

QUIBUS IN CASIBUS INTEGRALIUM ORDINA-
RIORUM QUAE AEQUATIONI DIFFERENTIALI:

$$x(x-1)\frac{d^2y}{dx^2} + ((\alpha+\beta+1)x-\gamma)\frac{dy}{dx} + \alpha\beta \cdot y = 0$$

SATISFACIUNT, ALTERUM AUT ALTERI
AEQUALE AUT INFINITUM EVADAT.

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TYPIS EXPRESSIT GUSTAVUS LANGE (PAUL LANGE).

MANIBUS PATRIS DILECTISSIMI

“SACRUM”



Praefatio.

Satisfacere solent aequationi differentiali de qua agimus et quam hoc modo:

$$x(x-1)y'' + ((\alpha + \beta + 1)x - \gamma)y' + \alpha\beta y = 0$$

scribere nobis licet, duae series gaussianae, quas convergentes habemus, pro omnibus quantitatis x valoribus qui sunt intra circulum descriptum circa unum ex punctis singularibus 0, 1, ∞ cum radio minore quam quantum distat centrum illius circuli a proximo puncto singulari, id est pro valoribus in circuitu puncti singularis:

$$\begin{aligned} x=0 & \left\{ \begin{array}{l} y = F(\alpha, \beta; 1/x) \\ y = x^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, x) \end{array} \right. \\ x=1 & \left\{ \begin{array}{l} y = F(\alpha, \beta, \alpha + \beta + 1 - \gamma, 1 - x) \\ y = (1-x)^{\gamma - \alpha - \beta} F(\gamma - \alpha, \gamma - \beta, \gamma - \alpha - \beta + 1, 1 - x) \end{array} \right. \\ x=\infty & \left\{ \begin{array}{l} y = x^{-\alpha} F\left(\alpha, \alpha - \gamma + 1, \alpha - \beta + 1, \frac{1}{x}\right) \\ y = x^{-\beta} F\left(\beta, \beta - \gamma + 1, \beta - \alpha + 1, \frac{1}{x}\right) \end{array} \right. \end{aligned}$$

quibus e seriebus quae ad puncta 0, 1 pertinent alteram earum quae unicuique punto sunt propriae, natura ipsa, alteram autem multiplicando demum per certam variabilis x potestatem uniformem et continuum evadere, deinde earum quae ad punctum $x = \infty$ pertinent, utramque eadem indole ac quam postremo commemoravimus, praeditam esse patet. — Ficeri autem potest, ut integralia quae ad idem punctum singulare pertinent, ad unum redigantur si aut x exponentes qui initio sunt, aequales fiunt, ideoque integralia ipsa congruant, quod fit: in puncti $x = 0$ circuitu si: $\gamma = 1$

$$\begin{array}{lll} " & x = 1 & " & \gamma = \alpha + \beta \\ " & x = \infty & " & \alpha = \beta \end{array}$$

aut cunctis numeris inter se differunt, quamobrem si quantitates $\alpha, \beta, \gamma - \alpha, \gamma - \beta$ non integri numeri sunt, semper integralium alterum infinitum evadit, atque si quantitates quas modo enumeravimus, integri

numeri sunt, exceptis singulis casibus idem evenit. Quamobrem alterum quoddam praeterea integrale, aequationi differentiali in illis casibus satisfaciens existere necesse est, et sane Ill. Spitzerus verificando probavit (cfr. Diarium Crellianum tom. 57.) et eodem loco Ill. Borchhardtius demonstravit, si $\alpha = \beta$ sumatur pro integrali quod pertinet ad punctum singulare $x = \infty$ hanc expressionem logarithmicam existere:

$$y = \int_0^1 (u-x)^{-\alpha} (1-u)^{\gamma-\alpha} u^\alpha \log \frac{u(1-u)}{u-x} du$$

qua ex cifra expressiones quoque ad puncta 0, 1 pertinentes, quantitatibus $\alpha, \beta, \gamma - \alpha, \gamma - \beta$ apte definitis, deduci possunt. Si constantes autem illo modo non coegerimus, ac si quae primum locum obtinent exponentes integris numeris inter se differunt, expressiones casus omnes complectentes haec ratio non praebet.

Deinde sicut ex integrali ordinario seriem gaussianam ita e logarithmico integrali seriem quandam evolvi posse necesse est.

Quas res in hac dissertatione pertractandas mihi proposui.

Apparebit ex iis quae supra diximus, dissertationem dividendam esse in partes duas:

Priore in parte

analytice methodo generali quam primus Eulerus adhibuit iisdem conditionibus ac quas Ill. Spitzerus indicavit integrale logarithmicum in puncti $x = \infty$ circuitu convergens iterum deducere nobis proposuimus, deinde eadem ratione, si constantes satis coercuerimus, integralia quoque ad puncta 0; 1 pertinentia derivabimus, ex illis denique integralibus integralia quae ad casus generales attinent, deducenda sunt.

Altera in parte

integralia quae supra invenimus in series gaussianarum modo evolvenda sunt.

Pars prima.

Ex iis quae supra diximus hanc partem in quatuor sectiones dividendam esse appetet:

- a. Integralium derivatio in punctorum 0, 1 ∞ cirenit si quidem qui primum locum habent exponentes in utroque integrali, postquam costantes certo modo conreuerimus, aequales aestimabuntur.

- b. Reductio casuum in quibus illae conditiones locum non habent ad casus quos supra attulimus.
 - c. Reductio casuum in quibus exponentes qui primum locum habent integris numeris inter se differunt et in quibus expressiones logarithmicae locum habent ad illos qui praecedunt.
 - d. Integralia quae reperimus per indices exposita.
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- a. Integralium derivatio in punctorum 0, 1 ∞ circuitu si quidem qui primum locum habent exponentes in utroque integrali, postquam constantes certo modo coercuimus, aequales aestimantur.

Primum integrale quod Ill. Spitzerus in puncti $x = \infty$ indicavit circuitu, deinde integralia quoque ad puncta 0, 1 attinentia hæc methodo fingenda sunt:

Ponamus pro y functionem in forma integralis definiti cum constantibus accuratius demum destinatis limitibus e duobus summandis, logarithmico et alogarithmico compositam. Quae jam cifra, ut aequationi differentiali satisfaciat, in illam inducta primum ejus membrum in completum integrale convertat necesse est, ita ut limitibus g, h in integrali illo positis identice nihilcum evadat. Quamobrem primum membrum aequationis differentialis, cum cifram pro y in eam induxerimus cum differentiali completo ejusdem formæ comparabimus. Quod differentiale ut primo membro aequationis differentialis identice aequaliter fiat, coefficientes potestatum variabilis x et in parte logarithmica et alogarithmica aequales esse debent, quoniam parametrum variabilem x nullo modo coercere nobis licet. Inde manat certus numerus aequationum e quibus quantitates quaesitas computabimus. Sunt denique limites ita definiendi ut iis inductis primum membrum aequationis differentialis vel cifra cum ea aequalis facta identice evanescat.

§ I.

Integrale pro puncto $x = \infty$ quaesitum ex iis quae supra commemoravimus in forma sumatur:

$$y = \int_g^h (x - u)^\lambda (U + U_1 \log(x - u))$$

qua in cifra quantitates U, U_1 functiones unius variabilis u , λ exponentem nondum destinatum g, h quantitates demum definiendas significant.

Statuamus primam et secundam derivationem:

$$\frac{dy}{dx} = \int_g^h (x-u)^{\lambda-1} (\lambda U + U_1 + \lambda U_1 \log(x-u)) du$$

$$\frac{d^2y}{dx^2} = \int_g^h (x-u^{\lambda-2}) (\lambda(\lambda-1) U_1 \log(x-u) + \lambda(\lambda-1) U + (2\lambda-1) U_1) du$$

e quibus, cum in aequationem differentialem contulimus, membris logarithmicis ab alogarithmicis separatis et per potestates variabilis x dispositis manat:

$$\begin{aligned} & \int_g^h \left\{ \log(x-u) U_1 [x^2(\lambda^2 + \lambda(\alpha+\beta) + \alpha\beta) - x(\lambda(\lambda+\gamma-1) \right. \\ & \quad \left. + (\lambda(\alpha+\beta+1) + 2\alpha\beta) u) + \gamma\lambda u + \alpha\beta u^2] \right. \\ & + x^2 [(\lambda(\lambda+\alpha+\beta) + \alpha\beta) U + (2\lambda+\alpha+\beta) U_1] \\ & - x [(\lambda(\lambda+\gamma-1) + ((\alpha+\beta+1)\lambda + 2\alpha\beta) u) U + (2\lambda+\gamma-1 \\ & \quad \left. + (\alpha+\beta+1) u) U_1] \right. \\ & \left. + (\gamma\lambda + \alpha\beta u) Uu + \gamma u U_1 \right\} (x-u)^{\lambda-2} du = 0. \end{aligned}$$

