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QUAE VOCANTUR SPHAERICAЕ,
PROGREDIENTIBUS.

DISSERTATIO INAUGURALIS

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CONSENSU ET AUCTORITATE

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MATRI OPTIMAE DILECTISSIMAE

HASCE STUDIORUM PRIMITIAS
PIO GRATIQUE ANIMO

OFFERT

AUCTOR.

Sit Fz functio argumenti z , quae loco a $z = -1$ usque ad $z = 1$ abscisso definita nusquam in infinite magnum excurrat. Quam functionem ex Cel. Dirichleti disquisitionibus notissimum est in huiusmodi seriem evolvi posse:

$$Fz = \sum_0^{\infty} \frac{2n+1}{2} \int_{-1}^1 Fz P^{(n)} z dz \cdot P^{(n)} z,$$

denotante $P^{(n)} z$ *n—tam* functionem Legendrianam

$$P^{(n)} z = \frac{1}{\pi} \int_0^{\pi} (z + \cos \phi \sqrt{z^2 - 1})^n d\phi = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{1 \cdot 2 \cdot 3 \cdots n} \left\{ z^n - \frac{n(n-1)}{2(2n-1)} z^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot (2n-1)(2n-3)} z^{n-4} - \dots \right\}.$$

Quam doctrinam vir Cel. C. Neumann in functiones complexi argumenti extendit in commentatione, quam inscripsit: Ueber die Entwicklung einer Function mit imaginärem Argument nach den Kugelfunctionen erster und zweiter Art. Halle 1862. Demonstrationem ab eo datam paucis verbis explicare nobis liceat.

In libro inscripto: Handbuch der Kugelfunctionen (p. 104) vir Cel. Heine seriem tractavit:

$$\frac{1}{v-u} = \sum_0^{\infty} (2n+1) Q^{(n)} v P^{(n)} u = \sum_0^{\infty} (2n+1) \int_0^{\infty} \frac{dt}{(v + \cos it \sqrt{v^2 - 1})^{n+1}} \cdot \frac{1}{\pi} \int_0^{\pi} (u + \cos \phi \sqrt{u^2 - 1})^n d\phi,$$

ubi radicum signa ita eligenda sunt, ut sit

$$\begin{aligned} \text{mod. } & (v + \sqrt{v^2 - 1}), \\ \text{nec non mod. } & (u + \sqrt{u^2 - 1}) \end{aligned} \left\{ > 1,$$

et $Q^{(n)} z$, quae functio extra rectam inter $z = -1$ et $z = 1$ abscissam naturam functionis integrae habet, et $P^{(n)} z$ significant integralia particularia aequationis differentialis:

$$(1-z^2) \frac{d^2 F}{dz^2} - 2z \frac{dF}{dz} + n(n+1)F = 0;$$

quam seriem ex hac conditione pendere efficit, ut sit

$$\text{mod. } (v + \sqrt{v^2 - 1}) > \text{mod. } (u + \sqrt{u^2 - 1}).$$

Cui conditioni non satisficeri Cel. Neumann l. c. demonstravit, nisi

$\frac{u}{v}$ versetur intra
 $\frac{v}{v}$ extra } ellipsis, cuius foci sint $z = \pm 1$.

Idem evoluto secundum seriem commemoratam integrali in Cel. Cauchii formula:

$$Fz = \frac{1}{2\pi i} \int \frac{Fv}{v-z} dv$$

deduxit theorema:

Functio Fz , quae intra ellipsis cum focus $z = \pm 1$ naturam functionis integrae habet, exprimi potest hac serie:

$$Fz = \sum_0^{\infty} \frac{2n+1}{2\pi i} \int Fv Q^{(n)} v dv \cdot P^{(n)} z,$$

ubi integrandum est per curvam rectae focus limitatae intra ellipsis circumscripam a positivo latere x -axis ad positivum latus y -axis versus, si quidem $z = x + yi$.

Simili modo e formula

$$F_1 z = \frac{1}{2\pi i} \left\{ \int \frac{F_1 v}{v-z} dv + \int \frac{F_1 u}{z-u} du \right\},$$

ubi signum \int respondet exteriori linea, per quam integratio peragenda est, \int interiori, evolvit theorema:

Functio $F_1 z$, quae intra duas ellipses confocales, cum focus quidem $z = \pm 1$, naturam functionis integrae habet, explicari potest serie duplice:

$$F_1 z = \sum_0^{\infty} \frac{2n+1}{2\pi i} \left\{ \int F_1 v Q^{(n)} v dv \cdot P^{(n)} z + \int F_1 u P^{(n)} u du \cdot Q^{(n)} z \right\},$$

ubi item per curvam intra confocales ellipses in se recurrentem integrandum est.

In secundo theoremate hoc alterum inest: Functio $F_1 z$, quae extra ellipsis cum focus $z = \pm 1$ naturam functionis integrae habet et, si mod. z satis magnum valorem assumit, in potestatum seriem $C + \sum_0^{\infty} a_{-(n+1)} z^{-(n+1)}$ evolvitur, hac serie exprimi potest:

$$F_1 z = C + \sum_0^{\infty} \frac{2n+1}{2\pi i} \int F_1 u P^{(n)} u du \cdot Q^{(n)} z.$$

In formula enim

$$F_1 z = \frac{1}{2\pi i} \left\{ \int \frac{F_1 v}{v-z} dv + \int \frac{F_1 u}{z-u} du \right\}$$

ad hunc casum applicata $\int \frac{F_1 v}{v-z} dv$ per circulum peragi licet, unde sequitur

$$\frac{1}{2\pi i} \int \frac{F_1 v}{v-z} dv = \sum_0^{\infty} \frac{1}{2\pi i} \int \frac{F_1 v}{v^{n+1}} dv \cdot z^n = \sum_0^{\infty} a_n z^n,$$

$$a_n = \frac{r^{-n}}{2\pi} \int_0^{2\pi} F_1(re^{2i}) e^{-n\vartheta_i} d\vartheta.$$

Quoniam autem a_1, a_2, \dots evanescunt, nec non, si mod. (z) infinite magnus fit, $\frac{1}{2\pi i} \int_i^F \frac{F_1 u}{z-u} du$, sequitur, ut sit $a_0 = C$ et

$$F_1 z = C + \frac{1}{2\pi i} \int_i^F \frac{F_1 u}{z-u} du = C + \sum_0^{\infty} \frac{2n+1}{2\pi i} \int F_1 u P^{(n)} u du \cdot Q^{(n)} z.$$

Quam evolutionem functionis $F_1 z$ secundum $Q^{(n)} z$ uno solum modo fieri posse, demonstrandum est.

Si, Cel. Neumannum sequentes in seriem

$$Fz = \sum_0^{\infty} \frac{2n+1}{2\pi i} \int Fv Q^{(n)} v dv \cdot P^{(n)} z$$

loco Fz introducimus $P^{(n)} z$, exhibemus formulas

$$\int P^{(n)} z Q^{(n)} z dz = \frac{2\pi i}{2n+1}$$

$$\int P^{(n)} z Q^{(m)} z dz = 0 \quad n \geq m.$$

Iam ponatur

$$F_1 z = K + \sum_0^{\infty} K_n Q^{(n)} z;$$

formularum praecedentium ope deducitur:

$$\int F_1 z \cdot P^{(n)} z dz = K_n \frac{2\pi i}{2n+1};$$

$$K_n = \frac{2n+1}{2\pi i} \int F_1 z P^{(n)} z dz, \text{ unde } K = C.$$

In hac commentatione recta via proficiscimur a seriebus ipsis

$$\sum_0^{\infty} A_n P^{(n)} z, \quad \sum_0^{\infty} B_n Q^{(n)} z,$$

quarum coefficientes A, A, B, B qualescumque valores accipiunt, et methodo. analytica accuratius explorare nobis proponimus, quae sint leges circa huiusmodi series, imprimis quae iis propriae et convergentiae et differentiationis conditiones valeant.

