2)|\langle \nabla_{\overline{y}}\overline{V}_1(\nabla_{\overline{y}}[\Xi^{-1}_{0,\tilde{z}_0}]_2)^{-1}, \Psi\circ\Xi^{-1}_{0,\tilde{z}_0}\rangle_d \mathrm{d}\overline{y}\mathrm{d}\overline{x} \\
&= \int_{\Omega\times Y} \langle \nabla_y V_1, \Psi\rangle_d \mathrm{d}y\mathrm{d}x.
\end{aligned}$$

3. We recall that up to now we know that the convergence of (9.19) holds with respect to the weak two-scale topology. According to (9.20) for all $\varepsilon > 0$ we have $\|\mathcal{T}_\varepsilon\nabla v_\varepsilon\|_{\mathrm{L}^2(\mathbb{R}\times Y)^d} = \|\nabla v_\varepsilon\|_{\mathrm{L}^2(\Omega)^d} \leq \|\nabla_y V_1\|_{\mathrm{L}^2(\Omega\times Y)^d}$ which combined with the inequality $\liminf_{\varepsilon\to 0}\|\mathcal{T}_\varepsilon\nabla v_\varepsilon\|_{\mathrm{L}^2(\mathbb{R}\times Y)^d} \geq \|\nabla_y V_1\|_{\mathrm{L}^2(\Omega\times Y)^d}$ resulting from the weak convergence yields $\lim_{\varepsilon\to 0}\|\mathcal{T}_\varepsilon\nabla v_\varepsilon\|_{\mathrm{L}^2(\mathbb{R}\times Y)^d} = \|\nabla_y V_1\|_{\mathrm{L}^2(\Omega\times Y)^d}$. Since weak convergence combined with convergence of the norms gives strong convergence, this verifies (9.19b). Thus, (9.19a) is an easy consequence of choosing $\varphi_\varepsilon = v_\varepsilon \in \mathrm{H}^1_0(\Omega\backslash\mathcal{C}_\varepsilon(\widetilde{z}_\varepsilon))$ as the testfunction in (9.18) and exploiting the convergence (9.19b).

4. Now we are going to show that there exists $(\widehat{v}_0, \widehat{V}_1) \in \mathrm{H}^1_0(\Omega){\times}\mathrm{L}^2(\Omega; \mathrm{H}^1_\mathrm{av}(\mathcal{Y}\backslash L^\mathrm{C}(\mathbf{0})))$ and a subsequence of $(\varepsilon)_{\varepsilon>0}$ such that (9.23a) and (9.23b) hold. For this purpose, we apply the transformation $\Xi_{\varepsilon,\tilde{z}_\varepsilon} : \Omega\backslash\mathcal{C}_\varepsilon(\widetilde{z}_\varepsilon) \to \Omega\backslash\mathcal{C}_\varepsilon(\mathbf{0})$ to the norm $\|\overline{v}_\varepsilon\|^2_{\mathrm{H}^1_0(\Omega\backslash\mathcal{C}_\varepsilon(\mathbf{0}))}$ in the following way.

$$\begin{aligned}
\|\overline{v}_\varepsilon\|^2_{\mathrm{H}^1_0(\Omega\backslash\mathcal{C}_\varepsilon(\mathbf{0}))} &= \int_\Omega |\overline{v}_\varepsilon|^2 + |\nabla_{\overline{x}}\overline{v}_\varepsilon|^2_d \mathrm{d}\overline{x} \\
&= \int_\Omega |\det(\nabla_x\Xi_{\varepsilon,\tilde{z}_\varepsilon})|(|v_\varepsilon|^2 + |\nabla_x v_\varepsilon(\nabla_x\Xi_{\varepsilon,\tilde{z}_\varepsilon})^{-1}|^2_d)\mathrm{d}x \\
&\leq C\|v_\varepsilon\|^2_{\mathrm{H}^1_0(\Omega\backslash\mathcal{C}_\varepsilon(\widetilde{z}_\varepsilon))}
\end{aligned}$$

Here, $v_\varepsilon = \overline{v}_\varepsilon \circ \Xi_{\varepsilon,\tilde{z}_\varepsilon} \in \mathrm{H}^1_0(\Omega\backslash\mathcal{C}_\varepsilon(\tilde{z}_\varepsilon))$ and in the last line we exploited assumption (9.2g). Hence, according to the a priori estimate (9.20) the norm $\|\overline{v}_\varepsilon\|_{\mathrm{H}^1_0(\Omega\backslash\mathcal{C}_\varepsilon(\mathbf{0}))}$ is uniformly bounded and Theorem 9.3 states that there exists a subsequence of $(\varepsilon)_{\varepsilon>0}$ and a function $(\widehat{v}_0,\widehat{V}_1) \in \mathrm{H}^1_0(\Omega)\times\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(\mathbf{0})))$ such that (9.23a) and (9.23b) hold for this subsequence.

5. Since the transformation $\Xi^{-1}_{\varepsilon,\tilde{z}_\varepsilon} : \Omega\backslash\mathcal{C}_\varepsilon(\mathbf{0}) \to \Omega\backslash\mathcal{C}_\varepsilon(\tilde{z}_\varepsilon)$ for every $x \in \Omega^-_\varepsilon$ is defined by $\Xi^{-1}_{\varepsilon,\tilde{z}_\varepsilon}(x) := \mathcal{N}_\varepsilon(x) + \varepsilon(T^{\mathrm{C}}_{\tilde{z}_\varepsilon(x)})^{-1}(\{\frac{x}{\varepsilon}\}_Y)$, we have $\nabla\Xi^{-1}_{\varepsilon,\tilde{z}_\varepsilon}(x) = \nabla(T^{\mathrm{C}}_{\tilde{z}_\varepsilon(x)})^{-1}(\{\frac{x}{\varepsilon}\}_Y)$. Exploiting that the given function $\tilde{z}_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega;[0,1]^m)$ is piecewise constant, for all $(x,y) \in \Omega^-_\varepsilon\times Y$ it holds $\mathcal{T}_\varepsilon(\nabla\Xi^{-1}_{\varepsilon,\tilde{z}_\varepsilon})(x,y) = \nabla_y(T^{\mathrm{C}}_{\tilde{z}_\varepsilon(x)})^{-1}(y)$. Therefore, due to assumption (9.2i) for almost every $(x,y) \in \Omega^-_\varepsilon\times Y$ we find

$$\lim_{\varepsilon\to 0}\mathcal{T}_\varepsilon(\nabla\Xi^{-1}_{\varepsilon,\tilde{z}_\varepsilon})(x,y) = \nabla_y(T^{\mathrm{C}}_{\tilde{z}_0(x)})^{-1}(y). \tag{9.29}$$

Since $[\Xi^{-1}_{0,\tilde{z}_0}]_2 : Y\backslash L^{\mathrm{C}}(\mathbf{0})$ for $(x,y) \in \Omega\times Y \to \bigcup_{x\in\Omega}\{x\}\times Y\backslash L^{\mathrm{C}}(\tilde{z}_0(x))$ is defined by $[\Xi^{-1}_{0,\tilde{z}_0}]_2(x,y) = (T^{\mathrm{C}}_{\tilde{z}_0(x)})^{-1}(y)$, combining (9.29) and (9.2g) yields the convergence (9.23c). The convergence (9.23d) is an immediate consequence of (9.23c).

6. To show $\mathcal{F}^{(2)}_\varepsilon(\nabla_y V_1)^{\mathrm{ex}} \circ \Xi^{-1}_{\varepsilon,\tilde{z}_\varepsilon} \xrightarrow{s} (\nabla_{\overline{y}}\overline{V}_1)(\nabla_{\overline{y}}[\Xi^{-1}_{0,\tilde{z}_0}]_2)^{-1}$ in $\mathrm{L}^2(\Omega\times Y)^d$ we start by exploiting Proposition 3.3(d) to obtain

$$\mathcal{T}_\varepsilon(\mathcal{F}^{(2)}_\varepsilon(\nabla_y V_1)^{\mathrm{ex}} \circ \Xi^{-1}_{\varepsilon,\tilde{z}_\varepsilon}) = (\mathbb{1}_{[\Omega\times Y]_\varepsilon} \circ \widetilde{\Xi}^{-1}_{0,\tilde{z}_\varepsilon})\mathcal{P}_\varepsilon(\nabla_y V_1 \circ \widetilde{\Xi}^{-1}_{0,\tilde{z}_\varepsilon})^{\mathrm{ex}},$$

