

459
DE
**AEQUATIONIBUS NONNULLIS
DIFFERENTIALIBUS.**

**DISSERTATIO
INAUGURALIS MATHEMATICA
QUAM
CONSENSU ET AUCTORITATE
ORDINIS AMPLISSIMI PHILOSOPHORUM
IN
ALMA LITERARUM
UNIVERSITATE FRIDERICA GUILELMA
UT SUMMI
IN PHILOSOPHIA HONORES
RITE SIBI CONCEDANTUR
DIE XXX. M. APRILIS A. MDCCXLII
H. I. L. S.
PUBLICE DEFENDET
AUCTOR
HENRICUS EDUARDUS HEINE
BEROLINENSIS.**

PARTES CONTRARIAS TUEBUNTUR:

G. HEINE, theol. lic.
W. GALLENKAMP, phil. cand.
L. KRONECKER, phil. stud.

BEROLINI
TYPIS ACADEMICIS.



VIRO

ILLUSTRISSIMO DOCTISSIMO HUMANISSIMO

G. LEJEUNE-DIRICHLET

PHILOS. DOCT. MATH. IN UNIVES. LITER. FRID. CUIL. PROF. PUBL. ORDIN. MAGIST. IN SCHOL. MILIT.
ACAD. REG. LITER. BEROL. SOD. ACAD. PETROPOL. ET PARIS. SOC. CORRESP.
ORD. DE AQUIL. RUBRA EQUIT. IN CLASSE QUARTA

PRAECEPTORI DILECTISSIMO USQUE VENERANDO

HASCE PRIMITIAS

D. D. D.

AUCTOR.

§. 1.

Disquisitiones nostrae vertuntur in investigatione aequationis differentialis, cui subiecta est fractio

$$R = \frac{1}{V(x-a)^2 + (y-b)^2 + (z-c)^2} \quad (1)$$

quantitatibus x, y, z sive realibus sive imaginariis. Differentiatione facilima invenitur

$$\frac{d^2 R}{d x^2} + \frac{d^2 R}{d y^2} + \frac{d^2 R}{d z^2} = 0. \quad (2)$$

Posito ergo

$$\begin{aligned} x &= \alpha \sqrt{1-\xi^2}, & y &= \alpha \xi \sqrt{1-\omega^2}, & z &= \alpha \xi \omega \\ a &= \sqrt{1-\xi^2}, & b &= \xi \sqrt{1-\omega^2}, & c &= \xi \omega \end{aligned} \quad (3)$$

aequationem (2) sic licet transformare, ut non amplius contineat dx, dy, dz . Est enim

$$\frac{dR}{dz} = \frac{dR}{d\alpha} \cdot \frac{d\alpha}{dz} + \frac{dR}{d\xi} \cdot \frac{d\xi}{dz} + \frac{dR}{d\omega} \cdot \frac{d\omega}{dz},$$

denique

$$\begin{aligned} \frac{d^2 R}{dz^2} &= \frac{d^2 R}{d\alpha^2} \left[\left(\frac{d\alpha}{dz} \right)^2 \right] + \frac{d^2 R}{d\xi^2} \left[\left(\frac{d\xi}{dz} \right)^2 \right] + \frac{d^2 R}{d\omega^2} \left[\left(\frac{d\omega}{dz} \right)^2 \right] + \frac{dR}{d\alpha} \cdot \frac{d^2 \alpha}{dz^2} + \frac{dR}{d\xi} \cdot \frac{d^2 \xi}{dz^2} + \frac{dR}{d\omega} \cdot \frac{d^2 \omega}{dz^2} + \\ &+ 2 \left\{ \frac{d^2 R}{d\alpha \cdot d\xi} \cdot \frac{d\alpha}{dz} \cdot \frac{d\xi}{dz} + \frac{d^2 R}{d\alpha \cdot d\omega} \cdot \frac{d\alpha}{dz} \cdot \frac{d\omega}{dz} + \frac{d^2 R}{d\xi \cdot d\omega} \cdot \frac{d\xi}{dz} \cdot \frac{d\omega}{dz} \right\}. \end{aligned} \quad (4)$$

Litera z mutata in y et x similes prodeunt aequationes, ita ut (2) induat formam

$$\begin{aligned} \frac{d^2 R}{d\alpha^2} \left[\left(\frac{d\alpha}{dy} \right)^2 + \left(\frac{d\alpha}{dx} \right)^2 \right] &+ \frac{d^2 R}{d\xi^2} \left[\left(\frac{d\xi}{dy} \right)^2 + \left(\frac{d\xi}{dx} \right)^2 \right] + \frac{d^2 R}{d\omega^2} \left[\left(\frac{d\omega}{dy} \right)^2 + \left(\frac{d\omega}{dx} \right)^2 \right] + \\ &+ \frac{dR}{d\alpha} \left[\frac{d^2 \alpha}{dy^2} + \frac{d^2 \alpha}{dx^2} + \frac{d^2 \alpha}{dz^2} \right] + \frac{dR}{d\xi} \left[\frac{d^2 \xi}{dy^2} + \frac{d^2 \xi}{dx^2} + \frac{d^2 \xi}{dz^2} \right] + \frac{dR}{d\omega} \left[\frac{d^2 \omega}{dy^2} + \frac{d^2 \omega}{dx^2} + \frac{d^2 \omega}{dz^2} \right] + \\ &+ 2 \frac{d^2 R}{d\alpha \cdot d\xi} \left[\frac{d\alpha}{dy} \cdot \frac{d\xi}{dy} + \frac{d\alpha}{dx} \cdot \frac{d\xi}{dx} + \frac{d\alpha}{dz} \cdot \frac{d\xi}{dz} \right] + 2 \frac{d^2 R}{d\alpha \cdot d\omega} \left[\frac{d\alpha}{dy} \cdot \frac{d\omega}{dy} + \frac{d\alpha}{dx} \cdot \frac{d\omega}{dx} + \frac{d\alpha}{dz} \cdot \frac{d\omega}{dz} \right] + \\ &+ 2 \frac{d^2 R}{d\xi \cdot d\omega} \left[\frac{d\xi}{dy} \cdot \frac{d\omega}{dy} + \frac{d\xi}{dx} \cdot \frac{d\omega}{dx} + \frac{d\xi}{dz} \cdot \frac{d\omega}{dz} \right] = 0. \quad (5) \end{aligned}$$

Ex aequationibus (3) statim valores emanant tam $\frac{dw}{dx} = 0$ quam $\frac{d^2w}{dx^2} = 0$; nec sine calculo valores $\frac{d\alpha}{dx}$, $\frac{d^2\alpha}{dx^2}$, $\frac{d\beta}{dx}$, etc. reperiuntur. Differentiatione enim secundum x instituta proferuntur aequationes tres

$$(A) \quad \begin{cases} 1 = \sqrt{1-\beta^2} \cdot \frac{d\alpha}{dx} - \frac{\alpha\beta}{\sqrt{1-\beta^2}} \cdot \frac{d\beta}{dx} \\ 0 = \sqrt{1-w^2} \cdot \beta \cdot \frac{d\alpha}{dx} + a\sqrt{1-w^2} \cdot \frac{d\beta}{dx} - \frac{\alpha \cdot \beta \cdot w}{\sqrt{1-w^2}} \frac{dw}{dx} \\ 0 = \beta w \frac{d\alpha}{dx} + \alpha w \frac{d\beta}{dx} + \alpha \beta \frac{dw}{dx} \end{cases}$$

quae (cum sit $\frac{dw}{dx} = 0$) statim reducuntur ad duas

$$\begin{aligned} 1 &= \sqrt{1-\beta^2} \cdot \frac{d\alpha}{dx} - \frac{\alpha\beta}{\sqrt{1-\beta^2}} \cdot \frac{d\beta}{dx} \\ 0 &= \beta \frac{d\alpha}{dx} + \alpha \frac{d\beta}{dx} \end{aligned}$$

e quibus perinde $\frac{d\alpha}{dx}$ atque $\frac{d\beta}{dx}$ invenire possumus. Differentiatione secundum x iterum facta tam $\frac{d^2\alpha}{dx^2}$ quam $\frac{d^2\beta}{dx^2}$ reperimus. Differentialia quantitatum α , β , w secundum y et z evadunt ex aequationibus

$$(B) \quad \begin{cases} 0 = \sqrt{1-\beta^2} \cdot \frac{d\alpha}{dy} - \frac{\alpha\beta}{\sqrt{1-\beta^2}} \cdot \frac{d\beta}{dy} \\ 1 = \beta \sqrt{1-w^2} \cdot \frac{d\alpha}{dy} + a\sqrt{1-w^2} \cdot \frac{d\beta}{dy} - \frac{\alpha \cdot \beta \cdot w}{\sqrt{1-w^2}} \frac{dw}{dy} \\ 0 = \beta w \frac{d\alpha}{dy} + \alpha w \frac{d\beta}{dy} + \alpha \beta \frac{dw}{dy} \end{cases}$$

et

$$(C) \quad \begin{cases} 0 = \sqrt{1-\beta^2} \cdot \frac{d\alpha}{dz} - \frac{\alpha\beta}{\sqrt{1-\beta^2}} \cdot \frac{d\beta}{dz} \\ 0 = \beta \sqrt{1-w^2} \cdot \frac{d\alpha}{dz} + a\sqrt{1-w^2} \cdot \frac{d\beta}{dz} - \frac{\alpha \cdot \beta \cdot w}{\sqrt{1-w^2}} \frac{dw}{dz} \\ 1 = \beta w \frac{d\alpha}{dz} + \alpha w \frac{d\beta}{dz} + \alpha \beta \frac{dw}{dz}. \end{cases}$$

Calculo finito valor obtinetur

$$\left(\frac{d\alpha}{dz}\right)^2 + \left(\frac{d\alpha}{dy}\right)^2 + \left(\frac{d\alpha}{dx}\right)^2 = 1.$$

Reliquae expressiones in (5) contentae statim prodeunt, ubi comparaveris aeq. (5) cum

$$\frac{d^2R}{d\zeta^2}(1-\zeta^2) + \frac{dR}{d\zeta} \frac{1-2\zeta^2}{\zeta} + \frac{1}{\zeta^2} \cdot \frac{d^2R}{d\omega^2}(1-\omega^2) - \frac{\omega}{\zeta^2} \cdot \frac{dR}{d\omega} + \alpha \frac{d^2(\alpha R)}{d\alpha^2} = 0. \quad (6)$$

Uberius autem de coordinatarum transformatione hoc loco egimus, cum tales res in dissertatione nostra saepius obviam fiant.

§. 2.

