

DE
FUNCTIONUM ANALYTICARUM UNIUS VARIABILIS
PER SERIES INFINITAS REPRAESENTATIONE.

DISSERTATIO INAUGURALIS MATHEMATICA

QUAM

CONSENSU ET AUCTORITATE

AMPLISSIMI PHILOSOPHORUM ORDINIS

IN

ALMA LITTERARUM UNIVERSITATE

FRIDERICA GUILELMA

AD SUMMOS

IN PHILOSOPHIA HONORES

RITE CAPESENDOS

DIE XXVIII. M. JULII ANNO H. S. LXX.

H. L. Q. S.

PUBLICE DEFENDET

AUCTOR

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§ 1.

Cogitanti mihi de evolvendis functionibus analyticis unius variabilis in series secundum propositas functiones progredientes tria potissimum problemata se obtulerunt, primum, omnes functiones, quae evolvi possunt, investigandi, alterum, rationem, quae inter omnes eiusdem functionis evolutiones intercedit, indagandi, tertium, fines, quibus serierum convergentia continetur, assignandi. Quarum quaestionum tractationem duobus exemplis simplicibus illustrare in hac mihi commentatione proposui.

Manifestum est secundum eas functiones, quibus fractiones rationales omnes repraesentari possunt, ope integralis Cauchyani eas etiam functiones, quae rationalium characterem habent, evolvi posse. Methodos, quibus fractionum rationalium secundum functiones Laplaceanas et Besselianas evolutio absolvitur, cum diligentius examinarem, animadverti, omnes illas evolutiones multasque novas eadem ratione maxime elementari, qua series Tayloriana derivari solet, perfici posse. Quae analogia quo clarius illucescat, notam hanc demonstrationem paucis, si placet, explicemus.

§ 2.

Ponamus functionem $f(x)$ intra circulum $C(\rho)$ radio ρ circa initium coordinatarum descriptum neque latius uniformem continuamque esse, punctum x intra circulum $C(\rho)$ iacere, punctum y circulum $C(\rho')$, cuius radius ρ' et ipso

ϱ minor, et valore absoluto quantitatis x maior est, percurrere. Constat, locum habere aequationem Cauchyanam:

$$f(x) = \frac{1}{2\pi i} \int \frac{f(y) dy}{y - x}$$

Iam denotante S summam seriei

$$S = \sum_{v=0}^{n-1} \frac{x^v}{y^{v+1}}$$

est

$$xS = \sum_{v=0}^{n-1} \frac{x^{v+1}}{y^{v+1}} \quad yS = \sum_{v=0}^{n-1} \frac{x^v}{y^v}$$

ergo

$$(y - x)S = 1 - \frac{x^n}{y^n}$$

vel

$$S = \frac{1}{y - x} \left(1 - \frac{x^n}{y^n} \right)$$

et

$$f(x) = \sum_{v=0}^{n-1} c_v x^v + \frac{1}{2\pi i} \int \frac{f(y) dy}{y - x} \cdot \frac{x^n}{y^n}$$

si ponitur

$$c_v = \frac{1}{2\pi i} \int \frac{f(y) dy}{y^{v+1}}$$

Quia x minor est, quam y ,*) residuum

$$\frac{1}{2\pi i} \int \frac{f(y) dy}{y - x} \frac{x^n}{y^n}$$

crescente n ultra omnem limitem decrescit. Est ergo

$$f(x) = \sum_{v=0}^{\infty} c_v x^v$$

si x intra circulum $C(\varrho)$, qui nullum functionis $f(x)$ punctum singulare continet, situm est. Si iacente x extra $C(\varrho)$ series reperta convergere non desineret, etiam $f(x)$ ultra hos fines uniformis et continua esset, quod suppositionibus nostris repugnat. Facile denique demonstratur, duas series $\sum_{v=0}^{\infty} c_v x^v$ et $\sum_{v=0}^{\infty} c'_v x^v$, nisi sit $c_v = c'_v$, toto

convergentiae circulo aequales esse non posse.

*) Ex duabus quantitatibus complexis eam dicimus altera maiorem, quae valorem absolutum maiorem habet.

I.

§ 3.

Videamus nunc, quomodo eadem argumentatio ad eas functiones adhiberi possit, quarum n ta ex n primis factoribus producti infiniti constat. Si statuitur

$$P_0(x) = 1 \quad P_n(x) = (x - a_0)(x - a_1)\dots(x - a_{n-1})$$

est

$$P_{n+1}(x) + a_n P_n(x) = x P_n(x)$$

$$\frac{1}{P_n(x)} + \frac{a_n}{P_{n+1}(x)} = \frac{x}{P_{n+1}(x)}$$

Iam denotante S summam seriei

$$S = \sum_0^{n-1} \frac{P_\nu(x)}{P_{\nu+1}(y)}$$

est

$$xS = \sum_0^{n-1} \frac{x P_\nu(x)}{P_{\nu+1}(y)} = \sum_0^{n-1} \frac{P_{\nu+1}(x)}{P_{\nu+1}(y)} + \sum_0^{n-1} \frac{a_\nu P_\nu(x)}{P_{\nu+1}(y)}$$

$$yS = \sum_0^{n-1} \frac{y P_\nu(x)}{P_{\nu+1}(y)} = \sum_0^{n-1} \frac{P_\nu(x)}{P_\nu(y)} + \sum_0^{n-1} \frac{a_\nu P_\nu(x)}{P_{\nu+1}(y)}$$

$$(y - x)S = 1 - \frac{P_n(x)}{P_n(y)}$$

vel

$$\sum_0^{n-1} \frac{P_\nu(x)}{P_{\nu+1}(y)} = \frac{1}{y - x} \left[1 - \frac{P_n(x)}{P_n(y)} \right]$$

Sed priusquam ex hac formula functionum analyticas evolutiones derivamus, de convergentia harum serierum disserendum est. Et quoniam haec quaestio generliter absolvvi non potest, plures casus memorabiles tractabimus. Qua in disquisitione maxime distinguendum est, utrum quantitates $a_0, a_1\dots$ intra regionem finitam coegerantur, an ultra omnes limites crescant.

§ 4.

Sint $b_0, b_1 \dots b_{m-1} m$ quantitates diversae et finitae, ponamusque absolute convergere m series

$$\sum_r^{\infty} (a_{\kappa+mr} - b_{\kappa})$$

denotante κ unum ex numeris $0, 1 \dots m-1$. Si

$$n = \lambda m + \mu \quad \text{et} \quad \mu < m$$

est, $P_n(x)$ in $m+1$ factores resolvimus, quorum est κ tus

$$(x - a_{\kappa})(x - a_{\kappa+m}) \dots [x - a_{\kappa+(\lambda-1)m}]$$

ultimus autem

$$(x - a_{\lambda m})(x - a_{\lambda m+1}) \dots (x - a_{\lambda m+\mu-1})$$

Atque ut verborum abundantiam, quantum fieri potest, evitemus, sequenti scribendi compendio semper abhinc utemur: Si $h_n(x)$ crescente n ad functionem exceptis singulis punctis finitam neque evanescentem continuo convergit et pro omnibus indicis n valoribus $\frac{f_n(x)}{g_n(x)} = h_n(x)$ est,

$f_n(x) \propto g_n(x)$ ponemus: Hae enim functiones, quod ad convergentiam serierum secundum eas progredientium attinet, aequivalentes sunt, exclusis iis argumenti valoribus, qui functionem $h_n(x)$ infinite vel parvam vel magnam reddunt et peculiares considerationes requirunt. Velut ex aequatione

$$(x - a_{\kappa})(x - a_{\kappa+m}) \dots (x - a_{\kappa+(\lambda-1)m}) \\ = (x - b_{\kappa})^{\lambda} \left(1 - \frac{a_{\kappa} - b_{\kappa}}{x - b_{\kappa}} \right) \dots \left(1 - \frac{a_{\kappa+(\lambda-1)m} - b_{\kappa}}{x - b_{\kappa}} \right)$$

quoniam una cum serie $\sum_r^{\infty} (a_{\kappa+mr} - b_{\kappa})$ etiam productum

$$H_0^{\infty} \left(1 - \frac{a_{\kappa+mr} - b_{\kappa}}{x - b_{\kappa}} \right)$$

absolute convergit, derivatur aequivalentia

$$(x - a_{\kappa}) \dots [x - a_{\kappa+(\lambda-1)m}] \propto (x - b_{\kappa})^{\lambda}$$

Itaque si

$$(x - b_0) \dots (x - b_{m-1}) = (\varphi x)^m$$

ponitur, facile perspicitur esse

$$P_n(x) \propto (\varphi x)^n$$

cum quotiens $\frac{P_n(x)}{(\varphi x)^n}$ in punctis $b_0, b_1 \dots b_{m-1}$ infinitus, in punctis $a_0, a_1 \dots a_{n-1}$ nihilo aequalis sit. Et quia differentiarum $b_x - a_x, b_x - a_{x+m} \dots$ finitus tantum numerus quantitatem dataam ε quamvis parvam superat, perspicitur esse

$$P_n(b_x) \propto \varepsilon_n^m \cdot [(b_x - b_1) \cdot (b_x - b_{x-1}) \cdot (b_x - b_{x+1}) \cdot (b_x - b_m)]^m$$

denotante ε_n quantitatem ultra omnem limitem decrescentem.

Quodsi radius convergentiae seriei $\sum_v^\infty c_v x^v$ per ϱ denotatur, series $\sum_v^\infty c_v P_v(x)$ convergit intra, divergit extra curvam, in qua functionis $\varphi(x)$ valor absolutus quantitati ϱ aequalis est.