where the transformation $\widetilde{\Xi}^{-1}_{0,\tilde{z}_\varepsilon} : (\mathbb{R}^d\times Y)\backslash\mathrm{Im}(\mathcal{T}_\varepsilon\mathbb{1}_{\mathcal{C}_\varepsilon(\mathbf{0})}) \to (\mathbb{R}^d\times Y)\backslash\mathrm{Im}(\mathcal{T}_\varepsilon\mathbb{1}_{\mathcal{C}_\varepsilon(\tilde{z}_\varepsilon)})$ is defined by $\widetilde{\Xi}^{-1}_{0,\tilde{z}_\varepsilon}|_{(\mathbb{R}^d\backslash\Omega^-_\varepsilon)\times Y} := \mathrm{id}$ and $\widetilde{\Xi}^{-1}_{0,\tilde{z}_\varepsilon}|_{\Omega^-_\varepsilon\times Y} := \Xi^{-1}_{0,\tilde{z}_\varepsilon}|_{\Omega^-_\varepsilon\times Y}$. This description enables us to simplify the proof of (9.23e) in the following way: Since $\mathbb{1}_{[\Omega\times Y]_\varepsilon} \circ \widetilde{\Xi}^{-1}_{0,\tilde{z}_\varepsilon} \to \mathbb{1}_{\Omega\times Y}$ in $\mathrm{L}^1(\mathbb{R}^d\times Y)$ and since $\mathbb{1}_{[\Omega\times Y]_\varepsilon} \circ \widetilde{\Xi}^{-1}_{0,\tilde{z}_\varepsilon} \leq 1$, for $V_\varepsilon := V_1 \circ \widetilde{\Xi}^{-1}_{0,\tilde{z}_\varepsilon}$ it is sufficient to verify

$$\mathcal{P}_\varepsilon(\nabla_y V_1 \circ \widetilde{\Xi}^{-1}_{0,\tilde{z}_\varepsilon})^{\mathrm{ex}} = \mathcal{P}_\varepsilon(\nabla_{\tilde{y}} V_\varepsilon)^{\mathrm{ex}}((\nabla_{\tilde{y}}[\widetilde{\Xi}^{-1}_{0,\tilde{z}_\varepsilon}]_2)^{-1})^{\mathrm{ex}} \to (\nabla_{\overline{y}}\overline{V}_1)^{\mathrm{ex}}((\nabla_{\overline{y}}[\Xi^{-1}_{0,\tilde{z}_0}]_2)^{-1})^{\mathrm{ex}}$$

with respect to the strong topology of $\mathrm{L}^2(\mathbb{R}^d\times Y)^d$. Due to assumption (9.2i) it holds $\lim_{\varepsilon\to 0}\|(\nabla_{\overline{y}}[\widetilde{\Xi}^{-1}_{0,\tilde{z}_\varepsilon}]_2)^{-1} - (\nabla_{\overline{y}}[\Xi^{-1}_{0,\tilde{z}_0}]_2)^{-1}\|_{\mathrm{L}^\infty(\Omega\times Y)^{d\times d}} = 0$ such that $\mathcal{P}_\varepsilon(\nabla_{\tilde{y}}V_\varepsilon)^{\mathrm{ex}} \to (\nabla_{\overline{y}}\overline{V}_1)^{\mathrm{ex}}$ in $\mathrm{L}^2(\mathbb{R}^d\times Y)^d$ remains to be shown. Observe that for a sequence $(W_\varepsilon)_{\varepsilon>0} \subset \mathrm{L}^2(\mathbb{R}^d\times Y)^d$ and a function $W_0 \in \mathrm{L}^2(\mathbb{R}^d\times Y)^d$ with $W_\varepsilon \to W_0$ in $\mathrm{L}^2(\mathbb{R}^d\times Y)^d$ it holds $\mathcal{P}_\varepsilon(W_\varepsilon) \to W_0$ in $\mathrm{L}^2(\mathbb{R}^d\times Y)^d$. Therefore, if we can guarantee $\nabla_{\tilde{y}}V_\varepsilon \to \nabla_{\overline{y}}\overline{V}_1$ in $\mathrm{L}^2(\Omega\times Y)^d$ we immediately obtain $\mathcal{P}_\varepsilon(\nabla_{\tilde{y}}V_\varepsilon)^{\mathrm{ex}} \to (\nabla_{\overline{y}}\overline{V}_1)^{\mathrm{ex}}$ in $\mathrm{L}^2(\mathbb{R}^d\times Y)^d$.

To prove $\nabla_{\tilde{y}}V_\varepsilon \to \nabla_{\overline{y}}\overline{V}_1$ in $\mathrm{L}^2(\Omega\times Y)^d$, we exploit the description $V_\varepsilon = \overline{V}_1 \circ \Xi_{0,\tilde{z}_0} \circ \widetilde{\Xi}^{-1}_{0,\tilde{z}_\varepsilon}$, which according to the chain rule results in

$$\nabla_{\tilde{y}}V_\varepsilon = \Big(\nabla_{\overline{y}}\overline{V}_1 \circ \Xi_{0,\tilde{z}_0} \circ \widetilde{\Xi}^{-1}_{0,\tilde{z}_\varepsilon}\Big)\Big(\nabla_{\overline{y}}[\Xi_{0,\tilde{z}_0}]_2 \circ \widetilde{\Xi}^{-1}_{0,\tilde{z}_\varepsilon}\Big)\Big(\nabla_{\overline{y}}[\widetilde{\Xi}^{-1}_{0,\tilde{z}_\varepsilon}]_2\Big).$$

Due to assumption (9.2i) for $\varepsilon \to 0$ it holds

$$(\nabla_{\overline{y}}[\Xi_{0,\tilde{z}_0}]_2) \circ \widetilde{\Xi}^{-1}_{0,\tilde{z}_\varepsilon}(\nabla_{\overline{y}}[\widetilde{\Xi}^{-1}_{0,\tilde{z}_\varepsilon}]_2) \to (\nabla_{\overline{y}}[\Xi_{0,\tilde{z}_0}]_2) \circ \Xi^{-1}_{0,\tilde{z}_0}(\nabla_{\overline{y}}[\Xi^{-1}_{0,\tilde{z}_0}]_2) = \mathrm{id} \quad \text{in } \mathrm{L}^\infty(\Omega\times Y)^{d\times d}.$$

Summarizing step 6, up to now the proof of convergence (9.23e) is reduced to the verification of $\nabla_{\overline{y}}\overline{V}_1 \circ \Xi_{0,\tilde{z}_0} \circ \widetilde{\Xi}^{-1}_{0,\tilde{z}_\varepsilon} \to \nabla_{\overline{y}}\overline{V}_1$ in $\mathrm{L}^2(\Omega\times Y)^d$. To verify this convergence, we recall that according to [70] the linear span of $\mathrm{C}_\mathrm{c}^\infty(\Omega)\times(\mathrm{C}^\infty(\mathcal{Y}\backslash L^\mathrm{C}(\mathbf{0}))\cap \mathrm{H}^1(\mathcal{Y}\backslash L^\mathrm{C}(\mathbf{0})))$ is a dense subset of the function space $\mathrm{L}^2(\Omega;\mathrm{H}^1(\mathcal{Y}\backslash L^\mathrm{C}(\mathbf{0})))$. Therefore, for the function $\overline{V}_1\in \mathrm{L}^2(\Omega;\mathrm{H}^1_\mathrm{av}(\mathcal{Y}\backslash L^\mathrm{C}(\mathbf{0})))\subset \mathrm{L}^2(\Omega;\mathrm{H}^1(\mathcal{Y}\backslash L^\mathrm{C}(\mathbf{0})))$ there exists a sequence $(\overline{V}_n)_{n\in\mathbb{N}}$ in this subset such that $\nabla_{\overline{y}}\overline{V}_n\to\nabla_{\overline{y}}\overline{V}_1$ in $\mathrm{L}^2(\Omega\times Y)^d$.

