Quadric-Line Configurations Degenerating Plane Picard Einstein Metrics I-II

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dedicated to 60-th birthday of Herbert Kurke

Abstract

We define Picard-Einstein metrics on complex algebraic surfaces as Kähler-Einstein metrics with negative constant sectional curvature pushed down from the complex unit ball allowing degenerations along cycles. We demonstrate how the tool of orbital heights, especially the Proportionality Theorem presented in [H98], works for detecting such orbital cycles on the projective plane. The simplest cycle we found on this way is supported by a quadric and three tangent lines (Apollonius configuration) with at most 3 cusp points sitting on the double points of the configuration. We determine precisely the uniformizing ball lattices in the case of 3, 2, 1 or 0 cusp(s) respectively. The corresponding orbital planes are (leveled) Shimura surfaces corresponding to Jacobian varieties of certain families of plane genus 3, 6, 5 or 13 genus respectively. We present many examples of plane orbital surfaces with quadrics, and determine for them precisely the uniformizing ball lattices. By the way we check that some of them are Galois quotients of celebrated 27 orbital planes with line arrangements occurring in the PTDM-list (Picard-Terada-Mostow-Deligne) which we will call also BHH-list (Barthel-Hirzebruch-Höfer) because it is most convenient to get it from [BHH]. The others are quotients of Mostow's [M2] half-integral arrangements. Proofs are based on the Proportionality Theorem and classification results for hermitian lattices and algebraic surfaces.

Contents

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Introduction

The main purpose of this article is to show that the world of complex algebraic surfaces is Picard-Einstein with a universal degeneration lifted finitely from a quadric and three tangents on the complex projective plane. The three tangent points are "points at infinity" (cusp points) from the non-euclidean metric viewpoint. We call a hermitian metric on a smooth complex surface $\overset{\circ}{\mathrm{X}}$ Picard-Einstein (in a wide sense), if it is Kähler-Einstein with negative constant sectional curvature. If, moreover, $\overset{\circ}{\mathrm{X}}$ is a Zariski open part of an algebraic surface X, then one says that X is Picard-Einstein (with Picard- Einstein metric) *degenerating* (at most) *along* $\mathrm{X} \setminus \overset{\circ}{\mathrm{X}}$. The Bergman metric on the two-dimensional complex unit ball $\mathbb{B}$ is Picard-Einstein, see [BHH], Appendix B, for a short approach. For a ball lattice $\Gamma \subset Aut_{hol}\,\mathbb{B}$ the (quasiprojective) quotient surface $X = X_\Gamma = \mathbb{B}/\Gamma$ (also any compactification $\hat{X}$ of X) is Picard-Einstein degenerating along the branch locus of the canonical quotient map $p = p_\Gamma\ :\ \mathbb{B} \longrightarrow \mathbb{B}/\Gamma$ (and along the compactification cycle). The Picard-Einstein property lifts to each finite cover $\hat{Y}$ of $\hat{X}$ degenerating (at most) along the preimages of branch loci of p_Γ and $\hat{Y} \longrightarrow \hat{X}$. We call $\hat{Y}$ *Picard-Einstein*, if it is finitely lifted (that means via finite covering) from a ball quotient surface $\mathbb{B}/\Gamma$ such that the Baily-Borel compactification $\widehat{\mathbb{B}/\Gamma}$ of $\mathbb{B}/\Gamma$ is the complex projective plane $\mathbb{P}^2$. If one finds a ball lattice with this property, then each complex projective surface is Picard-Einstein in the narrow sense because each such surface is a finite covering of $\mathbb{P}^2$, e.g. via general projections.

The first proof for the fact that $\mathbb{P}^2$ is Picard-Einstein (degenerating along six lines) can be found in [H86]. There we used the Picard modular group of Eisenstein numbers. The main result of this paper is to show that $\mathbb{P}^2$ is Picard-Einstein degenerating along the Apollonius configuration (Apoll-3) with precisely 3 cusp points, see theorem 9.1. The corresponding group $\Gamma(1+i)$ is the congruence sublattice of $\Gamma\ := \mathbb{SU}(diag(1,1,-1),\mathfrak{O})$, $\mathfrak{O} = \mathbb{Z} + \mathbb{Z}i$, $i = \sqrt{-1}$, belonging to the ideal $\mathfrak{O}(1+i)$. This is a Picard modular group of Gauß numbers.

Some papers of other authors have to be mentioned which come - with other methods -already near to this result, or present useful preparations: Terada [T], Deligne & Mostow [DM1, DM2], Matsumoto [Mat], van Geemen [vGm], Shvartsman [Sv1], [Sv2], Hashimoto [Has].

The most natural way for finding a configuration (reduced cycle Z) on an orbiface (two-dimensional orbifold), which could be the degenerate locus of a Picard-Einstein metrics has been described in [H98]. Beside of quotient singularities we allow also cusp singularities on the surface. The irreducible components of the configuration (points and irreducible curves) are endowed with natural numbers or ∞ (weights) in an admissible manner. Then one gets an *orbital cycle*. The surface X together with the orbital cycle $\mathbf{Z}$ is called an *orbital surface*. The orbital surface germs around points are irreducible components of the orbital cycle are called *orbital points* or *orbital curves*, respectively. Points or curves with weight ∞ are called *cusp points* or *cusp curves*, respectively. They form a subcycle Z_∞ of Z whose support is denoted by X_∞. The finitely weighted points are *quotient (triple) points*. For details we refer to [H98], where we corresponded rational numbers to our orbital objects called *orbital heights*. The orbital surface heights (global heights H) generalize volumes of Γ- fundamental domains on $\mathbb{B}$ of arbitrary ball lattices Γ. The orbital curve heights (local heights h) do the same for the complex unit disc $\mathbb{D}$ and $\mathbb{D}$-lattice groups. Euler form and signature form define on this way two different orbital heights H_e, H_τ and h_e, h_τ called *Euler* or *signature heights*, respectively. A *finite uniformization* $Y \longrightarrow \mathbf{X}$ of an orbital surface $\mathbf{X} = (X, \mathbf{Z})$ is a finite Galois covering $Y \longrightarrow X$ such that Y is smooth (outside cusp points) and the weights of the components of $\mathbf{Z}$ coincide with corresponding ramification indices. A *ball uniformization* of $\mathbf{X}$ is a (locally finite) infinite Galois covering (quotient map by a ball lattice) $\mathbb{B} \longrightarrow X_f := X \setminus X_\infty$ again with weights equal to corresponding ramification indices. We announce the following

Theorem 0.1. *For an orbital surface $\mathbf{X} = (X, \mathbf{Z})$ the following conditions are equivalent:*

(i) $\mathbf{X}$ *has a ball uniformization*

(ii) The proportionality conditions

(Prop 2) $H_e(\mathbf{X}) = 3H_\tau(X) > 0$

(Prop 1) $h_e(\mathbf{C}) = 2h_\tau(\mathbf{C}) < 0$ *for all orbital curves* $\mathbf{C} \subset \mathbf{Z}$

are satisfied, and there exists a finite uniformization Y of $\mathbf{X}$, which is of general type.

The direction (i) $\Rightarrow$ (ii) has been proved in [H98], see Proportionality Theorem IV.9.2. Notice that our h_τ is 3 times h_τ of [H98]. The other direction follows from the degree homogenity of the global heights and a well-known theorem of R.Kobayashi-Miyaoka-Yau applied to Y. Namely, it is easy to see that the (Prop 2)-condition lifts to the logarithmic Chern number condition $\bar{c}_1^2 = 3\bar{c}_2$ for Y. $\square$

In section 2 we use the explicit orbital height machine for detecting suitable weights for points and curves on the Apollonius configuration on $\mathbb{P}^2$ such that the corresponding orbital surface satisfies the proportionality conditions. This has been done for demonstrating and understanding a general approach to detect Picard-Einstein metrics on surfaces. Any orbital configuration (X, Z) defines a system $Dioph(X, Z)$ of diophantine equations. It comes out from a system of a quadratic and some linear equations with rational coefficients closely related with (Prop 2) or (Prop 1), respectively, for which we have to determine inverse of natural numbers as solutions (the inverse of the weights we look for). There are at most finitely many solutions, see [H98], IV.10.

In section 3 we discus the weights obtained in 2. This is a connection between classical approach and the proportionality technic. Then we give generators of the modular group and prepare the result for the second part of the paper.

In the next section we transform the detected weights to seven properties $(i), \ldots, (vii)$ of a uniformizing ball lattice Γ' we look for using the Proportionality Theorem via the system $Dioph(X, Z)$ again, this time in converse direction: We know the weights but the data (Chern numbers, selfintersections) of X, Z are unknown. With the seven postulated properties we are able to determine these data and to classify surface and curves to get $\widehat{\mathbb{B}/\Gamma'} = \mathbb{P}^2$ and the Apollonius configuration back. In the sections 9,10,11 we prove that the congruence lattice $\Gamma(1+i)$ has all the seven properties.

The second author wrote a detecting algorithm on MAPLE based on constructive proof of the Finiteness Theorem in [H98]. It proves that there are precisely 4 possibilities (Apoll-k), k=0,1,2,3 number of cusp points, of Picard-Einstein metrics on $\mathbb{P}^2$ degenerated along Apollonius configuration. Here we present another proof which is more analytic. We add some examples with two quadrics and some lines, where we used a MAPLE programm from the first author which is able to check the proportionality conditions for any orbital surface.

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Part I

Proportionality and Monodromy

1 The basic orbital surface: Plane with Apollonius cycle

We consider an orbital surface

$$\widehat{\mathbf{X}} = (\hat{X}; \widehat{\mathbf{C}}_0 + \widehat{\mathbf{C}}_1 + \widehat{\mathbf{C}}_2 + \widehat{\mathbf{C}}_3 + \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 + \mathbf{K}_1 + \mathbf{K}_2 + \mathbf{K}_3) \tag{1}$$

with smooth compact complex algebraic surface $\hat{X}$ supporting the orbital cycle

$$\mathbf{Z}(\widehat{\mathbf{X}}) = \widehat{\mathbf{C}}_0 + \widehat{\mathbf{C}}_1 + \widehat{\mathbf{C}}_2 + \widehat{\mathbf{C}}_3 + \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 + \mathbf{K}_1 + \mathbf{K}_2 + \mathbf{K}_3, \tag{2}$$

which consists of four orbital curves $\widehat{\mathbf{C}}_j$, $j = 0,1,2,3$, on $\widehat{\mathbf{X}}$ with weights v_j, three (finite) orbital abelian points $\mathbf{P}_j$, $j = 1,2,3$, of type $\mathbb{C}^2/Z_{v_s} \times Z_{v_t}$ where $Z_{v_s} \times Z_{v_t} \subset \mathbb{G}l_2(\mathbb{C})$ denotes the abelian group generated by 2 opposite reflections of orders v_s, v_t, and $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3$ are precisely the special points (cusp or quotient). For the surface $\hat{X}$ and the reduced cycle

$$Z(\hat{\mathbf{X}}) = \hat{C}_0 + \hat{C}_1 + \hat{C}_2 + \hat{C}_3 + P_1 + P_2 + P_3 + K_1 + K_2 + K_3 \tag{3}$$

we claim the following conditions:

(i) The surface $\hat{X}$ is the projective plane $\mathbb{P}^2$

(ii) a) $\hat{C}_0$ is a quadric on $\mathbb{P}^2$;

b) $\hat{C}_1, \hat{C}_2, \hat{C}_3$ are projective lines on $\mathbb{P}^2$;

c) P_1, P_2, P_3 are the three different intersection points of these lines;

d) $\hat{C}_j$ is the tangent line of $\hat{C}_0$ at K_j, $j = 1,2,3$;

e) The *configuration divisor* $\hat{C}_0 + \hat{C}_1 + \hat{C}_2 + \hat{C}_3$ is *symmetric*. This means that there is an effective action of the symmetric group S_3 on $\mathbb{P}^2$ preserving $\hat{C}_0 + \hat{C}_1 + \hat{C}_2 + \hat{C}_3$.

Definition 1.1. *If these conditions are satisfied we call $\hat{C}_0 + \hat{C}_1 + \hat{C}_2 + \hat{C}_3$ a plane Apollonius configuration or Apollonius configuration on $\mathbb{P}^2$, the cycle $Z(\hat{\mathbf{X}})$ a reduced plane Apollonius cycle and each effective cycle with this support a plane Apollonius cycle.*

The properties a),b),c),d) mean that the Apollonius configuration on $\mathbb{P}^2$ consists of a plane quadric and three different tangent lines of it. We will see below that e) is automatically satisfied with a unique S_3-action. The following graphic describes the corresponding configuration together with three additional lines L_j joining P_j and K_j. For the rest of this section we work on $\hat{X} = \mathbb{P}^2$ and omit the hats over C_j (see Figure 1).

Without loss of generality we can chose the S_3-symmetric

Normalized Model 1.2.

$$C_0 : (X + Y - Z)^2 - 4XY = X^2 + Y^2 + Z^2 - 2XY - 2XZ - 2YZ = 0;$$

$$\begin{array}{rclrclrcl}
C_1 : Y &=& 0 & C_2 : Z &=& 0 & C_3 : X &=& 0\,, \\
P_1 &=& (0:1:0) & P_2 &=& (0:0:1) & P_3 &=& (1:0:0)\,; \\
K_1 &=& (1:0:1) & K_2 &=& (1:1:0) & K_3 &=& (0:1:1)\,; \\
L_1 : X &=& Z & L_2 : X &=& Y & L_3 : Y &=& Z\,.
\end{array}$$

By elementary projective geometry the following facts are easy to check.

Proposition 1.3. *Up to $\mathbb{P}\mathbb{G}l_3$-equivalence the Apollonius configuration is uniquely determined. All Apollonius configurations are S_3-symmetric.*

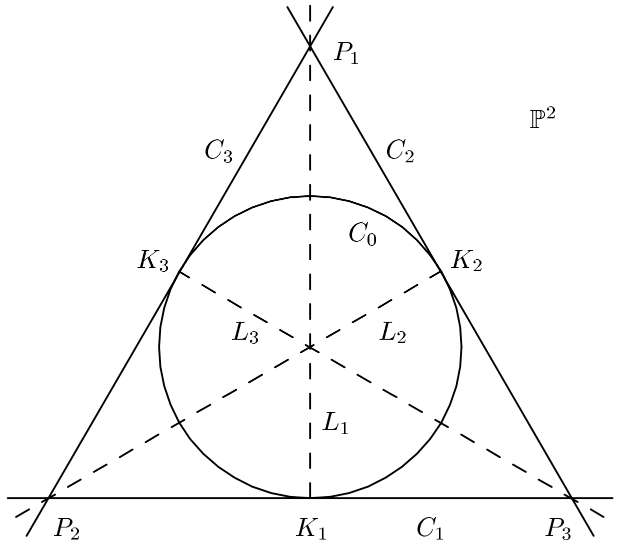


Figure 1: divisor $C_0 + C_1 + C_2 + C_3$

Corollary 1.4. *The action of the symmetric group S_3 on $\mathbb{P}^2$ preserving the configuration $C_0 + C_1 + C_2 + C_3$ is unique. It is determined by extending permutations of points $\pi : K_i \mapsto K_{\pi(i)}$, $P_i \mapsto P_{\pi(i)}$, $i = 1, 2, 3$, $\pi \in S_3$, to $\Pi \in Aut\,\mathbb{P}^2 = \mathbb{P}\mathbb{G}l_3(\mathbb{C})$. Especially for the normalized model 1.2 the group S_3 acts by permutation of canonical projective coordinates $(x : y : z)$ on $\mathbb{P}^2$.*

Remark 1.5. *The lines L_1, L_2, L_3 defined in (1.4) have a common point.*

Lemma 1.6. *For three projective lines C_1, C_2, C_3 on $\mathbb{P}^2$ intersecting each other in different points and for a given subgroup $\Sigma_3 \cong S_3$ of $\mathbb{P}\mathbb{G}l_3$ permuting them there is precisely one quadric C_0 with tangents C_1, C_2, C_3. For the canonical coordinate axes $X = 0$, $Y = 0$, $Z = 0$ of $\mathbb{P}^2$ the corresponding quadric (see 1.2, normalized model) has equation*

$$X^2 + Y^2 + Z^2 - 2XY - 2XZ - 2YZ = (X + Y - Z)^2 - 4XY = 0.$$

Definition 1.7. *An Apollonius cycle on a smooth compact complex algebraic surface Y is a cycle*

$$\mathbf{Z} = v_0 L_0 + v_1 L_1 + v_2 L_2 + v_3 L_3 + P_1 + P_2 + P_3 + K_1 + K_2 + K_3,$$

where the v_i's are positive integers, $P_1, P_2, P_3, K_1, K_2, K_3$ are points on Y, and the L_i's are smooth complete algebraic curves on Y with the following intersection behaviour:

$$L_0 \cdot L_j = 2K_j \; for\, j = 1, 2, 3; \; L_i \cdot L_j = P_k \; for\, \{i, j, k\} = \{1, 2, 3\}.$$

The supporting reduced curve $L_0 + L_1 + L_3 + L_3$ is called an Apollonius configuration on Y. The Apollonius cycle (configuration) is called symmetric, iff there is an algebraic S_3-action on Y, which preserves the cycle $\mathbf{Z}$ permuting effectively its components $\hat{C}_j$, P_j, K_j, $j = 1, 2, 3$, respectively.

Remark 1.8. *Obviously, $v_1 = v_2 = v_3$ holds in the symmetric case. If Y is the projective plane, then the Apollonius configuration is automatically of the (symmetric) plane Apollonius cycle consisting of a quadric and three tangent lines as defined in 1.1 .*

2 Proportionality

Turn back now to the more precise notations of 1.1 not assuming in this section the symmetry condition (ii) e). We blow up each of the the points K_j twice such that the proper transforms of $\hat{C}_i$ for $i = 1, 2, 3$ on the resulting surface $\tilde{X}$ do not intersect the proper transform of $\hat{C}_0$. The exceptional divisor

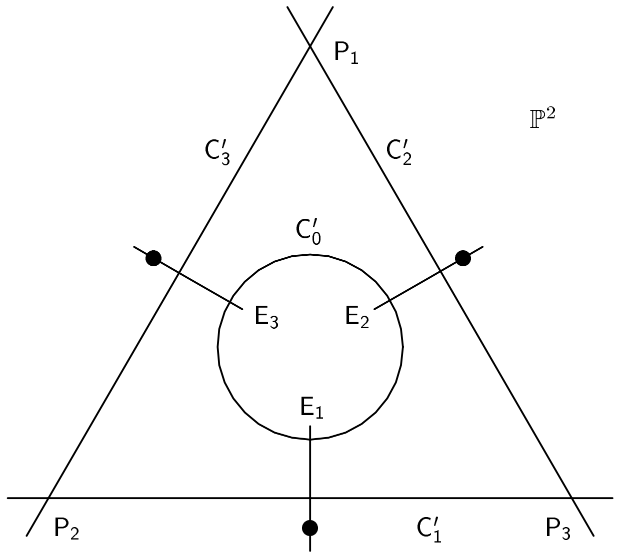


Figure 2: $\bullet$ singularity of type $\langle 2,1\rangle$

$E(\tilde{X} \longrightarrow \hat{X})$ on $\tilde{X}$ consists of three connected components. Each of them is a pair of transversally crossing smooth rational curves with selfintersection -1 or -2, respectively. Then we contract the three -2-curves to get a surface X' with three quotient singularities of type $\mathbb{C}^2/\pm\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ lying on exceptional curves $E_1, E_2, E_3 \subset X'$. On this way we get an orbital birational morphism $\mathbf{X}' \longrightarrow \hat{\mathbf{X}}$ being isomorphic outside $X'_\infty = E_1 + E_2 + E_3$ and $\hat{X}_\infty = K_1 + K_2 + K_3$. The proper transforms of the $\hat{C}_j$ are denoted by C_j', $j = 0,1,2,3$, respectively. On this way we get a complete orbital surface

$$\mathbf{X}' = (X';\ \mathbf{C}_0' + \mathbf{C}_1' + \mathbf{C}_2' + \mathbf{C}_3' + \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 + \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3)$$

called the canonical locally abelian model of $\hat{\mathbf{X}}$. The finite part supported by $X = X'_f = X' \setminus X'_\infty$ is the open orbital surface

$$\mathbf{X} = (X;\ \mathbf{C}_0 + \mathbf{C}_1 + \mathbf{C}_2 + \mathbf{C}_3 + \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3)$$

with supporting non-compact curves $C_j = C'_{jf} = C_j' \setminus X'_\infty$. The orbital cycle $Z(\mathbf{X}')$ is described in the Figure 2. The open orbital curves can be written as

$$\mathbf{C}_0 = v_0 C_0, \quad \mathbf{C}_1 = (v_1 C_1;\ \mathbf{P}_2 + \mathbf{P}_3), \quad \mathbf{C}_2 = (v_2 C_2;\ \mathbf{P}_1 + \mathbf{P}_3), \quad \mathbf{C}_3 = (v_3 C_3;\ \mathbf{P}_1 + \mathbf{P}_2)\ .$$

The corresponding atomic graphs of the four orbital curves and the three exceptional curves look like Figure 3. The molecular graph of the whole orbital cycle is the Figure 4.

In [H98], IV, Theorem 4.9.2, we proved that there are rather strong proportionality conditions for an orbital surface to be an orbital ball quotient. For this purpose we defined orbital heights for orbital curves and surfaces, which are rational numbers. First one has to draw the graph of an orbital curve $\hat{\mathbf{C}}$ on an arbitrary $\mathbb{B}-$orbital surface $\hat{\mathbf{X}}$ ($\mathbb{B}-$orbital means that only ball cusp singularities are allowed "at infinity"). On the open "finite" part $\mathbf{X}$ of $\hat{\mathbf{X}}$ at most quotient singularities are admitted. In our examples cusp and triple (fraction) singularities are possible. These singularities are classified in [H98], III, Definition 3.5.6 and Corollary 3.5.9. We denote the cusp and triple singularities with squares and triangles respectively. In the following shortly *special point* is cusp or triple singularity. In our examples K_1, K_2, K_3 are special points with weights k_1, k_2, k_3 respectively. The special points on the Figure 3 are cusps (squares), but every square can also be a triangle.

The (atomic) graph of the orbital curve $\hat{\mathbf{C}} = (v\hat{C};\ \sum \mathbf{P}_i + \sum \mathbf{K}_j)$ looks star-like – Figure 5. The center represents the curve $\hat{C}$ weighted with $v \in \mathbb{N}_+$ and s is the selfintersection number (C'^2) on the minimal singularity resolution $\tilde{X} \longrightarrow X'$ of the canonical locally abelian resolution $X' \longrightarrow \hat{X}$, which replaces each special point K by an irreducible curve E_K (finite quotient of an elliptic curve) supporting (at most 4) cyclic surface singularities.

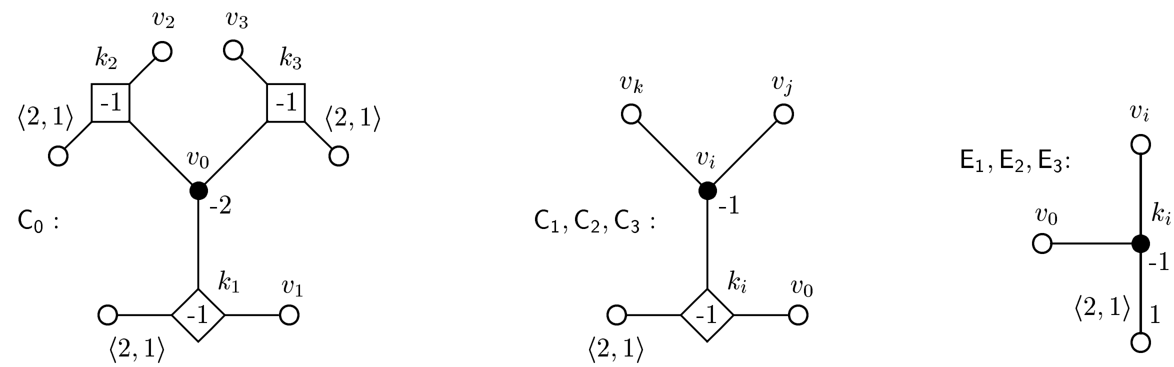


Figure 3: atomic graphs of $\mathsf{C}_0, \mathsf{C}_1, \mathsf{C}_2, \mathsf{C}_3,\ \mathsf{E}_1, \mathsf{E}_2, \mathsf{E}_3$

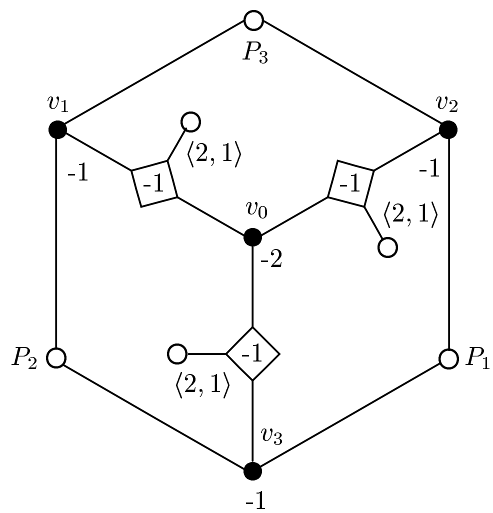


Figure 4: molecular graph of Apoll-3

The proper transform of $\hat{C}$ on X' or $\tilde{X}$ is denoted by C'. The arrows to small circles represent cyclic surface singularities P_i of type $\langle d_i, e_i\rangle$ of X' lying on C' and the circle itself represents the curve germ of weight v_i crossing C' at P_i. The abelian point $\mathbf{P}_i$: $\underset{v}{\circ}\overset{\langle d_i,e_i\rangle}{\longrightarrow}\underset{v_i}{\circ}$ consist of the crossing curve germs of C, C_i with weights v, v_i, respectively. The small boxes represent cusp points lying on $\hat{C}$, and the arrow to the box represent the intersection point of E_K and C' on X' being a cyclic singularity of type $\langle d_j, e_j\rangle$ isomorphic, by definition, to the singularity of $\mathbb{C}^2/\langle\begin{pmatrix}\zeta & 0\\ 0 & \zeta^e\end{pmatrix}\rangle$, where ζ denotes a primitive d-th unit root. Similarly the triangle represent triple points with weight v_k and the arrow to the triangle represent the intersection point of E_K and C' on X' being a cyclic singularity of type $\langle d_k, e_k\rangle$. The weight t at the box or triangle is the selfintersection of (the proper transform of) E_K on $\tilde{X}$. We omit the arrow orientation and $\langle\ ,\ \rangle$, if $\langle d_i, e_i\rangle$, $\langle d_j, e_j\rangle$ or $\langle d_k, e_k\rangle = \langle 1, 0\rangle$. This means that the corresponding intersection point is non-singular. The arrow orientation is also omitted, if the singularity of type $\langle d, e\rangle$ is symmetric. This means that its minimal resolution (linear tree of smooth rational curve with selfintersection numbers read off from the continued fraction of $\frac{d}{e}$) is symmetric. Examples are given in Figure 3. For more details we refer to [H98]. There we defined (see IV, Definition 4.7.3 and restrict to our situation) the *Euler height* of $\mathbf{C}$ by

(4) $$h_e(\mathbf{C}) = e(C') - \sum(1 - \frac{1}{v_i d_i}) - \#C'_\infty - \sum(1 - \frac{1}{v_k d_k}),$$

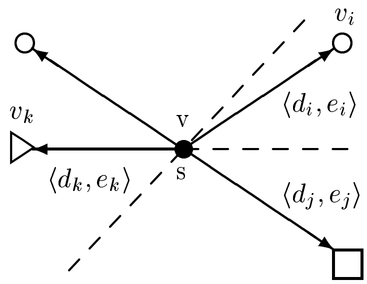


Figure 5:

and the signature (or selfintersection) height

$$h_\tau(\mathbf{C}) = \frac{1}{v}\Big[(C'^2) + \sum \frac{e_i}{d_i} + \sum \frac{e_j}{d_j}\Big] + \sum \Big[\frac{e_k}{v\, d_k} + \frac{1}{d_k v_k}\Big] \tag{5}$$

(which is $3\boldsymbol{\tau}_f(\hat{\mathbf{C}})$ in the notations of [H98]). The first sum runs over all abelian points $\mathbf{P}_i$ on $\mathbf{C}$, the second sum in (5) over all arrows $\overset{\langle d_j,e_j\rangle}{\circ\!\!\longrightarrow\!\diamond}$ joining the center with a cusp box and the last sum in (4-5) over all triple points (all arrows $\overset{\langle d_k,e_k\rangle}{\circ\!\!\longrightarrow\!\triangleleft}$ joining the center with a triple triangle), see picture 5.