Fingamus functionem R esse explicatam in seriem ascendentem secundum potestates ipsius α ita, ut sit

$$R = P_0 + P_1 \alpha + P_2 \alpha^2 + \dots + P_n \alpha^n + \dots$$

denotante P_n coefficientem non pendentem ex quantitate α ; cum sit adhibitis valoribus (3)

$$R = \frac{1}{\sqrt{1-2\alpha t+\alpha^2}},$$

posito brevitatis causa

$$t = \sqrt{1-\zeta^2} \sqrt{1-\zeta_i^2} + \zeta \zeta_i (\omega \omega_i + \sqrt{1-\omega^2} \sqrt{1-\omega_i^2}),$$

accipies, ut satis constat, valorem

$$P_n = \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{1 \cdot 2 \cdot 3 \dots n} \left\{ t^n - \frac{n \cdot (n-1)}{2 \cdot (2n-1)} t^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot (2n-1)(2n-3)} t^{n-4} - \dots \right\} \quad (7)$$

neque minus

$$P_n = 1 \text{ pro } t = 1. \quad (7)^*$$

Cum saepenumero pro t quantitates aliae sint ponendae, variabilem adiiciamus et ita quidem, ut aequiparemus seriem (7) multiplicatam in $\frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{1 \cdot 2 \cdot 3 \dots n}$ functioni $P_n[t]$. Facile intelligitur, cum R sive $\sum_{n=0}^{n=\infty} \{\alpha^n P_n\}$ data sit aequatione (6), functionem P_n subiectam esse aequationi

$$\frac{d^2P_n}{d\zeta^2}(1-\zeta^2) + \frac{dP_n}{d\zeta} \frac{1-2\zeta^2}{\zeta} + \frac{1}{\zeta^2} \frac{d^2P_n}{d\omega^2}(1-\omega^2) - \frac{\omega}{\zeta^2} \frac{dP_n}{d\omega} + n(n+1)P_n = 0. \quad (8)$$

Primo casum speciale contemplamur, posito quidem $\omega = \sin \phi$; tunc erit

$$(1-\omega^2) \frac{d^2P_n}{d\omega^2} = (1-\omega^2) \frac{d^2P_n}{d\phi^2} \left(\frac{d\phi}{d\omega} \right)^2 + (1-\omega^2) \frac{dP_n}{d\phi} \frac{d^2\phi}{d\omega^2} \\ - \omega \frac{dP_n}{d\omega} = -\omega \frac{dP_n}{d\phi} \frac{d\phi}{d\omega}.$$

Deinde habemus

$$\frac{d\phi}{d\omega} = \frac{1}{\cos \phi} = \frac{1}{\sqrt{1-\omega^2}}$$

$$\frac{d^2\phi}{d\omega^2} = \frac{\omega}{(1-\omega^2)^{\frac{3}{2}}}$$

unde

$$(1-\omega^2) \frac{d^2 P_n}{d\omega^2} - \omega \frac{dP_n}{d\omega} = \frac{d^2 P_n}{d\phi^2},$$

quare positio

$$(9) \quad t_1 = \sqrt{1-\xi^2} \sqrt{1-\xi_1^2} + \xi \xi_1 (\cos \phi \sqrt{1-\omega_1^2} + \sin \phi \omega_1)$$

prodit aequatio

$$(10) \quad \frac{d^2 P_n[t_1]}{d\xi^2} (1-\xi^2) + \frac{dP_n[t_1]}{d\xi} \frac{1-2\xi^2}{\xi} + \frac{1}{\xi^2} \frac{d^2 P_n[t_1]}{d\phi^2} + n(n+1) P_n[t_1] = 0.$$

Advocato theoremate, quamecumque functionem continuam unius variabilis ϕ a $\phi = 0$ usque ad $\phi = 2\pi$ una ratione in seriem explicari posse progradientem secundum sinus cosinusque multiplorum anguli ϕ , videmus coefficientem P_n hanc speciem induentem:

$$(11) \quad P_n[t_1] = g_{n,0} V_{n,0} + (g_{n,1} \cos \phi V_{n,1} + h_{n,1} \sin \phi W_{n,1}) + \dots \\ + (g_{n,n} \cos n\phi V_{n,n} + h_{n,n} \sin n\phi W_{n,n})$$

designantibus V et W functiones ipsius ϕ ; g et h functiones ipsarum ξ et ω_1 . Fieri non potest, ut $P_n[t_1]$ altius quam n^{th} multiplum ipsius ϕ contineat, cum sit functio integra n^{th} gradus ipsorum $\sin \phi$ et $\cos \phi$; $\sin \phi$ et $\cos \phi$ in series explicari queunt progradientes secundum sinus et cosinus multiplorum anguli ϕ , neque altius tenent quam n^{th} . Functione $P_n[t_1]$ subiecta aequationi (10) perspicitur facile, functiones V et W subiectas esse aequationibus differentialibus

$$(12) \quad \begin{cases} \frac{d^2 V_{n,m}}{d\xi^2} (1-\xi^2) + \frac{dV_{n,m}}{d\xi} \frac{1-2\xi^2}{\xi} + V_{n,m} \left(n \cdot \overline{n+1} - \frac{m^2}{\xi^2} \right) = 0 \\ \frac{d^2 W_{n,m}}{d\xi^2} (1-\xi^2) + \frac{dW_{n,m}}{d\xi} \frac{1-2\xi^2}{\xi} + W_{n,m} \left(n \cdot \overline{n+1} - \frac{m^2}{\xi^2} \right) = 0 \end{cases}$$

quarum duo integralia particularia sint $P_{n,m}$ et $Q_{n,m}$. Denotantibus $a_{n,m}$, $b_{n,m}$, $A_{n,m}$, $B_{n,m}$ constantes, i. e. quantitates non suspensas ex ξ , valores generales functionum V et W speciem hancce accipient:

$$V_{n,m} = a_{n,m} P_{n,m} + b_{n,m} Q_{n,m}$$

$$W_{n,m} = A_{n,m} P_{n,m} + B_{n,m} Q_{n,m}$$

unde aeq. (11)

$$\left. \begin{aligned} P_n[t_1] &= g_{n,0} P_{n,0} + (g_{n,1} \cos \phi + h_{n,1} \sin \phi) P_{n,1} + \dots \\ &\quad + (g_{n,n} \cos n\phi + h_{n,n} \sin n\phi) P_{n,n} + \\ &+ G_{n,0} Q_{n,0} + (G_{n,1} \cos \phi + H_{n,1} \sin \phi) Q_{n,1} + \dots \\ &\quad + (G_{n,n} \cos n\phi + H_{n,n} \sin n\phi) Q_{n,n}, \end{aligned} \right\} \quad (11)^*$$

designantibus g , h , G , H quantitates non dependentes a ϕ et ϕ , continentes tam g , quam w , quarum indolem perinde ac functionum P et Q deinceps investigabimus, nonnulla theorematum adiicientes.

§. 3.

Solutiones igitur partiales aequationum nostrarum differentialium (12) investigantes per series integramus, ascendentes illas, si modulus ipsius ρ (sive ρ , quantitate ρ reali) unitate non sit maior, descendentes autem, si ρ unitate non sit minor; pro $\rho = 1$ hae series multiplicatae in constantes rite creatas coincident.— Cl. EULER (*) uberioris formulas quasdam generales evolvit ut obiter nobis liceat tractare casum earum specialem. Statuto igitur

$$V_{n,m} = a_\lambda \rho^\lambda + a_{\lambda-2} \rho^{\lambda-2} + \dots = \sum_{p=0}^{p=\infty} (a_{\lambda-2p} \rho^{\lambda-2p})$$

differentiatione consequimur tam

$$\frac{dV_{n,m}}{d\rho} = \sum_{p=0}^{p=\infty} a_{\lambda-2p} (\lambda-2p) \rho^{\lambda-2p-1}$$

quam

$$\frac{d^2 V_{n,m}}{d\rho^2} = \sum_{p=0}^{p=\infty} a_{\lambda-2p} (\lambda-2p) (\lambda-2p-1) \rho^{\lambda-2p-2}.$$

Hisce valoribus positis in (12) invenitur

$$\begin{aligned} a_{\lambda-2p} \{(\lambda-2p)(\lambda-2p-1) + 2(\lambda-2p) - n(n+1)\} \\ = a_{\lambda-2p+2} \{(\lambda-2p+2)(\lambda-2p+1) + (\lambda-2p+2) - m^2\} \end{aligned}$$

sive

$$a_{\lambda-2p} = a_{\lambda-2p+2} \frac{(\lambda-2p+2+m)(\lambda-2p+2-m)}{(\lambda-2p)(\lambda-2p+1)-n(n+1)},$$

porro, ut generaliter loquar,

$$a_\lambda = 0, \text{ unde } a_{\lambda-2p} = 0.$$

Ne amplius habeamus $a_{\lambda-2p} = 0$, statuimus

$$(\lambda - 2p)(\lambda - 2p + 1) - n(n+1) = 0$$

unde valores prodeunt sive $(\lambda - 2p) = n$ sive $(\lambda - 2p) = -(n+1)$. Tunc erit, quia omnes termini $a_\lambda, a_{\lambda-1}, a_{\lambda-2}, \dots, a_{\lambda-2p+2}$ evanescunt, $a_{\lambda-2p} = \frac{0}{0}$. Itaque ubi posueris $a_{\lambda-2p} = k$ in altero casu, in altero $= k_1$, resultabit

$$a_n = k, \quad a_{-(n+1)} = k_1$$

eademque ratione

$$a_{n-2r} = a_{n-2r+2} \frac{(n-2r+m+2)(n-2r-m+2)}{(n-2r)(n-2r+1)-n(n+1)}$$

$$a_{-(n+1+2r)} = a_{-(n+1+2r-2)} \frac{(n-1+2r-m)(n-1+2r+m)}{(n+1+2r)(n+2r)-n(n+1)}.$$

Quoniam est

$$z(z+1) - n(n+1) = (z-n)(z+n+1)$$

hae aequationes subeunt formam

$$a_{n-2r} = a_{n-2r+2} \frac{\left(r - \frac{n+m+2}{2}\right) \left(r - \frac{n-m+2}{2}\right)}{r \left(r - \frac{2n+1}{2}\right)}$$

$$a_{-(n+1+2r)} = a_{-(n+1+2r-2)} \frac{\left(r + \frac{n-1-m}{2}\right) \left(r + \frac{n-1+m}{2}\right)}{r \left(r + \frac{2n+1}{2}\right)},$$

unde sequitur

$$\begin{aligned} V_{n,m} &= k \varrho^n \left\{ 1 - \frac{\left(\frac{n+m}{2}\right)\left(\frac{n-m}{2}\right)}{1 \cdot \left(\frac{2n-1}{2}\right)} \varrho^{-2} + \frac{\left(\frac{n+m}{2}\right)\left(\frac{n+m-1}{2}\right)\left(\frac{n-m}{2}\right)\left(\frac{n-m-1}{2}\right)}{1 \cdot 2 \cdot \left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)} \varrho^{-4} - \dots \right\} + \\ &+ k_1 \varrho^{-(n+1)} \left\{ 1 + \frac{\left(\frac{n+1-m}{2}\right)\left(\frac{n+1+m}{2}\right)}{1 \cdot \left(\frac{2n-1}{2}\right)} \varrho^{-2} + \frac{\left(\frac{n+1-m}{2}\right)\left(\frac{n+3-m}{2}\right)\left(\frac{n+1+m}{2}\right)\left(\frac{n+3+m}{2}\right)}{1 \cdot 2 \cdot \left(\frac{2n+3}{2}\right)\left(\frac{2n+5}{2}\right)} \varrho^{-4} + \dots \right\} \end{aligned}$$

sive posito, ut fieri solet,

$$F(\alpha, \beta, \gamma, x) = 1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)} x^2 + \dots$$

$$(13) \quad \begin{aligned} V_{n,m} &= k \varrho^n F\left(-\frac{n+m}{2}, -\frac{n-m}{2}, -\frac{2n-1}{2}, \varrho^{-2}\right) + \\ &+ k_1 \varrho^{-(n+1)} F\left(\frac{n+1-m}{2}, \frac{n+1+m}{2}, \frac{2n+3}{2}, \varrho^{-2}\right). \end{aligned}$$

Nec est praetermittendum, utrumque seriem vel pro $\rho = 1$ convergere, cum $\alpha + \beta - \gamma$ sit quantitas negativa (*).

§. 4.

Exhibitae a Cl. KUMMER (**) relationes inter illas series hypergeometricas, quarum elementa differunt, haud inutiles erunt ad seriem nostram in formam aptiorem redigendam; conditio, ut ρ unitate sit minor, evanescet, quippe cum series altera evadat finita, altera integrali exprimatur, quod aequo secundum ascendentes ac secundum descendentes protestates ipsius ρ explicari potest.

