Dynamics of D-branes in curved backgrounds

D I S S E R T A T I O N

zur Erlangung des akademischen Grades
doctor rerum naturalium
(Dr. rer. nat.)
im Fach Physik

eingereicht an der
Mathematisch-Naturwissenschaftlichen Fakultät I
der Humboldt-Universität zu Berlin

von
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geboren am 3.10.1974 in Hamburg

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eingereicht am:	4. Juli 2002
Tag der mündlichen Prüfung:	16. September 2002

Abstract

In recent years, the study of branes has led to many new insights into string and M-theory. Much of this study was done in the large-volume regime where geometric techniques provide reliable information. The extrapolation into the stringy regime usually requires new methods from boundary conformal field theory.

Branes on group manifolds give us a good handle on this issue. Although they describe non-trivial backgrounds leading to many interesting effects, they are still tractable. They also serve as building blocks in the coset and orbifold constructions of essentially all known conformal models.

The present thesis investigates the dynamics of branes on group manifolds and coset models. In some limiting regime, the dynamics are governed by non-commutative gauge theories. Many of the processes can be extrapolated to the stringy regime. They manifest themselves as renormalization group flows on the two-dimensional worldsheet theories with boundaries. Such flows are of interest also in condensed matter theory where they describe boundary phenomena in one-dimensional systems.

Essential data on these dynamical processes are encoded in D-brane charges. We will compare the obtained results on processes between brane configurations with the conjecture that the charges take their values in twisted K-groups.

Keywords:
String theory, D-branes, Boundary conformal field theory, Non-commutative field theory

Zusammenfassung

In den letzten Jahren hat die Erforschung von Branen zu vielen neuen Einsichten in String- und M-Theorie geführt. Ein Großteil dieser Forschung behandelte den Fall großen Volumens, wo geometrische Methoden zuverlässige Informationen liefern. Die Extrapolation in den Bereich, wo die endliche Ausdehnung des Strings wichtig wird ('stringy regime'), erfordert gewöhnlich neue Methoden aus der konformen Feldtheorie mit Randbedingungen.

Branen auf Gruppenmannigfaltigkeiten ermöglichen einen guten Zugang zu diesem Problem. Obwohl sie nichttriviale Hintergründe beschreiben, was zu vielen interessanten Effekten führt, sind sie immer noch gut beherrschbar. Sie dienen auch als Bausteine bei den Restklassen- und Orbifoldkonstruktionen von im Wesentlichen allen bekannten konformen Modellen.

Die vorliegende Arbeit untersucht die Dynamik von Branen auf Gruppenmannigfaltigkeiten und Restklassenmodellen. In einem bestimmten Grenzfall wird die Dynamik von nichtkommutativen Eichtheorien regiert. Viele der Prozesse lassen sich in den Bereich extrapolieren, wo Stringeffekte eine Rolle spielen. Sie äußern sich als Renormierungsgruppenflüsse auf den zweidimensionalen Weltflächentheorien mit Rändern. Solche Flüsse sind auch von Interesse in der Festkörpertheorie, wo sie Randphänomene in eindimensionalen Systemen beschreiben.

Wesentliche Daten über diese dynamischen Prozesse sind in Ladungen von D-Branen kodiert. Wir werden die Resultate, die wir über Prozesse zwischen verschiedenen Brankonfigurationen erhalten, mit der Vermutung vergleichen, dass die Ladungen Werte in getwisteten K-Gruppen annehmen.

Sclagwörter:
Stringtheorie, D-Branen, Konforme Feldtheorie mit Rand, Nichtkommutative Feldtheorie

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Chapter 1

Introduction

A consistent theory describing quantum systems as well as gravity is the dream of many physicists. In this theory, there would be a regime, some limit, in which it reduces to a quantum field theory on a (maybe curved) four-dimensional space-time. The dynamics of the geometric background would be governed by Einstein's field equations. We expect this classical geometry to emerge as an effective concept from our quantum gravity or 'Theory of Everything'.

String theory has many features that make it our best candidate for a unified theory of gravity and particle physics as described by the standard model. If string theory is the correct theory of nature, we should understand how to extract an effective geometry from it and how string theory determines this effective background.

The current picture we get from string theory is encouraging but not satisfactory. First, there is a way of extracting an effective geometry from string theory. Defining perturbative string theory not on a given geometric background, but as an abstract conformal field theory, we can read off geometric notions in certain limiting regimes [1]. This gives us some idea of how we can understand geometry as a derived concept.

Secondly, strings determine the background in which they propagate to some extent. Trying to quantize a string theory in a geometric background, we find, to lowest order, conditions that restrict the space-time to be a solution of the field equations of ten-dimensional (super-)gravity.

Still, the background dependence and the immense number of possible backgrounds is a severe unsolved problem in string theory. It is not even clear why four macroscopic dimensions are favorable. Without fixing this problem, string theory hardly has any predictive power.

The best arguments for string theory certainly do not come from its successes in describing nature. The strength of string theory is rather its unifying power. It provides a theory in which we can ask 'What are black holes?' as well as 'What are particles?'. It gives us new insights on how we can view geometry. And its large moduli space and the dualities acting on it show us that seemingly unrelated concepts could be closely connected.

The 'discovery' of D-branes [2] pushed string theory a large step forward. It helped in understanding dualities, in calculating black hole entropy [3], and led to the fascinating relation between gravity and gauge theories [4]. Furthermore, the gauge theories appearing on branes as low energy limits can help to construct backgrounds similar to the standard model. Therefore, it is desirable to learn more about properties of D-branes, about their stability and in particular their dynamics.

Weakly curved D-branes in a weakly curved background can be described as geometric objects: namely, as submanifolds bearing a gauge bundle. Their dynamics is governed by the Dirac-Born-Infeld action [5]. We can call this the 'geometric regime'. Eventually, we want to understand arbitrary D-branes in arbitrary backgrounds, because this would be the first step towards a background-independent theory. Furthermore, the compactified part of ten-dimensional space-time is probably curved, so that also for string phenomenology it is necessary to deal with non-trivial backgrounds. And thirdly, by exploring strings and branes in strongly curved spaces, we can hope to get a better insight how the notion of geometry changes on small length scales.

If backgrounds are strongly curved, the geometric description breaks down because the 'stringy' nature of the fundamental constituents becomes important. The perturbative quantization of the non-linear σ-model describing the motion of a string in such a background is not sensible; what we need is an exact conformal field theory containing all stringy effects.

Non-trivial exact conformal field theory backgrounds are rare. One class of such theories is given by Wess-Zumino-Witten (WZW) models [6, 7] that describe strings moving on group manifolds. The large symmetry of these models makes them rather tractable while, on the other hand, they already display many new interesting features that are not present in flat spaces. Most of these models, however, do not appear as string backgrounds because their dimension is too large. One exception is the WZW model of $SU(2)$ which occurs in the near-horizon geometry of Neveu-Schwarz 5-branes or Dirichlet-3-branes.

We can, however, use the WZW models as starting point for the construction of a much larger class of conformal field theories. They serve as basic building blocks for all the coset and orbifold constructions of exactly solvable string backgrounds. A lot of the structures and properties of WZW models survive in this process of 'model building'.

The present thesis deals with the dynamics of branes on group manifolds and coset models. Its focus is on branes with a maximal amount of symmetry both in the decoupling limit and deep in the stringy regime. Most of the results of this thesis have been published [8, 9, 10, 11, 12].

A thorough understanding of maximally symmetric branes on group manifolds G has been achieved over the last years. Cardy [13] first proposed a set of elementary boundary conditions in WZW models which we will call 'untwisted branes' or 'Cardy branes'. Later, a new set of branes, 'twisted branes', were constructed by Birke, Fuchs and Schweigert [14]. The 'twisting' involves an (outer) automorphism ω of the group G. The open string spectrum has the same high amount of symmetry as for the untwisted branes. Twisted and untwisted branes constitute the whole set of 'maximally symmetric' branes.

In the geometric regime, when the volume of the group manifold and the extension of the branes is large in units of the string length, we can find submanifolds on which the branes are localized. The geometric interpretation was given by Alekseev and Schomerus [15] and by Fuchs, Felder, Fröhlich and Schweigert [16]: they wrap (twisted) conjugacy classes on the group manifold.

When the background group manifold is large and weakly curved, we are in the 'decoupling limit' where the dynamics of the open string modes on the brane decouple from the bulk modes. The effective theory on the brane's world-volume is a gauge theory. If the brane is small, at least in some directions, the world-volume of the brane is no longer described by a classical geometric space, but rather involves notions of non-commutative geometry [17, 12]. The dynamics of branes is then governed by non-commutative gauge theories [18]. For untwisted branes where the extension in all directions can become small, the algebra of functions on the world-volume is even finite-dimensional, and the effective gauge theory is a matrix theory.

These non-commutative gauge theories have been carefully investigated in the past [18, 12]. In this thesis, we present the complete classification of symmetric solutions for twisted and untwisted branes on simple, simply-connected compact group manifolds G. The solutions describe the formation of bound states of brane configurations. Let us briefly specify the outcome of the analysis. Brane configurations X can be labeled by representations (modules) V^X of the subgroup $G^\omega \subset G$ consisting of elements which are invariant under the automorphism ω. The configurations X and Y are connected by a dynamical process precisely if the representations V^X and V^Y have the same dimension. The theory of solutions has a nice geometric interpretation in terms of certain vector bundles over G/G^ω. The result quoted above has been derived through an investigation of the structure of these bundles.

When we go away from the decoupling limit, some of the identified processes will persist, others will disappear and new processes may turn up. There is strong evidence that a certain class of solutions corresponding to a constant gauge field can be extrapolated deep into the stringy regime [19, 8]. These issues are closely related to boundary phenomena in one-dimensional condensed matter systems. The boundary renormalization group (RG) flows in such systems correspond to flows from one brane configuration into another. The processes under consideration in WZW models are described by the 'absorption of the boundary spin'-principle of Affleck and Ludwig [20] for systems containing a magnetic impurity.

The results of string theory on group manifolds can be used to derive results in theories which contain WZW models as building blocks. Coset models G/H [21] are the quotient of the WZW model on a group G and the WZW model on a subgroup $H \subset G$. The geometric background is the space $G/\mathrm{Ad}(H)$ of orbits of G under the adjoint action of H. The set of maximally symmetric branes contains again untwisted 'Cardy' branes and twisted branes [22]. Their geometry descends from the product of (twisted) G- and H-conjugacy classes in G [23, 24, 10] .

Many properties of branes on group manifolds carry over to coset models. In the decoupling limit, the effective theory on a coset brane can be derived from the non-commutative gauge theory of a brane in a group manifold $G \times H$ by putting certain constraints on the fields [9, 10]. The symmetric solutions can be classified completely for untwisted coset branes. For twisted branes, we lack a complete understanding, but we are still able to find a large class of solutions. Brane configurations X can be labeled by representations V^X of the invariant subgroup $G^\omega \times H^\omega$. Two configurations X and Y that are connected by a solution coincide on the diagonal $H^\omega_{\mathrm{diag}} \subset G^\omega \times H^\omega$,

$$V^X\big|_{H^\omega_{\mathrm{diag}}} \;\cong\; V^Y\big|_{H^\omega_{\mathrm{diag}}} \quad .$$

The theory of solutions has again a nice geometric interpretation in terms of H-equivariant vector bundles.

When we move away from the decoupling limit, we may again extrapolate some processes as in the case of group manifolds. This leads to a proposal which generalizes the 'absorption of boundary spin'-principle of Affleck and Ludwig to coset models [11]. Evidence for the conjectured principle comes from the comparison with renormalization group flows in unitary minimal models that have been obtained by different means. In particular, we will describe here in detail the critical and tricritical Ising model, and the three-states Potts model. Although these models are far away from the 'geometric regime', the pictorial description of branes in these models will turn out to be convenient to organize and visualize the flows.

Having a comprehensive understanding of brane processes at our disposal, we can look for some general structure which underlies the dynamics of branes. One approach is to find appropriate conserved quantities (charges) that encode the essential features of how the branes interact. From the remarks above, we infer that e.g. for brane configurations X on group manifolds in the decoupling limit the dimension $\dim V^X$ of the corresponding G^ω-module is a good candidate for such a conserved charge.

D-branes in string theory on a background M carry so called Ramond-Ramond (RR) charges [2]. They are assigned to arbitrary configurations of branes, stable and unstable, and they are conserved during all dynamical processes. The classification of conserved quantities would directly yield a classification of RR charge groups.

The charge groups are discrete abelian groups and can only depend on the background M. It is therefore natural to search for the charge group among the various discrete abelian groups that mathematicians assign to a background geometry M. Indeed, one has made a find: by now there is convincing evidence that K-theory is the right candidate to classify RR-charges. There exist different K-theories that one uses depending on the string theory under consideration. In type IIA/IIB string theory, the relevant groups are given by the usual complex topological K-groups $K^*(M)$. Of course, this can only hold as long as we consider a purely geometric background without non-trivial background fields. In dealing with the case of a non-vanishing Neveu-Schwarz 3-form $H \in H^3(M, \mathbb{Z})$, Bouwknegt and Mathai [25] proposed to employ the so called twisted K-groups $K^*_H(M)$.

The value of this proposal was confirmed by the explicit evaluation of charge groups on group manifolds $G = SU(n)$ by a CFT analysis of RG invariants [8], and the subsequent determination of the twisted K-groups [26]. A physical interpretation to the mathematical algorithm for computing K-theory in terms of D-brane instantons has also been given in [26].

For coset and orbifold models one expects equivariant K-theory to be the right setting. Unfortunately, little is known about equivariant K-theories relevant for coset models, nor are there many results from conformal field theory.

The outline of this thesis is as follows. In the next chapter we give an introduction to the main points in the context of brane dynamics. Chapter 3 presents the geometrical and conformal field theoretical aspects of maximally symmetric branes on groups and coset models. In particular, we will provide a complete list of such branes and their associated open string spectra. The focus of Chapter 4 will be on the dynamics of branes in the decoupling limit. We will construct the algebra of functions on these branes and introduce the non-commutative gauge theories governing the dynamics of open string modes. The main part of this chapter is devoted to the theory of classical solutions and their interpretation as brane processes. The chapter is rounded off by a series of examples. In Chapter 5 we will leave the decoupling regime and discuss the original version of Affleck and Ludwig's 'absorption of boundary spin'-principle as well as its proposed generalization to coset models. The unitary minimal models serve then as 'testing ground' for a detailed application of the conjectured rule. Finally, Chapter 6 deals with charge groups and the relation to K-theory. The results of the preceding chapters are used to determine groups of conserved charges. We obtain rather explicit results for the groups $G = SU(n)$ and compare them with those found in twisted K-theory.

* * *

This thesis is based on the publications [8, 9, 10, 11, 12]. The results on the dynamics of twisted branes in coset models, and the study of RG flows in the three-states Potts model, however, are new.

Chapter 2

Brane dynamics

This chapter wants to give a brief overview of the field in which this thesis is situated. We start in Section 2.1 with some basic concepts in the theory of strings and branes in the 'geometric' regime. The subsequent section discusses abstract non-geometric string backgrounds from conformal field theory. The description of the dynamics of branes by low energy effective field theories is dealt with in Section 2.3. This is followed in Section 2.4 by the example of flat branes in ten-dimensional Minkowski space in the presence of a constant magnetic B-field where the brane's world-volume is naturally described by non-commutative geometry. The chapter ends with a discussion of brane charges and their description by K-theory in Section 2.5.

2.1 What are D-branes?

D-branes are extended non-perturbative objects that appear in string theory as solitonic solutions of supergravity. They are characterized by their property that open strings can end on them. D-branes play an important role in the exploration of non-perturbative aspects in string theory. We will first give a short introduction into the basic features of string theory, before we come to the discussion of branes.

The standard way of introducing strings is as quantization of a classical string moving in a flat Minkowski target space. On the classical level there are different equivalent descriptions, most prominent the Nambu-Goto action, essentially the volume of the two-dimensional surface the string sweeps out in space-time, and the Polyakov action, where the world-sheet of the strings is equipped with an own metric γ_{ab},

$$\mathcal{S}_{\mathrm{P}}[X,\gamma] \;=\; -\frac{1}{4\pi\alpha'}\int_{\Sigma}\mathrm{d}\tau\, d\sigma\,(-\gamma)^{1/2}\gamma^{ab}\partial_a X^{\alpha}\partial_b X_{\alpha} \quad .$$

Here σ, τ are the coordinates on the world-sheet Σ, $X^\alpha(\sigma,\tau)$ is the embedding of the world-sheet into the D-dimensional Minkowski space. The pre-factor contains the string tension $T = (2\pi\alpha')^{-1}$.

Although classically equivalent, these two actions differ in the severity of quantizing them. The Nambu-Goto action involves a square-root which turns its quantization into an extremely hard job (nevertheless there is work on this subject initiated in [27], see [28] for recent progress). The Polyakov action, on the other hand, can be quantized in different approaches: by canonical covariant quantization, light-cone quantization, and BRST quantization. In all these approaches, the conformal symmetry which ensures independence from the local dynamics of the world-sheet metric survives the quantum corrections only if the dimension of the target space is $D = 26$ for the bosonic string. When we consider a supersymmetric version of the string including fermions, the critical dimension is $D = 10$. In total, one can formulate five different superstring theories in 10 dimensions: Type I, Type IIA and IIB, Heterotic $E_8 \times E_8$ and Heterotic $SO(32)$.

A theory which cannot be formulated in our four-dimensional space-time seems to be an odd object to study. But let us not be too fast in our judgment and let us first, patiently, discuss some interesting properties of string theory.

Each physical state in the spectrum of the freely moving superstring in 10-dimensional flat space-time can be understood as an excitation of the string moving with a momentum p_α. We can interpret $m^2 = p_\alpha p^\alpha$ as the mass-squared from the space-time point of view. It turns out that there is a simple formula determing m^2 in terms of the level N of excitation of the string,

$$m^2 \;=\; \frac{4}{\alpha'}(N-1) \quad .$$

Here, N is the total number of (left-moving) excited oscillator modes of the string. One observes immediately that the lowest lying level $N = 0$ leads to a negative mass-squared, interpreted as a tachyonic field.

In superstring theory, there is a way of consistently projecting out these tachyonic modes, called GSO-projection. On the next level $N = 1$, the modes correspond to massless fields in space-time. These can be decomposed into a scalar contribution, interpreted as a dilaton field Φ, a spin 2 particle, interpreted as the graviton $G_{\alpha\beta}$, and an antisymmetric 2-form $B_{\alpha\beta}$, called the Kalb-Ramond tensor. In addition, we find so-called Ramond-Ramond n-form fields. On the next level, the mass is of the order of the string tension $\sqrt{T}$ which is high against accessible energies, it may be as high as the Planck mass. The appearance of the massless spin 2 particle is the first sign that string theory might have something to do with gravity.

Now, a quantized theory of a freely moving string in a 10 dimensional space-time might not seem to be very exciting, especially if we want string theory to describe gravity in the end. We should at least understand how to deal with string theory in a curved background, not to mention that eventually we would like to have a background-independent formulation.

For a given background consisting of a metric and maybe some further background fields, we start to describe a string by a non-linear σ-model involving the Polyakov action (coupling to the metric) supplemented by couplings to the other fields. From the claim for conformal invariance, we get conditions on the background. The change of the couplings under scale transformations is described by the β-functions which can be calculated perturbatively. In a conformally invariant theory they have to vanish, $\beta = 0$. This equation can be read as the equation of motion for the background fields. It turns out that in first order it coincides with the equations of motion of a 10-dimensional supergravity theory. In some sense, the strings determine the background in which they propagate. This is the second signal for gravity in string theory.

The effective supergravity theory has classical solutions. One class are solitonic solutions involving the RR-fields where the energy is concentrated around a $(p+1)$-dimensional plane in the 10-dimensional space-time. Because of this property, such solutions are called p-branes. Their mass or tension $T_p \sim 1/g_s$ scales with the inverse of the string coupling constant, hence a p-brane is a non-perturbative object. We are familiar with classical solutions in quantum field theories. There, we interpret them as describing different vacua around which we can perturbatively quantize our theory. We are following the same approach in string theory. The p-brane solution is thought of being a background on which we can formulate a quantized string theory.

From the closed string perspective, these solitonic branes are defects in space-time to which the closed strings can couple (see fig. 2.1). There is a dual point of view. We can re-interpret the picture as describing a closed string which opens up to form an open string with its ends on the brane (see fig. 2.1). It was Polchinski's idea [2] to model the solitonic p-branes microscopically by a theory of open strings whose ends are constrained on the p-brane. Interpreted this way, the brane is called a D(irichlet)-p-brane, because we are imposing Dirichlet boundary conditions on the open string in directions transverse to the brane.

Different checks have been carried out to verify this picture, to verify that the Dp-branes are the right model for the solitonic p-branes. Especially, it was checked that they coincide in tension and coupling to RR-fields.

Quantizing open strings in a flat space-time with Dirichlet boundary conditions in some directions and Neumann boundary conditions in the other directions is again

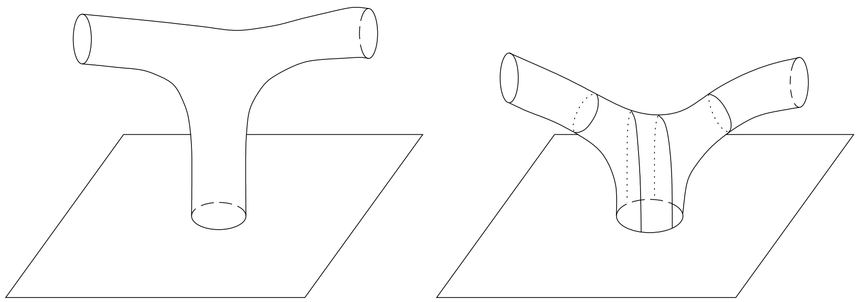


Figure 2.1: On the left: a closed string couples to a lower-dimensional defect in space-time. On the right: a closed string approaches the brane, opens up and interacts with the brane as open string and leaves the brane again.

not difficult. We interpret the excitation modes as space-time fields as we did before for the closed string. The massless fields consist of a vector field for the directions in the plane of the brane, and scalars describing the transverse fluctuations of the brane.

The effective theory of the fields on the world-volume of a brane are gauge theories. This is, on the one hand, an important property for string phenomenology. Indeed, it is possible to find configurations of intersecting branes s.t. the effective theory is a gauge theory with the gauge group of the standard model of particle physics (see e.g. [29] and references therein).

The appearance of gauge theories on branes led also to the famous AdS/CFT correspondence [4], stating that string theory on $\mathrm{AdS}_5 \times S^5$ is equivalent to a conformal super Yang Mills theory in four dimensions. Many other results in string theory over the past years are connected to D-branes, e.g. the calculation of entropy of extremal black holes [3]. Furthermore, D-branes play a fundamental role in the exploration of the web of dualities connecting the five string theories, and this has led to the conjecture of a unifying M-theory [30].

2.2 Exact string backgrounds

The starting point for the formulation of a perturbative string theory is a conformal field theory describing a single string moving in some target space. There is only a limited number of backgrounds where we have an exact quantization of string theory. The simplest possibility is a flat target space where the σ-model action is a massless quadratic action in the coordinates of the string. A quadratic action containing mass terms occurs in the case of a pp-wave background. Over the last months, much attention has been directed to this background (initiated by [31]).

If the background is weakly curved, a conformal σ-model can be formulated perturbatively. In the last section we saw that this leads at first order to the condition that the background is a solution of the supergravity action. Note that this perturbative expansion (expansion in α') is completely different from the string loop expansion in the coupling g_s.

When we consider strongly curved backgrounds, the described approach cannot be trusted any more, because we would need higher and higher corrections modifying the β-functions. Furthermore, non-perturbative effects are likely to appear.

Let us instead take a different point of view. The theory of a superstring in 10-dimensional space-time has central charge $c = 15$. Let us replace the σ-model describing the matter part by an *abstract* (Super) conformal field theory (CFT) of the same central charge $c = 15$. Analogously, for open strings, we replace the geometrical description of strings ending on a brane in space by an abstract CFT on a world-sheet with boundaries, called boundary CFT (BCFT). So, the question 'What D-branes can exist in a given background?' is now turned into 'What conformal boundary conditions can we impose in a given CFT?'.

Strongly curved backgrounds are interesting for a number of reasons. In string phenomenology, one studies compactifications of the ten-dimensional string theory on very short length scales, and unless torus compactifications are considered, the curvature is usually very large. On short length scales we expect that classical geometry gets modified and that it turns into some stringy or quantum geometry. By studying string theory on such backgrounds, we can explore these modifications and learn how to deal with geometry as an effective concept derived from the underlying theory. There are ideas on how to extract geometric data from an abstract CFT using non-commutative geometry (see e.g. [1, 32]). We will meet similar ideas when we explore the geometry of branes on short length scales.

How many exact non-trivial CFT backgrounds are there? One class of tractable conformal field theories are Wess-Zumino-Witten (WZW) models. Geometrically, they describe strings moving on a group manifold G. Most of the WZW models, however, cannot be part of a string background because their central charge is too large. One exception is the $SU(2)$ WZW model that appears in the near-horizon geometry of a stack of NS5-branes or in backgrounds containing $\mathrm{AdS}_3 \times S^3$.

Still, it makes sense to study WZW models in general. On the one hand, one can learn about phenomena which rely on the CFT nature of the models and could also occur in string backgrounds. On the other hand, WZW models are the starting point for the construction of essentially all known rational CFT's by 'conformal model building'. The tools in this 'model building' are the tensor product of two theories (central charges add up, $c = c_1 + c_2$), the coset construction (central charge is the difference $c = c_1 - c_2$) and orbifolding by a discrete group Γ (central charge is not affected). Many properties of the WZW models descend in the process of model building.

In the large class of coset models there are far more candidates that could take part in a string background. The Kazama-Suzuki models [33] are particularly interesting as they exhibit $N = 2$ supersymmetry. The simplest representatives of these models are the $N = 2$ minimal models. They are, on the other hand, the building blocks of the most prominent exact CFT backgrounds, the Gepner models [34].

Before we conclude this section, let us say a few words on the relation between the large volume limit and the abstract CFT description.

Often, to make contact with our world, one considers backgrounds of the form $\mathbb{R}^{1,3} \times M^6$ where the space-time consists of a four-dimensional Minkowski space on the one hand, and a small compact six-dimensional space M^6, or a CFT with central charge $c = 9$, on the other hand. In the limit of large volume and small curvature, the geometry of this six-dimensional compact space is restricted by conformal invariance and $N = 2$ supersymmetry to be a Calabi-Yau space, a Ricci-flat Kähler manifold with $SU(3)$ holonomy. In this regime, we can understand D-branes in Calabi-Yau manifolds as geometric objects. The D-branes preserving the highest amount of supersymmetry come in two classes: bundles on (special) Lagrangian submanifolds (type A) and bundles on holomorphic submanifolds (type B). Actually, it turns out that one has to take also singular bundles into consideration. This leads to the description of D-branes as objects in the category of coherent sheaves on the Calabi-Yau manifold.

A Calabi-Yau background is not isolated, we can change some parameters and deform the background, and we are still left with a Calabi-Yau manifold. These parameters are called moduli. We can now change moduli and ask what happens to the D-branes. There are some special points in the moduli space, points of higher symmetry. At the so-called Gepner point, the geometry is highly singular, but we have an abstract description as a CFT. Here, D-branes are described as conformal boundary conditions in Gepner models. Having descriptions of D-branes at different points of moduli space, it is of interest to study their relation and to explore how the geometric results are modified when we leave the large volume regime. See [35] for a discussion of this issue.

2.3 Dynamics and effective field theories

A first quantized string theory is given by a conformal quantum field theory which is defined on Riemann surfaces of arbitrary genus. To include open strings, one has to consider in addition conformal field theories on world-sheets with boundaries. Such a (B)CFT describes a background for the strings.

In this background, we have a perturbative prescription of how to calculate scattering amplitudes of strings by the Polyakov expansion. To any string mode we can associate a vertex operator in the CFT. The scattering amplitude for string modes is then calculated as a sum over correlation functions of the corresponding vertex operators on arbitrary Riemann surfaces, where the contributions are weighted according to the topology of the world-sheet, integrated over insertion points and moduli of the surfaces.

When we interpret the string modes as space-time fields, it is suggestive to look for an effective action for these modes resulting in the same scattering amplitudes.

In the CFT picture, at tree level, the equations of motion of these effective theories are nothing but the equations $\beta = 0$, the vanishing of the β-functions. The effective fields describing string modes take the role of couplings to operators by which the CFT is perturbed.

We can view it as follows: the fluctuations of strings in a given CFT background are governed by an effective action. The stationary points correspond to consistent conformally invariant backgrounds. The effective potential knows also something about the behavior of strings between these conformal fixed-points. One expects that there are transitions between string-vacua governed by flows or tunneling processes between them. String theory should be thought of a theory involving all two-dimensional field theories, the conformal ones being stationary points of some potential.

The flows along the gradient of the potential between two extrema are described by renormalization group (RG) flows between the two CFTs. RG flows thus describe certain aspects of dynamics in string theory.

Let us elaborate further on the calculation of the effective action. We usually have a perturbation expansion in two parameters: the expansion in the string coupling g_s (string loop expansion) which corresponds in quantum field theory to the Feynman graph expansion, and the expansion in string length (α' expansion) which controls the ('stringy') effects of the one-dimensional extension of a string.

As an example, we consider the effective theory describing a D-brane in a flat background. The massless modes of open strings on the brane are vector fields $\mathrm{A}_\alpha(\xi)$ living on the world-volume of the brane. The vertex operator associated to a mode

$\hat{\mathrm{A}}_\alpha(k)$ of momentum k is of the form

$$: \hat{\mathrm{A}}_\alpha(k) J^\alpha(x) e^{ik\cdot X(x)} : \tag{2.1}$$

where the colons denote normal ordering, and $J^\alpha = \partial X^\alpha$. The physical state conditions in string theory read $\hat{\mathrm{A}}_\alpha(k)k^\alpha = 0$ and $k^2 = 0$.