Quodsi cifram illam cum completo differentiali comparamus:

$$\frac{d}{du} [(x-u)^{\lambda-1} (V + V_1 \log(x-u))] =$$

$$\begin{aligned} & (x-u)^{\lambda-2} \left\{ \left(x \frac{dV_1}{du} + (1-\lambda) V_1 - u \frac{dV_1}{du} \right) \log(x-u) + x \frac{dV}{du} \right. \\ & \left. + x(1-\lambda) V - V_1 - u \frac{dV}{du} \right\} \end{aligned}$$

quantitatibus V, V_1 functiones unius variabilis u significantibus, ratione habita pro $\alpha = \beta$ coefficientem potestatis x^0 in parte alogarithmica evanescere quinque aequationes evadunt pro conditionibus identitatis utriusque cifrae seu ejusmodi ut primum membrum aequationis differentialis in completum differentiale convertatur. Quarum aequationum e prima exponens λ , e ceteris functiones V, V_1, U, U_1 prodeunt. Sunt autem illae aequationes:

$$1) \lambda^2 + \lambda(\alpha+\beta) + \alpha\beta = 0$$

$$2) (-\lambda(\lambda+\gamma-1) - (\lambda(\alpha+\beta+1) + 2\alpha\beta) u) U_1 = \frac{dV_1}{du}$$

$$3) (\gamma\lambda + \alpha\beta u) U_1 u = (1-\lambda) V_1 - u \frac{dV_1}{du}$$

$$4) -[\lambda(\lambda+\gamma-1) + ((\alpha+\beta+1)\lambda + 2\alpha\beta) u] U_1 \\ - [2\lambda+\gamma-1 + (\alpha+\beta+1) u] U_1 = \frac{dV}{du}$$

$$5) (\gamma\lambda + \alpha\beta u) Uu + \gamma u U_1 = (1-\lambda) V - V_1 - u \frac{dV}{du}$$

E 1) sequitur:

$$\text{aut } \lambda = -\alpha \text{ aut } \lambda = -\beta$$

Valorem $\lambda = -\alpha$ acceptum cum in quattuor aequationes 2) — 5) in-

duximus, functiones $U U_1$, $V V_1$ id est e 2) et 3) U_1 , V_1 deinde illis quantitatibus in 4) et 5) positis, U , V evadunt.

Primum $\frac{dV_1}{du}$ e 2) et 3) eliminando invenitur aequatio:

$$U_1 u^\alpha (u - 1) = V_1$$

qua ex aequatione quantitatem U_1 obtentam in 3) ponendo reperitur:

$$\frac{dV_1}{V_1} = \frac{u + \gamma - \alpha - 1}{u(u - 1)}$$

$$\log V_1 = \log(u - 1) - (\gamma - \alpha - 1) \log \frac{u}{u - 1}$$

$$V_1 = \frac{1}{\alpha} (u - 1)^{\gamma - \alpha} u^{\alpha - \gamma + 1}$$

ergo:

$$U_1 = \frac{1}{\alpha} (u - 1)^{\gamma - \alpha - 1} u^{\alpha - \gamma}$$

Quodsi quantitatem V_1 quam supra per U_1 expressam habuimus in aequationem 5) posuerimus eliminando $\frac{dV}{du}$ ex illa et ex 4) reductione apte perfecta colligitur:

$$U \alpha u (u - 1) = V + u (u - 1) U_1$$

quam cifram quantitatis U cum in 4) induximus, ratione habita valoris U_1 invenimus:

$$\frac{dV}{du} - \frac{V(u + \gamma - \alpha - 1)}{u(u - 1)} + (2u - 1)(u - 1)^{\gamma - \alpha - 1} u^{\alpha - \gamma} = 0$$

unde evenit integrando secundum methodum qua generaliter ad aequationes lineares primi ordinis integrandas uti solemus:

$$V = -e^{\int \frac{u + \gamma - \alpha - 1}{u(u - 1)} du} \left[\int \frac{2u - 1}{u} \left(\frac{u - 1}{u} \right)^{\gamma - \alpha - 1} e^{-\int \frac{u + \gamma - \alpha - 1}{u - 1} du} du + c \right]$$

quantitate c constantem integrationis significante.

Itaque est:

$$V = -(u - 1)^{\gamma - \alpha} u^{\alpha - \gamma + 1} (\log u (u - 1) + c)$$

$$U = -\frac{1}{\alpha u} \left(\frac{u - 1}{u} \right)^{\gamma - \alpha - 1} \log u (u - 1) + \left(\frac{u - 1}{u} \right)^{\gamma - \alpha - 1} \cdot \frac{1}{\alpha^2 u} - \frac{c}{\alpha u} \left(\frac{u - 1}{u} \right)^{\gamma - \alpha - 1}$$

Accipiamus, quo simpliciores cifras obtineamus:

$$c = \frac{1}{\alpha}$$

summandis secundo et tertio ex U sublatis restat:

$$U = \frac{1}{\alpha} (1 - u)^{\gamma - \alpha - 1} u^{\alpha - \gamma} \log u (u - 1)$$

Itaque cum cifrae quas supra sumpsimus identificatae sint primum membrum aequationis differentialis completum differentiale fit, jamque limites g, h sunt ita definiendi ut illis in aequationem inductis identice nihilum evadat.

Ponendum est igitur:

$$\{[V + V_1 \log(x-u)] (x-u)^{\lambda-1}\}_g^h = \\ \{(x-u)^{-\alpha-1} (u-1)^{\gamma-\alpha} u^{\alpha-\gamma+1} \left[\log \frac{x-u}{u(u-1)} - \frac{1}{\alpha} \right]\}_g^h = 0$$

unde ad limites constantes g, h definiendos tres casus locum habere posse patet:

1) $g = 0 \quad h = 1$ conditione $0 < \gamma - \alpha < 1$

2) $g = 0 \quad h = \infty$ „ aut $0 < \alpha; \gamma - \alpha < 1$ aut:
 $0 > \alpha; \gamma - \alpha > 1$

3) $g = 1 \quad h = \infty$ „ aut $0 < \alpha; 0 < \gamma - \alpha$ aut:
 $0 > \alpha; 0 > \gamma - \alpha$

quarum cifrarum conditione 1) evadens cum ea quae ab III. Spitzero indicata est, fere congruit:

$$y = \int_0^1 (x-u)^{-\alpha} (u-1)^{\gamma-\alpha-1} u^{\alpha-\gamma} \log \frac{u(u-1)}{x-u} du$$

§ 2.

Eadem ratione qua supra integralia quoque ad puncta $x=0$, $x=1$ pertinentia fingi possunt, scilicet pro $x=0$ et $0 < \beta < 1$

$$y = \int_0^1 (1-ux)^{-\alpha} (1-u)^{-\beta} u^{\beta-1} \log \frac{u(1-u)x}{1-ux} du$$

pro $x=1$ et $0 < \beta < 1$

$$y = \int_0^\infty (1-ux)^{-\alpha} (1-u)^{\alpha-1} u^{\beta-1} \log \frac{u(1-x)}{(1-u)(1-ux)} du$$

§ 3.

In cifris pro integralibus supra inventis logarithmus utriusque variabilis u, x functio erat. Quodsi unius variabilis x functio est, simpliciores cifrae obtinentur, quoniam in hoc casu logarithmum ex integrali tollere nobis licet. Qua de causa methodo supra adhibita inventitur:

Sub conditionibus $\alpha = \beta$

$\gamma = 1$

limitibus $g = 0, h = 1$ et $0 < \alpha < 1$

$$y = \int_0^1 (u-x)^{-\alpha} (1-u)^{-\alpha} u^{\alpha-1} \log \frac{x}{u^2} du; \text{ ratione analoga:}$$

$$y = \int_0^1 (1-ux)^{-\alpha} (1-u)^{\alpha-1} u^{-\alpha} \log \frac{1-x}{(1-u)^2 x} du$$

$$y = \int_0^1 (u-x)^{-\alpha} (1-u)^{\alpha-1} u^{-\alpha} \log \frac{1-x}{(1-u)^2} du$$

e quibus primum integrale et in puncti $x=0$ et in $x=\infty$, alterum in $x=0$, $x=1$, tertium in $x=1$, $x=\infty$ circuta valet, si bini qui primum locum habent exponentes integralium, non solum quae ad unum sed etiam quae ad alterum ex tribus punctis singularibus pertinent simul inter se aequales fiunt. Quodsi in cifris logarithmicis pro y inventis in § 1—2 easdem conditiones statuimus ac supra, comparando cum illis, relationes inter bina integralia evadunt, quae altera in parte diligentius nobis investigandae sunt.

b. Reductio casuum in quibus quantitates $\alpha \beta \gamma$ iisdem conditionibus ac supra non coercuimus.