Brevitatis causa mod. $(a+bi)$ signo $M(a+bi)$ denotari liceat.

Seriem $\sum_0^{\infty} (a_n + b_n i)$ dicemus absolute (unbedingt) convergere, si convergit series $\sum_0^{\infty} (M(a_n) + M(b_n))$. $\sum_0^{\infty} (a_n + b_n i)$ convergit absolute, dum converget $\sum_0^{\infty} M[a_n + b_n i]$, quod $M(a_n) + M(b_n) < 2M[a_n + b_n i]$; et vicissim: $\sum_0^{\infty} M[a_n + b_n i]$ convergit, dum converget absolute $\sum_0^{\infty} (a_n + b_n i)$, quod $M[a_n + b_n i] < M(a_n) + M(b_n)$.

§. 1.

Iam instituamus disquisitionem seriei

$$\sum_0^{\infty} A_n P^{(n)} z = \sum_0^{\infty} A_n \cdot \frac{1}{\pi} \int_0^{\pi} (z + \cos \phi \sqrt{z^2 - 1})^n d\phi.$$

Radicis signum, cum utrum libet esse constet, Cel. Heinium sequens ita definio, ut pars realis argumenti z eodem signo praedita sit, quo pars realis radicis $\sqrt{z^2 - 1}$; unde fit, ut etiam partes imaginariae signis correspondeant. Excipitur casus, ubi valor z realis est et $M(z) < 1$, quod subinde nullius momenti esse agnosces. Sequitur, ut sit

$$\begin{aligned} M[z + \sqrt{z^2 - 1}] &\geq 1 \quad (\text{Heine l. c. p. 75}) \\ \text{et } M[z + \sqrt{z^2 - 1}] &\geq M[z + \cos \phi \sqrt{z^2 - 1}]. \end{aligned}$$

Si convergit absolute

$$\sum_0^{\infty} A_n (z + \cos \phi \sqrt{z^2 - 1})^n,$$

dum ϕ migrat per valores $0, \dots, \pi$, appareat convergere

$$\sum_0^{\infty} A_n \frac{1}{\pi} \int_0^{\pi} (z + \cos \phi \sqrt{z^2 - 1})^n d\phi.$$

Si divergit

$$\sum_0^{\infty} A_n (z + \cos \phi \sqrt{z^2 - 1})^n$$

ϕ accipiente valorem $\phi_1 \begin{cases} > 0 \\ \leq \pi \end{cases}$,

contendimus divergere

$$\sum_0^{\infty} A_n \frac{1}{\pi} \int_0^{\pi} (z + \cos \phi \sqrt{z^2 - 1})^n d\phi,$$

dum argumentum z sit $\begin{cases} \text{reale et } M(z) \geq 1 \\ \text{mere imaginarium.} \end{cases}$

Demonstratio.

1) Sit argumentum z reale et $M(z) \geq 1$.

Diverget $\sum_0^{\infty} A_n (z + \cos \phi \sqrt{z^2 - 1})^n$ a $\phi = \phi_1$ usque ad $\phi = 0$.

Inquiramus haecce integralia:

$$\int_0^{\phi_1 - \varepsilon} A_n (z + \cos \phi \sqrt{z^2 - 1})^n d\phi, \quad \int_{\phi_1 - \varepsilon}^{\pi} A_n (z + \cos \phi \sqrt{z^2 - 1})^n d\phi,$$

ubi ε quamvis parvum est.

$$M \left[\int_0^{\phi_1 - \varepsilon} A_n (z + \cos \phi \sqrt{z^2 - 1})^n d\phi \right] = M \left[A_n (z + \cos(0 \dots \phi_1 - \varepsilon) \sqrt{z^2 - 1})^n \right] (\phi_1 - \varepsilon).$$

Sin autem divergit $\sum_0^{\infty} A_n (z + \cos \phi \sqrt{z^2 - 1})^n$ a $\phi = \phi_1$, ex notissima potestatum serierum proprietate consequitur, ut $M[A_n (z + \cos(\phi_1 - \varepsilon) \sqrt{z^2 - 1})^n]$ limite finito inferior permanere non possit; eoque minus $M[A_n (z + \cos(0 \dots \phi_1 - \varepsilon) \sqrt{z^2 - 1})^n]$.

$\int_{\phi_1-\varepsilon}^{\pi} A_n(z + \cos \phi \sqrt{z^2 - 1})^n d\phi$ eodem signo praeditum est, quo $\int_0^{\phi_1-\varepsilon} A_n(z + \cos \phi \sqrt{z^2 - 1})^n d\phi$;

ergo $M \left[\int_0^{\pi} A_n(z + \cos \phi \sqrt{z^2 - 1})^n d\phi \right]$ limite finito inferior non manet.

Idecirco $\sum_0^{\infty} A_n \frac{1}{\pi} \int_0^{\pi} (z + \cos \phi \sqrt{z^2 - 1})^n d\phi$ convergere nequit.

2) Sit argumentum z mere imaginarium.

$\sum_0^{\infty} A_n (z + \cos \phi \sqrt{z^2 - 1})^n$ $\begin{cases} \text{divergat a } \phi = 0 \text{ usque ad } \phi = \phi_1 \\ \text{convergat a } \phi = \phi_1 \text{ usque ad } \phi = \phi_2 \\ \text{divergat a } \phi = \phi_2 \text{ usque ad } \phi = \pi. \end{cases}$

Eadem methodo, qua supra, disquisitis integralibus:

$$\int_0^{\phi_1+\varepsilon} A_n (z + \cos \phi \sqrt{z^2 - 1})^n d\phi = \int_0^{\phi_1-\varepsilon} A_n (z + \cos \phi \sqrt{z^2 - 1})^n d\phi + \int_{\phi_1-\varepsilon}^{\phi_1+\varepsilon} A_n (z + \cos \phi \sqrt{z^2 - 1})^n d\phi,$$

$$\int_{\phi_2-\varepsilon}^{\pi} A_n (z + \cos \phi \sqrt{z^2 - 1})^n d\phi = \int_{\phi_2-\varepsilon}^{\phi_2+\varepsilon} A_n (z + \cos \phi \sqrt{z^2 - 1})^n d\phi + \int_{\phi_2+\varepsilon}^{\pi} A_n (z + \cos \phi \sqrt{z^2 - 1})^n d\phi,$$

reperitur:

$$\text{nec } M \left[\int_0^{\phi_1+\varepsilon} A_n (z + \cos \phi \sqrt{z^2 - 1})^n d\phi \right], \quad \text{nec } M \left[\int_{\phi_2-\varepsilon}^{\pi} A_n (z + \cos \phi \sqrt{z^2 - 1})^n d\phi \right]$$

limite finito inferiore manere.