The scattering amplitudes of open string modes are then calculated from correlation functions of vertex operators (2.1). This calculation involves the operator product expansion (OPE) of the currents

$$J^\alpha(x_1)\ J^\beta(x_2)\ \sim\ \eta^{\alpha\beta}\frac{\alpha'}{(x_1-x_2)^2}\quad ,$$

where $\eta^{\alpha\beta}$ is the flat Minkowski metric, the OPE between currents and vertex operators,

$$J^\alpha(x_1)\ \ : e^{ik\cdot X(x_2)} : \sim\ \frac{\alpha'}{x_1-x_2}\, k^\alpha\ \ : e^{ik\cdot X(x_2)} :\ \ ,$$

and the OPE of vertex operators,

$$: e^{ik_1\cdot X(x_1)} :: e^{ik_2\cdot X(x_2)} : \sim\ \frac{1}{(x_1-x_2)^{\alpha' k_1\cdot k_2}} : e^{i(k_1+k_2)\cdot X(x_2)} : \quad . \tag{2.2}$$

At lowest order, the resulting effective action for the scattering of massless open string modes is a dimensionally reduced $U(1)$ gauge theory,

$$\mathcal{S}\ \sim\ \frac{1}{g_s}\int \mathrm{d}^{p+1}\xi\ \mathrm{F}_{\alpha\beta}(\xi)\mathrm{F}^{\alpha\beta}(\xi)\ +\ \text{fermions and scalars}$$

where $\mathrm{F}_{\alpha\beta} = \partial_\alpha \mathrm{A}_\beta - \partial_\beta \mathrm{A}_\alpha$.

When we consider a stack of n branes, corresponding to a multi p-brane solution, we have more degrees of freedom for the ends of the open string, and we can assign so called Chan-Paton charges to them. Consequently, the fields $\mathrm{A}_\alpha(\xi)$ are matrix-valued and the effective theory now becomes a $U(n)$ Yang-Mills theory. The field strength is then defined as $\mathrm{F}_{\alpha\beta} = \partial_\alpha \mathrm{A}_\beta - \partial_\beta \mathrm{A}_\alpha + i[\mathrm{A}_\alpha, \mathrm{A}_\beta]$. Let us mention that for a single brane it is possible to sum all α' corrections not containing derivatives of the field strength $\mathrm{F}_{\alpha\beta}$. The result is the Dirac-Born-Infeld action [36],

$$\mathcal{S}\ \sim\ \int \mathrm{d}^{p+1}\xi\, e^{-\Phi}\sqrt{\det(G_{\alpha\beta}+B_{\alpha\beta}+2\pi\alpha'\mathrm{F}_{\alpha\beta})}\ \ ,$$

where G is the induced metric on the brane and B is the pullback of the Kalb-Ramond tensor. It is valid for zero string coupling and as long as corrections from derivatives of $\mathrm{F}_{\alpha\beta}$ are negligible.

2.4 Non-commutative geometry

In the last section we saw that for low energies we have a description in terms of an effective field theory on a classical geometric background.

One expects, however, that in general at small scales the geometric description is not applicable, that we have to modify our notions of geometry.

We should not ask 'What geometry is there?' but rather 'What geometry do we experience?'. Or in terms of string theory 'What geometry do strings or branes see?'. We want to view the geometry of the target space as being an effective geometry.

From the string point of view, we should look at the effective action for string modes, and see whether it can be described as an effective field theory on a classical space.

In the example of the last section we got an action functional involving the couplings $\hat{\mathrm{A}}_\alpha(k)$ which can be viewed as Fourier components of a space-time field $\mathrm{A}_\alpha(\xi)$,

$$\mathrm{A}_\alpha(\xi) \;=\; \frac{1}{(2\pi)^{(p+1)/2}} \int \mathrm{d}k\; \hat{\mathrm{A}}_\alpha(k)\, e^{ik\cdot\xi} \quad .$$

The reason why this is possible can be found in the OPE (2.2) of the vertex operators. For low energies compared to the string scale, the conformal weights of the vertex operators approach zero, and the OPE of vertex operators takes the same form as the multiplication of Fourier modes. To be more precise, we introduce the formal vertex operators

$$V[\mathrm{A}](x) \;=\; \frac{1}{(2\pi)^{(p+1)/2}} \int \mathrm{d}k\; \hat{\mathrm{A}}(k)\; : e^{ik\cdot X(x)} : \quad .$$

In the limit $\alpha' \to 0$, the OPE (2.2) becomes independent from worldsheet coordinates, and the OPE of the operators $V[\mathrm{A}]$ is given by the usual multiplication of functions,

$$V[\mathrm{A}](x_1)\, V[\mathrm{B}](x_2) \;\sim\; V[\mathrm{A}\cdot\mathrm{B}](x_2) \quad . \tag{2.3}$$

Let us now consider a flat background with constant B-field. The OPE of boundary vertex operators changes to

$$: e^{ik_1\cdot X(x_1)} :: e^{ik_2\cdot X(x_2)} : \sim \; \frac{1}{(x_1-x_2)^{\alpha' k_1\cdot k_2}}\, e^{-i\frac{\pi}{2}\, k_1^T \Theta k_2} \; : e^{i(k_1+k_2)\cdot X(x_2)} : \tag{2.4}$$

where Θ is an anti-symmetric matrix depending on the B-field. This modification affects also (2.3). We can, however, sustain the structure of (2.3) if we deform the pointwise product of two space-time fields $\mathrm{A}(\xi)$ and $B(\xi)$ to the Moyal-Weyl product

$$(\mathrm{A}\star\mathrm{B})(\xi) \;=\; e^{\frac{i\pi}{2}\Theta^{\alpha\beta}\partial^\xi_\alpha\partial^\eta_\beta}\mathrm{A}(\xi)\mathrm{B}(\eta)\big|_{\eta=\xi} \quad .$$

Thus we see that it is more appropriate to think of the world-volume of the brane to be non-commutative rather than classical in this case. The low energy effective action is a Yang-Mills theory on a non-commutative space.

Non-commutativity in the case of flat branes has been observed in various ways [37, 38, 39]. Although in this example it is possible to map the non-commutative gauge theory to a gauge theory on a commutative space (Seiberg-Witten map, see [40]), the natural notion for the world-volume of a brane is as non-commutative space.

2.5 D-brane charges and K-theory

In superstring theory, D-branes carry Ramond-Ramond (RR) charges, they couple to the RR-fields. The RR charge of a D-brane configuration does not change under smooth deformations or dynamical processes. A classification of RR charges would then also encode a lot of information on the possible dynamics of branes.

If we have found a number of processes, we could ask more generally whether we can assign charges to branes s.t. they are conserved. By construction, the charges take values in an abelian group, and this group should be determined by the physical background. There is a conjecture that D-brane charges are classified by K-theory.

What is K-theory? K-theory is a generalized cohomology theory. Let us briefly explain one type of K-theory, namely topological K-theory. Consider the set of isomorphism classes of vector bundles on a manifold M. This set has a semi-ring structure, we can add bundles (Whitney sum) and we can multiply bundles (tensor product), but there is no notion of an additive inverse of a bundle. This is reminiscent of the set of natural numbers $\mathbb{N}$ which also has the structure of a semi-ring. And as we can pass from the natural numbers $\mathbb{N}$ to the ring $\mathbb{Z}$ of integers, we pass from vector bundles to K-theory. This means that we consider pairs of bundles (E, F) and identify $(E, F) \sim (E', F')$ if there is a bundle G s.t.

$$E \oplus F' \oplus G \;\cong\; E' \oplus F \oplus G \quad . \tag{2.5}$$

Because of this identification it is now possible to find for every equivalence class $[(E, F)]$ an inverse $[(F, E)]$. The resulting ring $K^0(M)$ is called the K-ring of M. Its non-torsion part coincides with the non-torsion part of de Rham cohomology of even cohomology classes.

One can define higher K-theories $K^i(M)$ turning K-theory into a generalized cohomology theory which satisfies all axioms of a cohomology theory except that higher cohomology classes of points may be non-trivial. For complex vector bundles, only two different K-groups appear due to Bott periodicity, namely $K^0(M)$ and

$K^1(M)$. We can think of the latter as being the analog of the cohomology group of odd cohomology classes.

In a more abstract way, we can think of vector bundles as projective modules of the algebra $\mathcal{F}(M)$ of (smooth) functions on M. An obvious extension of topological K-theory of a manifold is then the K-theory $K(\mathcal{A})$ of an arbitrary algebra $\mathcal{A}$ as the group completion of the semi-group of finite projective $\mathcal{A}$-modules.

Why should K-theory classify D-brane charges? Massless RR-fields are differential forms. It would appear most natural for RR-charges to be cohomology classes which are measured by integrating the differential forms over the cycles wrapped by the D-brane. By now it is well known that this naive expectation is not correct. For example, there are stable D-brane states that would not exist if D-brane charges were classified by cohomology (e.g. the non-supersymmetric D0-branes in type I string theory). There are several reasons why we should use K-theory instead to classify D-brane charges. D-branes are described not only by a submanifold, but by gauge and vector bundles, a structure which is natural in K-theory. Furthermore, in the case of type IIB string theory, we can directly interpret the equivalence relation (2.5) in terms of a physical process, namely brane-antibrane annihilation. Consider a stack of D9-branes bearing the gauge bundle E together with a stack of anti $\overline{\text{D9}}$-branes with gauge bundle F. The process of creation or annihilation of a set of D9- and $\overline{\text{D9}}$-branes with the same gauge bundle G relates the two configurations

$$(E, F) \;\leftrightarrow\; (E \oplus G, F \oplus G) \quad .$$

Starting from configurations of D9 and $\overline{\text{D9}}$ branes, any other configuration can be reached by a suitable tachyon condensation. This leads to the conclusion that D-brane charges in type IIB theory are classified by the topological K-group $K^0(M)$.

Which K-theory should be used? We just argued that D-brane charges in type IIB theory are classified by topological K-theory. This holds as long as there is a vanishing NSNS 3-form H. In type IIA, D-brane charges (in absence of an H-field) are classified by $K^1(M)$, for D-branes in type I theory one has to use K-theory over real vector bundles instead. When we consider orbifold models where a discrete symmetry group is divided out, it is natural to consider an equivariant version of K-theory.

In the presence of a nontrivial H-field $H \in H^3(M, \mathbb{Z})$, it was proposed by Bouwknegt and Mathai to employ the twisted K-groups $K^*_H(M)$. They are defined as K-groups of an algebra whose elements are sections of a bundle on M taking values in compact operators on a separable Hilbert space. Such algebras are classified by elements of $H^3(M, \mathbb{Z})$, and therefore they provide a natural candidate for a classification of charges in a background given by M and H. Furthermore, if H vanishes, the algebra of sections factorizes into functions on M and compact operators and it becomes Morita equivalent to the algebra of functions on M. As K-groups of Morita equivalent algebras are isomorphic, the twisted K-group $K^*_{H=0}(M)$ coincides with the ordinary topological K-group. When the H-field is torsion class, i.e. some integer multiple of it vanishes in $H^3(M, \mathbb{Z})$, the proposal of Bouwknegt and Mathai reduces to K-groups suggested in [41] (see also [42] for an extensive discussion).

We have seen in this chapter that there are essentially two ways of approaching D-branes: from the geometric side and from the conformal field theory side. In this thesis, we always want to think of a D-brane as a boundary conformal field theory. We even drop the conditions on supersymmetry and the value of the central charge. This can be motivated, on the one hand, by applications of boundary conformal field theory in statistical physics and condensed-matter physics where these constraints are absent. On the other hand, we hope to understand some generic features of branes by analyzing a larger class of conformal field theories which do not directly appear as string backgrounds. Although the results in this thesis are derived from CFT, we will make extensively use of the geometrical interpretation of branes, including their classical geometry as well as non-commutative, fuzzy structures.

Chapter 3

Branes in group manifolds and coset models

This chapter presents a description of maximally symmetric branes on group manifolds and coset models. We put emphasis on the open string spectrum and the geometry of branes, these are the most important data that will be needed in the subsequent chapters.

At the beginning, there is a short introduction into WZW models which are used to describe strings on group manifolds, followed by a discussion of the classical geometry of branes in such backgrounds. A description of the corresponding boundary conformal field theory concludes the first section.

Then, in the second section, we will concentrate on coset models. Their construction out of WZW models is briefly reviewed. Thereafter it is discussed how boundary conditions in WZW models descend to the quotient theory. At the end we give a short description of the geometric interpretation of branes in coset models.

3.1 D-branes on group manifolds

3.1.1 Geometry of branes on group manifolds

Strings on the group manifold of a simple and simply connected compact group G are described by a WZW-model. Its action is evaluated on fields $g : \Sigma \mapsto G$ on a world-sheet Σ taking values in G. It is a non-linear σ-model supplemented by a topological Wess-Zumino term. In string theory, the latter corresponds to a non-trivial background H-field. The action involves one (integer) coupling constant k, which is known as the 'level'. For our purposes it is most convenient to think of k as controlling the size (in string units) of the background. Large values of k correspond to a large volume of the group manifold. On the other hand, k determines the value of the 3-form H-field.

The theory on the world-sheet Σ contains the chiral currents

$$J(z) \;=\; k\, g^{-1}(z,\bar{z})\partial g(z,\bar{z}) \quad , \quad \bar{J}(\bar{z}) \;=\; -k\, \bar{\partial} g(z,\bar{z})\, g^{-1}(z,\bar{z}) \quad . \tag{3.1}$$

Note that J and $\bar{J}$ take values in the finite dimensional Lie algebra $\mathfrak{g}$ of the group G. Denoting by T_α the generators of $\mathfrak{g}$, we can expand the currents in the form $J(z) = \mathrm{T}_\alpha\, J^\alpha(z)$. In the quantum theory, the OPE of these currents reads

$$J^\alpha(z_1)J^\beta(z_2) \;\sim\; \frac{k\,\kappa^{\alpha\beta}}{(z_1-z_2)^2} + \frac{i}{(z_1-z_2)} f^{\alpha\beta}{}_\gamma\, J^\gamma(z_2) \tag{3.2}$$

where $f^{\alpha\beta}{}_\gamma$ are the structure constants of $\mathfrak{g}$ and $\kappa^{\alpha\beta}$ is the Killing form.

The Laurent modes J^α_n of the currents satisfy the commutation relations of the infinite-dimensional affine Lie algebra $\widehat{\mathfrak{g}}_k$,

$$[J^\alpha_m\,,\, J^\beta_n] \;=\; i f^{\alpha\beta}{}_\gamma\, J^\gamma_{m+n} + k\, m\, \kappa^{\alpha\beta}\, \delta_{m+n,0} \quad ,$$

and they generate the chiral symmetry algebra of the model. The stress energy tensor is obtained through the Sugawara construction. Due to the large symmetry, WZW models are rational conformal field theories, meaning that there is only a finite set of irreducible representations of the chiral algebra. They have central charge

$$c = \frac{k}{k+g^\vee} \dim G$$

where $g^\vee$ denotes the dual Coxeter number of $\mathfrak{g}$. In the limit $k \to \infty$, the central charge approaches the dimension of the group G.

When we want to describe open strings at tree level, the 2-dimensional world-sheet Σ is taken to be the upper half plane $\Sigma = \{z \in \mathbb{C} | \Im z \geq 0\}$.

Along the boundary of this world sheet we need to impose some boundary condition. To ensure conformal invariance we need to glue the two non-vanishing components $T, \overline{T}$ of the stress energy tensor according to

$$T(z) \;=\; \overline{T}(\bar{z}) \quad \text{for all } z = \bar{z} \quad . \tag{3.3}$$

In general, the problem of finding all conformal boundary conditions is not tractable (see however [43] for recent progress). Here we will analyze instead boundary conditions that preserve the full bulk symmetry of the model[1], i.e. the affine algebra $\widehat{\mathfrak{g}}_k$.

[1] Boundary conditions preserving a smaller symmetry algebra have been considered in [44, 45]

Along the real line, the holomorphic and the anti-holomorphic currents are glued by imposing

$$J(z) \;=\; \Lambda \bar{J}(\bar{z}) \quad \text{for all} \quad z = \bar{z} \tag{3.4}$$

where Λ is an automorphism of the current algebra $\widehat{\mathfrak{g}}_k$ s.t. $\Lambda \overline{T} = \overline{T}$ (see e.g. [46]).

The allowed automorphisms Λ of the affine Lie algebra $\widehat{\mathfrak{g}}_k$ are easily classified. They are all of the form

$$\Lambda \;=\; \Omega \circ \mathrm{Ad}_{\tilde{g}} \quad \text{for some} \;\; \tilde{g} \in G \;\; . \tag{3.5}$$

Here, $\mathrm{Ad}_{\tilde{g}}$ denotes the adjoint action of the group element $\tilde{g}$ on the current algebra $\widehat{\mathfrak{g}}_k$. It is induced in the obvious way from the adjoint action of G on the finite dimensional Lie algebra $\mathfrak{g}$. The automorphism Ω does not come from conjugation with some element g. More precisely, it is an outer automorphism of the current algebra. Such outer automorphisms $\Omega = \Omega_\omega$ come with symmetries ω of the Dynkin diagram of the finite dimensional Lie algebra $\mathfrak{g}$. One may show that the choice of ω and $\tilde{g} \in G$ in eq. (3.5) exhausts all possibilities for the gluing automorphism Λ (see e.g. [47]).

So far, our discussion of the admissible types of gluing automorphisms Λ has been fairly abstract. It is possible, however, to associate some concrete geometry with each choice of Λ. There are different approaches to obtain a geometric description of the branes. One can analyze the gluing conditions (3.4) as classical equations for the open strings to see whether these employ Dirichlet conditions in some directions and force the string ends to stay on some submanifold of G. This was done in [15] for $\omega = \mathrm{id}$. On the other hand, it is possible to use closed strings as probes for the geometry. Such an analysis has been performed for a general gluing automorphism ω in [16] (see also [48], [49]).

Let us assume first that the element $\tilde{g}$ in eq. (3.5) coincides with the group unit $\tilde{g} = e$. This means that $\Lambda = \Omega = \Omega_\omega$ is determined by ω alone. The diagram symmetry ω induces an (outer) automorphism ω of the finite dimensional Lie algebra $\mathfrak{g}$ through the unique correspondence between vertices of the Dynkin diagram and simple roots. After exponentiation, ω furnishes an automorphism of the group G which we will also denote by ω to keep notations simple. One can show that the classical geometry of a brane with gluing condition (3.4) is that of a ω-twisted conjugacy class,

$$C^\omega_g \;:=\; \{\, g' g\, \omega(g')^{-1} \mid g' \in G \,\} \;\; .$$

The subsets $C^\omega_g \subset G$ are parametrized by equivalence classes of group elements g where the equivalence relation between two elements $g_1, g_2 \in G$ is given by: $g_1 \sim_\omega g_2$ iff $g_2 \in C^\omega_{g_1}$. Note that this parameter space $\mathcal{U}^\omega$ of equivalence classes is not a manifold, i.e. it contains singular points.

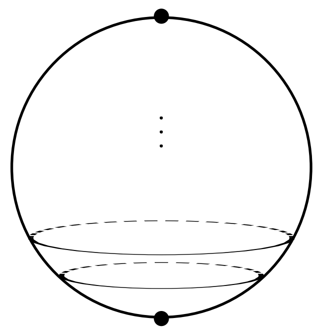


Figure 3.1: The geometry of maximally symmetric branes on $SU(2) \simeq S^3$ corresponding to the trivial gluing automorphism. The generic branes are 2-spheres S^2, and they degenerate to points at $\pm e$. Note that the picture only shows the analog situation in one dimension less.

Before we continue the more general description of the geometry, we want to consider the simplest example, namely branes in the group manifold $SU(2)$. In this case all possible automorphisms are inner, therefore we are restricted to $\Omega = \mathrm{id}$. The conjugacy classes C_g^{id} are 2-spheres $S^2 \subset S^3 \cong SU(2)$ for generic points g and they consist of a single point when $g = \pm e$ in the center of $SU(2)$ (see fig. 3.1). The branes are labeled by a one-dimensional parameter e.g. (locally) by the size of the two-sphere.

To describe the topology of C_g^ω and the parameter space $\mathcal{U}^\omega$ (at least locally) in the general case, we need some more notation. By construction, the action of ω on $\mathfrak{g}$ can be restricted to an action on the Cartan subalgebra $\mathcal{T}$. We shall denote the subspace of elements which are invariant under the action of ω by $\mathcal{T}^\omega \subset \mathcal{T}$. Elements in $\mathcal{T}^\omega$ generate a torus $T^\omega \subset G$. One may show that the generic ω-twisted conjugacy class C_g^ω looks like the quotient G/T^ω. Hence, the dimension of the generic submanifolds C_g^ω is $\dim G - \dim T^\omega$ and the parameter space has dimension $\dim \mathcal{T}^\omega$ almost everywhere. In other words, there are $\dim \mathcal{T}^\omega$ directions transverse to a generic twisted conjugacy class. This implies that the branes associated with the trivial diagram automorphism $\omega = \mathrm{id}$ have the largest number of transverse directions. It is given by the rank of the Lie algebra.

As we shall see below, not all these submanifolds C_g^ω can be wrapped by branes on group manifolds. There exists some integrality requirement that can be understood in various ways, e.g. as quantization condition within a semiclassical analysis [15] of the brane's stability [50, 51, 52]. This implies that there is only a finite set of allowed branes (if k is finite). The number of branes depends on the volume of the group measured in string units.

Let us become somewhat more explicit for $G = SU(n)$. We already discussed the simplest case with $n = 2$. For general $SU(n)$, the formulas $\dim SU(n) = (n-1)(n+1)$ and $\mathrm{rank}\, SU(n) = (n-1)$ show that the generic submanifolds C_g^{id} have dimension $\dim C_g^{\mathrm{id}} = (n-1)n$. In addition, there are singular cases, n of which are associated with elements g in the center $\mathbb{Z}_n \subset SU(n)$. The corresponding submanifolds C_g^{id} are 0-dimensional. Note that all the submanifolds C_g^{id} are even dimensional. Similarly, the generic manifolds C_g^{ω} for the non-trivial diagram symmetry ω have dimension $\dim C_g^{\omega} = (n-1)(n+1/2)$ for odd n and $\dim C_g^{\omega} = n^2 - n/2 - 1$ whenever n is even. For some exceptional values of g, the dimension can be lower. A complete illustrated discussion for branes in $SU(3)$ can be found in [53].

So far we restricted ourselves to $\Lambda = \Omega_\omega$ being a diagram automorphism. As we stated before, the general case is obtained by admitting an additional inner automorphism of the form $\mathrm{Ad}_{\tilde{g}}$. Geometrically, the latter corresponds to rigid translations on the group induced from the left action of $\tilde{g}$ on the group manifold (see e.g. [54, 55]). The freedom of translating branes on G does not lead to any new charges or to essentially new physics and we shall not consider it any further, i.e. we shall assume $\tilde{g} = e$ in what follows.

3.1.2 Conformal field theory description

The branes we considered in the previous section may be described through an exactly solvable conformal field theory[2]. In particular, there exists a detailed knowledge about their open string spectra based on the work of Cardy [13] and of Birke, Fuchs and Schweigert [14].

We shall use $\lambda, \mu, \ldots \in \mathcal{B}^\omega(\widehat{\mathfrak{g}}_k)$ to label different elementary conformal boundary conditions of the conformal field theories associated with the gluing conditions eq. (3.4) on the currents. The set $\mathcal{B}^\omega(\widehat{\mathfrak{g}}_k)$ depends on the choice of the diagram automorphism ω and on the level k. In principle, it depends also on the bulk spectrum, but we will always consider WZW models with charge-conjugated modular invariant[3].

Our main goal now is to explain the open string spectra that come with the maximally symmetric branes on group manifolds. For a pair of boundary labels $\lambda, \mu \in \mathcal{B}^\omega(\widehat{\mathfrak{g}}_k)$ associated with the same diagram automorphism ω, the open string

[2] A brief introduction into (boundary) conformal field theory can be found in Appendix A.
[3] We will encounter one exception in an example in Section 5.2.4.

Hilbert space can be decomposed into representations $\mathcal{H}^L$ of the current algebra $\widehat{\mathfrak{g}}_k$

$$\mathcal{H}_\lambda{}^\mu \;=\; \bigoplus_{L\in\mathrm{Rep}(\widehat{\mathfrak{g}}_k)} n_{L\lambda}{}^\mu \, \mathcal{H}^L \quad . \tag{3.6}$$

Obviously, the numbers $n_{L\lambda}{}^\mu$ have to be non-negative integers.

There exists a very simple argument due to Behrend et al. [56, 57] which shows that the matrices $(n_L)_\lambda{}^\mu$ give rise to a representation of the fusion algebra of $\widehat{\mathfrak{g}}_k$. This means that they obey the relations

$$\sum_{\mu\in\mathcal{B}^\omega(\widehat{\mathfrak{g}}_k)} (n_J)_\lambda{}^\mu \, (n_K)_\mu{}^\nu \;=\; \sum_{L\in\mathrm{Rep}(\widehat{\mathfrak{g}}_k)} N_{JK}{}^L (n_L)_\lambda{}^\nu \quad , \tag{3.7}$$

where $N_{JK}{}^L$ are the fusion rules of the current algebra $\widehat{\mathfrak{g}}_k$.

The argument of [56, 57] starts from a general Ansatz for the boundary state assigned to $\lambda \in \mathcal{B}^\omega(\widehat{\mathfrak{g}}_k)$. Using world sheet duality as in the derivation of the Cardy condition, one can express the matrices n_L in terms of the coefficients of the boundary states, which we denote by S^ω, and the modular matrix S of the current algebra $\widehat{\mathfrak{g}}_k$. The result resembles the familiar Verlinde formula,

$$(n_L)_\lambda{}^\mu \;=\; \sum_{J\in\mathrm{Spec}^\omega} \frac{\bar{S}^\omega_{J\mu} S^\omega_{J\lambda} S_{JL}}{S_{J0}} \quad \text{for} \quad \lambda,\mu\in\mathcal{B}^\omega(\widehat{\mathfrak{g}}_k) \quad , \quad L\in\mathrm{Rep}(\widehat{\mathfrak{g}}_k) \quad . \tag{3.8}$$

Here, $J\in\mathrm{Spec}^\omega$ runs over labels in $\mathrm{Rep}(\widehat{\mathfrak{g}}_k)$ that correspond to closed string modes which couple to boundary conditions with the gluing automorphism ω (see eq. (A.7) in Appendix A for a definition of Spec^ω). For a charge-conjugated modular invariant, this set simply consists of ω-symmetric weights

$$\mathrm{Spec}^\omega \;=\; \mathrm{Rep}^\omega(\widehat{\mathfrak{g}}_k) \;=\; \left\{ J\in\mathrm{Rep}(\widehat{\mathfrak{g}}_k) | \omega(J) = J \right\} \quad .$$

Assuming that we have found a complete set of elementary boundary conditions, the matrices S^ω have to be unitary. With these informations, the relations (3.7) can be directly checked using the expression (3.8).

The relations (3.6), (3.7) give the general form of the open string spectrum of maximally symmetric branes, and this result is valid not only in WZW models. In the remainder of this section, we will present concrete answers for the set of boundary labels $\mathcal{B}^\omega(\widehat{\mathfrak{g}}_k)$ and the integer coefficients n_L which have been achieved over the last years [13, 58, 14].

Let us start with the trivial diagram automorphism $\omega = \mathrm{id}$. Here, the set of boundary labels $\mathcal{B}^{\mathrm{id}}(\widehat{\mathfrak{g}}_k) = \mathrm{Rep}(\widehat{\mathfrak{g}}_k)$ coincides with the set of primaries of the affine Kac-Moody algebra $\widehat{\mathfrak{g}}_k$. The latter is well known to form a subset in the space $\text{Rep}(\mathfrak{g})$ of dominant integral weights which label equivalence classes of finite-dimensional irreducible representations for the finite dimensional Lie algebra $\mathfrak{g}$. To keep notations simple, we will not distinguish in notation between elements of $\text{Rep}(\mathfrak{g})$ and $\text{Rep}(\widehat{\mathfrak{g}}_k)$ and denote them both by capital letters $J, K, L, \ldots$.

The unitary matrix S^{id}_{JL} is given by the modular S-matrix. Using eq. (3.8) and the Verlinde formula (see eq. (A.3)) one finds that the matrices ${(n_L)_J}^K$ are just the fusion rules ${(N_L)_J}^K$. This is known as the 'Cardy case' [13].

We will now turn to the discussion of non-trivial automorphisms. In this case, branes can be labeled by certain fractional symmetric weights out of the set

$$\mathcal{B}^\omega(\mathfrak{g}) \;=\; \left\{\lambda = \sum \lambda_i \omega_i \,\middle|\, \lambda_i = \lambda_{\omega(i)}\;,\; l_i\lambda_i \in \mathbb{N}_0\right\} \;. \tag{3.9}$$

Here, ω_i are the fundamental weights of $\mathfrak{g}$, and the numbers l_i denote the length of the orbit of the i-th Dynkin node under the action of the automorphism ω. For finite k, the set of boundary labels is a subset $\mathcal{B}^\omega(\widehat{\mathfrak{g}}_k) \subset \mathcal{B}^\omega(\mathfrak{g})$ of the set of all fractional symmetric weights which is truncated by the level k. The precise form of the truncation will not be important in the following (see [14] for details). Let us briefly mention that the number of boundary conditions $\lambda \in \mathcal{B}^\omega(\widehat{\mathfrak{g}}_k)$ is equal to the number of symmetric weights in $\text{Rep}(\widehat{\mathfrak{g}}_k)$, this is nothing but the statement that $\mathcal{B}^\omega(\widehat{\mathfrak{g}}_k)$ is a complete set of elementary boundary conditions.