On this way we obtain for each Apoll-k separate system of diophantine equations. Effective Finiteness Theorem 4.10.3 [H98] say that there are only fininitely many possibilities of weighting the $\mathbb{B}$–orbital surface $\hat{\mathbf{X}}$. We prefer to consider all Apoll-k together and have only one system of diophantine equations. Any solution of this system is a candidate for appropriate weight. For this purpose we introduce "universal weights":

$$v_k := \begin{cases} v_i \text{ -- orbital curve} \\ \infty \text{ -- cusp point} \\ -v_k \text{ -- triple point (negative weight)}. \end{cases}$$

With the new universal weights we use also new Euler height ($v_k < 0$)

$$u_e(\mathbf{C}) = e(C') - \sum(1 - \frac{1}{v_i d_i}) - \sum(1 - \frac{1}{\infty\, d_j}) - \sum(1 - \frac{1}{v_k d_k}), \tag{6}$$

and the signature (or selfintersection) height

$$u_\tau(\mathbf{C}) = \frac{1}{v}\Big[(C'^2) + \sum \frac{e_i}{d_i} + \sum \frac{e_j}{d_j} + \sum \frac{e_k}{d_k}\Big]\ . \tag{7}$$

Obviously the connection between two heights is:

$$h_e(\mathbf{C}) = u_e(\mathbf{C}) + 2\sum_{v_k<0} \frac{1}{|v_k| d_k}\,, \qquad\qquad h_\tau(\mathbf{C}) = u_\tau(\mathbf{C}) + \sum_{v_k<0} \frac{1}{|v_k| d_k}\,. \tag{8}$$

For a special point K we define Euler and signature heights $u_e(E_K)$ and $u_\tau(E_K)$ using the exceptional curve E_K on X'.

Remark 2.1. *It is easy to redefine cusp and triple points:*
if $u_e(\mathbf{E}_K) = 0$ *and* $\lim_{v_j\to\infty} u_\tau(\mathbf{E}_K)v_j < 0$, *then* K *is cusp point;*
if exist $v_k < 0$ *such that* $u_e(\mathbf{E}_K) = 2u_\tau(\mathbf{E}_K) > 0$ *then* K *is triple point.*

For an orbital curve $\mathbf{C}$ using Proportionality [H98], IV, Theorem 4.9.2 we have $h_e(\mathbf{C}) = 2h_\tau(\mathbf{C})$. Applying (8) and Remark 2.1 we obtain

$$u_e(\mathbf{C}) = 2u_\tau(\mathbf{C}), \qquad \mathbf{C} \in \{\mathbf{C}_0, \mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3,\, \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}\ . \tag{9}$$

This is a system of seven diophantine equations connected with Apollonius configuration. In terms of [H98] and [HPV] these equations are Prop 1, Prop 0, Prop ∞.

Let as look at the Figure 3 and apply (9)

$$\left|\begin{array}{llcr} \mathbf{C}_0: & 2-\left(1-\frac{1}{k_1}\right)-\left(1-\frac{1}{k_2}\right)-\left(1-\frac{1}{k_3}\right) & = & 2\frac{-2}{v_0} \\ \mathbf{C}_1: & 2-\left(1-\frac{1}{v_2}\right)-\left(1-\frac{1}{v_3}\right)-\left(1-\frac{1}{k_1}\right) & = & 2\frac{-1}{v_1} \\ \mathbf{E}_1: & 2-\left(1-\frac{1}{v_0}\right)-\left(1-\frac{1}{v_1}\right)-\left(1-\frac{1}{2}\right) & = & 2\frac{-1+\frac{1}{2}}{k_1} \end{array}\right.$$

One write the equations for $\mathbf{C}_2, \mathbf{C}_3$, $\mathbf{E}_2, \mathbf{E}_3$ after cyclic permutation $1 \to 2 \to 3$. The last system is equivalent to

$$(10) \qquad \left|\begin{array}{rll} \frac{4}{v_0}+\frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3} & =1 & \\ \frac{1}{v_i}+\frac{1}{k_i}+\frac{1}{v_1}+\frac{1}{v_2}+\frac{1}{v_3} & =1, & i=1,2,3 \\ \frac{1}{v_0}+\frac{1}{v_i}+\frac{1}{k_i} & =\frac{1}{2}, & i=1,2,3. \end{array}\right.$$

We are looking for solutions of this system in $\infty \cup \mathbb{Z}\backslash 0$. If v_1, v_2, v_3 are known then v_0, k_1, k_2, k_3 are determined uniquely. The symmetric group S_3 acts on v_1, v_2, v_3 and after lifting on all variables $v_0, v_1, v_2, v_3, k_1, k_2, k_3$. Up to S_3 symmetry ($v_1 \leq v_2 \leq v_3$) the system (10) has 58 solutions (Table 1) and 41 from them are hyperbolic[1].

We want to connect the solutions of (10) with the classical theory. Let us consider the configuration divisor ([BHH, DM1]) $xyz(x-y)(y-z)(z-x)$ (see Figure 6). In this case special points are P_1, P_2, P_3, P_4.

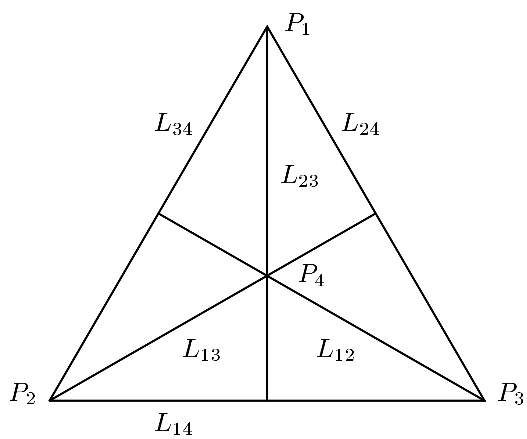


Figure 6: $xyz(x-y)(x-z)(x-z)=0, \qquad (x\!:\!y\!:\!z) \in \mathbb{P}^2$

We blow up the four special points and the resulting divisor have 10 lines with selfintersection -1. Let exceptional lines are L_{0j}, $j=1,2,3,4$. The proportionality equations (9) are:

$$(11) \qquad \left|\begin{array}{llcr} \mathbf{L}_{34}: & 2-\left(1-\frac{1}{l_{01}}\right)-\left(1-\frac{1}{l_{02}}\right)-\left(1-\frac{1}{l_{12}}\right) & = & 2\frac{-1}{l_{34}} \\ \mathbf{L}_{01}: & 2-\left(1-\frac{1}{l_{34}}\right)-\left(1-\frac{1}{l_{24}}\right)-\left(1-\frac{1}{l_{23}}\right) & = & 2\frac{-1}{l_{01}}. \end{array}\right.$$

The symmetric group $S_4 = S\{1,2,3,4\}$ acts on this system changing the indexes. Under the action of (13),(14), (23),(24),(34) on the first and (12), (13), (14) on the second equation we obtain ten equations for lines $\mathbf{L}$ respectively. We assume that $l_{ij} = l_{ji}$. The last diophantine system has 38 solutions in $\infty \cup \mathbb{Z}\backslash 0$ and 27 from them are hyperbolic (see [BHH], page 201, [DM1] page 86, [T] page 465).

[1]We say that a solution is hyperbolic if it satisfy the proportionality conditions (see proposition 2.3).

	v_0	v_1	v_2	v_3	k_1	k_2	k_3	H_e		v_0	v_1	v_2	v_3	k_1	k_2	k_3	H_e
1	18	-18	2	9	2	-18	3	13/216	30	5	2	10	10	-5	5	5	3/20
2	4	-12	2	3	3	-4	-12	1/24	31	6	2	12	12	-6	4	4	7/48
3	12	-12	2	6	2	-12	4	7/96	32	9	2	18	18	-9	3	3	13/108
4	12	-12	3	3	2	12	12	7/48	33	∞	2	∞	∞	∞	2	2	$\times$
5	10	-10	2	5	2	-10	5	3/40	34	2	3	3	3	-3	-3	-3	$\square$
6	8	-8	2	4	2	-8	8	9/128	35	3	3	3	6	-6	-6	∞	1/12
7	-3	-6	-6	2	1	1	3	$\times$	36	4	3	3	12	-12	-12	6	7/48
8	3	-6	2	2	3	-3	-3	$\square$	37	6	3	3	∞	∞	∞	3	1/6
9	6	-6	2	3	2	-6	∞	1/24	38	3	3	4	4	-6	-12	-12	1/12
10	4	-4	2	2	2	-4	-4	$\square$	39	4	3	4	6	-12	∞	12	17/96
11	-4	-4	4	4	1	2	2	$\square$	40	6	3	4	12	∞	12	4	11/48
12	-6	-3	-3	1	1	1	-3	$\times$	41	8	3	4	24	24	8	3	11/48
13	-6	-3	2	6	1	6	2	$\square$	42	10	3	5	15	15	5	3	37/150
14	-6	-3	3	3	1	3	3	$\square$	43	6	3	6	6	∞	6	6	1/4
15	∞	-2	1	∞	1	-2	2	$\times$	44	12	3	6	12	12	4	3	1/4
16	∞	-2	2	2	1	∞	∞	$\times$	45	∞	3	6	∞	6	3	2	1/6
17	2	1	d	-d	-1	-d	d	$\square$	46	18	3	9	9	9	3	3	13/54
18	1	2	2	2	-1	-1	-1	$\square$	47	4	4	4	4	∞	∞	∞	3/16
19	2	2	2	∞	-2	-2	∞	$\times$	48	5	4	4	5	20	20	10	99/400
20	2	2	3	6	-2	-3	-6	$\square$	49	6	4	4	6	12	12	6	13/48
21	3	2	3	∞	-3	-6	6	1/24	50	8	4	4	8	8	8	4	9/32
22	2	2	4	4	-2	-4	-4	$\square$	51	12	4	4	12	6	6	3	13/48
23	3	2	4	12	-3	-12	12	7/96	52	∞	4	4	∞	4	4	2	3/16
24	4	2	4	∞	-4	∞	4	3/32	53	12	4	6	6	6	4	4	7/24
25	4	2	5	20	-4	20	5	99/800	54	-12	4	12	12	3	2	2	7/48
26	3	2	6	6	-3	∞	∞	1/12	55	10	5	5	5	5	5	5	3/10
27	4	2	6	12	-4	12	6	13/96	56	∞	6	6	6	3	3	3	1/4
28	6	2	6	∞	-6	6	3	1/8	57	-8	8	8	8	2	2	2	9/64
29	4	2	8	8	-4	8	8	9/64	58	-2	∞	∞	∞	1	1	1	$\times$

Table 1: solutions of diophantine equations (10) $\in \mathbb{Z}\backslash 0 \cup \infty$

Theorem 2.2. *If $v_0, v_1, v_2, v_3, k_1, k_2, k_3$ is a solution on system (10) such that v_0 is even then*

$$l_{14} = l_{34} = v_3, \qquad l_{23} = l_{12} = v_1, \qquad l_{24} = k_2, \qquad l_{13} = \frac{v_0}{2},$$
$$l_{04} = k_1, \qquad l_{01} = l_{03} = v_2, \qquad l_{02} = k_3.$$

is a solution on (11). For v_0 even (∞ is also "even" number) the map

$$T: \quad (v_0,\ v_1, v_2, v_3,\ k_1, k_2, k_3) \longrightarrow (l_{12}, l_{13}, l_{23},\ l_{14}, l_{24}, l_{34},\ l_{01}, l_{02}, l_{03}, l_{04})$$

is surjective in the sets of solution of (11).

Proof. The condition for v_0 even is obvious. In the section 3 we will give geometric proof. $\square$

We are looking for solutions of (10) which are hyperbolic weights of Apollonius configuration. In this case $1 < v_0, v_1, v_2, v_3 \in \mathbb{N}$ and $k_1, k_2, k_3 \in \mathbb{Z}_- \cup \infty$, $\mathbb{Z}_-$ – negative integer numbers. The restrictions for k_1, k_2, k_3 are because K_1, K_2, K_3 are special points. Only solutions 20, 22, 26, 34, 35, 38 and 47 satisfy the above conditions.

We calculate Euler and signature height using (8)

$$\begin{aligned} h_e(\mathbf{C}_0) &= -1 + \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + 2\left(\frac{1}{|k_1|} + \frac{1}{|k_2|} + \frac{1}{|k_3|}\right), \\ h_e(\mathbf{C}_i) &= -1 + \frac{1}{v_1} + \frac{1}{v_2} + \frac{1}{v_3} - \frac{1}{v_i} + \frac{1}{k_i} + 2\frac{1}{|k_i|}, \qquad i = 1, 2, 3 \end{aligned} \tag{12}$$

$$
(13) \qquad \begin{aligned}
h_\tau(\mathbf{C}_0) &= \frac{-2}{v_0} + \frac{1}{|k_1|} + \frac{1}{|k_2|} + \frac{1}{|k_3|}\,,\\
h_\tau(\mathbf{C}_i) &= \frac{-1}{v_i} + \frac{1}{|k_i|}\,, \qquad i = 1,2,3
\end{aligned}
$$

and the result is the next table

(14)

	v_0	v_1	v_2	v_3	k_1	k_2	k_3	$h_e:$ $\mathbf{C}_0$	$\mathbf{C}_1$	$\mathbf{C}_2$	$\mathbf{C}_3$
20	2	2	3	6	-2	-3	-6	0	0	0	0
22	2	2	4	4	-2	-4	-4	0	0	0	0
26 Apoll-2	3	2	6	6	-3	∞	∞	-2/3	-1/3	-1/3	-1/3
34	2	3	3	3	-3	-3	-3	0	0	0	0
35 Apoll-1	3	3	3	6	-6	-6	∞	-2/3	-1/3	-1/3	-1/3
38 Apoll-0	3	3	4	4	-6	-12	-12	-2/3	-1/3	-1/3	-1/3
47 Apoll-3	4	4	4	4	∞	∞	∞	-1	-1/2	-1/2	-1/2

In this table we write only $h_e(\mathbf{C})$ because in all cases $h_e(\mathbf{C}) = 2h_\tau(\mathbf{C})$. According to relative proportionality ([H98], Proposition 4.7.4) an orbital curve $\mathbf{C}$ is such that $h_e(\mathbf{C}) = 2h_\tau(\mathbf{C}) < 0$. Then only the solutions 26, 35, 38 and 47 can be hyperbolic weights of Apollonius configuration.

Until now we did not prove that the points K_1, K_2, K_3 are allowed to be considered as special points. For the corresponding orbital curves $\mathbf{E} = \mathbf{E}_K$, $K = K_i$, $i \in \{1,2,3\}$, on X' we have (see Figure 3)

$$
\begin{aligned}
h_e(\mathbf{E}_i) = u_e(\mathbf{E}_i) &= 2 - \Big(1 - \frac{1}{v_0}\Big) - \Big(1 - \frac{1}{v_i}\Big) - \Big(1 - \frac{1}{2}\Big)\\
h_\tau(\mathbf{E}_i) = -u_\tau(\mathbf{E}_i) &= -\frac{-1 + \frac{1}{2}}{k_i}\,, \qquad k_i < 0,\\
\lim_{w \longrightarrow \infty} h_\tau(\mathbf{E}_i) w = \lim_{w \longrightarrow \infty} \frac{1}{w}\Big(-1 + \frac{1}{2}\Big) w &= -\frac{1}{2} < 0\,, \qquad k_i = \infty
\end{aligned}
$$

The explicit calculation give

(15) $\qquad h_e, h_\tau$ type :

	$\mathbf{E}_1$	$\mathbf{E}_2$	$\mathbf{E}_3$
38 Apoll-0	1/6, -1/12 (2, 3, 3)	1/12, -1/24 (2, 3, 4)	1/12, -1/24 (2, 3, 4)
35 Apoll-1	1/6, -1/12 (2, 3, 3)	1/6, -1/12 (2, 3, 3)	0, -1/2 (2, 3, 6)
26 Apoll-2	1/3, -1/6 (2, 2, 3)	0, -1/2 (2, 3, 6)	0, -1/2 (2, 3, 6)
47 Apoll-3	0, -1/2 (2, 4, 4)	0, -1/2 (2, 4, 4)	0, -1/2 (2, 4, 4)

This table shows that Apollonius configuration have triple points of type $(2,2,3)$, $(2,3,3)$, $(2,3,4)$ and cusp points of type $(2,3,6)$ and $(2,4,4)$. The graphs of these special points appears in the graphical classification list in [H98], III, Figure 3.5.2 and 3.5.3. So we can change to the graph of special points $\mathbf{K}$, which looks like Figure 7.

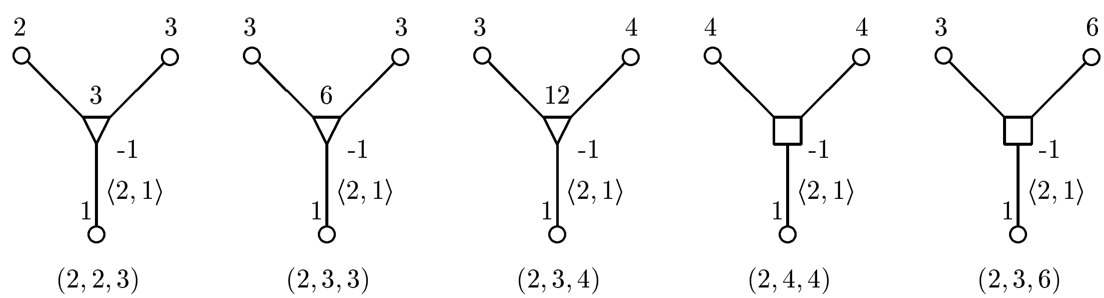


Figure 7: graphs of the special points $\mathbf{K}$

Now we calculate the *heights* of $\mathbf{X}$ using Proposition 4.10.2 in [H98], chapter IV, as definition. The local contributions appear in [H98], IV, Table 10.2, the global ones in (4.10.2), (4.10.3) there. Since the open surface X is smooth the formulas for the *Euler height* and the *signature height* simplify to

$$H_e(\mathbf{X}) = e(X') - \sum(1-\frac{1}{v_i})h_e(\mathbf{C}_i) - \sum h_e(\mathbf{P}_k) - \sum h_e(\mathbf{K}_m) \tag{16}$$

$$H_\tau(\mathbf{X}) = \tau(X') - \frac{1}{3}\sum(v_i-\frac{1}{v_i})h_\tau(\mathbf{C}_i) - \sum h_\tau(\mathbf{P}_k) - \sum h_\tau(\mathbf{K}_m) \tag{17}$$

with

$$\begin{aligned} e(X') &= \mathit{Euler\,number\,of\,} X' \\ &= e(\tilde{X}) - \#\{\mathit{components\,of\,} E(\tilde{X}\longrightarrow X')\} = e(\tilde{X})-3, \\ e(\tilde{X}) &= \sum(-1)^i \mathit{dim}\, H^i(\tilde{X},\mathbb{C}) = \mathit{Euler\,number\,of\,}\tilde{X} \end{aligned}$$

and

$$\begin{aligned} \tau(X') &= \tau(\tilde{X}) + \#\{\mathit{components\,of\,} E(\tilde{X}\longrightarrow X')\} = \tau(\tilde{X})+3, \\ \tau(\tilde{X}) &= \mathit{signature\,of\,}\tilde{X} = \mathit{signature\,of\,} H_2(\tilde{X},\mathbb{R}). \end{aligned}$$

The sums in (16), (17) run over all orbital curves $\mathbf{C}_i$, all abelian points $\mathbf{P}_k$ on $\mathbf{X}$ and all special points $\mathbf{K}_m$ (cusp or triple) on $\hat{\mathbf{X}}$. The point contributions can be read off from the molecular graph of the orbital cycle $\mathbf{Z}(\hat{\mathbf{X}})$ connecting the graphs of orbital curves and points as demonstrated in our example in Figure 4. Namely, for abelian points $\mathbf{P}$ we have

$$\begin{aligned} h_e(\mathbf{P}) &= 1-\frac{1}{vd}-\frac{1}{v'd}+\frac{1}{v'vd} \quad (\mathbf{P}:\ \underset{v}{\circ}\xrightarrow{\langle d,e\rangle}\underset{v'}{\circ} \text{ in general}) \\ &= 1-\frac{1}{v_i}-\frac{1}{v_j}+\frac{1}{v_iv_j} \quad \text{for our points } \mathbf{P}=\mathbf{C}_i\cap\mathbf{C}_j\,, \\ 3h_\tau(\mathbf{P}) &= Tr(P)+3l_P-\frac{e}{d}-\frac{e'}{d} \quad \text{(in general)} \\ &= 0+3\cdot 0-0-0 = 0 \text{ for our points } \mathbf{P}\,. \end{aligned} \tag{18}$$

Thereby l_P denotes the length of a resolution curve E_P (number of irreducible components of the linear tree E_P of rational curves) of the cyclic singularity P, $Tr(P)$ the trace of the intersection matrix of these components.

We define $h_e(\mathbf{K})$ and $h_\tau(\mathbf{K})$ for a special point $\mathbf{K}$ using the resolution orbital curve $\mathbf{E}_K$.

$$\begin{aligned} h_e(\mathbf{K}) &= 2 && \text{rational cusp point}\,, \\ h_e(\mathbf{K}) &= \Big(1-\frac{1}{2v}\Big)h_e(\mathbf{E}_K)+\sum_{i=1}^{3}h_e(\mathbf{T}_i) && \text{triple point}\,, \end{aligned} \tag{19}$$

where $\mathbf{T}_i$ are the abelian points of intersection between exception curve and orbital curves.

$$\begin{aligned} 3h_\tau(\mathbf{K}) &= Tr(\mathbf{K})+\sum_{j=1}^{4}(3l_j-\frac{e_j}{d_j}) && \text{cusp point}\,, \\ 3h_\tau(\mathbf{K}) &= \Big(v+\frac{2}{v}\Big)h_\tau(\mathbf{E}_K)+\sum_{i=1}^{3}h_\tau(\mathbf{T}_i)-\sum_{i=1}^{3}(v_i-\frac{1}{v_i})\frac{1}{v\,d_i} && \text{triple point}\,. \end{aligned} \tag{20}$$

$Tr(\mathbf{K})$ is the trace of the intersection matrix of $\tilde{E}_K$ being the preimage of $E_K\subset X'$ on $\tilde{X}$. The numbers l_j are the lengths of minimal resolutions of the cyclic surface singularities $T_j\in X'$ of type $\langle d_j,e_j\rangle$ sitting on E_K. The formulas (19) and (20) can be read from [H98], IV, Table 4.8.1.

We give explicitly the calculation for Apoll-2. For our main example Apoll-3 all calculation are written in [HPV]. By (18) it holds

$$h_e(\mathbf{P}_1) = 1-\frac{1}{6}-\frac{1}{6}+\frac{1}{6\cdot 6}=\frac{25}{36}\,,\qquad h_e(\mathbf{P}_2)=h_e(\mathbf{P}_3)=1-\frac{1}{2}-\frac{1}{6}+\frac{1}{2\cdot 6}=\frac{5}{12}\,.$$

Let T_1, T_2, T_3 be the cyclic singularities of $\mathbf{E}_1$, $T_1 \in C_0$, $T_2 \in C_1$, and T_3 is of type $\langle 2,1\rangle$ (see Figure 3 and 7 type $(2,2,3)$). Again by (18) we have

$$h_e(\mathbf{T}_1) = 1 - \frac{1}{3} - \frac{1}{3} + \frac{1}{3\cdot 3} = \frac{4}{9}\,, \qquad h_e(\mathbf{T}_2) = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{2\cdot 3} = \frac{1}{3}\,,$$
$$h_e(\mathbf{T}_3) = 1 - \frac{1}{2\cdot 3} - \frac{1}{2\cdot 1} + \frac{1}{2\cdot 3\cdot 1} = \frac{1}{2}\,,$$
$$h_\tau(\mathbf{T}_1) = h_\tau(\mathbf{T}_2) = 0\,, \qquad h_\tau(\mathbf{T}_3) = \frac{1}{3}\Big(-2 + 3\cdot 1 - \frac{1}{2} - \frac{1}{2}\Big) = 0\,.$$

We obtain the Euler and signature heights of the special points $\mathbf{K}$ using (19) and (20)

$$h_e(\mathbf{K}_2) = h_e(\mathbf{K}_3) = 2\,, \qquad h_e(\mathbf{K}_1) = \Big(1 - \frac{1}{2\cdot 3}\Big)\frac{1}{3} + \frac{4}{9} + \frac{1}{3} + \frac{1}{2} = \frac{14}{9}\,,$$
$$\text{(21)} \qquad h_\tau(\mathbf{K}_2) = h_\tau(\mathbf{K}_3) = \frac{1}{3}\Big(-3 + 3\cdot 1 - \frac{1}{2}\Big) = -\frac{1}{6}\,,$$
$$h_\tau(\mathbf{K}_1) = \frac{1}{3}\left[\Big(3 + \frac{2}{3}\Big)\frac{-1}{6} + 3\cdot 0 - \Big(3 - \frac{1}{3}\Big)\frac{1}{3} - \Big(2 - \frac{1}{2}\Big)\frac{1}{3} - \Big(1 - \frac{1}{1}\Big)\frac{1}{3\cdot 2}\right] = \frac{-2}{3}\,.$$

Knowing $\hat{X} = \mathbb{P}^2$ we get $e(\hat{X}) = 3$, $\tau(\hat{X}) = 1$, hence

$$e(\tilde{X}) = 3 + 6 = 9 \quad , \quad \tau(\tilde{X}) = 1 - 6 = -5$$

and

$$\text{(22)} \qquad e(X') = 9 - 3 = 6 \quad , \quad \tau(X') = -5 + 3 = -2.$$

Now we are able to calculate the heights of $\mathbf{X}$ explicitly substituting the local heights (12), (13), (18), (21) and $e(X')$, $\tau(X')$ into (16), (17), respectively:

$$\text{(23)} \qquad \begin{aligned} H_e(\mathbf{X}) &= 6 - \Big(1 - \frac{1}{3}\Big)\frac{-2}{3} - \Big(1 - \frac{1}{2}\Big)\frac{-1}{3} - 2\Big(1 - \frac{1}{6}\Big)\frac{-1}{3} - \frac{25}{36} - 2\cdot\frac{5}{12} - 2\cdot 2 - \frac{14}{9} = \frac{1}{12},\\ H_\tau(\mathbf{X}) &= -2 - \frac{1}{3}\left[(3 - \frac{1}{3})\frac{-1}{3} + (2 - \frac{1}{2})\frac{-1}{6} + 2(6 - \frac{1}{6})\frac{-1}{6}\right] - 3\cdot 0 - 2\frac{-1}{6} - \frac{-2}{3} = \frac{1}{36}\,. \end{aligned}$$

For other Apoll-k we give the heights of the special points $\mathbf{K}$ and $\mathbf{X}$ in the table

(24)

	$\mathbf{K}_1$	$\mathbf{K}_2$	$\mathbf{K}_3$	$\mathbf{X}$
Apoll-0	127/72, -17/36	541/288, -25/72	541/288, -25/72	1/12, 1/36
Apoll-1	127/72, -17/36	127/72, -17/36	2, -1/6	1/12, 1/36
Apoll-2	14/9, -2/3	2, -1/6	2, -1/6	1/12, 1/36
Apoll-3	2, -1/6	2, -1/6	2, -1/6	3/16, 1/16

Summarizing (15), (14) and (24) we proved in this section the following

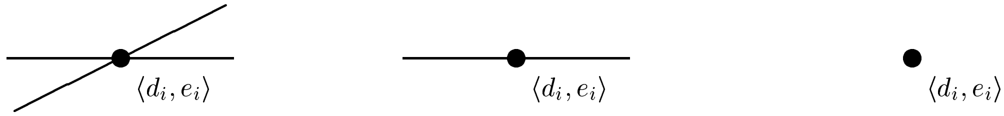
Proposition 2.3. *The orbital surface $\hat{\mathbf{X}}$ of (1) with $\hat{X} = \mathbb{P}^2$ and orbital locus (2) supported by any Apollonius configuration 1.1, (ii) a),b),c),d) satisfies the proportionality conditions for ball quotient surfaces described in [H98] (IV.9, Theorem 4.9.2):*

$$\text{(25)} \qquad \begin{array}{lr} h_e(\mathbf{E}_K) = 0\ ,\ h_\tau(\mathbf{E}_K) < 0\ ,\ \mathbf{K}\ -\ \textit{cusp point;} & (\textit{Prop}\ \infty)\\ h_e(\mathbf{E}_K) = -2h_\tau(\mathbf{E}_K) > 0\ ,\ \mathbf{K}\ -\ \textit{triple point;} & (\textit{Prop}\ 0)\\ h_e(\mathbf{C}_i) = 2h_\tau(\mathbf{C}_i) < 0\ ,\ i = 0,1,2,3; & (\textit{Prop}\ 1)\\ H_e(\mathbf{X}) = 3H_\tau(\mathbf{X}) > 0\,; & (\textit{Prop}\ \ 2) \end{array}$$

only in four cases Apoll-k, $k = 0,1,2,3$. The corresponding weights are given in (14) solutions 26, 35, 38 and 47. □

The universal heights introduced in (6) and (7) are an easy way to find all orbital hyperbolic weights for a given curves configuration (compare [H98], IV, Effective Finiteness Theorem 4.10.3). In this section we have presented the following algorithm:

Step 1. Resolve all special points and obtain model $\mathbf{X}' \longrightarrow \hat{\mathbf{X}} \longrightarrow \mathbf{X}$ (for Apollonius that is Figure 2) on which all orbital points are abelian or separate and the corresponding graphs look like



On $\mathbf{X}'$ we know the Euler number and selfintersection on each curve and also type of possible cyclic singularities $\langle d_i, e_i\rangle$. Now write $u_e(\mathbf{C}) = 2u_\tau(\mathbf{C})$ for each curve $\mathbf{C}$ on $\mathbf{X}'$. This is a system of diophantine equations (compare with (10) and (11)) with variables v_i–the weight of the orbital curve $\mathbf{C}_i$.