Habemus autem

$$\int_0^{\pi-(\phi_2-\varepsilon)} A_n (z + \cos \phi \sqrt{z^2 - 1})^n d\phi + \int_{\phi_2-\varepsilon}^{\pi} A_n (z + \cos \phi \sqrt{z^2 - 1})^n d\phi =$$

$$\int_0^{\pi-(\phi_2-\varepsilon)} A_n (z + \cos \phi \sqrt{z^2 - 1})^n d\phi - \int_{\pi}^{\phi_2-\varepsilon} A_n (z + \cos \phi \sqrt{z^2 - 1})^n d\phi,$$

quam summam appetit prioris integralis signo affectam esse. Eodem signo praeditum est

$$\int_{\pi-(\phi_2-\varepsilon)}^{\phi_1+\varepsilon} A_n (z + \cos \phi \sqrt{z^2 - 1})^n d\phi,$$

cuius modulus limite finito inferior non permanet. Quoniam denique

$$\lim_{\infty} \int_{\phi_1+\varepsilon}^{\phi_2-\varepsilon} A_n (z + \cos \phi \sqrt{z^2 - 1})^n d\phi = 0,$$

sequitur, ut

$$M \left[\int_0^{\pi} A_n (z + \cos \phi \sqrt{z^2 - 1})^n d\phi \right]$$

limite finito inferior manere nequeat.

Ergo $\sum_0^{\infty} A_n \cdot \frac{1}{\pi} \int_0^{\pi} (z + \cos \phi \sqrt{z^2 - 1})^n d\phi$ convergere nequit. Quibus expositis propositi demonstrata est.

Iam R nominetur radius convergentiae circuli potestatum seriei $\sum_0^{\infty} A_n z^n$ proprii. Ex iis, quae demonstravimus, sequitur, ut

$$\sum_0^{\infty} A_n \cdot \frac{1}{\pi} \int_0^{\pi} (z + \cos \phi \sqrt{z^2 - 1})^n d\phi$$

convergat, idque absolute, si $M[z + \sqrt{z^2 - 1}] < R$;

sed divergat, si $M[z + \sqrt{z^2 - 1}] > R$, dum argumentum z sit $\begin{cases} \text{reale et } M(z) \geq 1, \\ \text{mere imaginarium.} \end{cases}$

Itaque linea determinanda est, in qua

$$M[z + \sqrt{z^2 - 1}] = R.$$

$$z + \sqrt{z^2 - 1} = w, \quad z^2 - 1 = (w - z)^2, \quad z = \frac{1}{2} \left(w + \frac{1}{w} \right),$$

$$z = \frac{1}{2} \left(R + \frac{1}{R} \right) \cos \theta + i \frac{1}{2} \left(R - \frac{1}{R} \right) \sin \theta.$$

$R > 1$: Prodit ellipsis cum semi-axibus $\frac{1}{2} \left(R + \frac{1}{R} \right)$ et $\frac{1}{2} \left(R - \frac{1}{R} \right)$, unde foci $z = \pm 1$, quae ellipsis simul cum R crescit et decrescit.

$R = 1$: Ellipsis in rectam degenerat focus limitatam.

$R < 1$: Relationi $M[z + \sqrt{z^2 - 1}] = R$ z respondere nequit.

Quibus explicatis haec efficiuntur:

$R > 1$: Series $\sum_0^{\infty} A_n P^{(n)} z = \sum_0^{\infty} A_n \frac{1}{\pi} \int_0^{\pi} (z + \cos \phi \sqrt{z^2 - 1})^n d\phi$,

convergit, idque absolute, intra ellipsin cum axe principali $R + \frac{1}{R}$ et focus $z = \pm 1$;

divergit extra hanc ellipsin, dum argumentum z sit $\begin{cases} \text{reale,} \\ \text{mere imaginarium.} \end{cases}$

$R = 1$: Series convergit absolute, dum z migrat per valores $-1 \dots +1$, modo potestatum series $\sum_0^{\infty} A_n z^n$ absolute convergat, etiam si $z = 1$;

divergit, si $z \begin{cases} \text{reale et } M(z) > 1, \\ \text{mere imaginarium.} \end{cases}$

$R < 1$: Series divergit, si $z \begin{cases} \text{reale et } M(z) \geq 1, \\ \text{mere imaginarium.} \end{cases}$

Series

§. 2.

$$\sum_0^{\infty} A_n P^{(n)} z, \quad R > 1,$$

intra convergentiae ellipsin ex functionum $P^{(n)}z$ continuitate continua derivatas habet functiones omnium ordinum, quae singulis propositae seriei terminis differentiatis evadunt series pariter absolute convergentes atque continuae.

Notissimo theoremate, quo series convergens inter limites definitos integratur singulis seriei terminis integrandis, demonstratio propositionis nostrae eo redigitur, ut seriem $\sum_0^{\infty} A_n \frac{d^n}{dz^n} P^{(n)}z$ convergere demonstretur.

$$\sum_0^{\infty} A_n \frac{d^n}{dz^n} P^{(n)}z = \sum_0^{\infty} A_n \cdot \frac{1}{\pi} \int_0^{\pi} \frac{d^n}{dz^n} (z + \cos \phi \sqrt{z^2 - 1})^n d\phi,$$

convergit absolute, quoniam $\sum_0^{\infty} A_n \frac{d^n}{dz^n} (z + \cos \phi \sqrt{z^2 - 1})^n$, absolute convergit, id quod facile cognoscet.

Casus ille accuratius contemplandus est, quo z valorem induit ± 1 et $z^2 - 1$ in denominatoribus evanescit.

E formula, quam indicavit vir Cel. Cristoffel:

$$\frac{d}{dz} P^{(n)}z = (2n-1) P^{(n-1)}z + (2n-5) P^{(n-3)}z + (2n-9) P^{(n-5)}z + \dots,$$

deducitur

$$\frac{d^n}{dz^n} P^{(n)}z = a P^{(n-m)}z + b P^{(n-m-2)}z + \dots,$$

ubi a, b, \dots positivi, integri numeri sunt.

$P^{(n)}z$ autem aequari posse notissimum est alterutri formarum:

$$P^{(n)}z = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{1 \cdot 2 \cdot 3 \cdots n} \begin{cases} (z^2 - \alpha_{n,1}^2) z^2 - \alpha_{n,2}^2 \cdots (z^2 - \alpha_{n,\frac{n}{2}}^2) \\ z(z^2 - \beta_{n,1}^2)(z^2 - \beta_{n,2}^2) \cdots (z^2 - \beta_{n,\frac{n-1}{2}}^2) \end{cases},$$

ubi valores $\alpha, \alpha, \beta, \beta$, reales sunt et intercepti inter -1 et $+1$.

Itaque quoniam demonstratum est $\sum_0^{\infty} M \left[A_n \frac{d^n}{dz^n} P^{(n)}z \right]$ convergere, si argumentum z

reale sit et $\begin{cases} > 1 \\ < 1 \end{cases}$, apparel, seriem $\sum_0^{\infty} A_n \frac{d^n}{dz^n} P^{(n)}z$ utique absolute convergere, si sit $z = \pm 1$.

§. 3.