Neque enim ad ea punctorum $a_0, a_1 \dots$, quae forte extra fines convergentiae seriei $\sum_v^\infty c_v P_v(x)$ iacent, in quibus haec series vel convergere vel divergere potest, si de functionum per hasce series repraesentatione agitur, respiciendum nobis est. Lineae, in quibus $\varphi(x)$ valorem absolutum ϱ habet, per $C(\varrho)$ designandae, quarum quodvis punctum ab m punctis fixis $b_0 \dots b_{m-1}$ constans distanciarum productum habet, crescente ϱ a 0 ad ∞ magnam figurarum varietatem praebent. Primum ex m partibus diversis compositae sunt focos $b_0, b_1 \dots b_{m-1}$ cingentibus. Crescente ϱ duae harum partium confluunt, curvamque lemniscatae modo duos focos amplectentem efficiunt. Paulatim omnibus deinceps ramis in unum collectis linea magis magisque circuli speciem praebet.*)

*) Toto hoc curvarum confocalium systemate $n = 1$ puncta duplia inveniuntur, quorum centrum gravitatis idem est, quod n focorum, radices aequationis $\frac{1}{x - b_0} + \dots + \frac{1}{x - b_{m-1}} = 0$. Lineae has curvas sub angulis rectis secantes ab Ill. Gauss consideratae sunt. (Tom III, pag. 27.)

Iam denotante ϱ radium convergentiae seriei $\sum_0^\infty \frac{c_r}{y^{r+1}}$, series $\sum_0^\infty \frac{c_r}{P_{r+1}(y)}$ convergit extra $C(\varrho)$, divergit intra. In punctis $b_0, b_1 \dots b_{m-1}$ quomodo se habeat, ex indole functionis repraesentatae facilius, quam ex lege coefficientium colligitur. Si a unum ex punctis $a_0, a_1 \dots$ est, extra $C(\varrho)$ situm, quod in serie $a_0, a_1 \dots k$ vicibus reperitur, series $\sum_0^\infty \frac{c_r (y - a)^k}{P_{r+1}(y)}$ circa punctum a finita et continua est. Itaque $\sum_0^\infty \frac{c_r}{P_{r+1}(y)}$ sive finita, sive infinita est, ad iustum functionis repraesentatae valorem convergit.

§ 5.

Functio $f(y)$ extra curvam $C(\varrho)$ characterem functionis rationalis habeat, valoremque finitum servet, et in infinito evanescat. Variabilis y extra curvam $C(\varrho)$ iaceat, punctum x lineam $C(\varrho')$ percurrat [$\varrho < \varrho' < \varphi(y)$], quae punctum infinitum sensu positivo circumpleteatur. Est

$$f(y) = \sum_0^{n-1} \frac{c_r}{P_{r+1}(y)} + \frac{1}{2\pi i} \int \frac{f(x) dx}{y - x} \cdot \frac{P_n(x)}{P_n(y)}$$

si ponitur

$$c_r = \frac{1}{2\pi i} \int f(x) P_r(x) dx$$

Et quoniam $\frac{P_n(x)}{P_n(y)}$ crescente n in infinitum decrescit est

$$f(y) = \sum_0^\infty \frac{c_r}{P_{r+1}(y)}$$

quae series extra $C(\varrho)$ certe convergit. E. g., si $\varphi(x) < \varphi(y)$ est, fit

$$\frac{1}{y - x} = \sum_0^\infty \frac{P_r(x)}{P_{r+1}(y)}$$

Addi potest harum evolutionum coefficientes prorsus determinatos esse. Nam si series non ubique divergens

$\sum_{r=0}^{\infty} \frac{c_r}{P_{r+1}(y)}$ identice evanescit, certe punctum infinitum intra limites convergentiae iacet. Multiplicando igitur per y et ponendo $y = \infty$ invenitur $c_0 = 0$ etc. Iam series $\sum_{r=0}^{\infty} \frac{c_r}{P_{r+1}(y)}$, si extra curvam $C(\varrho)$ convergit, extra quam ex punctis $a_0, a_1 \dots$ iacent $a_\alpha, a_\beta \dots$, reprezentat functionem charactere rationalis praeditam, quae extra lineam $C(\varrho)$ non nisi in punctis $a_\alpha, a_\beta \dots$ infinite magna certi ordinis fieri potest. Itaque si $f(y)$ in curva $C(\varrho)$ aut functionis rationalis characterem servare desinit, aut in puncto a , quod inter $a_0, a_1 \dots$ minus quam k vicibus invenitur, k ti ordinis infinite magna evadit, seriei $\sum_{r=0}^{\infty} \frac{c_r}{P_{r+1}(y)}$ convergentia latius patere nequit. Si autem a inter $a_0, a_1 \dots$ non minus, quam k vicibus reperitur, dico seriem $\sum_{r=0}^{\infty} \frac{c_r}{P_{r+1}(y)}$ aliis punctis non impedientibus ultra curvam $C(\varrho)$ convergere. Est enim

$$f(y) = \frac{g_0}{(y-a)^z} + \dots + \frac{g_{k-1}}{y-a} + g(y)$$

et series, quae functionem $g(y)$ reprezentat convergit latius et series, in quam $\frac{1}{(y-a)^z}$ ($z = 1, 2 \dots k$) evolvi potest, finito terminorum numero constat. Nam sit inter quantitates $a_0, a_1 \dots, a_{n-1}$ z ta, quae ipsi a aequalis est, aequationem

$$\frac{1}{y-x} \left[1 - \frac{P_n(x)}{P_n(y)} \right] = \sum_{r=0}^{n-1} \frac{P_r(x)}{P_{r+1}(y)}$$

($z-1$) vicibus secundum x differentiando et $x=a$ ponendo obtinetur formula

$$\frac{1}{(y-a)^z} = \frac{1}{1 \cdot 2 \dots z-1} \sum_{r=0}^{z-1} \frac{P_r^{(z)}(a)}{P_{r+1}(y)}$$

Cognita igitur functione $f(y)$, de veris convergentiae seriei $\sum_{r=0}^{\infty} \frac{c_r}{P_{r+1}(y)}$ finibus facile est iudicium.

§ 6.

Si $f(x)$ intra curvam $C(\varrho)$ neque latius characterem functionis integrae habet, x intra hos fines iacet, y lineam $C(\varrho')$ [$\varrho > \varrho' > \varphi(x)$] percurrit neque est ullum punctorum $a_0, a_1 \dots$ aut in $C(\varrho')$ aut inter $C(\varrho)$ et $C(\varrho')$ situm, est

$$f(x) = \sum_{r=0}^{n-1} c_r P_r(x) + \frac{1}{2\pi i} \int \frac{f(y) dy}{y - x} \frac{P_n(x)}{P_n(y)}$$

$$c_r = \frac{1}{2\pi i} \int \frac{f(y) dy}{P_{r+1}(y)}$$

et

$$f(x) = \sum_{r=0}^{\infty} c_r P_r(x)$$

quae series intra curvam $C(\varrho)$ convergit, extra divergit. Sit non solum $\varrho > \varrho' > \varphi x$, sed etiam $\varrho > \varrho'' > \varphi x$, sintque ϱ' et ϱ'' ita electae, ut in anulari plani parte curvis $C(\varrho')$ et $C(\varrho'')$ terminata, nonnulla ex punctis $a_0, a_1 \dots$ iaceant. Denique sit integrale

$$\frac{1}{2\pi i} \int \frac{f(y) dy}{P_{r+1}(y)}$$

per curvam $C(\varrho')$ extensum $= c'_r$, per $C(\varrho'')$ autem $= c''_r$, quae quantitates, si ad paucas exceptiones non respicimus, diversae sunt. Ex aequationibus

$$f(x) = \sum_{r=0}^{\infty} c'_r P_r(x) \quad f(x) = \sum_{r=0}^{\infty} c''_r P_r(x)$$

ponendo $c'_r - c''_r = c_r$ perspicitur esse

$$\sum_{r=0}^{\infty} c_r P_r(x) = 0$$

quae series intra minorem curvarum $C(\varrho')$ et $C(\varrho'')$ certe convergit. Quia ex formula quia pluribus modis diversis omnem functionem evolvi posse sequitur, oritur quaestio gravissima, omnes eiusdem functionis evolutiones inveniendi, quae nullo negotio reducitur ad problema omnes cifrae evolutiones investigandi. Si plures series identice evanescentes repertae sunt multiplicando eas per constantes arbitrarias et addendo nova eiusdem generis series obtinetur. Itaque plures cifrae evolutiones $S, S' \dots$ dicimus esse

inter se independentes, si constantes $h, h' \dots$ ita determinari non possunt ut in serie $hS + h'S' + \dots$ omnes coefficientes evanescant. Quare problema propositum transformatur in quaestionem elegantiorum, sistema completum evolutionum cifrae inter se independentium indagandi. Quodsi series $\sum_{\nu}^{\infty} c_{\nu} P_{\nu}(x) = 0$ intra curvam $C(\varrho)$ convergit puncta $a_0, a_1 \dots$ omnia circumPLICANTEM, ponendo deinceps $x = a_0, a_1 \dots$ invenitur $c_0 = 0, c_1 = 0 \dots$ Quoniam igitur non existit series $\sum_{\nu}^{\infty} c_{\nu} P_{\nu}(x) = 0$, intra hanc lineam convergens, nascitur problema, omnes curvas $C(\varrho)$ quaerendi ita comparatas, ut habeatur series $\sum_{\nu}^{\infty} c_{\nu} P_{\nu}(x) = 0$ intra convergens, extra divergens.

§ 7.