Dedit Vir Cl. formulas hasce

$$(n^o. 49) F(\alpha, \beta, \alpha + \beta + \frac{1}{2}, x) = F\left(2\alpha, 2\beta, \alpha + \beta + \frac{1}{2}, \frac{1 - \sqrt{1-x}}{2}\right)$$

$$(n^o. 55) F\left(\alpha, \beta, \frac{\alpha + \beta + 1}{2}, x\right) = (1-2x)^{-\alpha} F\left(\frac{\alpha}{2}, \frac{\alpha+1}{2}, \frac{\alpha+\beta+1}{2}, \frac{4x^2-4x}{4x^2-4x+1}\right).$$

Quibus aequationibus adhibitis primum mutatur

$$F\left(-\frac{n+m}{2}, -\frac{n-m}{2}, -\frac{2n-1}{2}, \frac{\xi - \sqrt{\xi^2 - 1}}{2}\right) \text{ cum } F\left(-\frac{n+m}{2}, -\frac{n-m}{2}, -\frac{2n-1}{2}, \xi^{-2}\right),$$

deinde ope aequationis (n^o. 55) aequationes duas accipimus, si quidem primum pro α , tunc pro β valorem $-(n+m)$ substituimus,

$$\left. \begin{aligned} F\left(-\frac{n+m}{2}, -\frac{n-m}{2}, -\frac{2n-1}{2}, \xi^{-2}\right) &= \left(\frac{\sqrt{\xi^2 - 1}}{\xi}\right)^{n+m} F\left(-\frac{n+m}{2}, -\frac{n+m-1}{2}, -\frac{2n-1}{2}, \frac{1}{1-\xi^2}\right) \\ F\left(-\frac{n+m}{2}, -\frac{n-m}{2}, -\frac{2n-1}{2}, \xi^{-2}\right) &= \left(\frac{\sqrt{\xi^2 - 1}}{\xi}\right)^{n-m} F\left(-\frac{n-m}{2}, -\frac{n-m-1}{2}, -\frac{2n-1}{2}, \frac{1}{1-\xi^2}\right). \end{aligned} \right\} (14)$$

Functionem $F\left(-\frac{n+m}{2}, -\frac{n-m}{2}, -\frac{2n-1}{2}, \xi^{-2}\right)$ multiplicatam in ξ^n esse solutionem particularem aequationis (12), iam vidimus (13): quare $\kappa \xi^n F\left(-\frac{n+m}{2}, -\frac{n-m}{2}, -\frac{2n-1}{2}, \xi^{-2}\right)$, significante κ constantem arbitriam, aequationem eandem solvet. Hanc solutionem cum aequiparaveris literae $P_{n,m}$, erit $Q_{n,m}$ (sive altera solutio) ea pars aequationis (13), quae est multiplicata in k_1 ; huic quoque valori constantem arbitriam λ adiiciamus, ita ut sit $\lambda \xi^{-(n+1)} F\left(\frac{n+1+m}{2}, \frac{n+1-m}{2}, \frac{2n+3}{2}, \xi^{-2}\right) = Q_{n,m}$. Interdum ipsis

(*) Disquisitiones generales circ. ser. infin. auctore Gauss p. 19.

(**) Crelle, Journal für Mathematik. Vol. XV.

$P_{n,m}$ et $Q_{n,m}$ variabilem addimus, et ita quidem, ut posito $\kappa = \frac{1}{(-1)^{\frac{n-m}{2}}}$ sit

$$(-1)^{\frac{n-m}{2}} P_{n,m}[\sqrt{1-\varrho^2}] = \varrho^n F\left(-\frac{n+m}{2}, -\frac{n-m}{2}, -\frac{2n-1}{2}, \varrho^{-2}\right) = \\ = (1-\varrho^2)^{\frac{n-m}{2}} \varrho^m (-1)^{\frac{n-m}{2}} F\left(-\frac{n-m}{2}, -\frac{n-m-1}{2}, -\frac{2n-1}{2}, \frac{1}{1-\varrho^2}\right)$$

sive

$$(15) \quad P_{n,m}[\sqrt{1-\varrho^2}] = \\ = \varrho^m \left\{ \sqrt{1-\varrho^2}^{n-m} - \frac{(n-m)(n-m-1)}{2 \cdot (2n-1)} \sqrt{1-\varrho^2}^{n-m-2} + \frac{(n-m)(n-m-1)(n-m-2)(n-m-3)}{2 \cdot 4 \cdot (2n-1)(2n-3)} \sqrt{1-\varrho^2}^{n-m-4} - \text{etc.} \right\}.$$

$Q_{n,m}$ sine opera integrali exprimi licet (*) sic, ut sit

$$(16) \quad Q_{n,m}[\sqrt{1-\varrho^2}] = \varrho^{-m} \int_0^1 u^{\frac{n+m-1}{2}} (1-u)^{\frac{n-m}{2}} (\varrho^2 - u)^{-\frac{n-m+1}{2}} du,$$

statuta constante

$$\lambda = \frac{\Pi\left(\frac{n+m-1}{2}\right) \Pi\left(\frac{n-m}{2}\right)}{\Pi\left(\frac{n+1}{2}\right)}.$$

Calculum ipsum non adscribimus, quoniam mera substitutione formulae nostrae e generalibus deducuntur.

Notatu est dignum aequationem (7) sumto $t = \sqrt{1-\varrho^2}$ transire in hanc:

$$(17) \quad P_n[\sqrt{1-\varrho^2}] = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{1 \cdot 2 \cdot 3 \cdots n} P_{n,0}[\sqrt{1-\varrho^2}]$$

unde repetimus (7)*

$$(18) \quad P_{n,0}[1] = \frac{1 \cdot 2 \cdot 3 \cdots n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}.$$

Docet formula (15) esse

$$(18)^* \quad P_{n,m}[1] = 0 \quad (\text{casus } m=0 \text{ debet excipi})$$

unde concludimus esse $P_{n,m}[\sqrt{1-\varrho^2}]$, statuto $\varrho=0$, quantitatem finitam sive 0.

Formula (16), utpote quae tam secundum ascendentes quam secundum descendentes potestates ipsius ϱ explicari potest, non est inepta, valore ϱ sive superante sive non superante unitatem. Pro $\varrho=0$ est $Q_{n,m}=\infty$ cum ϱ^{-m} sit $= 0^{-m}=\infty$; pro $\varrho=0$, $m=0$ functio $Q_{n,m}$ valorem suscipit

(*) Cf. opus supra laudatum Cl. Kummer p. 142. Pro valore $m < n$ quantitas $\gamma - \beta$ est positiva.

$$(-1)^{-\frac{n+1}{2}} \int_0^1 u^{-1} (1-u)^{\frac{n}{2}} du = \infty$$

quod alia quoque ratione demonstrari potest, adhibitis quidem formulis a Cl. Euler (*) datis pro tali variabili, quae unitate sit minor.

§. 5.

Reversi ad contemplationes §. 2, ϱ , aequamus cifrae, unde (9) t , capit valorem $\sqrt{1-\varrho^2}$; functio $P_{n,0}$ aequat. (11)* (quae est $P_{n,0}[\sqrt{1-\varrho^2}]$) valorem $P_{n,0}[1] = \frac{1 \cdot 2 \cdot 3 \cdots n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$ (18); reliquae functiones $P_{n,m}$ valorem $P_{n,m}[1] = 0$ (18)*. Habemus idcirco pro (11)* aequationem

$$P_n[\sqrt{1-\varrho^2}] = g_{n,0} \frac{1 \cdot 2 \cdot 3 \cdots n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} + G_{n,0} Q_{n,0}[1] + (G_{n,1} \cos \phi + H_{n,1} \sin \phi) Q_{n,1}[1] + \cdots + (G_{n,n} \cos n\phi + H_{n,n} \sin n\phi) Q_{n,n}[1].$$

Omnis functiones $Q_{n,m}[1]$ sunt infinitae; $P_n[\sqrt{1-\varrho^2}]$ non involvit ϕ , unde argumentamur esse

$$G_{n,0} = 0, G_{n,1} = 0, \dots, G_{n,n} = 0; H_{n,1} = 0, \dots, H_{n,n} = 0.$$

Quare nobis est aequatio

$$g_{n,0} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{1 \cdot 2 \cdot 3 \cdots n} P_n[\sqrt{1-\varrho^2}] \quad (18)**$$

neque minus

$$\begin{aligned} P_n[\sqrt{1-\varrho^2} \sqrt{1-\varrho^2} + \varrho \varrho_1 (\cos \phi \sqrt{1-\omega_1^2} + \sin \phi \omega_1)] &= \\ &= g_{n,0} P_{n,0}[\sqrt{1-\varrho^2}] + (g_{n,1} \cos \phi + h_{n,1} \sin \phi) P_{n,1}[\sqrt{1-\varrho^2}] + \\ &\quad + \cdots + (g_{n,n} \cos n\phi + h_{n,n} \sin n\phi) P_{n,n}[\sqrt{1-\varrho^2}]. \end{aligned} \quad (19)$$

Iam statuto $\omega_1 = \sin \phi_1$ patet esse

$$\begin{aligned} P_n[\sqrt{1-\varrho^2} \sqrt{1-\varrho^2} + \varrho \varrho_1 \cos(\phi - \phi_1)] &= \\ &= \sum_{m=0}^{m=n} \{(g_{n,m} \cos m\phi + h_{n,m} \sin m\phi) P_{n,m}[\sqrt{1-\varrho^2}]\}. \end{aligned} \quad (19)*$$

(*) Institutiones calculi integr. Vol. II. p. 184 et 188. Substitutionem ipsam nullis difficultatibus implicitam non adscribimus, cum demonstratio exhibita sufficiat. Aequatio Cl. Euler $\lambda(\lambda-1)\alpha + \lambda c + f = 0$ transit in $\lambda = \pm m$, unde altera solutio involvat ϱ^{-m} , ϱ^{-m+2} , ..., id est, sit infinita pro $\varrho = 0$. Pro $m = 0$ evenit, ut binae series in unam coalescant, quare p. 188 laudavimus, ubi docetur, alteram solutionem logarithmum involvere ipsius ϱ , id est, esse infinitam sumto pro ϱ valore 0.