Jam nobis demonstrandum est, quomodo substituendo et derivando ex integralibus quae supra explicavimus et quae constantibus $\alpha \beta \gamma$ certo modo definitis nacti sumus, integralia quoque, illis conditionibus de sumtis fingi possint. Ad reductionem illam explicandam duobus theorematis nobis utendum est:

1) Ex integrali $F(x)$ quae satisfacit aequationi propositae cuius sunt coefficientes $\alpha \beta \gamma m^{\text{ies}}$ derivando integrale aequationi differentiali ejusdem formae satisfaciens fingi potest cuius eodem numero integro coefficientes aucti sunt: m^{ta} enim integralis $F(x)$ derivatio satisfacit aequationi quam ex m^{ies} derivanda proposita aequatione egressuram habemus, quam ejusdem formae esse ac propositam noscimus, et cuius sunt coefficientes: $\alpha + m$, $\beta + m$, $\gamma + m$.

2) Ex integrali $F(x)$ aequationi propositae satisfaciens, integrale aequationi differentiali ejusdem formae atque illa satisfaciens, fingi potest, quae habet coefficientes eodem numero integro diminutos.

Demonstratio.

Aequationem differentialem propositam cum coefficientibus $\alpha \beta \gamma$ substituendo:

$$y = x^{1-\gamma} (1-x)^{\gamma-\alpha-\beta} n$$

in ejusdem formae aequationem converti noscimus, cujus sunt coeffi-

cientes: $1 - \alpha$, $1 - \beta$, $2 - \gamma$ (cfr. Kummer, de serie hypergeometrica, diarium Crellianum tom. 15) unde, habita ratione primi theorematis, illius theorematis demonstratio secundum hoc schema patet, ubi in prima columna coefficientes aequationis propositae, in reliquis ii qui ex operationibus supra scriptis procedunt, quique ad novas aequationes differentiales pertinent, notati sunt:

	1. substituendo:	2. m^{ies} derivando:	3. substituendo:
y	$y = x^{1-\gamma}(1-x)^{\gamma-\alpha-\beta}\zeta$	$\frac{d^m\zeta}{dx^m} = \zeta^{(m)}$	$\zeta^{(m)} = x^{\gamma-m-1}(1-x)^{\alpha-\beta-\gamma-m}\cdot\gamma$
α	$1 - \alpha$	$1 - \alpha + m$	$\alpha - m$
β	$1 - \beta$	$1 - \beta + m$	$\beta - m$
γ	$2 - \gamma$	$2 - \gamma + m$	$\gamma - m$

Itaque operationibus illis deinceps peractis pro integrali $F(x)$ invenitur cifra:

$$\eta = x^{1+m-\gamma}(1-x)^{\gamma+m-\alpha-\beta} \frac{d^m}{dx^m} [x^{\gamma-1}(1-x)^{\alpha+\beta-\gamma} F(x)]$$

quod erat demonstrandum.

Quorum theorematum ope reductionem quam supra indicavimus, jam absolvemus.

§ 1.

Sumamus ex integralibus supra inventis quae valent pro omnibus valoribus x in puncti $x = 0$ circuitu id, cuius sunt limites: 0, 1.

Posito:

$$\gamma = 1; 0 < \beta < 1$$

integrale:

$$y = \int_0^1 (1 - ux)^{-\alpha} (1-u)^{-\beta} u^{\beta-1} \log \frac{u(1-u)}{1-ux} du$$

aequationi differentiali propositae satisfacere jam cognovimus. Praeterea habemus casus in quibus $\gamma = 1$ et:

$$\text{aut } 0 < \beta - m < 1$$

$$\text{aut } 0 < \beta + m < 1$$

investigandos.

$$\text{a)} 0 < \beta - m < 1$$

Ex iis quae supra exposuimus ab aequatione differentiali proficiendum nobis est cujus sunt coefficientes:

$$\begin{matrix} \alpha - m \\ \beta - m \\ 1 \end{matrix}$$

Cui, ut explicavimus, sub a. § 2. conditione dicta satisfacit integrale

$$\eta = \int_0^1 (1 - ux)^{-\alpha + m} (1 - u)^{-\beta + m} u^{\beta - m - 1} \log \frac{u(1-u)x}{1-ux} du = F(x)$$

qua ex cifra, ut integrale obtineamus aequationi propositae satisfaciens, ab aequatione jam nominata proficiscentes, operationes in hoc schemate descriptas deinceps absolvemus, unde denique pervenimus ad propositam:

η	1. m^{ies} derivando:	2. substituendo:	3. m^{ies} derivando:
	$\frac{d^n \eta}{dx^n} = \eta^{(n)}$	$\eta^{(m)} = x^{-m} \zeta$	$\frac{d^m \zeta}{dx^m} = y$
$\alpha - m$	α	$\alpha - m$	α
$\beta - m$	β	$\beta - m$	β
1	$1 + m$	$1 - m$	1

unde, operationibus quae ad integrale $\eta = F(x)$ referuntur, peractis aequationi differentiali propositae pro valore $\gamma = 1$ et conditione supra significata integrale satisfacere reperitur:

$$y = \frac{d^n}{dx^n} \left(x^m \frac{d^m}{dx^m} (F(x)) \right)$$

$$\text{b) } 0 < \beta + m < 1$$

Hic ducere debemus rationes secundum ea quae supra commemoravimus ab aequatione differentiali cuius coefficientes sunt:

$$\begin{matrix} \alpha + m \\ \beta + m \\ 1 \end{matrix}$$

cui ut sub a. § 2. demonstratum conditione modo allata b) integrale satisfacit:

$$\eta = \int_0^1 (1 - ux)^{-\alpha - m} (1 - u)^{-\beta - m} u^{\beta + m - 1} \log \frac{u(1-u)x}{1-ux} du = \Phi(x)$$

unde operationibus in hocce schemate perscriptis et conditione supra indicta integrale aequationis propositae efficitur:

	1. Adhibito theoremate 2.	2. substit.:
η	$\zeta = x^m (1-x)^{1-\alpha-\beta} \frac{d^m}{dx^m} \left((1-x)^{\alpha+\beta+2m-1} \eta \right)$	$\vartheta = x^{-m} \zeta$
$\alpha+m$	α	$\alpha+m$
$\beta+m$	β	$\beta+m$
1	$1-m$	$1+m$

3. Adhibito theoremate 2.

$$y = (1-x)^{1-\alpha-\beta} \frac{d^m}{dx^m} \left(x^m (1-x)^{\alpha+\beta+m-1} \vartheta \right)$$

α

β

1

quas operationes pro integrali $\eta = \Phi(x)$ susceptas cum deinceps persequanur aequationis propositae conditione supra data in casu $\gamma=1$ accipimus integrale:

$$y = (1-x)^{1-\alpha-\beta} \frac{d^m}{dx^m} \left(x^m \frac{d^m}{dx^m} ((1-x)^{\alpha+\beta+2m-1} \Phi(x)) \right)$$

§ 2.

Ratione analoga, si ex tribus inventis sub § 3. a. integralibus sumimus id quod substituendo: $x = 1 - \zeta$ in integrali supra (§ 1) pro $x = 0$ accepto nascitur, id quidem cuius sunt limites 0∞ invenimus in casibus:

a) $0 < \beta - m < 1$

$$y = \frac{d^m}{dx^m} \left((1-x)^m \frac{d^m}{dx^m} (F(1-x)) \right)$$

ubi ponendum est:

$$F(1-x) = \int_0^x (1-ux)^{-\beta+m} (1-u)^{\beta-m-1} u^{x-m-1} \log \frac{u(1-x)}{(1-u)(1-ux)} du$$

b) $0 < \beta + m < 1$

$$y = x^{1-\alpha-\beta} \frac{d^m}{dx^m} \left[(1-x)^m \frac{d^m}{dx^m} \left(x^{\alpha+\beta+2m-1} \Phi(1-x) \right) \right]$$

ubi ponendum est:

$$\Phi(1-x) = \int_0^\infty (1-ux)^{-\beta-m} (1-u)^{\beta+m-1} u^{\alpha+m-1} \log \frac{u(1-x)}{(1-u)(1-ux)} du$$

Eadem ratione si integralium trium sub a. § 1. conditione $\alpha = \beta$ positorum accipimus id quod ex supra accepto substitutione:

$$x = \frac{1}{t}; y = t^\alpha \eta$$

evadit cui e tribus integralibus id cuius sunt limites 0, 1 respondet, reperitur in casibus:

a) $0 < \gamma - \alpha - m < 1$

$$y = x^{\alpha+m+1} \frac{d^m}{dx^m} \left(x^m \frac{d^n}{dx^n} (x^{\alpha-1} F(x)) \right)$$

ubi est:

$$F(x) = \int_0^1 (x-u)^{-\alpha-m} (1-u)^{-\alpha+\gamma+m-1} u^{\alpha-\gamma-m} \log \frac{u(1-u)}{x-u} du$$

b) $0 < \gamma - \alpha + m < 1$

$$y = x^{\alpha-\gamma+m+1} (x-1)^{\gamma-2\alpha} \frac{d^m}{dx^m} \left[x^m \frac{d^n}{dx^n} (x^{\gamma-\alpha+m-1} (x-1)^{2\alpha-\gamma+m} \Phi(x)) \right]$$

ubi ponendum est:

$$\Phi(x) = \int_0^1 (x-u)^{-\alpha-m} (1-u)^{-\alpha+\gamma+m-1} u^{\alpha-\gamma+m} \log \frac{u(1-u)}{x-u} du$$

quae sunt integralia reperta notae et facile verificandae transformationis formulae ope:

$$\frac{d^m y}{dt^m} = (-1)^m x^{m+1} \frac{d^m}{dx^m} (x^{\alpha+m-1} \eta)$$

c) Reductio casuum in quibus exponentes qui primum locum habent cunctis numeris inter se differant ad illos qui praecedunt.