We will mainly deal with branes in the regime where the level k is large. In this case, we can label boundary conditions by the set $\mathcal{B}^\omega(\mathfrak{g})$ which can be obtained as inductive limit of the $\mathcal{B}^\omega(\widehat{\mathfrak{g}}_k)$ for $k \to \infty$.

The spectrum of twisted branes is determined, once we now the unitary matrix S^ω. An expression for this matrix has been given in [14]

$$S^\omega_{J\lambda} \;\sim\; \sum_{w\in W_\omega} \epsilon_\omega(w) \exp\Big(-\frac{2\pi i}{k+g^\vee}\big(w(J+\rho), \lambda+\rho_\omega\big)\Big) \;. \tag{3.10}$$

Here, $W_\omega \subset W$ is the ω-invariant part in the Weyl group of $\mathfrak{g}$. As W_ω can be considered as the Weyl group of another Lie algebra [59] it comes with a natural sign function ϵ_ω. The fractional weight $\rho_\omega = (1/l_1, 1/l_2, \ldots)$ is the twisted counterpart of the Weyl vector $\rho = (1, \ldots, 1)$.

The integers ${n_{L\lambda}}^\mu$ can now be calculated by eq. (3.8), resulting in a complicated expression involving sums over the Weyl group W_ω. It is possible to rewrite the n_L as simple linear combinations of fusion matrices belonging to some other affine Lie algebras [60, 61].

This completes our discussion of the spectrum of maximally symmetric branes on group manifolds. Before we move on to coset models, we want to say a few words about the boundary fields in the theory.

There is a generalized state-field correspondence that associates to every highest weight state $L \in \text{Rep}(\widehat{\mathfrak{g}}_k)$ in the Hilbert space (3.6) of the (λ, μ) boundary conformal field theory a boundary primary field $\psi_L^{(\lambda\mu)}$ living between boundaries $\lambda, \mu \in \mathcal{B}^\omega(\widehat{\mathfrak{g}}_k)$. The general structure of the boundary operator product expansion (OPE) is given by

$$\psi_J^{(\lambda\mu)}(x_1)\, \psi_K^{(\mu\nu)}(x_2) \ \sim \ \sum_L (x_1 - x_2)^{h_L - h_J - h_K}\, C_{JK}^{(\lambda\mu\nu)\,L} \psi_L^{(\lambda\nu)}(x_2) \quad \text{for} \quad x_1 < x_2 \tag{3.11}$$

where the numbers h_L denote the conformal weights of the fields which, in the case at hand, are given by

$$h_L \ = \ \frac{C_L}{2(k + g^\vee)} \quad . \tag{3.12}$$

Here, C_L is the quadratic Casimir of the representation $L \in \text{Rep}(\widehat{\mathfrak{g}}_k)$. For a consistent conformal field theory, there are conditions on the structure constants, in particular the associativity of the OPE has to be warranted. In the case of boundary CFT, these conditions, the sewing constraints, were first considered by Lewellen [62] (see also [63, 64, 65]).

Solutions to the sewing constraints are known for the case of trivial gluing automorphism $\omega = \text{id}$ (see [64, 17, 16, 56]). For non-trivially twisted boundary conditions, the boundary OPE is still to be found.

3.2 Boundary coset models

3.2.1 Coset construction

From now on let $H \subset G$ denote some simple simply connected subgroup of G. We want to study the associated G/H coset model. A more precise formulation of this theory requires a bit of preparation (more details can be found e.g. in [66]).

Induced from the embedding of H in G, there is an embedding of the affine Lie algebra $\widehat{\mathfrak{h}}_{k'}$ into $\widehat{\mathfrak{g}}_k$. The level k' is related to k by the embedding index x_e, $k' = kx_e$. We shall label the sectors $\mathcal{H}_{\mathfrak{h}}^{L'}$ of the affine Lie algebra $\widehat{\mathfrak{h}}_{k'}$ with labels $L' \in \text{Rep}(\widehat{\mathfrak{h}}_{k'})$. Note that the sectors of the numerator theory carry an action of the denominator algebra $\widehat{\mathfrak{h}}_{k'} \subset \widehat{\mathfrak{g}}_k$ and under this action each sector $\mathcal{H}_{\mathfrak{g}}^L$ decomposes according to

$$\mathcal{H}_{\mathfrak{g}}^L \ = \ \bigoplus_{L'} \ \mathcal{H}^{(L,L')} \otimes \mathcal{H}_{\mathfrak{h}}^{L'} \quad . \tag{3.13}$$

Here we have introduced the infinite dimensional spaces $\mathcal{H}^{(L,L')}$ which we want to interpret as sectors of the coset chiral algebra. The latter is usually hard to describe explicitly, but at least it is known to contain a Virasoro field with modes

$$L_n \;=\; L_n^{\mathfrak{g}} - L_n^{\mathfrak{h}} \quad . \tag{3.14}$$

One may easily check that they obey the usual exchange relations of the Virasoro algebra with central charge given by $c = c^{\mathfrak{g}} - c^{\mathfrak{h}}$.

Note that some of the spaces $\mathcal{H}^{(L,L')}$ may vanish simply because a given sector $\mathcal{H}_{\mathfrak{h}}^{L'}$ of the denominator theory may not appear as a subsector in a given $\mathcal{H}_{\mathfrak{g}}^{L}$. This allows to introduce the set

$$\mathcal{E} \;=\; \{\,(L,L') \in \mathrm{Rep}(\widehat{\mathfrak{g}}_k) \times \mathrm{Rep}(\widehat{\mathfrak{h}}_{k'})\,|\,\mathcal{H}^{(L,L')} \neq 0\,\} \quad .$$

Furthermore, some of the coset spaces labeled by different pairs (L,L') and (M,M') correspond to the same sector of the coset theory. Therefore when we label coset sectors by pairs of labels (L,L'), we have to take selection and identification rules into account.

There is an elegant formalism to describe these rules which is applicable in almost all coset models. It involves the so-called identification group $\mathcal{G}_{\mathrm{id}}$ which contains pairs $(\mathcal{J},\mathcal{J}')$ of simple currents. It is a subgroup of the direct product of the simple current groups of $\widehat{\mathfrak{g}}_k$ and $\widehat{\mathfrak{h}}_{k'}$. A simple current $\mathcal{J}$ of $\widehat{\mathfrak{g}}_k$ is an element in $\mathrm{Rep}(\widehat{\mathfrak{g}}_k)$ which has the property that the fusion product of $\mathcal{J}$ with any other representation L contains exactly one sector $K = \mathcal{J}L \in \mathrm{Rep}(\widehat{\mathfrak{g}}_k)$,

$$N_{\mathcal{J}L}{}^{K} \;=\; \delta_{K,\mathcal{J}L} \quad .$$

To formulate the selection rules in coset models, we introduce the monodromy charge $Q_{\mathcal{J}}(L)$ of L with respect to $\mathcal{J}$ in terms of conformal weights,

$$Q_{\mathcal{J}}(L) \;=\; h_{\mathcal{J}} + h_L - h_{\mathcal{J}L} \qquad \mathrm{mod} \;\; \mathbb{Z} \quad .$$

The monodromy charge appears when a simple current $\mathcal{J}$ acts on the modular S-matrix,

$$S_{\mathcal{J}L\,K} \;=\; e^{2\pi i Q_{\mathcal{J}}(K)}\, S_{L\,K} \quad .$$

We are now prepared to formulate selection and identification rules in terms of the identification group $\mathcal{G}_{\mathrm{id}}$ of simple currents:

- A pair (L,L') is allowed, i.e. $(L,L') \in \mathcal{E}$, if $Q_{\mathcal{J}}(L) = Q_{\mathcal{J}'}(L')$ for all $(\mathcal{J},\mathcal{J}') \in \mathcal{G}_{\mathrm{id}}$

- Two pairs (L,L') and $(\mathcal{J}L,\mathcal{J}'L')$ label the same sector, i.e.

$$\mathcal{H}^{(L,L')} \;\cong\; \mathcal{H}^{(\mathcal{J}L,\mathcal{J}'L')} \quad .$$

At this point we want to make one assumption, namely that all the equivalence classes we find in $\mathcal{E}$ contain the same number $N_0 = |\mathcal{G}_{\rm id}|$ of elements, in other words $\mathcal{G}_{\rm id}$ acts fixed-point free. This holds true for many important examples and it guarantees that the sectors of the coset theory are simply labeled by the equivalence classes[4], i.e. $\text{Rep}(\widehat{\mathfrak{g}}/\widehat{\mathfrak{h}}) = \mathcal{E}/\mathcal{G}_{\rm id}$. It is then also easy to spell out explicit formulas for the fusion rules and the S-matrix of the coset model. These are given by

$$N_{(J,J')(K,K')}{}^{(L,L')} \;=\; \sum_{(M,M')\sim(L,L')} N^{\mathfrak{g}}_{J\,K}{}^{M}\, N^{\mathfrak{h}}_{J'\,K'}{}^{M'} \quad , \tag{3.15}$$

$$S_{(L,L')(K,K')} \;=\; N_0\, S^{\mathfrak{g}}_{L\,K}\; \bar{S}^{\mathfrak{h}}_{L'\,K'} \tag{3.16}$$

where the bar over the second S-matrix denotes complex conjugation.

3.2.2 Boundary conditions from WZW models

Let us turn now to coset models with a boundary. We want to impose conditions along the boundary gluing left moving and right moving fields together with a suitable automorphism of the coset chiral algebra. In what follows, we will consider automorphisms that are induced by an automorphism of the numerator theory $\widehat{\mathfrak{g}}_k$. Whenever an automorphism of $\widehat{\mathfrak{g}}_k$ can be restricted to $\widehat{\mathfrak{h}}_{k'}$, it can also be restricted to the coset chiral algebra.

In the last section, we discussed what maximally symmetric boundary conditions are possible in WZW models. We label them by $\mathcal{B}^\omega(\widehat{\mathfrak{g}}_k)$ and $\mathcal{B}^\omega(\widehat{\mathfrak{h}}_{k'})$, respectively. The natural guess would be that branes in coset models are then labeled by pairs

$$(\lambda,\lambda') \;\in\; \mathcal{B}^\omega(\widehat{\mathfrak{g}}_k)\times\mathcal{B}^\omega(\widehat{\mathfrak{h}}_{k'})$$

with the spectrum

$$\mathcal{H}_{(\lambda,\lambda')}^{(\mu,\mu')} \;=\; \bigoplus_{(L,L')} n_{L\lambda}{}^{\mu}\; n_{L'\lambda'}{}^{\mu'}\; \mathcal{H}^{(L,L')} \quad . \tag{3.17}$$

This idea is essentially correct, but as for the sectors we have to deal with identification and selection rules for pairs of boundary labels. Only if the identification

[4]For more general cases, there are further sectors that cannot be constructed within the sectors of the numerator theory.

acts without fixed-points, the formula (3.17) gives the correct result. Note that fixed-points in boundary labels may occur even in coset models where there are no fixed-points in the sectors, i.e. where $\mathcal{G}_{\rm id}$ acts without fixed-points on $\mathcal{E}$.

To formulate identification and selection rules along the same lines as before, we have to introduce an action of the identification group $\mathcal{G}_{\rm id}$ on boundary labels, and we should have a notion of monodromy charge of a boundary condition. These concepts have been introduced in [22]. The action of a simple current on a boundary condition is defined with help of the matrix S^ω,

$$S^\omega_{L\,\mathcal{J}\lambda} \;=\; e^{2\pi i Q_{\mathcal{J}}(L)}\, S^\omega_{L\lambda} \;\;\text{for all}\;\; \lambda \in \mathcal{B}^\omega(\hat{\mathfrak{g}}_k) \quad .$$

Although not obvious at first sight, $\mathcal{J}\lambda$ is well-defined in this way. Similarly, the notion of a monodromy charge $Q_{\mathcal{J}}(\lambda)$ can be introduced by the formula

$$S^\omega_{\mathcal{J}L\,\lambda} \;=\; e^{2\pi i Q_{\mathcal{J}}(\lambda)} S^\omega_{L\,\lambda} \quad . \tag{3.18}$$

Again, the definition is only implicit, and in this case it is not completely clear that a well-defined charge $Q_{\mathcal{J}}(\lambda)$ can be extracted [22]. Note, however, that in all examples discussed in this thesis, the definition (3.18) works fine.

With the help of the introduced concepts, the selection rules and coset rules are formulated in a straightforward way. Details can be found in [22].

3.2.3 Geometry of coset branes

The geometry of maximally symmetric branes in coset models was uncovered in [23] (see also [24, 10]). To describe the answer we need some more notation. Recall first that geometrically the quotient G/H is formed with respect to the adjoint action of H on G, i.e. two points on G are identified if they are related by conjugation with an element of $H \subset G$. We denote the projection from G to the space G/H of H orbits by

$$\pi^G_{G/H} \;:\; G \longrightarrow G/H \quad .$$

Furthermore, we use $C^{G;\omega}_\lambda$ to refer to the conjugacy class of G along which the brane with label λ is localized and similarly for $C^{H;\omega}_{\lambda'}$. The latter is a conjugacy class in H. Through the embedding of H into G, we can regard it as a subset of G. Now we construct the set $C^{G;\omega}_\lambda (C^{H;\omega}_{\lambda'})^{-1}$ of all elements in G which are of the form gh^{-1} where $g \in C^{G;\omega}_\lambda$ and $h \in C^{H;\omega}_{\lambda'}$. This set is left invariant by conjugation with elements of H,

$$\begin{aligned} h\, C^{G;\omega}_\lambda (C^{H;\omega}_{\lambda'})^{-1}\, h^{-1} \;&=\; \big(h\, C^{G;\omega}_\lambda\, \omega(h)^{-1}\big)\,\big(h\, C^{H;\omega}_{\lambda'}\, \omega(h)^{-1}\big)^{-1} \\ &=\; C^{G;\omega}_\lambda (C^{H;\omega}_{\lambda'})^{-1} \quad , \end{aligned}$$

and hence it can be projected down to G/H. The brane (λ, λ') is then localized along the resulting subset $C^{G/H;\omega}_{(\lambda,\lambda')}$ of G/H,

$$C^{G/H;\omega}_{(\lambda,\lambda')} \;=\; \pi^{G}_{G/H}\Big(C^{G;\omega}_{\lambda} \; (C^{H;\omega}_{\lambda'})^{-1} \Big) \;\subset\; G/H \quad . \tag{3.19}$$

This is the straightforward generalization of a claim in [23] for the trivial automorphism $\omega = \mathrm{id}$. The result of [23] has been confirmed by an analysis of the 1-point functions of bulk fields in the presence of an untwisted coset brane [10].

We will present examples for this geometric descriptions for minimal models and parafermion theories in Chapter 4 and 5.

Chapter 4

Brane dynamics in WZW and coset models

In this chapter, we will analyze the dynamics of maximally symmetric branes in WZW and coset models by studying the low-energy effective theory describing excitations of open string modes on the brane. The effective theory is derived in a perturbative approach around the limit of large levels k. This limit can be understood as going to large volume of the target space. Note, however, that we allow the brane to be arbitrary small, and we cannot expect that the world-volume of the brane is just given by classical geometry. Indeed, it turns out that the effective theory can be described by a gauge theory on a non-commutative world-volume.

We will start in Section 4.1 to investigate the non-commutative geometry of the branes in WZW models. In Section 4.2, the effective gauge theory on the non-commutative world-volume will be introduced. Classical solutions of this theory are studied in Section 4.3, and an interpretation of these solutions in terms of brane processes is provided. Section 4.4 describes how results from WZW models can be used to construct an effective field theory of branes in coset models along with its solutions. Finally, we will discuss in Section 4.5 the results on coset models in some explicit examples.

4.1 Non-commutative geometry

4.1.1 Quantum geometry of branes

In the last chapter we got an idea about the shape of branes in a group manifold. We saw that their classical geometry is described by (twisted) conjugacy classes.

How do open strings experience the world-volume of a brane? We expect to find the classical picture of generic regular conjugacy classes when the brane is large and weakly curved, but when the brane's volume becomes small and when we are close to degeneration points, open strings will see a non-commutative world-volume.

In Chapter 2 we already saw in the example of a D-brane in the presence of a constant B-field, how the effective geometry felt by open strings can change. According to our procedure there, the world-volume geometry of branes can be read off from the correlators of boundary operators in the decoupling regime $k \to \infty$. Note that the conformal dimensions (3.12) of the boundary fields vanish in this limit so that the operator product expansion (3.11) becomes independent of the world sheet coordinates.

The world-volume algebra $\mathcal{A}^\lambda$ of a brane λ should be such that there is a map V sending $\mathrm{A} \in \mathcal{A}^\lambda$ to a boundary field $V[\mathrm{A}](x)$ respecting the OPE, i.e.

$$V[\mathrm{A}_1](x_1)\, V[\mathrm{A}_2](x_2) \;\sim\; V[\mathrm{A}_1\, \mathrm{A}_2](x_2) \quad . \tag{4.1}$$

In the case of WZW models, we have in addition the OPE involving the currents J^α. They induce on $\mathcal{A}^\lambda$ an action of the Lie algebra $\mathfrak{g}$ by

$$J^\alpha(x_1)\, V[\mathrm{A}](x_2) \;\sim\; \frac{1}{x_1 - x_2} V[\mathrm{L}^\alpha \mathrm{A}](x_2) \quad . \tag{4.2}$$

The action $\mathrm{L}^\alpha : \mathcal{A}^\lambda \to \mathcal{A}^\lambda$ of the generators T^α of $\mathfrak{g}$ turns $\mathcal{A}^\lambda$ into a G-module. More generally, we can also describe the vertex operators for open strings stretching between two branes λ, μ as a G-module $\mathcal{A}^{(\lambda,\mu)}$.

Let us take as an example $G = SU(2)$. There are no outer automorphisms in this case, hence $\omega = \mathrm{id}$. The generic conjugacy classes are 2-spheres and the space of functions thereon, the classical world-volume algebra, is spanned by spherical harmonics $Y^j_m, |m| \le j$ and $j = 0, 1, 2 \ldots$. There is an obvious action of $SU(2)$ on this function space where the Y^j_m transform in the spin j representation.

On the other hand, we can look at the spectrum of open string modes in the boundary theory $\lambda \in \mathcal{B}^{\mathrm{id}}(\widehat{su}(2)_k) \cong \mathrm{Rep}(\widehat{su}(2)_k) = \{0, 1, \ldots, k\}$ in the limit $k \to \infty$,

$$\mathcal{H}_\lambda{}^\lambda \;=\; \bigoplus_{j=0}^{2\lambda} \mathcal{H}^j \quad .$$

In each sector $\mathcal{H}^j$ we find an $SU(2)$-multiplet of ground states ψ^j_m, $m = -j, \ldots, j$, the corresponding primary fields have conformal weight $h = 0$ in the limit $k \to \infty$. On the space of ground states, we have a natural $SU(2)$ action, given by the zero modes of the currents. In the limit $\lambda \to \infty$, we find complete agreement with the $SU(2)$-module of spherical harmonics. As expected, the open strings 'see' the classical 2-sphere geometry in this limit.

When we take the boundary label λ to be finite, the angular momentum j is cut off at a finite value $j = \min(\lambda, k - \lambda) \leq \lambda$. This means that the brane's world-volume is 'fuzzy' since resolving small distances would require large angular momenta. Branes in $SU(2)$ have the geometry of non-commutative fuzzy 2-spheres which have been described earlier [67, 68, 69, 70]. This relation was first discussed in [17]. The analysis of [17] goes much beyond the study of open string spectra as it employs detailed information on the operator product expansions of open string vertex operators based on [64]. It was the first example for a non-commutative brane geometry besides branes in flat space. Using the results in [16, 71, 72] it is easy to generalize all these remarks on ordinary conjugacy classes (i.e. $\omega = \mathrm{id}$) to other groups (see also [67] for more details and explicit formulas on fuzzy conjugacy classes).

The geometry of twisted branes is more difficult to understand. We saw in the last chapter that they are described by twisted conjugacy classes. In contrast to ordinary conjugacy classes, they are never 'small'. More precisely, it is not possible to fit a generic twisted conjugacy class into an arbitrarily small neighborhood of the group identity unless the twist ω is trivial. This implies that the spectrum of angular momenta in ${\mathcal{H}_\lambda}^\lambda$ is not cut off before it reaches the obvious large momentum cut-off that is set by the volume of the group, i.e. by the level k. For large λ (and large k), it was found in [16] that the ground states in the boundary theory λ span the space of functions on the generic twisted conjugacy classes C_u^ω. The non-commutative geometry associated with twisted conjugacy classes with finite λ was unrevealed only recently in [12].

In the following we will give a description of the non-commutative geometry of twisted and untwisted branes close to the group unit e in arbitrary WZW models in the limit $k \to \infty$. First, the world-volume algebra will be formulated. Then we will compare it with the CFT results of the last chapter.

4.1.2 World-volume algebra

In Section 3.2 we explained that in the limit $k \to \infty$ all possible D-branes belonging to a gluing automorphism ω are labeled by the set $\mathcal{B}^\omega(\mathfrak{g})$ of (fractional) symmetric weights of $\mathfrak{g}$. In this section we propose an alternative in which the same boundary conditions are labeled by representations $\mathrm{Rep}(G^\omega)$ of the invariant subgroup $G^\omega = \{g \in G \mid \omega(g) = g\}$. For untwisted branes, where $\omega = \mathrm{id}$, the invariant subgroup is the group itself, $G^{\mathrm{id}} = G$, and we recover the familiar statement that the Cardy boundary conditions are labeled by representations of G. For nontrivial automorphisms ω, however, it is a priori not guaranteed that such a labeling is pos-

$\mathfrak{g}$	order	$\mathfrak{g}^\omega$	x_e	G	G^ω
A_2	2	A_1	4	$SU(3)$	$SO(3)$
A_{2n-1}	2	C_n	1	$SU(2n)$	$Sp(2n,\mathbb{C}) \cap SU(2n)$
A_{2n}	2	B_n	2	$SU(2n{+}1)$	$SO(2n{+}1)$
D_4	3	G_2	1	$\mathrm{Spin}(8) = \widetilde{SO}(8)$	$\widetilde{G}_2$
D_n	2	B_{n-1}	1	$\mathrm{Spin}(2n) = \widetilde{SO}(2n)$	$\mathrm{Spin}(2n{-}1) = \widetilde{SO}(2n{-}1)$
E_6	2	F_4	1	$\widetilde{E}_6$	$\widetilde{F}_4$

Table 4.1: This table lists all simple Lie algebras which admit a diagram automorphism ω. It provides the order of the automorphism, the invariant subalgebra $\mathfrak{g}^\omega$, the embedding index x_e of $\mathfrak{g}^\omega$ in $\mathfrak{g}$, the simply-connected group G with Lie algebra $\mathfrak{g}$ and the invariant subgroup G^ω. The tilde over a group denotes its universal cover.

sible in a canonical way[1]. In Table 4.1 we have listed the invariant subgroups of all simple, simply-connected compact groups under a diagram automorphism.

The correspondence between boundary conditions and representations of G^ω is provided by a bijective map $\Psi : \mathrm{Rep}(G^\omega) \to \mathcal{B}^\omega(\mathfrak{g})$. At the moment, we do not need any specific properties of Ψ or even a detailed construction. These will become crucial for the comparison with the CFT results later, but for now we just accept Ψ as a nice new way of labeling boundary conditions. Before we proceed, let us explain our notations. We use lower case labels $j, l, m, \ldots$ to denote representations of the invariant subgroup G^ω, but capital letters $J, L, M, \ldots$ for representations of G. For boundary labels we will use greek letters $\lambda, \mu, \ldots$.

We are ready to formulate the world-volume algebra of a brane $\Psi(l)$. Let V^l be the representation space for the irreducible representation $l \in \mathrm{Rep}(G^\omega)$. Then the world-volume algebra is given by

$$\mathcal{A}^{(l)} \cong \Big(\mathcal{F}(G) \otimes \mathrm{End}(V^l)\Big)^{G^\omega} \quad . \tag{4.3}$$

Here $\mathcal{F}(G)$ denotes the algebra of (smooth) functions on the group G and $\mathrm{End}(V^l)$ is the vector space of linear maps from V^l to itself. The space $\mathcal{F}(G) \otimes \mathrm{End}(V^l)$

[1] Just a few words on a slightly subtle point: we usually start with simply-connected groups, so that there is a one-to-one correspondence for representations of the group and the Lie algebra. This does not guarantee that the invariant subgroup is simply-connected as well, indeed, in the case of $G = SU(2n+1)$ the invariant subgroup $SO(2n+1)$ is not simply-connected. This forces us to distinguish between $\mathrm{Rep}(\mathfrak{g}^\omega)$ and $\mathrm{Rep}(G^\omega)$. For $G = SU(2n+1)$, we find two different set of branes close to the group unit in the limit $k \to \infty$ depending on whether k is even or odd. We will take k to be even in what follows, the correct labeling of boundary conditions is then done by the representations of the *group* G^ω. For the case of odd k see the remarks in Appendix B.

can be regarded as a vector space of matrix valued functions on the group G. The superscript G^ω restricts this space to functions $\mathrm{A}(g)$ which transform covariantly under right translations generated by G^ω,

$$\mathrm{A}(gg^\omega) \;=\; \mathrm{R}_l(g^\omega)^{-1}\mathrm{A}(g)\mathrm{R}_l(g^\omega) \text{ for all } g^\omega \in G^\omega \quad , \tag{4.4}$$

where $\mathrm{R}_l(h) \in \mathrm{End}(V^l)$ forms the representation labeled by l.

We can define an action of the group G on $\mathcal{A}^{(l)}$ by left translations; an element $g \in G$ acts as

$$(g\mathrm{A})(g') = \mathrm{A}(g^{-1}g') \quad . \tag{4.5}$$

$\mathcal{A}^{(l)}$ thus carries the structure of a G-module. This is precisely the usual construction of induced representations; we say that $\mathcal{A}^{(l)}$ is the G-module induced by the G^ω-representation on $V^l \otimes V^{l^+}$ where V^{l^+} is the representation conjugate to V^l.

Before we move on to formulate the world-volume algebra of an arbitrary brane configuration, we want to present an equivalent description of the G-module $\mathcal{A}^{(l)}$. Let us introduce the endomorphism bundle

$$G \times_{G^\omega} \mathrm{End}(V^l)$$

over the base space G/G^ω, associated to the principal G^ω-bundle G over G/G^ω. We can understand it as the direct product $G \times \mathrm{End}(V^l)$ modulo the equivalence relation

$$(g, \mathrm{A}) \;\sim\; (gg^\omega, \mathrm{R}_l(g^\omega)^{-1}\,\mathrm{A}\,\mathrm{R}_l(g^\omega)) \quad \text{for } g^\omega \in G^\omega \quad .$$

The world-volume algebra $\mathcal{A}^{(l)}$ is then isomorphic to the space of (smooth) sections in this bundle

$$\mathcal{A}^{(l)} \;\cong\; \Gamma\Big(G \times_{G^\omega} \mathrm{End}(V^l)\Big) \tag{4.6}$$

where the action of the group G corresponding to (4.5) is given by

$$g \cdot (g', \mathrm{A}) \;=\; (g^{-1}g', \mathrm{A}) \quad .$$

We have discussed the algebra for a single brane corresponding to open strings with both ends on the same brane. Eventually, we are interested in arbitrary brane configurations consisting of several branes. Let us introduce a notation for a stack of branes,

$$X \;=\; \bigoplus X_\lambda\ (\lambda) \quad .$$

It represents a superposition of $\sum X_\lambda$ D-branes in which X_λ branes of type $\lambda \in \mathcal{B}^\omega(\mathfrak{g})$ are placed on top of each other. Thanks to the map Ψ, we find a corresponding representation of G^ω, namely

$$V^X \;=\; \bigoplus X_{\Psi(l)} V^l \quad .$$

The G-module $\mathcal{A}^X$ describing strings on the configuration X is constructed analogously to (4.3),

$$\mathcal{A}^X \;\cong\; \Big(\mathcal{F}(G)\otimes \mathrm{End}(V^X)\Big)^{G^\omega} \quad . \tag{4.7}$$

In other words, $\mathcal{A}^X$ is the algebra of functions on G taking values in $\mathrm{End}(V^X)$ with the covariance property

$$\mathrm{A}(gg^\omega) \;=\; \mathrm{R}_X(g^\omega)^{-1}\mathrm{A}(g)\mathrm{R}_X(g^\omega) \tag{4.8}$$

for all $g^\omega \in G^\omega$.

Let us apply this general construction to a stack of N identical branes of type $\lambda = \Psi(l)$. We can extract the information on the single brane λ and find the typical Chan Paton factors,

$$\mathcal{A}^{N(l)} \;\cong\; \mathcal{A}^{(l)} \otimes \mathrm{Mat}(N) \quad .$$

The group G only acts on $\mathcal{A}^{(l)}$, not on $\mathrm{Mat}(N)$.

The world-volume algebra of an arbitrary configuration must contain the world-volume algebras of the constituents, and in addition we should find G-modules $\mathcal{A}^{(l,m)}$ describing strings stretching between two branes $\Psi(l), \Psi(m)$. Let us forget the algebra structure of $\mathcal{A}^X$ and decompose it as an induced G-module,

$$\mathcal{A}^X \;\cong\; \bigoplus_{l,m} X_{\Psi(l)}X_{\Psi(m)}\mathcal{A}^{(l,m)} \quad .$$

where

$$\mathcal{A}^{(l,m)} \cong \Big(\mathcal{F}(G)\otimes \mathrm{Hom}(V^l, V^m)\Big)^{G^\omega} \quad . \tag{4.9}$$

Here, $\mathrm{Hom}(V^l, V^m)$ is the vector space of linear transformations from V^l to V^m; the covariance property (4.4) is generalized to

$$\mathrm{A}(gg^\omega) \;=\; \mathrm{R}_m(g^\omega)^{-1}\mathrm{A}(g)\mathrm{R}_l(g^\omega) \text{ for all } g^\omega \in G^\omega \quad . \tag{4.10}$$

$\mathcal{A}^{(l,m)}$ is the G-module induced by $V^m \otimes V^{l^+}$ where l^+ labels the representation conjugate to V^l. Note that $\mathcal{A}^{(l,m)}$ does not form an algebra: one cannot combine two strings stretching both from l to m. Having one end on the brane labeled by l, the strings rather form a $\mathcal{A}^{(l)}$-(right)module, and analogously a $\mathcal{A}^{(m)}$-(left)module.