Mutato ϱ , cum ϱ , ϕ , cum ϕ neque minus ϱ cum ϱ , et ϕ cum ϕ , eadem ratione consequimur

$$(20) \quad P_n[\sqrt{1-\varrho^2} \sqrt{1-\xi_1^2} + \varrho \xi_1 \cos(\phi - \phi_1)] = \\ = \sum_{m=0}^{\infty} \{(\mathfrak{g}'_{n,m} \cos m\phi_1 + \mathfrak{h}'_{n,m} \sin m\phi_1) P_{n,m}[\sqrt{1-\xi_1^2}]\}$$

denotantibus $\mathfrak{g}'_{n,m}$ et $\mathfrak{h}'_{n,m}$ valores non suspensos ex ξ_1 , ϕ_1 . Cum hisce aequationibus tertiam coniungimus, ortam illam ex observatione §.2 de seriebus trigonometricis. Designante enim $\mathfrak{g}''_{n,m}$ valorem non pendentem e ($\phi - \phi_1$) accipimus

$$(21) \quad P_n[\sqrt{1-\varrho^2} \sqrt{1-\xi_1^2} + \varrho \xi_1 \cos(\phi - \phi_1)] = \sum_{m=0}^{\infty} \{\mathfrak{g}''_{n,m} \cos m(\phi - \phi_1)\} = \\ = \sum_{m=0}^{\infty} \{\mathfrak{g}''_{n,m} \cos m\phi \cos m\phi_1 + \mathfrak{g}''_{n,m} \sin m\phi \sin m\phi_1\}.$$

Adscito iterum theoremate §.2 concludimus coefficientes primum ipsorum $\cos m\phi$ in (19)* et (21), deinde ipsorum $\cos m\phi_1$ in (20) et (21) esse eosdem, unde primum summatio in (21) non usque ad $m = \infty$, sed ad $m = n$ extenditur, tunc quatuor prodeunt aequationes

$$(22) \quad \mathfrak{g}_{n,m} P_{n,m}[\sqrt{1-\xi_1^2}] = \mathfrak{g}''_{n,m} \cos m\phi_1; \quad \mathfrak{h}_{n,m} P_{n,m}[\sqrt{1-\xi_1^2}] = \mathfrak{g}''_{n,m} \sin m\phi_1$$

$$(23) \quad \mathfrak{g}'_{n,m} P_{n,m}[\sqrt{1-\xi_1^2}] = \mathfrak{g}''_{n,m} \cos m\phi; \quad \mathfrak{h}'_{n,m} P_{n,m}[\sqrt{1-\xi_1^2}] = \mathfrak{g}''_{n,m} \sin m\phi$$

sive

$$P_{n,m}[\sqrt{1-\xi_1^2}] \sqrt{\mathfrak{g}_{n,m}^2 + \mathfrak{h}_{n,m}^2} = P_{n,m}[\sqrt{1-\xi_1^2}] \sqrt{\mathfrak{g}'_{n,m}^2 + \mathfrak{h}'_{n,m}^2}.$$

Quia $\mathfrak{g}_{n,m}$ et $\mathfrak{h}_{n,m}$ tantum continent ξ_1 , ϕ_1 statuimus (declarantibus $a_{n,m}$ et $b_{n,m}$ constantes numericas)

$$\mathfrak{g}_{n,m} = a_{n,m} P_{n,m}[\sqrt{1-\xi_1^2}] \cos m\phi_1; \quad \mathfrak{h}_{n,m} = b_{n,m} P_{n,m}[\sqrt{1-\xi_1^2}] \sin m\phi_1.$$

Quantitates $a_{n,m}$ et $b_{n,m}$ non amplius tenere neque ξ_1 , neque ϕ_1 facile probatur. Posito enim

$$\mathfrak{g}'_{n,m} = a'_{n,m} P_{n,m}[\sqrt{1-\xi_1^2}] \cos m\phi; \quad \mathfrak{h}'_{n,m} = b'_{n,m} P_{n,m}[\sqrt{1-\xi_1^2}] \sin m\phi$$

habemus, neglectis indicibus

$$\sqrt{a^2 \cos^2 m\phi_1 + b^2 \sin^2 m\phi_1} = \sqrt{a'^2 \cos^2 m\phi + b'^2 \sin^2 m\phi}$$

deinde (22, 23)

$$a \cos m\phi \cos m\phi_1 = a' \cos m\phi \cos m\phi_1 \\ a = a'.$$

Quod a tantum continet ρ_1, ϕ_1 ; a' tantum ρ, ϕ ; a et a' valores sunt numerici. Deinde commonstrari potest esse $a = b$, cum ex aeq. (22) efficiatur et

$$\rho'' = aP[\sqrt{1-\rho^2}] P[\sqrt{1-\rho_1^2}] \text{ et } \rho'' = bP[\sqrt{1-\rho^2}] P[\sqrt{1-\rho_1^2}].$$

Quibus omnibus adhibitis reperitur

$$\begin{aligned} P_n[\sqrt{1-\rho^2}\sqrt{1-\rho_1^2} + \rho\rho_1 \cos(\phi - \phi_1)] &= \\ &= \sum_{m=0}^{n-1} \{a_{n,m} \cos m(\phi - \phi_1) P_{n,m}[\sqrt{1-\rho^2}] P_{n,m}[\sqrt{1-\rho_1^2}]\}. \end{aligned} \quad (24)$$

Posito

$$t_2 = \sqrt{1-\rho^2}\sqrt{1-\rho_1^2} + \rho\rho_1 \cos(\phi - \phi_1)$$

statim intelliges $P_n(t_2)$ eidem aequationi differentiali subiectam esse, variabilibus et ρ et ϕ , cui $P_n(t_1)$, i. e.

$$\frac{d^2 P_n[t_2]}{d\rho^2}(1-\rho^2) + \frac{dP_n[t_2]}{d\rho} \frac{1-2\rho^2}{\rho} + \frac{1}{\rho^2} \frac{d^2 P_n[t_2]}{d\phi^2} + n(n+1)P_n[t_2] = 0. \quad (25)$$

Casum speciale aequationum (24) et (25) hoc loco tractemus, quia saepe usu venit, i. e.

$$\rho = \sin \theta, \quad \rho_1 = \sin \theta_1$$

quibus valoribus statutis t_2 formam subit

$$t_2 = \cos \theta \cos \theta_1 + \sin \theta \sin \theta_1 \cos(\phi - \phi_1).$$

Fingamus in globo triangulum, cuius latera sint θ, θ_1, γ ; angulus lateribus θ et θ_1 inclusus sit $\phi - \phi_1$; tunc erit

$$\cos \gamma = \cos \theta \cos \theta_1 + \sin \theta \sin \theta_1 \cos(\phi - \phi_1). \quad (26)$$

Quibus valoribus positis eruitur ex aequatione (24)

$$P_n[\cos \gamma] = \sum_{m=0}^{n-1} \{a_{n,m} \cos m(\phi - \phi_1) P_{n,m}[\cos \theta] P_{n,m}[\cos \theta_1]\}. \quad (24)^*$$

Cum sit

$$\begin{aligned} \frac{dP_n}{d\rho} &= \frac{dP_n}{d\theta} \frac{d\theta}{d\rho} \\ \frac{d^2P_n}{d\rho^2} &= \frac{d^2P_n}{d\theta^2} \left(\frac{d\theta}{d\rho} \right)^2 + \frac{dP_n}{d\theta} \frac{d^2\theta}{d\rho^2} \\ \frac{1}{\cos \theta} &= \frac{d\theta}{d\rho} = \frac{1}{\sqrt{1-\rho^2}} \\ \frac{d^2\theta}{d\rho^2} &= \frac{\rho}{(1-\rho^2)^{3/2}} \end{aligned}$$

aeq. (25) induit speciem

$$\frac{d^2 P_n}{d\theta^2} + \frac{d P_n}{d\theta} \left\{ \frac{\rho}{V_1 - \rho^2} + \frac{1 - \rho^2}{\rho V_1 - \rho^2} \right\} + \frac{1}{\rho^2} \frac{d^2 P_n}{d\phi^2} + n(n+1)P_n = 0$$

sive

$$\frac{d^2 P_n}{d\theta^2} + \cot \theta \frac{d P_n}{d\theta} + \frac{1}{\sin^2 \theta} \frac{d^2 P_n}{d\phi^2} + n(n+1)P_n = 0$$

denique

$$(25)^* \frac{1}{\sin \theta} \frac{d \left(\sin \theta \frac{d P_n [\cos \gamma]}{d\theta} \right)}{d\theta} + \frac{1}{\sin^2 \theta} \frac{d^2 P_n [\cos \gamma]}{d\phi^2} + n(n+1)P_n [\cos \gamma] = 0.$$

Ex aequatione (24) sine difficultate deducitur formula

$$(27) \quad \frac{1}{\pi} \int_0^{2\pi} P_n [V_1 - \rho^2 V_1 - \rho_1^2 + \rho \rho_1 \cos(\phi - \phi_1)] \cos(m\phi) d\phi = \\ = a_{n,m} P_{n,m} [V_1 - \rho_1^2] P_{n,m} [V_1 - \rho_1^2] \cos m\phi_1,$$

pro $m = 0$ dimidia parte integralis sumta. Hoc casu constans $a_{n,0}$ haud ignota est; cum sit (18)**

$$g_{n,0} = \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{1 \cdot 2 \cdot 3 \dots n} P_n [V_1 - \rho_1^2] = \left(\frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{1 \cdot 2 \cdot 3 \dots n} \right)^2 P_{n,0} [V_1 - \rho_1^2] \text{ (aeq. 17)}$$

$$g_{n,m} = a_{n,m} P_{n,m} [V_1 - \rho_1^2] \cos m\phi_1$$

invenitur

$$a_{n,0} = \left(\frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{1 \cdot 2 \cdot 3 \dots n} \right)^2$$

unde

$$(27)^* \frac{1}{\pi} \int_0^{2\pi} P_n [V_1 - \rho^2 V_1 - \rho_1^2 + \rho \rho_1 \cos(\phi - \phi_1)] d\phi = \\ = a_{n,0} P_{n,0} [V_1 - \rho_1^2] P_{n,0} [V_1 - \rho_1^2] = P_n [V_1 - \rho^2] P_n [V_1 - \rho_1^2].$$

Integrale inter limites ϕ et 2π in duo integralia distribuimus, alterum intra limites ϕ et π , alterum intra limites π et 2π . In altero cum statueris

$$\phi = 2\pi - \psi$$

reperies limites π et ϕ ipsius ψ ; $\cos(\phi - \phi_1)$ transit in $\cos(2\pi - \psi - \phi_1)$ $= \cos(\psi + \phi_1)$; $d\phi$ in $-d\psi$. Restituto igitur ϕ pro ψ habebis formam alteram aequationis (27)*

$$(27)** P_n [V_1 - \rho^2] P_n [V_1 - \rho_1^2] = \\ = \frac{1}{2\pi} \int_0^\pi d\phi \{ P_n [V_1 - \rho^2 V_1 - \rho_1^2 + \rho \rho_1 \cos(\phi - \phi_1)] + P_n [V_1 - \rho^2 V_1 - \rho_1^2 + \rho \rho_1 \cos(\phi + \phi_1)] \}.$$

§. 6.

Speciali casu, tam ω quam ω_1 non assequente unitatem, finito ad aequationem (19)

$$P_n[t_1] = \sum_{m=0}^{n-1} \{(g_{n,m} \cos m\phi + b_{n,m} \sin m\phi) P_{n,m}[V_{1-\xi_1^2}]\}$$

revertimus, de quo casu neque minus de $P_n[t]$ breviter tantum dissere-mus, quoniam omnibus, quae antecedunt, perspectis harum functionum forma haud difficile cognoscitur. Mutato ξ cum ξ_1 , ξ_1 cum ξ accipimus

$$P_n[t_1] = \sum_{m=0}^{n-1} \{(g'_{n,m} \cos m\phi + b'_{n,m} \sin m\phi) P_{n,m}[V_{1-\xi^2}]\}$$

unde

$$g_{n,m} P_{n,m}[V_{1-\xi^2}] = g'_{n,m} P_{n,m}[V_{1-\xi_1^2}],$$

postremo

$$g_{n,m} = a_{n,m} P_{n,m}[V_{1-\xi_1^2}],$$

denotante $a_{n,m}$ valorem non dependentem ab ξ , ξ_1 , ϕ , continentem ω , et valores numericos. Habemus itaque

$$P_n[t_1] = \sum_{m=0}^{n-1} \{(a_{n,m} \cos m\phi + b_{n,m} \sin m\phi) P_{n,m}[V_{1-\xi_1^2}] P_{n,m}[V_{1-\xi_1^2}]\} \quad (28)$$

$b_{n,m}$ continent ω_1 et valores numericos; valores functionum a et b dati sunt aequatione differentiali statim definienda. Quo consilio dicimus functionem $P_n[t_1]$ secundum ξ_1 et ω_1 differentiatam eandem aeq. differentiale exhibere quam $P_n[t]$, cum sin ϕ nonnisi sit valor specialis ipsius ω . Igitur consequimur

$$\begin{aligned} \frac{d^2 P_n[t_1]}{d\xi_1^2} (1 - \xi_1^2) + \frac{d P_n[t_1]}{d\xi_1} \frac{1 - 2\xi_1^2}{\xi_1} + \frac{1}{\xi_1^2} \frac{d^2 P_n[t_1]}{d\omega_1^2} (1 - \omega_1^2) - \frac{\omega_1}{\xi_1^2} \frac{d P_n[t_1]}{d\omega_1} + \\ + n(n+1) P_n[t_1] = 0. \end{aligned}$$