Ex iis quae rationes praecedentes tradiderunt et iisdem principiis adhibendis reductio illa peragi potest, exceptis modo quibusdam casibus in quibus logarithmica integralia non obtinentur, quas quidem res pro numerorum integrorum valoribus quantitatam $\alpha, \beta, \gamma - \alpha, \gamma - \beta$ fieri posse supra exposuimus. Quoniam secundum ea quae pronuntiavimus pro quibuslibet quantitatatum α, β valoribus, si qui primum locum habent exponentes inter se aequales fiunt, integrale logarithmicum iudagari potest, ad quaestionem propositam peragendam eodem utroque theoremate principali iterum modo nobis utendum est.

Quae integralia ad omnes casus enumeratos spectantia per nostram methodum subduximus tabula hacce comparavimus, quorum cifrae ad punctum $x = \infty$ pertinentes ejusdem transformationis formulae ope obtinentur, postquam ibidem pro quantitate α exponentem singulos casus declarantem posuimus.

d. Integralia quae reperimus per indices exposita.

Tabula integralium logarithmicorum.

Punctum $x = 0$		
$\gamma - n = 1$	$0 < \beta - m - n < 1$	$y = \frac{d^{m+n}}{dx^{m+n}} \left(x^m \frac{d^n}{dx^n} (F(m, n)) \right)$
"	$0 < \beta + m - n < 1$	$y = \frac{d^n}{dx^n} \left[(1-x)^{1-\alpha-\beta+2n} \frac{d^m}{dx^m} \left(x^m \frac{d^n}{dx^n} ((1-x)^\alpha + \beta + 2m - 2n - 1 F(-m, n)) \right) \right]$
$\gamma + n = 1$	$0 < \beta - m + n < 1$	$y = x^n (1-x)^{1-\alpha-\beta-n} \frac{d^n}{dx^n} \left[(1-x)^{\alpha+\beta+2n-1} \frac{d^m}{dx^m} \left(x^m \frac{d^n}{dx^n} (F(m, -n)) \right) \right]$
"	$0 < \beta + m + n < 1$	$y = x^n (1-x)^{1-\alpha-\beta-n} \frac{d^{m+n}}{dx^{m+n}} \left(x^m \frac{d^m}{dx^m} ((1-x)^\alpha + \beta + 2m + 2n - 1 F(-m, -n)) \right)$
Punctum $x = 1$		
$\alpha + \beta = \gamma + n$	$0 < \beta - m - n < 1$	$y = (-1)^n \frac{d^{m+n}}{dx^{m+n}} \left((1-x)^m \frac{d^n}{dx^n} (\Phi(m, n)) \right)$
"	$0 < \beta + m - n < 1$	$y = (-1)^n \frac{d^n}{dx^n} \left[x^{1-\alpha-\beta-2n} \frac{d^m}{dx^m} \left((1-x)^m \frac{d^n}{dx^n} (x^{\alpha+\beta+2m-2n-1} \Phi(-m, n)) \right) \right]$
$\alpha + \beta = \gamma - n$	$0 < \beta - m + n < 1$	$y = (-1)^n (1-x)^n x^{1-\alpha-\beta-n} \frac{d^n}{dx^n} \left[x^{\alpha+\beta+2n-1} \frac{d^m}{dx^m} \left((1-x)^m \frac{d^n}{dx^n} (\Phi(m, -n)) \right) \right]$
"	$0 < \beta + m + n < 1$	$y = (-1)^n (1-x)^n x^{1-\alpha-\beta-n} \frac{d^{m+n}}{dx^{m+n}} \left((1-x)^m \frac{d^m}{dx^m} (x^{\alpha+\beta+2m+2n-1} \Phi(-m, -n)) \right)$

Punctum $x = \infty$

$$\beta = z - n \quad 0 < \gamma - z + m + n < 1$$

$$y = (-1)^n x^{-z+m+n+1} \frac{d^{m+n}}{dx^{m+n}} \left(x^{m+n} \frac{d^n}{dx^n} (x^{z-n-1} \Psi(m, n)) \right)$$

$$y = (-1)^n x^{-z+n+1}.$$

$$0 < \gamma - z - m + n < 1 \quad \frac{d^n}{dx^n} \left[x^{2z-\gamma+m-n} (x-1)^{\gamma-2z+2n} \frac{d^m}{dx^m} \left(x^m \frac{d^n}{dx^n} (x^{\gamma-z+n-1} (x-1)^{2z-\gamma+2m-2n} \Psi(-m, n)) \right) \right]$$

$$y = (-1)^n x^{z-\gamma+n+1} (x-1)^{\gamma-2z-n}.$$

$$\beta = z + n \quad 0 < \gamma - z + m - n < 1$$

$$\frac{d^n}{dx^n} \left[x^{\gamma-2z+m-n} (x-1)^{2z-\gamma+2n} \frac{d^m}{dx^m} \left(x^m \frac{d^n}{dx^n} (x^{z+n-1} \Psi(m, -n)) \right) \right]$$

$$y = (-1)^n x^{z-\gamma+m+n+1} (x-1)^{\gamma-2z-n}.$$

$$0 < \gamma - z - m - n < 1 \quad \frac{d^{m+n}}{dx^{m+n}} \left[x^{m+n} \frac{d^m}{dx^m} \left(x^{\gamma-z-n-1} (1-x)^{2z-\gamma+2m+2n} \Psi(-m, n) \right) \right]$$

Quo simpliciores fuerint cifrae adhibitae sunt significaciones haec:

$$F(\pm m, \pm n) = \int_0^1 (1-ux)^{-\alpha \pm m \mp n} (1-u)^{-\beta \pm m \mp n} u^{\beta \mp m \mp n - 1} \log \frac{u(1-u)x}{1-ux} du$$

$$\Phi(\pm m, \pm n) = \int_0^\infty (1-ux)^{-\alpha \pm m \mp n} (1-u)^{-\beta \pm m \mp n - 1} u^{\beta \mp m \mp n - 1} \log \frac{u(1-x)}{(1-u)(1-ux)} du$$

$$\Psi(\pm m, \pm n) = \int_0^1 (x-u)^{-\alpha \pm m \mp n} (1-u)^{-\alpha \pm m \mp n - 1} u^{\alpha \mp m \mp n - 1} \log \frac{u(1-u)}{x-u} du$$

Pars secunda.

Serierum evolutiones.

III. Fuchsium generaliter demonstravisse (cfr. Diarium Crellianum tom. 66) jam praefati sumus, pro aequationum differentialium specie quarum nobis proposita singularis est, si duo qui primum locum habent exponentes aut aequalis sint, aut numeris integris inter se differant, pro integralibus duas series, alteram logarithmicam, alteram e duobus summandis, logarithmico et alogarithmico compositam existere. Prior autem series, ut jam e praefatione patet, si constante quodam factore disceditur, est series gaussiana $F(\alpha \beta 1 x)$. Itaque ut alteram seriem pertractemus relinquimus, id quod duabus methodis fieri potest.

a) qua methodo ad formandam ex integrali ordinario seriem gaussianum uti solent.

b) indefinitorum coefficientium methodo.

Hic satis est, pro uno ex tribus punctis singularibus seriem plane evolvi, quia methodus, pro ceteris adhibenda eadem est atque illa, substituendoque ab altero ab alterum transire licet. A puncto igitur $x=0$ in posterioribus proficiamur.

a. Evolutio directa in series, formandi ex ordinario integrali seriem gaussianam methodo.

Proficiamur, secundum ea quae supra commemoravimus ab integrali quod ad punctum $x=0$ pertinens praemissione $\gamma=1$, $0 < \beta < 1$ jam invenimus:

$$1. \quad y = \int_0^1 (1-ux)^{-\alpha} (1-u)^{-\beta} u^{\beta-1} \log \frac{u(1-u)x}{1-ux} du$$

quoniam ceteros casus, ut ex tabula partis prioris patet, ad illos semper redigi licet. Primum integrale in seriem evolvendum e duobus summandis compositum esse perspicitur: quorum alter, secundum Fuchsii theorema supra memoratum, neglecto constante quodam factore, est logarithmus, multiplicatus per seriem gaussianam, alter autem e duabus partibus constat:

$$2. \quad \int_0^1 (1-ux)^{-\alpha} (1-u)^{-\beta} u^{\beta-1} \log \frac{u(1-u)}{1-ux} du = \\ \int_0^1 (1-ux)^{-\alpha} (1-u)^{-\beta} u^{\beta-1} \log (1-u) du - \\ \int_0^1 (1-ux)^{-\alpha} (1-u)^{-\beta} u^{\beta-1} \log (1-ux) du$$

Quodsi x valores satis parvi accipiuntur, sub integrali expressiones: $(1-ux)^{-\alpha}$, $(1-ux)^{-\alpha} \log (1-ux)$ per potestates ux evolvilicet. Quibus in seriebus ex integralibus illis evolvendis pro coefficientibus constantes prodeunt transcendentes qui aptarum formularum recursionis ope ad duos transcendentes rediguntur quorum alter ad alterum nondum revocari potest: ita quidem ut denique duae series oriuntur, duobus illis transcendentibus qui omnium cifrarum factores sunt communes, separatis, quarum coefficientes e constantibus aequationis differentialis solum compositi sunt.