This concludes our presentation of the algebra of functions on twisted D-branes. We will give support for our claim in the next section.

The description of the world-volume algebra above is valid for both twisted and untwisted branes. It is instructive to see how structures simplify when we specialize to untwisted branes. The invariant subgroup G^ω is the group itself. The covariance property (4.10) fixes the function $\mathrm{A}(g)$ completely in terms of its value at the group unit $\mathrm{A}(e)$, and the module $\mathcal{A}^{(l,m)}$ becomes finite-dimensional. It is isomorphic to the G-module

$$\mathcal{A}^{(l,m)} \;\cong\; \mathrm{Hom}(V^l, V^m) \quad . \tag{4.11}$$

To conclude, we find a finite-dimensional world-volume algebra for untwisted branes. This agrees with our discussion of the example of $G = SU(2)$ in Section 4.1.1.

4.1.3 Comparison to CFT

In the last section, we proposed a formulation for the algebra of boundary fields in the large k limit. We will now compare it with the CFT results described in Chapter 3. Essentially, we will show that our proposed G-module (4.9) reproduces the correct spectrum of boundary fields (3.6). Note that in the limit $k \to \infty$ the sewing constraints reduce to associativity which is manifest in our proposal.

The CFT description provides an expression (3.6) for the spectrum $\mathcal{H}_\lambda{}^\mu$ of strings stretching between D-branes of type $\lambda, \mu \in \mathcal{B}^\omega(\widehat{\mathfrak{g}}_k)$ in terms of irreducible representations of $\widehat{\mathfrak{g}}_k$ which explicitly shows that the space of ground states carries the structure of a unitary G-module.

We claim that in the limit $k \to \infty$, the G-module of ground states is isomorphic to the unitary G-module $\overline{\mathcal{A}}^{(l,m)}$ with $l = \Psi^{-1}(\lambda)$ and $m = \Psi^{-1}(\mu)$. Here, $\overline{\mathcal{A}}^{(l,m)}$ is the unitary module that is obtained by considering square-integrable functions instead of smooth functions.

We will prove this claim by decomposing $\overline{\mathcal{A}}^{(l,m)}$ into irreducibles. To do so, we note that there is a canonical isomorphism $\mathrm{Hom}(V, W) \cong W \otimes V^*$. Furthermore, we may apply the Peter-Weyl theorem to decompose the Hilbert space $\overline{\mathcal{F}}(G) \cong L_2(G)$ of square-integrable functions on G with respect to the regular action of $G \times G$ into

$$\overline{\mathcal{F}}(G) \;\cong\; \bigoplus_L U^{L^+} \otimes U^L$$

where L runs over all irreducible representations of G, and the two factors of $G \times G$ act on the two vector spaces U^{L^+}, U^L, respectively. To make contact with our definition of $\mathcal{A}^{(l,m)}$, we have to restrict the right regular G action to the subgroup G^ω, which leaves us with the $G \times G^\omega$-module

$$\overline{\mathcal{F}}(G) \;\cong\; \bigoplus_{L,n} b_{Ln}\, U^{L^+} \otimes V^n \quad .$$

The numbers $b_{Ln} \in \mathbb{N}_0$ are the so-called *branching coefficients* which count the multiplicity of the G^ω-module V^n in U^L. Combining these remarks we arrive at

$$\overline{\mathcal{F}}(G) \otimes \mathrm{Hom}(V^l, V^m) \;\cong\; \bigoplus_{L,n} b_{Ln}\, U^{L^+} \otimes V^n \otimes V^m \otimes V^{l^+} \quad .$$

It remains now to find the invariants under the G^ω-action. Note that G^ω acts on the last three tensor factors. The number of invariants in the triple tensor product of irreducible representations is simply given by the Clebsch-Gordan coefficients ${N^{nm}}_l$ of G^ω. Hence, as a G-module, we have shown that

$$\overline{\mathcal{A}}^{(l,m)} \;\cong\; \bigoplus_{L,n} b_{Ln} {N^{nm}}_l\, U^{L^+} \quad .$$

This expression has to be compared with the decomposition (3.6) of the CFT partition functions. Here, we encounter our first condition on the map Ψ. It has to be such that in the limit $k \to \infty$ the equation

$$(n^\omega_L)_\lambda{}^\mu \;=\; \sum_n b_{Ln}\; {N_{nl}}^m \tag{4.12}$$

is fulfilled for $\Psi(l) = \lambda, \Psi(m) = \mu$.

A careful analysis shows that there indeed exists an appropriate identification map Ψ. We discuss this issue for the cases of Table 4.1 in Appendix B. The proof of (4.12), however, is quite technical and not relevant for the further developments in this thesis. It can be found in [12] and is based on results from [73, 74].

As a simple cross-check we consider the case of trivial automorphism $\omega = \mathrm{id}$ where we can make contact to well known results (see [17]). First we observe that the construction above simplifies considerably since $G^\omega \cong G$. All the lower case labels can be replaced by capital letters. In particular, the boundary conditions are labeled by representations of G, and the identification map Ψ is trivial. The corresponding G-module structure is now given by

$$\mathcal{A}^{(LM)} \;\cong\; \bigoplus_J {N_{JL}}^M\, U^J \quad .$$

The spectrum is determined by the fusion rules of $\widehat{\mathfrak{g}}$. This is in complete agreement with the known CFT results in *Cardy's case* [13].

4.2 Effective theory in WZW models

In the last section, we proposed an expression for the algebra of boundary fields in the limit $k \to \infty$. These results together with the OPE of the currents can be used to derive the effective theory in leading order $1/\alpha' k$. We will formulate the resulting effective gauge theory on the non-commutative world-volume of a brane below.

The computation of the effective action for massless open string modes involves the determination of correlation functions of vertex operators

$$: J^\alpha V[\mathrm{A}_\alpha] : (x) \quad .$$

In leading order, it turns out that only 3- and 4-point functions contribute. The information encoded in our world-volume algebra provides us with the OPE (4.1) of primary fields and with the OPE (4.2) of currents and primary fields. The action of the Lie algebra $\mathfrak{g}$ on $\mathcal{A}^X$ appearing in the OPE (4.2) is the infinitesimal version of the action of the group G,

$$\mathrm{L}^\alpha \mathrm{A}(g) \;=\; \frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{A}(e^{-it\mathrm{T}^\alpha} g)\Big)\Big|_{t=0} \quad \text{for } \mathrm{A} \in \mathcal{A}^X \quad . \tag{4.13}$$

Using in addition the OPE (3.2) of the currents, the correlation functions can be computed to leading order [18]. The calculation is very similar to the case of flat branes, but the 3-point function gets an additional term involving the Lie algebra structure constants. Consequently, the resulting effective action is not given by a Yang-Mills theory on a non-commutative space alone, but involves also a Chern-Simons like term.

For a brane configuration $X = \bigoplus X_\lambda(\lambda)$, the fields A^α are hermitian elements of $\mathcal{A}^X$, i.e. they are functions on G with values in the hermitian endomorphisms $\mathrm{End}_{\mathrm{H}}(V^X)$ and equivariance property as formulated in eq. (4.8). The results of the computation [18] for the case of untwisted branes may easily be transferred to the situation of arbitrary maximally symmetric branes [12] and can be summarized in the following action

$$\mathcal{S}_X \;=\; \mathcal{S}_{\mathrm{YM}} + \mathcal{S}_{\mathrm{CS}} \;=\frac{\pi^2}{k^2}\left(\frac{1}{4}\int \mathrm{tr}\left(\mathrm{F}_{\alpha\beta}\,\mathrm{F}^{\alpha\beta}\right) - \frac{i}{2}\int \mathrm{tr}\left(f^{\alpha\beta\gamma}\,\mathrm{CS}_{\alpha\beta\gamma}\right)\right) \tag{4.14}$$

where we defined the 'curvature form' $\mathrm{F}_{\alpha\beta}$ by the expression

$$\mathrm{F}_{\alpha\beta}(\mathrm{A}) \;=\; i\,\mathrm{L}_\alpha \mathrm{A}_\beta - i\,\mathrm{L}_\beta \mathrm{A}_\alpha + i\,[\mathrm{A}_\alpha\,,\,\mathrm{A}_\beta] + f_{\alpha\beta}{}^\gamma \mathrm{A}_\gamma \tag{4.15}$$

and a non-commutative analogue of the Chern-Simons form by

$$\mathrm{CS}_{\alpha\beta\gamma}(\mathrm{A}) \;=\; \mathrm{L}_\alpha \mathrm{A}_\beta\,\mathrm{A}_\gamma + \frac{1}{3}\,\mathrm{A}_\alpha\,[\mathrm{A}_\beta\,,\,\mathrm{A}_\gamma] - \frac{i}{2}\,f_{\alpha\beta}{}^\rho\,\mathrm{A}_\rho\,\mathrm{A}_\gamma \quad . \tag{4.16}$$

The trace is always normalized s.t. $\mathrm{tr}\,\mathbf{1} = 1$.

Let us note that the curvature tensor $\mathrm{F}_{\alpha\beta}$ obeys a non-commutative analogue of the Bianchi identity

$$i\mathrm{L}_\alpha \mathrm{F}_{\beta\gamma} + i[\mathrm{A}_\alpha, \mathrm{F}_{\beta\gamma}] + f_{\alpha\beta}{}^{\delta}\,\mathrm{F}_{\delta\gamma} + \text{ cyclic permutations } = 0 \quad . \tag{4.17}$$

The action (4.14) is invariant under the infinitesimal gauge transformations

$$\delta \mathrm{A}_\alpha \;=\; i\mathrm{L}_\alpha \Lambda \;+\; i\,[\,\mathrm{A}_\alpha\,,\,\Lambda\,] \quad \text{for} \quad \Lambda \in \mathcal{A}^X\ ,\ \Lambda = \Lambda^\dagger \quad . \tag{4.18}$$

The field strength $\mathrm{F}_{\alpha\beta}$ transforms covariantly under gauge transformations. Note that the 'mass term' in the Chern-Simons form (4.16) guarantees the gauge invariance of $\mathcal{S}_{\mathrm{CS}}$. On the other hand, the effective action (4.14) is the unique combination of $\mathcal{S}_{\mathrm{YM}}$ and $\mathcal{S}_{\mathrm{CS}}$ in which mass terms cancel.

Invariance under infinitesimal transformations does not ensure invariance under 'large' gauge transformations that cannot be smoothly deformed into the trivial gauge $U \equiv 1$, because our action involves a Chern-Simons term. A finite gauge transformation is mediated by a unitary element U of $\mathcal{A}^X$,

$$\mathrm{A}_\alpha \;\longrightarrow\; U\mathrm{A}_\alpha U^{-1} - (\mathrm{L}_\alpha U)U^{-1} \quad .$$

It can be shown that the action (4.14) is invariant under these transformations exactly if the action vanishes for pure gauge configurations. For a trivial automorphism $\omega = \mathrm{id}$ and for automorphisms of order 2, this is always fulfilled[2]. For the automorphism of order 3, gauge invariance under 'large' transformations has not been establish yet.

Let us make an interesting side-remark: one could define a theory with the same action (4.14) on *all* matrix-valued functions on G without imposing any covariance condition. This action is not gauge-invariant under global transformations in general; only when we restrict the allowed gauge transformations to those transforming covariantly under G^ω, we obtain gauge symmetry.

Before we move on to the discussion of solutions to this non-commutative gauge theory, we want to show how it simplifies in the case of untwisted branes where $\omega = \mathrm{id}$. We saw in Section 4.1.2 that the world-volume algebra for untwisted branes is a finite-dimensional matrix algebra,

$$\mathcal{A}^X \;\cong\; \mathrm{End}(V^X) \;\cong\; \mathrm{Mat}(\dim V^X) \quad .$$

The derivatives L^α (see eq. (4.13)) implementing the infinitesimal $\mathfrak{g}$ action are realized as commutators with the representation matrices $\mathrm{R}_X(\mathrm{T}^\alpha)$,

$$\mathrm{L}^\alpha \mathrm{A} \;=\; [\mathrm{R}_X(\mathrm{T}^\alpha), \mathrm{A}] \quad \text{for } \mathrm{A} \in \mathcal{A}^X \quad .$$

[2]This result follows from a computation that we will perform in Section 4.3.4.

Similar gauge theories on matrix ('fuzzy') geometries [67, 68] have been studied before they were shown to appear in string theory (see e.g. [75, 76, 77, 78, 79, 80]).

4.3 Brane dynamics in WZW models

In the last sections we presented the low energy effective theory for symmetric (twisted or untwisted) branes in WZW models. The aim of this section is to investigate classical solutions of this field theory, along with their interpretation as condensation processes.

First, we will discuss some general properties of solutions introducing the classification into symmetric and non-symmetric solutions. We will give some examples of non-symmetric solutions and focus then on symmetric solutions and their interpretation.

4.3.1 General properties

A simple variation of the action (4.14) with respect to the gauge fields allows us to derive the following equations of motion,

$$\mathrm{L}^{\alpha}\mathrm{F}_{\alpha\beta} + [\mathrm{A}^{\alpha}, \mathrm{F}_{\alpha\beta}] = 0 \quad , \tag{4.19}$$

which mean that the curvature has to be covariantly constant. Note that the space of gauge fields, and hence the equations of motions, depends on the brane configuration we are looking at.

As a warm-up example we consider a rather trivial class of solutions. Any set of commuting constant matrices A_α is a solution to (4.19). Note that constant fields A_α can only fulfill the covariance property (4.8) if they commute with the representation matrices R_X. For every constituent brane in our configuration we find $\dim(G)$ degrees of freedom. These solutions have vanishing action and allow for an easy geometric interpretation [54, 18]: they describe translations of the branes in the group manifold (see fig. 4.1). This interpretation nicely explains the degrees of freedom we observed. In general, the translation of the single branes will break the G-symmetry of the symmetric configuration we started with.

The only symmetric translation is the uniform translation of the whole configuration (see fig. 4.2). It is simply obtained by setting the fields A_α to constant multiples of the identity matrix. This operation is a rather trivial one, and we would like to reduce our analysis of solutions to those whose 'center of mass' is not translated. To make this idea more precise, we state the following observation:

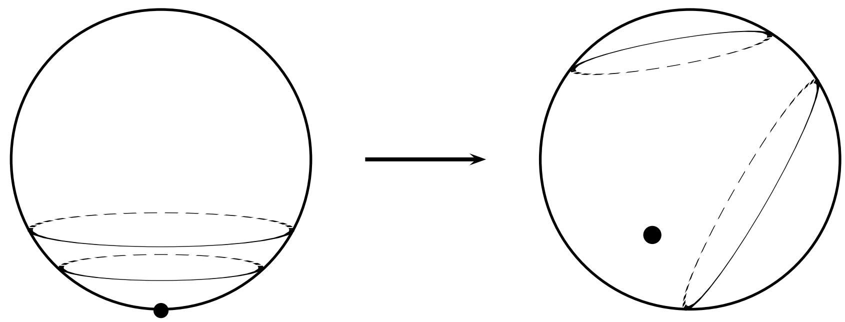


Figure 4.1: Independent translations of the individual branes break the symmetry.

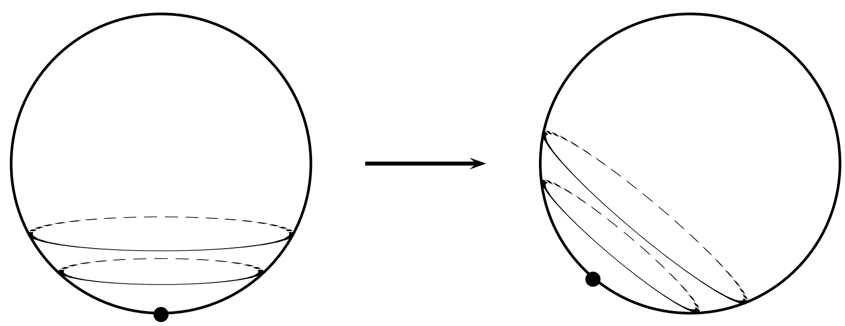


Figure 4.2: A uniform translation of the whole brane configuration preserves the symmetry.

Any solution A_α of (4.19) can be decomposed into a constant uniform translation $\mathrm{A}_\alpha^T \propto \mathbf{1}$ and a solution A_α^0 whose curvature has vanishing trace,

$$\mathrm{A}_\alpha \;=\; \mathrm{A}_\alpha^0 \;+\; \mathrm{A}_\alpha^T \;\;,\;\; \mathrm{tr}\,\mathrm{F}_{\alpha\beta}(\mathrm{A}^0) \;=\; 0 \;\;. \tag{4.20}$$

Although the equations of motion are nonlinear in A_α, both A_α^T and A_α^0 are solutions; they decouple because their commutator vanishes.

Let us sketch how this result can be derived. First we can use the Bianchi-identity (4.17) to prove that any solution to (4.19) has the property

$$\mathrm{tr}(\mathrm{F}_{\alpha\beta}) \;=\; \text{constant} \;\;.$$

Given a solution A_α, we can then define its constant translation part A^T_α by

$$A^T_\alpha \ := \ \frac{1}{2g^\vee}\ f_\alpha{}^{\beta\gamma}\ \mathrm{tr}(F_{\beta\gamma})\cdot \mathbf{1} \quad .$$

Note that we normalized the trace s.t. $\mathrm{tr}\,\mathbf{1} \ = \ 1$. Again by using (4.17) one can finally show that $A^0 = A - A^T$ is a solution with vanishing trace.

We already saw that some solutions preserve the symmetry and some solutions do not. This suggests to classify the solutions according to their symmetry properties. We will call a solution

- symmetric if $[F_{\alpha\beta}(A),\,\cdot\,] \ = \ 0$
- non-symmetric if $[F_{\alpha\beta}(A),\,\cdot\,] \ \neq \ 0$.

At the moment we can only see that these definitions give the right results in the case of translation solutions. When we discuss the interpretation of general solutions in Section 4.3.4, we will see that symmetric solutions as defined above describe processes connecting maximally symmetric brane configurations.

4.3.2 Non-symmetric solutions

Before we dive into the discussion of symmetric solutions, let us mention some solutions of (4.19) which break the G-symmetry.

We already encountered one type of solutions breaking the G-symmetry, the independent translations of the brane constituents in our configuration (see fig. 4.1). In this case the curvature is given by $F_{\alpha\beta} = f_{\alpha\beta}{}^\gamma A_\gamma$ which is not a multiple of the identity matrix in general. Only the uniform translations of the whole stack preserve the G-symmetry.

One particular non-trivial type of non-symmetric solutions can be obtained for each choice of a subalgebra $\mathfrak{h}$ in $\mathfrak{g}$. We label the subalgebra generators by $i, j, \ldots$, the directions orthogonal to $\mathfrak{h}$ by $a, b, \ldots$. Now let V^l be a representation of $\mathfrak{h}$. Then the equations of motion for a stack of $\dim V^l$ identical branes possess a constant solution of the form

$$A^i = \mathbf{1}\otimes R_l(T^i) \ \ \text{and} \ \ A^a = 0$$

where $\mathbf{1}$ is the identity matrix in the space-time degrees of freedom, and R_l are representation matrices acting on V^l. The curvature of these solutions is given by

$$F_{ij} = 0 \quad , \quad F_{ia} = 0 \quad , \quad F_{ab} = f_{ab}{}^i\, R_l(T_i) \quad .$$

Obviously, $[\mathrm{F}_{\alpha\beta}(\mathrm{A}), \cdot\,] \neq 0$ in general, so according to our classification this solution is non-symmetric. Note that the components of the curvature corresponding to directions of the subalgebra $\mathfrak{h}$ are zero, we expect therefore that there is an unbroken $\mathfrak{h}$-symmetry. Such symmetry considerations and a comparison of brane tensions give strong evidence that these solutions correspond to some of the symmetry-breaking branes constructed in [45].

4.3.3 Symmetric solutions

After this digression we will concentrate on symmetric solutions, i.e. solutions where $\mathrm{F}_{\alpha\beta}$ is proportional to the identity matrix $\mathbf{1}$. Splitting off a uniform translation as described in (4.20), we observe that we can restrict our analysis without loss to solutions with vanishing curvature $\mathrm{F} = 0$. In the following we will give a complete classification of these solutions.

Let us start with an initial configuration X. The fields are hermitian functions $\mathrm{A}_\alpha : G \to \mathrm{End}_{\mathrm{H}}(V^X)$ satisfying the covariance property (4.8). The following proposition reduces the classification of symmetric solutions to the problem of the existence of certain covariant matrix-valued functions K. We denote by $U(V^Y, V^X)$ the set of unitary transformations from V^Y to V^X.

The fields A_α form a solution to $\mathrm{F}(\mathrm{A}) = 0$ if and only if they are constructed out of a function $\mathrm{K} : G \to U(V^Y, V^X)$ satisfying

$$\begin{aligned} \mathrm{K}(gg^\omega) &= \mathrm{R}_X(g^\omega)^{-1}\mathrm{K}(g)\mathrm{R}_Y(g^\omega) \qquad \text{for all} \qquad g^\omega \in G^\omega \qquad \text{by} \\ \mathrm{A}_\alpha(g) &= -(\mathrm{L}_\alpha\mathrm{K})(g)\mathrm{K}(g)^{-1} = \mathrm{K}(g)(\mathrm{L}_\alpha\mathrm{K}^{-1})(g) \end{aligned} \tag{4.21}$$

with R_Y being a representation of G^ω of the same dimension as R_X,

$$\dim V^X = \dim V^Y \quad .$$

Furthermore, two functions K and K′ with the same covariance property lead to gauge-equivalent solutions.

One direction of this proposition is easy to prove. Given a function K with the desired covariance, we construct the fields A_α as in (4.21). The covariance of A_α,

$$\mathrm{A}_\alpha(gg^\omega) = \mathrm{R}_X(g^\omega)^{-1}\mathrm{A}_\alpha(g)\mathrm{R}_X(g^\omega) \quad , \tag{4.22}$$

is ensured by the behavior of K under right translations by $g^\omega \in G^\omega$. Moreover, the curvature of these gauge fields vanishes because

$$\begin{aligned}
\mathrm{F}_{\alpha\beta} &= i\mathrm{L}_\alpha \mathrm{A}_\beta - i\mathrm{L}_\beta \mathrm{A}_\alpha + i[\mathrm{A}_\alpha, \mathrm{A}_\beta] + f_{\alpha\beta}{}^\gamma \mathrm{A}_\gamma \\
&= -i\mathrm{L}_\alpha\big(\mathrm{L}_\beta \mathrm{K}\,\mathrm{K}^{-1}\big) + i\mathrm{L}_\beta\big(\mathrm{L}_\alpha \mathrm{K}\,\mathrm{K}^{-1}\big) \\
&\qquad -i\big(\mathrm{L}_\alpha \mathrm{K}\,\mathrm{L}_\beta \mathrm{K}^{-1} - \mathrm{L}_\beta \mathrm{K}\,\mathrm{L}_\alpha \mathrm{K}^{-1}\big) - f_{\alpha\beta}{}^\gamma \mathrm{L}_\gamma \mathrm{K}\,\mathrm{K}^{-1} \\
&= -i[\mathrm{L}_\alpha, \mathrm{L}_\beta]\mathrm{K}\,\mathrm{K}^{-1} - f_{\alpha\beta}{}^\gamma \mathrm{L}_\gamma \mathrm{K}\,\mathrm{K}^{-1} \;=\; 0 \quad .
\end{aligned}$$

Note that the solutions are not pure gauge because the function K is not an allowed gauge transformation as it has the wrong covariance property. Only if $X = Y$, the solution is gauge equivalent to the trivial solution $\mathrm{A} = 0$.

To prove the other direction contained in the proposition, we have to construct a function K out of a given solution of $\mathrm{F}_{\alpha\beta}(\mathrm{A}) = 0$. Such a function can be defined as the solution to the linear differential equation

$$\mathrm{L}^\alpha \mathrm{K}(g) = -\mathrm{A}^\alpha(g)\mathrm{K}(g) \quad .$$

The local integrability condition of this equation is precisely $\mathrm{F} = 0$, and as our group manifold is simply connected, we can always find a unique global solution $\mathrm{K}(g)$ for a given unitary $\mathrm{K}(e)$. The fields $\mathrm{A}_\alpha(g)$ are hermitian matrices, therefore the matrices $\mathrm{K}(g)$ are unitary and, consequently, invertible. It can be shown that the solution $\mathrm{K}(g)$ transforms under right translations by $g^\omega \in G^\omega$ as

$$\mathrm{K}(gg^\omega) \;=\; \mathrm{R}_X(g^\omega)^{-1}\mathrm{K}(g)\mathrm{R}_Y(g^\omega) \quad . \tag{4.23}$$

The representation R_Y of G^ω is determined by

$$\mathrm{R}_Y(\mathrm{T}_i) \;=\; \mathrm{K}(e)^{-1}\,\mathrm{A}_i(e)\,\mathrm{K}(e)$$

where T_i are the generators of the Lie algebra $\mathfrak{g}^\omega$.

Now let K, A and K', A' be solutions as in (4.21) for the same X, Y. To verify the last statement of our proposition, we have to show that A and A′ are gauge-equivalent. Indeed,

$$\mathrm{A}'_\alpha \;=\; U\mathrm{A}_\alpha U^{-1} - (\mathrm{L}_\alpha U)U^{-1} \tag{4.24}$$

with $U = \mathrm{K}'\,\mathrm{K}^{-1}$, i.e. A and A′ are related by a gauge transformation U. Note that

$$U(gg^\omega) = \mathrm{R}_X(g^\omega)^{-1}U(g)\mathrm{R}_X(g^\omega) \;\; \text{for all } g^\omega \in G^\omega \quad ,$$

and thus it is an allowed gauge transformation. This completes the proof of our proposition.

We have translated our problem of finding symmetric solutions to the question of existence of suitable functions K. The latter point has a beautiful geometric interpretation. To see that, let us consider the complex vector bundle

$$E^X \;=\; G \times_{G^\omega} V^X_{\mathbb{C}}$$

over the base manifold G/G^ω which consists of equivalence classes of pairs $(g, v) \in G \times V^X_{\mathbb{C}}$ where the equivalence relation is given by

$$(g, v) \sim (gg^\omega, \mathrm{R}_X(g^\omega)^{-1}v) \quad .$$

This bundle is associated to the principal G^ω-bundle $G \to G/G^\omega$. A function $\mathrm{K} : G \to U(V^Y, V^X)$ with the covariance property (4.23) exists if and only if the bundles associated to V^X and V^Y are isomorphic,

$$E^X \quad \simeq \quad E^Y \quad .$$

Given a function K we can use it to construct a bundle isomorphism. Consider the map

$$G \times V^Y_{\mathbb{C}} \ni (g, v) \;\mapsto\; (g, \mathrm{K}(g)\, v) \in G \times V^X_{\mathbb{C}} \quad .$$

For fixed g this map is a vector space isomorphism. As it is invariant under the action of G^ω, it descends to a bundle isomorphism when we divide out G^ω. The opposite is also true: given a bundle isomorphism, we can use it to construct a function K with the desired properties.

It turns out that all bundles that occur for the cases listed in Table 4.1 are trivial. The proof proceeds in two steps. First one shows that all bundles are stably equivalent to trivial bundles, i.e. that they can be made trivial by adding (Whitney sum) a trivial bundle of sufficiently high rank. In terms of (reduced) topological K-theory, we say that these bundles are mapped to the zero element in the complex K-group $\tilde{K}_{\mathbb{C}}(G/G^\omega)$. As there is a product in the K-group giving it the structure of a ring which is consistent with the tensor product of bundles, it is enough to show that bundles associated to fundamental representations have trivial K-class. This can be done case by case. In a second step, one shows that the rank of the bundles in question is high enough so that stable equivalence implies isomorphism. Details of this proof are provided in Appendix D.

The triviality of our bundles makes it possible to find functions $\mathrm{K}_Y : G \to U(V^Y, \mathbb{C}^{\dim V^Y})$ for a given representation V^Y satisfying

$$\mathrm{K}_Y(gg^\omega) \;=\; \mathrm{K}_Y(g)\mathrm{R}_Y(g^\omega) \quad .$$

These can be used to build functions K relating bundles associated to V^X and V^Y by

$$\mathrm{K}(g) \;=\; \mathrm{K}_X^{-1}(g)\mathrm{K}_Y(g) \quad .$$

Now let us discuss two situations where we can explicitly construct the function K.

In the case when V^X and V^Y are restrictions of representations of the group G, the construction of a suitable function K is particularly simple, namely we can choose

$$\mathrm{K}(g) \;=\; \mathrm{R}_X(g)^{-1}\mathrm{R}_Y(g) \quad . \tag{4.25}$$

The corresponding field A is given by

$$\mathrm{A}^\alpha(g) \;=\; \mathrm{R}_X(g)^{-1}\big(\mathrm{R}_Y(\mathrm{T}^\alpha) - \mathrm{R}_X(\mathrm{T}^\alpha)\big)\mathrm{R}_X(g) \tag{4.26}$$

where T^α are generators of the Lie algebra $\mathfrak{g}$.

There is another case, where we can obtain explicit expressions for the functions K. We start with a stack of $\dim V^S$ branes of type $\lambda = \Psi(l)$. V^S is a representation space of G. As a vector space, we can write $V^X = V^S \otimes V^l$, but note that the G^ω action on V^X has effects solely on the second factor V^l. Then we can construct a solution via the function $\mathrm{K} : G \to U(V^S \otimes V^l, V^S \otimes V^l)$ defined by

$$\mathrm{K}(g) \;=\; \mathrm{R}_S(g) \otimes \mathbf{1} \quad . \tag{4.27}$$

The solution takes the form

$$\mathrm{A}_\alpha(g) \;=\; \mathrm{R}_S(\mathrm{T}_\alpha) \otimes \mathbf{1} \quad . \tag{4.28}$$

We thus have found a class of constant solutions! These solutions, given by constant representation matrices $\mathrm{R}_S(\mathrm{T}_\alpha)$, will play an important role when we discuss brane dynamics at finite level k in Chapter 5.