Sit $\sum_{\nu}^{\infty} c_{\nu} P_{\nu}(x)$ ulla cifrae evolutio intra curvam $C(\varrho)$ convergens, extra divergens: Si puncta $a_0, a_1 \dots$ omnia intra $C(\varrho)$ iacerent, aequatio $\sum_{\nu}^{\infty} c_{\nu} P_{\nu}(x) = 0$, ut modo docuimus, locum habere non posset; si infinite multa non intra essent sita, series $\sum_{\nu}^{\infty} (a_{\kappa+m\nu} - b_{\kappa})$ non omnes convergerent. Sit igitur a_n punctum non intra situm, cuius index maximus est. Statuendo

$$R(x) = - \sum_{\mu}^n c_{\mu} \frac{P_{\mu}(x)}{P_{n+1}(x)} \quad Q_{\nu}(x) = \frac{P_{n+\nu}(x)}{P_{n+1}(x)}$$

functiones $Q_1(x), Q_2(x) \dots$ prorsus simile sistema functionum constituunt atque $P_0(x), P_1(x) \dots$ et fit

$$R(x) = \sum_{\nu}^{\infty} c_{n+\nu} Q_{\nu}(x)$$

quae series intra $C(\varrho)$ convergit neque latius. Quodsi inter puncta $a_0, a_1 \dots a_n$ sunt, quae intra $C(\varrho)$ iaceant, $R(x)$ in his valorem finitum habet: alias enim seriei $\sum_{\nu}^{\infty} c_{n+\nu} Q_{\nu}(x)$ convergentia arctioribus limitibus coerceretur. Iam si in ipsa linea $C(\varrho)$ nullum ex punctis $a_0, a_1 \dots a_n$ inveniretur

designato per a_α puncto extra iacente, in quo functionis $\varphi(a_\alpha)$ valor absolutus ϱ' quam minimus est, reperiri posset series $R(x) = \sum_v^\infty c_{n+v} Q_v(x)$ intra $C(\varrho')$ convergens; neque differentiae $c_{n+v} - c'_{n+v}$ omnes evanescerent, quia $\varrho' > \varrho$ esset; haberetur igitur series $\sum_v^\infty (c_{n+v} - c'_{n+v}) Q_v(x) = 0$, certe convergens intra lineam $C(\varrho)$, intra quam $a_{n+1}, a_{n+2} \dots$ omnia essent sita; quod fieri non potest. Itaque nisi curva $C(\varrho)$ per unum ex punctis $a_0, a_1 \dots$ transit, non existit series $\sum_v^\infty c_v P_v(x) = 0$ intra $C(\varrho)$ convergens, extra divergens.

Si autem $C(\varrho)$ per a_n transgreditur, iaceat x intra hanc lineam, sit $\varrho > \varrho' > \varphi(x)$, percurrat y circulum circa a_n descriptum extra $C(\varrho')$ iacentem neque ullum ex punctis $a_0, a_1 \dots$ ab a_n diversum continentem. Si ex quantitatibus $a_0, a_1 \dots a_{n-1} k-1$ ipsis a_n aequales sunt, aequationem

$$\frac{1}{y-x} = \sum_v^0 \frac{P_v(x)}{P_{v+1}(y)}$$

per $\frac{1}{2\pi i} (y-a_n)^{k-1}$ multiplicando et secundum y integrando invenitur

$$S_n = \sum_v^\infty c_{nv} P_v(x) = 0$$

$$c_{nv} = \left[\frac{1}{P_{v+1}(y)} \right] (y-a_n)^{-k}$$

denotante $[f(x)]_{x^n}$ coefficientem ipsius x^n in evolutione functionis $f(x)$ secundum potestates argumenti progrediente. Hanc cifrae representationem S_n intra $C(\varrho)$ convergentem, extra divergentem, ad a_n pertinere dicimus.

Serierum $S_0, S_1 \dots$ quia primi sunt termini $P_0(x), P_1(x) \dots$ constantes $h_0, h_1 \dots$ ita determinari non possunt ut expressionis $h_0 S_0 + h_1 S_1 + \dots$ omnes coefficientes evanescant. Ceterae autem cifrae evolutiones omnes in formam $h_0 S_0 + h_1 S_1 + \dots$ redigi possunt. Nam si series $\sum_v^\infty c_v P_v(x) = 0$ intra curvam $C(\varrho)$ convergit nec latius,

sit a_n ultimum punctorum $a_0, a_1 \dots$ intra eam non situm et

$$G(x) = - \sum_{\mu}^n c_{\mu} P_{\mu}(x),$$

fit

$$\frac{G(x)}{P_{n+1}(x)} = \sum_{\nu}^{\infty} c_{n+\nu} Q_{\nu}(x)$$

Punctorum $a_0, a_1 \dots, a_x, a_{\beta}, a_{\gamma}$ non intra $C(\rho)$, $a_{\lambda}, a_{\mu}, a_{\nu} \dots$ ($\lambda < \mu < \nu \dots$) intra $C(\rho)$ iaceant. Ex quantitatibus $c_{\alpha}, c_{\beta}, c_{\gamma} \dots$ quae manent arbitrariae, coefficientes

$c_{\lambda}, c_{\mu}, c_{\nu} \dots$ ita determinandi sunt, ut $\frac{G(x)}{P_n(x)}$ in punctis

$a_{\lambda}, a_{\mu} \dots$ valorem finitum servet; quod una tantum ratione fieri posse, facile perspicitur, cum ponitur primum $G(a_{\lambda})=0$, unde c_{λ} invenitur, deinde, si non $a_{\lambda}=a_{\mu}$, $G(a_{\mu})=0$, sin $a_{\lambda}=a_{\mu}$, $G'(a_{\mu})=0$, unde c_{μ} reperitur etc. Quantitates $c_{n+1}, c_{n+2} \dots$ ex $c_0, c_1 \dots c_n$ computantur ponendo deinceps in aequatione

$$\frac{G(x)}{P_{n+1}(x)} = \sum_{\nu}^{\infty} c_{n+\nu} Q_{\nu}(x)$$

et in iis, quae per differentiationem ex hac obtinentur, $x=a_{n+1}, a_{n+2} \dots$ quod per harum serierum convergentiam licet. Duae igitur cifrae evolutiones, intra $C(\rho)$ convergentes, in quibus $c_{\alpha}, c_{\beta}, c_{\gamma} \dots$ congruunt, diversae esse nullo modo possunt. Perspicitur autem, constantes $h_{\alpha}, h_{\beta} \dots$ ita posse determinari, ut $P_{\alpha}(x), P_{\beta}(x) \dots$, in expressione $h_{\alpha} S_{\alpha} + h_{\beta} S_{\beta} \dots$ eosdem habeant coefficientes, atque in serie $\sum_{\nu}^{\infty} c_{\nu} P_{\nu}(x)$. Itaque $S_0, S_1 \dots$ sistema completum evolutionum cifrae inter se independentium efficiunt.

§ 8.

Quantitatibus $a_0, a_1 \dots$ una cum n ultra omnem limitem crescentibus, commoditatis causa ponamus

$$P_n(x) = \left(1 - \frac{x}{a_0}\right) \left(1 - \frac{x}{a_1}\right) \dots \left(1 - \frac{x}{a_{n-1}}\right)$$

Convergente serie $\sum_r \frac{1}{a_r}$ quia convergit etiam productum

$$P(x) = \prod_r^\infty \left(1 - \frac{x}{a_r}\right)$$

ex aequatione

$$\frac{1}{x-y} \left(1 - \frac{P_n x}{P_{n+1} y}\right) = \sum_r^\infty \frac{P_r(x)}{a_r P_{r+1}(y)}$$

derivatur formula

$$\frac{1}{x-y} \left(1 - \frac{Px}{Py}\right) = \sum_r^\infty \frac{P_r(x)}{a_r P_{r+1}(y)}$$

Itaque ne deficere nostra nos methodus videatur, ostendamus, in seriem absolute convergentem et secundum has functiones $P_n(x)$ progredientem, expressionem $\frac{1}{x-y}$ omnino evolvi non posse. Quoniam $P_n(x)$ pro finitis argumenti x et omnibus indicis n valoribus quantitas finita est, nec nisi in punctis $a_0, a_1 \dots$ evanescit, series $\sum_r^\infty c_r P_r(x)$ una cum serie $\sum_r^\infty c_r$ aut convergit aut divergit. Quare evolutio $\sum_r^\infty c_r P_r(x)$ aut pro nullo aut pro omnibus finitis ipsius x valoribus valet. Nisi igitur $f(x)$ ubique in finito functionis integrae characterem praebet, aequatio $f(x) = \sum_r^\infty c_r P_r(x)$ non existit, unde fractionem $\frac{1}{x-y}$ evolvi non posse patet. Iam formulae

$$\frac{1}{2\pi i} \int \frac{P_r(x)}{P_{n+1}(x)} dx = 0 \quad (r > n) \quad \frac{1}{2\pi i} \int \frac{P_n(x)}{a_n P_{n+1}(x)} dx = 1$$

integrationis via puncta $a_0, a_1 \dots a_{n-1}$ simpliciter sensu negativo circumplicante, facile confirmantur. Itaque si

$$f(x) = \sum_r^\infty c_r P_r(x)$$

est, debet esse

$$c_r = \frac{1}{2\pi i} \int \frac{f(y) dy}{a_r P_{r+1}(y)}$$

et

$$\sum_r^\infty c_r P_r(x) = \frac{1}{2\pi i} \int f(y) dy \left[\frac{1}{x-y} - \frac{1}{x-y} \frac{P_n(x)}{P_n(y)} \right]$$

ergo

$$\sum_r^\infty c_r P_r(x) = \frac{1}{2\pi i} \int \frac{f(y) dy}{x-y} \frac{P_n x}{P_n y}$$

Functio igitur $f(x)$ ubique in finito charactere integrac praedita in seriem $\sum_r^\infty c_r P_r(x)$ evolvi potest, si $\int \frac{f(y) dy}{(y-x) P_n(y)}$ integrationis linea puncta $a_0 \dots a_{n-1}$ circumplexente, crescente n ad nihilum tendit, si non, non potest.

§ 9.