Step 2. Solve the system of diophantine equations in $\mathbb{Z}\backslash 0 \cup \infty$.

Step 3. Check every solution obtained in Step 2. For a solution one have to see:

1. if $v_i = 1$ then $\mathbf{C}_i$ is not "pure" orbital curve ($p:\ \mathbb{B} \longrightarrow \mathbb{B}/\Gamma$ is not ramified over $\mathbf{C}_i$). An example is solution 18 for Apollonius configuration.
2. if $v_i, v_j \in \mathbb{Z}_- \cup \infty$ then C_i and C_j does not intersect on X'.
3. if $v_i < 0$ or $v_i = \infty$ then $\mathbf{C}_i$ is triple or cusp point resolution curve and one apply Remark 2.1.
4. if $v_i > 1$ then $\mathbf{C}_i$ is a pure orbital curve. We check if $h_e(\mathbf{C}_i) = 2h_\tau(\mathbf{C}_i) < 0$.
5. calculate and see if $H_e(\mathbf{X}) = 3H_\tau(\mathbf{X}) > 0$.

If the above holds for any solution of our diophantine system, then this solutions satisfy the proportionality conditions ([H98], IV, Theorem 4.9.2) and is hyperbolic.

At this place we define similar to (6) and (7) "universal heights" for orbital surface $\mathbf{X}'$ and abelian points $\mathbf{P} \in X'$. The universal height $u(\mathbf{P})$ for abelian point $\mathbf{P}$ is the same as $h(\mathbf{P})$ in (18) but $v, v' \in \mathbb{Z}\backslash 0 \cup \infty$. Let us remember that $\mathbf{X}'$ has only abelian points. The universal heights of $\mathbf{X}$ are

$$U_e(\mathbf{X}) = e(X') - \sum (1 - \frac{1}{v_i}) u_e(\mathbf{C}_i) - \sum u_e(\mathbf{P}_k)\,,$$
$$U_\tau(\mathbf{X}) = \tau(X') - \frac{1}{3} \sum (v_i - \frac{1}{v_i}) u_\tau(\mathbf{C}_i) - \sum u_\tau(\mathbf{P}_k)\,,$$

where the fist sum runs over all curves on X' and the second over all abelian points on X'. We assume that **Step 3** condition 2 is satisfied. Then there exist connection between heights

$$H_e(\mathbf{X}) = U_e(\mathbf{X}) - 3 \sum_{v_k<0} \frac{1}{2v_k} u_e(\mathbf{E}_k)\,, \qquad H_\tau(\mathbf{X}) = U_\tau(\mathbf{X}) - \sum_{v_k<0} \frac{1}{v_k} u_\tau(\mathbf{E}_k)\,. \tag{26}$$

Using proportionality $u_e(\mathbf{E}_k) = 2u_\tau(\mathbf{E}_k)$ we have

$$H_e(\mathbf{X}) = 3H_\tau(\mathbf{X}) \quad \longleftrightarrow \quad U_e(\mathbf{X}) = 3U_\tau(\mathbf{X})\,.$$

For Apollonius configuration the universal heights are

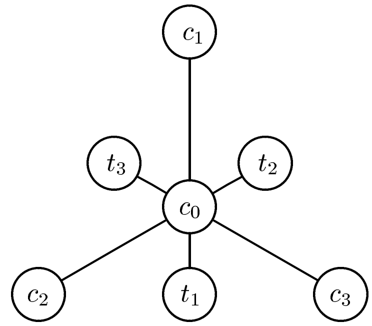
$$U_e(\mathbf{X}) = 1 - \frac{1}{v_0} + \frac{1}{v_1 v_2} + \frac{1}{v_2 v_3} + \frac{1}{v_3 v_1} + \sum_{i=1}^{3} \Big(-\frac{1}{v_i} - \frac{1}{2k_i} + \frac{1}{v_i k_i} + \frac{1}{v_0 k_i} \Big)\,,$$
$$U_\tau(\mathbf{X}) = \frac{1}{6} - \frac{2}{3v_0^2} - \sum_{i=1}^{3} \Big(\frac{1}{3v_i^2} + \frac{1}{6k_i^2} \Big)\,.$$

Remark 2.4. *It is not difficult to see that from (10) follows* $U_e(\mathbf{X}) = 3U_\tau(\mathbf{X}) \longleftrightarrow H_e(\mathbf{X}) = 3H_\tau(\mathbf{X})$.

Now we are ready to consider any of our 58 solution connected with Apollonius configuration. The solutions which don't satisfy **Step 3**–2 are denoted with $\times$, these which don't satisfy **Step 3**–4 are denoted with $\Box$. Only 41 solutions are hyperbolic and $H_e(\mathbf{X})$ for them is given in the table. The calculations are made by using (26). From these 41 hyperbolic solutions four are such that $v_i > 1,\ i = 0, 1, 2, 3,\ \ k_i < 0$ or $k_i = \infty,\ \ i = 1, 2, 3$, and they are 26, 35, 38, 47.

Until now it is not generally known that the four proportionality conditions (25) are sufficient for $\hat{\mathbf{X}}$ to be a ball quotient. In second part we prove it for our special plane orbital Apoll-3 surface solution 47. This will be prepared in section 8 translating precise heights and local conditions to geometric lattice conditions on the ball. For this purpose one has to read backwards the proof of the Proportionality Theorem 4.9.2 in [H98], well-prepared in the book parts before. In the section after we find an arithmetic ball lattice satisfying all these conditions.

We give a geometric interpretation of Table 1.



Problem. Find all magic triangles with elements $p/q \in \mathbb{Q}$, $p = 0$ or 1 such that:

$$-c_0 + c_1 + c_2 + c_3 = \frac{1}{2}$$
$$c_0 + c_i + t_i = \frac{1}{2}\,, \qquad i = 1, 2, 3\,.$$

Answer: $c_i = \dfrac{1}{v_i}$, $\quad t_i = \dfrac{1}{k_i}$ (see Table 1).

At the end of this section we give a table with the introduced new orbital heights on model $\mathbf{X}'$.

	e	τ
$u(\mathbf{P})$	$1 - \frac{1}{vd} - \frac{1}{v'd} + \frac{1}{v'vd}$ $\langle d, e\rangle$ v v'	$\frac{1}{3}\left(Tr(P) + 3\,l_P - \frac{e}{d} - \frac{e'}{d}\right)$ $\bullet \to$
$u(\mathbf{C})$	$e(C') - \sum\left(1 - \frac{1}{v_k d_k}\right)$ C C_k	$\frac{1}{v}\left((C'^2) + \sum \frac{e_k}{d_k}\right)$ C C_k
$U(\mathbf{X})$	$e(X') - \sum\left(1 - \frac{1}{v_i}\right)u_e(\mathbf{C}_i) - \sum u_e(\mathbf{P}_k)$	$\tau(X') - \frac{1}{3}\sum\left(v_i - \frac{1}{v_i}\right)u_\tau(\mathbf{C}_i) - \sum u_\tau(\mathbf{P}_k)$
$h(\mathbf{C})$	$u_e(\mathbf{C}) - 2\sum_{v_k<0} \frac{1}{v_k d_k}$ C C_k	$u_\tau(\mathbf{C}) - \sum_{v_k<0} \frac{1}{v_k d_k}$ C C_k
$H(\mathbf{X})$	$U_e(\mathbf{X}) - 3\sum_{v_k<0} \frac{1}{2v_k} u_e(\mathbf{E}_k)$	$U_\tau(\mathbf{X}) - \sum_{v_k<0} \frac{1}{v_k} u_\tau(\mathbf{E}_k)$

3 Monodromy

Let $\mu := (\mu_0, \dots, \mu_4)$ be rational numbers satisfying

(27) $$0 < \mu_k < 1\,, \qquad \sum_{i=0}^{4} \mu_k = 2$$

and consider families of plane curves

(28) $$w^d = u^{m_0}(u-1)^{m_1}(u-x)^{m_2}(u-y)^{m_3}\,, \qquad (x, y) \in \mathbb{C}^2$$

where
$$\mu_0 = \frac{\alpha_0}{d}, \ \cdots, \ \mu_4 = \frac{\alpha_4}{d}$$
and d is the least common multiple of denominators of the μ_k. Shortly we denote such a sequence with $[d;\ m_0, m_1, m_2, m_3, m_4] \longleftarrow \mu$. We suppose (x, y) is pair of parameters running though

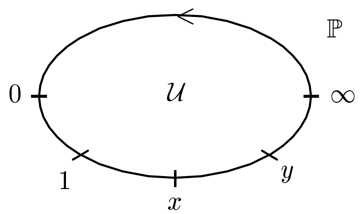
$$\Lambda := \{(x, y) \in \mathbb{C}^2 \mid xy(x-1)(y-1)(x-y) \neq 0\}.$$

Under this condition (w, u)-curve (28) is Riemann surface R and the projection $p: \ (w, u) \longrightarrow u$ is d-cover of $\mathbb{P}^1$ ramified in $0, 1, x, y, \infty$. We denote $0, 1, x, y, \infty$ with u_k, $k = 0, 1, 2, 3, 4$ respectively.

If $(d, m_k) > 1$ then R is singular surface at $(p^{-1}(u_k), u_k)$. Let R' is non-singular model of R obtained by blowing up the singularities of R. The geometric genus g of R' by Riemann-Hurwitz formula is

$$2 - 2g = d \cdot 2 - \sum_{k=0}^{4} (m_k - r_k), \qquad r_k = (d, m_k). \tag{29}$$

The aim of this section is to study the (x, y) moduli (28) and to connect to each solution of Table 1 six numbers $[d;\ m_0, m_1, m_2, m_3, m_4]$.



Let (x, y) are fixed and $\mathcal{U}$ is an simply connected domain on the Riemann sphere and the ordered points $\{u_0, u_1, u_2, u_3, u_4\} \in \partial\mathcal{U}$. We suppose $\partial\mathcal{U}$ have positive orientation with respect to $\mathcal{U}$. The map $p: \ (w, u) \longrightarrow u$ is d-cover of $\mathbb{P}$ and by (27) one can chose branch w such that $w = \varphi(u) := u^{\mu_0}(u-1)^{\mu_1}(u-x)^{\mu_2}(u-y)^{\mu_3}$ is well defined single valued holomorphic function when $u \in \mathcal{U}$.
We consider integrals

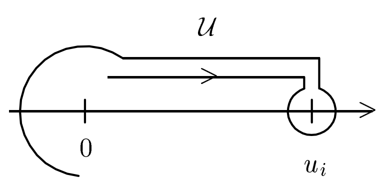
$$I_k := c_k \int_0^{u_k} \frac{du}{\varphi(u)} = c_k \int_0^{u_k} \frac{du}{w}, \qquad k = 1, 2, 3, \tag{30}$$

where the oriented path of integration $(0, u_k) \subset \mathcal{U}$, $c_k = 1 - \exp(-2\pi i \mu_k)$. The integrals I_k do not depend from the path of integration because du/w is holomorphic differential in $\mathcal{U}$.

Until now $(x, y) =: (x_0, y_0)$ are fixed. For general $(x, y) \in \Lambda$ we obtain the domain $\mathcal{U}(x, y)$ taking a path s joining (x_0, y_0) and (x, y) and define $\mathcal{U}(x, y)$ by continuation of $\mathcal{U}(x_0, y_0)$ along s; it is possible since the family (28) is locally trivial fiber space over Λ if μ is fixed. Notice that this choice of $\mathcal{U}$ depends from the path s.

We assume that $(d, m_0) = 1$ and let ϕ be the projective map

$$\phi: \ \Lambda \longrightarrow \mathbb{P}^2, \qquad \phi(x, y) := (I_1 : I_2 : I_3),$$



We will connect the map ϕ with periods of Jacobian variety $J(R')$. Let consider the homology group $H_1(R', \mathbb{Z})$ and let σ be the automorphism of R' defined by $\sigma(w, u) := (\varepsilon w, u)$, $\varepsilon = \exp(2\pi i/d)$.
We take three cycles $A_k \in H_1(R', \mathbb{Z})$, $k = 1, 2, 3$. They start from a point in $\mathcal{U}$ near to 0 go near to u_k, make a positive loop around u_k and then come back to starting point. The definition of these cycles[2] is correct since $(d, m_0) = 1$. There is a relation between periods of the holomorphic differential du/w on cycles A_k and (30)

$$\int_{A_k} \frac{du}{w} = \big(1 - \exp(-2\pi i \mu_k)\big) \int_0^{u_k} \frac{du}{w} = I_k. \tag{31}$$

Proposition 3.1. *For each $\gamma \in H_1(R', \mathbb{Z})$ there exist $c_k \in \mathbb{Z}[\varepsilon]$ such that*

$$\int_\gamma \frac{du}{w} = c_1 I_1 + c_2 I_2 + c_3 I_3.$$

[2]If $(d, m_0) > 1$ one cannot take cycles A_k as described above. In this case Pochhammer cycles on Jacobian variety work fine and then we change c_k to $c_k c_0 = c_k(1 - \exp(-2\pi i \mu_0))$.

Proof. The automorphism σ acts on each $\delta \in H_1(R',\mathbb{Z})$ and let denote the lifting by $\sigma(\delta)$. It is easy to see that the cycles $\sigma^k(A_j)$, $k = 0,\dots,d-1$, $j = 1,2,3$, generate $H_1(R',\mathbb{Z})$. On the other side

$$\int_{\sigma^k(A_j)} \frac{du}{w} = \int_{A_j} \frac{du}{\sigma^k(w)} = \int_{A_j} \frac{du}{\varepsilon^k w} = \varepsilon^{d-k} \int_{A_j} \frac{du}{w}\,.$$

Now by (31) the proof is complete. □

Let for general $(x,y) \in \Lambda$ cycles $A_k(x,y)$, $k = 1,2,3$, and basis of $H_1(R'(x,y),\mathbb{Z})$ are defined as continuation on the same path s as $\mathcal{U}(x,y)$. Any element δ of $\pi_1(\Lambda\,,(x,y))$ induces an automorphism of $H_1(R',\mathbb{Z})$.

Definition 3.2. *Let $\delta \in \pi_1(\Lambda\,,(x,y))$. By proposition 3.1 the periods I_1, I_2, I_3 are transformed as*

$$^t(I_1,I_2,I_3) \longrightarrow g(\delta)\,^t(I_1,I_2,I_3)\,, \qquad g(\delta) \in GL(3,\mathbb{Z}[\varepsilon])\,.$$

Picard modular group for μ and ϕ is

$$\Gamma(\mu) := \left\{ g(\delta) \in GL(3,\mathbb{Z}[\varepsilon]) \mid \delta \in \pi_1(\Lambda\,,(x,y)) \right\}.$$

The map ϕ is called developing map.

Remark 3.3. *The integrals I_k are three linearly independent solutions of the system of Fuchsian partial differential equations for Appell hypergeometric function F_1 (see [T]). The group $\Gamma(\mu)$ is also the monodromy group of these solutions.*

There are many papers about the group $\Gamma(\mu)$ (see Picard [P], Shimura [Sm64], Terada [T], Shiga [Shg], Deligne and Mostow [DM1, DM2], Yoshida [Y97]). Here we presented briefly $\Gamma(\mu)$. For some μ, the closure of the image of the map ϕ is projectively equivalent to the unit two dimensional complex ball and this map gives an isomorphism of $\mathbb{C}^2\backslash\Lambda$ into $\mathbb{B}_2/\Gamma(\mu)$.

We can take generators $\{\delta_s\}$ of $\pi_1(\Lambda,(x,y))$ and then $\{g(\delta_s)\}$ are generators for $\Gamma(\mu)$ (see [Mat]). Here we give different presentation similar to [T], [Y97].

Let us consider five different ordered points u_k on Riemann sphere $\mathbb{P}^1$. We assume u_k are fixed and the domain $\mathcal{U}$ and holomorphic differential du/w, $u \in \mathcal{U}$, are as above. The following 10 paths A_{ij}, $0 \le i < j \le 4$, are similar to cycles A_k of Jacobian variety. Namely A_{ij} is closed path starting from u_i, go near to u_j inside $\mathcal{U}$, make a positive loop around u_j and then come back to u_i inside $\mathcal{U}$. On the same way as we defined $\mathcal{U}$ for general (x,y) we define action of A_{ij} on I_k by continuation.

Definition-Proposition 3.4. *The action of A_{ij}, $0 \le i < j \le 3$, on I_k are generators of Picard modular group $\Gamma(\mu)$.*

Next definitions are from [DM2]. Let $S := \{0,1,2,3,4\}$. We say μ satisfies condition INT iff for all $s,t \in S$ with $\mu_s + \mu_t < 1$ and $s \ne t$,

$$\lambda_{st} := (1-\mu_s-\mu_t)^{-1} \in \mathbb{Z}\,. \tag{32}$$

We say that μ satisfies condition $\Sigma INT(S_1)$ iff $S_1 \subset S$, $\mu_s = \mu_t$ for all $s,t \in S_1$ and for all $s,t \in S$ with $s \ne t$ and $\mu_s + \mu_t < 1$,

$$\lambda_{st} := (1-\mu_s-\mu_t)^{-1} \in \begin{cases} \frac{1}{2}\mathbb{Z} & \text{if } s,t \in S_1 \\ \mathbb{Z} & \text{otherwise.} \end{cases} \tag{33}$$

We say that μ satisfies ΣINT if μ satisfies $\Sigma INT(S_1)$ for some $S_1 \subset S$.

In [DM1] is proved that $\Gamma(\mu)$ is a lattice in $\mathbb{PU}(2,1)$ if μ satisfies condition INT. In [M1] this hypothesis is weakened to condition ΣINT. List of all μ satisfying condition INT and ΣINT is given in [M2]. When $\mu_s + \mu_t = 1$ we substitute $\lambda_{st} = \infty$. Using list of [M2] one see that when $\mu_s + \mu_t > 1$ then $(1-\mu_s-\mu_t)^{-1} \in \mathbb{Z}$. So λ_{st} is well defined in all cases.

Theorem 3.5 ([BHH], page 197). *The diophantine equations (11) and (32) have the same solutions in the sets $\mathbb{Z}\backslash 0 \cup \infty$ and $\sum_{j=0}^4 \mu_j$ respectively. More precisely*

$$\lambda_{st} = l_{st}\,, \qquad 0 \le s < t \le 4\,,$$

If μ satisfy $0 < \mu_s < 1$ for $0 \le s \le 4$ then μ is hyperbolic solution. There are only 27 such solutions.

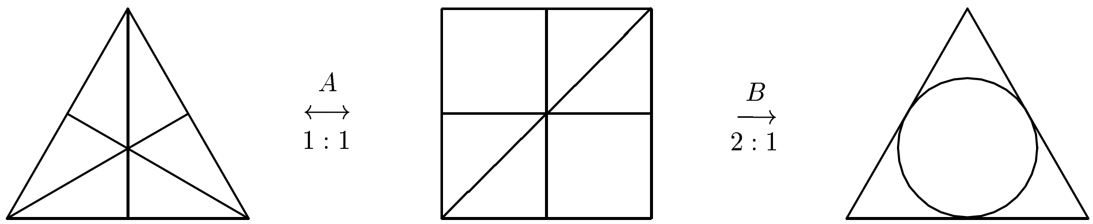
We will connect Figure 6, Λ and Figure 1. These are spaces with configuration divisors

$$\begin{array}{rcl} \mathbb{P}^2 & \longrightarrow & xyz(x-y)(x-z)(y-z)=0 \\ \mathbb{P}^1\times\mathbb{P}^1 & \longrightarrow & p_0p_1q_0q_1(p_0-p_1)(q_0-q_1)(p_0q_1-p_1q_0)=0 \\ \mathbb{P}^2 & \longrightarrow & p^2+q^2+r^2-2pq-2pr-2qr=0 \end{array}$$

respectively. Between them the rational maps

$$(34) \qquad A: \left| \begin{array}{l} \dfrac{x}{z}=\dfrac{p_0}{p_1} \\[2mm] \dfrac{y}{z}=\dfrac{q_0}{q_1}\,, \end{array}\right. \qquad\qquad B: \left| \begin{array}{l} \dfrac{p}{r}=\dfrac{p_0}{p_1}\cdot\dfrac{q_0}{q_1} \\[2mm] \dfrac{q}{r}=\Big(1-\dfrac{p_0}{p_1}\Big)\Big(1-\dfrac{q_0}{q_1}\Big) \end{array}\right.$$

are defined. Notice that B is 2-cover ramified above the quadric $(p+q-r)^2-4pq$.



$(x:y:z)\in\mathbb{P}^2$ $\qquad$ $(p_0:p_1;q_0:q_1)\in\mathbb{P}^1\times\mathbb{P}^1$ $\qquad$ $(p:q:r)\in\mathbb{P}^2$

We blow up the four points $P_k\in\mathbb{P}^2$ and $(0,0)$, $(1,1)$, $(\infty,\infty)\in\mathbb{P}^1\times\mathbb{P}^1$ and obtain the same model. We denote this surface with Y'. Let denote the 7 lines of $\mathbb{P}^1\times\mathbb{P}^1$ by X_0,X_1,X_∞, Y_0,Y_1,Y_∞, D_{xy} and special points by Q_0,Q_1,Q_∞. So we obtain for the corresponding weights

$$(35) \qquad \begin{array}{lllll} x_0=l_{34}\,, & x_1=l_{23}\,, & x_\infty=l_{03}\,, & y_0=l_{14}\,, & y_1=l_{12}\,, \\ y_\infty=l_{01}\,, & q_0=l_{02}\,, & q_1=l_{04}\,, & q_\infty=l_{24}\,, & d_{xy}=l_{13}\,. \end{array}$$

The symmetric group $S_5=S\{0,1,2,3,4\}$ acts on the curves L_{ij} of Y' an on μ_k by changing the indexes. This action lifts on the weights x_k,y_k,q_k,d_{xy}, $k=0,1,\infty$, in obvious manner. When $\mu_1=\mu_3$ then $x_k=y_k$ for $k=0,1,\infty$ and we have

$$(36) \qquad \begin{array}{llll} \lambda_{34}=x_0=v_3\,, & \lambda_{23}=x_1=v_1\,, & \lambda_{03}=x_\infty=v_2\,, & 2\lambda_{13}=2d_{xy}=v_0\,, \\ \lambda_{02}=q_0=k_3\,, & \lambda_{04}=q_1=k_1\,, & \lambda_{24}=q_\infty=k_2\,. & \end{array}$$

By theorem 3.5 and from the table [BHH] page 199 one see that for each solution of (11) there exist s and t with $\mu_s=\mu_t$. Since S_5 acts on μ we can permute the indexes so that $(s,t)\longrightarrow(1,3)$. Sometimes there are many such possibilities. Examples are $[4;1,2,1,2,2]$, $[4;2,1,2,1,2]$, $[4;1,2,2,2,1]$.

We have obtained:

Theorem 3.6. *i) If one apply the algorithm for finding proportional weights for $\mathbb{P}^2$ minus 6 lines and $\mathbb{P}^1\times\mathbb{P}^1$ minus seven lines as described at the end of section 2 then Prop. 1 for both models are diophantine equations (11). Both weights are connected with equations (35).*

ii) Diophantine equations (11) and (32) are equivalent.

iii) When $\mu_s=\mu_t$ after permuting the indexes $(s,t)\longrightarrow(1,3)$ and using (36) one obtain a solution of (10).

If we look at Table 1 we see that there are solutions with v_0 odd (examples are Apoll-k, $k=0,1,2$). They come from "pure" ΣINT solutions μ of (33) with $S_1=\{1,3\}$. In this case $2\lambda_{13}=2d_{xy}=v_0$ is odd number. At the end we give a table with corresponding μ's for Table 1.

The last two column of Table 2 are the number in [BHH] and [M2]. Looking at tables 1 and 2 we see

Theorem 3.7. *i) If v_s,k_s,v_0 is solution of (10) then by (36) one obtain μ which satisfy INT or ΣINT condition. For INT μ the last two column contain numbers and for ΣINT μ only the last column contain a number (see Table 2).*

ii) Any solution of (10) is hyperbolic iff the corresponding μ is such that $0<\mu_s<1$ for $0\le s\le 4$.

iii) If μ satisfy "pure" ΣINT condition with $\#S_1>2$ then this μ is not solution of (10).

	d	m_0	m_1	m_2	m_3	m_4	BHH	M2
1	18	1	8	11	8	8	16	79
2	12	3	3	10	3	5	14	68
3	12	1	5	8	5	5	12	62
4	12	3	5	8	5	3	12	70
5	10	1	4	7	4	4	6	57
6	8	1	3	6	3	3	4	53
7	6	2	5	2	5	-2		
8	6	2	1	6	1	2		
9	6	1	2	5	2	2	25	49
10	4	1	1	4	1	1	31	
11	4	0	3	2	3	0	30	
12	3	2	2	2	2	-2	37	
13	6	-1	4	4	4	1	38	
14	3	0	2	2	2	0	29	
15	2	-1	1	2	1	1	36	
16	2	0	1	2	1	0	34	
17	d	d-1	0	0	0	d+1	28	
18	2	2	-1	2	-1	2		
19	2	1	0	1	0	2	34	
20	6	4	0	3	0	5	32	
21	6	3	1	2	1	5		46
22	4	3	0	2	0	3	30	
23	12	7	2	4	2	9		65
24	4	2	1	1	1	3	21	42
25	20	11	5	5	5	14	17	85
26	6	4	1	2	1	4		47
27	12	7	3	3	3	8	13	69
28	6	3	2	1	2	4	24	50
29	8	5	2	2	2	5	3	54
30	10	6	3	2	3	6		59
31	12	7	4	2	4	7	10	67
32	18	10	7	2	7	10		81
33	2	1	1	0	1	1	35	
34	3	2	0	2	0	2	29	
35	6	3	1	3	1	4		48
36	12	5	3	5	3	8	11	70
37	3	1	1	1	1	2	19	41
38	12	7	2	6	2	7		66
39	12	6	3	5	3	7	27	71
40	12	5	4	4	4	7	26	73
41	24	9	9	7	9	14	18	89
42	15	6	6	4	6	8	15	78
43	6	3	2	2	2	3	22	52
44	12	5	5	3	5	6	9	72
45	6	2	3	1	3	3	23	51
46	9	4	4	2	4	4	5	56
47	4	2	1	2	1	2	20	43
48	20	9	6	9	6	10		87
49	12	5	4	5	4	6	8	74
50	8	3	3	3	3	4	2	55
51	12	4	5	4	5	6	8	74
52	4	1	2	1	2	2	20	43
53	12	5	5	4	5	5	7	75
54	12	4	7	2	7	4	10	67
55	5	2	2	2	2	2	1	44
56	6	2	3	2	3	2	22	52
57	8	2	5	2	5	2	3	54
58	1	0	1	0	1	0		

Table 2: correspondence $\mu \longleftrightarrow$ equations (10)

4 Matrix representation

Let us remember diophantine equations (11) with unknown variables l_{ij}. If μ satisfy INT or ΣINT then $l_{ij} := \lambda_{ij} \in \frac{1}{2}\mathbb{Z}$ is a solution of (11). Notice that some l_{ij} can be "half" integer. The symmetric group S_5 acts on l_{ij} and on this way on the corresponding curve (28) permuting m_s. We apply a permutation on each curve on our Apoll-k curves and calculate the genus by (29) and the result is:

(37)

	permutation	d	m_0	m_1	m_2	m_3	m_4	ΣINT	genus	symmetry
38 A-0	(01324)	12	7	7	2	2	6	{2, 3}	13	$Z_2 \times Z_2$
35 A-1	(03)	6	1	1	3	3	4	{0, 1}	5	$Z_2 \times Z_2$
26 A-2	(03)(24)	6	1	1	4	4	2	{0, 1}	6	$Z_2 \times Z_2$
47 A-3	(03)(14)	4	1	2	2	2	1		3	$Z_2 \times S_3$

Here we give generators for monodromy groups of our Apoll-k. One can use [T] formulas or calculate them explicitly using proposition 3.4. We denote them with M_{st}, $q_k = \exp(-2\pi i \mu_k)$.

$$M_{01} = \begin{bmatrix} q_0 q_1 & 0 & 0 \\ q_0 q_2 - q_0 & 1 & 0 \\ q_0 q_3 - q_0 & 0 & 1 \end{bmatrix} \quad M_{13} = \begin{bmatrix} 1 - q_1 + q_1 q_3 & 0 & q_1 - q_1^2 \\ (1-q_2)(1-q_3) & 1 & (q_1 - 1)(1 - q_2) \\ 1 - q_3 & 0 & q_1 \end{bmatrix} \quad M_{03} = \begin{bmatrix} 1 & 0 & q_1 - 1 \\ 0 & 1 & q_2 - 1 \\ 0 & 0 & q_0 q_3 \end{bmatrix}$$

$$M_{12} = \begin{bmatrix} 1 - q_1 + q_1 q_2 & q_1 - q_1^2 & 0 \\ 1 - q_2 & q_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad M_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 - q_2 + q_3 q_2 & q_2 - q_2^2 \\ 0 & 1 - q_3 & q_2 \end{bmatrix} \quad M_{02} = \begin{bmatrix} 1 & q_1 - 1 & 0 \\ 0 & q_0 q_2 & 0 \\ 0 & q_0 q_3 - q_0 & 1 \end{bmatrix}$$

The projective multiplicative group generated by M_{st} is $\Gamma(\mu)$.