When we specialize to the case of untwisted branes, things simplify dramatically. The world-volume algebra is finite dimensional,

$$\mathcal{A}^X \;\cong\; \mathrm{End}(V^X) \quad ,$$

because we can restrict the fields to the group unit. For any representation V^Y of the same dimension as V^X, we find a solution simply by evaluating (4.26) at the group unit,

$$\mathrm{A}_\alpha \;=\; \mathrm{R}_Y(\mathrm{T}_\alpha) - \mathrm{R}_X(\mathrm{T}_\alpha)$$

where we identified V^X and V^Y as vector spaces.

Let us recapitulate what we have shown. We started with a configuration corresponding to a representation V^X. We saw that any solution to $\mathrm{F}_{\alpha\beta} = 0$ is constructed out of a suitable function K involving a second representation V^Y. Such functions do exist because the complexified vector bundles associated to V^X and V^Y are isomorphic. Different functions K belonging to the same V^Y lead to gauge-equivalent solutions. To summarize:

> For a given ω-twisted brane configuration X on G with corresponding G^ω-module V^X we find up to gauge equivalence *one* solution for any representation V^Y satisfying
>
> $$\dim V^X \;=\; \dim V^Y$$
>
> by the construction (4.21).

4.3.4 Interpretation

On the last pages, we found a large number of solutions. We called them 'symmetric', but we have not shown yet that they really correspond to processes connecting maximally symmetric brane configurations. In this section we will first discuss how we can identify solutions with brane processes. We will analyze what condition a solution has to fulfill in order to describe a symmetric process and show that this condition coincides with our definition of symmetric solutions in Section 4.3.1. We will then discuss the interpretation of the solutions obtained in Section 4.3.3, and we will give evidence that they correspond to processes of the type

$$\text{Configuration } X \quad \longleftrightarrow \quad \text{Configuration } Y \quad . \tag{4.29}$$

Whether our solution is a process from X to Y or the other way round can only be decided by comparing the tensions of the configurations.

Solutions to the equations of motion (4.19) describe a decay or condensation process of the initial brane configuration X into some final configuration Y. How do we identify it? We have two different types of information at our disposal. On the one hand, we can compare the tension of D-branes in the final configuration Y with the value of the action $\mathcal{S}_X(\mathrm{A})$ at the classical solution A. On the other hand, we can look at fluctuations $\mathrm{A} + \delta\mathrm{A}$ around the chosen solution A and compare their dynamics with the low energy effective theory $\mathcal{S}_Y$ of the brane configuration Y. In formulas, this means that

$$\mathcal{S}_X(\mathrm{A}+\delta\mathrm{A}) \;\stackrel{!}{=}\; \mathcal{S}_X(\mathrm{A}) + \mathcal{S}_Y(\Phi_{XY}(\delta\mathrm{A})) \quad \text{with} \quad \mathcal{S}_X(\mathrm{A}) \;\stackrel{!}{=}\; \ln\frac{g_Y}{g_X} \quad . \tag{4.30}$$

The second requirement expresses the comparison of tensions in terms of the g-factors of the involved conformal field theories (see e.g. [18] for more details). We denote by Φ_{XY} an algebra isomorphism mapping the fluctuation fields $\delta\mathrm{A} \in \mathcal{A}^X$ to the world-volume algebra $\mathcal{A}^Y$ of the configuration Y. All equalities must hold to the order in $(1/k)$ that we used when constructing the effective actions.

We are especially interested in processes that connect maximally symmetric configurations of D-branes, i.e. those configurations for which the spectrum decomposes into representations of $\widehat{\mathfrak{g}}_k$. For such configurations we know the low energy effective actions so that we are able to compare them with the dynamics of fluctuations around the solution.

The isomorphism Φ_{XY} induces a G-module structure on $\mathcal{A}^X$ different from the original one,

$$\Phi_{XY}^{-1} \circ \mathrm{L}_\alpha^Y \circ \Phi_{XY} \;:\; \mathcal{A}^X \;\longrightarrow\; \mathcal{A}^X \quad . \tag{4.31}$$

When we do the comparison for third-order terms in $\delta\mathrm{A}$ in eq. (4.30) we find that

$$\Phi_{XY}^{-1} \circ \mathrm{L}_\alpha^Y \circ \Phi_{XY} \;\stackrel{!}{=}\; \tilde{\mathrm{L}}_\alpha^X \;:=\; \mathrm{L}_\alpha^X + [\mathrm{A}_\alpha,\,\cdot\,] \quad . \tag{4.32}$$

We have to verify that this is compatible with the Lie algebra commutation relations of $\tilde{\mathrm{L}}_\alpha^X$, and we also have to check that the second-order and fourth-order terms on both sides of (4.30) coincide. One can easily show that the shifted derivatives (4.32) behave as follows,

$$[\tilde{\mathrm{L}}_\alpha^X, \tilde{\mathrm{L}}_\beta^X] \;=\; i\; f_{\alpha\beta}{}^{\gamma}\; \tilde{\mathrm{L}}_\gamma^X - i\,[\mathrm{F}_{\alpha\beta}(\mathrm{A}),\,\cdot\,] \quad .$$

Furthermore, using the relation (4.32), the expansion of the action around the solution A reads

$$\mathcal{S}_X(\mathrm{A}+\delta\mathrm{A}) \;=\; \mathcal{S}_X(\mathrm{A}) + \mathcal{S}_Y(\Phi_{XY}(\delta\mathrm{A})) + \frac{i}{2}\mathrm{tr}\big([\mathrm{F}_{\alpha\beta}(\mathrm{A}), \delta\mathrm{A}^\alpha]\delta\mathrm{A}^\beta\big) \quad .$$

It can immediately be seen that we can comply with our requirements only if our solution has curvature proportional to the unit matrix,

$$[\mathrm{F}_{\alpha\beta}(\mathrm{A}),\,\cdot\,] \;=\; 0 \quad . \tag{4.33}$$

This condition coincides with our classification into symmetric and non-symmetric solutions in Section 4.3.1.

We derived (4.33) as a necessary condition for a solution that connects maximally symmetric configurations. In Section 4.3.3 we classified all solutions with this property, and now we want to show that they really describe symmetric brane processes. We claim that a solution on a configuration X of the form (4.21) involving a representation V^Y corresponds to a process between the configurations X and Y.

To verify this conjecture on brane dynamics, we have on the one hand to provide an algebra isomorphism Φ_{XY} which relates the derivatives in the right way (see eq. (4.32)) and on the other hand, we have to compare the value of the action at the solution to the g-factors coming from the CFT-results.

The isomorphism $\Phi_{XY} : \mathcal{A}^X \to \mathcal{A}^Y$ can be formulated with the help of the function K from which we constructed our solution,

$$\Phi_{XY}(\delta \mathrm{A})(g) \;=\; \mathrm{K}^{-1}(g)\,\delta\mathrm{A}(g)\,\mathrm{K}(g) \;\in \mathcal{A}^Y \;.$$

One can explicitly check that this isomorphism maps the derivatives L^Y_α to shifted derivatives on $\mathcal{A}^X$,

$$\Phi^{-1}_{XY} \circ \mathrm{L}^Y_\alpha \circ \Phi_{XY} \;=\; \mathrm{L}^X_\alpha + [-\mathrm{L}^X_\alpha \mathrm{K}\,\mathrm{K}^{-1},\, \cdot\,]$$

where the shift is given by the solution of the equations of motion $\mathrm{A}_\alpha = -\mathrm{L}^X_\alpha \mathrm{K}\,\mathrm{K}^{-1}$.

Our second check involves a comparison between the value of the action at the solution and the g-factors of a CFT description. For technical reasons, we restrict ourselves to $\omega = \mathrm{id}$ or automorphisms ω of order 2. This assumption implies strong constraints on the form of the structure constants which arise from the grading on $\mathfrak{g}$ induced by ω. To be specific, we may choose a basis in which only the numbers f_{ijk}, f_{abi} and cyclic permutations thereof do not vanish. Here, $i, j, k, \ldots$ denote indices for elements in the invariant subalgebra $\mathfrak{g}^\omega$ and $a, b, c, \ldots$ label directions orthogonal to $\mathfrak{g}^\omega$. With this in mind, we are now able to compute the action using no more than the properties of the solution we have specified. For a solution with $\mathrm{F} = 0$, the action reduces to

$$\mathcal{S}_X(\mathrm{A}) \;=\; \eta f^{\alpha\beta\gamma} \int \mathrm{tr}\, \mathrm{A}_\alpha \mathrm{A}_\beta \mathrm{A}_\gamma$$

with $\eta = i\pi^2/6$. To proceed further, we have to introduce new derivatives L^R that act by infinitesimal translations from the right,

$$\mathrm{L}^\mathrm{R}_\alpha \mathrm{K}(g) \;=\; \frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{K}\big(g e^{it\mathrm{T}_\alpha}\big)\Big)\Big|_{t=0} \quad . \tag{4.34}$$

The connection between the different derivatives L (4.13) and L^R (4.34) is given by the adjoint representation of G,

$$\mathrm{L}_\alpha \mathrm{K}(g) \;=\; -\mathrm{Ad}(g^{-1})_\alpha{}^\beta\, \mathrm{L}^\mathrm{R}_\beta \mathrm{K}(g) \quad . \tag{4.35}$$

We can now re-express the solution $\mathrm{A}_\alpha = -(\mathrm{L}_\alpha \mathrm{K})\mathrm{K}^{-1}$ as

$$\mathrm{A}_\alpha = \mathrm{Ad}(g^{-1})_\alpha{}^\beta \big(\mathrm{L}^{\mathrm{R}}_\beta \mathrm{K}\big)\mathrm{K}^{-1} \quad .$$

The action of the adjoint representation drops out when we perform the contraction with $f^{\alpha\beta\gamma}$, and we obtain

$$\mathcal{S}_X(\mathrm{A}) \;=\; \eta f^{\alpha\beta\gamma} \int \mathrm{tr}\,\big(\mathrm{L}^{\mathrm{R}}_\alpha \mathrm{K}^{-1}\big)\big(\mathrm{L}^{\mathrm{R}}_\beta \mathrm{K}\big)\mathrm{K}^{-1}\big(\mathrm{L}^{\mathrm{R}}_\gamma \mathrm{K}\big) \quad .$$

Due to the graded form of the structure constants one of the superscripts of $f^{\alpha\beta\gamma}$ has to lie in the direction of $\mathfrak{g}^\omega$, thus we can write

$$\begin{aligned}\mathcal{S}_X(\mathrm{A}) \;=\; & -3\eta f^{i\alpha\beta} \int \mathrm{tr}\,\mathrm{K}^{-1}\big(\mathrm{L}^{\mathrm{R}}_i \mathrm{K}\big)\big(\mathrm{L}^{\mathrm{R}}_\alpha \mathrm{K}^{-1}\big)\big(\mathrm{L}^{\mathrm{R}}_\beta \mathrm{K}\big) \\ & -2\eta f^{ijk} \int \mathrm{tr}\,\mathrm{K}^{-1}\big(\mathrm{L}^{\mathrm{R}}_i \mathrm{K}\big)\big(\mathrm{L}^{\mathrm{R}}_j \mathrm{K}^{-1}\big)\big(\mathrm{L}^{\mathrm{R}}_k \mathrm{K}\big) \quad . \end{aligned} \tag{4.36}$$

We can compute the right-derivatives of K in the directions of $\mathfrak{g}^\omega$ thanks to the covariance property (4.23),

$$\mathrm{L}^{\mathrm{R}}_i \mathrm{K}(g) \;=\; \mathrm{K}(g)\mathrm{R}_Y(\mathrm{T}_i) \;-\; \mathrm{R}_X(\mathrm{T}_i)\mathrm{K}(g) \quad .$$

We substitute this into the first term of (4.36) and obtain

$$\begin{aligned} & -3\eta f^{i\alpha\beta} \int \Big\{ \mathrm{tr}\,\Big(\mathrm{R}_Y(\mathrm{T}_i)\big(\mathrm{L}^{\mathrm{R}}_\alpha \mathrm{K}^{-1}\big)\big(\mathrm{L}^{\mathrm{R}}_\beta \mathrm{K}\big)\Big) \;-\; \mathrm{tr}\,\Big(\mathrm{R}_X(\mathrm{T}_i)\big(\mathrm{L}^{\mathrm{R}}_\alpha \mathrm{K}\big)\big(\mathrm{L}^{\mathrm{R}}_\beta \mathrm{K}^{-1}\big)\Big)\Big\} \\ =\; & \frac{3\eta}{2} f^{i\alpha\beta} \int \Big\{ \mathrm{tr}\,\Big(\mathrm{K}^{-1}[\mathrm{L}^{\mathrm{R}}_\alpha, \mathrm{L}^{\mathrm{R}}_\beta]\mathrm{K}\mathrm{R}_Y(\mathrm{T}_i)\Big) \;-\; \mathrm{tr}\,\Big(\mathrm{K}[\mathrm{L}^{\mathrm{R}}_\alpha, \mathrm{L}^{\mathrm{R}}_\beta]\mathrm{K}^{-1}\mathrm{R}_X(\mathrm{T}_i)\Big)\Big\} \\ =\; & \frac{3i\eta}{2} f^{i\alpha\beta} f^j{}_{\alpha\beta} \Big\{ \mathrm{tr}\,\Big(\mathrm{R}_Y(\mathrm{T}_i)\mathrm{R}_Y(\mathrm{T}_j)\Big) \;-\; \mathrm{tr}\,\Big(\mathrm{R}_X(\mathrm{T}_i)\mathrm{R}_X(\mathrm{T}_j)\Big)\Big\} \quad . \end{aligned}$$

Analogously we can treat the second term in (4.36) to find the result

$$\mathcal{S}_X(\mathrm{A}) \;=\; -\frac{\pi^2}{6k^2}\big(3g^\vee - \frac{2}{x_e} g^\vee_\omega\big)\Big\{\mathrm{tr}\Big(\mathrm{R}_Y(\mathrm{T}_i)\mathrm{R}_Y(\mathrm{T}_i)\Big) - \mathrm{tr}\Big(\mathrm{R}_X(\mathrm{T}_i)\mathrm{R}_X(\mathrm{T}_i)\Big)\Big\} \;, \tag{4.37}$$

where $g^\vee$ and $g^\vee_\omega$ are the dual Coxeter numbers of $\mathfrak{g}$ and $\mathfrak{g}^\omega$, respectively. The constant x_e denotes the embedding index of the embedding $\mathfrak{g}^\omega \hookrightarrow \mathfrak{g}$. It appears due to different normalization of the quadratic forms $\kappa_{\mathfrak{g}^\omega}$ and $\kappa_{\mathfrak{g}}|_{\mathfrak{g}^\omega}$. The computation relied heavily on the restriction to automorphisms of order 1 or 2. For all these cases (see Table 4.1) it can be shown that

$$3g^\vee - \frac{2}{x_e} g^\vee_\omega \;=\; \frac{\dim \mathfrak{g}}{\dim \mathfrak{g}^\omega} g^\vee \quad .$$

To proceed further, we expand the configurations X and Y into the constituents and finally arrive at

$$\mathcal{S}_X(\mathrm{A}) \;=\; -\frac{\pi^2}{6k^2}\frac{\dim\mathfrak{g}}{\dim\mathfrak{g}^\omega}\frac{g^\vee}{\dim V^X}\sum_l \big(Y_{\Psi(l)}\;-\;X_{\Psi(l)}\big)\dim(V^l)\frac{C_l}{x_e} \quad . \tag{4.38}$$

The number C_l is the quadratic Casimir of the representation l of $\mathfrak{g}^\omega$.

Let us pause for a moment to observe what our computation implies for $X = Y$. The function $\mathrm{K}(g)$ is then an allowed gauge transformation, and what we have just shown is that the action vanishes for pure gauge configurations. This is the result we anticipated in the footnote on page 40. It completes, finally, our argument for the invariance of the action (4.14) under global gauge transformations.

After this short intermission, we continue with checking the interpretation of the solutions as brane processes. As we recalled before, the result (4.38) for the value of the action must be compared with the difference between two logarithms of the g-factors [81] in the CFT-description. For a single brane λ, the g-factor is given by

$$g_\lambda \;=\; \frac{S^\omega_{0\lambda}}{\sqrt{S_{00}}} \quad . \tag{4.39}$$

It is shown in Appendix B that the ratio $g_\lambda/g_{\Psi(0)}$ can be expressed by an ordinary character of the representation $l = \Psi^{-1}(\lambda)$ of $\mathfrak{g}^\omega$,

$$\frac{g_{\Psi(l)}}{g_{\Psi(0)}} \;=\; \chi_l\Big(-\frac{2\pi i}{k+g^\vee}\frac{1}{x_e}\mathcal{P}(\rho)\Big) \quad ,$$

where $\mathcal{P}$ is the projection matrix specifying the embedding $\mathfrak{g}^\omega \subset \mathfrak{g}$ (see Appendix B).

The asymptotic expression for $k \to \infty$ can be found using (a generalized version of) the asymptotic expansion in [66, eq. (13.175)],

$$\frac{g_{\Psi(l)}}{g_{\Psi(0)}} \;=\; \dim(V^l)\Big(1 - \frac{\pi^2}{6(k+g^\vee)^2}\frac{\dim\mathfrak{g}}{\dim\mathfrak{g}^\omega}g^\vee\frac{C_l}{x_e} + \mathcal{O}(\frac{1}{k^4})\Big) \quad .$$

Details are provided in Appendix B.

Now we are prepared to do the comparison of the value of the action (4.38) with the g-factors. As we anticipated before, we get the result

$$\ln\frac{g_Y}{g_X} \;=\; \frac{\sum Y_\lambda\, g_\lambda}{\sum X_\lambda\, g_\lambda} \;=\; \mathcal{S}_X(\mathrm{A})$$

at order $1/k^2$.

This ends our discussion of brane dynamics in WZW models in the limit $k \to \infty$. We have provided a complete classification of processes accessible in this limit between maximally symmetric (twisted and untwisted) branes. In Chapter 5 we will extend some of these results to models at finite level k. The remaining two sections of this chapter are devoted to the discussion of branes in coset models.

4.4 Brane dynamics in coset models

After the analysis of branes in WZW models, we now want to turn to coset models. These are built as quotient theories of WZW models, so we expect that the insights we gained into brane dynamics in WZW models will help us a great deal in the investigation of the behavior of branes in coset models.

We will first derive the low energy effective theory on coset branes from the results of Section 4.2. Subsequently, solutions of this theory will be discussed and an interpretation as brane processes will be provided.

4.4.1 Reduction to coset models

We have introduced the effective action for branes on group manifolds in Section 4.2. The result has been discussed for a WZW model involving a single affine Lie algebra $\widehat{\mathfrak{g}}_k$. For our purposes below, we need the action for cases where the underlying affine Lie algebra is a direct sum of algebras with different levels k_r. From the resulting action we will then obtain the effective field theory of coset branes by reduction.

A coset model involves two chiral algebras $\widehat{\mathfrak{g}}$ and $\widehat{\mathfrak{h}}$ in the numerator and the denominator, respectively. In general, these possess decompositions of the form $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_1 \oplus \cdots \oplus \widehat{\mathfrak{g}}_R$ and $\widehat{\mathfrak{h}} = \widehat{\mathfrak{h}}_1 \oplus \cdots \oplus \widehat{\mathfrak{h}}_{R'}$ with possibly different levels $k_1, \ldots, k_R; k'_1, \ldots, k'_{R'}$ appearing in each summand. We will study a regime of the model in which some of the levels are sent to infinity while others stay finite. Let us assume that the decompositions above have been arranged such that $k_1, \ldots, k_S$ and $k'_1, \ldots k'_{S'}$ become large.

In this limiting regime we intend to study symmetric brane configurations $X = \sum X_{(\lambda,\lambda')}(\lambda, \lambda')$ where λ, λ' are multi-labels of the form $\lambda = (\lambda_1, \ldots, \lambda_S, 0, \ldots, 0)$ and $\lambda' = (\lambda'_1, \ldots, \lambda'_{S'}, 0, \ldots, 0)$ in which the labels for the small directions are chosen to be trivial[3]. As in the case of branes in WZW models, we associate a certain

[3] In the limit of large k the theory is essentially independent of the labels $\lambda_{S+1}, \ldots, \lambda_R$, $\lambda'_{S'+1}, \ldots, \lambda'_{R'}$

representation to such a configuration, namely the $G^\omega \times H^\omega$ module

$$V^X \;=\; \bigoplus_{l,l'} X_{(\Psi(l),\Psi(l'))}(V^l \otimes V^{l'^+}) \quad .$$

Note the appearance of the conjugate representation l'^+ in the $\mathfrak{h}^\omega$ part.

The field theory we are going to spell out now involves a number of gauge fields A_α where α labels a basis in $\mathfrak{g}\oplus\mathfrak{h}$, i.e. it runs through the values $1,\ldots,\dim\mathfrak{g}+\dim\mathfrak{h}$. The gauge fields A_α are elements of the space $\mathcal{A}^X$ which depends on the choice of our initial brane configuration X. We introduce slightly changed derivatives $\hat{\mathrm{L}}_\alpha$,

$$\hat{\mathrm{L}}_\alpha \mathrm{A} \;=\; \begin{cases} \mathrm{L}_\alpha \mathrm{A} & \text{for} \quad \alpha \leq \dim\mathfrak{g} \\ i\,\mathrm{L}_\alpha \mathrm{A} & \text{for} \quad \alpha > \dim\mathfrak{g} \end{cases} \tag{4.40}$$

where we have absorbed an extra factor $\sqrt{-1}$ into the definition of $\hat{\mathrm{L}}_\alpha, \alpha > \dim\mathfrak{g}$. This will turn out to be rather convenient in the following. We are now able to introduce the action,

$$\mathcal{S}_X^{\mathrm{WZW}}(\mathrm{A}) \;=\; \mathcal{S}_{\mathrm{YM}}(\mathrm{A}) + \mathcal{S}_{\mathrm{CS}}(\mathrm{A}) \;=\; 4\pi^2\left(\frac{1}{4}\mathrm{tr}(\mathrm{F}_{\alpha\beta}\mathrm{F}^{\alpha\beta}) - \frac{i}{2}\mathrm{tr}(\hat{f}^{\alpha\beta\gamma}\mathrm{CS}_{\alpha\beta\gamma})\right) \tag{4.41}$$

where $\hat{f}^{\alpha\beta\gamma} \;=\; f^{\alpha\beta\gamma}$ if $\alpha,\beta,\gamma \leq \dim\mathfrak{g}$ and $\hat{f}^{\alpha\beta\gamma} \;=\; if^{\alpha\beta\gamma}$ otherwise.

Our original action (4.14) for WZW branes has a pre-factor $1/k^2$ containing the level k. In our case here, different levels are involved, therefore it is convenient to absorb the level dependence by introducing the open string metric

$$G^{\alpha\beta} \;=\; \frac{2}{k(\alpha)}\,\kappa^{\alpha\beta} \;. \tag{4.42}$$

which is used to raise and lower the indices α. Here, the function $k(\alpha)$ has been introduced such that it takes the value k_r (or $k'_{r'}$) if α refers to a basis element in the Lie-algebra $\mathfrak{g}_r$ (or $\mathfrak{h}_{r'}$). $\kappa^{\alpha\beta}$ denotes the Killing form.

We use the effective action $\mathcal{S}^{\mathrm{WZW}}$ as a master theory from which we descend to the effective description of branes in coset conformal field theory. For the reduction it is convenient to switch to a new basis of $\mathfrak{g}\oplus\mathfrak{h}$ which makes reference to the embedding $\mathfrak{h}\subset\mathfrak{g}$. We shall employ $a = 1,\ldots,\dim\mathfrak{g}-\dim\mathfrak{h}$ when we label directions perpendicular to $\mathfrak{h}\subset\mathfrak{g}$ while labels $i = 1,\ldots,\dim\mathfrak{h}$ and $\tilde{\imath} = 1,\ldots,\dim\mathfrak{h}$ stand for directions along $\mathfrak{h}\subset\mathfrak{g}$ and $\mathfrak{h}$, respectively.

The effective theory for the coset brane configuration X involves the fields A_a and is given by the WZW action (4.41)

$$\mathcal{S}^{\text{coset}}(\mathrm{A}_a) \;=\; \mathcal{S}^{\text{WZW}}(\mathrm{A}_a, \mathrm{A}_i, \mathrm{A}_{\bar{\imath}})$$

along with the following constraints

$$i\mathrm{A}_i \;=\; \mathrm{A}_{\bar{\imath}} \;=\; 0 \tag{4.43}$$

$$(i\hat{\mathrm{L}}_i + \hat{\mathrm{L}}_{\bar{\imath}})\mathrm{A}_a + f_{ia}{}^b \mathrm{A}_b \;=\; 0\ . \tag{4.44}$$

From the coset model's point of view, the fields $\mathrm{A}_i, \mathrm{A}_{\bar{\imath}}$ play the role of auxiliary fields, appearing in the master WZW theory. The constraints (4.43,4.44) on them follow directly from the derivation of the coset effective theory that we will present soon. It is, however, possible to relax the constraints, and this will prove to be very useful when we discuss the theory of solutions. Indeed, one can show that

$$\mathcal{S}^{\text{coset}}(\mathrm{A}_a) \;=\; \mathcal{S}^{\text{WZW}}(\mathrm{A}_a, \mathrm{A}_i, \mathrm{A}_{\bar{\imath}})$$

where the fields obey the constraints

$$i\mathrm{A}_i + \mathrm{A}_{\bar{\imath}} \;= 0 \tag{4.45}$$

$$i\mathrm{F}_{i\alpha} + \mathrm{F}_{\bar{\imath}\alpha} \;= 0\ \ . \tag{4.46}$$

For $\alpha = a$ the second constraint is equivalent to the original constraint (4.44). These conditions allow to eliminate A_i and $\mathrm{A}_{\bar{\imath}}$ completely from the action (4.41), the action does only depend on A_a. Indeed, we are free to choose any $\mathrm{A}_i = i\mathrm{A}_{\bar{\imath}}$ satisfying the second constraint (4.46) for $\alpha = j$. In this sense, the choice (4.43) of setting the auxiliary fields to zero, is just a particular easy way to obtain the effective theory for A_a, but for our discussion of classical solutions of the coset model, it is more convenient to use the less restrictive constraint (4.45).

Note that our second formulation of the constraints is much more suitable to discuss gauge invariance. The constraint (4.46) is already in a gauge invariant form, whereas the other one (4.45) is not. We thus would obtain an equivalent theory by imposing instead of (4.45) the condition

$$i\mathrm{A}_i + \mathrm{A}_{\bar{\imath}} \;=\; -(i\hat{\mathrm{L}}_i + \hat{\mathrm{L}}_{\bar{\imath}})U\,U^{-1}$$

for some gauge transformation U.

This concludes the presentation of the effective theory. In the following, we will sketch how this result can be derived.

Let us start with the product $\mathcal{H}^{\mathfrak{g}\,\mu}_{\lambda} \otimes \mathcal{H}^{\mathfrak{h}\,\lambda'}_{\mu'}$ of state spaces for boundary theories of the G and H WZW theory describing strings stretched between the boundaries λ and μ, μ' and λ', respectively. Within such a space we want to find the state space $\mathcal{H}^{(\mu,\mu')}_{(\lambda,\lambda')}$ of the coset theory by imposing suitable constraints on the vectors $\psi \in \mathcal{H}^{\mathfrak{g}\,\mu}_{\lambda} \otimes \mathcal{H}^{\mathfrak{h}\,\lambda'}_{\mu'}$. Let us first project to the ground states for the actions of $\widehat{\mathfrak{h}} \subset \widehat{\mathfrak{g}}$ on the first factor and of $\widehat{\mathfrak{h}}$ on the second by demanding

$$J^i_n\,\psi \;=\; J^{\tilde\imath}_n\,\psi \;=\; 0 \qquad \text{for all} \;\; n > 0 \;\;, \tag{4.47}$$

where $i, \tilde\imath$ run through the usual range. From the formula (3.6) and the expansion (3.13) we can conclude that the resulting subspace of states ψ fulfilling (4.47) has the form

$$\bigoplus_{l,l',m'} n^{\mathfrak{g}\;\mu}_{l\,\lambda}\; n^{\mathfrak{h}\;\;\lambda'}_{m'\mu'}\; \mathcal{H}^{(l,l')} \otimes V^{l'} \otimes V^{m'} \;\;. \tag{4.48}$$

Here, $V^{m'}$ denotes the space of ground states in the $\widehat{\mathfrak{h}}$-sector labeled by m'. We now require the invariance under the diagonal $\mathfrak{h} \subset \mathfrak{g} \oplus \mathfrak{h}$ acting on the ground states,

$$(J^i_0 + \tilde J^i_0)\;\psi \;=\; 0 \;\;. \tag{4.49}$$

The only contribution in the sum comes from $l' = m'^{+}$, and the invariant part of $V^{l'} \otimes V^{m'^{+}}$ is one-dimensional. Finally, we are left with the space

$$\bigoplus_{l,l'} n^{\mathfrak{g}\;\mu}_{l\,\lambda}\; n^{\mathfrak{h}\;\;\mu'}_{l'\lambda'}\; \mathcal{H}^{(l,l')} \tag{4.50}$$

which is isomorphic to the state space $\mathcal{H}^{(\mu,\mu')}_{(\lambda,\lambda')}$ of the boundary coset model. In this way we have prepared states of the coset theory from states of the product of boundary WZW models.