Restat casus valde memorabilis, quo cum $a_0, a_1 \dots$ in infinitum crescant, series $\sum_r^\infty \frac{1}{a_r^n}$ divergit. Quem saepe numero absolvere licet ope theorematis ex lectionibus, quas Ill. Weierstrass de functionum ellipticarum theoria habuit, sumpti:

„Si numeri integri positivi n' existunt ita comparati, ut series $\sum_r^\infty \frac{1}{a_r^{n'}}$ convergat et inter eos n minimus est, posito

$$g(x, a) = \frac{x}{a} + \frac{x^2}{2a^2} + \dots + \frac{x^{n-1}}{(n-1)a^{n-1}}$$

productum

$$\prod_r^\infty \left(1 - \frac{x}{a_r}\right) e^{-g(x, a_r)}$$

pro omnibus finitis ipsius x valoribus convergit.“

Quod ut probetur, demonstrandum est, posito

$$1 - \varphi_r(x) = \left(1 - \frac{x}{a_r}\right) e^{-g(x, a_r)}$$

seriem $\sum_r^\infty \varphi_r(x)$ convergere. Potest $\varphi_r(x)$ secundum integras positivas argumenti x potestates in seriem ubique in finito convergentem evolvi, quae, quia $\varphi_r(0) = 0$ est membro constante caret. Est autem

$$\varphi'_r(x) = \frac{x^{n-1}}{a_r^n} e^{-g(x, a_r)}$$

ergo

$$\varphi_v(x) \sim \frac{1}{a_v^n}$$

Convergente igitur $\sum_v^\infty \frac{1}{a_v^n}$, convergit etiam $\sum_v^\infty \varphi_v(x)$.

Quod ut exemplo illustremus, sit

$$P_n(x) = \left(1 - \frac{x}{1}\right) \left(1 - \frac{x}{2}\right) \cdots \left(1 - \frac{x}{n}\right)$$

est

$$P_n(x) = \left[\left(1 - \frac{x}{1}\right) e^{\frac{x}{1}} \cdots \left(1 - \frac{x}{n}\right) e^{\frac{x}{n}} \right] e^{-x \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)}$$

vel

$$P_n(x) \sim e^{-x \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)}$$

Posito

$$\varphi(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \lg(n)$$

est

$$\varphi(n) - \varphi(n-1) = \frac{1}{n} + \lg \frac{n-1}{n} = \lg \left(1 - \frac{1}{n}\right) e^{\frac{1}{n}}$$

et

$$\varphi(n) - \varphi(1) = \lg \prod_2^n \left(1 - \frac{1}{v}\right) e^{\frac{1}{v}}$$

unde patet, ut notum est, $\varphi(\infty)$ esse finitam. Quare est

$$P_n(x) \sim \frac{1}{n^x}$$

seriesque $\sum_v^\infty c_v P_v(x)$ eandem convergentiae regionem habet

ac series $\sum_v^\infty \frac{c_v}{v^x}$ excepto puncto infinito. Dicimus punctum x ad dextram puncti y iacere, si pars realis differentiae $x - y$ positiva est. Iacente igitur x ad dextram ipsius y , est

$$\frac{1}{x-y} = \sum_v^\infty \frac{P_v(x)}{(v+1) P_{v+1}(y)}$$

Manifestum est, functionem integrum n gradus in

seriem $g(x) = \sum_{\nu}^n c_{\nu} P_{\nu}(x)$ evolvi posse. Quod si $h(x)$ circa punctum infinitum functionis rationalis characterem habet est $h(x) = g(x) + f(x)$ et $f(\infty) = 0$. Iaceat x ad dextram omnium punctorum singularium functionis $f(x)$, a quibus per rectam C , axi ordinatarum parallelam seiungatur, percurrat y curvam ad laevam lineae C iacentem et puncta singularia cuncta sensu positivo circumeuntum. Habemus aequationem

$$f(x) = \sum_{\nu}^{n-1} c_{\nu} P_{\nu}(x) + \frac{1}{2\pi i} \int \frac{f(y) dy}{x - y} \frac{P_n(x)}{P_n(y)}$$

si est

$$c_{\nu} = \frac{1}{2\pi i} \int \frac{f(y) dy}{(\nu + 1) P_{\nu+1}(y)}$$

et

$$f(x) = \sum_{\nu=0}^{\infty} c_{\nu} P_{\nu}(x).$$

Unde patet omnem functionem circa punctum infinitum charactere rationalis praeditam in seriem evolvi posse secundum functiones integras $P_n(x)$ progredientem et ad dextram omnium punctorum singularium convergentem.

II.

§ 10.

Venio nunc ad ea functionum systemata, quae ex functionis propositae in fractionem continuam evolutione originem ducunt. Si fractio continua

$$R_0(x) = \cfrac{1}{a_0 x - 1} - \cfrac{1}{a_1 x - 1} - \cfrac{1}{a_2 x - \dots}$$

in qua nec ullam quantitatum $a_0, a_1, a_2 \dots$ evanescere, nec infinite magnam evadere pro finitis indicis valoribus ponimus, in aliqua plani parte convergens est, n tam fractionem approximatam per $\frac{P_n(x)}{Q_n(x)}$ et residuum per $\frac{R_n(x)}{Q_n(x)}$ designo, ita, ut sit

$$\begin{aligned} P_{n+1}(x) + P_{n-1}(x) &= a_n x P_n(x) \quad (n = 1, 2.) \quad P_0 = 0, \quad P_1 = 1. \\ Q_{n+1}(x) + Q_{n-1}(x) &= a_n x Q_n(x) \quad (n = 0, 1.) \quad Q_{-1} = 0, \quad Q_0 = 1. \\ R_{n+1}(x) + R_{n-1}(x) &= a_n x R_n(x) \quad (n = 0, 1.) \quad R_{-1} = 1. \end{aligned}$$

Itaque si ponitur

$$S = \sum_0^n a_r Q_r(x) R_r(y)$$

est

$$\begin{aligned} x S &= \sum_r [a_r x Q_r(x)] R_r(y) = \sum_r [Q_{r-1}(x) + Q_{r+1}(x)] R_r(y) \\ y S &= \sum_r Q_r(x) [a_r y R_r(y)] = \sum_r Q_r(x) [R_{r-1}(y) + R_{r+1}(y)] \\ (y - x) S &= 1 + Q_n(x) R_{n+1}(y) - Q_{n+1}(x) R_n(y) \end{aligned}$$

Hac via satis elementari hoc invenitur formularum systema:

$$\begin{aligned}
 \sum_v^n a_v P_v(x) P_v(y) &= \frac{P_n(x) P_{n+1}(y) - P_{n+1}(x) P_n(y)}{y - x} \\
 \sum_v^n a_v P_v(x) Q_v(y) &= \frac{1}{y - x} + \frac{P_n(x) Q_{n+1}(y) - P_{n+1}(x) Q_n(y)}{y - x} \\
 \sum_v^n a_v P_v(x) R_v(y) &= \frac{R_0(y)}{y - x} + \frac{P_n(x) R_{n+1}(y) - P_{n+1}(x) R_n(y)}{y - x} \\
 \sum_v^n a_v Q_v(x) Q_v(y) &= \frac{Q_n(x) Q_{n+1}(y) - Q_{n+1}(x) Q_n(y)}{y - x} \\
 \sum_v^n a_v Q_v(x) R_v(y) &= \frac{1}{y - x} + \frac{Q_n(x) R_{n+1}(y) - Q_{n+1}(x) R_n(y)}{y - x} \\
 \sum_v^n a_v R_v(x) R_v(y) &= \frac{R_0(x) - R_0(y)}{y - x} + \frac{R_n(x) R_{n+1}(y) - R_{n+1}(x) R_n(y)}{y - x} \\
 \sum_v^n a_v (P_v(x))^2 &= P_n(x) P'_{n+1}(x) - P_{n+1}(x) P'_n(x) \\
 \sum_v^n a_v P_v(x) Q_v(x) &= P_n(x) Q'_{n+1}(x) - P_{n+1}(x) Q'_n(x) \\
 \sum_v^n a_v P_v(x) R_v(x) &= R'_0 x + P_n(x) R'_{n+1}(x) - P_{n+1}(x) R'_n(x) \\
 \sum_v^n a_v (Q_v(x))^2 &= Q_n(x) Q'_{n+1}(x) - Q_{n+1}(x) Q'_n(x) \\
 \sum_v^n a_v Q_v(x) R_v(x) &= Q_n(x) R'_{n+1}(x) - Q_{n+1}(x) R'_n(x) \\
 \sum_v^n a_v (R_v(x))^2 &= -R'_0(x) + R_n(x) R'_{n+1}(x) - R_{n+1}(x) R'_n(x)
 \end{aligned}$$

Eaedem formulae valent, si $P_n(x)$, $Q_n(x)$, $R_n(x)$ ad fractionem generaliorem

$$\frac{1}{a_0 x + b_0 - \frac{1}{a_1 x + b_1 - \frac{1}{a_2 x + b_2 - \dots}}}$$

referuntur.

Sunt $P_n(x)$ et $Q_n(x)$ functiones integrae ($n = 1$) ti et n ti gradus. $R_n(x)$ circa punctum infinitum in seriem secundum potestates descendentes ipsius x progredientem evolvi potest, quae a $-(n + 1)$ ta potestate incipit et est

$$P_n(-x) = (-1)^{n-1} P_n(x)$$

$$Q_n(-x) = (-1)^n Q_n(x)$$

$$R_n(-x) = (-1)^{n+1} R_n(x)$$

Denique inter has functiones habetur relatio:

$$P_n(x) + R_n(x) = R_0(x) \cdot Q_n(x)$$

unde patet, pro finitis indicis n valoribus seriem $R_n(x)$ eundem certe convergentiae circulum habere, ac seriem $R_0(x)$.