Theorem 4.1 (Terada [T], Deligne and Mostow [DM1]). *Up to multiplicative real constant there exist unique hermitian matrix*[3]

$$H = \begin{bmatrix} (\overline{q} - q_1 q)/(1-q_1) & q_1 q & q_1 q_2 q \\ \overline{q_1 q} & (\overline{q} - q_2 q)/(1-q_2) & q_2 q \\ \overline{q_1 q_2 q} & \overline{q_2 q} & (\overline{q} - q_3 q)/(1-q_3) \end{bmatrix}$$

satisfying ${}^t\overline{M} H M = H$ *for all* $M \in \Gamma(\mu)$*, where* $q = -\sqrt{\mu_4} = -\exp(-\pi i \mu_4)$*. The matrix* H *has signature* $(1,1,-1)$ *and define the projective unit ball* $\mathbb{B}_H := \{w \in \mathbb{P}^2 \mid \overline{w} H\, {}^t w < 0\}$*. The image* $\phi(\Lambda)$ *of developing map is dense subset of* $\mathbb{B}_H$*.*

Definition 4.2. *Arithmetic group for* μ *is* $A(\mu) := PU\big(\mathbb{Z}[\varepsilon], H\big) = \{Q \in GL(3, \mathbb{Z}[\varepsilon]) \mid {}^t\overline{Q} H Q = H\}$*. Any normal subgroup of* $A(\mu)$ *of finite index is called arithmetic group for hermitian form* H *and* $\mathbb{Z}[\varepsilon]$*.*

It is obvious from the generators of $\Gamma(\mu)$ that it is subgroup of $A(\mu)$. Is it true that $\Gamma(\mu)$ is arithmetic group connected with hermitian form H and $\mathbb{Z}[\varepsilon]$?

Proposition 4.3 (Mostow [M2] page 582, [DM1] page 76). *Let* μ *satisfies* ΣINT *condition. Then* $\Gamma(\mu)$ *is arithmetic iff for each integer* s *relative prime to* d *with* $1 < s < d-1$ *then* $\sum \langle s\mu_j \rangle = 1$ *or* 4*, where* $\langle b \rangle$ *denote fractional part of* b*.*

Looking at (37) and after some calculations we see that all Apoll-k are arithmetic groups. There is an extension of Picard modular group which is equal to arithmetic group $A(\mu)$. We need some definition more.

From (37) it follows that the curves Apoll-k for different $(x,y) \in \Lambda$ can be isomorphic as Jacobian varieties. For Apoll-k we define automorphisms of Λ by

$$a_{01}: \ (x,y) \longrightarrow (1-x, 1-y)\,, \qquad a_{23}: \ (x,y) \longrightarrow (y,x)\,,$$
$$a_{04}: \ (x,y) \longrightarrow \left(\frac{1}{x}, \frac{1}{y}\right), \qquad a_{12}: \ (x,y) \longrightarrow \left(\frac{1}{x}, \frac{y}{x}\right).$$

Let T_{012} and T_3 be the groups generated by $\langle a_{01}, a_{23} \rangle$ and $\langle a_{04}, a_{12}, a_{23} \rangle$ respectively and they are isomorphic to $Z_2 \times Z_2$ and $Z_2 \times S_3$. Two curves (28) for different (x,y) and $(x',y') \in \Lambda$ are isomorphic if (x,y) is equivalent to (x',y') under T_{012} or T_3 for Apoll-0,1,2 or Apoll-3. Since the corresponding Jacobian varieties are isomorphic and by proposition 3.1 there exist matrix $Q \in \mathbb{Z}[\varepsilon]$ such that

$$\big(I_1, I_2, I_3\big)(x', y') = Q\, {}^t\big(I_1, I_2, I_3\big)(x,y)\,.$$

This matrix Q belongs to arithmetic group $A(\mu)$.

As we defined paths the A_{ij} in section 3 let for $\mu_i = \mu_j$ A'_{ij} be "half" from A_{ij}, namely that is path from u_i to u_j. It changes u_i and u_j.

Definition-Proposition 4.4. *The action of* A_{ij}*,* $0 \le i < j \le 3$*, and* A'_{st} *for all* $\mu_s = \mu_t$ *on* I_k *are generators of full Picard modular group* $F\Gamma(\mu)$*. If* $\mu_s = \mu_t$ *then* $\big(g(A'_{st})\big)^2 = g(A_{st})$*.*

So we need to take as generators for Apoll-0,1,2 the permutations $(0,1)$, $(2,3)$ and for Apoll-3 $(0,4)$, $(1,2)$, $(2,3)$, $(1,3)$. They are

$$M'_{01} = \begin{bmatrix} -q_1 & 0 & 0 \\ (q_1 - q_1 q_2)/(q_1 - 1) & 1 & 0 \\ (q_1 - q_1 q_3)/(q_1 - 1) & 0 & 1 \end{bmatrix} \qquad M'_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 - q_2 & q_2 \\ 0 & 1 & 0 \end{bmatrix}$$

For Apoll-3 we give them explicitly

$$(38) \quad M'_{04} = \begin{bmatrix} i & 1-i & -1+i \\ -1+i & 2-i & -1+i \\ -1+i & 1-i & i \end{bmatrix} \quad M'_{12} = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad M'_{13} = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 0 & 0 \end{bmatrix} \quad M'_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

[3]The matrix H and the ball come from Riemann periods relations on Jacobian variety $\Pi J {}^t\Pi = J$, $\Pi J {}^t\overline{\Pi} < 0$, J is the intersection matrix.

From Deligne and Mostow ([DM2] page 76) results follows $\Gamma(\mu)$ is normal subgroup of $F\Gamma(\mu)$ and more precisely

$$F\Gamma(\mu)/\Gamma(\mu) \approx Z_2 \qquad \text{for Apoll-0,1,2}\,,$$
$$F\Gamma(\mu)/\Gamma(\mu) \approx Z_2 \times S_3 \qquad \text{for Apoll-3}\,.$$

Terada proved ([T] page 182 example 8) that for Apoll-3 $A(\mu) = F\Gamma(\mu)$.

In the following we consider in more details Apoll-3. In this case we substitute $q_1 = q_2 = q_3 = -1$, $q_0 = q_4 = -i$ and obtain generators of Picard modular group $\Gamma(\mu)$

$$\begin{array}{lll} M_{01} = \begin{bmatrix} i & 0 & 0 \\ 2i & 1 & 0 \\ 2i & 0 & 1 \end{bmatrix} & M_{02} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & i & 0 \\ 0 & 2i & 1 \end{bmatrix} & M_{03} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & i \end{bmatrix} \\ M_{12} = \begin{bmatrix} 3 & -2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & M_{13} = \begin{bmatrix} 3 & 0 & -2 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{bmatrix} & M_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & 2 & -1 \end{bmatrix} \end{array} \tag{39}$$

Lemma 4.5. *The multiplicative set of matrices* $\left[\begin{smallmatrix} 1&0&0\\0&1&0\\0&0&1 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 0&1&0\\1&0&0\\0&0&1 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 1&0&0\\0&0&1\\0&1&0 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 0&0&1\\0&1&0\\1&0&0 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 0&1&0\\0&0&1\\1&0&0 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 0&0&1\\1&0&0\\0&1&0 \end{smallmatrix}\right]$, *is isomorphic to the symmetric group* S_3 *and they correspond to permutations:* identity, (12), (13), (23), (123) *and* (132) *respectively.*

Let $\mathcal{A}\Gamma$ be the multiplicative group generated by M'_{04} and M_{st}, $0 \leq s < t \leq 3$. This is *Apollonius modular group*[4] for Gauß numbers.

Theorem 4.6. *i)* $\mathcal{A}\Gamma$ *is normal subgroup of* $F\Gamma$ *and* $F\Gamma/\mathcal{A}\Gamma = S_3$.
ii) $\mathcal{A}\Gamma = \left\{ Q \in GL(3,\ \mathbb{Z}[i]) \mid {}^t\overline{Q}HQ = H,\ Q \approx E \bmod (1+i) \right\}$. $E = \left(\begin{smallmatrix} 1&0&0\\0&1&0\\0&0&1 \end{smallmatrix}\right)$.

By (38) and (39) we obtain the inclusion $\subset$ for ii). The opposite direction is the aim of the second part of the paper.

The first part i) is known for μ which satisfy INT condition. Notice that if we replace E with any of the matrices from lemma 4.5 then we obtain factor classes of $F\Gamma/\mathcal{A}\Gamma$. $\square$

For the Hermitian form H we have $H = \begin{bmatrix} 1 & -1+i & 1-i \\ -1-i & 1 & -1+i \\ 1+i & -1-i & 1 \end{bmatrix}$.

Proposition 4.7. *There exist matrix* $d \in SL(3, \mathbb{Z}[i])$ *such that* ${}^t\overline{d}Hd = \mathrm{diag}(1,1,-1)$.
For example $d := \begin{bmatrix} -1-i & -1 & 1+i \\ -i & -2i & 2i \\ 1-i & -2i & -1+2i \end{bmatrix}$. $\square$

If we take as generators Picard modular group Γ and some matrices from (38) then we will obtain other modular group and orbital surfaces. In section 6 we shall give more examples.

5 The ball quotient as moduli surface of curves of special type

For Apoll-3 there is an exact sequence (see Lemma 9.2)

$$0 \longrightarrow Z_2 = \mathcal{A}\Gamma/\Gamma \longrightarrow F\Gamma/\mathcal{A}\Gamma \longrightarrow S_3 = F\Gamma/\mathcal{A}\Gamma \longrightarrow 1$$

with an index-2 subgroup Γ of $\mathcal{A}\Gamma$. It comes from the double cover of $\hat{X} = \mathbb{P}^2$ branched precisely along the quadric $\hat{C}_0$. In second part we prove that $\mathbb{B}/\mathcal{A}\Gamma$ is $\mathbb{P}^2$. The compactification branch locus of the quotient map $p:\ \mathbb{B} \longrightarrow \mathbb{B}/\mathcal{A}\Gamma$ is precisely Apollonius configuration. Here we use this result and classify the ball quotient surface $\hat{Y} = \widehat{\mathbb{B}/\Gamma}$. On this way we demonstrate how the Proportionality help to classify the covering surface if one already knows the corresponding quotient surface.

[4]In case Apoll-0,1,2, Apollonius modular group is equal to Picard modular group $\Gamma(\mu)$.

The degree formulas for orbital heights applied to the finite orbital double covering $\mathbf{f}: \hat{Y} \longrightarrow (\mathbb{P}^2, 2\hat{C}_0)$, see [H98], compare with (16), (17), yield

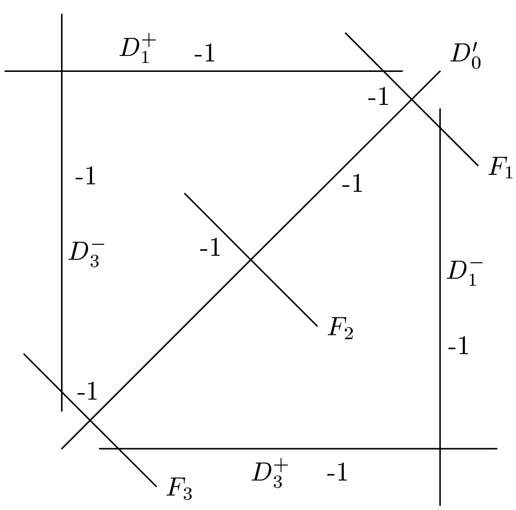
$$
\begin{aligned}
e(\hat{Y}) &= H_e(\hat{Y}) = 2H_e(\mathbb{P}^2, 2\hat{C}_0) = 2[e(\mathbb{P}^2) - (1-\frac{1}{2})e(\hat{C}_0)] = 2[3 - \frac{1}{2}\cdot 2] = 4,\\
\tau(\hat{Y}) &= H_\tau(\hat{Y}) = 2H_\tau(\mathbb{P}^2, 2\hat{C}_0) = 2[\tau(\mathbb{P}^2) - \frac{1}{3}(2-\frac{1}{2})\cdot\frac{1}{2}(\hat{C}_0^2)] = 2[1 - \frac{1}{2}\cdot 2] = 0.
\end{aligned}
\tag{40}
$$

Now we calculate Euler numbers and selfintersections of irreducible preimage curves $\hat{D}_i$ of $\hat{C}_i$, $i = 0,1,2,3$, respectively, by the degree formulas for local orbital heights, see [H98], compare with (4),(5). Since $\hat{C}_0$ is the branch locus we get immediately $\hat{D}_0 \cong \hat{C}_0 \cong \mathbb{P}^1$. We have to change to the double covering $Y' \longrightarrow X'$ for getting a locally abelian situation. With the ramification indices $v_0 = 2$, $v = v_j = 1$, $j = 1,2,3$, of f along D'_j covering C'_j, we get

$$
e(D'_j) = h_e(D'_j) = [D'_j : C'_j]\cdot h_e(\mathbf{C}'_j) = [e(C'_j) - 2(1-\frac{1}{v})]\cdot[D'_j : C'_j] = 2\cdot[D'_j : C'_j],
$$

hence

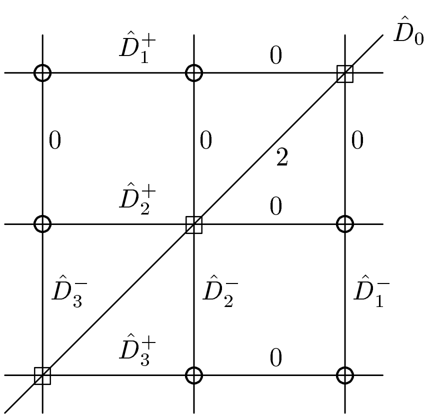
$$
\begin{aligned}
&e(D'_i) = 2,\ D'_i \cong \mathbb{P}^1,\ [D'_i : C'_i] = 1,\ i = 0,1,2,3;\\
&(D'^2_j) = h_\tau(D'_j) = [D'_j : C'_j]\cdot h_\tau(\mathbf{C}'_j) = (C'^2_j) = -1,\ j = 1,2,3;\\
&(D'^2_0) = h_\tau(D'_0) = [D'_0 : C'_0]\cdot h_\tau(\mathbf{C}'_0) = \frac{1}{2}\cdot(C'^2_0) = -1.
\end{aligned}
$$



Since $f' : Y' \longrightarrow X'$ is not ramified and not inert at D'_j, each of the curves C'_j has precisely two irreducible preimage curves D_j^+ and D_j^-. Let $F_j \subset Y'$ denote the preimage of $E_j \subset X'$. Locally Z_2 acts around each fixed point on Y' with smooth image on X' as a reflection group. Starting from a preimage of the intersection point of E_j and C'_0 we see that Z_2 acts effectively on F_j because it acts trivially on D'_0, see Figure 2. This means that $[F_j : E_j] = 2$. We calculate

$$
\begin{aligned}
e(F_j) &= h_e(F_j) = [F_j : E_j]\cdot h_e(\mathbf{E}_j)\\
&= 2\cdot[e(E_j) - 2(1-\frac{1}{2})] = 2;\\
(F_j^2) &= h_\tau(F_j) = [D'_j : C'_j]\cdot h_\tau(\mathbf{E}_j)\\
&= 2\cdot[(E_j^2) + \frac{1}{2}] = -1.
\end{aligned}
$$

Forgetting for a moment D_2^+ and D_2^- we get the above configuration on Y'. From (40) follows that



$$
\chi(\hat{Y}) = \frac{1}{4}(e+\tau) = 1,\ c_1^2(\hat{Y}) = 12\chi - e = 8. \tag{41}
$$

Blowing down the curves F_1, F_2, F_3 we get two crossing smooth rational curves with selfintersection 0 on $\hat{Y}$, for instance D_1^+ and D_3^-. There is up to isomorphism only one smooth compact surface with Chern numbers $\chi = 1$ and $c_1^2 = 8$ and such crossing curve pair, namely $\mathbb{P}^1 \times \mathbb{P}^1$. For this fact we refer to [H98], end of V.2 (blow up the intersection point of the curves and blow down the two curves to get a smooth rational surface with $c_1^2 = 9$, which must be $\mathbb{P}^2$, see [H98], V.2, Proposition 5.2.4). Taking in consideration now also D_2^+ and D_2^- we get the left branch configuration on $\hat{Y} = \widehat{\mathbb{B}/\Gamma} = \mathbb{P}^1 \times \mathbb{P}^1$.

The six circles mark the (abelian) quotient points (images of all Γ_2-elliptic points on $\mathbb{B}$); the three boxes mark the compactifying cusp points. $\hat{D}_0$ crosses each of the three marked horizontal and three vertical fibers in precisely one point. Therefore $\hat{D}_0$ is a section for both canonical projections of $\mathbb{P}^1 \times \mathbb{P}^1$. Blowing up the central cusp point in last figure and blowing down the two $\hat{D}_2^+$ and $\hat{D}_2^-$ after, then $\hat{D}_0$ becomes a smooth rational curve on $\mathbb{P}^2$ with selfintersection 1, hence a projective line. It is uniquely determined as line through the remaining two cusp points. Therefore $\hat{D}_0$ coincides with the diagonal line on $\mathbb{P}^1 \times \mathbb{P}^1$.

Altogether we get the following

Theorem 5.1. *The compactified ball quotient surface $\widehat{\mathbb{B}/\Gamma}$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The compactified branch locus of the quotient map $p : \mathbb{B} \longrightarrow \mathbb{B}/\Gamma$ consists of three horizontal fibres $\hat{D}_j^+$, three vertical fibres $\hat{D}_j^-$ and the diagonal $\hat{D}_0$. The configuration is $Z_2 \times S_3$-invariant, where the generator of Z_2 changes the $\mathbb{P}^1$-components of each point $(P,Q) \in \mathbb{P}^1 \times \mathbb{P}^1$ and S_3 acts by simultaneous permutation of natural homogeneous $\mathbb{P}^2$-coordinates $(x : y : z)$ with sum zero ($x + y + z = 0$), on both components. The cusp points are the three intersection points of the diagonal curve $\hat{D}_0$ with the other curves $\hat{D}_j^\pm$. The ramification indices of p at $\hat{D}_0$ or $\hat{D}_j^\pm$ are 2 or 4, respectively, for $j = 1, 2, 3$. The uniformizing ball lattice of BHH-20 is Γ with generators (39).*

The $2S_3$-invariance comes from the factor group $F\Gamma/\Gamma$. Cusp points and the branch indices are simply lifted from those of

$$\mathbb{B} \longrightarrow \mathbb{B}/\mathcal{A}\Gamma = Y/Z_2 = \mathbb{P}^2 \setminus \{K_1, K_2, K_3\},$$

with obvious notation. Only at D_0 we loose the factor 2, while the other branch indices remain to be 4.

$\square$

We want to interpret the ball quotient surface $\widehat{\mathbb{B}/\Gamma} = \mathbb{P}^1 \times \mathbb{P}^1/2S_3 = \mathbb{P}^2/S_3$ as compactified moduli space of a special curve family. Following Shimura [Sm64] we consider plane curves of affine equation type $w^4 = p_2(u)\, p_3(u)^2$, where $p_n(u) \in \mathbb{C}[u]$ denotes a normalized polynomial of degree n.

Similarly to (28) let us consider again the families of plane (w,u) curves

$$\begin{aligned} C_{\mu a} \ :\ \ & w^d = (u-a_0)^{m_0}(u-a_1)^{m_1}(u-a_2)^{m_2}(u-a_3)^{m_3}(u-a_4)^{m_4}\,, \qquad a \in \hat{\Lambda}, \\ & \hat{\Lambda} := \left\{ a = (a_0, \ldots, a_4) \mid a \in (\mathbb{P}^1)^5,\ a_s \neq a_t \right\}. \end{aligned} \tag{42}$$

If some $a_s = \infty$ then we substitute $(u - a_s)^{m_s} \equiv 1$. We denote with $\tilde{C}_{\mu a}$ the normalization of projective closure $C_{\mu a} \subset \mathbb{P}^2$. The genus of $\tilde{C}_{\mu a}$ for general $a \in \hat{\Lambda}$ is known by (29) and if μ is fixed it is independent of a. The projective group $PGL(2,\mathbb{C})$ acts on a and this action preserves genus of $C_{\mu a}$. If $g \in PGL(2,\mathbb{C})$ and $a' = ga$ then the curves $C_{\mu a}$ and $C_{\mu a'}$ are projective equivalent. We can consider moduli space

$$C_\mu := \{C_{\mu a} \mid a \in \Lambda\}\,, \qquad \Lambda := PGL(2,\mathbb{C})\backslash\hat{\Lambda}\,.$$

Without loss of generality we choose $a_0 = 0$, $a_1 = 1$, $a_4 = \infty$, $a_2 = x$, $a_3 = y$ and vary only $x, y \neq 0, 1, \infty$, $x \neq y$. In other words $\Lambda = (x,y)$ can be identified with the complement of seven lines on $\mathbb{P}^1 \times \mathbb{P}^1$ as on the picture near to (41).

Proposition 5.2. *The symmetric group $S_5(0,1,2,3,4)$ has exact representation on Λ given by transpositions (st) in the next table.*

(43) $\quad (x,\, y) \longrightarrow$

(01)	$1-x,\ 1-y$	(13)	$x/y,\ 1/y$
(02)	$x/(x-1),\ (x-y)/(x-1)$	(14)	$x/(x-1),\ y/(y-1)$
(03)	$(y-x)/(y-1),\ y/(y-1)$	(23)	$y,\ x$
(04)	$1/x,\ 1/y$	(24)	$1-x,\ y(1-x)/(y-x)$
(12)	$1/x,\ y/x$	(34)	$x(1-y)/(x-y),\ 1-y$

The symmetric group S_5 acts on μ and on $C_{\mu a}$ permuting m_t. Let H be subgroup of S_5 and μ is H-invariant (for example $[4;\ 2,2,1,1,2]$ is $H = \langle(01),(23)\rangle$ invariant). Then we define

$$C_{\mu H} := \{C_{\mu a} \mid a \in \Lambda_H\}\,, \qquad \Lambda_H := \Lambda/H := PGL(2,\mathbb{C})\backslash\hat{\Lambda}/H\,,$$

where H acts on $a \in (\mathbb{P}^1)^5$ by permuting a_s. If two curves f, g are elements of $C_{\mu H}$ they define Jacobian varieties which are isomorphic.

Following Deligne and Mostow [DM2] page 76, let $\Gamma_{\mu H}$ be the extension of Γ_μ corresponding to H (μ is H invariant). We do not give here general definition of $\Gamma_{\mu H}$, since for all our examples we will give generators of this group.

Our main idea is the correspondce:

$$(\mu,\, H) \longrightarrow \Lambda_H \longrightarrow C_{\mu H} \longrightarrow \Gamma_{\mu H} \longrightarrow \mathbb{B}/\Gamma_{\mu H} \longrightarrow \text{Proportionality} \longrightarrow \text{Surface} \tag{44}$$

Example 5.3. $\mu = [4;\ 1,2,2,2,1]$, $H = \mathrm{id}$ or (04). This is Apoll-3 and the curve is:

$$C_\mu :\ w^4 = u(u-1)^2(u-x)^2(u-y)^2, \qquad (x,y) \in \Lambda\,.$$

Matsumoto [Mat] and van Geemen [vGm] work with the following family:

$$C_{[4;\ 2,2,1,1,2]} :\ w^4 = u^2(u-1)^2(u-\gamma_1)(u-\gamma_2), \qquad (\gamma_1,\gamma_2) \in \Lambda\,.$$

Since $(03)(24)[4;\ 1,2,2,2,1] = [4;\ 2,2,1,1,2]$ and by (43) we obtain the relation between both families:

$$(x,y) \longrightarrow \gamma(x,y) = (\gamma_1,\gamma_2) = \left(\frac{1-x}{1-y},\ \frac{y(1-x)}{x(1-y)}\right). \tag{45}$$

We prefer to work with $\mu = [4;\ 1,2,2,2,1]$ since in this case the generators of full Picard modular group $F\Gamma$ are simple.

If $H = (04)$ then μ is H invariant and $\Gamma_{\mu H} = \Gamma_{\mu(04)} = \mathcal{A}\Gamma = \langle \Gamma,\, M'_{04}\rangle$ (see (38)). Now we want to find the quotient surface $\Lambda/(04) = \Lambda/((x,y) \sim (1/x,\, 1/y))$. We do this in terms of γ. Using (43) and (45) we see that the (04) action on (x,y) goes down to the transposition of γ_1 and γ_2. In S_5 language that is $(03)(24)\cdot(04)\cdot(03)(24) = (23)$. So instead of (x,y) quotient $\Lambda/(04)$ we must to find (γ_1,γ_2) quotient $\Lambda/(23)$. This surface is $\mathbb{P}^2$ and we know this by (34) and the map B defined there.

Let define the maps

$$H :\ \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^2\,, \qquad (\gamma_1,\gamma_2) \longrightarrow (p:q:r) := (\gamma_1\gamma_2 : (1-\gamma_1)(1-\gamma_2) : 1)\,,$$

and write our curve as

$$\begin{aligned} w^4 &= u^2(u-1)^2(u-\gamma_1)(u-\gamma_2) = u^2(u-1)^2(u^2-(\gamma_1+\gamma_2)u+\gamma_1\gamma_2) \\ &= u^2(u-1)^2(u^2+(q-p-1)u+p). \end{aligned}$$

The last (w,u) curve has genus 3 precisely when $pq\left((q-p-1)^2-4p)\right) \neq 0$. So we have obtained our Normalized model 1.2 again. We denote last (p,q) moduli space with $C_{\mu(04)}$.

We have a commutative *moduli diagram* of algebraic morphism

$$\begin{array}{ccccccc} C_\mu & \longrightarrow & \Lambda & \hookrightarrow & \mathbb{P}^1\times\mathbb{P}^1 & = & \widehat{\mathbb{B}/\Gamma} \\ \downarrow & & \downarrow & & \downarrow & & \\ C_{\mu(04)} & \longrightarrow & \Lambda/(04) & \hookrightarrow & \mathbb{P}^2 & = & \widehat{\mathbb{B}/\mathcal{A}\Gamma} \\ \downarrow & & \downarrow & & \downarrow & & \\ & \longrightarrow & \Lambda/2S_3 & \hookrightarrow & \mathbb{P}^2/S_3 & = & \widehat{\mathbb{B}/F\Gamma} \end{array} \tag{46}$$

The second author don't know how to complete the last diagram. In next section we obtain similar diagrams for other subgroups H of S_5.

$\Lambda/2S_3$ is the moduli space of the curve family $\tilde{C}_\mu$ by Piñeiro's result in the first appendix of [HPV]. But $\mathbb{P}^2/S_3 = \widehat{\mathbb{B}/F\Gamma}$ is also the moduli space of abelian 3-folds with $\mathbb{Q}(i)$-multiplication of type $(2,1)$, see [Sm63]. The Jacobians of the above curves $\tilde{C}_\mu$ are obviously abelian threefolds of this type, see [Sm64]. It follows that

Theorem 5.4. *The compactified moduli spaces of of the curve families $\tilde{C}_\mu$ and of (principally polarized) abelian 3-folds with $\mathbb{Q}(i)$-multiplication of type $(2,1)$ coincide with $\mathbb{P}^2/S_3 \cong \widehat{\mathbb{B}/F\Gamma}$.*

$\square$

We have two families of curves

$$\begin{aligned} C_\mu : \quad w^4 &= u^2(u-1)^2(u-\gamma_1)(u-\gamma_2) \\ C_{\mu(40)} : \quad w^4 &= u^2(u-1)^2(u^2+(q-p-1)u+p). \end{aligned}$$

If γ_1 and γ_2 are known one reconstruct corresponding (x,y) using (45). For given (p,q) we reconstruct γ_1 and γ_2 as roots of quadratic equation $Z^2+(q-p-1)Z+p$ but we loose the order. Observe that the order of γ_1, γ_2 determines the order of the linear factors of C_μ. Forgetting the order of γ_1, γ_2 means to forget the order of the two linear factors and this is the case $C_{\mu(04)}$. Then we say that our curves are (only simply) distinguished.

Theorem 5.5. *The surfaces $\mathbb{P}^1\times\mathbb{P}^1 = \widehat{\mathbb{B}/\Gamma}$ and $\mathbb{P}^2 = \widehat{\mathbb{B}/\mathcal{A}\Gamma}$ are the (compactified) moduli spaces of double distinguished respectively distinguished curves of Shimura equation type. More precisely: The correspondence*

$$C_\mu \mapsto (x,y) \mapsto (\gamma_1,\gamma_2)$$

defines a map to the moduli space $\mathbb{P}^1\times\mathbb{P}^1 \supset \Lambda$ of distinguished curves, which restricts to the set of curves $w^4=u^2(u-1)^2(u-\gamma_1)(u-\gamma_2)$ and $w^4=u^2(u-1)^2(u-\gamma_1)(u-\gamma_2)$. Via Z_2 equivalence interchanging these curves by changing γ_1 and γ_2 we get a map to the moduli space $\mathbb{P}^2=\mathbb{P}^1\times\mathbb{P}^1/(04)\supset\Lambda/(04)$ of distinguished curves, which restricts in isomorphy-compatible manner to the curves $C_{\mu(04)}=u^2(u-1)^2(u^2+(q-p-1)u+p)$, where $p=\gamma_1\gamma_2$, $q=(1-\gamma_1)(1-\gamma_2)$. □

6 More proportional orbital planes with quadrics

Here we present more examples to demonstrate how (44) works.