By the state-field correspondence we can build boundary operators in the product theory. For large level k these can be reduced to boundary operators in the coset theory [10]. Now, the result for the 3- and 4-point function in the coset model can be read off from the result in the WZW model for G and H by restricting the boundary fields according to our constraints (4.47) and (4.49).

The second constraint (4.49) leads directly to the condition (4.44) on the fields A_a. On the other hand, the first constraint (4.47) leads to a strong suppression of terms involving the fields $\mathrm{A}_i, \mathrm{A}_{\tilde\imath}$. A careful analysis shows that this suppression is strong enough to neglect these terms in the considered order of $1/k$. Consequently, we can set $\mathrm{A}_i, \mathrm{A}_{\tilde\imath}$ to zero, and this explains the first condition (4.43) on the fields.

We still have to explain the factor $\sqrt{-1}$ in the definition of the derivatives (4.40). This has to do with the quadratic terms in the effective action which are given by the conformal dimensions. A mode (l, l') of the coset model contributes a quadratic term proportional to $h_{(l,l')}$. From the product theory, however, we find the mode (l, l') accompanied with two fields of weight $h_{l'}$ coming from the two $\widehat{\mathfrak{h}}$ parts in the product theory. Thus, in the product theory we find $h_{(l,l')} + 2h_{l'}$ which does not give the desired result unless $h_{l'} = 0$. The introduction of an extra factor $\sqrt{-1}$ guarantees that the two contributions to the quadratic term proportional to $h_{l'}$ cancel instead of adding up. The higher order terms in the constrained theory are not affected by the redefinition of the derivatives.

More details about this derivation can be found in [10].

4.4.2 Solutions and brane processes

Having found the effective theory describing branes in coset models, we can now start to study solutions. Some of the solutions from WZW branes descend to coset branes. We will provide an interpretation for them as brane processes.

First we have to determine the equations of motions for the fields A_a. We vary the WZW action (4.41) under the constraints (4.45,4.46). One can show that the variation vanishes away from field configurations satisfying the constraints, and so we do not have to introduce Lagrange multipliers. The equations of motion take the same form as in the unconstrained case,

$$\hat{\mathrm{L}}^\alpha \mathrm{F}_{\alpha b} + [\mathrm{A}^\alpha, \mathrm{F}_{\alpha b}] \; = \; 0 \quad . \tag{4.51}$$

Although it is not obvious at first sight, this equation does not depend on the $\mathrm{A}_i, \mathrm{A}_{\bar{\imath}}$ because of the constraints.

One can perform an analysis similar to the one in Section 4.3 to obtain a notion of 'symmetric' and 'non-symmetric' solutions. It turns out that any symmetric solution A_a to the equations of motions can be supplemented by some auxiliary fields $\mathrm{A}_i = i\mathrm{A}_{\bar{\imath}}$ s.t. $\mathrm{F}_{\alpha\beta} = 0$. This means that we can restrict our search for symmetric solutions to the candidates provided by our classification of symmetric solutions in WZW models.

Recall that we constructed the symmetric solutions for WZW branes by means of a function K. In our case this is a function $\mathrm{K} : G \times H \to U(V^Y, V^X)$ on the product $G \times H$ with values in the unitary homomorphisms from V^Y to V^X, where V^Y is a $G^\omega \times H^\omega$ module with $\dim V^Y = \dim V^X$. The corresponding solution (4.21) of the WZW model

$$\mathrm{A}_\alpha(g) \; = \; -(\hat{\mathrm{L}}_\alpha \mathrm{K})(g)\mathrm{K}(g)^{-1}$$

automatically satisfies the coset equations of motion (4.51) as well as the constraints (4.46). We only have to impose the remaining constraint (4.45) i.e. we have to look for functions K satisfying

$$(i\hat{\mathrm{L}}_i + \hat{\mathrm{L}}_{\bar{\imath}})\mathrm{K} \equiv 0 \quad .$$

The function K has to be invariant under the action of the diagonal $H_{\mathrm{diag}} \subset G \times H$ by left translations.

Let us reformulate our quest for symmetric solutions as an existence problem of a function K with certain properties.

> To construct a symmetric solution on a coset brane configuration X, we have to find a function $\mathrm{K} : G \times H \to U(V^Y, V^X)$ satisfying
>
> $$\mathrm{K}(gg^\omega, hh^\omega) = \mathrm{R}_X(g^\omega, h^\omega)^{-1}\mathrm{K}(g,h)\mathrm{R}_Y(g^\omega, h^\omega) \quad \text{and} \tag{4.52}$$
> $$\mathrm{K}(h'g, h'h) = \mathrm{K}(g,h) \quad \text{for } h' \in H \quad . \tag{4.53}$$

As for branes on group manifolds, there is a geometric interpretation for this existence problem. Consider again the vector bundles

$$E^X = (G \times H) \times_{G^\omega \times H^\omega} V^X_{\mathbb{C}} \quad .$$

In addition to the vector bundle structure, we need the action of the diagonal group H on E^X induced by

$$\begin{array}{rccc} H \ni h_0 : & (G \times H) \times V^X_{\mathbb{C}} & \to & (G \times H) \times V^X_{\mathbb{C}} \\ & \Cup & & \Cup \\ & (g, h\,; v) & \mapsto & (h_0^{-1}g, h_0^{-1}h\,; v) \end{array} \quad .$$

This action turns E^X into a H-vector bundle[4] E^X_H over the H-space

$$G \times H \,/\, G^\omega \times H^\omega \quad .$$

Now, a function $\mathrm{K} : G \times H \to U(V^Y, V^X)$ with the properties (4.52,4.53) exists if and only if the two H-spaces E^X_H and E^Y_H are isomorphic,

$$E^X_H \simeq E^Y_H \quad . \tag{4.54}$$

A general classification of these induced H-vector bundles seems to be much harder than the classification of ordinary vector bundles in the case of WZW models.

[4] A H-space M_H is a manifold which admits a (smooth) action of a (Lie) group H. A H-vector bundle E_H over a base manifold M_H is a H-space s.t. the fibre-wise action of H is a vector space isomorphism, and the projection to the base commutes with the action of H.

Nevertheless, it is possible to find a large number of solutions to this problem. As in the WZW model, the interpretation of these solutions is a transition between the brane configurations X and Y. We will come back to this point and show explicitly how the arguments for this interpretation descend from the results in WZW models. Before we do so, let us present two types of solutions where we have explicit expressions for K. We encountered the corresponding solutions in WZW models already at the end of Section 4.3.3.

First, we want to consider solutions for a configuration X whose representation V^X is a restriction of a representation of $G \times H$. Note that for untwisted branes, $\omega = \mathrm{id}$, this is of course always the case. We assume further that the $G^\omega \times H^\omega$-module V^Y is a restriction of a $G \times H$-module. If we restrict it instead to the diagonal $H_{\mathrm{diag}} \subset G \times H$, it should coincide with the restricted representation V^X,

$$V^X\big|_{H_{\mathrm{diag}}} \;=\; V^Y\big|_{H_{\mathrm{diag}}} \quad .$$

We construct the function K as in (4.25),

$$\mathrm{K}(g,h) \;=\; \mathrm{R}_X(g,h)^{-1}\mathrm{R}_Y(g,h) \quad .$$

The covariance under the action of the diagonal H_{diag} can be explicitly checked

$$\begin{aligned} \mathrm{K}(h'g,h'h) &= \mathrm{R}_X(g,h)^{-1}\mathrm{R}_X(h',h')^{-1}\mathrm{R}_Y(h',h')\mathrm{R}_Y(g,h) \\ &= \mathrm{K}(g,h) \end{aligned}$$

where we used that the representation matrices R_X and R_Y coincide on the diagonal group H_{diag}.

Let us now consider our second class of solutions.

Assume that the configuration X has a corresponding $G^\omega \times H^\omega$-module which is the following product of a G^ω-module and a H^ω-module

$$V^X \;=\; V^l \otimes (V^S\big|_{H^\omega} \otimes_{H^\omega} V^{l'}) \quad ,$$

where V^S is a G-module. Then there is a solution describing a process to the configuration Y whose corresponding representation V^Y has also the form of a product,

$$V^Y \;=\; (V^l \otimes_{G^\omega} V^S\big|_{G^\omega}) \otimes V^{l'} \quad .$$

As vector spaces, we can identify V^X and V^Y both with $V^l \otimes V^S \otimes V^{l'}$. The solution is constructed with the help of the following function K acting on this vector space,

$$\mathrm{K}(g,h) \;=\; \mathbf{1} \otimes \mathrm{R}_S(h^{-1}g) \otimes \mathbf{1} \quad .$$

It is straightforward to check that K fulfills both conditions (4.52,4.53). The coset fields A_a take the form

$$\mathrm{A}_a(g,h) \;=\; \mathbf{1} \otimes \mathrm{R}_S(h)^{-1}\mathrm{R}_S(\mathrm{T}_a)\mathrm{R}_S(h) \otimes \mathbf{1} \quad . \tag{4.55}$$

We observe that the fields are constant in all directions of G. These solutions will play a special role when we discuss extrapolations to finite level k in Chapter 5.

Before we move on to the interpretation of the solutions, let us note one common property of these two types of solutions. The modules V^X and V^Y are in both cases equivalent when we restrict them to the diagonal $H^\omega_{\mathrm{diag}} \subset G^\omega \times H^\omega$,

$$V^X\big|_{H^\omega_{\mathrm{diag}}} \;\cong\; V^Y\big|_{H^\omega_{\mathrm{diag}}} \quad . \tag{4.56}$$

This is certainly a general feature and is valid for all symmetric solutions. In the case of untwisted branes $\omega = \mathrm{id}$, the converse is also true: whenever the representations of two configurations coincide on the diagonal H_{diag}, there is a solution connecting them. For branes with a non-trivial twist we have not enough control over the existence of solutions to decide whether such a converse statement is possible.

We finally want to discuss the interpretation for the formulated solutions. As already mentioned, the solutions correspond to processes of the form

$$\text{Configuration } X \quad \longleftrightarrow \quad \text{Configuration } Y \quad , \tag{4.57}$$

completely similar to (4.29).

The evidence for this interpretation comes again from considering fluctuations around the solution and from an analysis of brane tensions (g-factors). Luckily, we do not need to perform long calculations to arrive at this conclusion. Many results from WZW models (generalized to products of WZW models) carry over to the discussion here.

Let us recall from Section 4.3.4 what we have to show. The action expanded around a chosen solution A_a that is supposed to describe a process from X to Y should contain a constant part related to the g-factors, and the remaining part should coincide with the action of the configuration Y,

$$\mathcal{S}^{\mathrm{coset}}_X(\mathrm{A}_a + \delta\mathrm{A}_a) \;=\; \ln\frac{g_Y}{g_X} \;+\; \mathcal{S}^{\mathrm{coset}}_Y(\Phi_{XY}(\delta\mathrm{A}_a)) \quad . \tag{4.58}$$

The map Φ_{XY} relates the fluctuations A_a to fields on the configuration Y (compare eq. (4.30)). In addition, the map Φ_{XY} should also respect the constraint (4.44). From the analysis of Section 4.3.4 we know that the map Φ_{XY} essentially shifts the derivatives by a commutator with the fields A_α (see eq. (4.32)). The diagonal combination $i\hat{\mathrm{L}}_i + \hat{\mathrm{L}}_{\bar{\imath}}$ appearing in the constraint (4.44) is thus shifted by $i\mathrm{A}_i + \mathrm{A}_{\bar{\imath}}$ which is zero according to the constraint (4.45). We conclude that the constraints for the fluctuations $\delta\mathrm{A}_a$ are correctly translated to constraints on fields of the configuration Y.

It remains to verify eq. (4.58). We can use the results (4.30) on WZW models noting that the ratio of the g-factors g_Y/g_X is the same for coset branes and the corresponding WZW branes. The expansion of the coset action can be traced to the expansion of the action of the WZW model,

$$\begin{aligned}\mathcal{S}^{\text{coset}}_X(\mathrm{A}_a + \delta\mathrm{A}_a) &= \mathcal{S}^{\text{WZW}}_X(\mathrm{A}_a + \delta\mathrm{A}_a, \mathrm{A}_i, \mathrm{A}_{\bar{\imath}}) \\ &= \ln\frac{g_Y}{g_X} \;+\; \mathcal{S}^{\text{WZW}}_Y(\Phi_{XY}(\delta\mathrm{A}_a), 0, 0) \\ &= \ln\frac{g_Y}{g_X} \;+\; \mathcal{S}^{\text{coset}}_Y(\Phi_{XY}(\delta\mathrm{A}_a))\end{aligned}$$

leading to the desired result. This verifies the identification of the solution with the brane process $X \leftrightarrow Y$.

For a general twisted brane configuration, the presented processes will only provide a subset of all symmetric solutions. In the case of untwisted branes, however, all solutions and processes are of the described form. Let us report briefly on the consequences of specializing to the case of untwisted branes. The fields A_a take their values in the finite algebra $\mathrm{End}(V^X)$. For any configuration Y whose representation coincides with V^X on the diagonal H_{diag}, there is a solution given by

$$\mathrm{A}_a \;=\; \mathrm{R}_Y(\mathrm{T}_a) - \mathrm{R}_X(\mathrm{T}_a) \quad .$$

In particular, we can find a positive energy solution for an arbitrary brane. Let the representation corresponding to this brane be $V^L \otimes V^{L'}$. Then define the function K as

$$\mathrm{K}(g,h) \;=\; \mathrm{R}_L(g^{-1}h) \otimes \mathbf{1} \quad .$$

The resulting representation V^Y is $V^0 \otimes (V^L\big|_H \otimes_H V^{L'})$. In other words, any untwisted coset brane can be obtained as condensate of a configuration of branes with trivial label in the numerator $\widehat{\mathfrak{g}}$ theory.

Before we conclude this section and move on to some examples, let us add some remarks. First note that we tacitly assumed that the results from Section 4.3 for simple WZW models can be generalized to product theories. This is indeed possible without difficulties, details can be found in [10]. Secondly, we should note that it may happen that the coset action vanishes in the order k^2. This is the case if all 'large' directions[5] that are used in the construction of the solution are divided out. We will encounter such a case in the example of the minimal models. It can be shown, however, that in this case the relation between g-factors and the value of the action at the solution is fulfilled also in the order $1/k^3$.

4.5 Examples and geometric interpretation

In this final section we want to illustrate our very general results in three simple examples. It will become clear that the solutions we have constructed above are capable of describing brane processes with very different geometrical manifestations.

4.5.1 Parafermions

The parafermion theories can be constructed as a coset model $\widehat{su}(2)_k/\widehat{u}(1)_k$. The free bosonic $U(1)$ theory is embedded such that its current gets identified with the component J^3 of the $SU(2)$ current.

Before we apply our general results on brane dynamics, we want to have an intuitive geometric picture of this model and its branes. We use the following parametrization of $SU(2)$,

$$SU(2) \;=\; \left\{ g(z,\psi) \,\middle|\, z \in \mathbb{C}, |z| \leq 1, \psi \in [0, 2\pi[\right\}$$

with

$$g(z,\psi) \;=\; \begin{pmatrix} z & e^{i\psi}\sqrt{1-|z|^2} \\ -e^{-i\psi}\sqrt{1-|z|^2} & \bar{z} \end{pmatrix} \quad .$$

The embedded $U(1)$ can be described by restricting z to $|z| = 1$,

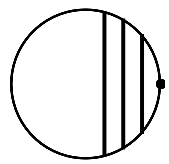
$$U(1) \;=\; \left\{ \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix},\ \alpha \in [0, 2\pi[\right\} \quad .$$

The adjoint action of $U(1)$ results in translations in ψ. We therefore conclude that the coset geometry is described by z alone, that means it has the topology of a disc

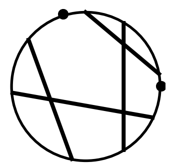
$$SU(2)\big/_{\mathrm{Ad}\, U(1)} \;\simeq\; D^2 \quad .$$

[5] by large directions we mean those which belong to a large level k_r

What about the branes? Conjugacy classes in $SU(2)$ are labeled by the real part of z (which is just half the trace of $g(z,\psi)$). In the coset geometry they appear as straight lines on the disc, degenerating to points for $\Re z = \pm 1$.



Conjugacy classes of $U(1)$ are points, their products with a conjugacy class of $SU(2)$ descend to rotated lines on the disc



A brane in the parafermion theory is thus described by the angular position and the opening angle of the line. We will see in the following CFT discussion that both labels are quantized and that there is no continuous $U(1)$ symmetry in the model.

The numerator CFT has sectors $\mathcal{H}^l_{su(2)}$ where $l = 0, 1, \ldots, k$, the sectors $\mathcal{H}^m_u$ of the denominator algebra $\widehat{u}(1)_k$ carry a label $m = -k+1, \ldots, k$. We can label the sectors $\mathcal{H}^{(l,m)}$ of the coset model by pairs (l,m) of numerator and denominator labels. The possible pairs (l,m) are restricted by a selection rule forcing the sum $l+m$ to be even. Furthermore some pairs label the same sector so that we have to identify the pairs $(l,m) \sim (k-l, m+k)$ where we take the label m to be $2k$-periodic. Note that this field identification has no fixed points.

Let us restrict our discussion here to untwisted branes, $\omega = \mathrm{id}$. Twisted branes in the parafermion theory have been constructed in [44] (they are called B-branes there), and we will discuss them for the special case $k = 3$ in Section 5.2.4. Untwisted branes have labels from the same set as sectors, and we will denote a brane by the pair (L, M). These labels have a direct interpretation in our geometrical picture as opening angle and angular position of the brane (see fig. 4.3). Before we start to investigate the dynamics of these branes, let us make some remarks on the absence of a $U(1)$ symmetry.

It seems surprising that the positions of the point-like branes are restricted to a discrete set. The evidence for that comes from looking at the spectrum of such a brane which does not contain any massless mode (conformal weight $h = 1$). Consequently, there is no possibility to deform the boundary condition smoothly by a marginal deformation.

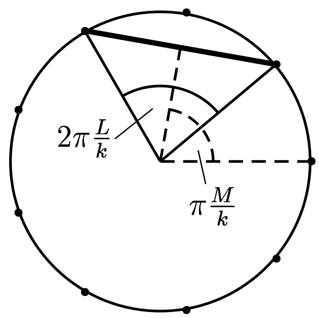


Figure 4.3: A generic untwisted brane (L, M) in the parafermion model and the geometric interpretation of the labels of the brane. The possible positions of the point-like branes of type $(0, M)$ are also indicated.

Let us have a look at some examples for special values of k. For $k = 2$ we find the Ising model (central charge $c = 1/2$). Obviously, we do not expect a $U(1)$ symmetry there.

For $k = 3$ we find the three-state Potts model (central charge $c = 4/5$) which does not have continuous moduli for D0-branes.

For $k = 4$ it becomes more interesting: the central charge is $c = 1$. We expect a continuous moduli space for D0-branes, so why do we not see it? The parafermion theory for $k = 4$ describes a boson on $S^1/\mathbb{Z}_2$. The four branes of the form $(0, M)$ are identified with D0-branes fixed on the orbifold singularities, in agreement with the absence of massless modes. We also find D0-branes along the interval realized as $(2, 0)$ and $(2, 2)$-branes. We expect to find a massless state in their open string spectrum as the branes can move along the interval, and indeed we find one in the sector $\mathcal{H}^{(4,0)}$. Note however, that this is special to $k = 4$.

Now we want to apply our general formalism to formulate the effective action for the parafermion branes. The master WZW theory involves four fields $\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3, \mathrm{A}_{\tilde{3}}$. A brane configuration X corresponds to a representation V^X of $SU(2) \times U(1)$. The fields are elements of $\mathrm{End}(V^X)$, and the Lie algebra $su(2) \oplus u(1)$ acts on them via the derivatives $\hat{\mathrm{L}}_\alpha$.

The constraints (4.43,4.44) read in the parafermion case

$$\mathrm{A}_3 \;=\; \mathrm{A}_{\tilde{3}} \;=\; 0 \tag{4.59}$$

$$(i\hat{\mathrm{L}}_3 + \hat{\mathrm{L}}_{\tilde{3}})\mathrm{A}_a + f_{3a}{}^{b}\mathrm{A}_b \;=\; 0 \quad a, b = 1, 2 \quad . \tag{4.60}$$

Eventually we arrive at the following effective action for the parafermion theory,

$$\mathcal{S}_X^{\text{Para}} \;=\; \frac{\pi^2}{k^2}\left(\frac{1}{4}\text{tr}\;\text{F}_{ab}\text{F}^{ab}+\frac{1}{2}f^{3ab}\,\text{tr}\,(\hat{\text{L}}_{\bar{3}}\text{A}_a)\,\text{A}_b\right) \quad . \tag{4.61}$$

When we consider configurations of branes with trivial label in the denominator part, the second term vanishes and we are left with a pure Yang-Mills term.

Let us now discuss some brane processes. Take a brane whose corresponding representation is $V^L \otimes V^M$. When we decompose this representation into irreducible representations of the diagonally embedded $U(1)$, we find

$$V^L\big|_{u(1)} \otimes_{u(1)} V^M \;\longrightarrow\; V^{-L+M} \oplus V^{-L+2+M} \oplus \cdots \oplus V^{L+M} \quad .$$

We discussed in Section 4.4 for untwisted coset branes that any two configurations whose representations coincide on the diagonal group H_{diag} are connected by a solution. In our case here, we could take for example the representation $(V^{L-1} \otimes V^{M-1}) \oplus (V^0 \otimes V^{L+M})$.

As mentioned at the end of Section 4.4, we can obtain any untwisted coset brane as a condensate of a configuration of branes of type $(0, M)$. In our special example, this configuration would correspond to the representation

$$V^0 \otimes V^{M-L} \,\oplus \cdots \oplus\, V^0 \otimes V^{M+L} \quad .$$

Geometrically, this configuration is a chain of adjacent point-like branes. This chain is unstable and forms as a bound state the brane connecting the starting point and the endpoint of the chain (see fig. 4.4). Thus, any untwisted brane in the parafermion model can be constructed out of the point-like branes sitting at the boundary.

4.5.2 N=2 Minimal models

Our results can easily be extended to the $N = 2$ super-symmetric minimal models. The latter are obtained as $\widehat{su}(2)_k \oplus \widehat{u}(1)_2/\widehat{u}(1)_{k+2}$ coset theories. Now we need three integers (l, s, m) to label sectors, where $l = 0, \ldots, k$, $s = -1, 0, 1, 2$ and $m = -k-1, \ldots, k+2$ are subjected to the selection rule $l + s + m =$ even. Maximally symmetric branes are labeled by triples (L, S, M) from the same set. We shall restrict our attention to the cases with $S = 0$.

The $U(1)$ factor in the numerator contributes an additional field B which enters the effective action (4.61) minimally coupled to the gauge fields $\text{A}_a, a = 1, 2$. The solution that we discussed in the parafermion theory carries over to the new theory if we set $\text{B} = 0$, and its interpretation is the same since the perturbation does not

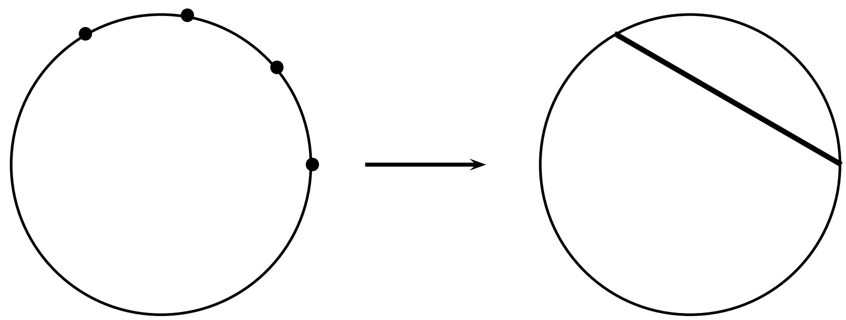


Figure 4.4: Any untwisted brane in the parafermion model can be obtained as a condensate of point-like branes on the boundary.

act in the $\widehat{u}(1)_2$ part. It means once more that a chain of P adjacent $(L = 0)$-branes decays into a single $(L = P - 1)$-brane. This process admits again for a very suggestive pictorial presentation. Using the geometric setting described in Section 3, we find the target space of the $N = 2$ minimal models as a disc with $k+2$ equidistant punctures at the boundary (this is the bosonic part of the geometry, the string scale circle corresponding to the $\widehat{u}(1)_2$ part is projected onto the disc). This was first described in [44]. Let us label the punctures by a $k+2$-periodic integer $q = 0, \ldots, k+1$. A brane (L, M) is then represented through an oriented straight line stretching between the points $q_1 = M - L - 1$ and $q_2 = M + L + 1$. In the described process, a chain of branes, each of minimal length, decays to a brane forming a straight line between the ends of the chain (see fig. 4.5). In [82] similar pictures occur in a geometric description using the realization of $N = 2$ minimal models as Landau-Ginzburg models.

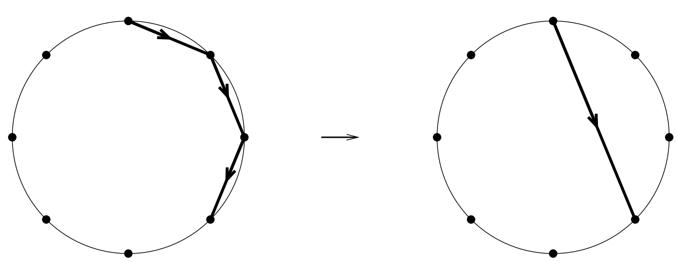


Figure 4.5: A chain of branes can decay into a single brane.

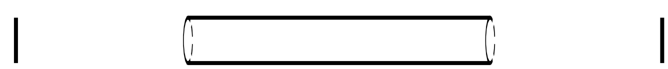
In fig. 4.5 we have tacitly assumed that the processes we identified in the large k regime persist to finite values of k. We will come back to this point of extrapolating such flows away from the decoupling limit in Chapter 5.

4.5.3 Minimal models

The minimal models are constructed as a $\widehat{su}(2)_k \oplus \widehat{su}(2)_1 / \widehat{su}(2)_{k+1}$ coset. The embedding of the denominator theory is diagonal.

Let us try to get a geometrical picture of this model. The group elements $(g(z_1, \psi_1), g(z_2, \psi_2))$ of $SU(2) \times SU(2)$ from the numerator are parametrized by $z_1, z_2 \in D^2$ and $\psi_1, \psi_2 \in [0, 2\pi[$ as in Section 4.5.1. The group $SU(2) \times SU(2)$ has to be divided by the adjoint action of the diagonal denominator $SU(2)$. Let us act on the first factor of $SU(2) \times SU(2)$. The orbits of the adjoint action are conjugacy classes, and they are parametrized by $\Re z_1 \in [-1, 1]$. To obtain these orbits, we do not need the full $SU(2)$ of the numerator. For $|z_1| < 1$, there is still a $U(1)$ part left. The orbits of the adjoint action of this $U(1)$ on the second factor in $SU(2) \times SU(2)$ are parametrized by $z_2 \in D^2$ (compare the discussion in the parafermion case, Section 4.5.1). For $|z_1| = 1$, the orbit in the first $SU(2)$ factor is a point, so we can act with the full denominator $SU(2)$ on the second factor, leaving us with orbits parametrized by $z_2 \in [-1, 1]$. To conclude, the geometrical picture we get is

$$\{-1\} \times [-1, 1] \quad \sqcup \quad]-1, 1[\times D^2 \quad \sqcup \quad \{1\} \times [-1, 1]$$



which can be viewed as a solid cylinder with squeezed ends. Furthermore, if we interpret the level as describing the size of the group, this cylinder will look more like a long, thin rod for high level k. We will come back to this picture to analyze the geometric description of branes after we have discussed the CFT results on boundary states in minimal models.

The sectors of the numerator theory are labeled by two integers (l, m) where $l = 0 \ldots k, m = 0, 1$. Together with a label $l' = 0 \ldots k+1$ from the denominator we label the sectors of the coset model by triples (l, m, l'). From the coset construction we find the selection rule that $l + m + l'$ has to be even and the field identification $(l, m, l') \sim (k - l, 1 - m, k + 1 - l')$. Because of the selection rule, m is determined by fixing l and l' so that we can label the sectors by pairs (l, l').

For the minimal models, we find all possible conformal boundary conditions just by considering the usual untwisted *Cardy* case. Thus, branes are labeled by triples (L, M, L'), or, taking advantage of the selection rules, by pairs (L, L').

Let us try to visualize these branes in our cylinder-like geometry in fig. 4.6. We first consider branes with $L' = 0$. These are described as descendants of conjugacy

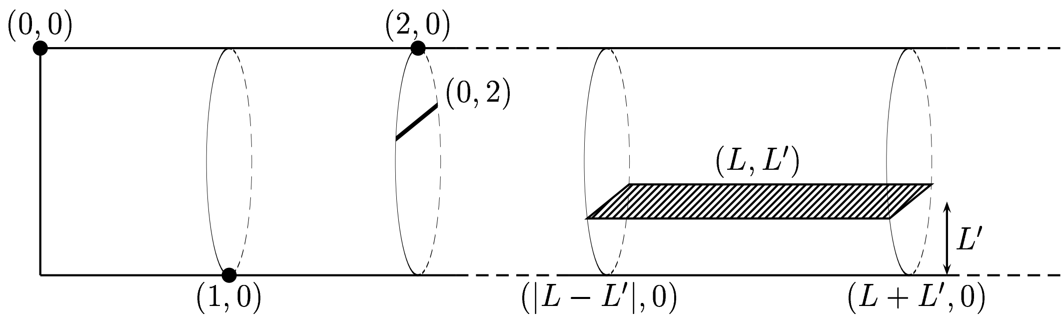


Figure 4.6: Geometric interpretation: The picture shows the underlying geometry of the minimal models together with the possible branes.

classes in the numerator theory, and they are labeled by the position on the axis of the cylinder (depending on the label L varying between 0 and k) and their vertical position (depending on the label M). As the label M can only take the values $M = 0, 1$ corresponding to point-like conjugacy classes, these branes are point-like objects sitting at the top and at the bottom of the cylinder depending on L being odd or even (see fig. 4.6). Let us now take the numerator labels to be trivial, $L = M = 0$. Then, the geometry is obtained by descending from a conjugacy class of the denominator $SU(2)$ embedded in the numerator to the coset geometry. The position along the cylinder axis as well as the vertical positions is fixed and given by the label L'. They extend in the remaining direction and form string-like branes (see fig. 4.6).