By applying the transformations $\Xi_{0,\tilde{z}_0}^{-1}:\Omega\times Y\backslash L^\mathrm{C}(\mathbf{0})\to\bigcup_{x\in\Omega}\{x\}\times Y\backslash L^\mathrm{C}(\tilde{z}_0(x))$ and $\widetilde{\Xi}_{0,\tilde{z}_\varepsilon}:(\mathbb{R}^d\times Y)\backslash\mathrm{Im}(\mathcal{T}_\varepsilon\mathbb{1}_{\mathcal{C}_\varepsilon(\tilde{z}_\varepsilon)})\to(\mathbb{R}^d\times Y)\backslash\mathrm{Im}(\mathcal{T}_\varepsilon\mathbb{1}_{\mathcal{C}_\varepsilon(\mathbf{0})})$ to the integral of the left hand side of (9.30) below, due to assumption (9.2g) there exists a constant $C>0$ such that the estimate (9.30) holds for all $n\in\mathbb{N}$ and every $\varepsilon>0$. Here, C is independent of $\varepsilon>0$.

$$\|(\nabla_{\overline{y}}\overline{V}_1-\nabla_{\overline{y}}\overline{V}_n)\circ\Xi_{0,\tilde{z}_0}\circ\widetilde{\Xi}_{0,\tilde{z}_\varepsilon}^{-1}\|^2_{\mathrm{L}^2(\Omega\times Y)^d}\le C^2\|\nabla_{\overline{y}}\overline{V}_1-\nabla_{\overline{y}}\overline{V}_n\|^2_{\mathrm{L}^2(\Omega\times Y)^d}\tag{9.30}$$

For an arbitrary but fixed $\Delta>0$ we now choose $n_\Delta\in\mathbb{N}$ such that

$$\|\nabla_{\overline{y}}\overline{V}_1-\nabla_{\overline{y}}\overline{V}_{n_\Delta}\|_{\mathrm{L}^2(\Omega\times Y)^d}\le\tfrac{\Delta}{3C}.\tag{9.31}$$

Thus, we are able to estimate $\|\nabla_{\overline{y}}\overline{V}_1\circ\Xi_{0,\tilde{z}_0}\circ\widetilde{\Xi}_{0,\tilde{z}_\varepsilon}^{-1}-\nabla_{\overline{y}}\overline{V}_1\|_{\mathrm{L}^2(\Omega\times Y)^d}$ as follows.

$$\begin{aligned}\|\nabla_{\overline{y}}\overline{V}_1\circ\Xi_{0,\tilde{z}_0}\circ\widetilde{\Xi}_{0,\tilde{z}_\varepsilon}^{-1}-\nabla_{\overline{y}}\overline{V}_1\|_{\mathrm{L}^2(\Omega\times Y)^d}\le{}&\|(\nabla_{\overline{y}}-\nabla_{\overline{y}}\overline{V}_{n_\Delta})\circ\Xi_{0,\tilde{z}_0}\circ\widetilde{\Xi}_{0,\tilde{z}_\varepsilon}^{-1}\|^2_{\mathrm{L}^2(\Omega\times Y)^d}\\&+\|\nabla_{\overline{y}}\overline{V}_{n_\Delta}\circ\Xi_{0,\tilde{z}_0}\circ\widetilde{\Xi}_{0,\tilde{z}_\varepsilon}^{-1}-\nabla_{\overline{y}}\overline{V}_{n_\Delta}\|_{\mathrm{L}^2(\Omega\times Y)^d}\\&+\|\nabla_{\overline{y}}\overline{V}_{n_\Delta}-\nabla_{\overline{y}}\overline{V}_1\|_{\mathrm{L}^2(\Omega\times Y)^d}\end{aligned}\tag{9.32}$$

According to the continuity of $\nabla_{\overline{y}}\overline{V}_{n_\Delta}$ and assumption (9.2i) there exists $\varepsilon_0>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$ it holds $\|\nabla_{\overline{y}}\overline{V}_{n_\Delta}\circ\Xi_{0,\tilde{z}_0}\circ\widetilde{\Xi}_{0,\tilde{z}_\varepsilon}^{-1}-\nabla_{\overline{y}}\overline{V}_{n_\Delta}\|_{\mathrm{L}^2(\Omega\times Y)^d}\le\frac{\Delta}{3}$. Combining this result with the estimates (9.30), (9.31), and (9.32) we end up with

$$\|\nabla_{\overline{y}}\overline{V}_1\circ\Xi_{0,\tilde{z}_0}\circ\widetilde{\Xi}_{0,\tilde{z}_\varepsilon}^{-1}-\nabla_{\overline{y}}\overline{V}_1\|_{\mathrm{L}^2(\Omega\times Y)^d}\le\tfrac{\Delta}{3}+\tfrac{\Delta}{3}+\tfrac{\Delta}{3C}<\Delta.$$

Since this holds true for all $\Delta>0$, the convergence (9.23e) is verified and the proof is concluded. □

9.2 Two-scale effective damage model based on unidirectional crack evolution

In this section we formulate the two-scale effective crack model $(\mathbf{S}^0_\mathrm{C})$ and $(\mathbf{E}^0_\mathrm{C})$ derived by performing the limit passage $\varepsilon\to0$ in $(\mathrm{S}^\varepsilon_\mathrm{C})$ and $(\mathrm{E}^\varepsilon_\mathrm{C})$ rigorously. For $\mathcal{Y}:=\mathbb{R}/_\Lambda$ denoting the periodicity cell, the limit function space $\mathbf{Q}_0^\mathrm{C}$ has the following structure:

$$\mathbf{Q}_0^\mathrm{C}:=\mathrm{H}^1_{\Gamma_\mathrm{Dir}}(\Omega)^d\times\mathrm{L}^2(\Omega;\mathrm{H}^1_\mathrm{av}(\mathcal{Y}\backslash L^\mathrm{C}(\mathbf{0})))^d\times\mathrm{W}^{1,p}(\Omega;[0,1]^m).$$

For given $(\ell_0,\ell_1)\in\mathrm{C}^1([0,T];\mathrm{L}^2(\Omega)^d\times\mathrm{L}^2(\Omega)^{d\times d})$ the definition of the external loading $\ell^{\ell_0,\ell_1}_{\mathrm{C},0}\in\mathrm{C}^1([0,T];(\mathrm{H}^1_{\Gamma_\mathrm{Dir}}(\Omega)^d\times\mathrm{L}^2(\Omega;\mathrm{H}^1_\mathrm{av}(\mathcal{Y}\backslash L^\mathrm{C}(\mathbf{0})))^d)^*)$ is motivated by the compactness result of Theorem 9.3. For all $t\in[0,T]$ and $(u_0,U_1)\in\mathrm{H}^1_{\Gamma_\mathrm{Dir}}(\Omega)^d\times\mathrm{L}^2(\Omega;\mathrm{H}^1_\mathrm{av}(\mathcal{Y}\backslash L^\mathrm{C}(\mathbf{0})))^d$ it reads as follows:

$$\langle\ell^{\ell_0,\ell_1}_{\mathrm{C},0}(t),(u_0,U_1)\rangle:=\langle\ell_0(t),u_0\rangle_{\mathrm{L}^2(\Omega)^d}+\langle E\ell_1(t),\nabla_xEu_0+\nabla_yU_1\rangle_{\mathrm{L}^2(\Omega\times Y\backslash L^\mathrm{C}(\mathbf{0}))^{d\times d}}.\tag{9.33}$$

Here, the two-scale function $E\ell_1(t) \in \mathrm{L}^2(\Omega{\times}Y)^{d\times d}$ for almost every $(x,y) \in \Omega{\times}Y$ is defined by $E\ell_1(t)(x,y) := \ell_1(t)(x)$. For $(u_0, U_1) \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d{\times}\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(\mathbf{0})))^d$ we set $\widetilde{\mathbf{e}}(u_0,U_1) := \mathbf{e}_x(u_0) + \mathbf{e}_y(U_1)$. Thus, the functional $\widetilde{\mathbf{E}}_0^{\mathrm{C}} : [0,T]{\times}\mathbf{Q}_0^{\mathrm{C}} \to \mathbb{R}$ building the foundation of the limit energy functional is given by

$$
\begin{aligned}
&\widetilde{\mathbf{E}}_0^{\mathrm{C}}(t,u_0,U_1,z_0)\\
&\quad := \tfrac{1}{2}\langle \mathbb{C}_{\mathrm{strong}}\widetilde{\mathbf{e}}(u_0,U_1), \widetilde{\mathbf{e}}(u_0,U_1)\rangle_{\mathrm{L}^2(\Omega\times Y\backslash L^{\mathrm{C}}(\mathbf{0}))^{d\times d}} + \|\nabla z_0\|^p_{\mathrm{L}^p(\Omega)^{m\times d}} - \langle \ell^{\ell_0,\ell_1}_{\mathrm{C},0}(t),(u_0,U_1)\rangle
\end{aligned}
$$

and the two-scale energy functional $\mathbf{E}_0^{\mathrm{C}} : [0,T]{\times}\mathbf{Q}_0^{\mathrm{H}} \to \mathbb{R}_\infty$ reads as follows:

$$
\mathbf{E}_0^{\mathrm{C}}(t,u_0,U_1,z_0) := \begin{cases} \widetilde{\mathbf{E}}_0^{\mathrm{C}}(t,u_0,U_1,z_0) & \text{if } U_1 \text{ satisfies (9.34) below,}\\ \infty & \text{otherwise,}\end{cases}
$$

$$
U_1(x,\cdot) \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(z_0(x))) \quad \text{for almost every } x \in \Omega. \tag{9.34}
$$

For the limit function $\kappa_0^{\mathrm{C}} \in \mathrm{L}^{q'}(\Omega;[0,\infty))^m$ (see (9.7)) the limit dissipation distance $\mathbf{D}_0^{\mathrm{C}} : \mathrm{W}^{1,p}(\Omega;[0,1]^m){\times}\mathrm{W}^{1,p}(\Omega;[0,1]^m) \to [0,\infty]$ of the sequence $(\mathcal{D}_\varepsilon^{\mathrm{C}})_{\varepsilon>0}$ is defined via

$$
\mathbf{D}_0^{\mathrm{C}}(z_1,z_2) := \begin{cases} \displaystyle\int_\Omega |\langle \kappa_0^{\mathrm{C}}(x), z_2(x)-z_1(x)\rangle_m|\mathrm{d}x & \text{if } z_1 \geq z_2,\\ \infty & \text{otherwise.}\end{cases}
$$

For given initial values $(u_0^0,U_1^0,z_0^0) \in \mathbf{Q}_0^{\mathrm{C}}$, where U_1^0 is assumed to satisfy (9.34) with respect to z_0^0, the rate-independent crack evolution is modeled by the energetic formulation $(\mathbf{S}^0_{\mathrm{C}})$ and $(\mathbf{E}^0_{\mathrm{C}})$.