Functio

$$P^{(n)}z = \sum_0^{\infty} A_n P^{(n)}z, \quad R > 1,$$

intra convergentiae ellipsin seriei propriam finita atque continua, cum secundum § antegressam functionem offerat derivatam iisdem proprietatibus praeditam, ex notissimo Cel. Cauchii theoremate intra limites definitos integrae functionis naturam habet.

Itaque convergentiae ellipsis amplior esse nequit amplissima ellipsi confocali, intra quam functio Fz quoquaversus uno, quo fieri potest, modo continuata ab integrae functionis natura non discedit.

Intra quam amplissimam ellipsis quoniam ex primo Cel. Neumann theoremate, quod in introductione nostra explicavimus, adiuncta illa doctrina, functionem argumenti z una solum ratione secundum $P^{(n)}z$ evolvi posse, series proposita certissime convergit, adipiscimur hoc theorema, quod valet, dummodo notionem convergentiae ellipsis supra constitutam respiciamus:

Series

$$Fz = \sum_0^{\infty} A_n P^{(n)} z$$

convergentiae ellipsis focus $z = \pm 1$ praeditam habet: aut rectam focus limitatam, aut, dum series intra propriam ellipsis converget, amplissimam ellipsis confocalem, intra quam functio integra Fz quoquaversus continuata integrae functionis naturam conservat.

§. 4.

Functiones de serie $Fz = \sum_0^{\infty} A_n P^{(n)} z$, $R > 1$, derivatae quia intra convergentiae ellipsis seriei propriam naturam functionum integrarum habent, seriebus secundum $P^{(n)} z$ procedentibus explicari possunt.

$$\text{Ponatur } F'z = \sum_0^{\infty} D_n P^{(n)} z.$$

$$A_n = \frac{2n+1}{2} \int_{-1}^1 Fz P^{(n)} z dz, \quad D_n = \frac{2n+1}{2} \int_{-1}^1 F'z P^{(n)} z dz.$$

Coefficiens D_n facile per valores A , A exprimitur. Nam

$$\int_{-1}^1 F'z P^{(n)} z dz = [Fz P^{(n)} z]_{-1}^1 - \int_{-1}^1 Fz \frac{d}{dz} P^{(n)} z dz.$$

$$\begin{aligned} \int_{-1}^1 Fz \frac{d}{dz} P^{(n)} z dz &= (2n-1) \int_{-1}^1 Fz P^{(n-1)} z dz + (2n-5) \int_{-1}^1 Fz P^{(n-3)} z dz + \\ &\quad + (2n-9) \int_{-1}^1 Fz P^{(n-5)} dz + \dots = 2A_{n-1} + 2A_{n-3} + 2A_{n-5} + \dots \end{aligned}$$

$$\text{Ergo } D_n = \frac{2n+1}{2} [F(1) - (-1)^n F(-1) - 2(A_{n-1} + A_{n-3} + A_{n-5} + \dots)].$$

$$\text{Item } \int_0^z Fz dz \text{ per eandem plani partem evolvi potest in seriem } \sum_0^{\infty} I_n P^{(n)} z.$$

$$I = \frac{2n+1}{2} \int_{-1}^1 \int_0^z Fz dz \cdot P^{(n)} z dz.$$

Valorem I_n per A , A redditum vir Cel. Bauer (Exelle's Journal für Math. Tom. 56) huius formulae ope:

$$\int_1^z P^{(n)} z \, dz = \frac{P^{(n+1)} z - P^{(n-1)} z}{2n+1}.$$

$$\int_{-1}^1 \int_0^z F_z dz \cdot P^{(n)} z \, dz = \left[\int_0^z F_z dz \cdot \int_1^z P^{(n)} z \, dz \right]_1^{-1} - \int_{-1}^1 F_z \int_1^z P^{(n)} z \, dz \cdot dz = - \int_{-1}^1 F_z \int_1^z P^{(n)} z \, dz \cdot dz.$$

$$\text{Ergo } I_n = - \frac{1}{2} \int_{-1}^1 (P^{(n+1)} z - P^{(n-1)} z) \, dz = - \left(\frac{A_{n+1}}{2n+3} - \frac{A_{n-1}}{2n-1} \right).$$

$$I_0 = \frac{1}{2} \int_{-1}^1 \int_0^z F_z dz \cdot dz = \frac{1}{2} \left[\int_0^z F_z dz \cdot z \right]_1^{-1} - \frac{1}{2} \int_{-1}^1 F_z \cdot z \, dz = \frac{1}{2} \left(\int_0^1 F_z dz + \int_0^{-1} F_z dz \right) - \frac{A_1}{3}.$$

§. 5.

Quaeruntur relationes, quae valeant inter coefficientes A , A_n , ipsos ab hac parte et functionem F_z ab illa.

Primum habemus relationem principalem:

$$A_n = \frac{2n+1}{2} \int_{-1}^1 F_z P^{(n)} z \, dz.$$

Deinde si functio F_z in potestatum seriem $\sum_0^\infty a_n z^n$ evolvi potest, quae, etiam si $z=1$, absolute convergat, coefficiens A_n aequalis est seriei absolute convergenti:
 $\frac{(2n+1) \cdot 1 \cdot 2 \cdot 3 \cdots n}{1 \cdot 3 \cdot 5 \cdots (2n+1)} a_n + \frac{(2n+1) \cdot 1 \cdot 2 \cdot 3 \cdots (n+2)}{2 \cdot 1 \cdot 3 \cdot 5 \cdots (2n+3)} a_{n+2} + \dots$
 $\dots + \frac{(2n+1) \cdot 1 \cdot 2 \cdot 3 \cdots (n+2r)}{2 \cdot 4 \cdots 2r \cdot 1 \cdot 3 \cdot 5 \cdots (2n+2r+1)} a_{n+2r} \dots$

Adhibita enim hac Cel. Legendrii formula:

$$z^n = \frac{1 \cdot 2 \cdot 3 \cdots n}{1 \cdot 3 \cdot 5 \cdots (2n+1)} \left\{ (2n+1) P^{(n)} z + (2n-3) \frac{2n+1}{2} P^{(n-2)} z + (2n-7) \frac{(2n+1)(2n+3)}{2 \cdot 4} P^{(n-4)} z + \dots \right\},$$

$$\text{ex } \sum_0^\infty a_n z^n \text{ fit } \sum_0^\infty a_n \frac{1 \cdot 2 \cdot 3 \cdots n}{1 \cdot 3 \cdot 5 \cdots (2n+1)} \left\{ (2n+1) P^{(n)} z + (2n-3) \frac{2n+1}{2} P^{(n-2)} z + \dots \right\}.$$

Habemus autem

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{1 \cdot 3 \cdot 5 \cdots (2n+1)} \left\{ (2n+1) M[P^{(n)} z] + (2n-3) \frac{2n+1}{2} M[P^{(n-2)} z] + \dots \right\} \equiv 1,$$

dum z migrat per valores $-1 \cdots +1$,

quod $M[P^{(n)} z] \equiv 1$, si $-1 \equiv z \equiv 1$, et $P^{(n)} 1 = 1$.

$$\text{Itaque cum convergat } \sum_0^\infty M[a_n] \frac{1 \cdot 2 \cdot 3 \cdots n}{1 \cdot 3 \cdot 5 \cdots (2n+1)} \left\{ (2n+1) \cdot M[P^{(n)} z] + \dots \right\},$$

dum $-1 \equiv z \equiv 1$,

$$\text{seriem } \sum_0^\infty a_n \frac{1 \cdot 2 \cdot 3 \cdots n}{1 \cdot 3 \cdot 5 \cdots (2n+1)} \left\{ (2n+1) P^{(n)} z + (2n-3) \frac{2n+1}{2} P^{(n-2)} z + \dots \right\} \text{ secundum}$$

$P^{(n)}z$ ordinari patet, quo coefficientes A, A series absolute convergentes formae supra indicatae fiunt.