For L, L' both non-zero, we encounter the generic situation where we descend from a product of non-trivial conjugacy classes. The branes then extend along the axis of the cylinder. The extension is between $|L - L'|$ and $L + L'$, the vertical position is controlled by L' (see fig. 4.6).

Now we want to formulate the effective action using our general formalism. We start with some brane configuration X where all labels belonging to the level 1 part are zero, $M = 0$. This corresponds to a representation V^X of $SU(2) \times SU(2)$. On such a configuration we have nine fields $\mathrm{A}_a, \mathrm{B}_a, \mathrm{C}_{\tilde{a}}$ in the master WZW model corresponding to directions in the first and the second $\widehat{su}(2)$-part of the numerator and to directions of the denominator part, respectively. The labels $a, \tilde{a}$ can take the values $1, 2, 3$. Note that we do not use the basis introduced in Section 4.4, the field A_i defined there would correspond to the diagonal combination $\mathrm{A}_a + \mathrm{B}_a$. The numerator fields $\mathrm{A}_{\tilde{i}}$ of Section 4.4 correspond to the fields $\mathrm{C}_{\tilde{a}}$ here. The fields are elements of a matrix algebra. The action governing the dynamics of these fields is constructed as in Section 4.4, the Lie algebras act via the derivatives $\mathrm{L}_a, \mathrm{L}_{\tilde{a}}$. From the numerator part we only get derivatives coming from the part with large level k. There are no degrees of freedom coming from the part with level 1, all fields are constant in the corresponding directions. Note that we did not alter the definitions of the derivatives in directions of the numerator by a factor $\sqrt{-1}$, hence we have omitted the hats over the L's.

The constraints (4.43,4.44) translate into

$$\mathrm{A}_a + \mathrm{B}_a \ = \ \mathrm{C}_{\tilde{a}} \ = \ 0 \quad \text{for all} \quad a \tag{4.62}$$

and

$$(i\mathrm{L}_a + i\mathrm{L}_{\tilde{a}})\mathrm{A}_b + f_{ab}{}^c \mathrm{A}_c \ = \ 0 \quad \text{for all} \quad a, c\ . \tag{4.63}$$

By the first of these relations we can eliminate B_a and C_a from the action. The action is expanded according to powers of the level k. As leading terms we find

$$\mathcal{S}(\mathrm{A}) \ = \ -\frac{1}{2k}\mathrm{tr}(\mathrm{L}_a\mathrm{A}_b\mathrm{L}^a\mathrm{A}^b) + \frac{1}{2k}\mathrm{tr}(\mathrm{L}_{\tilde{a}}\mathrm{A}_b\mathrm{L}^{\tilde{a}}\mathrm{A}^b) - \frac{2i}{3k}f^{abc}\mathrm{tr}(\mathrm{A}_a\mathrm{A}_b\mathrm{A}_c)\ . \tag{4.64}$$

Taking the k-dependent metric into account, we note that the action is of order $1/k^3$.

Solutions to this effective theory can be found according to our general formalism of Section 4.4. We have to find representations V^Y of $SU(2) \times SU(2)$ that coincide with V^X on the diagonal.

Let us go into an example by considering a single $(L, 0)$ brane, $L > 0$. The corresponding representation restricted to the diagonal $SU(2)$ is the spin-$L/2$ representation. The only configuration Y leading to the same representation consists of the single brane $(0, L)$. We can easily calculate the value of the action for this solution and obtain

$$\mathcal{S}_{(L,0)}(\mathrm{A}_a) \ = \ \frac{\pi^2}{3k^3}L(L+2) > 0\ . \tag{4.65}$$

The solution describes the flow from the single $(0, L)$-brane to the $(L, 0)$-brane. Such a process has been discussed already in [83].

Our next example will be a configuration of one $(L, 0)$-brane I and one $(L+2, 0)$ brane II. The fields A_a are then described by quadratic matrices of size $2L+4$ which we can understand as consisting of four blocks I-I, I-II, II-I, II-II where the block I-I describes modes of strings with both ends on brane I and so on.

$$A \;=\; \left(\begin{array}{c|c} \text{I-I} & \text{I-II} \\ \hline \text{II-I} & \text{II-II} \end{array}\right) \begin{array}{l} \}\, L+1 \\ \}\, L+3 \end{array} \tag{4.66}$$

Besides the solutions describing the decays $(0,L) \to (L,0)$ or $(0,L+2) \to (L+2,0)$ we find two more,

$$\begin{array}{rcl} (L+1,1) & \longrightarrow & (0,L) \oplus (0,L+2) \\ (1,L+1) & \longrightarrow & (0,L) \oplus (0,L+2) \quad . \end{array}$$

The described analysis of brane processes carries over to more general brane configurations. We find that any (L,L')-brane finally decays into a configuration with trivial denominator labels,

$$(L,L') \longrightarrow (\,|L-L'|,0) + (\,|L-L'|+2,0) + \cdots + (L+L',0)\ . \tag{4.67}$$

All branes with nontrivial label from the denominator part are unstable and decay into configurations of branes with trivial denominator part. Which branes appear in the decay product is determined by the rules of how a tensor product of representations is decomposed into irreducible representations. These are exactly the processes described in [83]. But our analysis shows more, namely that any two configurations $\sum X_{(L,L')}(L,L')$ and $\sum Y_{(L,L')}(L,L')$ are connected by a process if

$$\bigoplus X_{(L,L')}\, V^L \otimes_{su(2)} V^{L'} \;\cong\; \bigoplus Y_{(L,L')}\, V^L \otimes_{su(2)} V^{L'}\ .$$

For example, any brane (L,L') can be constructed as condensate from L=0-branes,

$$(0,\,|L-L'|\,) + (0,\,|L-L'|+2) + \cdots + (0,L+L') \longrightarrow (L,L')\ .$$

Recently there has been a study of RG flows in minimal models [84] extending the work of [83]. All fixed points discovered there by a thorough CFT-investigation can also be found from our general coset analysis.

Let us end this chapter by visualizing the analyzed processes in fig. 4.7. We can obtain any brane (L,L') by a condensation process from a configuration of string-like branes. A generic brane, however, is not stable and will decay into configurations of point-like branes as illustrated in fig. 4.7.

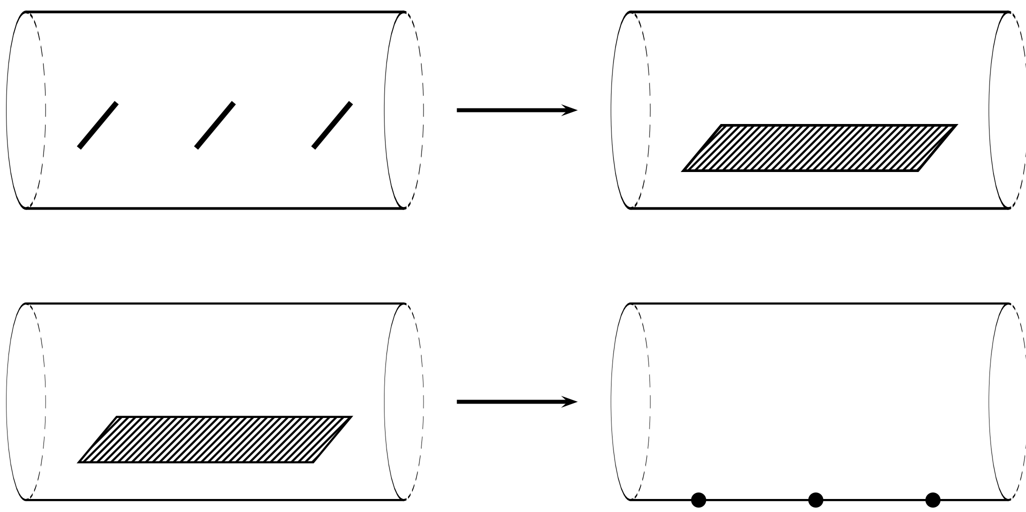


Figure 4.7: Processes in the minimal model geometry of fig. 4.6. Two processes are shown: (a) A configuration of string-like branes condenses into a two-dimensional brane. (b) A two-dimensional brane decays into point-like branes at the bottom of the cylinder.

Chapter 5

Stringy brane dynamics

This chapter deals with the analysis of brane dynamics in WZW and coset models at finite, small levels. It is based on the idea of extrapolating known RG-flows from the decoupling limit. This approach is supported by investigations in condensed matter theory, especially by the work on the Kondo model. The results obtained there were applied to study brane dynamics in several articles [19, 85, 8].

In Section 5.1 we will study the extrapolation of flows and present a simple rule for boundary RG flows in coset models which was obtained in [11]. We will apply this rule in Section 5.2 and will illustrate its capacities in several examples.

5.1 Boundary RG-flows at finite level

5.1.1 Motivation

We would like to understand the dynamics of branes in the stringy regime when the level k is finite. How can we approach this problem?

Proceeding along the lines of the previous chapter would force us to include all the higher order corrections to the effective action. Unfortunately, this problem is even more complicated than finding the non-abelian Born-Infeld action. Hence, we cannot hope to get a complete picture of the brane dynamics in the stringy regime.

We discussed in Chapter 2 that we can describe some aspects of brane dynamics by studying renormalization group flows caused by perturbations on the boundary. We can hope that some of the RG flows identified in the large k limit possess a deformation into the small volume theory, so that we can describe part of the dynamics by extrapolating the flows to finite k. The results of [86, 87] and the comparison with exact studies (see e.g. [88]) in particular models display a remarkable stability of the RG flows as we move away from the decoupling limit. We have, however,

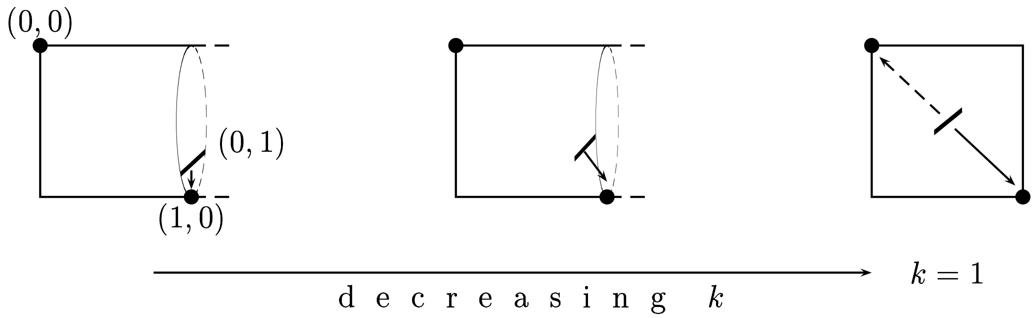


Figure 5.1: Extrapolation of the flow $(0, 1) \to (1, 0)$ from large k to $k = 1$ where we encounter a 'new' flow $(0, 1) \to (0, 0)$.

to take into consideration that in this way we may overlook new flows that are not visible in the large k analysis.

Let us illustrate the extrapolation of flows in the example of minimal models. We start with a model at large k and consider the brane $(0, 1)$ (labels are as in Section 4.5). In the geometric description, this is a string-like brane sitting close to the point-like brane $(1, 0)$. We identified a flow from the string-like brane to the point-like brane in Section 4.5. Now let k decrease. The position of the string-like brane changes, it moves upwards and towards the squeezed end of the cylinder. Finally, we arrive at $k = 1$, the Ising model. The string-like brane is now sitting exactly in the center of the cushion-like geometry[1] (see fig. 5.1). The extrapolated flow still connects $(0, 1)$ to $(1, 0)$, but we see immediately that there must also be a flow from $(0, 1)$ to $(0, 0)$ just because of symmetry arguments. Physically, these two flows are flows from the free boundary condition to either fixed spin up or fixed spin down at the boundary. Therefore, we find a 'new' flow at small k which looks absolutely similar to the other one, but it becomes too 'large' as k goes to infinity to be visible in a perturbative analysis. It would be nice to have a rule for boundary flows at small k including these 'extra' flows.

We now want to consider boundary flows in WZW models. When we decrease the level k, the conformal weights of most of the fields increase. In particular, the large number of fields that are marginal in the limit $k \to \infty$ and that we used in our effective gauge theory in Chapter 4 become irrelevant. Only the currents J^α have conformal weight $h = 1$ independent of k. Therefore, it seems most reliable to extrapolate flows that are induced by perturbations with the currents. In the

[1] We should not take the geometric picture too serious at small level, but just use it as a nice pictorial representation of the model.

language of Chapter 4, solutions that involve only the currents J are constant solutions, $\mathrm{L}_\alpha \mathrm{A}_\beta = 0$. We found a class of constant symmetric solutions in Section 4.3.3 for a stack of branes where the fields A_α form a representation of G (see eq. (4.28)). These fields A_α describe a perturbation of the form

$$S_{\text{pert}} \;=\; \int_{\partial\Sigma} dx\, \mathrm{A}_\alpha J^\alpha(x) \quad . \tag{5.1}$$

This situation also occurs in the work on the Kondo effect where a boundary spin couples to electrons in conduction bands. We will start to review some aspects of this work in the following section.

5.1.2 Absorption of boundary spin

The Kondo model is designed to understand the effect of magnetic impurities on the low temperature conductivity of a conductor. Usually a decreasing temperature will result in an increasing conductivity, because the scattering with phonons is reduced (Matthiesen's rule). In some cases, however, when magnetic impurities are present, the conductivity reaches a maximum and starts to decrease again. This phenomenon is explained by the coupling of the electrons to the magnetic impurities. The electrons tend to screen the impurity, and this coupling increases when temperatures become low.

Let us say that the conductor has electrons in k conduction bands. We can build several currents from the basic fermionic fields like charge current or flavor current. Among them is the spin current $\vec{J}(y)$ which gives rise to a $\widehat{su}(2)_k$ current algebra. The coordinate y measures the radial distance from a spin S impurity at $y = 0$ to which the spin current couples[2]. This coupling is

$$H_{\text{pert}} \;=\; \lambda\; \mathrm{R}_\alpha J^\alpha(0) \quad . \tag{5.2}$$

where $\mathrm{R}_\alpha = \mathrm{R}_S(\mathrm{T}_\alpha)$ $(\alpha = 1, 2, 3)$ is a $2S+1$ dimensional irreducible representation of $su(2)$, λ is the coupling constant.

The operator H_{pert} acts on the tensor product $V^S \otimes \mathcal{H}$ of the $2S+1$ -dimensional quantum mechanical state space of our impurity with the Hilbert space $\mathcal{H}$ for the unperturbed theory described by a Hamiltonian H_0. The formula (5.2) is simply the Hamiltonian formulation of the perturbations we would like to study, as one can see by comparison with formula (5.1) above.

Fortunately, a lot of techniques have been developed to deal with perturbations of the form (5.2). In fact, this problem is what Wilson's renormalization group

[2]We only consider the case of a single isolated impurity.

techniques were designed for. From the old analysis we know that there are two different cases to be distinguished. When $2S > k$ ('under-screening') the low temperature fixed point of the Kondo model appears only at infinite values of λ. On the other hand, the fixed point is reached at a finite value $\lambda = \lambda^*$ of the renormalized coupling constant λ if $2S \leq k$ (exact- or over-screening resp.). In the latter case, the fixed points are described by non-trivial (interacting) conformal field theories. Affleck and Ludwig [89, 20] found an elegant rule to determine these strong-coupling fixed-points.

Let us motivate their rule by some heuristic arguments. Take a closer look at the Hamiltonian $H_0 + H_{\text{pert}}$. It contains in particular the following part

$$H^{\text{spin}} \;=\; \frac{1}{k+2}\sum_{n=-\infty}^{+\infty} :J_n \cdot J_{-n}: \;+\; \lambda \sum_{n=-\infty}^{+\infty} J_n \cdot \mathrm{R} \quad ,$$

where the J_n are Laurent modes of the currents. For $\lambda = \lambda^* = 2/(k+2)$ we can rewrite this expression up to an infinite constant in terms of redefined currents

$$\tilde{J}_n = J_n + \mathrm{R}$$

which also satisfy the commutation relations of the affine Lie algebra $\widehat{su}(2)_k$. This looks like an unperturbed theory for the currents $\tilde{J}$, the boundary spin has been 'absorbed' by the conduction electrons. The reorganization of the state space as representation w.r.t. the redefined currents is given by the fusion rules of $\widehat{su}(2)_k$.

Detailed renormalization group analysis shows that this fixed-point is indeed reached. The spectrum at the fixed-point is given by

$$\mathrm{tr}_{V^S\otimes\mathcal{H}^L}\left(q^{H_0+H_{\text{pert}}}\right)^{\text{ren}}_{\lambda=\lambda^*} = \sum_J N_{SJ}{}^L \chi^J(q) \quad . \tag{5.3}$$

Here, $H_0 = L_0 + c/24$ is the unperturbed Hamiltonian, and the superscript $^{\text{ren}}$ stands for 'renormalized'. By S we label a dominant highest-weight representation of $\widehat{su}(2)$, and V^S denotes the corresponding module of the finite-dimensional Lie algebra $su(2)$. The formula (5.3) is the content of the 'absorption of boundary spin'-principle by Affleck and Ludwig [89, 20] .

It is straightforward to generalize these considerations to an arbitrary simple Lie algebra $\mathfrak{g}$. The space $\mathcal{H}^L$ can be any of the $\widehat{\mathfrak{g}}_k$-irreducible subspaces in the physical state space $\mathcal{H}$ of the theory. Formula (5.3) means that our perturbation with some irreducible representation S interpolates continuously between a building block $\dim(V^S)\chi^L(q)$ of the partition function of the UV-fixed point (i.e. $\lambda = 0$) and the sum of characters on the right hand side of the previous formula,

$$\dim(V^S)\;\chi^L(q) \;\longrightarrow\; \sum_J N_{SJ}{}^L \chi^J(q) \quad . \tag{5.4}$$

We would like to apply this result from the Kondo model to formulate brane processes in WZW models at finite k. Before we do that, we give a generalized version of (5.4) for coset models in the following section.

5.1.3 Generalization to coset models

In [11] it was proposed to generalize the 'absorption of boundary spin'-principle to coset models. The suggested rule is

$$\sum_{S',L'} b_{S^+S'}\ N_{S'L'}{}^{J'}\ \chi^{(L,L')}(q) \ \longrightarrow\ \sum_{J} N_{SJ}{}^{L}\ \chi^{(J,J')}(q) \quad . \tag{5.5}$$

Here, S, L and J' label dominant highest-weight representations of $\widehat{\mathfrak{g}}$ and $\widehat{\mathfrak{h}}$, respectively. The coefficients $b_{SS'}$ are the branching coefficients describing the decomposition of V^S, the corresponding representation of the finite Lie algebra $\mathfrak{g}$, into representations $V^{S'}$ of $\mathfrak{h}$,

$$V^S \ =\ \bigoplus b_{SS'}\ V^{S'} \quad .$$

The existence of an embedding of affine Lie algebras $\widehat{\mathfrak{h}} \subset \widehat{\mathfrak{g}}$ guarantees that these representations can again be identified with highest-weight representations $\mathcal{H}^{S'}$ of $\widehat{\mathfrak{h}}$.

The idea behind this generalization is essentially to do a perturbation with some field $\mathrm{A}_a J^a$ only involving directions orthogonal to the embedded $\mathfrak{h} \subset \mathfrak{g}$, directly motivated from the reduction procedure at infinite levels. This means that the flows (5.5) are generated by fields from the coset sectors

$$\mathcal{H}^{(0,L')} \quad , \quad \text{where} \quad V^{L'} \subset V^{\theta}|_{\mathfrak{h}} \quad . \tag{5.6}$$

Here, θ labels the integrable highest-weight representation which is built from the adjoint representation of the Lie algebra $\mathfrak{g}$. The adjoint representation $L' = \theta'$ of $\mathfrak{h}$ can be omitted from the list (5.6) if it occurs only once in the decomposition of θ.

To see that (5.5) is really a generalization of (5.4), we should recover the flows (5.4) when specializing to the trivial subgroup $\{e\}$ of $\mathfrak{g}$. The primed label can then be omitted and the branching coefficient is just the dimension of the representation V^S.

5.1.4 Application to brane processes

Now, having found a general formula (5.5) describing both WZW and coset models, we want to exploit it to gain some informations on brane processes. Consider configurations of symmetric branes belonging to the same gluing automorphism. We start with a configuration of the form

$$X \;=\; \bigoplus_{S',\mu'} b_{S^+S'}\; n_{S'\lambda'}{}^{\mu'}\,(\lambda,\mu') \tag{5.7}$$

with fixed λ, λ'. Now introduce a spectator brane κ, κ' and look at the partition function describing open strings stretching between this single brane and the configuration X,

$$Z_X^{(\kappa,\kappa')}(q) \;=\; \sum_{S',\mu'} b_{S^+S'}\; n_{S'\lambda'}{}^{\mu'}\, Z_{(\lambda,\mu')}^{(\kappa,\kappa')}(q) \quad .$$

We know the decomposition (see eq. (3.17)) of the single partition functions into coset characters, and obtain

$$\begin{aligned} Z_X^{(\kappa,\kappa')}(q) \;&=\; \sum_{S',\lambda',L,L'} b_{S^+S'}\; n_{S'\lambda'}{}^{\mu'}\; n_{L\lambda}{}^{\kappa}\; n_{L'\mu'}{}^{\kappa'}\; \chi^{(L,L')}(q) \\ &=\; \sum_{L,J'} n_{L\lambda}{}^{\kappa}\; n_{J'\lambda'}{}^{\kappa'}\; \Big(\sum_{S',L'} b_{S^+S'}\; N_{S'L'}{}^{J'}\; \chi^{(L,L')}(q)\Big) \quad . \end{aligned}$$

In the last step we used the fact that the $n_{L'\mu'}{}^{\kappa'}$ form a representation of the fusion algebra of $\widehat{\mathfrak{h}}$ (see (3.7)).

Now we have found a combination of characters on the r.h.s. that allows us to apply our rule (5.5). We find the flow

$$Z_X^{(\kappa,\kappa')}(q) \;\;\longrightarrow\;\; \sum_{L,J'} n_{L\lambda}{}^{\kappa}\; n_{J'\lambda'}{}^{\kappa'}\Big(\sum_{J} N_{SJ}{}^{L}\; \chi^{(J,J')}(q)\Big) \quad .$$

As the $n_{L\lambda}{}^{\kappa}$ form a representation of the fusion algebra of $\widehat{\mathfrak{g}}$, we can rewrite the r.h.s. as

$$Z_Y^{(\kappa,\kappa')}(q) \;=\; \sum_{\mu} n_{S\lambda}{}^{\mu}\Big(\sum_{J,J'} n_{J\mu}{}^{\kappa}\; n_{J'\lambda'}{}^{\kappa'}\; \chi^{(J,J')}(q)\Big)$$

with

$$Y \;=\; \bigoplus_{\mu} n_{S\lambda}{}^{\mu}\;(\mu,\lambda') \quad . \tag{5.8}$$

Independently of the spectator brane (κ,κ'), we always observe the flow

$$X \;\;\longrightarrow\;\; Y \quad .$$

Let us summarize the result in a compact form. Choose a representation S of the affine Lie algebra $\widehat{\mathfrak{g}}$, and boundary conditions λ, λ' of $\widehat{\mathfrak{g}}$ and $\widehat{\mathfrak{h}}$, respectively. Then our rule predicts the following flow between boundary conditions,

$$(\lambda, S^+|_{\mathfrak{h}} \,\hat{\times}\, \lambda') \quad \longrightarrow \quad (S \,\hat{\times}\, \lambda, \lambda') \ . \tag{5.9}$$

Here, $\hat{\times}$ denotes the (twisted) fusion product defined as the formal sum

$$S \,\hat{\times}\, \mu \ := \ \bigoplus_{\nu} \ n_{S\mu}{}^{\nu} \ \nu \quad ,$$

and analogously for $S' \,\hat{\times}\, \mu'$. The restriction $S|_{\mathfrak{h}}$ is defined as

$$S|_{\mathfrak{h}} \ = \ \bigoplus b_{SS'} \ S'$$

involving the branching coefficients of the embedding of finite-dimensional Lie algebras $\mathfrak{h} \subset \mathfrak{g}$. Note that in most cases, we find mixtures of elementary boundary conditions on both sides of the flow that can be identified by expanding the formal sums of boundary labels occurring in the (twisted) fusion rules.

We now would like to discuss the relation of the rule (5.9) with the perturbative results of Chapter 4. There, we considered coset models in a limiting regime in which some of the involved levels become large. For comparison, let us evaluate our rule (5.9) in this limit, assuming that the representation S is trivial in the directions belonging to small levels. Using our identification map ψ we can then replace the boundary labels $\lambda = \psi(l), \lambda' = \psi(l')$ by the representations[3] l, l'^{+} of G^ω and H^ω, and the (twisted) fusion products in rel. (5.9) become usual tensor products of Lie group representations,

$$V^l \otimes (V^S|_{H^\omega} \otimes_{H^\omega} V^{l'^+}) \quad \longrightarrow \quad (V^l \otimes_{G^\omega} V^S|_{G^\omega}) \otimes V^{l'^+} \quad .$$

These are precisely the processes obtained from the solution (4.55) of the effective field theory in the limit $k \to \infty$.

This ends the discussion and motivation of the conjectured rule (5.9) on boundary RG flows in coset models. We will see its broad applicability in the examples of the next section. Before, we want to mention another conjectured principle governing boundary RG flows, the ‘g-conjecture’ of Affleck and Ludwig [81]. It is model-independent and states that the ground-state degeneracy, given by the g-factors,

[3]Note that we have to take the conjugate representation for the H^ω part, compare Section 4.4.

decreases along the flows. In string theory, the g-factors are interpreted as tensions of branes, and it is natural to expect that brane configurations tend to lower their tension.

The 'g-conjecture' is motivated from a perturbative analysis which shows that the g-factors decrease to leading order along the RG flow. Further support given in [81] comes from the 'absorption of boundary spin'-principle for WZW models as we will see below.

We woul like to investigate whether the two conjectures, our rule for coset models and the decrease of g-factors, are compatible. This amounts to the check whether the g-factors of the configurations X (5.7) and Y (5.8) always obey $g_X > g_Y$. The g-factors can be calculated as a sum of g-factors of elementary boundary conditions,

$$\begin{aligned} g_X &= \sum_{S',\lambda'} b_{S^+S'}\; {n_{S'\mu'}}^{\lambda'}\, g_{(\lambda,\lambda')} \\ g_Y &= \sum_{\mu} {n_{S\lambda}}^{\mu}\, g_{(\mu,\mu')} \end{aligned}$$

with

$$g_{(\lambda,\lambda')} = \frac{S^{\omega}_{0\,(\lambda,\lambda')}}{\sqrt{S_{00}}} \quad .$$

It can be shown that the ratio of g_X and g_Y does not depend on λ and λ', but only on the 'boundary spin' S. The 'g-conjecture' and our rule are compatible if the following inequality for the quantum dimensions[4] of S and its restriction to $\widehat{\mathfrak{h}}$ is fulfilled,

$$\sum_{S'} b_{S^+S'}\, \frac{S^{\mathfrak{h}}_{S'0}}{S^{\mathfrak{h}}_{00}} > \frac{S^{\mathfrak{g}}_{S0}}{S^{\mathfrak{g}}_{00}} \quad . \tag{5.10}$$

In all examples we considered, this inequality holds, but it remains unclear whether it is satisfied for an arbitrary coset model.

When we go to the case of WZW models which are special coset models where the denominator theory is trivial, the proposed rule (5.9) simplifies to

$$\dim(V^S)\,\lambda \;\longrightarrow\; S\hat{\times}\lambda \quad . \tag{5.11}$$

The inequality (5.10) for the quantum dimensions reduces to

$$\dim(V^S) > \frac{S^{\mathfrak{g}}_{S0}}{S^{\mathfrak{g}}_{00}}$$

[4] The quantum dimension of a highest-weight representation L of an affine Lie algebra $\widehat{\mathfrak{g}}$ is given by the ratio $S^{\mathfrak{g}}_{L0}/S^{\mathfrak{g}}_{00}$ of modular S-matrices, see eq. (A.4).

which is fulfilled, because the quantum dimension is always smaller than the ordinary dimension. We conclude that these flows are compatible with the 'g-conjecture'. They have been used by Affleck and Ludwig in [81] for the case of $\mathfrak{g} = su(2)$ to support their claim.

5.2 Example: Minimal Models

5.2.1 General remarks

As an application of our rule, let us consider the unitary minimal models. They can be realized as diagonal coset models of the form

$$\widehat{su}(2)_k \oplus \widehat{su}(2)_1 / \widehat{su}(2)_{k+1} \quad .$$

Correspondingly, their sectors are labeled by three integers (l, m, l') in the range $l = 0 \ldots k$, $m = 0, 1$, $l' = 0 \ldots k+1$. Branching selection rules restrict $l + m + l'$ to be even, and there is an identification $(l, m, l') \sim (k - l, 1 - m, k + 1 - l')$ between admissible labels. Our rule (5.9) predicts flows for a large number of starting configurations. Many of them are superpositions of boundary conditions, but here we will concentrate on perturbations of a single boundary condition (J, M, J'). Let us assume that $1 \leq J' \leq k$. Then we choose the representation S (the 'boundary spin') of the numerator theory as $S = (J', 0)$. With this choice our rule becomes

$$(J, M, J') \longrightarrow \bigoplus_L N_{J\,J'}{}^L \; (L, M, 0) \tag{5.12}$$

where $N_{J\,J'}{}^L$ denote the fusion rules of $\widehat{su}(2)_k$. On the other hand, if we select S to be $(k + 1 - J', 0)$, we find

$$(J, M, J') \longrightarrow \bigoplus_L N_{J\,J'-1}{}^L \; (L, 1 - M, 0) \quad . \tag{5.13}$$

These two flows are illustrated in fig. 5.2.