Stability condition $(\mathbf{S}^0_{\mathrm{C}})$ and energy balance $(\mathbf{E}^0_{\mathrm{C}})$ for all $t \in [0,T]$:

$$
\mathbf{E}_0^{\mathrm{C}}(t,u_0(t),U_1(t),z_0(t)) \leq \mathbf{E}_0^{\mathrm{C}}(t,\widetilde{u},\widetilde{U},\widetilde{z}) + \mathbf{D}_0^{\mathrm{C}}(z_0(t),\widetilde{z}) \quad \text{for all } (\widetilde{u},\widetilde{U},\widetilde{z}) \in \mathbf{Q}_0^{\mathrm{C}}
$$

$$
\begin{aligned}
&\mathbf{E}_0^{\mathrm{C}}(t,u_0(t),U_1(t),z_0(t)) + \mathrm{Diss}_{\mathbf{D}_0^{\mathrm{C}}}(z_0;[0,t])\\
&\qquad\qquad = \mathbf{E}_0^{\mathrm{C}}(0,u_0^0,U_1^0,z_0^0) - \int_0^t \langle \dot{\ell}^{\ell_0,\ell_1}_{\mathrm{C},0}(s),(u_0(s),U_1(s))\rangle \mathrm{d}s
\end{aligned}
$$

Here, $\mathrm{Diss}_{\mathbf{D}_0^{\mathrm{C}}}(z_0;[0,t]) := \sup \sum_{j=1}^N \mathbf{D}_0^{\mathrm{C}}(z_0(t_{j-1}),z_0(t_j))$, where for $N \in \mathbb{N}$ the supremum is taken with respect to all finite partitions $\pi_N := \{0 = t_0 < t_1 < \ldots < t_N = t\}$ of the interval $[0,T]$. The following existence result extends Theorem 6.18 to the situation with cracks.

Theorem 9.8 (Existence of solutions)**.** *Assume that* (9.1), (9.2), *and* (9.3) *hold. Let* $\mathbf{E}_0^{\mathrm{C}} : [0,T]{\times}\mathbf{Q}_0^{\mathrm{C}} \to \mathbb{R}_\infty$ *and* $\mathbf{D}_0^{\mathrm{C}} : \mathrm{W}^{1,p}(\Omega;[0,1]^m){\times}\mathrm{W}^{1,p}(\Omega;[0,1]^m) \to [0,\infty]$ *be defined as described above. Moreover, let* $(u_0^0,U_1^0,z_0^0) \in \mathbf{Q}_0^{\mathrm{C}}$ *be given such that it is the limit of a stable sequence* $(u_\varepsilon^0,z_\varepsilon^0)_{\varepsilon>0}$ *with respect to* $0 \in [0,T]$ *in the sense of Definition* 6.12 (*Obviously, the sequence* $(u_\varepsilon^0)_{\varepsilon>0}$ *with* $u_\varepsilon^0 \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega\backslash\mathcal{C}_\varepsilon(z_\varepsilon))^d \not\subset \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ *is not assumed to converge weakly to* u_0^0 *in* $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$*; see Definition* 6.12). *If* $\nabla u_\varepsilon^0 \overset{s}{\to} \nabla_x E u_0^0 + \nabla_y U_1^0$

in $\mathrm{L}^2(\Omega{\times}Y)^{d\times d}$ *and* $R_{\frac{\varepsilon}{2}} z_\varepsilon^0|_\Omega \to \nabla z_0^0$ *in* $\mathrm{L}^p(\Omega)^{m\times d}$*, then there exists an energetic solution* $(u_0, U_1, z_0) : [0,T] \to \mathbf{Q}_0^{\mathrm{C}}$ *of the rate-independent system* $(\mathbf{Q}_0^{\mathrm{C}}, \mathbf{E}_0^{\mathrm{C}}, \mathbf{D}_0^{\mathrm{C}})$ *with initial condition* (u_0^0, U_1^0, z_0^0) *satisfying*

$$\begin{aligned}(u_0, U_1) &\in \mathrm{L}^\infty([0,T]; \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d{\times}\mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(\mathbf{0})))^d),\\ z_0 &\in \mathrm{L}^\infty([0,T]; \mathrm{W}^{1,p}(\Omega;[0,1]^m)) \cap \mathrm{BV}_{\mathbf{D}_0^{\mathrm{C}}}([0,T]; \mathrm{W}^{1,p}(\Omega;[0,1]^m)).\end{aligned}$$

Proof. The proof of this theorem is completely analog to that of Theorem 6.18. However, observe that whenever Proposition 3.7 or Proposition 3.8 is exploited in the proofs of the Theorems 6.15, 6.17, or 6.18, here it is replaced by Theorem 9.3 or Theorem 9.5, respectively. Thus, establishing Theorem 9.8 is straight forward. □

9.3 One-scale effective damage model based on unidirectional crack evolution

Similar to Chapter 8, by choosing $\ell_1 :\equiv \mathbf{0}$ in (9.33), we are able to formulate a one-scale model which is equivalent to that of Section 9.2. Let the state space $\mathcal{Q}_0^{\mathrm{C}}(\Omega)$ be given by

$$\mathcal{Q}_0^{\mathrm{C}}(\Omega) := \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d{\times}\mathrm{W}^{1,p}(\Omega;[0,1]^m).$$

The energy functional is based on a mapping $\mathbb{C}_{\mathrm{eff}}^{\mathrm{C}} : \mathrm{W}^{1,p}(\Omega;[0,1]^m) \to \mathcal{M}(\Omega)$, which for $\xi \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, for almost every $x \in \Omega$, and for $z_0 \in \mathrm{W}^{1,p}(\Omega;[0,1]^m)$ is defined via the unit cell problem

$$\langle \mathbb{C}_{\mathrm{eff}}^{\mathrm{C}}(z_0)(x)\xi, \xi\rangle_{d\times d} := \min\Big\{ I^{\mathrm{C}}(z_0(x), \xi, v) \,\Big|\, v \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(\mathbf{0}))^d \Big\}. \tag{9.35}$$

Here, for $\widehat{z} \in [0,1]^m$ the functional $I^{\mathrm{C}}(\widehat{z}, \xi, \cdot) : \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(\mathbf{0}))^d \to \mathbb{R}_\infty$ is given by

$$I^{\mathrm{C}}(\widehat{z}, \xi, v) := \begin{cases} \widetilde{I}^{\mathrm{C}}(\widehat{z}, \xi, v) & \text{if } v \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(\widehat{z}))^d,\\ \infty & \text{otherwise,}\end{cases}$$

where the continuous functional $\widetilde{I}^{\mathrm{C}}(\widehat{z}, \xi, \cdot) : \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(\mathbf{0}))^d \to \mathbb{R}$ is given by

$$\widetilde{I}^{\mathrm{C}}(\widehat{z}, \xi, v) := \int_{Y\backslash L^{\mathrm{C}}(\mathbf{0})} \Big\langle \mathbb{C}_{\mathrm{strong}}\Big(\xi + \mathbf{e}_y(v)(y)\Big), \xi + \mathbf{e}_y(v)(y)\Big\rangle_{d\times d} \mathrm{d}y.$$

Note that (9.35) alternatively might be expressed by the much shorter relation

$$\langle \mathbb{C}_{\mathrm{eff}}^{\mathrm{C}}(z_0)(x)\xi, \xi\rangle_{d\times d} = \min\Big\{ \widetilde{I}^{\mathrm{C}}(z_0(x), \xi, v) \,\Big|\, v \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(z_0(x)))^d \Big\}.$$