Advocata formula (28) et aequatione differentiali ipsius $P_{n,m}[V_{1-\xi_1^2}]$, i. e. (indicibus neglectis)

$$\frac{d^2 P}{d\xi_1^2} (1 - \xi_1^2) + \frac{d P}{d\xi_1} \frac{(1 - 2\xi_1^2)}{\xi_1} + n(n+1) P = \frac{m^2}{\xi_1^2} P$$

resultat

$$\begin{aligned} 0 = \sum_{m=0}^{n-1} \left\{ P_{n,m}[V_{1-\xi_1^2}] P_{n,m}[V_{1-\xi_1^2}] \left(\left(\frac{d^2 a_{n,m}}{d\omega_1^2} \frac{(1 - \omega_1^2)}{\xi_1^2} + \frac{m^2 a_{n,m}}{\xi_1^2} - \frac{\omega_1}{\xi_1^2} \frac{d a_{n,m}}{d\omega_1} \right) \cos m\phi + \right. \right. \\ \left. \left. + \left(\frac{d^2 b_{n,m}}{d\omega_1^2} \frac{(1 - \omega_1^2)}{\xi_1^2} + \frac{m^2 b_{n,m}}{\xi_1^2} - \frac{\omega_1}{\xi_1^2} \frac{d b_{n,m}}{d\omega_1} \right) \sin m\phi \right) \right\} \end{aligned}$$

pro quovis valore ipsius ϕ , unde deducitur

$$(29) \quad \begin{cases} \frac{d^2 a_{n,m}}{d\omega_1^2} (\omega_1^2 - 1) + \omega_1 \frac{da_{n,m}}{d\omega_1} - m^2 a_{n,m} = 0 \\ \frac{d^2 b_{n,m}}{d\omega_1^2} (\omega_1^2 - 1) + \omega_1 \frac{db_{n,m}}{d\omega_1} - m^2 b_{n,m} = 0. \end{cases}$$

Neminem fugiat aequat. (29) statuto $\omega_1 = \sin \phi_1$ in formam abire

$$\frac{d^2 a_{n,m}}{d\phi_1^2} = -m^2 a_{n,m} \quad \text{et} \quad \frac{d^2 b_{n,m}}{d\phi_1^2} = -m^2 b_{n,m}$$

unde evadit

$$\begin{aligned} a_{n,m} &= c \sin m\phi + c_1 \sin m\phi \\ b_{n,m} &= c' \sin m\phi + c'_1 \sin m\phi. \end{aligned}$$

Introductis igitur symbolis

$$(30) \quad \begin{cases} [m, \omega_1]^t = 1 - \frac{m^2}{1,2} \omega_1^2 + \frac{(m+2)m^2(m-2)}{1,2,3,4} \omega_1^4 - \frac{(m+4)(m+2)m^2(m-2)(m-4)}{1,2,3,4,5,6} \omega_1^6 + \dots \\ [m, \omega_1] = m\omega_1 - \frac{(m+1)m(m-1)}{1,2,3} \omega_1^3 + \frac{(m+3)(m+1)m(m-1)(m-3)}{1,2,3,4,5} \omega_1^5 - \dots \end{cases}$$

habebis $[m, \sin \phi_1]^t = \cos m\phi_1$; $[m, \sin \phi_1] = \sin m\phi_1$. Differentiatione instituta facili negotio comprobatur $[m, \omega_1]^t$ et $[m, \omega_1]$ complere aequationes (29). Numerus membrorum seriei alterius (30) erit finitus pro pari m , alterius pro impari, quare has adiicimus formulas

$$(30)^* \quad \begin{cases} [m, \omega_1]^t = \sqrt{1-\omega_1^2} \left\{ 1 - \frac{(m+1)(m-1)}{1,2} \omega_1^2 + \frac{(m+3)(m+1)(m-1)(m-3)}{1,2,3,4} \omega_1^4 - \dots \right\} \\ [m, \omega_1] = \sqrt{1-\omega_1^2} \left\{ m\omega_1 - \frac{(m+2)m(m-2)}{1,2,3} \omega_1^3 + \frac{(m+4)(m+2)m(m-2)(m-4)}{1,2,3,4,5} \omega_1^5 - \dots \right\}. \end{cases}$$

Quas formulas veras esse facile monstratur statuto pro $a_{n,m}$ sive $b_{n,m}$ valore $\pm \sqrt{1-\omega_1^2}$ unde aequatio ipsius \pm prodit. Habemus ergo

$$P_n[t_1] = \sum_{m=0}^{n-n} \left\{ P_{n,m} [\sqrt{1-\xi^2}] P_{n,m} [\sqrt{1-\xi_1^2}] \{ \cos m\phi (c_{n,m} [m, \omega_1] + c'_{n,m} [m, \omega_1]^t) + \right. \\ \left. + \sin m\phi (C_{n,m} [m, \omega_1] + C'_{n,m} [m, \omega_1]^t) \} \right\}$$

Posito $\omega_1 = \sin \phi_1$, ubi comparationem institueris cum (24), invenitur, restituta litera ω_1

$$(31) \quad P_n[t_1] = \sum_{m=0}^{n-n} \left\{ c_{n,m} P_{n,m} [\sqrt{1-\xi^2}] P_{n,m} [\sqrt{1-\xi_1^2}] (\cos m\phi [m, \omega_1]^t + \sin m\phi [m, \omega_1]) \right\}$$

declarante $c_{n,m}$ eandem constantem quam $a_{n,m}$ in (24). In summatione conficienda tam valoribus (30) quam (30)* utemur.

Sine ullo labore ex praecedentibus valor ipsius $P_n[t]$ deduci potest. Sumto enim in (31) pro ω_1 valore $\omega\sqrt{1-\omega_1^2} - \omega_1\sqrt{1-\omega^2} = \omega_2$ accepies

$$P_n[\sqrt{1-\xi^2}\sqrt{1-\xi_1^2} + \varrho\varrho_1((\sqrt{1-\omega^2}\sqrt{1-\omega_1^2} + \omega\omega_1)\cos\phi + (\omega\sqrt{1-\omega_1^2} - \omega_1\sqrt{1-\omega^2})\sin\phi)] = \\ = \sum_{m=0}^{n-m} \{c_{n,m} P_{n,m}[\sqrt{1-\xi^2}] P_{n,m}[\sqrt{1-\xi_1^2}] (\cos m\phi[m, \omega_2] + \sin m\phi[m, \omega_2])\},$$

etiam si $\phi = 0$ (*); tunc erit

$$P_n[t] = \sum_{m=0}^{n-m} \{c_{n,m} P_{n,m}[\sqrt{1-\xi^2}] P_{n,m}[\sqrt{1-\xi_1^2}]\}$$

continente $c_{n,m}$ et ω et ω_1 . Aequatione differentiali ut antea adhibita reperitur

$$P_n[t] = \sum_{m=0}^{n-m} \{c_{n,m} P_{n,m}[\sqrt{1-\xi^2}] P_{n,m}[\sqrt{1-\xi_1^2}] ([m, \omega_1]'[m, \omega] + [m, \omega_1][m, \omega])\}. \quad (32)$$

§. 7.

Hac data occasione nonnullas observationes expromamus de his functionibus, quas per P_n et $P_{n,m}$ designavimus. Similes de functionibus Q non adnectimus, si quidem nullius momenti nobis videntur, antequam quaestiones de valore ipsius $\frac{1}{P_{n,m}}$ profligatae sunt; ceterum formulae nonnullae finales, differentialibus cum integralibus mutatis, n cum $-(n+1)$ statim ad $Q_{n,m}$ referri possunt.

Differentiatione facta liquet nobis esse aequationem

$$P_{n,m}[\sqrt{1-\xi^2}] = \frac{\xi^m}{(n-m+1)(n-m+2)\dots(2n)} \frac{d^{n+m}((x^2-1)^n)}{dx^{n+m}} \quad (33)$$

posito differentiatione perfecta $x = \sqrt{1-\xi^2}$. Habemus enim

$$(x^2-1)^n = x^{2n} - nx^{2n-2} + \frac{n(n-1)}{1 \cdot 2} x^{2n-4} - \dots \\ \frac{d^p ((x^2-1)^n)}{dx^p} = 2n(2n-1)\dots(2n-p+1) \left\{ x^{2n-p} - \frac{(2n-p)(2n-p-1)}{(2n-1)^2} x^{2n-p-2} + \right. \\ \left. + \frac{(2n-p)(2n-p-1)(2n-p-2)(2n-p-3)}{(2n-1)(2n-3)^2 \cdot 4} x^{2n-p-4} - \dots \right\}. \quad (34)$$

Statuto itaque $p = n+m$

(*) Cf. exempli gratia Dove, Repert. d. Physik. Vol. I. 1837. p. 152-174.

$$\begin{aligned} & \frac{\varrho^m}{(n-m+1)(n-m+2)\dots 2n} \frac{d^{n+m}(x^2-1)^n}{dx^{n+m}} = \\ & = \varrho^m \left\{ x^{n-m} - \frac{(n-m)(n-m-1)}{2(n-1)} x^{n-m-2} + \frac{(n-m)(n-m-1)(n-m-2)(n-m-3)}{2 \cdot 4 \cdot (2n-1) \cdot (2n-3)} x^{n-m-4} - \dots \right\} \\ & = P_{n,m}[\sqrt{1-\varrho^2}]; \text{ posito } x = \sqrt{1-\varrho^2} \end{aligned}$$

$$(33)^* \quad P_{n,m}[x] = \frac{(1-x^2)^{\frac{m}{2}}}{(n-m+1)(n-m+2)\dots 2n} \frac{d^{n+m}(x^2-1)^n}{dx^{n+m}}.$$

Quia est

$$P_n[x] = \frac{1 \cdot 3 \dots (2n-1)}{1 \cdot 2 \dots n} P_{n,0}[x]$$

accipimus pro $m=0$

$$(33)^{**} \quad P_n[x] = \frac{1}{2^n 1 \cdot 2 \cdot 3 \dots n} \frac{d^n(x^2-1)^n}{dx^n}.$$

Aequationibus (14) aequiparatis eruitur

$$\begin{aligned} & \varrho^m F\left(-\frac{n-m}{2}, -\frac{n-m-1}{2}, -\frac{2n-1}{2}, \frac{1}{1-\varrho^2}\right) = \\ & = F\left(-\frac{n+m}{2}, -\frac{n+m-1}{2}, -\frac{2n-1}{2}, \frac{1}{1-\varrho^2}\right) (-1)^m (1-\varrho^2)^m \quad (*) \end{aligned}$$

sive cum statussemus

$$\begin{aligned} P_{n,m}[\sqrt{1-\varrho^2}] & = F\left(-\frac{n-m}{2}, -\frac{n-m-1}{2}, -\frac{2n-1}{2}, \frac{1}{1-\varrho^2}\right) \varrho^m (1-\varrho^2)^{\frac{n-m}{2}}, \\ F\left(-\frac{n+m}{2}, -\frac{n+m-1}{2}, -\frac{2n-1}{2}, \frac{1}{1-\varrho^2}\right) & = (-1)^m \varrho^m P_{n,m}[\sqrt{1-\varrho^2}] (1-\varrho^2)^{-\frac{n+m}{2}}. \end{aligned}$$

Cum hac comparamus formulam (34) sumto $p=n-m$, unde prodit ($x=\sqrt{1-\varrho^2}$)

$$\begin{aligned} \frac{d^{n-m}(x^2-1)^n}{dx^{n-m}} & = 2n(2n-1)\dots(n+m+1)(1-\varrho^2)^{\frac{n+m}{2}} F\left(-\frac{n+m}{2}, -\frac{n+m-1}{2}, -\frac{2n-1}{2}, \frac{1}{1-\varrho^2}\right); \\ \frac{1}{(n+m+1)(n+m+2)\dots 2n} \frac{d^{n-m}(x^2-1)^n}{dx^{n-m}} \frac{(1-\varrho^2)^{\frac{n+m}{2}}}{(1-\varrho^2)^{\frac{n+m}{2}}} & = (-1)^m \varrho^m P_{n,m}[\sqrt{1-\varrho^2}]; \\ \frac{1}{(n+m+1)(n+m+2)\dots 2n} \frac{d^{n-m}(x^2-1)^n}{dx^{n-m}} & = \frac{(-\varrho^2)^m}{(n-m+1)\dots 2n} \frac{d^{n+m}(x^2-1)^n}{dx^{n+m}}, \text{ sive} \end{aligned}$$

(*) Haec aequatio contradicere sibi videtur, quippe cum altera pars pro $\varrho=0$ sit 0. Sed accuratius partem alteram investigans inveniet $F\left(-\frac{n+m}{2}, -\frac{n+m-1}{2}, -\frac{2n-1}{2}, 1\right)=0$.

$$\frac{1}{(n-m)!} \frac{d^{n-m}(x^2-1)^n}{dx^{n-m}} = \frac{(x^2-1)^m}{(n+m)!} \frac{d^{n+m}(x^2-1)^n}{dx^{n+m}}. \quad (34)^*$$

Idem theorema paeclarum iam dudum a Cl. IACOBI (*) comprobatum est ratione prorsus discrepante; nos exhibuimus demonstrationem, quae facile e formulis emanet.