Primo igitur ex 2., quantum ad priorem secundi membra summandum pertinet, series haec nascitur:

$$\int_0^1 (1-ux)^{-\alpha} (1-u)^{-\beta} u^{\beta-1} \log u (1-u) du = \\ \int_0^1 (1-u)^{-\beta} u^{\beta-1} \log u (1-u) du + \frac{x \cdot \alpha}{1} \int_0^1 (1-u)^{-\beta} u^{\beta} \log u (1-u) du + \\ + x^2 \frac{x(\alpha+1)}{2!} \int_0^1 (1-u)^{-\beta} u^{\beta+1} \log u (1-u) du + \dots \\ + x^{\nu+1} \frac{\alpha(\alpha+1)\dots(\alpha+\nu)}{(\nu+1)!} \int_0^1 (1-u)^{-\beta} u^{\beta+\nu} \log u (1-u) du$$

Ad transcendentes reducendos hac formula utamur, quam unum quemque summandum e quibus cifram transcendentis generalem compositam habemus, integrando, ratione quidem habita, cifram quae est extra integrale praemissione $0 < \beta < 1$ integratione peracta in utroque casu evanescere, reductione apte adhibita obtinemus:

$$\int_0^1 (1-u)^{-\beta} u^{\beta+\nu-1} \log u (1-u) du = \\ \frac{\beta+\nu}{\nu+1} \int_0^1 (1-u)^{-\beta} u^{\beta+\nu-1} \log u (1-u) du - \\ \frac{2\beta+\nu-1}{(\nu+1)^2} \int_0^1 (1-u)^{-\beta} u^{\beta+\nu-1} du$$

Eandem formulam iteram ad omnes summandos logarithmicos adhibentes seriem reperimus e cifris compositam in quibus transcendentes eadem semper potestate u inter se differunt et una cifra excepta logarithmo vacant. Ad unam quamque cifram alogarithmicam recursionis formulam hanc adhibendo:

$$\int_0^1 (1-u)^{-\beta} u^{\beta+v} du = \frac{\beta+v}{v+1} \int_0^1 (1-u)^{-\beta} u^{\beta+v-1} du$$

series cifrarum nascitur, in qua duo solum transcendentes sunt:

$$\int_0^1 (1+u)^{-\beta} u^{\beta-1} \log u (1-u) du = C$$

$$\int_0^1 (1-u)^{-\beta} u^{\beta-1} du = C^1$$

quae est series:

$$\int_0^1 (1-u)^{-\beta} u^{\beta+v} \log u (1-u) du = \\ \frac{\beta(\beta+1)\dots(\beta+v)}{(v+1)!} C - \left[\frac{2\beta+v-1}{(v+1)^2} \frac{(\beta+v-1)\dots\beta}{v!} + \frac{2\beta+v-2}{v^2} \cdot \frac{(\beta+v)\dots(\beta+v-2)\dots\beta}{v+1(v-1)!} \right. \\ \left. + \frac{2\beta+v-3}{(v-1)^2} \frac{(\beta+v)(\beta+v-1)}{v(v+1)} \frac{(\beta+v-3)\dots\beta}{(v-2)!} + \dots + \frac{2\beta-1}{1^2} \frac{(\beta+v)\dots(\beta+1)}{(v+1)!} \right] C^1$$

Multiplicando per:

$$\frac{x(x+1)\dots(x+v)}{(v+1)!} x^v$$

seriem illam, cifra generalis seriei propositae oritur. Quia in serie factorem quantitatis C seriem gaussianam $F(\alpha \beta 1 x)$, factorem autem C^1 seriem minus simplicem repraesentare perspicuum est.

Aequationis 2. alteri quoque summando, si productum

$$(1-ux)^{-\alpha} \cdot \log(1-ux)$$

evolvimus scriei formam dare nobis licet hancce:

$$\int_0^1 (1-ux)^{-\alpha} (1-u)^{-\beta} u^{\beta-1} \log(1-ux) du = \\ -x \int_0^1 (1-u)^{-\beta} u^{\beta} du - x^2 \left(\frac{1}{2} + \frac{\alpha}{1} \right) \int_0^1 (1-u)^{-\beta} u^{\beta+1} du \\ - x^3 \left(\frac{1}{3} + \frac{\alpha}{1} \cdot \frac{1}{2} + \frac{\alpha(\alpha+1)}{2!} \cdot \frac{1}{1} \right) \int_0^1 (1-u)^{-\beta} u^{\beta+2} du - \dots \\ - x^v \left(\frac{1}{v} + \frac{\alpha}{1} \cdot \frac{1}{v-1} + \frac{\alpha(\alpha+1)}{2!} \cdot \frac{1}{v-2} + \dots + \frac{\alpha(\alpha+1)\dots(\alpha+v-2)}{(v-1)!} \right) \\ \int_0^1 (1-u)^{-\beta} u^{\beta+v-1} du$$

Per formulam recurrentem 4. ad unamquamque cifram adhibitam efficitur ut eonstans C factor cifrarum omnium proprius evadat, quare fit coefficiens x^v :

$$C^1 \frac{\beta(\beta+1)\dots(\beta+v-1)}{v!} \left(\frac{1}{v} + \frac{\alpha}{1} \cdot \frac{1}{v-1} + \frac{\alpha(\alpha+1)}{2!} \frac{1}{v-2} + \dots + \frac{\alpha(\alpha+1)\dots(\alpha+v-2)}{(v-1)!} \cdot \frac{1}{1} \right)$$

unde series quae pertinet ad illud integrale multiplicando per x^v et summando secundum v prodit. Quodsi, quam pro primo summando jam invenimus ad seriem illam addimus, pro primo aequationis 2. membro series haec existit:

$$\begin{aligned}
 & \int_0^1 (1 - ux)^{-\alpha} (1 - u)^{-\beta} u^{\beta-1} \log \frac{u(1-u)}{1-ux} du = \\
 & \quad C.F(\alpha \beta 1 x) + \\
 & + C' x \left\{ \frac{\alpha(1-2\beta)}{1-1^2} + \frac{\beta}{1} + x \left[\frac{\beta(\beta+1)}{1 \cdot 2} \left(\frac{1}{2} + \frac{\alpha}{1} \frac{1}{1} \right) + \right. \right. \\
 & \quad \left. \frac{\alpha(\alpha+1)}{1 \cdot 2} \left(\frac{2\beta}{2^2} \frac{\beta}{1!} + \frac{2\beta-1}{1^2} \frac{\beta+1}{2!} \right) \right] \\
 & + x^2 \left[\frac{\beta(\beta+1)(\beta+2)}{3!} \left(\frac{\alpha(\alpha+1)}{2!} + \frac{\alpha}{2} + \frac{1}{3} \right) - \frac{\alpha(\alpha+1)(\alpha+2)}{3!} \cdot \right. \\
 & \left. \left(\frac{2\beta+1}{3^2} \cdot \frac{\beta(\beta+1)}{2!} + \frac{2\beta}{2^2} \frac{\beta+2}{3} \cdot \frac{\beta}{1} + \frac{2\beta-1}{1^2} \frac{(\beta-1)(\beta+2)}{3!} \right) \right] + \dots \}
 \end{aligned}$$

Seriem illam cum ea quam altera methodo jam deducemus congruere in sequentibus demonstrabitur.

b. Seriei evolutio coefficientium indefinitorum methodo.

§ 1.

Evolutio formae generalis.

Ex iis quae in superioribus exposuimus seriem quam pro integrali logarithmico ad punctum $x=0$ pertinente conditione $\gamma=1$; $0<\beta<1$ invento evolvimus e duobus summandis: logarithmico et alogarithmico compositam esse patet quorum alter, ut factor $\log x$ seriem continet gaussianam, alter seriem minus simplicem praebet. Quo simpliciorem illique aequivalentem adipiscamur seriem, ab alio integrali nobis proficiscendum est, quod quidem simpliciore ratione evolvere licet. Sequentia id probabunt:

Pro y in puncti $x=0$ circuitu duo integralia particularia invenimus:

$$y_1 = F(\alpha \beta 1 x)$$

$$y_2 = F(\alpha \beta 1 x) \log x + \Psi$$

Ψ brevitatis causa seriem supra evolutam significante. Si ergo y_2 particulare est integrale, idem auctum etiam quavis serie gaussiana per constantem $-x$ multiplicata integrale esse debet; quare alterum integrale in forma sumere nobis licet:

$$y = -x F(\alpha \beta 1 x) \cdot \log x + \psi$$

ubi ψ secundum ea quae supra diximus notione est magis generali quam Ψ .