However, since we are going to refer to the notation of Chapter 8, we have to introduce the functionals $I^{\mathrm{C}}(\widehat{z}, \xi, \cdot)$ and $\widetilde{I}^{\mathrm{C}}(\widehat{z}, \xi, \cdot)$. For given functions $\ell_0 \in \mathrm{C}^1([0,T]; \mathrm{L}^2(\Omega)^d)$ and $z_0 \in \mathrm{W}^{1,p}(\Omega;[0,1]^m)$, here the external loading $\ell_{\mathrm{C},0}^{\ell_0} \in \mathrm{C}^1([0,T]; (\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$ for all $t \in [0,T]$ and every $u_0 \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ is modeled by

$$\langle \ell_{\mathrm{C},0}^{\ell_0}(t), u_0\rangle := \langle \ell_0(t), u_0\rangle_{\mathrm{L}^2(\Omega)^d}. \tag{9.36}$$

Note that in the case of $\ell_1 :\equiv \mathbf{0}$ the mapping $\ell^{\ell_0,\mathbf{0}}_{\mathrm{C},0}$ of (9.33) can be understood as an element of $\mathrm{C}^1([0,T];(\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$, too. Now, the one-scale limit energy functional $\mathcal{E}^{\mathrm{C}}_0 : [0,T]\times\mathcal{Q}^{\mathrm{C}}_0(\Omega) \to \mathbb{R}$ is defined via

$$\mathcal{E}^{\mathrm{C}}_0(t,u_0,z_0) := \tfrac{1}{2}\langle \mathbb{C}^{\mathrm{C}}_{\mathrm{eff}}(z_0)\mathbf{e}(u_0),\mathbf{e}(u_0)\rangle_{\mathrm{L}^2(\Omega\times Y)^{d\times d}} + \|\nabla z_0\|^p_{\mathrm{L}^p(\Omega)^{m\times d}} - \langle \ell^{\ell_0}_{\mathrm{C},0}(t),u_0\rangle.$$

For $\kappa^{\mathrm{C}}_0 \in \mathrm{L}^{q'}(\Omega;[0,\infty)^m)$ denoting the limit function mentioned in the definition of the microscopic dissipation distance defined in line (9.7), the limit dissipation distance $\mathcal{D}^{\mathrm{C}}_0 : \mathrm{W}^{1,p}(\Omega;[0,1]^m)\times\mathrm{W}^{1,p}(\Omega;[0,1]^m) \to [0,\infty]$ reads as follows:

$$\mathcal{D}^{\mathrm{C}}_0(z_1,z_2) := \begin{cases} \displaystyle\int_\Omega |\langle \kappa^{\mathrm{H}}_0(x), z_2(x)-z_1(x)\rangle_m|\mathrm{d}x & \text{if } z_1 \geq z_2,\\ \infty & \text{otherwise.}\end{cases}$$

For given initial values $(u^0_0,z^0_0) \in \mathcal{Q}^{\mathrm{C}}_0(\Omega)$ the existence of an energetic solution of the rate-independent system $(\mathcal{Q}^{\mathrm{C}}_0(\Omega),\mathcal{E}^{\mathrm{C}}_0,\mathcal{D}^{\mathrm{C}}_0)$ is implied by combining Theorem 9.8 with Theorem 9.9 below.

Stability condition $(\mathrm{S}^0_{\mathrm{C}})$ **and energy balance** $(\mathrm{E}^0_{\mathrm{C}})$ **for all** $t \in [0,T]$**:**

$$\mathcal{E}^{\mathrm{C}}_0(t,u_0(t),z_0(t)) \leq \mathcal{E}^{\mathrm{C}}_0(t,\widetilde{u},\widetilde{z}) + \mathcal{D}^{\mathrm{C}}_0(z_0(t),\widetilde{z}) \quad \text{for all } (\widetilde{u},\widetilde{z}) \in \mathcal{Q}^{\mathrm{H}}_0(\Omega)$$
$$\mathcal{E}^{\mathrm{C}}_0(t,u_0(t),z_0)(t)) + \mathrm{Diss}_{\mathcal{D}^{\mathrm{C}}_0}(z_0;[0,t]) = \mathcal{E}^{\mathrm{C}}_0(0,u^0_0,z^0_0) - \int_0^t \langle \dot{\ell}^{\ell_0}_{\mathrm{C},0}(s),u_0(s)\rangle\mathrm{d}s$$

Here, $\mathrm{Diss}_{\mathcal{D}^{\mathrm{C}}_0}(z_0;[0,t]) := \sup\sum_{j=1}^N \mathcal{D}^{\mathrm{C}}_0(z_0(t_{j-1}),z_0(t_j))$, where for $N \in \mathbb{N}$ the supremum is taken with respect to all finite partitions $\pi_N := \{0 = t_0 < t_1 < \ldots < t_N = t\}$ of the interval $[0,T]$.

Theorem 9.9 (Equivalence of the two-scale and one-scale model)**.** *Assume that the conditions* (9.1), (9.2), *and* (9.3) *hold. For* $\widehat{z} \in [0,1]^m$ *let* $\mathcal{L}^{\mathrm{C}}_{\widehat{z}} : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(\widehat{z}))^d$ *denote the linear operator, which for* $\xi \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ *is defined by*

$$\mathcal{L}^{\mathrm{C}}_{\widehat{z}}(\xi) := \mathrm{Argmin}\Big\{\langle \mathbb{C}_{\mathrm{strong}}(\xi+\mathbf{e}_y(v)),\xi+\mathbf{e}_y(v)\rangle_{\mathrm{L}^2(Y\backslash L^{\mathrm{C}}(\widehat{z}))^{d\times d}}\,\Big|\, v \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(\widehat{z}))^d\Big\}.$$

For $\ell_0 \in \mathrm{C}^1([0,T];(\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$ *and* $\ell_1 :\equiv \mathbf{0}$ *let* $\ell^{\ell_0,\mathbf{0}}_{\mathrm{C},0} \in \mathrm{C}^1([0,T];(\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$ *and* $\ell^{\ell_0}_{\mathrm{C},0} \in \mathrm{C}^1([0,T];(\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$ *be given by* (9.33) *and* (9.36).

Furthermore, let $z_0 \in \mathrm{L}^\infty([0,T];\mathrm{W}^{1,p}(\Omega;[0,1]^m)) \cap \mathrm{BV}_{\mathcal{D}^{\mathrm{C}}_0}([0,T];\mathrm{W}^{1,p}(\Omega;[0,1]^m))$ *and let* $(u_0,U_1) \in \mathrm{L}^\infty([0,T];\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d\times\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(\mathbf{0})))^d)$ *satisfying for all* $t \in [0,T]$ *and almost every* $x \in \Omega$ *the condition* $U_1(t,x,\cdot) \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(z_0(t,x)))^d$*. Then for* (u^0_0,U^0_1,z^0_0) *satisfying the stability condition* $(\mathbf{S}^0_{\mathrm{C}})$ *for* $t=0$ *the following two statements are equivalent:*

(a) *The function* $(u_0,U_1,z_0) : [0,T] \to \mathbf{Q}^{\mathrm{C}}_0$ *with* $(u_0(0),U_1(0),z_0(0)) = (u^0_0,U^0_1,z^0_0)$ *is a solution of* $(\mathbf{S}^0_{\mathrm{C}})$ *and* $(\mathbf{E}^0_{\mathrm{C}})$*.*

(b) *The function* $(u_0,z_0) : [0,T] \to \mathcal{Q}^{\mathrm{C}}_0(\Omega)$ *with* $(u_0(0),z_0(0)) = (u^0_0,z^0_0)$ *is a solution of* $(\mathrm{S}^0_{\mathrm{C}})$ *and* $(\mathrm{E}^0_{\mathrm{C}})$*, and* $U_1(t) := \mathcal{L}^{\mathrm{C}}_{z_0(t,\cdot)}(\mathbf{e}_x(u_0(t))(\cdot))$ *for all* $t \in [0,T]$*.*

Proof. By introducing the functional $I_0^{\mathrm{C}}(z_0, u_0, \cdot) : \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(\mathbf{0})))^d \to \mathbb{R}_\infty$ via

$$I_0^{\mathrm{C}}(z_0, u_0, U) := \begin{cases} \widetilde{I}_0^{\mathrm{C}}(z_0, u_0, U) & \text{if } U(x,\cdot) \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(z_0(x)))^d \text{ for almost all } x \in \Omega, \\ \infty & \text{otherwise,} \end{cases}$$

where the continuous functional $\widetilde{I}_0^{\mathrm{C}}(z_0, u_0, \cdot) : \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(\mathbf{0})))^d \to \mathbb{R}$ is given by

$$\widetilde{I}_0^{\mathrm{C}}(z_0, u_0, U) := \langle \mathbb{C}_{\mathrm{strong}} \widetilde{\mathbf{e}}(u_0, U), \widetilde{\mathbf{e}}(u_0, U) \rangle_{\mathrm{L}^2(\Omega \times Y \backslash L^{\mathrm{C}}(\mathbf{0}))^{d \times d}},$$

the proof of this statement is completely analog to that of Theorem 8.20. □

9.4 Discussion of the results

This chapter is devoted to the derivation of effective models for the propagation of microscopic cracks. Observe that the two-scale limit state space $\mathbf{Q}_0^{\mathrm{C}}$ is defined via $\mathbf{Q}_0^{\mathrm{C}} := \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \times \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(\mathbf{0})))^d \times \mathrm{W}^{1,p}(\Omega; [0,1]^m)$, i.e., the microscopic cracks appearing in the energetic formulation $(\mathrm{S}^\varepsilon_{\mathrm{C}})$ and $(\mathrm{E}^\varepsilon_{\mathrm{C}})$ are shifted to the second scale in the limit. Moreover, similar to Chapter 8 the external loading of the effective two-scale limit model depends on the microscopic scale. By assuming the term of the external loading which is responsible for this dependence on the microscopic scale to be zero, we are able to formulate an equivalent one-scale model. Since $\mathcal{Q}_0^{\mathrm{C}}(\Omega) := \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \times \mathrm{W}^{1,p}(\Omega; [0,1]^m)$ is the state space for the effective one-scale model and since cracks are modeled by the jump set of the displacement field, the considered body Ω contains no cracks at any time. Hence, $(\mathrm{S}^0_{\mathrm{C}})$ and $(\mathrm{E}^0_{\mathrm{C}})$ can be understood as a damage model for the linear elastic body Ω, where the constitutive relation is given by (9.35).