Example 6.1. $\mu=[12;\ 7,7,2,2,6]$, $H=\mathrm{id}$ or (23). Looking at table (37) we see that this is Apoll-3. Due to Deligne and Mostow ([DM2] page 76) we know know that $\Gamma_\mu=\mathcal{A}\Gamma_\mu:=\langle\Gamma_\mu,\,M'_{23}\rangle=\Gamma_{\mu(23)}$. The generators and hermitian form are given in section 4. We must substitute $q_0=q_1=\varepsilon^7$, $q_2=q_3=\varepsilon^2$, $q_4=\varepsilon^6$, $q=\varepsilon^3$, $\varepsilon=\exp(-2\pi i/12)$. This is arithmetic cocompact subgroup of index 2 of the full Picard modular group $F\Gamma_\mu$ and the curve we associate is:

$$C_\mu=C_{\mu(23)}: \quad w^{12}=u^7(u-1)^7(u^2+(q-p-1)u+p)^2\,,$$

where (p,q) are coordinates in $\mathbb{C}^2$, $pq\left((q-p-1)^2-4p)\right)\neq 0$. I section 2 we check that Proportionality conditions are satisfied. The orbital surface seems to be $\mathbb{B}/\Gamma_\mu=\Lambda/(23)\subset\mathbb{P}^2$ with Apollonius configuration divisor, but until now it is not generally known that the four proportionality conditions (25) are sufficient for $\hat{\mathbf{X}}$ to be a ball quotient.

Example 6.2. $\mu=[6;\ 1,1,3,3,4]$ or $[6;\ 1,1,4,4,2]$, $H=\mathrm{id}$ or (01). These are Apoll-1 and Apoll-2. Both cases are similar to 6.1. The table (37) show that ΣINT condition set is $\{0,1\}$. By the same argument we have $\Gamma_\mu=\mathcal{A}\Gamma_\mu:=\langle\Gamma_\mu,\,M'_{01}\rangle=\Gamma_{\mu(01)}$. To find curves we apply $\sigma=(02)(13)$ to μ and obtain similar to example 5.3 new coordinates (γ_1,γ_2),

$$(x,y)\longrightarrow\gamma(x,y)=(\gamma_1,\gamma_2)=\left(\frac{x}{x-y},\ \frac{x-1)}{x-y}\right).$$

In these new coordinates $(01)-(x,y)$ action goes to $(23)-(\gamma_1,\gamma_2)$ action, which is interchange of γ_1 and γ_2. From (γ_1,γ_2) coordinates in $\mathbb{P}^1\times\mathbb{P}^1$ we go to (p,q) coordinates on $\mathbb{P}^2$. We go further as in previous 6.1.

Example 6.3. $\mu=[3;\ 1,1,1,1,2]$, $H=\mathrm{id}$. This is classical Picard [P], BHH-19, modular group for the curve $w^3=u(u-1)(u-x)(u-y)$. The surface is $\mathbb{P}^2$ with 4 cusps and line arrangement as in Figure 6. It is known that BHH-19 has the uniformizing ball lattice $A(\mu)(1-\varepsilon)$, $\varepsilon=\exp(2\pi i/3)$, see [H86], Ch. I or [H98], V.2. Arithmetic group $A(\mu)$ is definition 4.2, $A(\mu)(1-\varepsilon):=\{g\in A(\mu)\mid g\sim E \bmod 1-\varepsilon\}$. Moreover, we have an exact sequence of group homomorphisms

$$1\longrightarrow\Gamma_\mu=A(\mu)(1-\varepsilon)\longrightarrow A(\mu)=F\Gamma_\mu\longrightarrow S_4\longrightarrow 1.$$

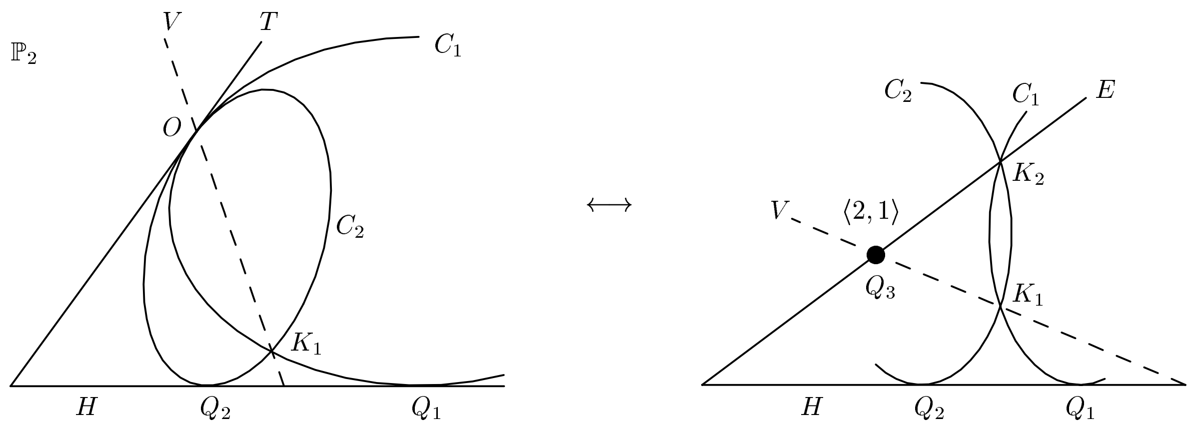
We give a table with proportional invariants as in BHH-19.

	01	02	03	04	12	13	14	23	24	34	surface
weights	∞	∞	∞	∞	3	3	3	3	3	3	
h_e	0	0	0	0	$-2/3$	$-2/3$	$-2/3$	$-2/3$	$-2/3$	$-2/3$	$1/3$

Example 6.4. $\mu = [3;\ 1,1,1,1,2]$, $H = (23)$. This is again as previous example (see Table 1 and 2 solution 37) but this time $\Gamma_{\mu(23)} = \langle \Gamma_\mu, M'_{23}\rangle$. The uniformizing ball lattice $\Gamma_{\mu(23)}$ is nothing else but the preimage of $(23) \subset S_4$ in $A(\mu)$. In other words, we have an exact subsequence

$$1 \longrightarrow A(\mu)(1-\varepsilon) = \Gamma_\mu \longrightarrow \Gamma_{\mu(23)} \longrightarrow \langle (23)\rangle \longrightarrow 1.$$

The curve is obvious $w^3 = u(u-1)(u^2+(q-p-1)u+p)$ but how to find the appropriate surface? One expect $\mathbb{P}^2/(23)$, where (23) action on $\mathbb{P}^2$ is $(a:b:c) \sim (b:a:c)$.



Here we use Proportionality again. We consider on $\mathbb{P}^2$ two quadric C_1, C_2 with two intersection points O, K_1, where at O the intersection has multiplicity 3. The tangents to both quadric are denoted by T, H and the line though O, K_1 with V (at this time we don't use V). Let us blow up two times $\mathbb{P}^2$ at O and then contract the line T. Now we contract the exceptional line with selfintersection -2 and obtain the model right with Q_3- the singularity of type $\langle 2,1\rangle$. This is the surface we look for. Before the last contraction one gets Hirzebruch surface $\mathbb{F}_2$. The orbital cycle is E, C_1, C_2, H with Q_1, Q_2, K_2 cusp points and Q_3 elliptic point.

We must check proportionality conditions and that $\mathbb{P}^2/(23)$ is isomorphic to $\hat{\mathbb{F}}_2$. Let start with proportionality. This is already done in section 2. This is because this orbital cycle and the Apollonius cycle (quadric and three tangents) have the some model $\mathbf{Y}'$. Really after blowing up Q_1, Q_2, K_2 two, two, one times and contracting exceptional lines with selfintersection -2 we get Figure 2. We give weights and heights in table:[5]

	Q_1	Q_2	K_2	H	E	C_1	C_2	surface
weight	∞	∞	∞	6	3	3	3	
h_e	0	0	0	$-1/3$	$-2/3$	$-2/3$	$-2/3$	$1/6$

From the other side it is not difficult to see that $\mathbb{P}^2/(23)$ is isomorphic to $\hat{\mathbb{F}}_2$. Looking at Figure 6 one can think the quotient map $\mathbb{P}^2 \longrightarrow \mathbb{P}^2/(23)$ as symmetries about L_{13} and the map is ramified (only) along this line. It goes down to $H \subset \hat{\mathbb{F}}_2$. The line pairs $\{L_{14}, L_{34}\}$, $\{L_{23}, L_{12}\}$ are mapped onto C_1, C_2, respectively. The line L_{24} projects onto E and the fixed point $(-1:1:0) \in L_{24}$ goes to elliptic point Q_3 of type $\langle 2,1\rangle$. Namely, by proportionality of heights and relations with surface invariants, one checks that the quotient surface has Euler number 3. Therefore the blowing up the only surface singularity Q_3 yields a smooth rational surface with Euler number 4, hence a Hirzebruch surface $\mathbb{F}_d/\mathbb{P}^1$. Again by proportionality H, C_1, C_2, E have selfintersection $2,0,0,0$ respectively. It follows immediately that $\mathbb{F}_d = \mathbb{F}_2$ (see [H98] Remark 5.2.7).

The weights from the previous example go obviously down to the weights in our case. We have proved the following

[5] Both authors have MAPLE packages for working with orbital invariants.

Proposition 6.5. *For a suitable choice of C_1, C_2 the open plane*

$$\mathbb{P}^2 \setminus supp\,(C_1 + C_2 + T + H)$$

has a Picard-Einstein metric. As uniformizing ball lattice one takes a suitable index-2 extension $\Gamma_{\mu(23)}$ of $A(\mu)(1-\varepsilon)$, where $A(\mu)$ the ring of integral Eisenstein numbers. More precisely, after a birational transformation β - the blowing up two times of O and contraction of the exceptional line with self-intersection -2 one gets the $\langle(23)\rangle$-quotient of the orbital surface BHH-19 of the table in [BHH], p. 201.

Let σ be an element of S_5. By proposition 5.2 with σ we associate an automorphism of Λ. This automorphism has fixed line in Λ if and only if when $\sigma = (ij)(kj), i, j, k, l$ different (see [DM2], Lemma 8.3.2). In the following the group H we consider have such elements.

Example 6.6. $\mu = [3;\ 1,1,1,1,2]$, $H = (01)(23)$. This is very closed to example 6.4 and we give only some results. The action $(01)(23)$ is $(x,y) \sim (1-y, 1-x)$ which goes to $\mathbb{P}^2$ as $(a:b:c) \sim (c-b:c-a:c)$. This is symmetries about $a+b-c=0$. Orbital surface is the same as example 6.4 with weight table (the line V is the image $a+b=c$):

	Q_1	Q_2	K_1	K_2	V	H	E	C_1	C_2	surface
weight	∞	∞	∞	∞	2	3	3	3	3	
h_e	0	0	0	0	$-2/3$	$-1/3$	$-1/3$	$-2/3$	$-2/3$	$1/6$

The group is $\Gamma_{\mu(01)(23)} = \langle \Gamma_\mu,\ M'_{01}M'_{23}\rangle$ (in other words we take as a new generator generator the multiplication of two matrices. As coordinates on the moduli space one can take $q = x+1-y$, $p = x(1-y)$. Similar proposition to 6.5 holds.

Example 6.7. $\mu = [3;\ 1,1,1,1,2]$, $H = \langle(01),(23)\rangle$. The modular group we associate is $\Gamma_{\mu\langle(01),(23)\rangle} = \langle\Gamma_\mu,\ M'_{01},\ M'_{23}\rangle$. The coordinates on the moduli space we consider $p = xy(1-x)(1-y)$, $q = (1-xy)\big(1-(1-x)(1-y)\big)$. We want to find the quotient $\mathbb{P}^2/\langle(01),(23)\rangle$. Let us consider the map

$$\pi:\ \mathbb{P}^2 \longrightarrow \mathbb{P}^2\,, \qquad (x:y:z) \longrightarrow (a:b:c) := \big((x-y)^2 : z^2 : (z-x-y)^2\big)\ .$$

Obviously this map is invariant under the action of $\langle(01),(23)\rangle$ and $\mathbb{P}^2$ is the quotient we look for. The image of the lines L_{ij} from Figure 6 is our Apollonius configuration. Namely the lines $\{L_{24}, L_{14}, L_{23}, L_{12}\}$ projects to the quadric; $\{L_{13}, L_{24}\}$ to $\{a=0,\ b=0\}$ respectively and the line $x+y=z$ goes to $c=0$. The quotient map π is $Z_2 \times Z_2$ cover ramified only along $z(x-y)(z-x-y)=0$. After calculating the heights we see that they are the same as Apoll-2. By Proportionality Theorem we get the following

Theorem 6.8. *The open plane*

$$\mathbb{P}^2 \setminus \{xyz((x-y-z)^2 - 4yz) = 0\}$$

has a Picard-Einstein metric. As uniformizing ball lattice one takes a suitable $Z_2 \times Z_2$ extension $\Gamma_{\mu\langle(01),(23)\rangle}$ of $A(\mu)(1-\varepsilon)$, where $A(\mu)$ the ring of integral Eisenstein numbers, $\mu = [3;\ 1,1,1,1,2]$. Apoll-2 is $Z_2 \times Z_2$ quotient of the orbital surface BHH-19.

Example 6.9. $\mu_1 = [6;\ 5,1,3,1,2]$, $\mu_2 = [4;\ 3,1,1,1,2]$, $H = id$ or (13). These are solutions 21 and 24 from Table 1, and 24 is BHH-21. They have special weights v_3, k_1, k_2 precisely as example 6.4. In case $H = (13)$ the orbital cycle is again C_1, C_2, E, H and the surface is $\mathbb{F}_2$.

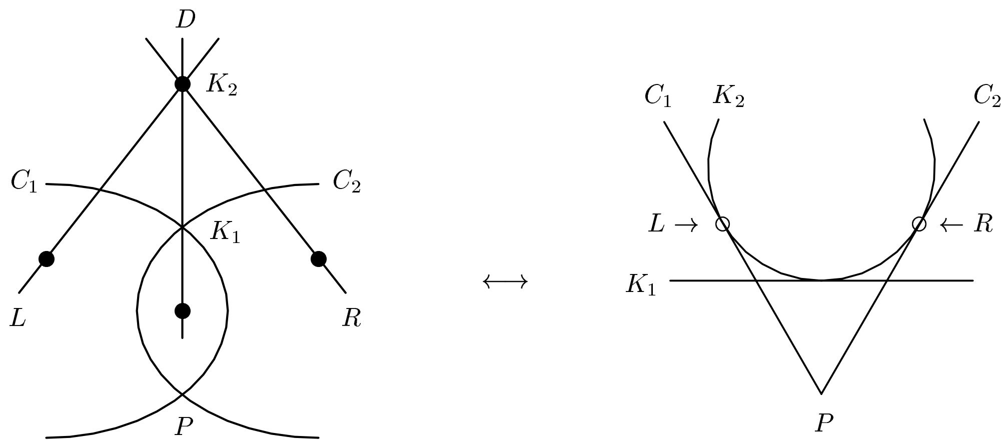
Since μ_1 has only Z_2 symmetrie and μ_1 satisfies pure ΣINT condition $\Gamma_{\mu_1} = F\Gamma_{\mu_1} = A(\mu_1)$.

For μ_2 and $H = id$ orbital cycle is Figure 6 with three cusps and one quotient point $(1:1:1) \in \mathbb{P}^2$. The exact sequence holds

$$1 \longrightarrow \Gamma_{\mu_2} \longrightarrow F\Gamma_{\mu_2} \longrightarrow S_3 \longrightarrow 1.$$

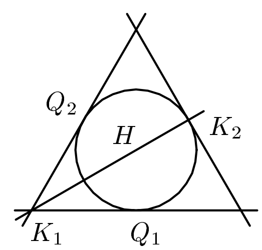
and from generators one gets $\Gamma_{\mu_2} \subset A(\mu_2)(1-i)$. This is again ring of Gauß integers but with different hermitian form.

Example 6.10. $\mu = [4;\ 1,2,2,2,1]$, $H = (23)$. This is solution 52. The curve is $w_4 = u(u-1)^2(u^2 + (q-p-1)u+p)^2$, $p = xy$, $q = (1-x)(1-y)$. Notice that this time we forget the order of quadratic terms instead of linear. The group is obviously $\Gamma_{\mu(23)} = \langle\Gamma_\mu, M'_{23}\rangle$. To obtain surface we prosed as in example 5.3 and apply $\sigma = (03)(24)$ to μ. Since $\sigma^{-1}(23)\sigma = (04)$ the (x,y) action on (23) goes



to (04) action on (γ_1, γ_2). Using (43) we see that (04) is $(\gamma_1, \gamma_2) \sim (1/\gamma_1, /\gamma_2)$ and so one have to gets the quotient $\mathbb{P}^1 \times \mathbb{P}^1/(\gamma_1, \gamma_2) \sim (1/\gamma_1, /\gamma_2)$. By proportionality of the heights we find that the quotient have Euler number 4, signature 0 and 4 singularities of type $\langle 2, 1\rangle$, namely the images of the fixed points $(1, 1)$, $(1, -1)$, $(-1, 1)$, $(-1, -1)$. All curves are rational with selfintersection 0, only D have selfintersection 1. Cusp points are K_1, K_2. We want to find nonsingular minimal model of this surface. After resolving all singularities and blowing up K_1 we find smooth surface with Euler number $4+5=9$. Now we contract -1-curves L, R, D and the exceptional curves to the down three singular points and obtain smooth surface with Euler number $9-6=3$. Since this surface have rational curves it is $\mathbb{P}^2$ and we obtain our Apollonius configuration. Cusp points K_1, K_2 goes to horizontal line and quadric. This explains very well line 52 to the Table 1.

Example 6.11. $\mu = [4;\ 1, 2, 2, 2, 1]$, $H = (04)(23)$. Again Gauß numbers. The (04)(23) action in γ_1, γ_2 coordinates is $(\gamma_1, \gamma_2) \leftrightarrow (1/\gamma_2, 1/\gamma_1)$ and is ramified only along quadric $\gamma_1\gamma_2 = 1$. The quotient of $\mathbb{P}^1 \times \mathbb{P}^1$ with respect to this action is $\mathbb{P}^2$. The proof is similar to example 6.4 where we find the quotient $\mathbb{P}^2/(23)$. The configuration divisor is $xyz(x-y)\big((x-y-1)^2 - 4y\big)$, namely Apollonius configuration with diagonal line $H:\ x = y$.



	Q_1	Q_2	K_1	K_2	H	C_0	C_1	C_2	C_2	surface
weight	-4	-4	∞	∞	2	2	4	4	4	
h_e	$1/4$	$1/4$	0	0	$-1/2$	-1	$-1/2$	$-1/2$	$-1/2$	$3/16$

Notice that H is the image of diagonal line and the quadric of $\gamma_1\gamma_2 = 1$. Now cusp points are K_1, K_2 and Q_1, Q_2 are quotient points.

Example 6.12. $\mu_0 = [12;\ 7, 7, 2, 2, 6]$, $\mu_1 = [6;\ 1, 1, 3, 3, 4]$, $\mu_2 = [6;\ 1, 1, 4, 4, 2]$, $H = \langle (01), (23)\rangle$. These are full Picard modular groups for Apoll-k, $k = 0, 1, 2$. From section 4 we know that they are arithmetic groups connected with hermitian forms H_k. To find the orbital surface one must find the quotient $\mathbb{P}^1 \times \mathbb{P}^1/(x, y) \sim (y, x) \sim (1-x, 1-y)$. We know that $\mathbb{P}^1 \times \mathbb{P}^1/(x, y) \sim (y, x) = \mathbb{P}^2$. Let the quotient map be $(x, y) \to (p, q) := (xy, (1-x)(1-y))$. In (p, q) coordinates the action $(x, y) \to (1-x, 1-y)$ is $(p, q) \to (q, p)$. But from example 6.4 we know this quotient and we obtain again $\hat{\mathbb{F}}_2$ with two quadrics and two lines.

7 Two Gauß ball lattices – commesurability

Example 7.1. $\mu_3 = [4;\ 1, 2, 2, 2, 1]$, $H_3 = \langle (04), (13)\rangle$. This is Apoll-3. We consider together also $\mu_4 = [4;\ 3, 1, 1, 1, 2]$, $H_4 = \langle (13)\rangle$. In both cases we connect the curves

$$C_3:\ w^4 = u(u-1)^2(u-x)^2(u-y)^2\,, \qquad C_4:\ w^4 = u^3(u-1)(u-x)(u-y)$$

having genus 3 and 4 respectively. We use (x, y) on C_4 and (γ_1, γ_2) coordinates on C_3. The birational relations between them is defined in (43). Shortly we denote these groups with

$$G_3 := \Gamma_{[4;\,1,2,2,2,1]\langle(04),(13)\rangle}\,, \qquad G_4 := \Gamma_{[4;\,3,1,1,1,2]\langle(13)\rangle}.$$

To find $\mathbb{B}/G_4$ we need to find the quotient $\mathbb{P}^2/(x,y) \sim (x/y, 1/y)$ which is symmetries about the vertical line L_{23} on Figure 6. We know from example 6.4 that this is $\hat{\mathbb{F}}_2$ and the orbital surface contain two quadrics and two lines.

Since $\sigma^{-1}\langle(04),(13)\rangle = \langle(01),(23)\rangle$ it follows $\mathbb{B}/G_3 = \mathbb{P}^1 \times \mathbb{P}^1/(\gamma_1,\gamma_2) \sim (\gamma_2,\gamma_1) \sim (1-\gamma_1, 1-\gamma_2)$. As in example 6.12 the surface is $\hat{\mathbb{F}}_2$. The weight and height table for both groups is the same:

	Q_1	Q_2	K_2	H	E	C_1	C_2	surface
weight	∞	-4	∞	4	4	4	2	
h_e	0	$1/4$	0	$-1/2$	$-1/4$	$-1/2$	$-1/2$	$3/32$

We want to prove that both groups are conjugate. We know the monodromy matrices from section 4. For C_4 and $\mu = [4; 3,1,1,1,2]$ we substitute $(q_0,q_1,q_2,q_3,q_4,q) = (i,-i,-i,-i,-1,-i)$ and obtain generators $\{T_{st}\}$ of G_4 and hermitian form $\mathcal{H}_4$.

$$T_{12} = \begin{bmatrix} i & 1-i & 0 \\ 1+i & -i & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad T'_{13} = \begin{bmatrix} 1+i & 0 & -i \\ 1+i & 1 & -1-i \\ 1 & 0 & 0 \end{bmatrix} \qquad T_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 1-i \\ 0 & 1+i & -i \end{bmatrix}$$

$$T_{01} = \begin{bmatrix} 1 & 0 & 0 \\ 1-i & 1 & 0 \\ 1-i & 0 & 1 \end{bmatrix} \qquad T_{02} = \begin{bmatrix} 1 & -1-i & 0 \\ 0 & 1 & 0 \\ 0 & 1-i & 1 \end{bmatrix} \qquad T_{03} = \begin{bmatrix} 1 & 0 & -1-i \\ 0 & 1 & -1-i \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathcal{H}_4 = \begin{bmatrix} 1 & -1 & i \\ -1 & 1 & -1 \\ -i & -1 & 1 \end{bmatrix}$$

Theorem 7.2. *The groups G_3 and G_4 presented in example 7.1 are conjugate. More precisely*

i) G_3 and G_4 are extensions

$$\Gamma_{[4;\ 1,2,2,2,1]} \xleftarrow{Z_2} \Gamma_{[4;\ 1,2,2,2,1](04)} \xleftarrow{Z_2} G_3 \;=\; G_4 \xrightarrow{Z_2} \Gamma_{[4;\ 3,1,1,1,2]}$$

of Picard modular groups corresponding to curves C_3, C_4 respectively. These curves appear in [BHH]-list as numbers 20 and 21.

ii) they are arithmetic groups

$$G_i = \{g \in GL(3,\mathbb{Z}[i]) \mid g \sim E \text{ or } E_a \mod(1+i),\ |\det(g)| = 1,\ {}^t\overline{g}\,\mathcal{H}_i g = \mathcal{H}_i\},$$
$$\Gamma_{[4;\ 3,1,1,1,2]} = \{g \in GL(3,\mathbb{Z}[i]) \mid g \sim E \mod(1+i),\ |\det(g)| = 1,\ {}^t\overline{g}\,\mathcal{H}_4 g = \mathcal{H}_4\},$$

where $E = \left[\begin{smallmatrix} 1&0&0\\0&1&0\\0&0&1 \end{smallmatrix}\right]$, $E_a = \left[\begin{smallmatrix} 0&0&1\\0&1&0\\1&0&0 \end{smallmatrix}\right]$;

iii) there exist matrix $Q \in GL(3,\mathbb{Z}[i])$, $\det Q = 1+i$, such that ${}^t\overline{Q}\mathcal{H}_3 Q = \mathcal{H}_4$, $Q^{-1}G_3Q = G_4$;

iv) the hermitian form $\mathcal{H}_3$ and $\mathcal{H}_4$ are similar to:

$${}^t\overline{g}_3\mathcal{H}_3 g_3 = \left[\begin{smallmatrix} 0&0&-i\\0&1&0\\i&0&0 \end{smallmatrix}\right], \qquad {}^t\overline{g}_4\mathcal{H}_4 g_4 = \left[\begin{smallmatrix} 0&0&-1-i\\0&1&0\\-1+i&0&0 \end{smallmatrix}\right], \qquad g_3, g_4 \in SL(3,\mathbb{Z}[i]);$$

v) let $\mathbb{B}_3$ and $\mathbb{B}_4$ be the projective two dimensional complex balls: $\mathbb{B}_i := \{w \in \mathbb{P}^2 \mid {}^t\overline{w}\mathcal{H}_i w < 0\}$. Then the quotient surfaces $\mathbb{B}_i/G_i$ are isomorphic to Hirzebruch surface $\mathbb{F}_2$ with one singularity of type $\langle 2,1\rangle$. The arrangement is as in example 6.4 and the weight table is given in example 7.1.

Proof. This is commesurability between two groups like [DM2] Corrolary 10.18.

iii-iv) We can take

$$Q := \begin{bmatrix} -1 & 1+i & -i \\ -1 & 1 & -1 \\ -1 & 1-i & -1 \end{bmatrix}, \qquad g_3 := \begin{bmatrix} 0 & -1 & i \\ i & -1 & 0 \\ i & -1 & -i \end{bmatrix}, \qquad g_4 := \begin{bmatrix} -i & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

We need to check also the inclusions $Q^{-1}G_3Q \subset G_4$ and $Q^{-1}G_3Q \supset G_4$. Since we know the generators of both groups it is suficient to check these inclusions only for them. The next equalities show that really $Q^{-1}G_3Q = G_4$. Some terms appear with negative degree – they are cusp generators. We consider projective groups and terms like $-1, \pm i$ are not important.

$$
\begin{aligned}
Q^{-1}\,M'_{04}\,Q &= i\,T_{02}\,T_{03}\,T_{23} \\
Q^{-1}\,M_{01}\,Q &= T'_{13}\,T_{23}\,T_{12} \\
Q^{-1}\,M_{02}\,Q &= i\,T_{13}\,T_{01}\,T_{03} \\
Q^{-1}\,M_{03}\,Q &= -T'_{13}\,T_{13}\,T_{03}^{-2}\,T_{02}^{-1} \\
Q^{-1}\,M_{12}\,Q &= T_{03}^{-1}\,T_{02}\,T_{03} \\
Q^{-1}\,M'_{13}\,Q &= T_{23}\,T_{03}\,T_{23} \\
Q^{-1}\,M_{23}\,Q &= T_{23}^{2}\,T_{02}
\end{aligned}
\qquad
\begin{aligned}
T_{01} &= Q^{-1}M_{01}^{3}\,M'_{13}\,M_{01}Q \\
T_{02} &= Q^{-1}M_{23}Q \\
T_{03} &= Q^{-1}M'^{-1}_{13}\,M_{12}^{-1}\,M_{13}\,M_{23}Q \\
T_{12} &= Q^{-1}M_{12}\,M'_{13}\,M'^{3}_{04}\,M_{01}^{3}\,M_{12}^{-1}\,M_{02}^{3}\,M_{01}Q \\
T'_{13} &= i\,Q^{-1}M_{02}\,M_{12}\,M_{01}Q \\
T_{23} &= -i\,Q^{-1}M'_{04}\,M'^{-1}_{13}\,M_{12}^{-1}Q
\end{aligned}
$$

i) Since $Q^{-1}G_3Q = G_4$ and from the construction of both groups i) follows immediately.

ii) The inclusions $\subset$ follows from the fact that the generators satisfy such inclusions. The other direction is not trivial. From the second part of the paper we know the exact group sequence

$$1 \longrightarrow \{g \in A(\mu_3) \mid g \sim E \bmod (1+i)\} = \Gamma_{\mu_3(04)} \longrightarrow F\Gamma_{\mu_3} = A(\mu_3) \longrightarrow S_3 \longrightarrow 1\,.$$

The ideal $(1+i)$ splits $A(\mu_3)$ onto 6 classes. They are characterized as $\{g \in A(\mu_3) \mid g \sim E_i \bmod(1+i)\}$, where E_i are defined in Lemma 4.5. In the following we will write only ~ 0 instead of $\sim 0 \bmod(1+i)$. Now it is easy to get the opposite inclusion $\supset$ for G_3, namely

$$G_3 = \{g \in A(\mu_3) \mid g \sim E \text{ or } E_a,\ {}^t\overline{g}\mathcal{H}_3 g = \mathcal{H}_3\}\,.$$

We need the following

Proposition 7.3. *Let Q be the matrix from Theorem 7.2, $\det Q = 1+i$. Then*

i) $Q\{g \in GL(3,\mathbb{Z}[i] \mid g \sim 0 \bmod(1+i)\}Q^{-1} \subset\ GL(3,\mathbb{Z}[i])$.

ii) $QE_aQ^{-1} \sim E \bmod(1+i)$.

iii) Let $\{E_s\}$ be the set of matrices from Lemma 4.5. Then only two of them E and E_a satisfy the inclusion $Q^{-1}E_sQ \subset GL(3,\mathbb{Z}[i])$.