Exempli causa

$$\frac{1}{v-u} = \sum_0^{\infty} \frac{u^n}{v^{n+1}} = \sum_0^{\infty} [(2n+1) Q^{(n)} v P^{(n)} u],$$

$$Q^{(n)} z = \sum_{r=0}^{\infty} \frac{1 \cdot 2 \cdot 3 \cdots (n+2r)}{2 \cdot 4 \cdot \cdots \cdot 2r \cdot 1 \cdot 3 \cdots (2n+2r+1)} \frac{1}{z^{n+2r+1}}, \text{ donec } M[z] > 1.$$

Quod attinet ad methodum, qua valores A, A in evolutione propositae cuius-piam functionis secundum $P^{(n)}z$ procedente simplicius determinentur, talem consequeris, dummodo functione natura sua functioni propositae propinquia diversis methodis secundum $P^{(n)}z$ evoluta relatio inter coefficientes A, A eveniat, per quam coefficientes methodo recurrente consequi possis (Confer Cel. Baueri commentationem supra memoratam). Exemplum:

$$e^{az} = \sum_0^{\infty} A_n P^{(n)} z$$

$$\frac{de^{az}}{dz} = ae^{az} = \sum_0^{\infty} \frac{2n+1}{2} \left\{ e^a - (-1)^n e^{-a} - 2(A_{n-1} + A_{n-3} + \dots) \right\} P^{(n)} z$$

$$A_n = \frac{2n+1}{2a} \left\{ e^a - (-1)^n e^{-a} - 2(A_{n-1} + A_{n-3} + \dots) \right\}$$

$$A_n = -(2n+1) \left\{ \frac{A_{n-1}}{a} - \frac{A_{n-2}}{2n-3} \right\}.$$

§. 6.

Procedamus ad explorationem seriei secundum $Q^{(n)}z$ progredientis

$$\sum_0^{\infty} B_n Q^{(n)} z = \sum_0^{\infty} B_n \int_0^{\infty} \frac{dt}{(z + \cos it \sqrt{z^2 - 1})^{n+1}};$$

radicis signum ita, ut in $P^{(n)}z = \frac{1}{\pi} \int_0^{\pi} (z + \cos \phi \sqrt{z^2 - 1})^n d\phi$ (§. 1.) definitur.

Donec z extra rectam focus $z = \pm 1$ limitatam versatur, $Q^{(n)}z$ naturam functionis integrae habet. (Si $z = \pm 1$, $Q^{(n)}z = \infty$; si $z = 0$, $Q^{(n)}z$ ambiguum.)

Nam

1) functio $Q^{(n)}z$ finita et continua manet.

$$Q^{(n)}z = \frac{1}{(\sqrt{z^2 - 1})^{n+1}} \cdot \int_0^{\infty} \frac{dt}{\left(\frac{z}{\sqrt{z^2 - 1}} + \cos it \right)^{n+1}}.$$

$\frac{z}{\sqrt{z^2 - 1}}$ est formae: $(x + yi)$, ubi x positivum; idcirco $M \left[\frac{z}{\sqrt{z^2 - 1}} + \cos it \right] = [(x + \cos it)^2 + y^2]^{\frac{1}{2}} > \cos it$.

$M[Q^{(n)}z] < \frac{1}{M[\sqrt{z^2-1}]^{n+1}} \cdot \int_0^\infty \frac{dt}{(\cos it)^{n+1}}$, quod integrale finitum manet.

Simul adipiscimur

$$\frac{1}{M[z + \cos it\sqrt{z^2-1}]} = \frac{1}{M\left[\sqrt{z^2-1}\left(\frac{z}{\sqrt{z^2-1}} + \cos it\right)\right]} < \frac{1}{M[z + \sqrt{z^2-1}]}.$$

Continuitas ex eo sequitur, quod functio sub signo integralis continua manet.

$$2) \frac{d^n}{dz^n} Q^{(n)}z = \int_0^\infty \frac{dt}{dz^n} \frac{1}{(z + \cos it\sqrt{z^2-1})^{n+1}} dt,$$

quo enim modo sub 1) usi sumus, eodem deduci potest, functionem dextrae partis finitam et continuam manere, unde demonstratio pendet.

Quibus peractis perscrutemur seriem

$$\sum_0^\infty B_n \int_0^\infty \frac{dt}{(z + \cos it\sqrt{z^2-1})^{n+1}}.$$

Ponimus, $M[B_n]$ limite finito B inferiorem manere.

Si convergit absolute

$$\sum_0^\infty B_n \frac{1}{(z + \cos it\sqrt{z^2-1})^{n+1}},$$

a $t=0$, ad $t=\infty$, efficimus convergere

$$\sum_0^\infty B_n \int_0^\infty \frac{dt}{(z + \cos it\sqrt{z^2-1})^{n+1}}.$$

Sine dubio convergit $\sum_0^\infty B_n \int_0^{t_1} \frac{dt}{(z + \cos it\sqrt{z^2-1})^{n+1}}$; porro seriem

$\sum_0^\infty B_n \int_{t_1}^\infty \frac{dt}{(z + \cos it\sqrt{z^2-1})^{n+1}}$ convergere hoc modo intelliges.

$$\begin{aligned} M\left[\sum_0^\infty B_n \int_{t_1}^\infty \frac{dt}{(z + \cos it\sqrt{z^2-1})^{n+1}}\right] &< B \sum_0^\infty \int_{t_1}^\infty \frac{dt}{M[z + \cos it\sqrt{z^2-1}]^{n+1}} < \\ &< B \sum_0^\infty \frac{1}{M[\sqrt{z^2-1}]^{n+1}} \int_{t_1}^\infty \frac{dt}{(\cos it)^{n+1}} = B \int_{t_1}^\infty \sum_0^\infty \frac{dt}{M[\sqrt{z^2-1} \cos it]^{n+1}} = \\ &= B \int_{t_1}^\infty \frac{dt}{M[\sqrt{z^2-1} \cos it] \left[1 - \frac{1}{M[\sqrt{z^2-1} \cos it]}\right]}, \end{aligned}$$

si t_1 ita assumptum sit, ut sit $\frac{1}{M[\sqrt{z^2-1} \cos it_1]} < 1$,

$$= B \frac{1}{1 - \frac{1}{M[\sqrt{z^2-1} \cos it(t_1 \dots \infty)]}} \cdot \int_{t_1}^\infty \frac{dt}{M[\sqrt{z^2-1} \cos it]}, \text{ quod quidem integrale finitum manet.}$$

Si divergit

$$\sum_0^{\infty} B_n \frac{1}{(z + \cos it\sqrt{z^2 - 1})^{n+1}},$$

t accipiente valorem $t_1 > 0$, contendimus divergere

$$\sum_0^{\infty} B_n \int_0^{\infty} \frac{dt}{(z + \cos it\sqrt{z^2 - 1})^{n+1}},$$

dum argumentum z sit $\begin{cases} \text{reale et } M[z] > 1, \\ \text{mere imaginarium.} \end{cases}$

Demonstratio.