Considering the static case the papers [12, 66] yield homogenization results for periodically distributed microscopic cracks. In [66] Γ-convergence techniques are used to derive effective formulas for microscopic bulk and surface energies. There, different scalings of the microscopic surface energy are investigated. On the other hand, in [12] the unfolding technique is used to provide effective formulas for periodically distributed closed and open cracks. Among other things, there for any bounded sequence of displacement fields with jump sets corresponding to the periodically distributed cracks compactness is shown. This compactness result is essential for identifying the limit model.

In our case, the microscopic crack propagation models allow for the individual evolution of every microscopic crack. For this reason, in Subsection 9.1.1 the following compactness result is shown: Due to the result for the periodic case, for a given sequence $(u_\varepsilon, z_\varepsilon)_{\varepsilon>0}$ with $z_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)$, $\sup_{\varepsilon>0} \|u_\varepsilon\|_{\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega\backslash \mathcal{C}_\varepsilon(z_\varepsilon))^d} < \infty$, and a function $z_0 \in \mathrm{L}^p(\Omega)^m$ such that $z_\varepsilon \to z_0$ in $\mathrm{L}^p(\Omega)^m$ there exists a subsequence of $(u_\varepsilon)_{\varepsilon>0}$ (not relabeled) and a function $(u_0, U_1) \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \times \mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(\mathbf{0})))^d$ such that $u_\varepsilon \xrightarrow{s} E u_0$ in $\mathrm{L}^2(\Omega \times Y)^d$ and $\nabla u_\varepsilon \xrightharpoonup{w} \nabla_x E u_0 + \nabla_y U_1$ in $\mathrm{L}^2(\Omega \times Y)^{d\times d}$. Furthermore, we showed that $U_1(x,\cdot) \in \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(z_0(x)))^d$ for almost every $x \in \Omega$. Since the damage variable $z_0 \in \mathrm{W}^{1,p}(\Omega; [0,1]^m)$ is intended to describe the crack propagation in the effective two-scale model, this relation of the jump set of U_1 and the function z_0 is crucial.

By assumption (9.3) the proof of this result is traced back to the results available in the setting of microscopic voids. In our opinion, starting with the assumptions (9.1) and (9.2) due to condition (9.3) there are no further restrictions on the admissible crack geometry. However, following the approach of [10] the compactness result might be proven without assuming condition (9.3) to hold. There, in the case of periodically distributed cracks compactness of $(u_\varepsilon)_{\varepsilon>0}$ is shown by decomposing every function u_ε into $\mathcal{Q}^*_\varepsilon(u_\varepsilon)$ and $\mathcal{R}^*_\varepsilon(u_\varepsilon)$, where $\mathcal{Q}^*_\varepsilon(u_\varepsilon)$ denotes a polynomial interpolant of degree less than or equal to one and $\mathcal{R}^*_\varepsilon(u_\varepsilon)$ is defined by $\mathcal{R}^*_\varepsilon(u_\varepsilon) := u_\varepsilon - \mathcal{Q}^*_\varepsilon(u_\varepsilon)$. This technique may be extended to the here required non-periodic case.

The second ingredient for identifying the limit models via Γ-convergence techniques is the construction of a recovery sequence. In the case of periodically distributed cracks this is easily done by exploiting density results for the space $\mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}\backslash L^{\mathrm{C}}(\mathbf{0}))^d$; see step 1 of the proof of Lemma 9.7. However, for the individual crack propagation models we need to improve this result in the following way (see Subsection 9.1.2): For given functions $(\widetilde{u}_0, \widetilde{U}_1, \widetilde{z}_0) \in \mathbf{Q}_0^{\mathrm{C}}$ and a sequence $(\widetilde{z}_\varepsilon)_{\varepsilon>0}$ with $\widetilde{z}_\varepsilon \in \mathrm{K}_{\varepsilon\Lambda}(\Omega; [0,1]^m)$ and $\widetilde{z}_\varepsilon \to \widetilde{z}_0$ in $\mathrm{L}^p(\Omega)^m$ we have to construct a sequence $(\widetilde{u}_\varepsilon)_{\varepsilon>0}$ with $\widetilde{u}_\varepsilon \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega\backslash\mathcal{C}_\varepsilon(\widetilde{z}_\varepsilon))^d$, $\widetilde{u}_\varepsilon \xrightarrow{s} E\widetilde{u}_0$ in $\mathrm{L}^2(\Omega\times Y)^d$, and $\nabla\widetilde{u}_\varepsilon \xrightarrow{s} \nabla_x E\widetilde{u}_0 + \nabla_y \widetilde{U}_1$ in $\mathrm{L}^2(\Omega\times Y)^{d\times d}$. To ensure that for any $\varepsilon > 0$ the jump set of $\widetilde{u}_\varepsilon$ is contained in $\mathcal{C}_\varepsilon(\widetilde{z}_\varepsilon)$ the assumptions (9.2g), (9.2h), and (9.2i) have to be exploited carefully; see the proof of Theorem 9.5.

10 Outlook

In this thesis we do not investigate the time-wise regularity of energetic solutions. In general, energetic solutions do have jumps with respect to time. By assuming for all $t \in [0,T]$ the energy functional $\mathcal{E}(t,\cdot,\cdot) : \mathcal{Q} \to \mathbb{R}_\infty$ to be strictly convex with respect to the variable $(u,z) \in \mathcal{Q} = \mathcal{U}\times\mathcal{Z}$, the authors of [61] showed continuity of the energetic solution with respect to time. Note that the first term of the microscopic energy functionals of the Sections 7.1–9.1 is a product of a linear function with respect to the damage variable and a quadratic function with respect to the displacement field. None of these microscopic energy functionals are jointly convex with respect to the displacement field and the damage variable. However, considering the scalar case $d = m = 1$ in Section 7.3, we find

$$\mathcal{E}_0^{\mathrm{In}}(t,u_0,z_0) = \int_\Omega \tfrac{1}{2}\Big((1-z_0)\mathbb{C}_{\mathrm{strong}}^{-1} + z_0\mathbb{C}_{\mathrm{weak}}^{-1}\Big)^{-1}\Big(\tfrac{\partial}{\partial x}u_0\Big)^2 + \Big|\tfrac{\partial}{\partial x}z_0\Big|^p \mathrm{d}x + \langle \ell(t), u_0\rangle$$

as an explicit expression for the effective one-scale limit energy functional. By calculating the second variation $D^2\mathcal{E}_0^{\mathrm{In}}(t,\cdot,\cdot)$ for every $t \in [0,T]$ one easily obtains that $\mathcal{E}_0^{\mathrm{In}} : \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)\times\mathrm{W}^{1,p}(\Omega;[0,1]) \to \mathbb{R}$ is strictly convex. Hence, the convexity of the effective one-scale energy functionals of the Sections 6.3–9.3 seems to be worthwhile of investigation. Since the constitutive relation of the effective material tensor and the limit damage variable is generally given by a unit cell problem, such an investigation might involve some kind of shape derivative.

In the context of modeling crack propagation, for most materials it is reasonable to assume that opposite lips of a fracture cannot interpenetrate at any time. Up to now the crack models of Chapter 9 do not take this physically reasonable assumption into account. In the static case of periodically distributed microscopic cracks the models in [12] and [67] do incorporate a non-interpenetration constraint preventing such a behavior. There, the jump of the displacement field multiplied with the normal vector of the crack surface is assumed to be greater than or equal to zero, which is modeled by an additional surface term entering the energy functional.