Percurrente z circulum paulo maiorem C , evanescit integrale $\int \frac{P_n(z) dz}{z - x}$, si x extra C iacet, quia $P_n(x)$ functio

integra est, et integrale $\int \frac{R_n(z) dz}{z - x}$, si x intra C iacet,

quia $R_n(x)$ extra C nullum punctum singulare habet et in infinito evanescit. Unde colligitur, integrale

$$\frac{1}{2\pi i} \int \frac{R_0(z) Q_n(z) dz}{z - x}$$

si x intra C iacet, functionem $P_n(x)$, si extra iacet, functionem $-R_n(x)$ exhibere.

§ 11.

Iam quid ex formulis inventis sequatur, videamus, si est

$$R_0(x) = \frac{F(\alpha, \beta+1, \gamma+1, x^{-2})}{xF(\alpha, \beta, \gamma, x^{-2})}$$

designante, ut solet, $F(\alpha, \beta, \gamma, x)$ functionem hypergeometricam serie

$$1 + \frac{\alpha \beta}{\gamma} x + \frac{\alpha(\alpha+1) \beta(\beta+1)}{\gamma(\gamma+1)} \frac{x^2}{1 \cdot 2} + \dots$$

definitam. Est $a_0 = 1$ et

$$a_{2n-1} = \frac{(\beta+1) \cdot (\beta+n-1) (\gamma-\alpha+1) \cdot (\gamma-\alpha+n-1)}{\alpha \cdot (\alpha+n-1) (\gamma-\beta) \cdot (\gamma-\beta+n-1)} \gamma(\gamma+2n-1)$$

$$a_{2n} = \frac{\alpha \cdot (\alpha+n-1) (\gamma-\beta) \cdot (\gamma-\beta+n-1)}{(\beta+1) \cdot (\beta+n) (\gamma-\alpha+1) \cdot (\gamma-\alpha+n)} \frac{(\gamma+2n)}{\gamma}$$

unde ope theorematis § 9 memorati, posito

$$N = n^{\alpha-\beta-\frac{1}{2}}$$

facile derivantur aequivalentiae

$$a_{2n-1} \sim N^{-2} \quad a_{2n} \sim N^2$$

Inter radices aequationis $1 - 2xz + z^2 = 0$, quarum productum = 1 est, ea, quae > 1 est, per $x + \sqrt{x^2 - 1}$, ea quae < 1 est, per $x - \sqrt{x^2 - 1}$ denotetur.

Ex iis, quae Ill. Thomé *) de fractione continua $R_0(x)$ demonstravit, sequitur, valente formula superiore, si n impar, inferiore, si n par est,

$$P_n(x) \sim (x + \sqrt{x^2 - 1})^n \frac{N}{N-1}$$

$$Q_n(x) \sim (x + \sqrt{x^2 - 1})^n \frac{N}{N-1}$$

$$R_n(x) \sim (x - \sqrt{x^2 - 1})^n \frac{N}{N-1}$$

exceptis in aequivalentia prima valoribus, pro quibus $R_0(x)$ evanescit, in altera et tertia iis, pro quibus infinite magna evadit. Vix opus est monere, per $R_0(x)$ eum tantum designari functionis $\frac{F(\alpha, \beta+1, \gamma+1, x^{-2})}{xF(\alpha, \beta, \gamma, x^{-2})}$ ramum, qui fractione continua repraesentatur et toto plano exclusa recta puncta -1 et $+1$ iungente, per $C(1)$ in posterum denotanda, functionis rationalis characterem habet, cum in linea $C(1)$ neque $R_0(x)$, neque $R_1(x), R_2(x) \dots$ sint definitae.

Ex aequatione differentiali, cui satisfacit $F(\alpha, \beta, \gamma, x)$, perspicitur, aequationem $F(\alpha, \beta, \gamma, x) = 0$ multiplices radices non admittere. Iam formula

$$\beta\gamma F(\alpha, \beta, \gamma, x) - \beta(\gamma - \alpha) F(\alpha, \beta+1, \gamma+1, x) = \gamma(1-x) \frac{dF(\alpha, \beta, \gamma, x)}{dx}$$

docet aequationes

$$F(\alpha, \beta, \gamma, x) = 0 \quad \text{et} \quad F(\alpha, \beta+1, \gamma+1, x) = 0.$$

nullam habere radicem communem. Sit u una ex radicibus aequationis $F(\alpha, \beta+1, \gamma+1, x^{-2}) = 0$, extra $C(1)$ sita, quales pro certis elementorum α, β, γ valoribus existere,

*) Diarium Crellianum, Tom. 66 et 67.

Ill. Thomé exemplis probavit. Ex aequatione $P_n(x) + R_n(x) = R_0(x) Q_n(x)$ sequitur $P_n(u) = -R_n(u)$. Sit v una ex radicibus aequationis $F(\alpha, \beta, \gamma, x^{-2}) = 0$ extra $C(1)$ sita, et $S_n(x) = \frac{R_n(x)}{R_0(x)}$; ex formula $\frac{P_n(x)}{R_0(x)} + S_n(x) = Q_n(x)$ sequitur $S_n(v) = Q_n(v)$. Demonstravit autem Ill. Thomé, quotientem

$$R_n(x) \cdot (x - v) : (x - \sqrt{x^2 - 1})^n \frac{N}{N-1}$$

ad talem functionem convergere, quae pro $x = v$ valorem finitum obtineat. Itaque

$$\lim S_n(x) : (x - \sqrt{x^2 - 1})^n \frac{N}{N-1}$$

pro $x = v$ finitus est.

Unde manant aequivalentiae

$$P_n(u) \sim (u - \sqrt{u^2 - 1})^n \frac{N}{N-1}$$

$$Q_n(v) \sim (v - \sqrt{v^2 - 1})^n \frac{N}{N-1}$$

Ellipsi, in qua $x + \sqrt{x^2 - 1}$ valorem absolutum constantem ρ servat, per $C(\rho)$ denotata, si seriei $\sum_v^\infty c_v x^v$ radius convergentiae ρ est, series $\sum_v^\infty c_v P_v(x)$ et $\sum_v^\infty c_v Q_v(x)$, si $\rho > 1$ est, intra ellipsin $C(\rho)$ convergunt, extra divergunt, series $\sum_v^\infty c_v R_v(x)$, si $\rho < 1$ est, extra ellipsin $C\left(\frac{1}{\rho}\right)$ convergit, intra divergit.

Ex formulis § 10 inventis manant series sequentes, quarum de convergentia facile est iudicium:

$$\sum_{r=1}^{\infty} a_r P_r(x) R_r(y) = \frac{R_0(y)}{y-x}$$

$$\sum_{r=1}^{\infty} a_r P_r(x) R_r(y) = - \frac{P_n(x) R_{n+1}(y) - P_{n+1}(x) R_n(y)}{y-x}$$

$$\sum_{r=1}^{\infty} a_r Q_r(x) R_r(y) = \frac{1}{y-x}$$

$$\sum_{r=1}^{\infty} a_r Q_r(x) R_r(y) = - \frac{Q_n(x) R_{n+1}(y) - Q_{n+1}(x) R_n(y)}{y-x}$$

$$\sum_{r=1}^{\infty} a_r R_r(x) R_r(y) = \frac{R_0(x) - R_0(y)}{y-x}$$

$$\sum_{r=1}^{\infty} a_r R_r(x) R_r(y) = - \frac{R_n(x) R_{n+1}(y) - R_{n+1}(x) R_n(y)}{y-x}$$

$$\sum_{r=1}^{\infty} \int_x^{\infty} a_r [R_r(x)]^2 dx = R_0(x)$$

$$\sum_{r=1}^{\infty} a_r [R_r(x)]^2 = - [R_n(x) R'_{n+1}(x) - R_{n+1}(x) R'_n(x)]$$

$$\sum_{r=1}^{\infty} a_r P_r(u) P_r(x) = 0$$

$$\sum_{r=1}^{\infty} a_r Q_r(v) P_r(x) = \frac{1}{v-x}$$

$$\sum_{r=1}^{\infty} a_r P_r(u) Q_r(x) = \frac{1}{x-u}$$

$$\sum_{r=1}^{\infty} a_r Q_r(v) Q_r(x) = 0.$$

$$\sum_{r=1}^{\infty} a_r P_r(u) R_r(x) = \frac{R_0(x)}{x-u}$$

$$\sum_{r=1}^{\infty} a_r Q_r(v) R_r(x) = \frac{1}{x-v}$$

Itaque si de functione $f(y)$ idem ponitur, quod § 5, est

$$f(y) = \sum_{r=1}^{\infty} c_r R_r(y) + \frac{1}{2\pi i} \int \frac{Q_n(x) R_{n+1}(y) - Q_{n+1}(x) R_n(y)}{y-x} f(x) dx$$

$$c_r = \frac{a_r}{2\pi i} \int Q_r(x) f(x) dx$$

$$f(y) = \sum_{r=1}^{\infty} c_r R_r(y)$$

quae series extra curvam $C(\rho)$ certe convergit, neque latius,

nisi forte $f(y)$ ultra $C(\varrho)$ characterem functionis rationalis habet atque in ipsa linea $C(\varrho)$ tantummodo pro $y = v$ primi ordinis infinita fit. Qua conditione quia series

$$\frac{1}{y - v} = \sum_{r=0}^{\infty} a_r Q_r(v) R_r(y) \text{ toto plano exclusa linea } C(1)$$

convergit, series functionem $f(y)$ repraesentans in curva $C(\varrho)$ convergere non desinit. Si de functione $f(x)$ idem statuitur, quod § 6, est

$$f(x) = \sum_{r=0}^{\infty} Q_r(x) + \frac{1}{2\pi i} \int \frac{Q_n(x) R_{n+1}(y) - Q_{n+1}(x) R_n(y)}{y - x} f(y) dy$$

$$c_r = \frac{a_r}{2\pi i} \int f(y) R_r(y) dy$$

$$f(x) = \sum_{r=0}^{\infty} c_r Q_r(x)$$

quae series intra ellipsin $C(\varrho)$ convergit, extra divergit. Haec omnia breviter enarravisse satis est, ad problemata § 6 proposita, quorum ad solutionem omnia nunc praeparata sunt, tractanda properamus.