Divergat series $\sum_0^{\infty} B_n \frac{1}{(z + \cos it\sqrt{z^2 - 1})^{n+1}}$ a $t = t_1$ ad $t = 0$.

Eodem modo, quo in §. 1, ostenditur

$M \left[B_n \int_0^{t_1-t} \frac{dt}{(z + \cos it\sqrt{z^2 - 1})^{n+1}} \right]$ finito valore inferiorem non permanere.

$B_n \int_{t_1-t}^{\infty} \frac{dt}{(z + \cos it\sqrt{z^2 - 1})^{n+1}}$ autem eodem signo affectum est, quo

$$B_n \int_0^{t_1-t} \frac{dt}{(z + \cos it\sqrt{z^2 - 1})^{n+1}};$$

itaque $M \left[B_n \int_0^{\infty} \frac{dt}{(z + \cos it\sqrt{z^2 - 1})^{n+1}} \right]$ limite finito inferior non manet.

Ergo $\sum_0^{\infty} B_n \int_0^{\infty} \frac{dt}{(z + \cos it\sqrt{z^2 - 1})^{n+1}}$ convergere non potest.

Iam R_1 nominemus radium convergentiae circuli seriei $\sum_0^{\infty} B_n z^n$ proprii.

Ex demonstratis sequitur, ut

$$\sum_0^{\infty} B_n \int_0^{\infty} \frac{dt}{(z + \cos it\sqrt{z^2 - 1})^{n+1}}$$

convergat, idque absolute, si $\frac{1}{M[z + \sqrt{z^2 - 1}]} < R_1$;

sed divergat, si $\frac{1}{M[z + \sqrt{z^2 - 1}]} > R_1$, dum argumentum z sit $\begin{cases} \text{reale et } M[z] > 1, \\ \text{mere imaginarium.} \end{cases}$

Aequationi $\frac{1}{M[z + \sqrt{z^2 - 1}]} = R_1$ respondet

$R_1 < 1$: ellipsis cum axe principali $\frac{1}{R_1} + R_1$ et focis $z = \pm 1$, quae quidem ellipsis

decrescit crescente R_1 et vice versa;

$R_1 = 1$: recta inter focos $z = \pm 1$;

$R_1 > 1$: nullus valor z .

Quibus haec efficiuntur:

$R_1 < 1$: Series

$$\sum_0^{\infty} B_n Q^{(n)} z = \sum_0^{\infty} B_n \int_0^{\infty} \frac{dt}{(z + \cos it \sqrt{z^2 - 1})^{n+1}}$$

convergit idque absolute extra ellipsin cum axe principali $\frac{1}{R_1} + R_1$ et focus $z = \pm 1$;

divergit intra hanc ellipsin, dum argumentum z } reale et $M[z] > 1$,
} mere imaginarium.

$R_1 \geq 1$: Series convergit idque absolute ubique extra rectam focus $z = \pm 1$ limitatam.

§. 7.

Series

$$\sum_0^{\infty} B_n \int_0^{\infty} \frac{dt}{(z + \cos it \sqrt{z^2 - 1})^{n+1}}$$

extra convergentiae ellipsin ex functionum $Q^{(n)} z$ continuitate continua derivatas habet functiones omnium ordinum, quae singulis propositae seriei terminis differentiatis producent series pariter absolute convergentes et continuas.

Nam

$$\sum_0^{\infty} B_n \frac{d^m}{dz^m} \int_0^{\infty} \frac{dt}{(z + \cos it \sqrt{z^2 - 1})^{n+1}} = \sum_0^{\infty} B_n \int_0^{\infty} \frac{d^m}{dz^m} \frac{dt}{(z + \cos it \sqrt{z^2 - 1})^{n+1}} dt$$

convergit absolute, quia $\sum_0^{\infty} B_n \frac{d^m}{dz^m} \frac{1}{(z + \cos it \sqrt{z^2 - 1})^{n+1}}$ absolute convergit et $M[B_n]$ finito valore inferiorem manere possum est.

Itaque series $F_1 z = \sum_0^{\infty} B_n Q^{(n)} z$ extra convergentiae ellipsin naturam functionis integrae habet; quamobcausam convergentiae ellipsis minor esse nequit minima ellipsi confocali, extra quam $F_1 z$ quoquaversus continuata naturam functionis integrae conservat.

Ex tertio autem theoremate in introductione deducto series extra hanc minimam ellipsin re vera convergit; ergo consequimur theorema:

Convergentiae ellipsis seriei $F_1 z = \sum_0^{\infty} B_n Q^{(n)} z$ minima ellipsis focus $z = \pm 1$ praedita est, extra quam functio $F_1 z$ quoquaversus continuata a functionis integrae natura non discedit.

§. 8.

Functiones de serie $F_1 z = \sum_0^{\infty} B_n Q^{(n)} z$ derivatae, cum extra convergentiae ellipsin propositae seriei naturam habeant functionum integrarum, secundum $Q^{(n)} z$ evolvi possunt in series, quae ex commemorato theoremate in introductione allato, huius formae sunt: $\sum_0^{\infty} D_n^{(r)} Q^{(n)} z$, ubi index r respondet r -tae functioni derivatae.

Coefficiens $D_n^{(1)}$ per valores B_n sic exprimitur.

$$D_n^{(1)} = \frac{2n+1}{2\pi i} \int F'_1 z P^{(n)} z \, dz,$$

ubi integrandum est sic, ut in introductione explicavimus.

$$\begin{aligned} & \frac{2n+1}{2\pi i} \int F'_1 z P^{(n)} z \, dz = -\frac{2n+1}{2\pi i} \int F_1 z \frac{d}{dz} P^{(n)} z \, dz = \\ & = -\frac{2n+1}{2\pi i} \left[(2n-1) \int F_1 z P^{(n-1)} z \, dz + (2n-5) \int F_1 z P^{(n-5)} z \, dz + (2n-9) \int F_1 z P^{(n-9)} z \, dz + \dots \right] \\ & = -(2n+1) [B_{n-1} + B_{n-3} + B_{n-5} + \dots]. \end{aligned}$$

$$\text{Ergo } D_n^{(1)} = -(2n+1) [B_{n-1} + B_{n-3} + B_{n-5} + \dots].$$

Quod attinet ad functionem

$$\int_{z_0}^z F_1 z \, dz,$$

hanc amonodromam esse appareat, nisi coefficiens a_{-1} in evolutione functionis $F_1 z = \sum_0^{\infty} a_{-(n+1)} z^{-(n+1)}$ evanescat.

§. 9.

Quaerendae sunt methodi, quibus valores coefficientium B, b in evolutione propositae cuiuspia functionis $F_1 z$ secundum $Q^{(n)} z$ progrediente

$$F_1 z = C + \sum_0^{\infty} B_n Q^{(n)} z$$

exhibeantur.