Ex aequationibus (33)* et (34)* repetitur

$$\frac{(n+m)!}{(n-m)!} \frac{d^{n-m}(x^2-1)^n}{dx^{n-m}} = (-1)^m (1-x^2)^{\frac{m}{2}} (n-m+1)(n-m+2)\dots 2n P_{n,m}[x];$$

$$P_{n,m}[x] = \frac{(-1)^m}{(n+m+1)(n+m+2)\dots 2n} (1-x^2)^{-\frac{m}{2}} \frac{d^{n-m}(x^2-1)^n}{dx^{n-m}}. \quad (33)^{***}$$

§. 8.

Theorema Cl. TAYLORI docet esse

$$(zi\varrho)^{-n} ((x+zi\varrho)^2 - 1)^n =$$

$$= (zi\varrho)^{-n} \left\{ (x^2-1)^n + zi\varrho \frac{d(x^2-1)^n}{dx} + \frac{(zi\varrho)^2}{1 \cdot 2} \frac{d^2(x^2-1)^n}{dx^2} + \dots + \frac{(zi\varrho)^{2n}}{1 \cdot 2 \dots n} \frac{d^{2n}(x^2-1)^n}{dx^{2n}} \right\},$$

statuto $i = \sqrt{-1}$. Cum $(x^2-1)^n$ sit polynomium $2n^{\text{ti}}$ gradus, exsistit $\frac{d^{n+r}(x^2-1)^n}{dx^{2n+r}} = 0$, significante r numerum integrum positivum. Seriei multiplicatione in $(zi\varrho)^{-n}$ perfecta statim intelligitur coeffidentes potestatum $(\varrho iz)^{-n+r}$ et $(\varrho i)^{-n+r} z^{n-r}$, i. e.

$$\frac{1}{1 \cdot 2 \cdot 3 \dots r} \frac{d^r(x^2-1)^n}{dx^r}$$

et

$$\frac{1}{1 \cdot 2 \cdot 3 \dots (2n-r)} (-1)^{n-r} \varrho^{2(n-r)} \frac{d^{2n-r}(x^2-1)^n}{dx^{2n-r}}$$

esse aequivalentes, posito $x = \sqrt{1-\varrho^2}$ (34)*, unde formulam nostram summatioriam patet offerre

$$(zi\varrho)^{-n} ((x+zi\varrho)^2 - 1)^n =$$

$$= (i\varrho)^{-n} \left\{ (x^2-1)^n (z^n + z^{-n}) + \frac{i\varrho}{1} \frac{d(x^2-1)^n}{dx} (z^{n+1} + z^{-n+1}) + \dots + \frac{(i\varrho)^n}{(n)!} \frac{d^n(x^2-1)^n}{dx^n} \right\}.$$

Sumto igitur $z = e^{i\varphi}$ facili eruitur negotio

(*) Crelle, Journal für Mathematik. Vol. II. p. 225.

$$2^n (\sqrt{1-\rho^2} + \rho i \cos \phi)^n = \frac{1}{1 \cdot 2 \cdots n} \frac{d^n (x^2 - 1)^n}{dx^n} + 2 \sum_{m=1}^{n-m} \left\{ \frac{(i\rho)^{-m}}{1 \cdot 2 \cdots (n-m)} \frac{d^{n-m} (x^2 - 1)^n}{dx^{n-m}} \cos m\phi \right\}$$

sive (*)

$$(35) 2^n (x + \cos \phi \sqrt{x^2 - 1})^n = \frac{1}{1 \cdot 2 \cdots n} \frac{d^n (x^2 - 1)^n}{dx^n} + 2 \sum_{m=1}^{n-m} \left\{ \frac{(x^2 - 1)^{-\frac{m}{2}}}{1 \cdot 2 \cdots (n-m)} \frac{d^{n-m} (x^2 - 1)^n}{dx^{n-m}} \cos m\phi \right\}.$$

Theorematibus (34)* et (33)* hae relationes dantur

$$(35)^* 2^n (x + \cos \phi \sqrt{x^2 - 1})^n = \frac{1}{1 \cdot 2 \cdots n} \frac{d^n (x^2 - 1)^n}{dx^n} + 2 \sum_{m=1}^{n-m} \left\{ \frac{(x^2 - 1)^{\frac{m}{2}}}{1 \cdot 2 \cdots (n+m)} \frac{d^{n+m} (x^2 - 1)^n}{dx^{n+m}} \cos m\phi \right\},$$

$$(35)^{**} (x + \cos \phi \sqrt{x^2 - 1})^n = P_n[x] + 2^{-(n-1)} \sum_{m=1}^{n-m} \left\{ i^m \frac{(n-m+1)(n-m+2) \cdots 2n}{1 \cdot 2 \cdots (n+m)} P_{n,m}[x] \cos m\phi \right\}.$$

Inde obtinemus et integrale Cl. LAPLACII

$$(36) P_n[x] = \frac{1}{\pi} \int_0^\pi (x + \cos \phi \sqrt{x^2 - 1})^n d\phi$$

et

$$(37) P_{n,m}[x] = \frac{1 \cdot 2 \cdots (n+m)}{(n-m+1)(n-m+2) \cdots (2n)} i^{-m} \frac{2}{\pi} \int_0^\pi (x + \cos \phi \sqrt{x^2 - 1})^n \cos m\phi d\phi.$$

Aequatione (37) theorema (27) induit speciem

$$\begin{aligned} & \frac{1}{a_{n,m}} \int_0^\pi d\psi \int_0^{2\pi} d\phi (t_2 + \sqrt{1-t_2^2} \cos \psi)^n \cos m(\phi - \phi') = \\ & = (-1)^m \left(\frac{1 \cdot 2 \cdots (n+m)}{(n-m+1)(n-m+2) \cdots 2n} \right)^{\frac{1}{2}} 2^{2n} \left(\int_0^\pi (\sqrt{1-\rho^2} + \cos \psi i\rho)^n d\psi \right) \times \\ & \quad \times \left(\int_0^\pi (\sqrt{1-\rho_1^2} + \cos \psi i\rho_1)^n d\psi \right). \end{aligned}$$

Coniunctis dein formulis (37) et (33)* reperies

$$(38) \frac{d^{n+m} (x^2 - 1)^n}{dx^{n+m}} = \frac{2}{\pi} (n+m)! (x^2 - 1)^{-\frac{m}{2}} \int_0^\pi (x + \cos \phi \sqrt{x^2 - 1})^n \cos m\phi d\phi$$

neque minus

$$(39) \frac{d^{n-m} (x^2 - 1)^n}{dx^{n-m}} = \frac{2}{\pi} (n-m)! (x^2 - 1)^{\frac{n}{2}} \int_0^\pi (x + \cos \phi \sqrt{x^2 - 1})^n \cos m\phi d\phi.$$

Omnis has formulas perscripsimus, ut lucem afferrent tam super functionibus P quam super differentialibus ipsius $(x^2 - 1)^n$.

(*) Euler, qui hisce seriebus vacuit (Instit. calc. integr. Vol. I. cap. vi) tam $(1 + n \cos \phi)^n$ quam $(1 + n \cos \phi)^{-n}$ tractavit, simplicem coefficientum ipsius $\cos m\phi$ legem non vidit.

§. 9.

Formulae (38) et (39) neque minus (34)* ex integralium contemplatione repeti possunt, quod quoniam non statim appetet, illas deducemus. Demonstrabimus igitur e. g. esse

$$\frac{d^m \left\{ \int_0^\pi (x + \cos \phi \sqrt{x^2 - 1})^n d\phi \right\}}{dx^m} = \\ = (n+1)(n+2)\dots(n+m)(x^2 - 1)^{-\frac{m}{2}} \int_0^\pi (x + \cos \phi \sqrt{x^2 - 1})^n \cos m\phi d\phi. \quad (39)^*$$

Quem ad finem statuimus $u_{n, \pm p}$ sive

$$u = (x^2 - 1)^{\mp \frac{p}{2}} \int_0^\pi (x + \cos \phi \sqrt{x^2 - 1})^n \cos(\pm p\phi) d\phi \\ v = \int_0^\pi (x + \cos \phi \sqrt{x^2 - 1})^n \cos(\pm p+1)\phi d\phi,$$

indicante p numerum integrum positivum. Differentiatione admissa exit

$$\sqrt{x^2 - 1} \frac{d(u(x^2 - 1)^{\pm \frac{p}{2}})}{dx} = n \int_0^\pi (x + \cos \phi \sqrt{x^2 - 1})^{n-1} (\sqrt{x^2 - 1} + x \cos \phi) \cos(p\phi) d\phi.$$

Adiicimus

$$-nv = -n \int_0^\pi (x + \cos \phi \sqrt{x^2 - 1})^{n-1} (x + \cos \phi \sqrt{x^2 - 1}) (\cos p\phi \cos \phi - \sin p\phi \sin \phi) d\phi.$$

Itaque consequimur

$$\sqrt{x^2 - 1} \frac{d(u(x^2 - 1)^{\pm \frac{p}{2}})}{dx} - nv = \\ = n \int_0^\pi (x + \cos \phi \sqrt{x^2 - 1})^{n-1} \{x \sin \phi \sin(\pm p\phi) + \sin(\pm p\phi + \phi) \sin \phi \sqrt{x^2 - 1}\} d\phi. \quad (40)$$