Primam dehinc factorem $\log x$, quoniam F est integrale particulare aequationis differentialis propositae, expressione illa ejusque derivationibus in eam inductis evanescere perspicitur, ita ut ad seriem ψ definiendam haec aequatio remaneat:

$$2. \quad x(x-1)\psi'' + [(\alpha+\beta+1)x-1]\psi' + \alpha\beta\psi = \\ x(\alpha+\beta)F(\alpha\beta 1 x) + 2x(x-1)F'(\alpha\beta 1 x).$$

Ponamus, ψ inveniendi causa seriem hancce cum coefficientibus indefinitis a:

$$\psi = a_0 + a_1 x + a_2 x^2 + \dots + a_v x^v + \dots \text{ (in inf.)}$$

unde in 2. inducta illa cifra et relatione quae pro serie gaussiana valet, spectata:

$$F'(\alpha\beta 1 x) = \frac{\alpha\beta}{1} F(\alpha+1, \beta+1, 2, x)$$

aequatio haec sequitur:

$$x(x-1)(1.2a_2 + 2.3a_3x + 3.4a_4x^2 + \dots) + [(\alpha+\beta+1)x-1] \\ (a_1 + 2a_2x + 3a_3x^2 + \dots) \\ + x\beta(a_0 + a_1x + a_2x^2 + \dots) = \\ (\alpha+\beta)x\left(1 + \frac{\alpha\beta}{1}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{2!}x^2 + \dots\right) + 2\alpha\beta \cdot x(x-1) \\ \left(1 + \frac{(\alpha+1)(\beta+1)}{1!2!}x + \frac{(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}{2!3!}x^2 + \dots\right)$$

qua in aequatione coefficientes singularium potestatum x natura ipsa evanescant necesse est.

Coefficientibus binomialibus ut solent significatis:

$$\frac{x(\alpha+1) \dots (\alpha+v-1)}{1 \cdot 2 \cdot 3 \dots v} = (\alpha+v-1)_v$$

coefficientem x^v cum nihilo comparando manat dehinc haec lex coefficientium:

$$3. \quad a_{v-1}(v+1)^2 - a_v(v+x)(v+\beta) = x \frac{(\alpha+v)(\beta-1)+(\beta+v)(\alpha-1)}{v+1} \\ (\alpha+v-1)_v \cdot (\beta+v-1)_v$$

Quo clarius illam legem perspiciamus primum membrum per unam quantitatem certo indice praeditam denotemus:

$$a_{v+1}(v+1)^2 - a_v(v+x)(v+\beta) = A_{v+1} \\ (v=0, 1, 2, \dots)$$

qua inducta in 3. legem illam transire patet in hancce:

$$4. \quad A_{v+1} = x \cdot \frac{(\alpha+v)(\beta-1)+(\beta+v)(\alpha-1)}{v+1} (\alpha+v-1)_v \cdot (\beta+v-1)_v$$

Jam quantitates a , e quibus solis series ψ composita est, per quantitates A exprimantur necesse est. Ad id propositum peragendum schemate hocce significationibus supra adhibitis nascente utemur:

$$A_1 = a_1 \cdot 1^2 - a_0 \alpha \beta$$

$$A_2 = a_2 \cdot 2^2 - a_1 (\alpha + 1) (\beta + 1)$$

$$\vdots$$

$$A_v = a_v \cdot v^2 - a_{v-1} (\alpha + v - 1) (\beta + v - 1)$$

Quantitates $a_{v-1} \dots a_0$ paullatim eliminando evadit a_v per quantitates A expressa:

$$a_v =$$

$$\frac{A_v + A_{v-1}}{v^2} \frac{(\alpha + v - 1)(\beta + v - 1)}{(v - 1)^2} + \frac{A_{v-2}}{v^2(v - 1)^2} \frac{(\alpha + v - 1)(\alpha + v - 2)(\beta + v - 1)(\beta + v - 2)}{(v - 1)^2(v - 2)^2} + \dots$$

$$\dots + A_1 \frac{(\alpha + v - 1) \dots (\alpha + 1)(\beta + v - 1) \dots (\beta + 1)}{v^2(v - 1)^2 \dots 1^2} +$$

$$A_0 \frac{(\alpha + v - 1) \dots (\alpha + 1) \alpha (\beta + v - 1) \dots (\beta + 1) \beta}{v^2(v - 1)^2 \dots 1^2}$$

ubi uniformitatis causa: $a_0 = A_0$ positum est.

Inde pro serie:

$$\psi = \sum_0^\infty a_v x^v$$

brevitatis causa significato:

$$F_v = 1 + \frac{(\alpha + v)(\beta + v)}{(v + 1)} x + \frac{(\alpha + v)(\alpha + v + 1)(\beta + v)(\beta + v + 1)}{(v + 1)^2(v + 2)^2} x^2 + \dots \text{ (in inf.)}$$

expressio haecce manat:

$$\psi = A_0 F_0 + A_1 F_1 \frac{x}{1^2} + A_2 F_2 \frac{x^2}{2^2} + \dots + A_v F_v \frac{x^v}{v^2} + \dots$$

vel inductis quantitatibus A valoribus qui manaut ex 4:

$$\begin{aligned} \psi &= a_0 F_0 + z \left\{ F_1 x + [(\alpha + 1)(\beta - 1) + (\beta + 1)(\alpha - 1)] \frac{x\beta}{1 \cdot 2} F_2 \frac{x^2}{2^2} \right. \\ &\quad + [(\alpha + 2)(\beta - 1) + (\beta + 2)(\alpha - 1)] \frac{\alpha(\alpha + 1)\beta(\beta + 1)}{1 \cdot 2^2 \cdot 3} F_3 \frac{x^3}{3^2} + \\ &\quad \dots + [(\alpha + v - 1)(\beta - 1) + (\beta + v - 1)(\alpha - 1)] \frac{\alpha(\alpha + 1) \dots (\alpha + v - 2)\beta(\beta + 1) \dots (\beta + v - 2)}{1 \cdot 2^2 \cdot 3^2 \dots (v - 1)^2 \cdot v} \\ &\quad \left. F_v \frac{x^v}{v^2} + \dots \right\} \end{aligned}$$

ubi F_0 secundum significationem quam supra scripsimus, series gaussiana $F(\alpha \beta 1 x)$ ceterae autem F — functiones, residua ejusdem seriei gaussiana sunt, cum relatio evadat:

$$7. F_0 = 1 + \frac{x^\beta}{1^2} x + \frac{x(x+1)\beta(\beta+1)}{1^2 \cdot 2^2} x^2 + \\ \dots + \frac{x(x+1)\dots(x+v-2)\beta(\beta+1)\dots(\beta+v-2)}{1 \cdot 2^2 \cdot \dots \cdot (v-1)^2} x^{v-1} + \\ + \frac{x(x+1)\dots(x+v-1)\beta(\beta+1)\dots(\beta+v-1)}{1 \cdot 2^2 \cdot \dots \cdot v^2} F_v \cdot x^v$$

Qua functionum F inde quia omnes a cifra 1 incipiunt, in usu communi pro satis parvis x valoribus seriem ψ ut simplicem potestatum x seriem considerare nobis licet quoniam in seriebus F potestates sequentes pro prima negligendae sunt.

Seriem ita evolutam cum ea quam supra deduximus congruere jam nobis demonstandum est.

Accipiamus duos constantes arbitrarios c_1, c_2 et ponamus quo simpliciores fiant expressiones:

$$a_0 = 1; x = -1$$

Ex iis quae supra contendimus secundum theorema illud notum, omne integrale particulare in generali inesse debere inter utramque seriem haec relatio locum habebit:

$$8. \int_0^1 (1-ux)^{-\alpha} (1-u)^{-\beta} u^{\beta-1} \log \frac{u(1-u)x}{1-ux} du = \\ c_1 (F(\alpha \beta 1 x) \log x + \psi) + c_2 F(\alpha \beta 1 x)$$

Profecto constantes ex illa relatione definiiri posse reperitur. Ac jam primum quia in factoribus $\log x$ singularium potestatum x coefficientes aequales esse debent in utroque membro aequationem hancce habemus:

$$c_1 = \int_0^1 (1-u)^{-\beta} u^{\beta-1} du$$

Eodem arguento ad potestates x in parte alogarithmico adhibito, si in primum membrum aequationis 8. induximus seriem sub a. evolutam in alterum eam quam pro ψ invenimus, sequitur:

$c_2 = \int_0^1 (1-u)^{-\beta} u^{\beta-1} \log u (1-u) du - \int_0^1 (1-u)^{-\beta} u^{\beta-1} du$
 quibus constantium valoribus in 8. inductis singulas x potestates inter se comparando relatio illa verificatur. Itaque quas methodo utraque series retulimus inter se congruere probatum est.