Primum habemus formulam:

$$B_n = \frac{2n+1}{2\pi i} \int F_1 z P^{(n)} z \, dz.$$

Deinde seriem $\sum_0^{\infty} B_n Q^{(n)} z$, dum absolute convergat, introductis loco $Q^{(n)} z$ series in §. 5 allatis:

$$Q^{(n)} z = \sum_{r=0}^{\infty} \frac{1 \cdot 2 \cdot 3 \cdots (n+2r)}{2 \cdot 4 \cdots 2r \cdot 1 \cdot 3 \cdots (2n+2r+1)} \frac{1}{z^{n+2r+1}}$$

secundum potestates quantitatis $\frac{1}{z}$ ordinari patet et comparatis aequalium potestatum coefficientibus in series $\sum_0^{\infty} B_n Q^{(n)} z$ et $\sum_0^{\infty} a_{-(n+1)} z^{-(n+1)}$ valor B_n per a , a exprimitur.

Ad cognoscendum, num $\sum_0^{\infty} B_n Q^{(n)} z$ absolute convergat, theoremate in §. 6 deducto uti non possumus, quod in eo praemittitur, $M[B_n]$ limite finito inferiore manere, id quod affirmare nequis, priusquam B_n determinatum sit, quod ipsum quaerimus.

Quamobrem recta via demonstrabimus functionem

$$F_1(z) - C = f(z) = \sum_0^{\infty} a_{-(n+1)} z^{-(n+1)}$$

in seriem, si valor z satis magnus sumitur, absolute convergentem $\sum_0^{\infty} B_n Q^{(n)} z$ evolvi posse, dum $M[a_{-(n+1)}]$ limite finito inferior maneat.

$$\text{E serie } \frac{1}{v-u} = \sum_0^{\infty} (2n+1) Q^{(n)} v P^{(n)} u \text{ sequitur, ut sit } \frac{m!}{v^{m+1}} = \sum_0^{\infty} (2n+1) Q^{(n)} v \frac{d^m}{dz^m} P^{(n)} u,$$

ubi in dextra parte post differentiationem u valori 0 aequandum est.

Unde Cel. Heinii formula

$$\frac{1}{z^{m+1}} = \frac{1 \cdot 3 \cdot 5 \cdots (2m-1)}{1 \cdot 2 \cdot 3 \cdots m} \left\{ (2m+1) Q^{(m)} z - (2m+5) \frac{2m+1}{2} Q^{(m+2)} z + \right. \\ \left. + (2m+9) \frac{(2m+1)(2m+3)}{2 \cdot 4} Q^{(m+4)} z - \dots \right\},$$

quae series absolute convergit (§. 2).

Cuius formulae ope series $\sum_0^{\infty} a_{-(n+1)} z^{-(n+1)}$ secundum $Q^{(n)} z$ ordinata transit in $\sum_0^{\infty} B_n Q^{(n)} z$,

$$\text{ubi } B_n = \frac{1 \cdot 3 \cdots (2n+1)}{1 \cdot 2 \cdots n} \left\{ a_{-(n+1)} - \frac{n(n-1)}{2(2n-1)} a_{-(n-1)} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot (2n-1)(2n-3)} a_{-(n-3)} - \dots \right\}.$$

Itaque quaeritur, quibus conditionibus convergat series:

$$\sum_0^{\infty} (2n+1) \frac{1 \cdot 3 \cdots (2n-1)}{1 \cdot 2 \cdots n} \left\{ M[a_{-(n+1)}] + \frac{n(n-1)}{2(2n-1)} M[a_{-(n-1)}] + \right. \\ \left. \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot (2n-1)(2n-3)} M[a_{-(n-3)}] + \dots \right\} \cdot M[Q^{(n)} z].$$

Habemus autem

$$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{1 \cdot 2 \cdots n} \left\{ 1 + \frac{n(n-1)}{2(2n-1)} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4(2n-1)(2n-3)} + \dots \right\} = M[P^{(n)} i] = \\ = \frac{1}{\pi} \int_0^{\pi} (1 + \cos \phi \sqrt{2})^n d\phi < (1 + \sqrt{2})^n;$$

et, si ponitur $M[z] > 1$,

$$Q^{(n)} z = \sum_{r=0}^{\infty} \frac{1 \cdot 2 \cdot 3 \cdots (n+2r)}{2 \cdot 4 \cdots 2r \cdot 1 \cdot 3 \cdots (2n+2r+1)} \frac{1}{z^{n+2r+1}};$$

cum vero $\frac{1 \cdot 2 \cdot 3 \cdots (n+2r)}{2 \cdot 4 \cdots 2r \cdot 1 \cdot 3 \cdots (2n+2r+1)} \leq 1$,

$$M[Q^{(n)} z] < \frac{1}{M[z]^{n+1}} + \frac{1}{M[z]^{n+3}} + \dots = \frac{1}{1 - \frac{1}{M[z]^2}} \frac{1}{M[z]^{n+1}}.$$

Quoniam positum est, $M[a_{-(n+1)}]$ limite finito (a) inferiorem manere, accipimus $\sum_0^{\infty} (2n+1) \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{1 \cdot 2 \cdot 3 \cdots n} \left\{ M[a_{-(n+1)}] + \frac{n(n-1)}{2(2n-1)} \cdot M[a_{-(n-1)}] + \dots \right\} M[Q^{(n)} z] <$

$$< \sum_0^{\infty} (2n+1) (1 + \sqrt{2})^n a \frac{1}{1 - \frac{1}{M[z]^2}} \frac{1}{M[z]^{n+1}},$$

quae series convergit, donec $M[z] > 1 + \sqrt{2}$.

Ergo consecuti sumus theorema:

Functio $fz = \sum_0^{\infty} a_{-(n+1)} z^{-(n+1)}$ in seriem, si $M[z] > 1 + \sqrt{2}$, absolute convergentem $\sum_0^{\infty} B_n Q^{(n)} z$ evolvi potest, dum $M[a_{-(n+1)}]$ limite finito inferior maneat.

Restat, ut determinemus B_n , B methodo supra indicata. Brevitatis causa coefficiens termini generalis in serie

$$Q^{(n)} z = \sum_{r=0}^{\infty} \frac{1 \cdot 2 \cdot 3 \cdots (n+2r)}{2 \cdot 4 \cdots 2r \cdot 1 \cdot 3 \cdots (2n+2r+1)} \frac{1}{z^{n+2r+1}}$$

denotetur signo $[n, r]$; ita nanciscimur:

$$a_{-1} = [n, r]_{\substack{n=0 \\ r=0}} B_0$$

$$a_{-3} = [n, r]_{\substack{n=0 \\ r=1}} B_0 + [n, r]_{\substack{n=2 \\ r=0}} B_2$$

$$a_{-5} = [n, r]_{\substack{n=0 \\ r=2}} B_0 + [n, r]_{\substack{n=2 \\ r=1}} B_2 + [n, r]_{\substack{n=4 \\ r=0}} B_4$$

$$\vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots$$

$$a_{-(2t+1)} = [n, r]_{\substack{n=0 \\ r=t}} B_0 + [n, r]_{\substack{n=2 \\ r=t-1}} B_2 + [n, r]_{\substack{n=4 \\ r=t-2}} B_4 \dots + [n, r]_{\substack{n=2t \\ r=0}} B_{2t}$$

$$a_{-2} = [n, r]_{\substack{n=1 \\ r=0}} B_1$$

$$a_{-4} = [n, r]_{\substack{n=1 \\ r=1}} B_1 + [n, r]_{\substack{n=3 \\ r=0}} B_3$$

$$a_{-6} = [n, r]_{\substack{n=1 \\ r=2}} B_1 + [n, r]_{\substack{n=3 \\ r=1}} B_3 + [n, r]_{\substack{n=5 \\ r=0}} B_5$$

$$\vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots$$

$$a_{-(2t+2)} = [n, r]_{\substack{n=1 \\ r=t}} B_1 + [n, r]_{\substack{n=3 \\ r=t-1}} B_3 + [n, r]_{\substack{n=5 \\ r=t-2}} B_5 \dots + [n, r]_{\substack{n=2t+1 \\ r=0}} B_{2t+1}$$