Let us assume that there is an element p of $\{g \in GL(3,\mathbb{Z}[i]) \mid g \sim E \text{ or } E_a,\ {}^t\overline{g}\,\mathcal{H}_4 g = \mathcal{H}_4\}$ and $p \notin G_4$. We want to get contradiction. We consider $r := QpQ^{-1}$. Using the proposition 7.3 we obtain $r \in GL(3,\mathbb{Z}[i]),\ {}^t\overline{r}\mathcal{H}_3 r = \mathcal{H}_3$. In other words r is an element of the full Picard modular group $F\Gamma_{\mu_3}$. If $r \in G_3$ then using Theorem 7.2 iii) we have $Q^{-1}rQ = p \in G_4$. So let us assume that $r \notin G_3$. Then r does not belongs to classes presented by E or E_a. But in this case $Q^{-1}rQ = p \not\subset GL(3,\mathbb{Z}[i])$ and we have contradiction.

Since $T'_{13} \sim E_a$ and $E_a^2 = E$ we get $\Gamma_{\mu_4} = \{g \in GL(3,\mathbb{Z}[i]) \mid g \sim E,\ {}^t\overline{g}\,\mathcal{H}_4 g = \mathcal{H}_4\}$.

Now v) follows from examples 6.9 and 7.1. $\square$

Remark 7.4. We have translated information from G_3 to G_4. On the similar way we can also connect classical Picard curve $w^3 = u(u-1)(u-x)(u-y)$ (example 6.7) with Apoll-2 curve $w^6 = u(u-1)(u-x)^4(u-y)^4$. This is another example of commesurability.

Part II

Arithmetic Lattices

In this part we work only with the lattice of Gauß integers and use different notations. They are shortly and more convenient in this case. We give the connection with the notations from Part I, $\mu = [4;\ 1,2,2,2,1]$, in the table:

part I	part II
$A(\mu) = F\Gamma_\mu$	Γ
$A(\mu)(1+i) = \mathcal{A}\Gamma_\mu = \Gamma_{\mu(04)}$	$\Gamma' = \Gamma(\pi) = \Gamma(1+i)$
Γ_μ	Γ_2

8 Ball lattice conditions

We look for an arithmetic ball lattice $\Gamma \subset \mathbb{SU}((2,1),\mathbb{C}) \subset \mathbb{G}l_3(\mathbb{C})$ acting effectively on the complex two-ball

$$\mathbb{B} = \{(z_1,z_2) = (z_1 : z_2 : 1) \in \mathbb{P}^2; |z_1|^2 + |z_2|^2 < 1\} \subset \mathbb{P}^2 = \mathbb{P}^2(\mathbb{C})$$

via projective (fractional linear) transformations with postulated data described in 8.1 below (for special Γ' instead of general Γ). For the sake of simplicity we assume that all our ball lattices Γ are arithmetical (arithmetic defined subgroup of $\mathbb{SU}((2,1),\mathbb{C})$) and that they act effectively on $\mathbb{B}$.

Furthermore we use the following notions, see [H98], especially chapter IV, for more details. A *reflection* is an element $1 \neq \sigma \in \Gamma$ of finite order fixing a subdisc $\mathbb{D} = \mathbb{D}_\sigma$ of $\mathbb{B}$ pointwise. The disc $\mathbb{D}_\sigma$ is uniquely determined by σ. It is called a Γ-*reflection disc*, if such $\sigma \in \Gamma$ exists. If Γ is fixed we omit the prefix Γ-, also for further notations depending on Γ. For given subdisc $\mathbb{D}$ of $\mathbb{B}$ we call σ a $\mathbb{D}$-*reflection*, if $\mathbb{D} = \mathbb{D}_\sigma$ for a reflection σ. The group of $\mathbb{D}$-reflections in Γ is finite cyclic. Its order is called the Γ-*reflection order* at/of $\mathbb{D}$. It coincides, say by definition, with the ramification index of the natural locally finite quotient map $p : \mathbb{B} \longrightarrow \mathbb{B}/\Gamma$ along $\mathbb{D}$, and appears as weight of the orbital image curve $\mathbb{D}/\Gamma$ on the orbital quotient surface $\mathbb{B}/\Gamma$.

A Γ-cusp is a boundary point $\kappa \in \partial\mathbb{B}$ of $\mathbb{B}$ such that the unipotent elements of the isotropy group Γ_κ form a lattice in the unipotent radical of the parabolic group $\mathbf{P}_\kappa(\mathbb{R})$ of all elements of $\mathbb{SU}((2,1),\mathbb{C})$ fixing κ. The set of all Γ-cusps is denoted by $\partial_\Gamma\mathbb{B}$. The quotient map p extends in a continuous manner to a unique surjective map $p^* : \mathbb{B}^* \twoheadrightarrow \widehat{\mathbb{B}/\Gamma}$ from $\mathbb{B}^* = \mathbb{B}^*(\Gamma) := \mathbb{B} \cup \partial_\Gamma\mathbb{B}$ onto the Baily-Borel compactification $\widehat{\mathbb{B}/\Gamma}$ of $\mathbb{B}/\Gamma$, which is a projective surface adding a finite number of normal points to $\mathbb{B}/\Gamma$.

An element $1 \neq \gamma \in \Gamma$ is called (honestly) *elliptic* if it has finite order and is not a reflection. It is equivalent to say that γ has precisely one fixed point Q on $\mathbb{B}$. In opposition we call $Q \in \mathbb{B}$ a Γ-*elliptic point*, if it is an *isolated fixed point* of Γ, which means that an elliptic element $\gamma \in \Gamma$ exists fixing Q.

Two subsets M, N of $\mathbb{B}^*$ are called Γ-*equivalent*, iff there is a $\gamma \in \Gamma$ such that $N = \gamma(M)$. Two points P, $Q \in \mathbb{B}^*$ are said to be Γ-*equivalent*, iff $\{P\}$ and $\{Q\}$ are. The Γ-equivalence classes of Γ-elliptic points, Γ-cusps or Γ-reflection discs are finite, see [H98].

We look for an arithmetic ball lattice Γ' satisfying seven special conditions. For the subdiscs $\mathbb{D}_i$ below we will use the following notation for the subgroup of all elements acting on $\mathbb{D}_i$:

$$\Gamma'_i := \{\gamma \in \Gamma';\ \gamma(\mathbb{D}_i) = \mathbb{D}_i\},\ i = 0,1,2,3.$$

Postulates 8.1. *for the ball lattice* Γ'

(i) There are precisely three Γ'*-inequivalent* Γ'*-cusps* $\kappa_1,\kappa_2,\kappa_3 \in \partial\mathbb{B}$. *The corresponding cusp points* K_1,K_2,K_3 *on* $\hat{X} := \widehat{\mathbb{B}/\Gamma'}$ *are nonsingular.*

(ii) There is up to Γ'*-equivalence precisely one* Γ'*-reflection disc* $\mathbb{D}_0 \subset \mathbb{B}$ *with reflection order 4 such that* $\kappa_1,\kappa_2,\kappa_3 \in \partial\mathbb{D}_0$ *is a complete set of* Γ'_0*-inequivalent cusps for the quotient curve* $\mathbb{D}_0/\Gamma'_0$.

(iii) Up to Γ'-equivalence there are precisely three Γ'-reflection discs $\mathbb{D}_1$, $\mathbb{D}_2$, $\mathbb{D}_3$ with reflection order 4 supporting at the boundary $\partial\mathbb{D}_j$ precisely one cusp up to Γ'_j-equivalence, namely κ_j, $j = 1, 2, 3$, respectively.

(iv) Each Γ'-reflection disc is Γ'-equivalent to one of the four discs above.

(v) Up to Γ'-equivalence there are precisely three Γ'-elliptic points $O_1, O_2, O_3 \in \mathbb{B}$. They coincide with the pairwise intersection points of $\mathbb{D}_1$, $\mathbb{D}_2$, $\mathbb{D}_3$ (for suitable choice of the three discs). The isotropy group Γ'_{O_j}, $O_j := \mathbb{D}_k \cap \mathbb{D}_l$, $\{j, k, l\} = \{1, 2, 3\}$ coincides with the abelian group of order 16 generated by the reflections of order 4 fixing the points of $\mathbb{D}_k$ or $\mathbb{D}_l$, respectively.

(vi) The Euler-Bergmann volume of a Γ'-fundamental domain is equal to $\frac{3}{16}$.

(vii) There is a subgroup Σ_3 of $Aut_{hol}\mathbb{B}$ isomorphic to S_3 normalizing Γ', which acts on $\mathbb{D}_0$ and permutes $\mathbb{D}_1$, $\mathbb{D}_2$, $\mathbb{D}_3$.

We illustrate the situation in Picture 8 with a mixed 2- or 3-dimensional imagination (the latter around $\mathbb{D}_0$ with boundary points $\kappa_1, \kappa_2, \kappa_3$) of the real 4-dimensional unit ball $\mathbb{B}$.

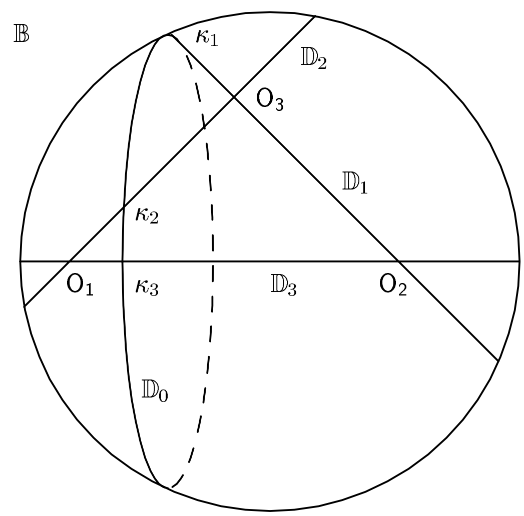


Figure 8: Representative Γ'-fixed point configuration on $\mathbb{B}$

Theorem 8.2. *Under the conditions (i) - (vii) it holds that $\hat{X} = \widehat{\mathbb{B}/\Gamma'}$ is the projective plane $\mathbb{P}^2$. The compactified branch locus of the quotient map*

$$p : \mathbb{B} \longrightarrow X = \mathbb{B}/\Gamma'$$

consists of a quadric $\hat{C}_0$ and three tangents $\hat{C}_j$, $j = 1, 2, 3$. These curves are the (compactified) images of the reflection discs $\mathbb{D}_0$ or $\mathbb{D}_j$, $j = 1, 2, 3$, respectively. There is up to $\mathbb{P}\mathbb{G}l_3$-equivalence an - up to S_3-symmetry - unique projective coordinate system on $\mathbb{P}^2$ such that the projective lines $\hat{C}_j$, $j \neq 0$, are the coordinate axes and the quadric has the equation

$$\hat{C}_0 : (X + Y - Z)^2 - 4XY = X^2 + Y^2 + Z^2 - 2XY - 2XZ - 2YZ = 0. \tag{47}$$

In orbital surface terms we will prove mainly that

8.3. *The orbital ball quotient surface $\hat{\mathbf{X}} = \widehat{\mathbb{B}/\Gamma'}$ coincides, up to projective equivalence, with*

$$\hat{\mathbf{X}} = (\hat{X}; \hat{\mathbf{C}}_0 + \hat{\mathbf{C}}_1 + \hat{\mathbf{C}}_2 + \hat{\mathbf{C}}_3 + \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 + \mathbf{K}_1 + \mathbf{K}_2 + \mathbf{K}_3)$$

described in section 1, (1), (2), (3) with properties 1.1 (i), (ii) a),b),c),d) (omitting the symmetry condition e) here).

The open curves $C_i = \hat{C}_i \backslash \{K_1, K_2, K_3\}$ are defined as images of the discs $\mathbb{D}_i$, $i = 0,1,2,3$, the points P_j are the images of the elliptic points O_j, and the cusp points K_j are the images of the cusps κ_j with respect to the extended quotient map $p^* : \mathbb{B}^* \longrightarrow \widehat{\mathbb{B}/\Gamma'}$.

We use again the height calculus for orbital surfaces developed in [H98] based on equivariant K-theory. The orbital heights of orbital ball quotient surfaces are links between differential geometric volumes of fundamental domains and algebraic-geometric invariants of surfaces or embedded curves. Mainly Euler heights h_e and signature heights h_τ are used. We dispose on the following strong

Theorem 8.4. *([H98], IV, Theorem 4.8.1, first part) For ball lattices $\Gamma \subset \mathbb{U}((2,1),\mathbb{C})$ with open orbital ball quotient $\mathbb{B}/\Gamma$ it holds that*

$$H_e(\mathbb{B}/\Gamma) = covol_{EB}(\Gamma) := vol_{EB}(\mathfrak{F}_\Gamma) := vol_{\gamma_2}(\mathfrak{F}_\Gamma).$$

□

Thereby $\mathfrak{F}_\Gamma$ denotes a Γ-fundamental domain on $\mathbb{B}$, and the volume is taken with respect to the $\mathbb{U}((2,1),\mathbb{C})$-invariant Euler-Bergmann (volume) form $\gamma_2 = \frac{1}{3}\gamma_1 \wedge \gamma_1$ on $\mathbb{B}$ with $\gamma_1 = -3\omega$ (Kähler-Einstein relation) the Ricci form and ω the Kähler form of the Bergmann metric on $\mathbb{B}$. For these details we refer to [BHH], Appendix B.

From this theorem and condition (vi) for Γ' we get

$$covol_{EB}(\Gamma') = \frac{3}{16} \tag{48}$$

The signature form on $\mathbb{B}$ can be proportionally defined to be $\sigma = \frac{1}{3}(\gamma_1 \wedge \gamma_1 - 2\gamma_2)$. As for Euler heights we have

Theorem 8.5. *([H98], IV, Theorem 4.8.1, second part) For ball lattices $\Gamma \subset \mathbb{U}((2,1),\mathbb{C})$ it holds that*

$$H_\tau(\mathbb{B}/\Gamma) = \frac{1}{3} covol_{EB}(\Gamma) = covol_\sigma(\Gamma) = vol_\sigma(\mathfrak{F}_\Gamma).$$

□

This is the origin of the proportionality relation ($Prop$ 2) for orbital ball quotient surfaces $\mathbb{B}/\Gamma$. From condition (vi) for Γ' we get now

$$H_\tau(\mathbb{B}/\Gamma') = \frac{1}{3} H_e(\mathbb{B}/\Gamma') = \frac{1}{16} \tag{49}$$

Now we change our attention to orbital curves coming from discs. Let $\mathbb{D} \subset \mathbb{B}$ be a (linearly embedded complete) disc whose image on $\mathbb{B}$ is an algebraic curve $\mathbb{D}/\Gamma$ on $\mathbb{B}/\Gamma$ (Γ-disc). For the finer object, the orbital curve $\mathbb{D}/\Gamma \subset \mathbb{B}/\Gamma$. Euler height and covolume are connected by

Theorem 8.6. *([H98], IV.7, first part of (4.7.7))*

$$h_e(\mathbb{D}/\Gamma) = covol_{EP}(\Gamma_\mathbb{D}) := vol_{EP}(\mathfrak{F}_{\Gamma_\mathbb{D}}) := vol_\eta(\mathfrak{F}_{\Gamma_\mathbb{D}}),$$

□

where

$$\Gamma_\mathbb{D} := N_\Gamma(\mathbb{D})/Z_\Gamma(\mathbb{D}) \tag{50}$$

is the effectivized subgroup of all elements of Γ acting on $\mathbb{D}$,

$$N_\Gamma(\mathbb{D}) = \{\gamma \in \Gamma; \gamma(\mathbb{D}) = \mathbb{D}\}, Z_\Gamma(\mathbb{D}) = \{\gamma \in \Gamma; \gamma_\mathbb{D} = id_\mathbb{D}\}. \tag{51}$$

The volume of a $\Gamma_\mathbb{D}$-fundamental domain $\mathfrak{F}_{\Gamma_\mathbb{D}}$ is taken with respect to the $\mathbb{U}((1,1),\mathbb{C})$-invariant Euler-Poincaré form η on $\mathbb{D}$. This explicitly well-known volume form is normalized in such a way that the height h_e of any compact quotient curve C of $\mathbb{D}$ by a torsion free $\mathbb{D}$-lattice N is nothing else but the Euler number $e(C) = 2 - 2g < 0$, g the genus of C. Assume for a moment that $N = N_G(\mathbb{D})$ for a torsion free cocompact ball lattice G and K is a canonical divisor of the smooth compact algebraic surface $\mathbb{B}/\Gamma$. By relative proportionality, see [BHH], appendix B.3.E, it holds that $3e(C) = -2(K \cdot C)$. Together with the adjunction formula $-(K \cdot C) = e(C) + (C^2)$ one gets $e(C) = 2(C^2)$. In order to calculate selfintersection numbers by means of volumes we define adequately the signature form on $\mathbb{D}$ to be $\frac{1}{2}\eta$. We proved also

Theorem 8.7. *([H98], IV.7, second part of (4.7.7))*

$$h_\tau(\mathbb{D}/\Gamma) = \frac{1}{2} covol_{EP}(\Gamma_\mathbb{D}) = \frac{1}{2} vol_\eta(\mathfrak{F}_{\Gamma_\mathbb{D}}).$$

□

This is the origin of proportionality condition (*Prop* 1) for orbital disc quotients on ball quotient surfaces. Especially we get

$$2h_\tau(\mathbb{D}/\Gamma'_i) = h_e(\mathbb{D}/\Gamma'_j)\ i = 0, 1, 2, 3. \tag{52}$$

Now we check the admissibility of our cusp conditions, see ($Prop\,\infty$). There are precisely 3 cusp points K_1, K_2, K_3 on $\widehat{\mathbb{B}/\Gamma'}$ coming from $\kappa_1, \kappa_2, \kappa_3$ (condition (i)). The possible graphs of the corresponding orbital cusp points $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3$ are classified in [H98], III.3.5. We denote one of these points, say the first, by κ, K or $\mathbf{K}$, respectively. In general, each cusp point is the quotient of an elliptic singularity by a cyclic group G_κ of order 1, 2, 3, 4 or 6, see [H98], IV.4.5. Since two 4-reflection discs go through our special κ and there are no 2-reflection discs (condition (iv) and (ii), (iii) before), the group G_κ is cyclic of order 4, and the graph of $\mathbf{K}$ must look like

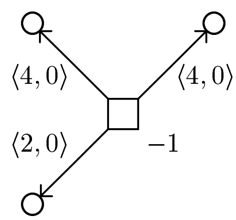


Figure 9: atomic graph of cusp point

(-1 in the box will be explained below, see (53). This means that K has a canonical smooth rational resolution curve E_κ supporting a surface singularity of cyclic quotient type $\langle 2, 1\rangle$. In [H98] we call it the cusp curve corresponding to the center of the resolution graph 9 of κ. Remember that we have three of them: E_1, E_2, E_3 corresponding to $\kappa_1, \kappa_2, \kappa_3$, which are contracted to K_1, K_2, K_3, respectively, along the birational morphism $X' \longrightarrow \hat{X} = \widehat{\mathbb{B}/\Gamma'}$. Resolving the three singularities of type $\langle 2, 1\rangle$ by rational -2-curves we get a birational morphism $\tilde{X} \longrightarrow X'$ with three connected exceptional curves $L_j + E_j$ on $\tilde{X}$ contracted to the nonsingular points K_j along $\tilde{X} \longrightarrow \hat{X}$ by the last part of condition (i). Omitting indices again, the smooth rational components L, E intersect each other transversally and $(L^2) = -2$. The contraction to a nonsingular point is only possible, if E has on $\tilde{X}$ selfintersection $(E^2)_{\tilde{X}} = -1$. So for all proper transforms of E_j on $\tilde{X}$ we get

$$(E_j^2)_{\tilde{X}} = -1,\ j = 1, 2, 3. \tag{53}$$

Proposition 8.8. *The compactified ball quotient surface $\hat{X} = \widehat{\mathbb{B}/\Gamma'}$ is smooth. Moreover, the closures $\hat{C}_i$ of $C_i := \mathbb{D}_i/\Gamma'$, $i = 0, 1, 2, 3$, on $\hat{X}$ are smooth curves.*

Proof. The singularities of any Baily-Borel compactified ball quotient surface come from (honest) elliptic points and cusps. The cusp points K_j are nonsingular by (i). By condition (v) there are only three points $P_j \in \mathbb{B}/\Gamma'$ with elliptic preimages, namely the images of O_j, $j = 1, 2, 3$. Let O be one of them. The corresponding isotropy group Γ'_O is generated by reflections, see condition (iv) again. Therefore the points P_j are nonsingular (Chevalley criterion [Bou], V.5 Theorem 4); for our application, see [H98], I.1, Lemma 1.1.1 and IV.5, proof of Lemma IV.5.9). Now it is clear that $\hat{X}$ has to be smooth.

We denote by $\hat{C}$ be an arbitrary one of the curves $\hat{C}_i \subset \hat{X}$ and by C' its proper transform on X'. Assume that $\hat{C}$ goes through one of our cusp points K with canonical resolution curve E on X'. Its preimage on $\mathbb{B}$ is one of the Γ'-reflection discs $\mathbb{D} = \mathbb{D}_j$. It corresponds to one of the $\langle 4, 0\rangle$ arrows in the cusp diagram 9. Looking down again to X' this diagram teaches us that C' intersects E locally transversal at (at most two) nonsingular surface points. By (ii) and (iii) E is intersected by precisely two of the reflection curves C'_j because the cusps κ_i are boundaries of precisely two of the corresponding reflection discs, see picture 8. So C' intersects E at one point only. Because of transversality this is a nonsingular point of C'. This point remains nonsingular on $\hat{C} \subset \hat{X}$ after contraction of E (or of $L+E$ starting from $\tilde{X}$). Locally around E_1, E_2, E_3 the intersection behaviour of these curves on X' with C'_j, $j = 0, 1, 2, 3$, is described in Picture 2.

It remains to be proved that the non-compact curves $C_j \subset \mathbb{B}/\Gamma'$ are smooth. In [H98], IV.4, we proved that for Γ'-rational discs $\mathbb{D}$ on $\mathbb{B}$ the natural map $\mathbb{D}/\Gamma'_{\mathbb{D}} \longrightarrow \mathbb{D}/\Gamma'$ is the normalization (singularity resolution) of the latter curve on $\mathbb{B}/\Gamma'$. Our Γ'-reflection discs are arithmetic because Γ' is. Curve singularities on $\mathbb{D}/\Gamma'$ come from (honest) Γ'-*cross points* Q on $\mathbb{D}$. Such a point Q is characterized by the property that through Q goes a Γ'-equivalent disc $\mathbb{D}'$ not being Γ'_Q-equivalent, see [H98], IV, Definition 4.4.5 and Proposition 4.4.6. Assume that Q is a Γ'-cross point of $\mathbb{D}$. Then it is the intersection point of two Γ'-reflection discs $\mathbb{D} = \mathbb{D}_\sigma$, $\mathbb{D}' = \mathbb{D}_\delta$ belonging to reflections σ, $\delta \in \Gamma'$, say. Then Q is an elliptic point because it is fixed also by the elliptic element $\sigma\delta$, which is not a reflection, because its representation on the tangent space $T_Q = T_Q(\mathbb{B})$ at $Q \in \mathbb{B}$ has two non-trivial eigenvalues, namely the non-trivial eigenvalue of σ and the non-trivial eigenvalue of δ. The only Γ'-elliptic points are the Γ'-orbits of O_1, O_2, O_3 by condition (iv). So we can assume without loss of generality that Q is one of these points, say $Q = O_3 = \mathbb{D}_1 \cap \mathbb{D}_2$, $\mathbb{D} = \mathbb{D}_1 = \mathbb{D}_\sigma$. The disc $\mathbb{D}'$ cannot coincide with $\mathbb{D}_2$ because the latter disc is not Γ'-equivalent with $\mathbb{D}_1$ by (iii). Therefore Q is the intersection point of three different reflection discs. But then the isotropy group Γ'_Q is not abelian because their elements produce at least three eigenlines in T_Q by the directions of the three reflection discs through Q. This contradicts to the second part of condition (iv). Hence, there is no Γ'-reflection disc $\mathbb{D}$ with Γ'-cross point; the image curves are smooth. This finishes the proof of the proposition. $\square$

It follows that the orbital quotient surface looks like

$$\widehat{\mathbb{B}/\Gamma'} = \hat{\mathbf{X}} = (\hat{X}; \hat{\mathbf{C}}_0 + \hat{\mathbf{C}}_1 + \hat{\mathbf{C}}_2 + \hat{\mathbf{C}}_3 + \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 + \mathbf{K}_1 + \mathbf{K}_2 + \mathbf{K}_3)$$

we started with in section 1 not knowing until now that $\hat{X} = \mathbb{P}^2$. Moreover, we have to prove the properties (i), (ii) a), ... ,d) before definition 1.1. Let us start with

c$'$) P_1, P_2, P_3 are the three different intersection points of the curves
C_1, C_2, C_3.

This follows now immediately from (iv), because an intersection point of two of these reflection curves is necessarily an image point of Γ'-elliptic point. Up to Γ'-equivalence there are only three of them, namely O_1, O_2, O_3.

d$'$) $\hat{C}_j$ and $\hat{C}_0$ touch each other at K_j (with local intersection number 2), $j = 1, 2, 3$.

$\hat{C}_0$ goes through each of the cusp points K_j by (ii). The other reflection curve through K_j is $\hat{C}_j$ by (iii), see Figure 8. From the intersection graph 9 we deduced the intersection behaviour of the curves C'_0, C'_j, E_j locally around E_j, which is described in picture 2. Going back to $\tilde{X}$ we blow down first the -1-curve E_j. On the corresponding surface the proper transforms of C'_0 and C'_j intersect each other transversally. The proper transform of the -2-curve L becomes a smooth rational -1-curve denoted by L again supporting this intersection point. The intersection of the two C-curves with L are transversal, too. Now blow down the -1-curve L to K_j to see that the local situation of touching we look for is well-described in picture 1.

Now we relate Euler numbers e_i with selfintersections s'_i of C'_i on X' for $i = 0, 1, 2, 3$ using geometric height formulas (12), (13) for orbital curves $\mathbf{C}$ on open orbital surfaces:

$$h_e(\mathbf{C}) = e(C') - \sum (1 - \frac{1}{v_i d_i}) - \#C'_\infty \ ,$$

$$h_\tau(\mathbf{C}) = \frac{1}{v}[(C'^2) + \sum \frac{e_i}{d_i} + \sum \frac{e_j}{d_j}] \ .$$

The sums on the right-hand side can be read off from the atomic graph of the orbital curve $\mathbf{C}$ (or compact orbital curve $\mathbf{C}' = (vC';\ \sum \mathbf{P}_i + \sum \mathbf{K}_m)$ which have been already described in Figure 3.