Incorporating such a constraint to the microscopic crack propagation models of Section 9.1 induces the following challenges: First of all, the asymptotic behavior of the microscopic constraints has to be investigated to identify the limit constraint entering the effective models. Observe that although this is already done in [12, 67] for the static case, here these results need to be generalized to the evolutionary case involving non-periodic distributions of microscopic cracks.

Secondly, the here presented homogenization technique, based on the evolutionary Γ-convergence introduced in [56], requires the construction of a mutual recovery sequence

$(\widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon)_{\varepsilon>0}$. Neglecting the non-interpenetration constraint, in the proof of Theorem 9.5 for a given $(\widetilde{z}_\varepsilon)_{\varepsilon>0}$ the displacement component $(\widetilde{u}_\varepsilon)_{\varepsilon>0}$ is constructed such that the jump set of $\widetilde{u}_\varepsilon$ is contained in the set of microscopic cracks $\mathcal{C}_\varepsilon(\widetilde{z}_\varepsilon)$ associated to the damage variable $\widetilde{z}_\varepsilon$. Now, by adding the non-interpenetration constraint to the microscopic models, this constraint needs to be respected while constructing the mutual recovery sequence. Referring to the already very technical proof of Theorem 9.5, this adaptation seems to be the most challenging part of incorporating a non-interpenetration constraint into the crack models of Chapter 9.

Bibliography

[1] Robert A. Adams and John J. F. Fournier. *Sobolev Spaces.* Elsevier, Oxford Amsterdam, 2003.

[2] Grégoire Allaire. Homogenization and two-scale convergence. *SIAM. Journal on Mathematical Analysis*, 23(6):1482–1518, 1992.

[3] Hans Wilhelm Alt. *Lineare Funktionalanalysis.* Springer, Berlin Heidelberg New York, 1999.

[4] Ahmed Benallal and Eric Lorentz. Gradient constitutive relations: numerical aspects and application to gradient damage. *Computer methods in applied mechanics and engineering*, 194:5191–5220, 2005.

[5] Guy Bouchitté, Alexander Mielke, and Tomáš Roubíček. A complete-damage problem at small strains. *ZAMP. Zeitschrift für angewandte Mathematik und Physik*, 60(2):205–236, 2009.

[6] Blaise Bourdin, Gilles A. Francfort, and Jean-Jacques Marigo. The variational approach to fracture. *Journal of Elasticity*, 91(1–3):5–148, 2008.

[7] Andrea Braides and Anneliese Defranceschi. *Homogenization of multiple integrals.* Oxford: Clarendon Press, 1998.

[8] Annalisa Buffa and Christoph Ortner. Compact embedding of broken Sobolev spaces and applications. *IMA. Journal of Numerical Analysis*, 29:827–855, 2009.

[9] Philippe G. Ciarlet. *Mathematical Elasticity. Volume I: Three-Dimensional Elasticity.* North-Holland Publishing Co., Amsterdam, 1988.

[10] Doina Cioranescu, Alain Damlamian, Patrizia Donato, Georges Griso, and Rachad Zaki. The periodic unfolding method in domains with holes. *SIAM. Journal on Mathematical Analysis*, 44(2):718–760, 2012.

[11] Doina Cioranescu, Alain Damlamian, and Georges Griso. Periodic unfolding and homogenization. *C. R. Math. Acad. Sci. Paris*, I 335:99–104, 2002.

[12] Doina Cioranescu, Alain Damlamian, and Julia Orlik. Homogenization via unfolding in periodic elasticity with contact on closed and open cracks. *Asymptotic Analysis*, 82(3–4), 2013.

[13] Doina Cioranescu and Patrizia Donato. *An introduction to homogenization.* Oxford: Oxford University Press, 1999.

[14] Bernard Dacorogna. *Direct methods in the calculus of variations.* Springer, Berlin Heidelberg New York, 2008.

[15] Gianni Dal Maso, Gilles A. Francfort, and Rodica Toader. Quasistatic crack growth in nonlinear elasticity. *Archive for Rational Mechanics and Analysis*, 176(2):165–225, 2005.

[16] Cristian Dascalu, Bertrand François, and Oumar Keita. A two-scale model for subcritical damage propagation. *International Journal of Solids and Structures*, 47(3–4):493–502, 2010.

[17] Messoud A. Efendiev and Alexander Mielke. On the rate-independent limit of systems with dry friction and small viscosity. *Journal of Convex Analysis*, 13(1):151–167, 2006.

[18] Jürgen Elstrodt. *Maß- und Integrationstheorie.* Springer, Berlin Heidelberg New York, 1998.

[19] Matteo Focardi, Maria S. Gelli, and Marcello Ponsiglione. Fracture mechanics in perforated domains: a variational model for brittle porous media. *M^3AS. Mathematical Models & Methods in Applied Sciences*, 19(11):2065–2100, 2009.

[20] Gilles A. Francfort and Adriana Garroni. A variational view of partial brittle damage evolution. *Archive for Rational Mechanics and Analysis*, 182(1):125–152, 2006.

[21] Gilles A. Francfort and Jean-Jacques Marigo. Stable damage evolution in a brittle continuous medium. *European Journal of Mechanics. A. Solids.*, 12(2):149–189, 1993.

[22] Michel Frémond and Boumediene Nedjar. Damage, gradient of damage and principle of virtual power. *International Journal of Solids and Structures*, 33(8):1083–1103, 1996.

[23] Artur Ganczarski and Jacek J. Skrzypek. *Modeling of material damage and failure of structures. Theory and applications. With a foreword by Prof. Holm Altenbach.* Springer, Berlin Heidelberg New York, 1999.

[24] Adriana Garroni and Christopher J. Larsen. Threshold-based quasi-static brittle damage evolution. *Archive for Rational Mechanics and Analysis*, 194(2):585–609, 2009.

[25] Alan A. Griffith. The phenomena of rupture and flow in solids. *Philosophical Transactions of the Royal Society, London*, A221:163–198, 1920.

[26] Klaus Hackl, Jan Saczuk, and Helmut Stumpf. Rate theory of nonlocal gradient damage-gradient viscoinelasticity. *International journal of solids and structures*, 19(5):675–706, 2003.

[27] Hauke Hanke. Homogenization in gradient plasticity. *M^3AS. Mathematical Models & Methods in Applied Sciences*, 21(8):1651–1684, 2011.

[28] Christian Heinemann and Christiane Kraus. Existence of weak solutions for Cahn-Hilliard systems coupled with elasticity and damage. *Advances in Mathematical Sciences and Applications*, 21(2):321–359, 2011.

[29] Christian Heinemann and Christiane Kraus. Existence results for diffuse interface models describing phase separation and damage. *European Journal of Applied Mathematics*, 24(2):179–211, 2013.

[30] Christian Heinemann and Christiane Kraus. *Phase separation coupled with damage processes. Analysis of phase field models in elastic media.* Springer, Berlin Heidelberg New York, 2014.

[31] Ulrich Hornung. *Homogenization and porous media.* Springer, Berlin Heidelberg New York, 1997.

[32] Lazar M. Kachanov. *The Theory of Creep.* National Lending Library for Science and Technology, Boston Spa, 1960.

[33] Lazar M. Kachanov. *Delamination Buckling of Composite Materials.* Mechanics of Elastic Stability. Dordrecht (Netherlands) etc.: Kluwer Academic Publishers, 1988.

[34] Lazar M. Kachanov. *Introduction to Continuum Damage Mechanics.* Mechanics of Elastic Stability. Dordrecht (Netherlands) etc.: Kluwer Academic Publishers, second edition edition, 1990.

[35] Michael Kaliske, Stefan Kolling, Ralf Mueller, and Mario Timmel. A micromechanical approach to simulate rubberlike materials with damage. *Computers, Materials and Continua*, 5(3):161–172, 2007.

[36] Michael Kaliske, Stefan Kolling, and Mario Timmel. Damage simulation based on a two scale approach. *Proceedings of the 5th LS-DYNA Forum, Ulm, Germany*, pages 33–47, 2006.

[37] Michael Kaliske, Stefan Kolling, and Mario Timmel. Modelling of microstructural void evolution with configurational forces. *ZAMM. Zeitschrift für Angewandte Mathematik und Mechanik*, 89(8):698–708, 2009.

[38] Alexander M. Khludnev and Jan Sokolowski. Griffith formulae for elasticity systems with unilateral conditions in domains with cracks. *European Journal of Mechanics. A. Solids*, 19(1):105–119, 2000.

[39] Evgueni Ya. Khruslov and Vladimir A. Marchenko. *Homogenization of partial differential equations. Translated from the original Russian by M. Goncharenko and D. Shepelsky.* Boston, MA: Birkhäuser, 2006.

[40] Dorothee Knees. Griffith-formula and J-integral for a crack in a power-law hardening material. *M^3AS. Mathematical Models & Methods in Applied Sciences*, 16(11):1723–1749, 2006.