§ 12.

Ad quamvis radicem v aequationis $R_0(x) = \infty$ extra $C(1)$ sitam pertinet cifrae evolutio

$$S = \sum_{r=0}^{\infty} a_r Q_r(v) Q_r(x) = 0$$

quae convergit, si x intra ellipsin, in qua v iacet, versatur, neque latius. Iam sit $\sum_{r=0}^{\infty} c_r Q_r(x)$ ulla cifrae evolutio intra lineam $C(\varrho)$ convergens. Punctorum v extra ullam ellipsin $C(\varrho)$ ($\varrho > 1$) iacentium finitus est numerus. Nam quia $R_0(x)$ circa punctum infinitum in seriem secundum descendentes argumenti potestates progredientem evolvi potest, omnes aequationis $R_0(x) = \infty$ radices in spatio finito iacent. Omnem autem functionem in spatio finito, in quo ubique rationalis characterem habet, nisi in finito punctorum numero non evadere infinitam, Ill. Weierstrass demon-

stravit.* Sint igitur $v_1, v_2 \dots v_n$ radices aequationis $R_0(x) = \infty$ non intra curvam $C(\varrho)$ sitae, $S_1, S_2 \dots S_n$ cifrae evolutiones ad eas pertinentes, $v_{n+1}, v_{n+2} \dots$ reliquae radices, intra $C(\varrho)$ iacentes. Consideremus functionem

$$R(x) = \sum_v^\infty c_v R_v(x)$$

quae series ubique extra $C(1)$ convergit. Quia

$$R_n(x)(x - v) : (x - \sqrt{x^2 - 1})^n \frac{N}{N-1}$$

convergit ad functionem, quae pro $x = v$ valorem finitum habet, $R(x)$ infinita fieri non potest, nisi primi ordinis in punctis $v_1, v_2 \dots v_n$ et $v_{n+1}, v_{n+2} \dots$

Iam series $\sum_v^\infty c_v P_v(x)$ et ipsa intra ellipsin $C(\varrho)$ convergit atque ex relatione

$$P_n(x) + R_n(x) = R_0(x) Q_n(x)$$

perspicitur esse

$$R(x) = \sum_v^\infty c_v R_v(x) = - \sum_v^\infty c_v P_v(x)$$

exceptis valoribus, in quibus $R_0(x) = \infty$ est. Sed quia $R(x) = \sum_v^\infty c_v R_v(x)$ certe extra lineam $C(1)$ rationalis

characterem habet, aequatio $R(x) = - \sum_v^\infty c_v P_v(x)$, si in aliqua plani parte valet, pro omnibus ipsius x valoribus locum habere debet, pro quibus series $\sum_v^\infty c_v P_v(x)$ conver-

git. Primum igitur in punctis $v_{n+1}, v_{n+2} \dots$ finita esse $R(x)$ debet; deinde intra ellipsin $C(\varrho)$ functionis rationalis characterem habet. Quamobrem toto plano functionis rationalis charactere est praedita: Itaque est $R(x)$ functio rationalis, quae quia nisi in punctis $v_1, v_2 \dots v_n$ infinita esse non potest et in infinito evanescit, necessario formam habet

$$R(x) = \frac{h_1}{x - v_1} + \frac{h_2}{x - v_2} + \dots + \frac{h_n}{x - v_n}$$

Si autem x intra $C(\varrho)$ iacet et v non intra, est

*) Diarium Crellianum, Tom. 52.

$$\frac{1}{x - v} = \sum_0^\infty a_r Q_r(v) R_r(x)$$

Est igitur

$$R(x) = \sum_v^\infty c_r R_r(x) = \sum_v^\infty a_r [h_1 Q_r(v_1) + \dots h_n Q_r(v_n)] R_r(x)$$

Pluribus autem modis secundum functiones $R_r(x)$ evolvi functio non potest. Est ergo

$$c_r = a_r [h_1 Q_r(v_1) + \dots h_n Q_r(v_n)]$$

Quare seriei $\sum_v^\infty c_r Q_r(x)$ coefficientes cum coefficientibus seriei $h_1 S_1 + \dots h_n S_n$ congruunt. Completum igitur sistema $S_1, S_2 \dots$ esse demonstravimus: Independentes autem inter se nisi essent hae cifrae evolutiones, constantes $h_1, h_2 \dots h_n$ ita determinari possent, ut omnes expressionis $h_1 S_1 + \dots h_n S_n$ coefficientes evanescerent, sive ut esset $h_1 Q_r(v_1) + \dots h_n Q_r(v_n) = 0$. Itaque quia

$$\frac{1}{x - v} = \sum_0^\infty a_r Q_r(v) R_r(x)$$

est, functio $\frac{h_1}{x - v_1} + \dots + \frac{h_n}{x - v_n}$ identice evanesceret, quod nisi $h_1 \dots h_n$ omnes evanescunt, fieri non potest. Itaque $S_1, S_2 \dots$ sistema completum evolutionum cifrae inter se independentium constituunt, quod hac etiam ratione inventitur: Transeat ellipsis $C(\varrho)$ per punctum v , iaceat x intra $C(\varrho)$, percurrat y circulum circa v descriptum, extra $C(\varrho')$ iacentem ($\varrho > \varrho' > v + \sqrt{v^2 - 1}$) neque ullam aliam radicem aequationis $R_0(x) = \infty$ continentem. Iam aequationem fundamentalem

$$\frac{1}{y - x} = \sum_0^\infty a_r Q_r(x) R_r(y)$$

secundum y integrando, obtinetur

$$\sum_0^\infty a_r [R_r(y)]_{(y-v)^{-1}} Q_r(x) = 0$$

vel quia

$$[R_r(y)]_{(y-v)^{-1}} = Q_r(v) [R_0(y)]_{(y-v)^{-1}}$$

est

$$\sum_{\nu}^{\infty} a_{\nu} Q_{\nu}(v) Q_{\nu}(x) = 0$$

Quodsi series $\sum_{\nu}^{\infty} c_{\nu} Q_{\nu}(x) = 0$ existit intra ellipsis $C(\varrho)$ convergens, extra divergens, in ipsa linea $C(\varrho)$ una ex quantitatibus v iacere debet. Alias enim, si $v_1, v_2 \dots v_n$ extra $C(\varrho)$ iacerent et $S_1 \dots S_n$ ad ea pertinerent, series $\sum_{\nu}^{\infty} c_{\nu} Q_{\nu}(x)$, ut modo docuimus, in formam $h_1 S_1 + \dots + h_n S_n$ redigi posset latiusque convergeret, quia series $S_1 \dots S_n$ extra $C(\varrho)$ convergunt. Observo denique, propter relationem $Q_n(-x) = (-1)^n Q_n(x)$ ex formula $\sum_{\nu}^{\infty} c_{\nu} Q_{\nu}(x) = 0$ manare aequationes $\sum_{\nu}^{\infty} c_{2\nu} Q_{2\nu}(x) = 0$ et $\sum_{\nu}^{\infty} c_{2\nu+1} Q_{2\nu+1}(x) = 0$.

In theoria serierum

$$\sum_{\nu}^{\infty} c_{\nu} P_{\nu}(x) \quad \text{et} \quad \sum_{\nu}^{\infty} c_{\nu} \frac{R_{\nu}(x)}{R_0(x)}$$

quae pars eodem modo absolvitur, ac serierum

$$\sum_{\nu}^{\infty} c_{\nu} Q_{\nu}(x) \quad \text{et} \quad \sum_{\nu}^{\infty} c_{\nu} R_{\nu}(x)$$

hoc loco non commoramus.

§ 13.

Functio $f(x)$ in anulari plani parte, curvis $C(\varrho_0)$ et $C(\varrho)$ ($\varrho > \varrho_0$) terminata nec latius characterem integrae habeat, et punctum x intra hos fines iaceat. Quantitates ϱ' et ϱ'_0 ita determinentur, ut

$$\varrho > \varrho' > x + \sqrt{x^2 - 1} > \varrho'_0 > \varrho_0$$

sit neque aut inter $C(\varrho)$ et $C(\varrho')$ aut inter $C(\varrho_0)$ et $C(\varrho'_0)$ aut in ipsis lineis $C(\varrho')$ et $C(\varrho'_0)$ ulla radix v aequationis $R_0(x) = \infty$ iaceat. Iam punctis y et y_0 curvas $C(\varrho')$ et $C(\varrho'_0)$ sensu positivo percurrentibus, est

$$f(x) = \frac{1}{2\pi i} \int \frac{f(y) dy}{y-x} + \frac{1}{2\pi i} \int \frac{f(y_0) dy_0}{x-y_0}$$

unde colligitur

$$f(x) = \sum_{\nu}^{\infty} [c_{\nu} Q_{\nu}(x) + c'_{\nu} R_{\nu}(x)]$$

si ponitur

$$c_\nu = \frac{a_\nu}{2\pi i} \int f(y) R_\nu(y) dy \quad c'_\nu = \frac{a_\nu}{2\pi i} \int f(y_0) Q_\nu(y_0) dy_0.$$

Quae series extra $C(\varrho)$ divergit neque intra $C(\varrho_0)$ convergit excepto casu, quem sub finem §. 11 commemo-ravimus.