Discedunt igitur cifrae illae serie gaussiana, multiplicata per constantem quendam factorem.

§ 2.

Convergentia series ψ .

Seriem ψ esse convergentem probaturi simplicitatis causa accipiamus:

$$a_0 = 1 \quad x = 1$$

unde oritur series in forma:

$$\begin{aligned} 1. \quad & \psi = \\ & F_0 + F_1 x + [(\alpha+1)(\beta-1) + (\beta+1)(x-1)] \frac{\alpha\beta}{1 \cdot 2^2} F_2 \frac{x^2}{2^2} \\ & + [(\alpha+2)(\beta-1) + (\beta+2)(x-1)] \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2^2 \cdot 3} F_3 \frac{x^3}{3^2} + \dots \\ & \dots + [(\alpha+v-1)(\beta-1) + (\beta+v-1)(\alpha-1)]. \\ & \frac{\alpha(\alpha+1)\dots(\alpha+v-2)\beta(\beta+1)\dots(\beta+v-2)}{1 \cdot 2^2 \cdot 3^2 \dots (v-1)^2 v} F_v \frac{x^v}{v^2} + \dots \end{aligned}$$

Quodsi pro quantitatibus quae sunt in illa serie, modulos inducimus ita ut ponatur:

$$\text{mod } x = \omega$$

$$\text{mod } (\alpha-1) = x$$

$$\text{mod } (\beta-1) = \lambda$$

$$\text{mod } F_v = \Phi_v$$

quoniam modulus summae aut minor summa modulorum aut maxime ei aequalis est, sequitur:

$$\text{mod } (\alpha+v) \geq x+v+1$$

$$\text{mod } (\beta+v) \geq \lambda+v+1$$

$$\text{mod } F_v = \Phi_v \geq 1 + \frac{(x+v+1)(\lambda+v+1)}{(v+1)^2} \omega$$

$$+ \frac{(x+v+1)(x+v+2)(\lambda+v+1)(\lambda+v+2)}{(v+1)^2(v+2)^2} \omega^2 + \dots$$

quibus cifris in 1. positis deducimus:

$$\begin{aligned} 2. \quad \psi & \leq \Phi_0 + \Phi_1 \frac{\omega}{1^2} + [(x+2)\lambda + (\lambda+2)x] \frac{(x+1)(\lambda+1)}{1 \cdot 2} \Phi_2 \frac{\omega^2}{2^2} + \dots \\ & + [(x+v)\lambda + (\lambda+v)x] \frac{(x+1)\dots(x+v-1)(\lambda+1)\dots(\lambda+v-1)}{1 \cdot 2^2 \cdot 3^2 \dots (v-1)^2 v} \Phi_v \frac{\omega^v}{v^2} + \dots \end{aligned}$$

Quodsi hujus inaequationis secundum membrum convergens esse demonstravimus et seriem ψ convergere patet. Quoniam $\Phi_0 = \text{mod } F_0$, pro serie gaussiana conditione $0 < \omega < 1$ convergentem habemus, ceteram secundi memtri partem, si x intra eosdem limites est, convergere demonstrandum est. Itaque quotientem n^{tae} cifrae, divisae per $n-1^{\text{tam}}$ pro $n=\infty$ fieri minorem 1 illustrare debemus. Illius autem

quotientis ut supra explicavimus, divisis numeratore et denominatore per n^3 forma est haec:

$$3. \frac{\Phi_n}{\Phi_{n-1}} \left(\frac{x}{n} + 1 \right) \lambda + \left(\frac{\lambda}{n} + 1 \right) x - \left(\frac{x-1}{n} + 1 \right) \left(\frac{\lambda-1}{n} + 1 \right) \left(1 - \frac{1}{n} \right) \omega$$

Prior quotiens:

$$\frac{\Phi_n}{\Phi_{n-1}}$$

cum relationem 7 in praecedente § statutam ad rem propositam adhibemus, transformari potest. Tali modo fit:

$$\begin{aligned} \Phi_0 = 1 &+ \frac{(x+1)(\lambda+1)}{1^2} \omega + \frac{(x+1)(x+2)(\lambda+1)(\lambda+2)}{1^2 \cdot 2^2} \omega^2 + \\ &+ \frac{(x+1) \dots (x+v)(\lambda+1) \dots (\lambda+v)}{1 \cdot 2^2 \dots (v-1)^2} \omega^{v-1} \\ &+ \frac{(x+1) \dots (x+v)(\lambda+1) \dots (\lambda+v)}{1 \cdot 2^2 \dots v^2} \Phi_v \cdot \omega^v \end{aligned}$$

quam seriem primo a cifra Φ_n , tum a cifra Φ_{n-1} abrumpendo si aequationes eo acceptas subtraximus reductione facta obtinemus:

$$\frac{\Phi_n}{\Phi_{n-1}} = \frac{1}{\omega \left(\frac{x}{n} + 1 \right) \left(\frac{\lambda}{n} + 1 \right)} \left[1 - \frac{1}{\Phi_{n-1}} \right]$$

ubi secundum ea quae supra commemoravimus:

$$\begin{aligned} \Phi_{n-1} = 1 &+ \left(1 + \frac{x}{n} \right) \left(1 + \frac{\lambda}{n} \right) \omega + \left(1 + \frac{x}{n} \right) \left(1 + \frac{x}{n+1} \right) \left(1 + \frac{\lambda}{n} \right) \\ &\quad \left(1 + \frac{\lambda}{n+1} \right) \omega^2 + \dots \end{aligned}$$

unde sequitur:

$$\begin{aligned} \Phi_{n-1} &\leq \sum_{v=0}^{\infty} \left(1 + \frac{x}{n} \right)^v \left(1 + \frac{\lambda}{n} \right)^v \omega^v \\ &\leq \sum_{v=0}^{\infty} \left(1 + \frac{x+\lambda}{u} + \frac{x\lambda}{n^2} \right)^v \cdot \omega^v \end{aligned}$$

Denotamus:

$$\frac{x+\lambda}{n} + \frac{x\lambda}{n^2} = \delta_n$$

qua significatione fit:

$$\Phi_{n-1} \leq \sum_{v=0}^{\infty} (1 + \delta_n)^v \omega^v = \frac{1}{1 - (1 + \delta_n) \omega}$$

ubi :

$$\lim \delta_n (n=\infty) = 0$$

Itaque est:

$$\frac{1}{\Phi_{n-1}} \gtrless 1 - (1 + \delta_n) \omega$$

ergo:

$$1 - \frac{1}{\Phi_{n-1}} \gtrless 1 - (1 - (1 + \delta_n) \omega)$$

$$\gtrless (1 + \delta_n) \omega$$

unde sequitur:

$$\frac{\Phi^n}{\Phi_{n-1}} \gtrless \frac{1 + \delta_n}{\left(\frac{x}{n} + 1\right)\left(\frac{\lambda}{n} + 1\right)}$$

Idecirco pro limite quotientis 3. invenitur:

$$\lim \left[\frac{1 + \delta_n}{\left(\frac{x}{n} + 1\right)\left(\frac{\lambda}{n} + 1\right)} \cdot \frac{\left(\frac{x}{n} + 1\right)\lambda + \left(\frac{\lambda}{n} + 1\right)x}{\left(\frac{x-1}{n} + 1\right)\lambda + \left(\frac{\lambda-1}{n} + 1\right)x} \right]$$

$$\left[\left(\frac{x-1}{n} + 1 \right) \left(\frac{\lambda-1}{n} + 1 \right) \left(1 - \frac{1}{n} \right) \omega \right]_{(n=\infty)} = \omega$$

Ergo convergit inaequationis 2. secundum membrum pro $0 < \omega < 1$
ac series ψ pro $0 < x < 1$.

§ 3.

De limite erroris quem serie a certo cifrarum numero
abrupta tenemus.

In iis qui in usu saepissime nobis occurrere solent casibus α β x quantitatum realium, si pro illis fractiones positivas verasque statuimus simpliciter limes erroris qui seriem ab n^{ta} cifra abrumpendo nascitur, definiri potest.