Per partes integrantes e valoribus primitivis ipsius u et v sequentes invenimus

$$u(x^2 - 1)^{\pm \frac{p}{2}} = \left[\frac{\sin(\pm p\phi)}{\pm p} (x + \cos \phi \sqrt{x^2 - 1})^n \right]_0^\pi + \\ + \frac{n}{\pm p} \sqrt{x^2 - 1} \int_0^\pi (x + \cos \phi \sqrt{x^2 - 1})^{n-1} \sin \phi \sin(\pm p\phi) d\phi \\ v = \left[\frac{\sin(\pm p+1)\phi}{\pm p+1} (x + \cos \phi \sqrt{x^2 - 1})^n \right]_0^\pi + \\ + \frac{n}{\pm p+1} \sqrt{x^2 - 1} \int_0^\pi (x + \cos \phi \sqrt{x^2 - 1})^{n-1} \sin \phi \sin(\pm p+1)\phi d\phi$$

denotante symbolo $[f(\phi)]_0^{\pi}$ differentiam valorum $f(\pi)$ et $f(0)$. Cum p sit numerus integer, partes liberae ab integratione evanescunt, unde pro (40) prodit

$$(x^2 - 1) \frac{d(u(x^2 - 1)^{\pm \frac{p}{2}})}{dx} - n\nu \sqrt{x^2 - 1} = (\pm p) u(x^2 - 1)^{\pm \frac{p}{2}} x + \sqrt{x^2 - 1} (\pm p) \nu$$

sive

$$(41) \quad (x^2 - 1)^{\pm \frac{p}{2} + 1} \frac{du}{dx} = \sqrt{x^2 - 1} \nu \{ \pm p + n + 1 \}$$

$$(x^2 - 1)^{\frac{1 \pm p}{2}} \frac{du}{dx} = \nu \{ \pm p + n + 1 \}.$$

Ex aequat. (41) formula (39)* tanquam casus specialis evadit. Sumtis enim primum signis superioribus habebis

$$(p+n+1) \int_0^\pi (x + \cos \phi \sqrt{x^2 - 1})^n \cos(p+1)\phi d\phi =$$

$$= (x^2 - 1)^{\frac{p+1}{2}} \frac{d \left\{ (x^2 - 1)^{-\frac{p}{2}} \int_0^\pi (x + \cos \phi \sqrt{x^2 - 1})^n \cos p\phi d\phi \right\}}{dx}$$

$$(p+n+1) u_{n,p+1} = \frac{du_{n,p}}{dx}$$

$$\frac{d^2 u_{n,p}}{dx^2} = \frac{du_{n,p+1}}{dx} (p+n+1) = (p+n+1) (p+n+2) u_{n,p+2},$$

etc.

$$(42) \quad \frac{d^n u_{n,p}}{dx^n} = (p+n+1) (p+n+2) \dots (p+n+m) u_{n,p+m}.$$

Pro $p = 0$ ex (42) aequat. (39)*

$$\frac{d^n u_{n,0}}{dx^n} = (n+1) (n+2) \dots (n+m) u_{n,m}$$

prodit. E contemplationibus huius paragraphi (omnino liberis neque suspensis ex antecedentibus paragraphis) valor §. 8 ipsius $u_{n,m}$ reperiri potest. Nam sumtis signis inferioribus aequat. (41) accipimus

$$(42)^* \quad \frac{du_{n,-p}}{dx} = (n-p+1) u_{n,-p+1}$$

$$\frac{d^n u_{n,-p}}{dx^n} = (n-p+1) (n-p+2) \dots (n-p+m) u_{n,-p+m}.$$

Iam statuimus $m = p = n$, unde exsistit

$$\frac{d^n u_{n,-n}}{dx^n} = (n)! u_{n,n}.$$

Valor ipsius $\frac{d^n u_{n,-n}}{dx^n}$ haud difficilis est inventu, quia est

$$\begin{aligned} u_{n,-n} &= (x^2 - 1)^{\frac{n}{2}} \int_0^\pi (x + \cos \phi \sqrt{x^2 - 1})^n \cos n\phi d\phi \\ &= (x^2 - 1)^{\frac{n}{2}} \int_0^\pi \cos^n \phi \cos n\phi d\phi = \frac{(x^2 - 1)^{\frac{n}{2}} \pi}{2^n}. \end{aligned}$$

Quare adipiscimur

$$u_{n,0} = \frac{\pi}{2^n(n)!} \frac{d^n((x^2 - 1)^n)}{dx^n}$$

$$u_{n,m} = \frac{\pi}{2^n(n+m)!} \frac{d^{n+m}((x^2 - 1)^n)}{dx^{n+m}}.$$

Ut et formulam (34)* deducamus, ponimus in (42)* $m = n - r$, $p = n$, unde consequimur ($r < n$)

$$\frac{d^{n-r} u_{n,-n}}{dx^{n-r}} = \frac{d^{n-r} ((x^2 - 1)^n)}{dx^{n-r}} \frac{\pi}{2^n} = (n-r)! u_{n,-r}$$

sive

$$u_{n,-m} = \frac{\pi}{2^n} \frac{1}{(n-m)!} \frac{d^{n-m}((x^2 - 1)^n)}{dx^{n-m}}.$$

Quod est secundum definitionem

$$u_{n,-m} = (x^2 - 1)^m u_{n,m}$$

sine difficultate reperitur

$$\frac{1}{(n-m)!} \frac{d^{n-m}((x^2 - 1)^n)}{dx^{n-m}} = \frac{(x^2 - 1)^m}{(n-m)!} \frac{d^{n+m}((x^2 - 1)^n)}{dx^{n+m}}.$$

§. 10.

Factae sunt hae disquisitiones (quibus occasione oblata plura adieciimus, quae non directe ad propositum nostrum pertinent), ut explicaremus statum caloris non pendentem e tempore in corporibus sphaeroidicis ellipticis homogeneis rotatione ellipsis ortis, in quorum superficie temperatura conservatur pariter non suspensa e tempore. Solvit quidem non solum casum nostrum speciale, sed etiam generale problema de corporibus sphaeroidicis ellipticis Cl. LAMÉ (*); sed cum fieri nequeat sine calculo molestissimo illo quidem, ut ex solutione generali specialis ducatur — generalem enim formulam casui speciali adaptans, quantitatem

(*) Journal de mathématiques par J. Liouville. 1839. p. 126 - 164 et 351 - 386.

$\frac{9}{6}$ invenies — unde hic casus methodum peculiarem possit: non inutile esse videtur, me rem aliter tractare, cum ratio mea, theorematibus infinitens tempore novissimo demonstratis, problema, de quo agitur, facilis solvat minoribusque ambagibus, nisi fallor, iisdemque formulis sphaeroides tractet ortas rotatione ellipsis seu circa axem maiorem seu circa minorem, qui casus reducentur in altero opusculo alter ad alterum.

Cum Cl. LAMÉ in casu generali censuit quodvis sphaeroidis punctum esse determinatum intersectione sphaeroidis ellipticae duarumque hyperboliarum, quarum altera sit elliptica, altera hyperbolica, quae habeant focos eosdem cum superficie sphaeroidis ellipticae datae, quam substitutionem casui speciali adaptans unum angulum duasque sphaeroides introducit: nos unam quidem superficiem sphaeroidicam ellipticam statuimus eamque eorundem focorum quorum data, duos autem angulos introducimus respondentes alterum latitudini, alterum longitudini in terra, id quod statim a principio plane mutat rationem problema tractandi.

§. 11.

Concipiatur quodvis punctum in spatio determinatum tribus coordinatis x, y, z , quae sint parallelae principalibus axis sphaeroidis datae; sphaeroidis centrum sit coordinatarum initium, axes principales sint $r_0, r_0, \sqrt{r_0^2 - e^2}$ sive e est quantitas realis, sive imaginaria, i. e. in sphaeroide orta rotatione ellipsis circa axem seu minorem seu maiorem. Tunc erit aequatio superficie sphaeroidis datae

$$(43) \quad \frac{x^2 + y^2}{r_0^2} + \frac{z^2}{r_0^2 - e^2} = 1$$

desideratque problema, ut status (u) non pendens a tempore corporis superficie sphaeroidica elliptica (43) circumdati sic definiatur, ut u in superficie aequet functionem quandam datam trium variabilium x, y, z ; functionem quidem finitam fingamus. Quo facilius exprimamus conditio-
nem, ponamus

$$(44) \quad \begin{cases} x = r \sin \theta \cos \phi \\ y = r \sin \theta \sin \phi \\ z = \sqrt{r^2 - e^2} \cos \theta \end{cases}$$

angulo θ crescente a $\theta = 0$ usque ad $\theta = \pi$, angulo ϕ a $\phi = 0$ usque ad $\phi = 2\pi$. Tunc transformatione rite (*) instituta aequatio notissima

$$\frac{d^2 u}{dx^2} + \frac{d^2 u}{dy^2} + \frac{d^2 u}{dz^2} = 0$$

transit in formam

$$\frac{d^2 u}{dr^2} (r^2 - e^2) + \frac{du}{dr} \frac{2r^2 - e^2}{r} + \frac{1}{\sin \theta} \frac{d(\sin \theta \frac{du}{d\theta})}{du} + \frac{1}{\sin^2 \theta} \frac{d^2 u}{d\phi^2} - \frac{e^2}{r^2} \frac{d^2 u}{d\phi^2} = 0, \quad (45)$$

ex qua functionem u , ipsam finitam, ita reperias, ut sumto valore r_0 pro r sit

$$u = f(\theta, \phi)$$

ubi functio $f(\theta, \phi)$ designat datum superficie statum. Haec forma satis digna, quam consideremus, mutari potest cum aequat. (45)

$$r \frac{d^2(ru)}{dr^2} + \frac{1}{\sin^2 \theta} \frac{d^2 u}{d\phi^2} + \frac{1}{\sin \theta} \frac{d(\sin \theta \frac{du}{d\theta})}{d\theta} = e^2 \left\{ \frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} + \frac{1}{r^2} \frac{d^2 u}{d\phi^2} \right\},$$

quae pro $e = 0$ in eam transit, quae simile nostri globi problema solvit, quod efficitur in cylindro altitudinis infinitae aequatione

$$\frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} + \frac{1}{r^2} \frac{d^2 u}{d\phi^2} = 0.$$

(*) Qua ratione huiusmodi transformationes fiant, invenitur in §. 4, cuius formula (5), mutatis α, β, ω in r, θ, ϕ hic in usum revocari potest. Cum calculus huius paragraphi molestior sit quam §. 4, haud ab re erit, differentialia prima, prodeuntia ex eliminatione aequationum respondentium ipsis (A), (B), (C) §. 4, hic adscribi. Hisce datis differentiationes altiores facillime instituti queunt. Habemus igitur

$$\begin{aligned} \frac{dr}{dx} &= \frac{(r^2 - e^2) \sin \theta \cos \phi}{r^2 - e^2 \sin^2 \theta}; & \frac{d\theta}{dx} &= \frac{r \cos \theta \cos \phi}{r^2 - e^2 \sin^2 \theta}; & \frac{d\phi}{dx} &= -\frac{\sin \phi}{r \sin \theta}; \\ \frac{dr}{dy} &= \frac{(r^2 - e^2) \sin \theta \sin \phi}{r^2 - e^2 \sin^2 \theta}; & \frac{d\theta}{dy} &= \frac{r \cos \theta \sin \phi}{r^2 - e^2 \sin^2 \theta}; & \frac{d\phi}{dy} &= \frac{\cos \phi}{r \sin \theta}; \\ \frac{dr}{dz} &= \frac{r \sqrt{r^2 - e^2} \cos \theta}{r^2 - e^2 \sin^2 \theta}; & \frac{d\theta}{dz} &= \frac{\sin \theta \sqrt{r^2 - e^2}}{r^2 - e^2 \sin^2 \theta}; & \frac{d\phi}{dz} &= 0. \end{aligned}$$

Tunc consequimur

$$\frac{dr}{dx} \frac{d\theta}{dx} + \frac{dr}{dy} \frac{d\theta}{dy} + \frac{dr}{dz} \frac{d\theta}{dz} = 0$$

$$\frac{dr}{dx} \frac{d\phi}{dx} + \frac{dr}{dy} \frac{d\phi}{dy} + \frac{dr}{dz} \frac{d\phi}{dz} = 0$$

$$\frac{d\theta}{dx} \frac{d\phi}{dx} + \frac{d\theta}{dy} \frac{d\phi}{dy} + \frac{d\theta}{dz} \frac{d\phi}{dz} = 0.$$

Inde dicamus lineam r , angulosque θ, ϕ datas esse coordinatas, ad quas quodvis punctum referatur; earum reductio ad x, y, z facilime exhibetur formulis (44); geometrice rem iam §. 10 interpretati sumus. Observandum autem est, quantitate e reali, $\frac{r}{e}$ superare unitatem — cum $z = \sqrt{r^2 - e^2} \cos \theta$ sit realis — quantitate e imaginaria, modulum ipsius $\frac{r}{e}$ esse situm intra limites 0 et $\frac{r}{e}$. Commoditatis causa statuimus et $\frac{r}{e} = \rho$ et $\frac{r}{e} = \varrho_0$.