Accedamus nunc ad rationem inter plures eiusdem functionis evolutiones considerandam, cuius rei causa hanc totam quaestionem aggressi sumus. Sit

$$\sum_{\nu=0}^{\infty} [c_\nu Q_\nu(x) - c'_\nu R_\nu(x)] = 0$$

ulla cifrae evolutio inter curvas $C(\varrho_0)$ et $C(\varrho)$ ($\varrho > \varrho_0$) con-vergēns. Posito

$$R(x) = \sum_{\nu=0}^{\infty} c_\nu Q_\nu(x)$$

est etiam

$$R(x) = \sum_{\nu=0}^{\infty} c'_\nu R_\nu(x)$$

Haec igitur functio, quoniam propter primam aequationem intra curvam $C(\varrho)$ propter alteram extra curvam $C(\varrho_0)$, atque ideo propter utramque toto plano characterem rationalis habet, esse debet functio rationalis, quae propter primam aequationem intra $C(\varrho)$ finita est propter alteram nisi in punctis v_1, v_2, \dots, v_n , non intra $C(\varrho)$ iacentibus, infinita esse non potest et in infinito evanescit. Ergo est

$$R(x) = \frac{h'_1}{x - v_1} + \dots + \frac{h'_n}{x - v_n}$$

et

$$\sum_{\nu=0}^{\infty} c'_\nu R_\nu(x) = \sum_{\nu=0}^{\infty} a_\nu [h'_1 Q_\nu(v_1) + \dots + h'_n Q_\nu(v_n)] R_\nu(x)$$

quare $c'_\nu = a_\nu [h'_1 Q_\nu(v_1) + \dots + h'_n Q_\nu(v_n)]$. Porro est:

$$\frac{1}{x - v} = \sum_{\nu=0}^{\infty} \left(\frac{a_\nu}{2\pi i} \int \frac{R_\nu(y) dy}{y - v} \right) Q_\nu(x),$$

sive posito

$$\bar{R}_r(v) = \frac{\gamma}{\beta(\gamma-\alpha)} \frac{v^2-1}{2} Q'_r(v) - P_r(v)$$

$$\frac{1}{x-v} = \sum_{r=0}^{\infty} a_r \bar{R}_r(v) Q_r(x) \quad \text{et}$$

$$R(x) = \sum_{r=0}^{\infty} c_r Q_r(x) = \sum_{r=0}^{\infty} a_r [h'_1 \bar{R}_r(v_1) + \dots + h'_n \bar{R}_r(v_n)] Q_r(x)$$

unde colligitur $c_r = a_r [h'_1 \bar{R}_r(v_1) + \dots + h'_n \bar{R}_r(v_n)]$.

Posito igitur

$$S = \sum_{r=0}^{\infty} a_r Q_r(v) Q_r(x) = 0,$$

$$S' = \sum_{r=0}^{\infty} a_r [\bar{R}_r(v) Q_r(x) - Q_r(v) R_r(x)] = 0$$

completum esse sistema $S_1, \dots, S_n, S'_1, \dots, S'_n$ manifestum est. Si autem constantes $h_1, \dots, h_n, h'_1, \dots, h'_n$ ita determinari possent, ut expressionis $h_1 S_1 + \dots + h_n S_n + h'_1 S'_1 + \dots + h'_n S'_n$ omnes coefficientes evanescerent, esset ipsius $R_v(x)$ coefficiens $h'_1 Q_r(v_n) + \dots + h'_n Q_r(v_n) = 0$, unde $h'_1 = 0, \dots, h'_n = 0$; omnes igitur seriei $h_1 S_1 + \dots + h_n S_n$ coefficientes evanescerent unde, $h_1 = 0, \dots, h_n = 0$. Constituunt igitur $S_1, S_2, \dots, S'_1, S'_2, \dots$ sistema completum evolutionum cifrae inter se independentium facileque ut supra probatur, si existat series $\sum_{r=0}^{\infty} [c_r Q_r(x) - c'_r R_r(x)] = 0$ inter $C(\varrho_0)$ et $C(\varrho)$ convergens nec latius, et $\varrho_0 = 1$ esse, et in linea $C(\varrho)$ unum ex punctis v iacere.

§. 14.

Ex functionibus, quas modo tractavimus, maxime memorabiles sunt eae, quae sphaericæ vocantur. A quibus profectus cum ad omnia quae exposui, pervenerim, haud alienum esse existimo paullo diligentius in earum theoriam inquirere. Aequationem hic fundamentalem

$$\frac{1}{y-x} = \sum_{r=0}^{\infty} (2r+1) P_r(x) Q_r(y)$$

al Ill. Heine repartam et duabus demonstrationibus munitam esse constat. Quarum prior ab Ill. Thomé rigorosa est facta. Alteram cognita vera integralium Laplaceano-

rum indole, aliquantum simplificare potui. Qua explicata tertiam ex methodo §. 10 adhibita manantem adiungam.

Ex identitatibus

$$\begin{aligned} - \int_0^{y - \sqrt{y^2 - 1}} \frac{dz}{dz} \sqrt{\frac{1 - 2yz + z^2}{1 - 2xz + z^2}} dz &= 1 \\ - \frac{d}{dz} \sqrt{\frac{1 - 2yz + z^2}{1 - 2xz + z^2}} &= \frac{(y-x)(1-z^2)}{(1-2yz+z^2)^{\frac{1}{2}}(1-2xz+z^2)^{\frac{1}{2}}} \\ \frac{1-z^2}{(1-2xz+z^2)^{\frac{1}{2}}} &= \sqrt{1-2xz+z^2} + 2z \frac{d}{dz} \frac{1}{\sqrt{1-2xz+z^2}} \end{aligned}$$

manifestum est esse

$$\frac{1}{y-x} = \int_0^{y - \sqrt{y^2 - 1}} \frac{dz}{\sqrt{1-2yz+z^2}} \left(\frac{1}{\sqrt{1-2xz+z^2}} + 2z \frac{d}{dz} \frac{1}{\sqrt{1-2xz+z^2}} \right)$$

Quod si $z < x - \sqrt{x^2 - 1}$ est, series

$$\frac{1}{\sqrt{1-2xz+z^2}} = \sum_0^\infty P_\nu(x) z^\nu$$

per quam functiones $P_0(x), P_1(x) \dots$ definimus, convergit.

Percurrente autem z rectam puncta 0 et $y - \sqrt{y^2 - 1}$ iungentum, semper est $z < x - \sqrt{x^2 - 1}$, si $y - \sqrt{y^2 - 1} < x - \sqrt{x^2 - 1}$ est. Hac igitur conditione est

$$\frac{1}{y-x} = \sum_0^\infty (2\nu+1) P_\nu(x) \int_0^{y - \sqrt{y^2 - 1}} \frac{z^\nu dz}{\sqrt{1-2yz+z^2}}$$

sive posito $Q_n(y) = \int_0^{y - \sqrt{y^2 - 1}} \frac{z^n dz}{\sqrt{1-2yz+z^2}}$

$$\frac{1}{y-x} = \sum_0^\infty (2\nu+1) P_\nu(x) Q_\nu(y).$$

Quare si $f(x)$ intra ellipsin $C(\varrho)$ characterem functionis integrae habet, x intra $C(\varrho)$ iacet, y lineam $C(\varrho')$ percurrit ($\varrho > \varrho' > x + \sqrt{x^2 - 1}$), est

$$\begin{aligned} f(x) &= \frac{1}{2\pi i} \int dy \int_0^{y - \sqrt{y^2 - 1}} \frac{dz}{\sqrt{1-2yz+z^2}} \\ &\quad \left(\frac{1}{\sqrt{1-2xz+z^2}} + 2z \frac{d}{dz} \frac{1}{\sqrt{1-2xz+z^2}} \right) \end{aligned}$$

cuius formulae ope $f(x)$ secundum functiones sphaericas evolvi potest.

In altera demonstratione, quam nunc aggredimur, symmetriae gratia etiam $P_n(x)$ per integrale repraesentemus. Si $z > x + \sqrt{x^2 - 1}$ est, series

$$\frac{1}{\sqrt{1 - 2xz + z^2}} = \sum_0^\infty P_r(x) z^{r-1}$$

convergit. Percurrente autem z ellipsin puncta $x - \sqrt{x^2 - 1}$ et $x + \sqrt{x^2 - 1}$ simpliciter circumeuntem, cuius omnibus in punctis $z > x + \sqrt{x^2 - 1}$ est, ex hac aequatione multiplicando per $z^n dz$ et integrando sequitur

$$P_n(x) = \frac{1}{2\pi i} \int \frac{z^n dz}{\sqrt{1 - 2xz + z^2}}.$$

Quod integrale non mutatur curva integrationis ita deformanda, ut puncta $x - \sqrt{x^2 - 1}$ et $x + \sqrt{x^2 - 1}$ complecti non desinat neque ullo negotio reducitur ad

$$P_n(x) = \frac{1}{i\pi} \int_{x - \sqrt{x^2 - 1}}^{x + \sqrt{x^2 - 1}} \frac{z^n dz}{\sqrt{1 - 2xz + z^2}}$$

percurrente z rectam puncta $x - \sqrt{x^2 - 1}$ et $x + \sqrt{x^2 - 1}$ iungentem. Itaque si t est variabilis realis a -1 ad $+1$ tendens, integrationis linea aequatione $z = x + t\sqrt{x^2 - 1}$ definitur. Unde concluditur

$$P_n(x) = \frac{1}{\pi} \int_{-1}^{+1} (x + t\sqrt{x^2 - 1})^n \frac{dt}{\sqrt{1 - t^2}}$$

quod est notum integrale Laplaceanum ex fonte genuino deductum *).