Unde ambae recursionis formulae:

$$\left\{ \begin{array}{l} a_{-(2t+1)} = \frac{1}{2t+1} B_0 + \frac{2t}{(2t+1)(2t+3)} B_2 + \frac{2t(2t-2)}{(2t+1)(2t+3)(2t+5)} B_4 \dots + \\ \qquad \qquad \qquad + \frac{2t(2t-2) \dots 2}{(2t+1)(2t+3) \dots (2t+2t+1)} B_{2t} \\ a_{-(2t+2)} = \frac{1}{2t+3} B_1 + \frac{2t}{(2t+3)(2t+5)} B_3 + \frac{2t(2t-2)}{(2t+3)(2t+5)(2t+7)} B_5 \dots + \\ \qquad \qquad \qquad + \frac{2t(2t-2) \dots 2}{(2t+3)(2t+5) \dots (2t+2t+3)} B_{2t+1} \end{array} \right.$$

§. 10.

Exemplum tractemus $\log(1 + \frac{1}{z})$.

$\log\left(1 + \frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)z^{n+1}}$ serie secundum $Q^{(n)}z$ procedente $\sum_0^{\infty} B_n Q^{(n)}z$ exprimitur, quae valet per universum planum loco inter $z = -1$ et $z = 1$ excepto.

Ex formulis in § antegressa deductis accipimus:

$$\log\left(1 + \frac{1}{z}\right) = Q^{(0)}z + \sum_{t=0}^{\infty} B_{2t+1} Q^{(2t+1)}z = Q^{(0)}z - \frac{1}{2}Q^{(1)}z + \frac{1}{3}Q^{(3)}z - \frac{1}{4}Q^{(5)}z + \frac{1}{5}Q^{(7)}z - \dots$$

Utamur deinde formula: $B_n = \frac{2n+1}{2\pi i} \int \log\left(1 + \frac{1}{z}\right) P^{(n)}z dz$.

$$\begin{aligned} B_{2t+1} &= \frac{2(2t+1)+1}{2\pi i} \int \log\left(1 + \frac{1}{z}\right) P^{(2t+1)}z dz = \\ &= \frac{2(2t+1)+1}{2\pi i} \int \frac{1}{z(z+1)} \int_P^z P^{(2t+1)}z dz \cdot dz = \frac{1}{2\pi i} \int \frac{P^{(2t+2)}z - P^{(2t)}z}{z(z+1)} dz, \end{aligned}$$

quam integrationem per infinite parvum circulum, puncto $z=0$ pro centro circumscriptum, peragi patet.

Itaque

$$\begin{aligned} \frac{1}{2\pi i} \int \frac{P^{(2t+2)}z - P^{(2t)}z}{z(z+1)} dz &= \frac{P(0)^{(2t+2)} - P(0)^{(2t)}}{2\pi i} \int \frac{dz}{z} = P(0)^{(2t+2)} - P(0)^{(2t)} = \\ &= (-1)^{t+1} \frac{1 \cdot 3 \cdot 5 \cdots (2t-1)}{2 \cdot 4 \cdot 6 \cdots 2t} \frac{4t+3}{2t+2}. \end{aligned}$$

$$\begin{aligned} \text{Ergo } \log\left(1 + \frac{1}{z}\right) &= Q^{(0)}z + \sum_0^{\infty} (-1)^{t+1} \frac{1 \cdot 3 \cdot 5 \cdots (2t-1)}{2 \cdot 4 \cdot 6 \cdots 2t} \frac{4t+3}{2t+2} Q^{(2t+1)}z = \\ &= Q^{(0)}z - \frac{1}{2}Q^{(1)}z + \frac{1}{3}Q^{(3)}z - \frac{1}{4}Q^{(5)}z + \dots \end{aligned}$$

quae cum supra deductis correspondent.

Cum $M[B_n]$ hic limite finito inferior maneat, series secundum §§. 6, 7, quiunque est valor argumenti z in convergentiae spatio, absolute convergit et derivatas functiones habet series differentiatias singulis propositae seriei terminis exortas.

Si z mutatur cum $\frac{1}{z_1}$, serie

$$Q^{(0)}\left(\frac{1}{z_1}\right) + \sum_0^{\infty} (-1)^{t+1} \frac{1 \cdot 3 \cdot 5 \cdots (2t-1)}{2 \cdot 4 \cdot 6 \cdots 2t} \frac{4t+3}{2t+2} Q^{(2t+1)}\left(\frac{1}{z_1}\right),$$

exprimitur $\log(1+z_1)$, ubi argumenti valori (0) respondet functionis valor (0), per universum planum, exceptis convergentiae limitibus, locis inter $z = \begin{cases} +1 & \dots +\infty \\ -1 & \dots -\infty \end{cases}$ abscissis.

VIT A.

Natus sum Ludovicus Guilelmus Thomé die XIII mensis Martii anni h. s. XLI Dollendorpii, in pago Rhenano prope Bonnam sito, patre Josepho, cuius obitum vehementer lugeo, matre Sabina e gente Fuchs, quam adhuc superstitem maxime veneror. Fidem confiteor catholicam.

Primitus literarum elementis imbutus decem annos natus gymnasium Fridericum Guilelmum Coloniense adii, quod septem per annos sub auspiciis Cel. Knebel frequentavi. Hinc testimonio maturitatis academicae munitus ineunte aestate anni h. s. LIX, ut maxime rebus physicis et mathesi operam darem, Bonnam discessi, ubi a rectore Magnifico Jahn civis academicus factus per quinque semestria disserentes audivi viros Ill. Bergemann, Bischof, van Calker, Dahlmann, Landolt, Noeggerath, Pluecker, Springer, Troschel. Deinde Munichium petivi, ubi unum per annum viris Ill. Naegeli, Radlkofer, a Siebold magistris academicis usus sum. Denique hanc almam literarum universitatem Berolinensem adii ibique scholis virorum Ill. Cel. Exp. Dove, Hoppe, Kronecker, Kummer, Quincke, Weierstrass per quinque semestria interfui. Exercitationibus seminarii mathematici, quas moderantur viri Ill. Kummer et Weierstrass per duo semestria affui.

Quibus omnibus viris, optime de me meritis, imprimis Ill. Kummer et Ill. Weierstrass summas quas possum gratias ago.

THESES.

- 1) **Matheseos natura vel maxime percipitur ex disciplinarum mathematicarum origine.**
 - 2) **Physicae molecularis doctrinam instituere non tam mathematicorum adhuc est, quam physicorum.**
 - 3) **Ratio nuper proposita, quae vult lumen quod vocatur septentrionale nasci fluido electrico per aërem in atmosphaerae limite admodum tenuem, ut per tubum Geisslerianum, migrante, maxime probabilis esse videtur.**
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