Filling these contributions in the height formulas we get together with (Prop 1), see (52),

$$
\begin{aligned}
&h_e(\mathbf{C}_0) = e_0 - 3,\\
&h_e(\mathbf{C}_j) = e_j - (1-\frac{1}{4}) - (1-\frac{1}{4}) - 1\ ,\ j = 1,2,3;\\
&h_\tau(\mathbf{C}_0) = \frac{1}{4}(s_0' + 0 + 0),\\
&h_\tau(\mathbf{C}_j) = \frac{1}{4}(s_j' + 0 + 0)\ ,\ j = 1,2,3.
\end{aligned}
\tag{54}
$$

It follows that

$$s_0' = 2e_0 - 6 \quad , \quad s_j' = 2e_j - 5\ ,\ j = 1,2,3.$$

Blowing down the three rational -1-curves and the three rational -2-curves on $\tilde{X}$ to the cusp points K_1, K_2, K_3 we get on $\hat{X}$ the selfintersections $s_0 = s_0' + 6$, $s := s_j = s_j' + 2$ for the curves $\hat{C}_i$, $i = 0,1,2,3$, because $\hat{C}_0$ goes through all three cusp points and each $\hat{C}_j$ only through one of them. It follows that

$$s_0 = 2e_0 \quad , \quad s = 2e - 3,\ e := e_j = e(\hat{C}_j), j > 0.$$

In a similar opposite use of height formulas in comparison with their calculation in the previous section we can calculate now the Euler number and signature of $\hat{X}$ using (16) and (17):

$$H_e(\mathbf{X}) = e(X') - \sum(1 - \frac{1}{v_i})h_e(\mathbf{C}_i) - \sum h_e(\mathbf{P}_j) - 2\#\{rational\, cusp\, points\}$$

$$H_\tau(\mathbf{X}) = \tau(X') - \frac{1}{3}\sum(v_i - \frac{1}{v_i})h_\tau(\mathbf{C}_i) - \sum h_\tau(\mathbf{P}_j) - \sum h_\tau(\mathbf{K}_m)$$

The point contributions have been already substituted in 2, see (23). The left-hand sides are known from (49). So we get with the above substitutions ($h_\tau(\mathbf{C}_j) = s'/4 = (2e-5)/4$ ...)

$$\frac{3}{16} = e(X') - (1 - \frac{1}{4})(e_0 - 3) - 3(1 - \frac{1}{4})(e - \frac{5}{2}) - 3 \cdot \frac{9}{16} - 2 \cdot 3$$

$$\frac{1}{16} = \tau(X') - \frac{1}{3}[(4 - \frac{1}{4})(2e_0 - 6)/4 + 3(4 - \frac{1}{4})(2e - 5)/4] - 3 \cdot 0 - 3 \cdot (-\frac{1}{6}).$$

Set $E := e(\hat{X}) = e(X') - 3$ and $S := \tau(\hat{X}) = \tau(X') + 3$. After substitution we obtain

$$
\begin{array}{ccccccc}
8E & + & 6(3-e_0) & + & 9(5-2e) & = & 39,\\
16S & + & 10(3-e_0) & + & 15(5-2e) & = & 41
\end{array}
\tag{55}
$$

Proposition 8.9. *Let Y be a smooth compact complex algebraic surface supporting a configuration $L_0 + L_1 + L_2 + L_3$ with smooth curves L_i, $i = 0,1,2,3$ intersecting pairwise in at least one point. Assume that the invariants $E = e(Y)$, $S = \tau(Y)$, $e_0 = e(L_0)$ and $e = e(L_j)$, $j = 1,2,3$ satisfy the relations (55). Then $Y = \mathbb{P}^2$, and the curves $L_i's$, $i = 0,1,2,3$, are rational.*

Proof. We need some basic facts of surface classification theory, which can be found in [BPV], for instance. Adding the first to the second equation of (55) we get the relation

$$64\chi + 22(3 - e_0) + 33(5 - 2e) = 119 \tag{56}$$

for the arithmetic genus $\chi = \chi(Y) = (E + S)/4$ of Y. The integers

$$3 - e_0 = 2g_0 + 1 \quad , \quad 5 - 2e = 4g + 1\,, \tag{57}$$

where g_0, g are the genera of L_0 or L_j, $j > 0$, respectively, are positive. From (56) we get $\chi < 0$ or

$$\chi(Y) = 1, g_0 = g(L_0) = 0, g = g(L_j) = 0. \tag{58}$$

We exclude the former case: Assume that $\chi < 0$. Then Y has negative Kodaira dimension. By surface classification theory Y must be a (blown up) ruled surface over a smooth compact curve B of genus q, say. The arithmetic genus of Y is equal to $\chi = 1 - q < 0$. The fibres of the fibration $Y \longrightarrow B$ are linear trees of rational curves. Since, by assumption, $L_1 + L_2 + L_3$ is a connected cycle it cannot belong to any finite union of fibres. Therefore one of the components covers B finitely. It follows that $g \geq q$. The identity (56) yields

$$64(1-q) + 22(1+2g_0) + 33(1+4g) = 119,$$

hence $11g_0 + 33g = 16q$, which contradicts to $g \geq q > 1$.

We proved that the relations (58) must be satisfied. Altogether we solve(d) the simple linear system (55) of diophantine equations coming from the Proportionality Theorem. We get the surface invariants

$$\chi = 1,\ E = 3,\ S = 1,\ (K^2) = 9,\ (K^2)/E = 3,$$

where $(K^2) = 12\chi - E$ is the selfintersection index of a canonical divisor K on Y. We proved also that L_i, $i = 0, 1, 2, 3$, is rational by (58).

The extreme Chern quotient $(K^2)/E = 3$ with positive Euler number E is only possible for $Y = \mathbb{P}^2$ or for compact ball quotient surfaces $\mathbb{B}/\Gamma$ for torsion free ball lattices Γ by a theorem of Miyaoka-Yau, Kodaira-classification of surfaces and fine classification of rational surfaces, see [H98], V.2, Proposition 5.2.4, and the references given there. But $\mathbb{B}$, hence also $\mathbb{B}/\Gamma$, is hyperbolic in the sense of Kobayashi. Therefore it does not support any rational curve. The compact ball quotient case is excluded by the rationality of $L_i \subset Y$. Therefore Y must be the projective plane. □

Corollary 8.10. *If Γ' satisfies the conditions (i),...,(vii), then $\hat{X} = \widehat{\mathbb{B}/\Gamma'}$ is the projective plane, $\mathbb{P}^2$, $\hat{C}_0$ is a quadric and $\hat{C}_1$, $\hat{C}_2$, $\hat{C}_3$ are tangent lines. In other words, $\hat{C}_0 + \hat{C}_1 + \hat{C}_2 + \hat{C}_3$ is a plane Apollonius configuration.*

Proof. We have only to summarize. $\hat{X}$ is a smooth surface by 8.8. Moreover, as Baily-Borel compactification $\hat{X}$ is projective, hence algebraic. We proved already that our four curves $\hat{C}_i$ are smooth, see Proposition 8.8. Together with c′), and (55) the assumptions of the proposition are satisfied. Therefore $hatX = \mathbb{P}^2$ and our curves are rational. More precisely, from Bezout's theorem and the intersection behaviour described in c′, d′ follows that the configuration is of Apollonius type. □

Now we finish the proof of 8.3 and Theorem 8.2. The projective lines $\hat{C}_j$ can be used as coordinate lines $X = 0$, $Y = 0$, $Z = 0$ of $\mathbb{P}^2$ such that the configuration divisor $\hat{C}_0 + \hat{C}_j + \hat{C}_1 + \hat{C}_1$ is S_3-invariant by Proposition 1.3 and Corollary 1.4 with the natural projective action of S_3 on $\mathbb{P}^2$ permuting coordinates. The uniqueness of the equation (47) of $\hat{C}_0$ comes from (the proof) of Lemma 1.6 verifying that this equation is the only S_3-symmetric possibility. Theorem 8.2 is proved.

The weights for the orbital cycle of a smooth orbital ball quotient surface come from reflection orders only, by definition. Therefore 8.3 follows now from these order postulates in 8.1 (ii),(iii) and from postulate (iv) forbidding other branch curves beside of C_i, $i = 0, 1, 2, 3$. □

9 The Gauß congruence ball lattice

Let $\mathbb{Q}(i)$, $i = \sqrt{-1}$, be the field of Gauss numbers and $\mathfrak{O} = \mathbb{Z}[i] = \mathbb{Z} + \mathbb{Z}i$ the (maximal) order of Gauss integers in it. The center Z of the unitary group

$$\tilde{\Gamma} := \mathbb{U}((2,1),\mathfrak{O}) = \{g \in \mathbb{G}l_3(\mathfrak{O});\ {}^t\bar{g} \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right) g = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)\}$$

with Gauss integers as coefficients is generated by $\left(\begin{smallmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{smallmatrix}\right)$. The ineffective kernel of the action of $\tilde{\Gamma}$ on the ball $\mathbb{B}$ coincides with Z. We concentrate our attention to the *special Gauss ball lattice* $\Gamma := \mathbb{SU}((2,1),\mathfrak{O})$, which is an arithmetic ball lattice acting effectively on $\mathbb{B}$. It holds that $\tilde{\Gamma} = Z \cdot \Gamma$. The isomorphisms

$$\tilde{\Gamma}/Z \cong \Gamma \cong \mathbb{PU}((2,1),\mathfrak{O}) \cong \mathbb{PSU}((2,1),\mathfrak{O})$$

allow us to identify (sometimes, if we want) these groups. The most important role plays the congruence subgroup $\Gamma' := \Gamma(1+i)$ (*Gauss congruence ball lattice*) of the prime ideal of $\mathbb{Z}[i]$ generated by the prime divisor $1 + i$ of 2 with residue field $\mathbb{F}_2$.

We want to prove the following

Theorem 9.1. *The arithmetic ball lattice Γ' satisfies all conditions (i),..,(vii) of 8.1. The Baily-Borel compactification $\widehat{\mathbb{B}/\Gamma'}$ is equal to $\mathbb{P}^2$ with Apollonius configuration 1 supporting the orbital cycle of $\widehat{\mathbb{B}/\Gamma'}$.*

An essential role in the proof plays the theory of hermitian lattices, which is not so difficult in the case of $\mathfrak{O}$ − *lattices* with small ranks, because $\mathfrak{O}$ is an euclidean ring. The basic lattice is $\Lambda := \mathfrak{O}^3$ endowed with the indefinite unimodular hermitian form

$$\langle\ ,\ \rangle\ :\ \Lambda\times\Lambda \longrightarrow \mathfrak{O}\ ,\quad \langle \begin{pmatrix} a_1\\ a_2\\ a_3\end{pmatrix}, \begin{pmatrix} b_1\\ b_2\\ b_3\end{pmatrix}\rangle = a_1\bar{b}_1 + a_2\bar{b}_2 - a_3\bar{b}_3.$$

We consider Γ as group of unimodular automorphisms of the hermitian $\mathfrak{O}$-lattice $\Lambda := \mathfrak{O}^3$. Then Γ' consists of all elements of Γ which restrict to an automorphism of the sublattice $\Lambda' := (1+i)\Lambda$. The factor group Γ/Γ' acts effectively on the residue space $\Lambda/\Lambda' \cong \mathbb{F}_2^3$. The hermitian structure on Λ reduces to the canonical non-degenerate bilinear form on $\mathbb{F}_2^3$. Therefore Γ/Γ' appears as subgroup of the corresponding orthogonal group $\mathbb{O}(3,\mathbb{F}_2)$. This group consists of permutation matrices only, because the canonical basis vectors of $\mathbb{F}_2^3$ are the only ones with $(\mathbb{F}_2)$-norm 1 and norm 1 vectors in its orthogonal complement. Hence $\Gamma/\Gamma' \subseteq \mathbb{O}(3,\mathbb{F}_2) \cong S_3$.

We want to prove that the inclusion is the identity. It suffices to find two non-commuting elements in Γ/Γ'. Let $\mathfrak{a}\in\mathfrak{O}^3$ be a vector whose hermitian norm $\mathfrak{a}^2 := \langle\mathfrak{a},\mathfrak{a}\rangle$ is equal to ±1 or ±2. We define the reflection $R_\mathfrak{a} : \mathfrak{O}^3 \longrightarrow \mathfrak{O}^3$ by

$$R_\mathfrak{a} : \mathfrak{z} \mapsto \mathfrak{z} - \frac{2}{\mathfrak{a}^2}\langle\mathfrak{z},\mathfrak{a}\rangle\mathfrak{a}.$$

It sends $\mathfrak{a}$ to $-\mathfrak{a}$ and each vector of the orthogonal complement

$$\Lambda_\mathfrak{a} := \{\mathfrak{u}\in\mathfrak{O}^3;\ \mathfrak{u}\perp\mathfrak{a}\}$$

to itself. Therefore $id \neq R_\mathfrak{a}$ is an isometry of Λ. Its reduction $\bar{R}_\mathfrak{a}$ (modulo $1+i$) is the reflection isometry

$$r_{\bar{\mathfrak{a}}}\ :\ \mathbb{F}_2^3 \longrightarrow \mathbb{F}_2^3\ ,\ \bar{\mathfrak{z}} \mapsto \bar{\mathfrak{z}} - (\bar{\mathfrak{z}},\bar{\mathfrak{a}})\bar{\mathfrak{a}},$$

where we overline by bar all kinds of reductions modulo $1+i$. This is a non-trivial isometry if and only if $\mathfrak{a}^2 = \pm2$ and $\mathfrak{a}\not\equiv\mathfrak{o}$ *modulo* $1+i$.

The following examples yield two such reflections. Take

$$\mathfrak{a} = \begin{pmatrix} 1+i\\ 1\\ 1\end{pmatrix}\ ,\ \mathfrak{b} = \begin{pmatrix} 1\\ i\\ 0\end{pmatrix}.$$

Both have norm 2. As reductions of the corresponding reflections we get $r_{(0,1,1)}$ or $r_{(1,1,0)}$ with matrix representations $\begin{pmatrix} 1&0&0\\ 0&0&1\\ 0&1&0\end{pmatrix}$ or $\begin{pmatrix} 0&1&0\\ 1&0&0\\ 0&0&1\end{pmatrix}$, respectively. Obviously, they generate $\mathbb{O}(3,\mathbb{F}_2)$.

Lemma 9.2. *We have an exact group sequence*

$$1 \longrightarrow \Gamma' \longrightarrow \Gamma \xrightarrow{red} S_3 \longrightarrow 1.$$

with a section $S_3 \longrightarrow \Gamma$ sending $S_3 \cong \mathbb{O}(3,\mathbb{F}_2)$ to the stationary group $\Gamma_{\mathbb{P}\mathfrak{c}} := \{\gamma\in\Gamma;\ \gamma(\mathfrak{c})\in\mathfrak{O}\mathfrak{c}\}$ for a vector $\mathfrak{c}\in\mathfrak{O}^3$ with negative norm $\mathfrak{c}^2 = -3$.

Proof. The left-exact part comes from the definition of Γ' as kernel of the reduction homomorphism

$$\Gamma = \mathbb{SU}((2,1),\mathfrak{O}(1+i)) \xrightarrow{red} \mathbb{SU}((2,1),\mathfrak{O}/i\mathfrak{O}) \cong \mathbb{O}(3,\mathbb{F}_2)\quad .$$

The surjectivity of the reduction homomorphism has just been verified. The reflections $R_\mathfrak{a}$ and $R_\mathfrak{b}$ ($\mathfrak{a},\mathfrak{b}$ as above), act trivially on the orthogonal complements $\Lambda_\mathfrak{a}$ or $\Lambda_\mathfrak{b}$, respectively, hence they fix $\mathfrak{c} = \begin{pmatrix} i\\ 1\\ 2-i\end{pmatrix}$

generating the rank one lattice $\Lambda_{\mathfrak{a}} \cap \Lambda_{\mathfrak{b}}$. The norm 2 vectors $\mathfrak{a}$, $\mathfrak{b}$ have been chosen in such a way that their Gram matrix is

$$\begin{pmatrix} \mathfrak{a}^2 & \langle \mathfrak{a}, \mathfrak{b} \rangle \\ \langle \mathfrak{b}, \mathfrak{a} \rangle & \mathfrak{b}^2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

hence

$$\begin{aligned} R_{\mathfrak{a}}: &\quad \mathfrak{a} \mapsto -\mathfrak{a},\ \mathfrak{b} \mapsto \mathfrak{b} - \mathfrak{a},\ \mathfrak{c} \mapsto \mathfrak{c}; \\ R_{\mathfrak{b}}: &\quad \mathfrak{a} \mapsto \mathfrak{a} - \mathfrak{b},\ \mathfrak{b} \mapsto -\mathfrak{b},\ \mathfrak{c} \mapsto \mathfrak{c}. \end{aligned}$$

Looking at the corresponding matrix representation it is clear that the subgroup of $\Gamma_{\mathbb{P}\mathfrak{c}}$ generated by $-R_{\mathfrak{a}}$, $-R_{\mathfrak{b}}$ is isomorphic to S_3. □

More generally we define *reflections* $\rho \in \mathbb{U}((2,1),\mathbb{C})$ as elements of finite order with precisely two different eigenvalues. The eigenspace $E(\rho)$ of the double eigenvalue of ρ is called the *reflection plane* of ρ. We call ρ a $\mathbb{B}$-*reflection*, iff $E(\rho)$ is an indefinite hermitian subspace of $\mathbb{C}^3$. In this case (only) $\mathbb{D}(\rho) := \mathbb{P}E(\rho) \cap \mathbb{B}$ is a complete (linear) subdisc of $\mathbb{B}$ called the *reflection disc* of ρ. The complete linear subdisc $\mathbb{D}$ of $\mathbb{B}$ is called a Γ-*reflection disc* iff there exists a $\mathbb{B}$-reflection $\rho \in \Gamma$ such that $\mathbb{D} = \mathbb{D}(\rho)$. Starting from $\mathbb{D}$ the $\mathbb{D}$-*reflection group* $Z_\Gamma(\mathbb{D})$ defined in (51) is finite and cyclic. Its order is called the *reflection order* of $\mathbb{D}$ w.r.t. Γ. The latter definitions apply to any ball lattice $\Gamma \subset \mathbb{U}((2,1),\mathbb{C})$.

Proof of Theorem 9.1(i). The second statement follows from the first by Theorem 8.2. So we have to check step by step the properties (i),...,(vii) of 8.1.

(i) By a result of Shvartsman [Sv1],[Sv2], the surface $\widehat{\mathbb{B}/\Gamma}$ has only one cusp point. We refer to [Zin] for the more general result, that the number of cusp points of Picard modular surfaces $\widehat{\mathbb{B}/\mathbb{U}((2,1),\mathfrak{O}_L)}$, L an arbitrary imaginary quadratic number field, coincides with the class number of L. It is also known that $\mathbb{B}^* = \mathbb{B} \cap \partial\mathbb{B}(L)$ setting $\partial\mathbb{B}(L) = \partial\mathbb{B} \cap \mathbb{P}^2(L)$ in this case.

With the above notations we get $\partial_\Gamma\mathbb{B} = \partial\mathbb{B}(\mathbb{Q}(i)) = \Gamma\kappa$ with $\kappa = \mathbb{P}\mathfrak{k}$, for each

$$\mathfrak{k} \in \Lambda_0 := \{\mathfrak{a} \in \Lambda; \mathfrak{a}^2 = 0\}$$

because $\partial_{\mathbb{B}}(\mathbb{Q}(i)) = \mathbb{P}\Lambda_0$. The set Λ_0 maps onto

$$\bar{V}_0 := \{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \} \subset \mathbb{F}_2^3$$

by reduction. The group $S_3 = \Gamma/\Gamma'$ acts effectively on $\bar{V}_0$ with bi-transitive restriction on the non-zero vectors. It follows that Γ acts bi-transitively on $\partial_\Gamma\mathbb{B}/\Gamma'$ completely represented by

$$k_1 = (0:1:1),\ k_2 = (1:0:1),\ k_3 = (1:1:0)$$

with ineffective kernel Γ'. Especially we get up to Γ'-equivalence precisely three cusps. This proves the first part of (i).

For the proof of the second part and later use we introduce the notations

$$X := \mathbb{B}/\Gamma' \subset \hat{X} := \widehat{\mathbb{B}/\Gamma'},\ Y := \mathbb{B}/\Gamma \subset \hat{Y} := \widehat{\mathbb{B}/\Gamma}.$$

We know that $Y = X/S_3 \subset \hat{Y} = \hat{X}/S_3$ considering $S_3 = \Gamma/\Gamma'$ now as subgroup of $Aut\,X = Aut\,\hat{X}$. If $z = \mathbb{P}\mathfrak{z}$, $\mathfrak{z} \in \mathbb{C}^3$, is a point of $\mathbb{B}^*$ we denote its image on X by Z. The quotient map of $\mathbb{B}$ onto $\mathbb{B}/\Gamma'$ is denoted by p'. These notations will be preserved also for the extensions of this projections to $\mathbb{B}^*$. Since the cusp points $K_i = p'(k_i)$ are S_3-equivalent, it suffices to show that an arbitrary one of them is non-singular. We move the ball inside of $\mathbb{P}^2$ such that $\infty := (0:0:1)$ becomes a $\mathbb{Q}(i)$-rational boundary point of the image ball $g\mathbb{B}$. For this purpose we choose $g \in \mathbb{G}l_3(\mathfrak{O})$ such that

$${}^t\bar{g} \begin{pmatrix} 0 & 0 & -i \\ 0 & 1 & 0 \\ i & 0 & 0 \end{pmatrix} g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Such choice is possible. Namely the $\mathbb{Z}$-lattices $(\mathbb{Z}^3, \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right))$ and $(\mathbb{Z}^3, \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right))$ are isometric because they are unimodular, indefinite and have same rank, signature and type (defined by norms modulo 8). We refer to ([Se70], V.2). The isometry can be extended to isometries of hermitian $\mathfrak{O}$-lattices

$$(\mathfrak{O}^3, I) \cong (\mathfrak{O}^3, \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right)) \cong (\mathfrak{O}^3, \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)) = \Lambda, \ I = \left(\begin{smallmatrix} 0 & 0 & -i \\ 0 & 1 & 0 \\ i & 0 & 0 \end{smallmatrix}\right),$$

where the added first one is obvious. We get the Siegel domain

$$g\mathbb{B} = \mathbb{P}V_- \ : \ 2Im\, u - |v|^2 > 0,$$

$$V = (\mathbb{C}^3, I) = \mathbb{C} \otimes (\mathfrak{O}^3, I) = \mathbb{C} \otimes g\Lambda \ , \ V_- = \{\mathfrak{x} \in V; \langle \mathfrak{x}, \mathfrak{x} \rangle_I < 0\}$$

On $g\mathbb{B}$ act $G := g\Gamma g^{-1} = \mathbb{SU}(I, \mathfrak{O})$ and its congruence subgroup $G' = G(1+i) = g\Gamma' g^{-1}$ with quotient group $G/G' = \Gamma/\Gamma' = S_3$. The stationary group of Γ at ∞ is generated by $\left(\begin{smallmatrix} i & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & i \end{smallmatrix}\right)$ and by its unipotent part

$$U_\infty(\mathfrak{O}) = \{ \begin{pmatrix} 1 & i\bar{a} & \frac{i}{2}|a|^2 + r \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} =: [a, r]; \ a \in \mathbb{C}, \ r \in \mathbb{R}\} \cap \mathbb{S}l_3(\mathfrak{O}),$$

see [H98], IV.2, also for the next considerations. As torsion free nilpotent group of rank 3 each unipotent ball lattice has three generators. As generators of the unipotent congruence subgroup $U_\infty(\mathfrak{O})'$ one finds $[1+i, 1]$, $[1-i, 1]$ and $[0, 2]$. The covolume of $\mathbb{Z}(1+i)+\mathbb{Z}(1-i)$ in $\mathbb{C}$ and the covolume of $2\mathbb{Z}$ in $\mathbb{R}$ are both equal to 2. The selfintersection of the elliptic curve $T_\infty = T_\infty(G')$ in the cusp bundle $F_\infty = F_\infty(G')$ coincides with the characteristic number t of the unipotent lattice. This number can be calculated as -2 times the covolume volume quotient $\frac{2}{2}$, hence $(T_\infty^2) = t = -2$.

Now consider T_∞ as embedded curve in F_∞ . Endowed with trivial weight 1 it is an orbital curve $\mathbf{T}_\infty$. In order to get the canonical partial resolution of a cusp point $\mathbf{K}$ of $\hat{X}$ we look at the canonical abelization $\mathbf{X}' \longrightarrow \hat{\mathbf{X}}$ of the orbital surface $\hat{\mathbf{X}} = \widehat{\mathbb{B}/\Gamma}$. Following [H98], IV.5, the canonical orbital resolution $\mathbf{E} = \mathbf{E}_K$ of $\mathbf{K}$ coincides with the orbital quotient curve $\mathbf{T}_\infty/Z_4$ with $Z_4 = \langle\sigma\rangle$ generated by the reflection $\sigma = \left(\begin{smallmatrix} i & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & i \end{smallmatrix}\right)$. From the classification of cusp points by resolution graphs in [H98], III.5, we know that $\mathbf{K}$ has to be of type (2,4,4), which means that $\mathbf{E}_K = (\mathbb{P}^1; \ \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3)$ with abelian points $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ of cyclic type $\langle 2, e_1\rangle$, $\langle 4, e_2\rangle$, $\langle 4, e_3\rangle$, respectively. We determine these types precisely together with the selfintersection (E^2) on the minimal resolution $\tilde{X}$ of X'. For this purpose we calculate the signature heights of our orbital curves, see (5). First we receive $h_\tau(\mathbf{T}_\infty) = (T_\infty^2) = -2$. Now we use the following

Proposition 9.3. *([H98], Theorems II.2.4, II.4.2). If $\mathbf{C} \longrightarrow \mathbf{D}$ is Galois-finite morphism of orbital curves and $h = h_e$ or $h = h_\tau$ denote Euler heights or signature heights, respectively, then it holds that*

$$h(\mathbf{C}) = [C : D]h(\mathbf{D}), \ [C : D] = deg(C \longrightarrow D).$$

□

Applied to the Galois-covering $\mathbf{T}_\infty \longrightarrow \mathbf{E}$ of degree 4 we get

$$h_\tau(\mathbf{E}) = \frac{1}{4} \cdot h_\tau(\mathbf{T}_\infty) = \frac{1}{4} \cdot (-2) = -\frac{1}{2}.$$

The explicit formula (5) for signature heights yields

$$-\frac{1}{2} = (E^2) + \frac{e_1}{2} + \frac{e_2}{4} + \frac{e_3}{4}, \ e_j \in \mathbb{N},$$

where the summands have to be smaller than 1. Since a Γ-reflection of order 4 belongs to the cusp group at least one abelian point on $\mathbf{E}$, say $\mathbf{P}_3$ has to be of type $\langle 4, 0\rangle$. The last identity reduces to

$$(E^2) = -\frac{1}{2} - \frac{e_1}{2} - \frac{e_2}{4} > -2,$$

hence $(E^2) = -1$ because the selfintersection must be negative (E is contractible to the cusp point K). Below we will see that there is no Γ'-reflection disc with Γ'-reflection order 2, see 11.10. Therefore $\mathbf{P}_1$ cannot be of type $\langle 2, 0\rangle$, hence $e_1 = 1$, $e_2 = 0$.

We proved that the graph of the orbital cusp point $\mathbf{K}$ is already drawn in figure 7. So E is a projective line supporting precisely one singular surface point $P = P_1$, $(E^2) = -1$, P of type $\langle 2, 1\rangle$ as it has been drawn already, for E_1 say, in figure 2. Therefore E contracts to the non-singular surface point K. The proof of property (i) is finished. □

10 Unimodular sublattices

In the next two sections we give basic definitions and results without proofs because the latter are of purely arithmetic nature, not so interesting for algebraic geometers. For detailed proofs we refer to the HU-preprint [HPV] available via INTERNET.

Let $K = \mathbb{Q}(i)$ be the Gauß number field, $\mathfrak{O} = \mathbb{Z}[i]$ the ring of Gauß integers, V a finite dimensional K-vector space of dimension n with a hermitian metric $<\ ,\ >$ with values in K. An $\mathfrak{O}$-module $\Lambda \subset V$, more precisely $(\Lambda, <,>_{|\Lambda})$, is called a sublattice of V, and a V-lattice, if moreover n coincides with the rank ($\mathfrak{O} - rank$) of Λ. A hermitian $\mathfrak{O}$-module Λ is a torsion free $\mathfrak{O}$-module of finite rank together with an hermitian form with values in K. It is a V-lattice in $V = K \otimes \Lambda$ endowed with the extended hermitian form. The dual lattice of Λ is the V-lattice

$$\Lambda^{\#} = \{\mathfrak{x} \in V = K \otimes \Lambda; <\mathfrak{x}, \mathfrak{l}> \in \mathfrak{O}\ for\, all\, \mathfrak{l} \in \Lambda\}.$$

Notice that $\Lambda \subseteq \Lambda^{\#}$ iff the hermitian form has (only) integral values on Λ. A hermitian $\mathfrak{O}$-lattice is called *unimodular* iff $\Lambda^{\#} = \Lambda$. This happens if and only if the hermitian form has integral values on Λ and the discriminant $d(\Lambda)$ is a unit (± 1). Two subsets M, N of a hermitian $\mathfrak{O}$-lattice are orthogonal, iff $<\mathfrak{m}, \mathfrak{n}>= 0$ for all $\mathfrak{m} \in M, \mathfrak{n} \in N$. We write $M \perp N$ in this case. The *orthogonal complement* of M in Λ is the sublattice

$$M^{\perp} = M^{\perp}_{\Lambda} = \{\mathfrak{l} \in \Lambda; \mathfrak{l} \perp M\}.$$

(We omit the index $_\Lambda$ if Λ is fixed and there is no danger of misunderstandings). Two sublattices M, N of Λ are called *orthogonal complementary* (in Λ), iff $M \cap N = O$, $M^{\perp} = N$ and $N^{\perp} = M$.

Proposition 10.1. *Let $(\Lambda, <\ ,\ >)$ be a unimodular hermitian $\mathfrak{O}$-lattice, M and N orthogonal complementary sublattices of Λ, then $M^{\#}/M \cong N^{\#}/N$ as $\mathfrak{O}$-modules.* □

Corollary 10.2. *Under the conditions of the proposition, M is unimodular if and only if its Λ-orthogonal complement N is unimodular.* □

For arbitrary hermitian $\mathfrak{O}$-lattices Λ and sublattices M we denote by $Aut\Lambda \subset End_{\mathfrak{O}}\Lambda$ the isometry group of Λ and by $Aut(\Lambda, M)$ its subgroup of isometries sending M to M.

Corollary 10.3. *Let Λ be a unimodular hermitian $\mathfrak{O}$-lattice, M a unimodular sublattice and N its orthogonal complement in Λ. Then $\Lambda = M \oplus N$ and*

$$Aut(\Lambda, M) = Aut(\Lambda, N).$$

□

We need classification results for unimodular lattices.

Proposition 10.4. *(see Hashimoto [Has], Prop. 3.8). Let $(V, <\ ,\ >)$ be a hermitian space of dimension r over K of signature (p_+, p_-) which contains a unimodular V-lattice ($\mathfrak{O}$-sublattice of V of rank r).*

(i) If r is odd, then there is only one genus of unimodular V-lattices.