Eadem methodus ope aequationis

$$\frac{1}{\sqrt{1 - 2xz + z^2}} = \sum_0^\infty P_r(x) z^r$$

*) Eodem modo, si $\frac{1}{(1 - 2xy + y^2)^n} = \sum_0^\infty F_r(x)y^r$ ponitur designante n quantitatem positivam, inveniuntur integralia functiones $F_r(x)$ repraesentantia, quae Ill. Heine via multo minus directa derivavit.

formulam

$$P_n(x) = \frac{1}{\pi} \int_{-1}^{+1} (x + t \sqrt{x^2 - 1})^{-n-1} \frac{dt}{\sqrt{1-t^2}}$$

exhibit. Itaque nunc functiones sphaericas utriusque generis ratione persymmetrica definimus per aequationes

$$P_n(x) = \frac{1}{i\pi} \int_{\frac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}}^{\frac{x+\sqrt{x^2+1}}{x-\sqrt{x^2-1}}} \frac{z^n dz}{\sqrt{1-2xz+z^2}} \quad Q_n(x) = \int_0^{\frac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}} \frac{z^n dz}{\sqrt{1-2xz+z^2}}$$

ex quibus primo aspectu perspicitur, functionem $Q_n(x)$ circumante x unum ex punctis $+1$ et -1 quantitate $i\pi P_n(x)$ augeri.

Denotantibus z_0 et z_1 duas ex quantitatibus

$$0, x - \sqrt{x^2 - 1}, x + \sqrt{x^2 - 1},$$

radicibus aequationis $z^n \sqrt{1-2xz+z^2} = 0$, est

$$\int_{z_0}^{z_1} \frac{d}{dz} (z^n \sqrt{1-2xz+z^2}) dz = 0$$

vel

$$\int_{z_0}^{z_1} \frac{(n+1)z^{n+1} - (2n+1)xz^n + nz^{n-1}}{\sqrt{1-2xz+z^2}} dz = 0$$

unde duae simul manant formulae

$$(n+1)P_{n+1}(x) + nP_{n-1}(x) = (2n+1)xP_n(x)$$

$$(n+1)Q_{n+1}(x) + nQ_{n-1}(x) = (2n+1)xQ_n(x)$$

Itaque posito

$$S = \sum_{\nu}^n (2\nu+1) P_{\nu}(x) Q_{\nu}(y)$$

est

$$xS = \sum_{\nu}^n [(\nu+1)P_{\nu+1}(x) + \nu P_{\nu-1}(x)] Q_{\nu}(y)$$

$$yS = \sum_{\nu}^n P_{\nu}(x) [(\nu+1)Q_{\nu+1}(y) + \nu Q_{\nu-1}(y)]$$

$$(y-x)S = 1 + (n+1) [P_n(x) Q_{n+1}(y) - P_{n+1}(x) Q_n(y)]$$

vel

$$\sum_{\nu}^{\infty} (2\nu+1) P_{\nu}(x) Q_{\nu}(y) = \frac{1}{y-x} + \frac{(n+1)[P_n(x)Q_{n+1}(y) - P_{n+1}(x)Q_n(y)]}{y-x}$$

Si $\rho < x - \sqrt{x^2 - 1}$ est, series $\sum_{\nu}^{\infty} P_{\nu}(x) \rho^{\nu}$ convergit, omniaque eius membra infra limitem finitum iacent unde $P_n(x) < \frac{g}{\rho^n}$ esse patet. Series autem

$$\sum_{\nu}^{\infty} Q_{\nu}(y) \sigma^{\nu} = \sum_{\nu}^{\infty} \int_0^{y - \sqrt{y^2 - 1}} \frac{(z\sigma)^{\nu} dz}{\sqrt{1 - 2yz + z^2}}$$

convergit, si pro omnibus ipsius z valoribus $z\sigma < 1$, vel si $\sigma < y + \sqrt{y^2 - 1}$ est. Itaque designante h quantitatem finitam $Q_n(y) < \frac{h}{\sigma^n}$ est. Unde facile colligitur, si

$$x + \sqrt{x^2 - 1} < y + \sqrt{y^2 - 1}$$

est, residuum

$$\frac{(n+1)[P_n(x)Q_{n+1}(y) - P_{n+1}(x)Q_n(y)]}{y-x}$$

ad nihilum convergere et esse

$$\frac{1}{y-x} = \sum_{\nu}^{\infty} (2\nu+1) P_{\nu}(x) Q_{\nu}(y). *$$

Iam coefficientes harum serierum ad formam notam simpliciorem revocemus. Si est

$R_0(x) = 0$, $R_1(x) = 1$, $(n+1)R_{n+1}(x) + nR_{n-1}(x) = (2n+1)xR_n(x)$ expressio $P_n(x)Q_0(x) - R_n(x)$ eidem recursionis formulae satisfacit atque $Q_n(x)$ et pro $n = o$ valorem $Q_0(x)$, pro $n = 1$ valorem $xQ_0(x) - 1 = Q_1(x)$ obtinet. Quare est

$$Q_n(x) = P_n(x)Q_0(x) - R_n(x)$$

denotante $R_n(x)$ functionem integrum. Est autem

*) Ex formulis §. 10 expositis totidem manant propositiones de functionibus sphaericis, quas brevitatis causa non perscribo, velut aequatio

$$\frac{1}{1-x^2} = \sum_{\nu}^{\infty} (2\nu+1)(Q_{\nu}(x))^2 = \sum_{\nu}^{\infty} (2\nu+1)Q_{\nu}(x^2).$$

$$Q_0(x) = \int_0^{x - \sqrt{x^2 - 1}} \frac{dz}{\sqrt{1 - 2xz + z^2}} = \frac{1}{2} \lg \frac{x-1}{x+1} = \frac{1}{2} \int_{-1}^{+1} \frac{dy}{y-x}$$

Iam posito

$$q_n(x) = \frac{1}{2} \int_{-1}^{+1} \frac{P_n(y) dy}{y-x}$$

est

$$q_n(x) = \frac{1}{2} \int_{-1}^{+1} \frac{P_n(x) dy}{y-x} + \frac{1}{2} \int_{-1}^{+1} \frac{(P_n(y) - P_n(x)) dy}{y-x}$$

vel

$$q_n(x) = P_n(x) Q_0(x) - r_n(x)$$

designante $r_n(x)$ functionem integrum. Itaque est

$$Q_n(x) - q_n(x) = r_n(x) - R_n(x)$$

Sed $Q_n(\infty) = 0$ et $q_n(\infty) = 0$; functio autem integra $r_n(x) - R_n(x)$ in infinito evanescere nequit, nisi identice evanescit. Quare est

$$Q_n(x) = \frac{1}{2} \int_{-1}^{+1} \frac{P_n(y) dy}{y-x}$$

Quae formula idcirco memorabilis est, quod eum tantum reprezentat functionis $Q_n(x)$ ramum, qui toto plano exclusa linea $C(1)$ uniformis et continuus est et in infinito evanescit.

Iam secundum functiones $Q_n(y)$ pluribus modis evolvi functionem non posse, per considerationes ex methodo coefficientium indeterminatorum haustas facile perspicitur. Itaque ex aequatione

$$Q_n(y) = \sum_{\nu=0}^{\infty} \left(\frac{1}{2\pi i} \int Q_n(x) P_{\nu}(x) dx \right) Q_{\nu}(y)$$

invenitur:

$$\frac{1}{2\pi i} \int Q_n(x) P_{\nu}(x) dx = 0 \quad (n > \nu) \quad \frac{1}{2\pi i} \int Q_n(x) P_n(x) dx = 1$$

Est igitur $0 = \frac{1}{2\pi i} \int Q_n(x) P_\nu(x) dx =$

$$\frac{1}{2\pi i} \int P_\nu(x) dx \frac{1}{2} \int_{-1}^{+1} \frac{P_n(y) dy}{y-x} = \frac{1}{2} \int_{-1}^{+1} P_n(y) dy \frac{1}{2\pi i} \int \frac{P_\nu(x) dx}{y-x}$$

et

$$\int_{-1}^{+1} P_n(y) P_\nu(y) dy = 0 \quad \text{et} \quad \int_{-1}^{+1} [P_n(y)]^2 dy = \frac{2}{2n+1}$$

Quarum formularum ope serierum secundum functiones $P_n(x)$ progredientium coefficientes determinari solent.

Denique etiam in seriem secundum functiones $R_n(x)$ progredientem ope formulae

$$\frac{1}{y-x} = \sum_\nu (2\nu+1) R_\nu(x) \frac{Q_\nu(y)}{Q_0(y)}$$

omnem functionem intra unam ellipsium $C(\varrho)$ charactere integrae praeditam evolvi posse, idque uno modo, hic commemorasse satis est.

VITA.

Natus sum Ferdinandus Georgius Frobenius Berolini anno 1849 die XXVI mensis Octobris patre Ferdinando, quem adhuc superstitem veneror, matre Elisabeth, e gente Friedrich, quam iam defunctam lugeo.

Fidei addictus sum evangelicae.

Gymnasium Joachimicum sub auspiciis Ill. Kiessling ab anno 1860 usque ad annum 1867 frequentavi.

Maturitatis testimonio munitus primum Universitatem Georgiam Augustanam Gottingensem adii, in qua per unum semestre ab Ill. Meyer et Stern in analysin et ab Ill. Weber in physicen introductus sum. Deinde numero civium Universitatis Fridericae Guilelmae a Rectore Magnifico de Langenbeck legitime adscriptus, per sex semestria disserentes audivi viros Ill. Dove, Kronecker, Kummer, Magnus, Quincke, Trendelenburg, Weierstrass.

Exercitationibus seminarii mathematici, quas moderantur Ill. Kummer et Weierstrass per quattuor semestria interfui.

Quibus omnibus viris optime de me meritis maximeque Ill. Kronecker, Kummer, Weierstrass, qui insignem semper benevolentiam in me contulerunt, gratias ago maximas.

THESES.

1. Kantius suam de spatio et tempore sententiam non satis gravibus argumentis confirmavit.
 2. Tractationem calculi differentialis integralium definitorum theoria anteire debet.
 3. Melius est, analysis superioris, quam geometriae recentioris syntheticae elementa in scholis doceri.
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