Fingamus expressionem u tanquam functionem duarum variabilium in seriem explicatam esse ita, ut sit

$$u = \sum_{n=0}^{n=\infty} X_n$$

X_n satisfaciente aequationi

$$(46) \quad \frac{1}{\sin \theta} \frac{d \left(\sin \theta \frac{d X_n}{d \theta} \right)}{d \theta} + \frac{1}{\sin^2 \theta} \frac{d^2 X_n}{d \phi^2} + n(n+1) X_n = 0,$$

quod semper fieri potest (*). Cum $\frac{d^2 X_n}{d \phi^2}$ sit e genere functionum X_n , i. e. in aequatione (46) poni queat pro X_n , integritate salva, quod statim monstrabitur, bis differentiando (46) secundum variabilem ϕ resultat ex aequat. (45)

$$(47) \quad \frac{d^2 X_n}{d \phi^2} \rho^2 (\rho^2 - 1) + \frac{d X_n}{d \rho} \rho (2\rho^2 - 1) - \frac{d^2 X_n}{d \rho^2} - n(n+1) \rho^2 X_n = 0.$$

Cum hac aequatione observationes haud ignotas in speciem generalem ipsius X_n coniungimus, est enim

$$(48) \quad X_n = \sum_{m=0}^{m=n} (\{g_{n,m} \cos m\phi + h_{n,m} \sin m\phi\} P_{n,m}[\cos \theta])$$

declarantibus g et h constantes. Ut has quantitates, non suspensas e ρ et ϕ , continentes ϱ_0 , definiamus, aequationem (47) cum (48) iungimus, unde prodit

$$\frac{d^2 g_{n,m}}{d \rho^2} \rho^2 (\rho^2 - 1) + \frac{d g_{n,m}}{d \rho} \rho (2\rho^2 - 1) + g_{n,m} (m^2 - n(n+1) \rho^2) = 0$$

$$\frac{d^2 h_{n,m}}{d \rho^2} \rho^2 (\rho^2 - 1) + \frac{d h_{n,m}}{d \rho} \rho (2\rho^2 - 1) + h_{n,m} (m^2 - n(n+1) \rho^2) = 0.$$

(*) Crelle, Journal für Mathematik. Vol. XVII. Lejeune-Dirichlet, sur les séries dont le terme général dépend de deux angles etc.

Quas formulas iam solvimus; denotantibus scilicet $k_{n,m}$ et $k'_{n,m}$, $l_{n,m}$ et $l'_{n,m}$ valores numericos, prodit

$$\begin{aligned} g_{n,m} &= k_{n,m} P_{n,m} [\sqrt{1-\xi^2}] + k'_{n,m} Q_{n,m} [\sqrt{1-\xi^2}] \\ h_{n,m} &= l_{n,m} P_{n,m} [\sqrt{1-\xi^2}] + l'_{n,m} Q_{n,m} [\sqrt{1-\xi^2}] \\ X_n &= \sum_{m=0}^{m=\infty} \{P_{n,m} [\sqrt{1-\xi^2}] (k_{n,m} \cos m\phi + l_{n,m} \sin m\phi) + \\ &\quad + Q_{n,m} [\sqrt{1-\xi^2}] (k'_{n,m} \cos m\phi + l'_{n,m} \sin m\phi)\} P_{n,m} [\cos \theta]. \end{aligned}$$

Sit primo ξ quantitas imaginaria, in quo casu valorem peculiarem o ei tribuere possumus; cum $Q_{n,m}[1] = \infty$, $P_{n,m}[1]$ quantitas finita sit (sive o) §. 4, $k'_{n,m}$ et $l'_{n,m}$ evanescere debent. Deinde vero sit ξ quantitas realis, statuto $\xi = 1$, $P_{n,m}[0]$ non erit valor infinitus. (E prima definitione sequitur, esse P_n coefficientem ipsius a^n in aequatione $\frac{1}{\sqrt{1+a^2}} = \sum_{n=0}^{n=\infty} a^n P_n$. Ceterum ex (15) facile intelliges, esse $P_{n,m}$ sive = 0, sive quantitatatem finitam). $Q_{n,m}$ (*) contra erit valor infinitus, unde aequae $k'_{n,m}$ et $l'_{n,m}$ evanescent; nec pro omnibus problematis solutio $Q_{n,m}[\sqrt{1-\xi^2}]$ reicienda erit; in nonnullis non solum theoriae caloris, verum etiam attractionis (magnetismi) non erit inepta. Ergo habemus

$$X_n = \sum_{m=0}^{m=\infty} \{P_{n,m} [\sqrt{1-\xi^2}] (k_{n,m} \cos m\phi + l_{n,m} \sin m\phi) P_{n,m} [\cos \theta]\}. \quad (49)$$

Ut et $k_{n,m}$ et $l_{n,m}$ reperias, esse animadvertis pro $r = r_0$ sive $\xi = \xi_0$ summam omnium X_n sive $u = f(\theta, \phi)$, unde valor ipsius X_n

$$= \frac{2n+1}{4\pi} \int_0^\pi d\theta_1 \sin \theta_1 \int_0^{2\pi} P_n [\cos \gamma] f(\theta_1, \phi_1) d\phi_1$$

evadit. Pro $\xi = \xi_0$ et $k_{n,m}$ et $l_{n,m}$ literae remanent invariatae, unde consequimur

$$\begin{aligned} &\frac{2n+1}{4\pi} \int_0^\pi d\theta_1 \sin \theta_1 \int_0^{2\pi} P_n [\cos \gamma] f(\theta_1, \phi_1) d\phi_1 = \\ &= \sum_{m=0}^{m=\infty} \{P_{n,m} [\sqrt{1-\xi_0^2}] (k_{n,m} \cos m\phi + l_{n,m} \sin m\phi) P_{n,m} [\cos \theta]\} \\ P_{n,m} [\cos \theta] P_{n,m} [\sqrt{1-\xi_0^2}] k_{n,m} &= \\ &= \frac{2n+1}{4\pi^2} \int_0^\pi d\theta_1 \sin \theta_1 \int_0^{2\pi} d\phi_1 f(\theta_1, \phi_1) \int_0^{2\pi} d\phi P_n [\cos \gamma] \cos m\phi \end{aligned}$$

(*) Est enim aut $(\frac{1}{\sqrt{2}})^m$ aut $\Gamma 0$ factor ipsius $Q_{n,m}[0]$. Ceterum functione Q_n sic transformata, ut P_n , facile perspicuit, eam factorem $(1-\xi^2)^{-\frac{1}{2}}$ involvere, i. e. esse infinitam pro $\xi = \pm 1$.

similemque valorem ipsius $I_{n,m}$ significante $\cos \gamma$ ut in (§. 5) $\cos \theta \cos \theta_1 + \sin \theta \sin \theta_1 \cos(\phi - \phi_1)$. Ceterum pro $m=0$ pars dimidia alterius lateris ponenda est. Statuto igitur

$$\cos \gamma_1 = \cos \theta \cos \theta_1 + \sin \theta \sin \theta_1 \cos(\phi_2 - \phi_1)$$

habebis, cum sit $u = \sum_{n=0}^{+\infty} X_n$

$$(50) \quad u = \frac{1}{4\pi^2} \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty}$$

$$\left\{ (2n+1) \frac{P_{n,m}[\sqrt{1-\rho^2}]}{P_{n,m}[\sqrt{1-\rho_0^2}]} \int_0^\pi d\theta_1 \sin \theta_1 \int_0^{2\pi} d\phi_1 f(\theta_1, \phi_1) \int_0^{2\pi} d\phi_2 P_n[\cos \gamma_1] \cos m(\phi - \phi_2) \right\}$$

pro $m=0$ dimidia parte sumpta. Formula (27) adhibita altera aeq. (50) forma existit

$$(50)^* \quad u = \frac{1}{4\pi} \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty}$$

$$\left\{ (2n+1) a_{n,m} P_{n,m}[\cos \theta] \frac{P_{n,m}[\sqrt{1-\rho^2}]}{P_{n,m}[\sqrt{1-\rho_0^2}]} \int_0^\pi d\theta_1 \sin \theta_1 \int_0^{2\pi} d\phi_1 f(\theta_1, \phi_1) P_{n,m}[\cos \theta_1] \cos m(\phi - \phi_1) \right\}.$$

Hanc dissertationem concludentes non praetermittimus, esse pro $e=0$ (globo) $\frac{P_{n,m}[\sqrt{1-\rho^2}]}{P_{n,m}[\sqrt{1-\rho_0^2}]} = \frac{r^n}{r_0^n}$, i. e. quantitatem haud pendentem ex m . Summationem igitur secundum m confidentes obtinemus

$$\sum_{m=0}^{+\infty} \{ a_{n,m} P_{n,m}[\cos \theta] P_{n,m}[\cos \theta_1] \cos m(\phi - \phi_1) \} = P_{n,m}[\cos \gamma]$$

$$u = \frac{1}{4\pi} \sum_{n=0}^{+\infty} \left(\frac{r}{r_0} \right)^n (2n+1) \int_0^\pi d\theta_1 \sin \theta_1 \int_0^{2\pi} d\phi_1 f(\theta_1, \phi_1) P_n[\cos \gamma],$$

quae formula cum nota globi congruit. Ceterum summa potest confici ita, uti duae tantummodo integrationes in aequatione resideant; generalem autem formulam ad speciem simpliciorem redigendi fecimus frustra periculum.



V I T A.

Natus sum ego Henricus Eduardus Heine Berolini Idibus Martii anno MDCCCXXI patre Carolo Henrico Heine beato, argentariam faciente, matre Henrietta e gente Mertens. Confessiōnē addictus sum evanđelicae. Primis literarum rudimentis domi imbutus, puer undecim annorum gymnasium Fridericianum adii, quod tunc directore Cl. Ribbeck florebat; postea gymnasium Coellnanum frequentavi, unde directore Cl. August maturitatis testimonio ornatus anno MDCCCXXXVIII discessi et ab Ill. Boeckh, t. t. rectore magnifico, in huius academiae cives receptus sum. Mense Maio anni MDCCCXXXIX Gottingam me contuli, ubi unum annum cum dimidio in literas incubui. In studiis autem magistri ducesque mihi fuere: in mathesi viri Ill. Dirksen, Encke, Gauss, Goldschmidt, Lejeune-Dirichlet, Steiner, Stern; in physice Cl. Dove, Magnus, Seebeck; in chemia Ill. Mitscherlich, in mineralogia Ill. Weiss, in philosophia Ill. Herbart et Steffens; in archaeologia beatus Odofredus Müller. Nec possum hoc loco omnibus illis Ill. viris, quorum egregia disciplina atque institutione sum usus, gratias non agere maximas.

THESES.

- 1) Infinitum eatenus tantum admittendum, quatenus ad limitum theoriam reduci potest.
 - 2) Cifra limes.
 - 3) Mathesis non eget philosophia.
 - 4) Functionum derivatarum methodus non adhibenda in calculo differentiali.
 - 5) Integrale limes summae.
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