(ii) If r is even, then the set of unimodular V-lattices consists of at most two genera. The cardinality of this set is 2 if and only if $p_- \equiv r/2$ modulo 2.

□

A genus consists, by definition, of all V-lattices which are locally $\mathbb{U}(V)$-isometric at all natural primes p. More precisely, two such lattices M, M' belong to the same genus iff for each natural prime p there exists

$$\gamma_p \in \mathbb{U}(V_p),\ V_p = V \otimes \mathbb{Q}_p = V \otimes K_p$$

endowed with the $<\ ,\ >$-extending form, sending $M_p = M \otimes \mathbb{Z}_p = M \otimes \mathfrak{O}_p$ to M'_p. The V-lattices M, M' belong to the same class if and only if $g(M) = M'$ for a suitable $g \in \mathbb{U}(V)$.

Proposition 10.5. *(see Hashimoto [Has], Theorem 3.9). If the hermitian metric on V is indefinite, then each genus of unimodular V-lattices consists of one class.* □

Corollary 10.6. *There are precisely two isometry classes of indefinite unimodular hermitian $\mathfrak{O}$-lattices of rank 2; one is odd and the other even. They are represented by $(\mathfrak{O}^2, \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right))$ or $(\mathfrak{O}^2, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))$, respectively.* □

Corollary 10.7. *All definite unimodular hermitian $\mathfrak{O}$-lattices Λ of rank 2 are isometric to the standard lattice $(\mathfrak{O}^2, \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right))$.* □

We say that two hermitian $\mathfrak{O}$-lattices with integral values have the same *parity*, iff they are both odd or both even, respectively. Let $\tilde{\Gamma} = \tilde{\Gamma}(\Lambda)$ be the automorphism group of a fixed hermitian $\mathfrak{O}$-lattice Λ, and Γ' a subgroup of finite index. A Γ'-class of sublattices of Λ is a Γ'-orbit of one (arbitrary) sublattice of Λ. The normal subgroup of elements with determinant 1 of any subgroup G of the linear group of a finite dimensional vector space is denoted by $\mathbb{S}G$. Usually we set

$$\Gamma = \Gamma(\Lambda) := \mathbb{S}\tilde{\Gamma} = \mathbb{S}\tilde{\Gamma}(\Lambda). \tag{59}$$

Theorem 10.8. *Let Λ be an indefinite unimodular $\mathfrak{O}$-lattice of signature (p_+, p_-) of odd rank $r = p_+ + p_-$. With the above notations it holds that:*

(i) If $p_+ \geq 2$, then there exists precisely one $\tilde{\Gamma}$ -class containing a definite unimodular sublattice of rank 2.

(ii) If $p_- \geq 2$ or $(p_+, p_-) = (2, 1)$, then there exist precisely two $\tilde{\Gamma}$-classes of indefinite unimodular rank-2 sublattices.

The parity and discriminant form under the conditions of (ii) a complete invariant system for $\tilde{\Gamma}$-classes of unimodular rank-2 sublattices of Λ. □

Let $\Lambda \cong \mathfrak{O}^r$ be an indefinite unimodular lattice of odd rank r as in the above theorem. For two unimodular rank-2 sublattices E, E' of Λ of same discriminant and parity we denote by $Isom(E, E')$ set of isometries of E onto E' and set

$$\Gamma'(E, E') = \{\gamma \in \Gamma';\ \gamma(E) = E'\}.$$

Corollary 10.9. *Under the conditions of the theorem the restriction maps*

$$\tilde{\Gamma}(E, E') \longrightarrow Isom(E, E')\ ,\ \Gamma(E, E') \longrightarrow Isom(E, E')$$

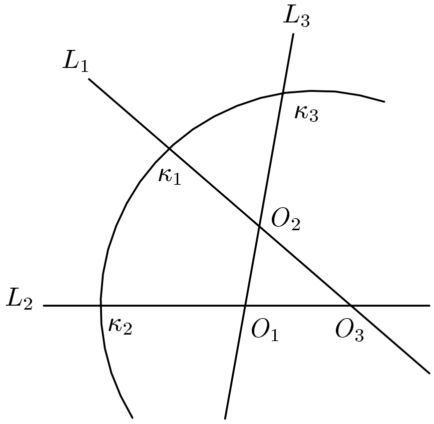
are surjective. The isometry class $Cl_\Lambda(E)$ of sublattices of Λ containing E, the $\tilde{\Gamma}$-class $\tilde{\Gamma} \cdot \{E\}$ and the Γ-class $\Gamma \cdot \{E\}$ coincide. □

Now come back to the Picard modular group $\Gamma = \mathbb{SU}((2,1), \mathfrak{O})$, the special automorphism group of the standard unimodular lattice $\Lambda = \mathfrak{O}^3$ of signature $(2, 1)$, and its congruence subgroup $\Gamma(\pi)$.

Proposition 10.10. *There are precisely three Γ-classes of unimodular rank-2 sublattices E of Λ completely represented by lattices with Gram matrices $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (definite), $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ (indefinite, odd) or $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ (even), respectively. The Γ-class splits into three $\Gamma(\pi)$-classes if and only if E is not even. In the even case we have only one class $\Gamma(\pi) \cdot \{E\} = \Gamma \cdot \{E\}$.* □

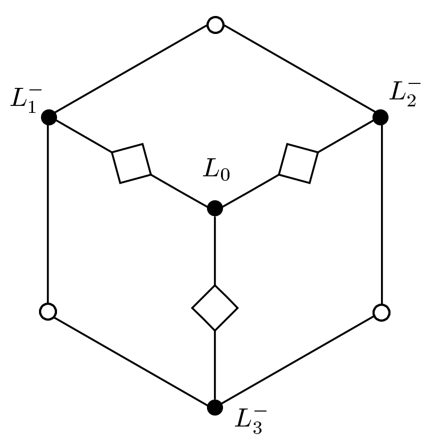
At the end of this section we draw a representative plane picture of projective images of unimodular rank-2 lattices E_i representing $\mathbb{E}_i$, $i = 0, 1, 2, 3$. We distinguish for $i = 1, 2, 3$ definite and indefinite representatives by upper index $+$ or $-$, respectively. By Proposition 10.10 we have a complete system of representatives $E_0, E_1^+, E_2^+, E_3^+, E_1^-, E_2^-, E_3^-$ of $\Gamma(\pi)$-classes. For the rest of this section we denote the subplanes $\mathbb{R} \otimes E_i^{\pm}$ of the canonical hermitian signature (2,1) space $\mathbb{C}^3$ by $E_i^{\pm}$ and the projective lines $\mathbb{P}E_i^{\pm} \subset \mathbb{P}^2 = \mathbb{P}^2(\mathbb{C})$ by $L_i^{\pm}$. The orthogonal complements of $\mathfrak{a} = \left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right)$, $\mathfrak{b} = \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)$, $\mathfrak{c} = \left(\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right)$, $\mathfrak{e} = \left(\begin{smallmatrix} -1 \\ 1 \\ 1 \end{smallmatrix}\right)$ in $\mathbb{C}^3$ yield the special representatives $E_1^- = \mathfrak{a}^{\perp}$, $E_2^- = \mathfrak{c}^{\perp}$, $E_3^+ = \mathfrak{a}^{\perp}$ and $E_0 = \mathfrak{e}^{\perp}$. Using projective coordinates $(x : y : z)$ the corresponding lines are described by linear equations:

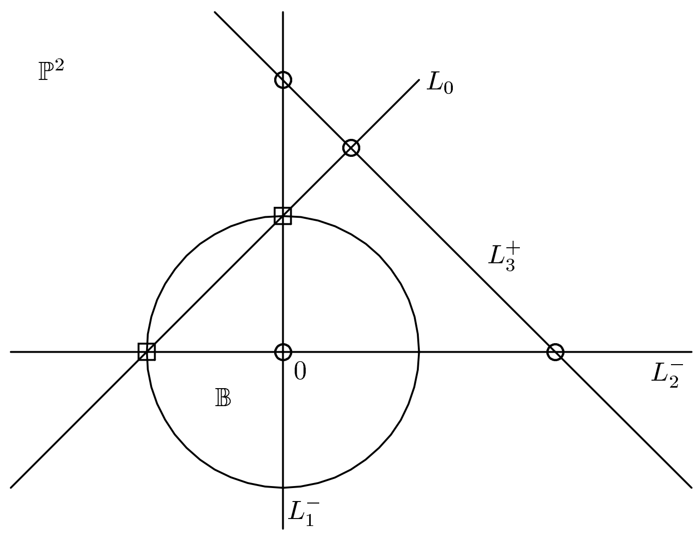
$$L_0 : X - Y + Z = 0, \; L_1^- : Y = 0, \; L_2^- : X = 0,$$
$$L_3^+ = L_\infty : Z = 0 \;\; \text{(the infinite line)}$$



on the (real) projective plane where the (real) ball points lay inside of the (unit) circle, the Γ-cusps sit on the circle. All intersection points of the lines are real, hence all visible in the real picture. Restricting to $\mathbb{B}$ we forget L_3^+ and the marked points. Then we get for the remaining lines and points the dual unweighted graph left down.

Applying $S_3 \subset \Gamma$ (the alternating subgroup $A_3 \subset S_3$ is sufficient) we get similar graphs including also $L_3^- = \mathbb{P}E_3^-$. Altogether we get the $\Gamma(\pi)$-graph of indefinite unimodular rank-2 lattices.





11 Elements of finite order

In this section we determine positive weights as reflection orders, calculate (negative) heights as Euler-Poincare volumes of fundamental domains in discs $\mathbb{D}_i$, $i = 1, 2, 3$, cutten out as intersections of L_i^- with the ball $\mathbb{B}$.

Let K be a number field, $\mathfrak{O} = \mathfrak{O}_K$ its ring of integers, Γ a subgroup of $\mathbb{G}l_m(\mathfrak{O})$, $\mathfrak{a} \subset \mathfrak{O}$ an ideal and $\Gamma(\mathfrak{a}) \subseteq \Gamma$ the corresponding congruent subgroup defined as kernel of the natural group homomorphism

$$\Gamma \longrightarrow \mathbb{G}l_m(\mathfrak{O}) \longrightarrow \mathbb{G}l_m(\mathfrak{O}/\mathfrak{a}).$$

Lemma 11.1. *If $\gamma^n = 1$ for $\gamma \in \Gamma(\mathfrak{a})$, Then $\mathfrak{a}$ divides $\zeta - 1$ in $\mathfrak{O}_L$, $L = K(\zeta)$, where ζ is an arbitrary eigenvalue of γ (a suitable n-th unit root).* $\square$

Corollary 11.2. *In the special case of the field $K = \mathbb{Q}(i)$ of Gauß numbers there are at most two possibilities for non-trivial ideals $\mathfrak{a} \subset \mathfrak{O}$ such that $\Gamma(\mathfrak{a})$ contains non-trivial elements of finite order,* *namely* $\mathfrak{a} = (\pi) = (1+i)$ *or* $\mathfrak{a} = (2)$. *The only orders of such elements are* 2 *and* 4. *Elements of order* 4 *belong to* $\Gamma(\pi) \setminus \Gamma(2)$. *Especially,* $\Gamma(\pi^3)$ *is a torsion free group.* $\square$

We concentrate our further considerations to subgroups of $\tilde{\Gamma} = \mathbb{U}((2,1), \mathfrak{O})$, $\mathfrak{O} = \mathfrak{O}_K$, $K = \mathbb{Q}(i)$, especially to $\Gamma = \mathbb{S}\tilde{\Gamma}$, again. Elements of order 4 in Γ belong to the $\mathbb{G}l_3(K)$-conjugation classes of $diag(1, i, -i)$, $diag(-1, i, i)$ or $diag(-1, -i, -i)$, and elements of order two are conjugated to $diag(1, -1, -1)$. The conjugacy classes of the latter three types exhaust the set of all semisimple elements $1 \neq \sigma \in \Gamma$ with a double eigenvalue. This follows easily from the fact that the characteristic polynomials $\chi_\gamma(T)$ have to lay in $\mathfrak{O}[T]$. Semisimple elements in $\tilde{\Gamma}$ with precisely two eigenvalues are called Λ-*reflections*. The *reflection lattice* $E(\sigma) \subset \Lambda$ is defined to be the intersection of $\Lambda = \mathfrak{O}^3$ with the eigenspace of the double eigenvalue of σ. Obviously, it has $\mathfrak{O}$-rank 2.

Proposition 11.3. *For each* Λ-*reflection* $\sigma \in \Gamma(\pi)$ *is the reflection lattice* $E(\sigma)$ *unimodular.* $\square$

For $\Gamma' \subseteq \tilde{\Gamma}$ and any pair of orthogonal complementary sublattices $\mathfrak{O}\mathfrak{a} \perp E$ of Λ we have a pair of restriction homomorphisms

$$Aut\, E \longleftarrow \Gamma'(E, E) \longrightarrow Aut\, \mathfrak{O}\mathfrak{a}$$

For unimodular E and $\Gamma' = \tilde{\Gamma}$ one gets a pair of cartesian projections

$$Aut\, E \longleftarrow \tilde{\Gamma}(E, E) = \tilde{\Gamma}(\mathfrak{O}\mathfrak{a}, \mathfrak{O}\mathfrak{a}) \cong Aut\, E \times Aut\, \mathfrak{O}\mathfrak{a} \longrightarrow Aut\, \mathfrak{O}\mathfrak{a} \cong \mathfrak{O}^*,$$

where the surjectivity on the left-hand side comes from Corollary 10.9 and the identity from Corollary 10.3. Restricting to Γ we get an exact sequence

$$\begin{array}{ccccc} 1 \longrightarrow \mathbb{S}Aut\, E \longrightarrow & \Gamma(E, E) = \Gamma(\mathfrak{O}\mathfrak{a}, \mathfrak{O}\mathfrak{a}) & \longrightarrow Aut\, \mathfrak{O}\mathfrak{a} \cong \mathfrak{O}^* \longrightarrow 1, \\ & \| & \\ & Aut\, E & \end{array}$$

where the vertical isomorphism sends ρ to $\rho \times det\rho$. It restricts via intersections with $\Gamma(\pi)$ to the obviously splitting exact sequence

$$\begin{array}{ccccc} 1 \longrightarrow (\mathbb{S}Aut\, E)(\pi) \longrightarrow & \Gamma(\pi)(E, E) = \Gamma(\pi)(\mathfrak{O}\mathfrak{a}, \mathfrak{O}\mathfrak{a}) & \longrightarrow Aut\, \mathfrak{O}\mathfrak{a} \cong \mathfrak{O}^* \longrightarrow 1, \\ & \| & \\ & (Aut\, E)(\pi) & \end{array} \tag{60}$$

Lemma 11.4. *Each maximal finite subgroup* G *of* $(\mathbb{S}Aut\, E)(\pi)$ *is cyclic of order* 4.

$\square$

Theorem 11.5. *(i) Each maximal finite subgroup* T *of* $\Gamma(\pi)$ *is isomorphic to* $\mathfrak{O}^* \times \mathfrak{O}^*$.

(ii) The set of all these groups coincides with the set of intersections

$$\Gamma(\pi)(E, E) \cap \Gamma(\pi)(E', E') = \Gamma(\pi)(\mathfrak{O}\mathfrak{c}) \cap \Gamma(\pi)(\mathfrak{O}\mathfrak{b}),$$

where $E = \Lambda_\mathfrak{c}$ *and* $E' = \Lambda_\mathfrak{b}$ *are two different unimodular rank-2 sublattices of* Λ *with orthogonal vectors* $\mathfrak{b}$, $\mathfrak{c}$ *of hermitian norms* ± 1.

(iii) Each element $\delta \in \Gamma(\pi)$ *of order* 2 *is a square of a reflection* $\rho \in \Gamma(\pi)$ *of order* 4.

(iv) Each element $\gamma \in \Gamma(\pi)$ *of finite order is a reflection or a product of two reflections.*

(v) A non-trivial element of finite order of $\Gamma(\pi)$ *has order* 2 *if and only if it belongs to* $\Gamma(\pi^2) = \Gamma(2)$. *It has order* 4 *if and only if it belongs to* $\Gamma(\pi) \setminus \Gamma(\pi^2)$.

$\square$

Notation. $E_\mathbb{R} := \mathbb{R} \otimes E$ for each $\mathfrak{O}$-lattice E.

Definition-Remark 11.6. *We call a* Λ-*reflection* δ *a* $\mathbb{B}$- *reflection, iff* $L(\delta) := \mathbb{P}E_\mathbb{R}(\delta)$ *intersects* $\mathbb{B}$. *We denote the corresponding (complete linear) subdisc* $\mathbb{D}(\delta) = L(\delta) \cap \mathbb{B}$ *or by* $\mathbb{D}_\mathfrak{a}$, *where* $\mathfrak{a} \in \mathbb{C}^3$ *is an arbitrary non-trivial vector orthogonal to* E. δ *is a* $\mathbb{B}$-*reflection if and only if* $a^2 > 0$ *or, equivalently,* $E(\delta)$ *is indefinite.*

Corollary 11.7. *Any three different projective lines*

$$L_j = \mathbb{P}E_{j\mathbb{R}} \subset \mathbb{P}^2 = \mathbb{P}(\Lambda_{\mathbb{R}})$$

of unimodular rank-2 sublattices of Λ have no common intersection point Q on $\mathbb{B}$. □

Proposition 11.8. *The surface $\mathbb{B}/\Gamma(\pi)$ is smooth. There are precisely three $\Gamma(\pi)$-orbits of $\Gamma(\pi)$-elliptic points on $\mathbb{B}$. Its union is the Γ-orbit of $O = (0:0:1) \in \mathbb{B}$ consisting of all $\mathfrak{q} \in \Lambda$ with $\mathfrak{q}^2 = -1$. Each subgroup $\Sigma \cong S_3$ of Γ acts transitively as permutation group on the three orbits via conjugation. The isotropy group $\Gamma(\pi)_Q$ of each $\Gamma(\pi)$- elliptic point Q is the product of two cyclic groups each generated by a reflection of order 4. $\mathfrak{O}\mathfrak{q}$ is the intersection of the corresponding reflection lattices E and E'. Both are unimodular, indefinite and odd.* □

Corollary 11.9. *Each $\mathfrak{q} \in \Lambda$ with hermitian norm $\mathfrak{q}^2 = -1$ extends uniquely, up to $\mathfrak{O}^*$-factors and order of enumeration, to an orthogonal basis $(\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{q})$ of Λ. The both unique unimodular (indefinite odd) rank-2 sublattices of Λ with intersection $\mathfrak{O}\mathfrak{q}$ are the reflection planes*

$$E_1 = \Lambda_{\mathfrak{a}_1} = \mathfrak{O}\mathfrak{a}_2 + \mathfrak{O}\mathfrak{q}, \quad E_2 = \Lambda_{\mathfrak{a}_1} = \mathfrak{O}\mathfrak{a}_1 + \mathfrak{O}\mathfrak{q}, \tag{61}$$

Moreover, we set

$$E_3 := \Lambda_{\mathfrak{q}} = \mathfrak{O}\mathfrak{a}_1 + \mathfrak{O}\mathfrak{a}_2.$$

The set of residue planes of the lattices E_j, $j = 1,2,3$, coincides with the set of the three unimodular odd subplanes in $\mathbb{F}_2^3 = \Lambda/\pi\Lambda$ with possibly other enumeration. Two -1-vectors $\mathfrak{q}, \mathfrak{q}'$ belong to the same $\Gamma(\pi)$-orbit if and only if $\mathfrak{q} \equiv \mathfrak{q}' \, mod \, \pi$. □

Proposition 11.10. *The irreducible components of the branch locus of the quotient map $\mathbb{B} \longrightarrow \mathbb{B}/\Gamma(\pi)$ are smooth. It consists of 4 curves*

$$C_0 = \mathbb{D}_0/\Gamma_0, \; C_1 = \mathbb{D}_1/\Gamma_1, C_2 = \mathbb{D}_2/\Gamma_2, \; C_3 = \mathbb{D}_3/\Gamma_3, \tag{62}$$

where $\mathbb{D}_j = \mathbb{B} \cap L_j$, $L_j = \mathbb{P}(E_j\mathbb{R})$, E_j an arbitrary unimodular indefinite rank-2 sublattice of Λ with (non- degenerate) residue subplane $\mathbb{E}_j$ of $\mathbb{F}_2^3$ and $\Gamma_j = \Gamma(\pi)(E_j, E_j)$, $j = 0,1,2,3$, respectively. The ramification index is 4 for all four components. The action of $S_3 = \Gamma/\Gamma(\pi)$ on $\mathbb{B}/\Gamma(\pi)$ permutes the curves C_1, C_2, C_3 and restricts to an effective action on C_0. C_k intersects C_l in precisely one point P_m for any triple $\{k,l,m\} = \{1,2,3\}$. The intersection is transversal. The points P_1, P_2, P_3 are the images of all $\Gamma(\pi)$-elliptic points on $\mathbb{B}$. The $\mathbb{B}$-reflection discs $\mathbb{D}_1, \mathbb{D}_2, \mathbb{D}_3$ do not intersect $\mathbb{D}_0$. □

Proposition 11.11. *Each Γ-cusp $\kappa = \mathbb{P}\mathfrak{k}$, $\mathfrak{k} \in \Lambda$ a primitive isotropy vector, is the intersection of precisely two $\Gamma(\pi)$-reflection lines L_0, L_1. Both come from unimodular indefinite lattices E_0, E_1, where the first one is even and the other odd. Each unimodular even (hence indefinite) lattice E_0 contains isotropy vectors $\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3$ representing all the possible non- trivial residue isotropy vectors*

$$\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix} \in \mathbb{F}_2^3.$$

They can be choosed as A_3-orbit of $\mathfrak{k}_1$, where A_3 is the alternating subgroup of a group (isomorphic to and identified with) $S_3 \subset \Gamma(E_0, E_0)$ acting on E such that the corresponding reflection lines $L_1, L_2, L_3 \neq L_0$ through κ_1, κ_2 or κ_3, respectively, intersect each other pairwise on $\mathbb{B}$. These three elliptic intersection points represent the three $\Gamma(\pi)$-orbits of all $\Gamma(\pi)$-elliptic points. □

Remark 11.12. *Restricting to reflection discs $\mathbb{D}_j = L_j \cap \mathbb{B}$ we realized the situation described in Figure 8.*

Proof of Theorem 8.2. It remains to check the properties (ii),...,(vii) postulated in 8.1 For $\Gamma' = \Gamma(\pi)$.

(ii) Let $\kappa_1, \kappa_2, \kappa_3 \in \partial_K \mathbb{D}_0$ be an S_3-orbit for $S_3 \subset \Gamma_0 = \Gamma(E_0, E_0)$, see 11.11, and $\kappa \in \partial_K \mathbb{D}_0$ arbitrary. We have to show that $\kappa \in \Gamma_0' \kappa_j$. Assume, for instance, that $\kappa \equiv \kappa_1 \, mod \, \pi$. Then $\kappa = L_0 \cap L_1'$, hence $\kappa = \gamma\kappa_1$ for a suitable $\gamma \in \Gamma$ with $\kappa_1 = L_0 \cap L_1$. Since pairs of reflection lines through one point are unique, γ acts on L_0 and transfers L_1 to L_1'. Therefore γ or $\gamma \circ (2,3)$ sends L_j to L_j', $j = 2,3$. This property can be assumed now for our γ. Then $\gamma \equiv E \, mod \, \pi$ which means that γ belongs to Γ'.

(iii) Take $\mathbb{D}_1 = \mathbb{B} \cap L_1$ described in 11.11. Then $\kappa_1 = L_1 \cap L_0 \in \partial_K \mathbb{D}_1$ is fixed by $(2,3) \in S_3 \subset \Gamma(E_0, E_0)$. For arbitrary $\kappa \in \partial_K \mathbb{D}_0$ take $\gamma \in \Gamma$ such that $\kappa = \gamma \kappa_1$. By the same argument as above, γ acts on L_1 and sends L_0, L_2 to L'_0 or L'_2, respectively (if not take $\gamma \circ (2,3)$). It follows again that γ belongs to Γ'_1.

(iv) see Proposition 10.10.

(v) see Proposition 11.8.

(vi) The proof of the following theorem is completely published in [H98].

Theorem 11.13. *([H98], V, Theorem 5A.4.7). Let K be an imaginary quadratic number field with ring of integers $\mathfrak{O}_K$, discriminant $D = D_{K/\mathbb{Q}} \neq -3$, Dirichlet character $\chi(n) = (\frac{D}{n})$ (generalized quadratic residue, Jacobi symbol) and corresponding Dirichlet series $L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$. Then for $\Gamma = \mathbb{SU}((2,1), \mathfrak{O}_K)$ with fundamental domain $\mathfrak{F}_\Gamma$ on $\mathbb{B}$ it holds that*

$$vol_{EB}(\Gamma) = \frac{3|D|^{5/2}}{32\pi^3} L(3, \chi).$$

It is now easy to calculate for $K = \mathbb{Q}(i)$ the $\mathfrak{F}_\Gamma$-volume $\frac{1}{32}$ in the case of Gauß numbers. This was first proved by Shvartsman [Sv1]. Since $\Gamma/\Gamma(\pi) \cong S_3$ it follows that $vol_{EB}(\Gamma(\pi)) = \frac{3}{16}$.

(vii) see Proposition 11.11.

Theorem 8.2 is proved.

□

References

[BHH] Barthel, G., Hirzebruch, F., Höfer, Th.: Geradenkonfigurationen und algebraische Flächen, Aspects of Mathematics D 4, Vieweg, Braunschweig -Wiesbaden, 1986

[Bou] Bourbaki, N.: Groupes et algèbre de Lie, Hermann, Paris, 1968

[BPV] Barth, W., Peters, C., van de Ven, A.: Compact complex surfaces, Erg. d. Mathem., Springer, Berlin, 1984

[DM1] Deligne, P., Mostow, G.D.: Monodromy of hypergeometric functions and non-lattice integral monodromy, Publ. Math. IHES, **63** (1986), 5-89

[DM2] Deligne, P., Mostow, G.D.: Commesurabilities among Lattices in $PU(1,n)$, Princeton Univ. Press, Princeton, 1993

[vGm] van Geemen, B.: Projective models of Picard modular varieties, Lect. Notes Math. **1515** (1991), 68 - 99

[Has] Hashimoto, K.: Elliptic conjugacy classes of the Siegel modular group and unimodular hermitian forms over the ring of cyclotomic integers, J. Fac. Sci. Univ. Tokyo **33** (1986), 57 - 82

[H86] Holzapfel, R.-P.: Geometry and Arithmetic around Euler partial differential equations, Dt. Verlag d. Wiss., Berlin/Reidel Publ. Comp., Dordrecht, 1986

[H98] Holzapfel, R.-P.: Ball and surface arithmetics, Aspects of Mathematics **E 29**, Vieweg, Braunschweig -Wiesbaden, 1998

[HPV] Holzapfel, R.-P., Piñeiro, A., Vladov, N.: Picard-Einstein metrics and class fields connected with Apollonius cycle, HU-Preprint **98-15**, Humboldt-University, 1998

[Mat] Matsumoto, K.: On modular functions in 2 variables attached to a family of hyperelliptic curves of genus 3, Sc. Norm. Sup. Pisa **16** No. 4 (1989), 557 - 578

[M1] Mostow, G.D.: Generalised Picard lattices arising from half-integral conditions, Publ. Math. IHES, **63** (1986), 91-106.

[M2] Mostow, G.D.: On discontinuous action of monodromy groups on the complex n-ball, J. of AMS **1** (1988), 555 - 586

[P] Picard, E.: Sur une extension aux fonctions de deux indépendantes analogues aux fonctions modularies, Acta Math. **2**, (1883), 114 - 135

[Se70] Serre, J.-P.: Course d'arithmétique, Presses Univ. France, Paris, 1970

[Shg] Shiga, H.: On the representation of the Picard modular function by Theta constants I-II, Publ. R.I.M.S. Kyoto Univ. **24** (1988), 311-360

[Sm63] Shimura, G.: On analytic families of polarized abelian varieties and automorphic functions, Ann. Math. **78** No. 1 (1963), 149-192

[Sm64] Shimura, G.: On purely transcendental fields of automorphic functions of several variables, Osaka Journ. Math. **1** No.1 (1964), 1 - 14
On analytic families of polarized abelian varieties and automorphic functions, Ann. Math. **78** No. 1 (1963), 149-192

[Sv1] Shvartsman, O.W.: On the factor space of an arithmetic group acting on the complex unit ball (russian), PhD thesis, MGU, Moscow, 1974

[Sv2] Shvartsman, O.W.: On the factor space of an arithmetic group acting on the complex ball (russian), Usp. Math. Nauk **28** No.4 (1974), 202 - 203

[T] Terada, T.: Fonctions hypergéoétriques F_1 et fonctions automorphes I, II, J. Math. Sos. Japan **35** (1983), 451 - 475, **37** (1885), 173 -185

[Y97] Yoshida, M.: Hypergeometric functions, my love; modular interpretations of configuration spaces, Aspects of Mathematics **E 32**, Vieweg, Braunschweig -Wiesbaden, 1997

[Zin] Zink, Th.: Über die Spitzen einiger arithmetischer Untergruppen unitärer Gruppen, Math. Nachr. **79** (1979), 315 - 320

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