[41] Dorothee Knees and Alexander Mielke. Energy release rate for cracks in finite-strain elasticity. *Mathematical Methods in the Applied Sciences*, 31(5):501–528, 2008.

[42] Dorothee Knees, Alexander Mielke, and Chiara Zanini. On the inviscid limit of a model for crack propagation. *M^3AS. Mathematical Models & Methods in Applied Sciences*, 18(9):1529–1569, 2008.

[43] Dorothee Knees, Alexander Mielke, and Chiara Zanini. Crack growth in polyconvex materials. *Physica D*, 239(15):1470–1484, 2010.

[44] Dorothee Knees, Riccarda Rossi, and Chiara Zanini. A vanishing viscosity approach to a rate-independent damage model. *M^3AS. Mathematical Models & Methods in Applied Sciences*, 23(4):565–616, 2013.

[45] Michal Kočvara, Alexander Mielke, and Tomáš Roubíček. A rate-independent approach to the delamination problem. *Mathematics and Mechanics of Solids*, 11(4):423–447, 2006.

[46] Serguei M. Kozlov, Olga A. Olejnik, and Vasilii V. Zhikov. *Homogenization of differential operators and integral functionals. Translated from the Russian by G. A. Yosifian.* Springer, Berlin Heidelberg New York, 1994.

[47] Valery I. Levitas, Alexander Mielke, and Florian Theil. A variational formulation of rate-independent phase transformations using an extremum principle. *Archive for Rational Mechanics and Analysis*, 162:137–177, 2002.

[48] Adrian Lew, Patrizio Neff, Michael Ortiz, and Deborah Sulsky. Optimal BV estimates for a discontinuous Galerkin method for linear elasticity. *AMRX. Applied Mathematics Research Express*, 3:73–106, 2004.

[49] Andreas Mainik and Alexander Mielke. Existence results for energetic models for rate-independent systems. *Calculus of Variations and Partial Differential Equations*, 22(1):73–99, 2005.

[50] Alexander Mielke. Finite elastoplasticity Lie groups and geodesics on $SL(d)$. In *Geometry, mechanics, and dynamics. Volume in honor of the 60th birthday of J. E. Marsden*, pages 61–90. Springer, Berlin Heidelberg New York, 2002.

[51] Alexander Mielke. Energetic formulation of multiplicative elasto-plasticity using dissipation distances. *Continuum Mechanics and Thermodynamics*, 15(4):351–382, 2003.

[52] Alexander Mielke. Evolution of rate-independent systems. In *Handbook of differential equations: Evolutionary equations. Vol. II*, pages 461–559. Elsevier, Oxford Amsterdam, 2005.

[53] Alexander Mielke. Complete-damage evolution based on energies and stresses. *Discrete and Continuous Dynamical Systems. Series S*, 4(2):423–439, 2011.

[54] Alexander Mielke. Differential, energetic, and metric formulations for rate-independent processes, A. Mielke, Chapter: Differential, energetic, and metric formulations for rate-independent processes, in: Nonlinear PDEs and applications, C.I.M.E. summer school, Cetraro, Italy 2008. *L. Ambrosio, G. Savaré, eds., vol. 2028 of Lecture Notes in Mathematics*, pages 87–167, 2011.

[55] Alexander Mielke and Tomáš Roubíček. Rate-independent damage processes in nonlinear elasticity. *M^3AS. Mathematical Models & Methods in Applied Sciences*, 16(2):177–209, 2006.

[56] Alexander Mielke, Tomáš Roubíček, and Ulisse Stefanelli. Γ-limits and relaxation for rate-independent evolutionary problems. *Calculus of Variations and Partial Differential Equations*, 31:387–416, 2008.

[57] Alexander Mielke, Tomáš Roubíček, and Marita Thomas. From damage to delamination in nonlinearly elastic materials at small strains. *Journal of Elasticity*, 109(2):235–273, 2012.

[58] Alexander Mielke, Tomáš Roubíček, and Jan Zeman. Complete damage in elastic and viscoelastic media and its energetics. *Computer Methods in Applied Mechanics and Engineering*, 199(21–22):1242–1253, 2010.

[59] Alexander Mielke and Florian Theil. A mathematical model for rate-independent phase transformations with hysteresis. *In H.-D. Alber, R. Balean, and R. Farwig, editors, Proceedings of the Workshop on "Models of Continuum Mechanics in Analysis and Engineering".*, pages 117–129, 1999.

[60] Alexander Mielke and Florian Theil. On rate-independent hysteresis models. *NoDEA. Nonlinear Differential Equations and Applications*, 11(2):151–189, 2004.

[61] Alexander Mielke and Marita Thomas. Damage of nonlinearly elastic materials at small strain – existence and regularity results. *ZAMM. Zeitschrift für Angewandte Mathematik und Mechanik*, 2:88–112, 2010.

[62] Alexander Mielke and Aida M. Timofte. Two-scale homogenization for evolutionary variational inequalities via the energetic formulation. *SIAM. Journal on Mathematical Analysis*, 39(2):642–668, 2007.

[63] Gabriel Nguetseng. A general convergence result for a functional related to the theory of homogenization. *SIAM. Journal on Mathematical Analysis*, 20(3):608–623, 1989.

[64] Malte A. Peter. *Coupled reaction-diffusion systems and evolving microstructure: mathematical modelling and homogenisation*. Logos Verlag Berlin, Berlin, 2007.

[65] Yurli N. Rabotnov. *Creep Problems of Strucural Members*. North-Holland Publishing Co., Amsterdam, 1969.

[66] Lucia Scardia. Damage as Γ-limit of microfractures in anti-plane linearized elasticity. *M^3AS. Mathematical Models and Methods in Applied Sciences*, 18(10):1703–1740, 2008.

[67] Lucia Scardia. Damage as the Γ-limit of microfractures in linearized elasticity under the non-interpenetration constraint. *Advances in Calculus of Variations*, 3(4):423–458, 2010.

[68] Luc Tartar. *The general theory of homogenization. A personalized introduction.* Springer, Berlin Heidelberg New York, 2009.

[69] Augusto Visintin. Some properties of two-scale convergence. *Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat., IX. Ser., Rend. Lincei, Mat. Appl.*, 15(2):93–107, 2004.

[70] Joseph Wloka. *Partial differential equations. Transl. from the German by C. B. and M. J. Thomas.* Cambridge etc.: Cambridge University Press, 1987.

Publikationen und Vorträge

- H. Hanke, D. Knees, Homogenization of elliptic systems with non-periodic, state dependent coefficients. *WIAS Preprint no.* 1880, (2013). (eingereicht)
- H. Hanke, D. Knees, Derivation of an effective damage model with evolving micro-structure. *WIAS Preprint no.* 1749, (2012).
- H. Hanke, Homogenization in gradient plasticity. *M^3AS. Mathematical Models & Methods in Applied Sciences.* 21 (2011) pp. 1651–1684.
- H. Hanke, Homogenization in gradient plasticity. GAMM-*Mitteilungen.* 34 (2011) pp. 102–106.

Vorträge

- Derivation of an effective damage model with evolving micro-structure. *Oberseminar zur Analysis.* Universität Duisburg-Essen, Fachbereich Mathematik, Essen, January 29, 2013.
- Derivation of an effective damage model with non-periodic evolving microstructure. 12*th* GAMM *Seminar on Microstructures.* February 8–9, 2013, Humboldt–Universität zu Berlin, Institut für Mathematik, February 9, 2013.
- Derivation of an effective damage evolution model. *A Matheon Multiscale Workshop.* Technische Universität Berlin, Institut für Mathematik, April 20, 2012.
- Derivation of an effective damage evolution model using two-scale convergence techniques. 83*th Annual Meeting of the International Association of Applied Mathematics and Mechanics* (GAMM 2012). Session on Damage Processes and Contact Problems, March 26–30, 2012, Technische Universität Darmstadt, Fachbereich Mathematik, March 29, 2012.
- Derivation of an effective damage evolution model using two-scale convergence techniques. *International Workshop on Evolution Problems in Damage, Plasticity, and Fracture: Mathematical Models and Numerical Analysis*, September 19–21, 2012, University of Udine, Department of Mathematics, Italy, September 19, 2012.
- Derivation of an effective damage model based on micro-structure evolution. *Workshop on Phase Separation, Damage and Fracture.* September 21–23, 2011, WIAS, September 22, 2011.

- Γ-limits in rate-independent evolutionary problems and homogenization in gradient plasticity. *DK-Seminar "Numerical Simulations in Technical Sciences".* Graz University of Technology, Faculty of Technical Mathematics and Technical Physics, Austria, November 18, 2009.
- Homogenization in gradient plasticity. ISIMM *Workshop on Mathematical Problems of Solid Mechanics.* October 8–9, 2009, Technische Universität Darmstadt, Fachbereich Mathematik, October 9, 2009.