Generalis est ε expressio secundum 6. § 1., simplicitatis causa $a_0 = x = 1$ posito, haecce:

$$1. \varepsilon = (\alpha + n - 1)_v (\beta + n - 1)_v \cdot \frac{1}{n+1} x^{n+1} \left\{ \frac{(\alpha+n)(\beta-1) + (\beta+n)(\alpha-1)}{n+1} F_{n+1} + \right.$$

$$\left. + \frac{(\alpha+n+1)(\beta-1) + (\beta+n+1)(\alpha-1)}{n+2} \frac{(\alpha+n)(\beta+n)}{(v+1)(n+2)} F_{n+2} + \dots \right\}$$

qua in expressione cifrae fibula inclusae suppositione de α β x prolata omnes negativae, functiones F ut ex aequatione:

$$F_v = 1 + \frac{(\alpha+v)(\beta+v)}{(v+1)^2} x + \frac{(\alpha+v)(\alpha+v+1)(\beta+v)(\beta+v+1)}{(v+1)^2(v+2)^2} x^2 + \dots$$

statim perspicitur, cumtae minores sunt quam:

$$F_n < 1 + x + x^2 + \dots = \frac{1}{1-x}$$

Est igitur quum $\varepsilon = -\varepsilon'$ ponimus unaque in secundum aequationis 1. membrum cifras positivas inducimus:

$$\begin{aligned} \varepsilon' &< (\alpha+n)_n (\beta+n)_n \cdot \frac{1}{n+1} \frac{x^{n+1}}{1-x} \left\{ \frac{(\alpha+n)(1-\beta) + (\beta+n)(1-\alpha)}{n+1} + \right. \\ &\quad \left. + \frac{(\alpha+n+1)(1-\beta) + (\beta+n+1)(1-\alpha)}{n+2} \cdot \frac{(\alpha+n)(\beta+n)}{(n+1)(n+2)} x + \dots \right\} \end{aligned}$$

qua in fibula primus cuiusque cifrae factor minor est quam : 2. cum enim α, β maxime = 1 fieri possint, est:

$$\frac{(\alpha+n+v)(1-\beta) + (\beta+n+v)(1-\alpha)}{n+v+1} \lesssim 2 - \alpha - \beta$$

ergo semper minor quam 2. Posito igitur cuiusvis ejusmodi cifrae loco numero illo vel magis inaequatio valet. Quo factore communi sejuncto in fibula series F_n remanet quam supra $< \frac{1}{1-x}$ invenimus.

Ergo observando:

$$(\alpha+n-1)_n (\beta+n-1)_n < 1$$

pro ε' evadit limes:

$$\varepsilon' < \frac{x^{n+1}}{(1-x)^2} \cdot \frac{2}{(n+1)^2}$$

c. Quibus in casibus singularibus simpliciores cifrae existant.

Pro singularibus x valoribus simpliciores cifrae evadere possunt quae hoc loco nominabuntur.

1. Primo quem jam priore in parte casum denotavimus absolvere volumus. Ibidem, § 4., si bini qui primum locum habent exponentes, non solum qui ad unum sed etiam qui ad alterum. trium punctorum singularium pertinent, inter se aequales fiebant, integralia logarithmica inveniebantur simpliciore forma quam quae e generali expressione nascuntur, postquam easdem conditions in eam induximus. Relatio ibidem denotata eo quod omne particulare integrale in generali continetur necesse est, fundita erat. Quare pro puncto $x = 0$ relatio oritur:

$$\int_0^1 (1-ux)^{-\alpha} (1-u)^{-\alpha} u^{\alpha-1} \log \frac{1}{u^2 x} du =$$

$$C_1 \int_0^1 (1-ux)^{-\alpha} (1-u)^{-\alpha} u^{\alpha-1} du +$$

$$+ C_2 \int_0^1 (1-ux)^{-\alpha} (1-u)^{-\alpha} u^{\alpha-1} \log \frac{u(1-u)x}{1-ux} du$$

C_1, C_2 constantibus qui comparando coefficientes x^0 et in parte logarithmica et alogarithmica sequente modo procedunt:

$$C_2 = -1$$

$$C_1 = -\frac{\int_0^1 (1-u)^{-\alpha} u^{\alpha-1} \log \frac{u}{1-u} du}{\int_0^1 (1-u)^{-\alpha} u^{\alpha-1} du}$$

Valoribus illis in aequationem praecedentem inductis fit:

$$\int_0^1 (1-u)^{-\alpha} u^{\alpha-1} du \cdot \int_0^1 (1-ux)^{-\alpha} (1-u)^{-\alpha} u^{\alpha-1} \log \frac{1-u}{u(1-ux)} du = \\ \int_0^1 (1-u)^{-\alpha} u^{\alpha-1} \log \frac{1-u}{u} du \cdot \int_0^1 (1-ux)^{-\alpha} (1-u)^{-\alpha} u^{\alpha-1} du$$

quae aequatio adhibitis recursionis formulis quas sub a. § 1 hujus partis statuimus: 3. 4. singulas x potestates comparando directe verificari potest.

2. Pro serie ex ψ nascente, conditione $\alpha = \beta$ forma expressionis fit simplicior: posito $a_0 = x = 1$ in hoc casu obtinetur:

$$\psi = F_0 + F_1 x + 2(\alpha-1) \left\{ \frac{\alpha^2(\alpha+1)}{1^2 \cdot 2} F_2 \frac{x^2}{2^2} + \right. \\ \left. + \frac{\alpha^2(\alpha+1)^2(\alpha+2)}{1^2 \cdot 2^2 \cdot 3} F_3 \frac{x^3}{3^2} + \dots \right\}$$

ubi functiones F formam habent quae attinet ad hunc casum.

3. Multo simplicior autem fit ratio, si α vel β negativo numero integro aequalia aestimantur, id quod pertinet ad casum in quo series gaussiana $F'(\alpha \beta 1 x)$ a certo cifrarum numero abrumptur ideoque functionem integrum repraesentat. Ita functiones F , usque ad cifram quae pro indice integrum illum numerum habet, non minus functiones integrae sunt: extra autem illam cifram simplex potestatum x series restat, id quod optime contendimus, si ad coefficientium legem 3. seriei ψ sub § 1 respicimus, quae fit pro $\beta = -n$:

$$a_{v+1}(v+1)^2 - a_v(v+\alpha)(v-n) = (-1)^v x (\alpha+v-1)_v (n-v+1)_v \\ \frac{(n-v)(1-x)-(n+1)(v+\alpha)}{v+1}$$

ex quo ab $n+1^{\text{ta}}$ cifra legem valere perspicitur:

$$a_{n+\mu+1} = a_{n+\mu} \cdot \frac{\mu(n+\mu+\alpha)}{(n+\mu+1)^2} \quad (\mu = 1, 2, 3, \dots)$$

cum ex n^{ta} et $n+1^{\text{ta}}$ cifra sequatur:

$$a_{n+1} = \frac{(-1)^n \alpha x}{(n+1)^2} \cdot (\alpha+n)_n$$

unde, cifra $G_{(x)}^n$ functionem integrum n^{ti} gradus significante, ψ forma esse debere patet:

$$\psi = G_{(x)}^n + \frac{(-1)^n \alpha x}{(n+1)^2} x + n_n x^{n+1} \left(1 + \frac{1 \cdot (\alpha+n+1)}{(n+2)^2} x + \frac{1 \cdot 2 \cdot (\alpha+n+1) (\alpha+n+2)}{(n+2)^2 (n+3)^2} x^2 + \dots \right)$$

In casu quam simplicissimo:

$$n = 1$$

nascitur præv $a_0 = 1$ $x = -1$ series haecce:

$$\begin{aligned} \psi = 1 + (2x-1) x + \frac{x(\alpha+1)}{2^2} x^2 + \frac{1 \cdot \alpha(\alpha+1)(\alpha+2)}{2^2 \cdot 3^2} x^3 \\ + \frac{1 \cdot 2 \cdot \alpha(\alpha+1)(\alpha+2)(\alpha+3)}{2^2 \cdot 3^2 \cdot 4^2} x^4 + \dots \end{aligned}$$

THESES.

1. Tractationem calculi differentialis integralium definitorum theoria anteire debet.
 2. Qui in literis vult versari carere ille non potest rerum logicarum cognitione.
 3. Usus communis verum pro omni in literis mathematicis progressu fundamentum praebet.
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V I T A.

Natus sum, Constantinus Winterberg Corbachii, capitalis principatus Gualdeccensis, anno 1841. die XXXI. mensis Martii patre Carolo praesidente ministerii gualdeccensis quem jam defunctum lugeo, matre Emma, e gente Rhode, quam adhuc superstitem veneror. Fidei addictus sum evangelicae. Progymnasium Arolsiensem postquam absolveram gymnasium Corbaccense sub auspiciis Ill. Curtze ab anno 1853 usque ad 1861 frequentavi. Maturitatis testimonio munitus, postquam per novem annos terra marique militia functus eram, ossis fracti causa ad dimissionem mihi rogandam commotus, vere 1870 Universitatem literariam Berolinensem adii qua, militari ordini cum jam addictus eram numero civium non quidem adscriptus, permissione tamen a Rectore Magnifico du-bois-Reymond benevolenter mihi data, per octo semestria disserentes audivi viros Ill: Dove, Foerster, Helmholtz, Harms, Kronecker, Kummer, Qincke, Thomé, Weierstrass. Exercitationibus seminarii mathematici quas moderantur Ill. Kummer et Weierstrass per quinque semestria interfui. Quibus omnibus viris optime de me meritis, maximeque Ill. Kronecker, Kummer, Weierstrass, qui insignem semper benevolentiam in me contulerunt, gratias ago maximas.