The first of these flows can be seen in perturbation theory for large level k [83, 84], whereas the second does not become 'small' in this limit. Nevertheless, both flows are known to exist [90, 88, 91]. They are generated by the $(0, 0, 2)$ field (in standard Kac labels $(1, 3)$) and differ by the sign of the perturbation. This is in agreement with our general statements on the boundary fields generating the flow (5.9).

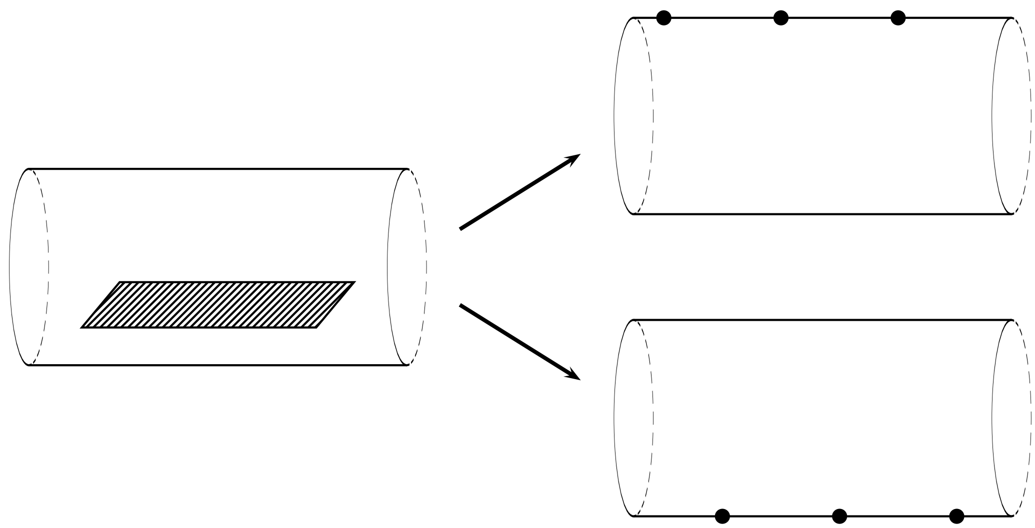


Figure 5.2: Two flows from a generic boundary condition to superpositions of boundary conditions with trivial denominator label $L' = 0$.

5.2.2 Ising model

In the simplest minimal model, the critical Ising model, there are three possible elementary boundary conditions: the free boundary condition $(0, 1, 1)$, and boundary conditions $(0, 0, 0), (1, 1, 0)$ in which the boundary spin is forced to be either up or down. Starting from the free condition, the system can be driven into a theory with fixed spin [92]. These are precisely the two flows (5.12), (5.13). They have been already discussed in Section 5.1.

5.2.3 Tri-critical Ising model

The second model in the unitary minimal series is the tricritical Ising model with central charge $c = 7/10$. Once more, the flows (5.12,5.13) triggered by the ϕ_{13} field [90] are correctly reproduced by (5.12) and (5.13). There are, however, more flows known which correspond to a perturbation with other fields [93]. As our rule depends on the specific coset construction, it is possible to find additional flows by choosing different coset realizations of the same theory. For the tricritical Ising model, such alternative realizations do exist. One is given by $(E_7)_1 \oplus (E_7)_1/(E_7)_2$. When we apply our rule to this coset construction, it reproduces the two known

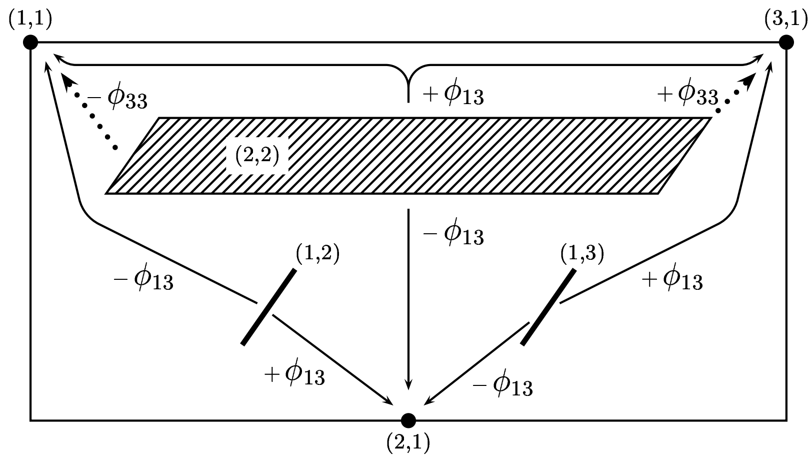


Figure 5.3: Boundary RG flows in the tricritical Ising model induced by the fields ϕ_{13} and ϕ_{33}.

flows caused by the ϕ_{33} field. In Kac labels they read

$$(2,2) \longrightarrow (3,1)\ , \quad (2,2) \longrightarrow (1,1)\ ,$$

and they are depicted together with the other flows in fig. 5.3. These two flows also appear in higher minimal models [86] where we do not know a coset realization for the ϕ_{33}-perturbations. This may be related to the observation that the tricritical Ising model seems to be the only theory in which the considered perturbations are integrable [86]. Nevertheless, recovering flows from the exceptional E_7 coset construction can be considered as an important check of the conjectured rule.

There are more realizations of the tricritical Ising model as coset model, but only for one of them our rule predicts flows from single boundary conditions. This is the construction as a $so(7)_1/(G_2)_1$ coset model. The flows found there coincide with the ϕ_{13}-flows (5.13), i.e. with those flows found in the $SU(2)$ construction that cannot be obtained from the perturbative approach.

5.2.4 Three-State Potts model

The 3-state Potts model is a square lattice model where at each site i there is an angular variable θ_i taking values $0, \pm 2\pi/3$. The interaction is given by the classical Hamiltonian

$$\beta H \;=\; -c \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j) \quad ,$$

the sum running over nearest neighbor pairs. When the model is at its critical coupling it can be described by a conformal field theory. Introducing a boundary into the problem, one can show that there are 8 possible boundary conditions [94, 95]. These are the free boundary condition, the three different fixed boundary conditions, three mixed boundary conditions (one of the three spin states is forbidden at the boundary) and one additional boundary condition whose interpretation in the classical Potts model is not as simple as for the others (see [94] for details). We use the nomenclature of [94] and call the boundary conditions 'free', A, B, C, AB, BC, AC and 'new', respectively.

The CFT describing the critical 3-state Potts model is a minimal model of central charge $c = 4/5$. It can be obtained by various coset constructions. We will review three of them below and determine flows between boundary conditions using the rule (5.9). In this section, we will see the rule in action in examples with twisted boundary conditions or non-charge conjugated modular invariants.

We will start with the construction as a

$$\frac{\widehat{su}(2)_3}{\widehat{u}(1)_3}$$

coset that we already encountered in the discussion of parafermion theories in Section 4.5. The untwisted branes are labeled by pairs (L, M) where the labels L and M lie in the range $L = 0, 1, 2, 3$ and $M = -2, -1, 0, 1, 2, 3$. Selection rules force the sum $L+M$ to be even, and the pairs (L, M) and $(3-L, M\pm3)$ label the same brane. These are the usual Cardy branes, and there are six of them in the model. We adopt the geometric interpretation from Section 4.5, but we are aware that it can only be a pictorial aid and should not be taken too seriously. In this interpretation, three branes are points on the boundary of the disc and correspond to the three fixed boundary conditions A, B, C. The other three describe mixed boundary conditions AB, BC, AC and are represented as lines (see fig. 5.4).

The remaining two boundary conditions can be constructed as twisted branes. The twist affects only the denominator $\widehat{u}(1)$ and acts there as

$$\Omega J(z) \;=\; -J(z) \quad .$$

These twisted branes are labeled by pairs $(L, \pm)$ where $L = 0 \ldots 3$ is an integer coming from the numerator part, and the sign $\pm$ comes from the twisted $U(1)$. Selection rules force L to be even in combination with the sign $+$, and odd if it comes with $-$. Furthermore, there is an identification between pairs, $(L, +) \sim (3 - L, -)$. All in all, we find two boundary conditions $(0, +)$ and $(1, -)$ as promised. In our geometric picture the brane $(0, +)$ appears as point-like object in the center of the

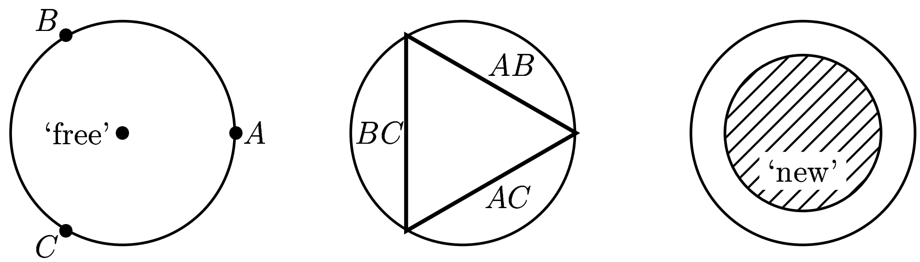


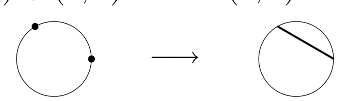
Figure 5.4: Pictorial representation of boundary conditions in the 3-states Potts model.

disc, and the brane $(1,-)$ is a two-dimensional disc placed at the origin (see fig. 5.4). They are the 'free' and the 'new' boundary condition, respectively. Table 5.1 gives an overview of boundary conditions in this particular model.

Now, we want to apply our rule (5.9) to determine RG-flows. We first observe that the rule does not describe flows starting from a single boundary condition. Instead, we will analyze all possible flows for superpositions of two boundary conditions. In all these cases the boundary spin triggering the flow is $S = 1$.

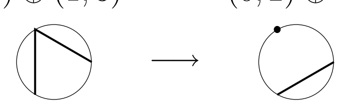
We start with untwisted branes. Applying the rule (5.9) for $\lambda = L = 0$ and $\lambda' = L' = 1$, we find the flow

$$A \oplus B \;=\; (0;0) \oplus (0;2) \;\longrightarrow\; (1;1) \;=\; AB \quad .$$



As one could already infer from symmetry arguments, there are also the flows $B \oplus C \to BC$ and $A \oplus C \to AC$. Starting instead with $\lambda = L = 1$ and $\lambda' = L' = 2$ we find

$$AB \oplus BC \;=\; (1;1) \oplus (1;3) \;\longrightarrow\; (0;2) \oplus (2;2) \;=\; B \oplus AC \quad ,$$



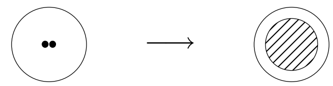
analogous results can be obtained for permutations of the letters A, B, C.

Let us now turn to twisted branes. The rule (5.9) involves the annulus coefficients ${n_{S'\mu'}}^{\nu'}$ for the twisted $U(1)$-branes. They read

$$ {n_{M-}}^{-} \;=\; {n_{M+}}^{+} \;=\; \begin{cases} 1 & M \text{ even} \\ 0 & M \text{ odd} \end{cases} \quad , \quad {n_{M+}}^{-} \;=\; {n_{M-}}^{+} \;=\; \begin{cases} 0 & M \text{ even} \\ 1 & M \text{ odd} \end{cases} \quad . $$

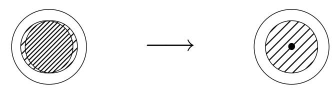
Choosing $\lambda = L = 0$ and $\lambda' = -$ in (5.9) yields the flow

$$2 \cdot \text{`free'} \;=\; 2 \cdot (0;+) \;\longrightarrow\; (1;-) \;=\; \text{`new'} \quad .$$



If we set $\lambda = L = 1$ and $\lambda' = +$, the resulting flow is

$$2 \cdot \text{`new'} \;=\; 2 \cdot (1;-) \;\longrightarrow\; (0;+) \oplus (2;+) \;=\; \text{`free'} \oplus \text{`new'} \quad .$$



These are all flows provided by the rule (5.9) for superpositions of two boundary conditions. The field responsible for the flows comes from the coset sectors $\mathcal{H}^{(0,\pm 2)}$ and has conformal weight $h = 2/3$. This can be concluded from our general prescription in Section 5.1.3 (see eq. (5.6)).

We now turn to the description of the Potts model as diagonal $SU(2)$ coset,

$$\frac{\widehat{su}(2)_3 \oplus \widehat{su}(2)_1}{\widehat{su}(2)_4}$$

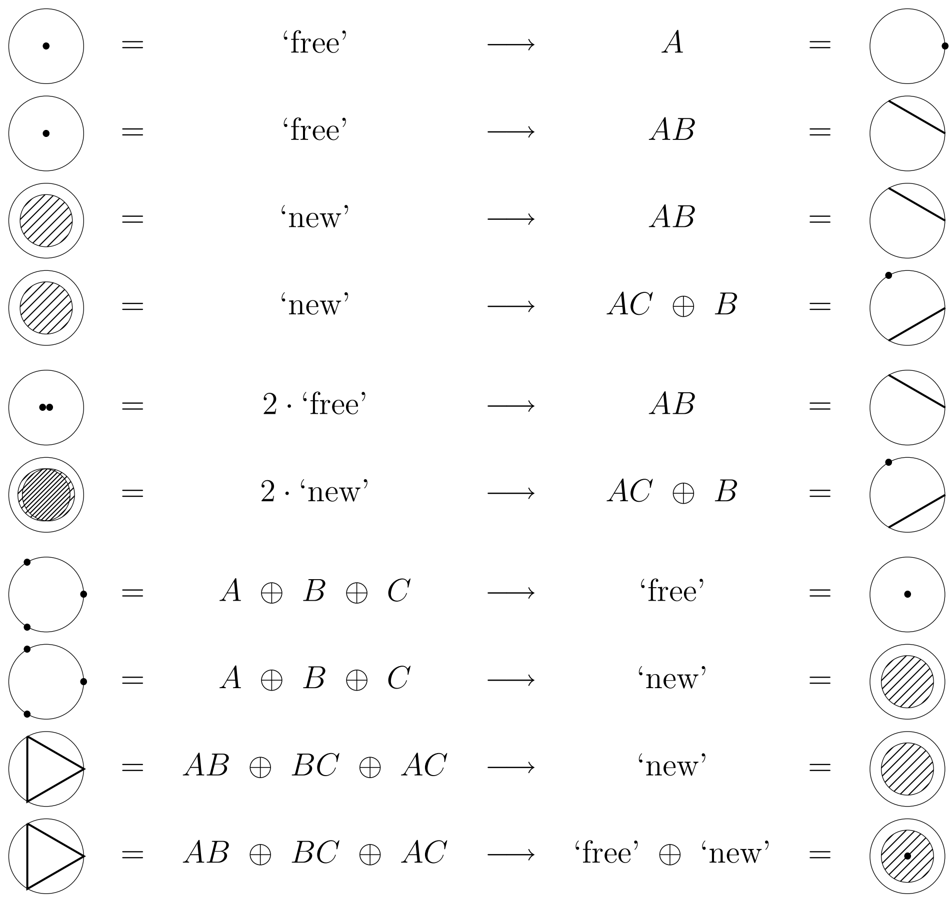
where the modular invariant is obtained from charge-conjugated modular invariants in the numerator, the denominator $su(2)_4$ contributes a D_4 modular invariant. The perturbing field here is identified as

$$(0, 0; 2)$$

having again conformal weight $h = 2/3$.

We find four boundary conditions $(0, 1, 2, 3)$ in the $su(2)_3$ part and two boundary conditions $(0, 1)$ in the $su(2)_1$ part. The $su(2)_4$ part has a D_4 modular invariant. There are four boundary conditions which we label by $0, 1, 2+, 2-$. The coefficients of the corresponding boundary states in terms of Ishibashi states can be found e.g. in [56].

Identification and selection rules leave us with eight boundary conditions for the 3-state Potts model. They are given in Table 5.1. Applying our rule, we observe first that we find the same flows involving superpositions of two boundary conditions that we discussed in the parafermion construction. In addition we find flows relating 'free' and 'new' boundary conditions with the others, namely (for superpositions of maximally three boundary conditions):



Let us finally discuss the construction of the Potts model as

$$\frac{\widehat{su}(3)_1 \oplus \widehat{su}(3)_1}{\widehat{su}(3)_2}$$

coset. Its sectors are labeled by three $su(3)$ weights

$$[(L_1, L_2), (M_1, M_2); (L'_1, L'_2)]$$

where L_i, M_i, L'_i are non-negative integers (Dynkin labels) obeying

$$0 \leq L_1 + L_2 \leq 1 \quad , \quad 0 \leq M_1 + M_2 \leq 1 \quad , \quad 0 \leq L'_1 + L'_2 \leq 2$$
$$2(L_1 + M_1 - L'_1) + L_2 + M_2 - L'_2 \; = \; 0 \quad \mod 3 \quad .$$

Boundary label from			g-factor	Notation from [94]
$\frac{\widehat{su}(2)_3}{\widehat{u}(1)_3}$	$\frac{\widehat{su}(2)_3 \oplus \widehat{su}(2)_1}{\widehat{su}(2)_4}$	$\frac{\widehat{su}(3)_1 \oplus \widehat{su}(3)_1}{\widehat{su}(3)_2}$		
$(0;0)$	$(0,0;0)$	$[(0,0),(0,0);(0,0)]$	N	A
$(0;2)$	$(0,0;2+)$	$[(0,0),(0,1);(2,0)]$	N	B
$(0;-2)$	$(0,0;2-)$	$[(0,0),(1,0);(0,2)]$	N	C
$(1;1)$	$(2,0;2-)$	$[(0,0),(1,0);(1,0)]$	$N\lambda^2$	AB
$(1;3)$	$(2,0;0)$	$[(0,0),(0,0);(1,1)]$	$N\lambda^2$	BC
$(1;-1)$	$(2,0;2+)$	$[(0,0),(0,1);(0,1)]$	$N\lambda^2$	AC
$(1;-)$	$(1,0;1)$	$[0,0;0;\omega]$	$N\lambda^2\sqrt{3}$	'new'
$(0;+)$	$(3,0;1)$	$[0,0;1;\omega]$	$N\sqrt{3}$	'free'

Table 5.1: Boundary conditions in the 3-state Potts model in three different coset constructions. The g-factors are given in terms of $N^4 = (5-\sqrt{5})/2$ and $\lambda^2 = (1+\sqrt{5})/2$.

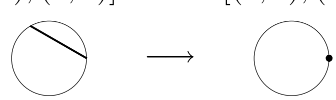
The sectors are identified according to the field identification

$$[(L_1,L_2),(M_1,M_2);(L_1',L_2')] \sim \\ \sim [(1-L_1-L_2,L_1),(1-M_1-M_2,M_1);(2-L_1'-L_2',L_1')] \quad .$$

What remains are 6 sectors. According to the standard Cardy construction, these give rise to 6 boundary conditions which are listed in Table 5.1 along with their g-factors. Before we go to construct the remaining two boundary conditions, we want to look for RG flows.

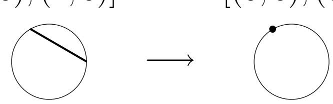
Let us start with the boundary condition AB and exhibit what flows are 'predicted' by (5.9). We choose the perturbation $S = [(0,0),(0,1)]$ and find the flow

$$AB \;=\; [(0,0),(1,0);(1,0)] \;\longrightarrow\; [(0,0),(0,0);(0,0)] \;=\; A \quad .$$



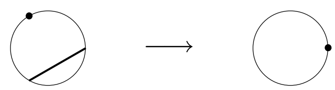
The spin $S = [(0,1),(0,0)]$ leads to

$$AB \;=\; [(0,0),(1,0);(1,0)] \;\longrightarrow\; [(0,0),(0,1);(2,0)] \;=\; B \quad .$$



Analogously, we find $BC \to B$, $BC \to C$ and $AC \to A$, $AC \to C$. These constitute all flows from single boundary conditions described by the rule. For a superposition of two boundary conditions we find flows of the form

$$AC \oplus B \quad \longrightarrow \quad A \quad .$$



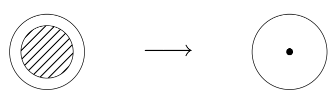
The two remaining boundary conditions can be obtained from twisted gluing conditions using an automorphism which interchanges the two Dynkin labels of the $su(3)$ theories. In the $su(3)_1$ there is only one sector left invariant under this automorphism, in the $su(3)_2$ theory there are two. In total we find two twisted boundary conditions

$$[0,0;0;\omega] \quad \text{and} \quad [0,0;1;\omega] \quad ,$$

there are no selection or identification rules in this example. We can calculate their g-factors (see Table 5.1) and identify the two boundary conditions as the 'new' and the 'free' boundary condition, respectively.

Again, we want to investigate what flows are described by the rule (5.9). Let us start with the 'new' boundary condition and try the perturbation $S = [(1,0),(0,0)]$. This leads to

$$\text{'new'} \;=\; [0,0;0;\omega] \quad \longrightarrow \quad [0,0;1;\omega] \;=\; \text{'free'} \quad .$$



We can identify the field that drives the described flows. From our general prescription (5.6), we conclude that the perturbing field is

$$\big((0,0),(0,0);(1,1)\big)$$

which has conformal weight $h = 2/5$.

Let us compare our results with the work of Affleck et al.[94]. They find several flows driven by fields of conformal weight $h = 2/3$ and $h = 2/5$. The flows they find are all reproduced by our rule. For single boundary conditions we find exact coincidence, for superpositions our rule suggests further flows that have not been analyzed in [94].

Figure 5.5 summarizes part of the results for boundary RG flows in the 3-states Potts-model obtained by the rule (5.9).

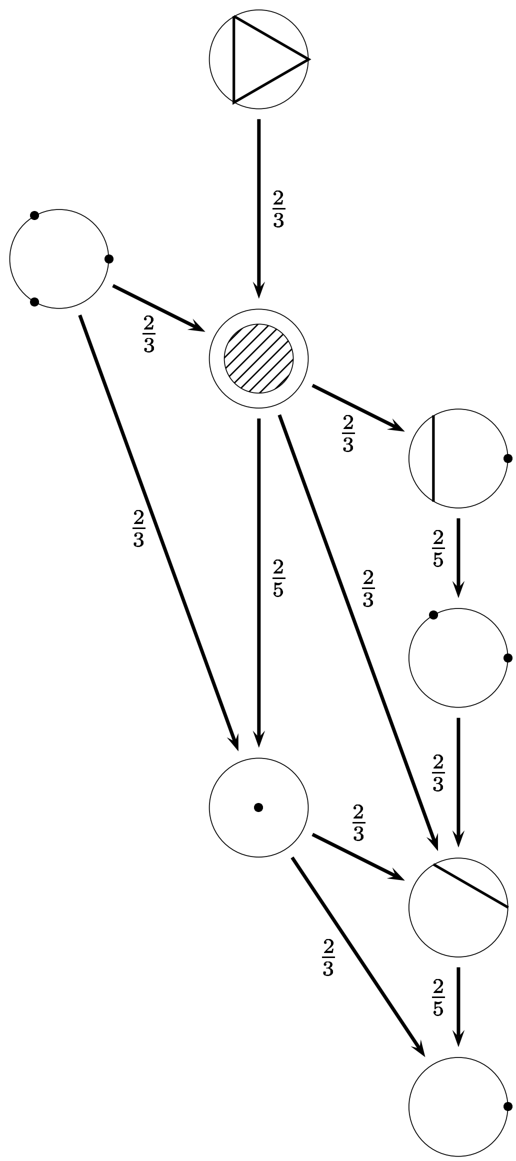


Figure 5.5: Some of the boundary RG-flows found in the 3-states Potts model. The vertical ordering of the configurations is done according to the g-factors. The conformal weight of the field responsible for a flow is quoted.

Chapter 6

Conserved charges and twisted K-theory

We would like to see whether the described brane dynamics obey some conservation laws, i.e. if we can assign charges to the branes that are conserved in physical processes. So we are looking for some discrete abelian group $C(M)$, where M denotes the physical background, and a map from arbitrary brane configurations to $C(M)$ such that the map is invariant under renormalization group flows. Note that we define charges not as couplings but as invariants under dynamical processes.

We should mention at this point that from the string-theoretic point of view not all perturbations and therefore not all RG-flows in the CFT should be considered. When our CFT is part of a consistent string background, some fields are projected out by the GSO projection and these are not available any more to perturb the system. We should take this into account when we want to relate the groups of RG-invariants to Ramond-Ramond charges carried by branes in string theory.

In the following, we will first analyze charges in the decoupling limit. Then, we will investigate how these charges groups are modified at finite levels. In the example of branes in $SU(n)$, we will give explicit answers for the charge groups and compare them with results obtained in twisted K-theory.

6.1 Charges in the decoupling limit

Let us analyze what charges are conserved in the processes that we observe in the decoupling limit of WZW models. We saw in Chapter 4 that maximally symmetric branes are labeled by representations of the invariant subgroup. The only invariant in the processes discussed there is the dimension of the representation. In particular, any brane configuration X can be obtained from a configuration corresponding to the trivial representation of dimension $\dim V^X$. Therefore, we find for every automorphism a charge group $\mathbb{Z}$, the charge of a brane is the dimension of the corresponding representation.

Now let us turn to untwisted branes in coset models. Here, again, we can associate a representation V^X of $G \times H$ to any configuration X. The processes found in Section 4.4 can be easily characterized: any two configurations X, Y are connected by a process precisely if the corresponding representation coincide when restricted on the diagonal H_{diag},

$$V^X\big|_{H_{\text{diag}}} \quad \cong \quad V^Y\big|_{H_{\text{diag}}} \ .$$

Therefore, we can associate to any configuration its representation on H_{diag} as charge. Charges then take their values in the representation ring of H,

$$C^{\text{id}}(G/H) \ = \bigoplus_{L' \ \in \text{Rep}(\mathfrak{h})} \mathbb{Z} \ . \tag{6.1}$$

As not all representations are allowed because of selection rules, the charge group is actually a subgroup of the representation ring. The irreducible representations occurring in the subgroup have the same conjugacy class as the trivial representation, i.e. their highest-weight vector is an element of the root lattice. Note that we get additional charges from the 'small' directions with small level k, because we do not see any processes changing labels from these parts in our perturbative analysis.

Let us mention that the occurence of the representation ring of H fits nicely with the structure of Ramond-Ramond charge lattices found in certain Kazama-Suzuki models [96].

For the twisted branes in coset models, we have much less control on the charge group. Twisted brane configurations are described by representations of $G^\omega \times H^\omega$. We lack of a complete classification of processes between twisted coset branes, but at least we know that all symmetric processes leave the restriction of the representation on the diagonal H^ω_{diag} invariant (see eq. (4.56)). Consequently, we find a possible charge group as (a subgroup of) the representation ring of H^ω, but there may be further charges.

6.2 Charges for finite level

When we consider WZW models or coset models at finite levels, we expect that the results for charge groups have to be modified. Let us illustrate this in the example of symmetric branes in $SU(2)$. For level k the labels L lie in the range $0, \ldots, k$. We have a geometrical understanding of what the possible D-branes are. They are given by conjugacy classes which form 2-spheres embedded in $SU(2) \cong S^3$. Their radius depends on L and for $L = 0, k$ they degenerate to a point. $L = 0$ describes a D0-brane at the origin e, $L = k$ a D0-brane at $-e$ (see fig. 3.1 in Chapter 3).

Now consider a stack of D0-branes at e. This stack is expected to condense into a D2-brane with finite volume. If we put more and more D0-branes together, the radius of the resulting D2-brane will first grow, then decrease, and finally a stack of $k+1$ D0-branes will decay to a D0-brane at $-e$ (see fig. 6.1). If we assign charge 1 to the D0-brane at e and want the charge to be conserved, the D0-brane at $-e$ must carry charge $k+1$. On the other hand we could just translate the D0-brane from e to $-e$. In the supersymmetric model, this translation also acts in the fermionic part, and the D0-brane at e is translated to an anti-D0-brane at $-e$ carrying charge -1. Thus we have to identify $k+1$ and -1 which means that charge is only well-defined modulo $k+2$ (for details see [19]).

We can obtain the same result in a different way not involving a translation. To this end, we consider a stack of $(k+1)$ branes with label $L = 1$, the smallest spherical brane close to the group unit. From the rule (5.11) of Chapter 5, we expect this configuration to condense into a brane of label $J = k-1$, the smallest spherical brane around $-e$. Associating the dimension of the representation as charge, we find for the stack of branes with label $L = 1$ the charge $2(k+1)$, for the single $J = k-1$ brane the value k. Again, the charge can only be well defined modulo $k+2$.

When we want to analyze charge conservation at finite levels, we have to investigate the flows discussed in Chapter 5 and see what charges they leave invariant. Let us denote the charge of a coset brane (λ, λ') by $q_{(\lambda,\lambda')} \in C(G/H)$. Invariance under the flows (5.9) translate into the condition

$$\sum_{S',\mu'} b_{S^+ S'}\; n_{S'\lambda'}{}^{\mu'}\; q_{(\lambda,\mu')} \;=\; \sum_{\mu} n_{S\lambda}{}^{\mu}\; q_{(\mu,\lambda')} \quad . \tag{6.2}$$

The general procedure to determine the charge group would be to start with the group

$$C_0(\widehat{\mathfrak{g}}_k/\widehat{\mathfrak{h}}_{k'}) \;=\; \bigoplus_{(\lambda,\lambda')} \mathbb{Z} \quad , \tag{6.3}$$

where we introduced one charge for each single brane, and then take the quotient by all relations given by (6.2).

When we specialize to WZW models, the general condition for charge conservation becomes

$$\dim(V^S)\; q_\lambda \;=\; \sum_{\mu} n_{S\lambda}{}^{\mu}\; q_\mu \tag{6.4}$$

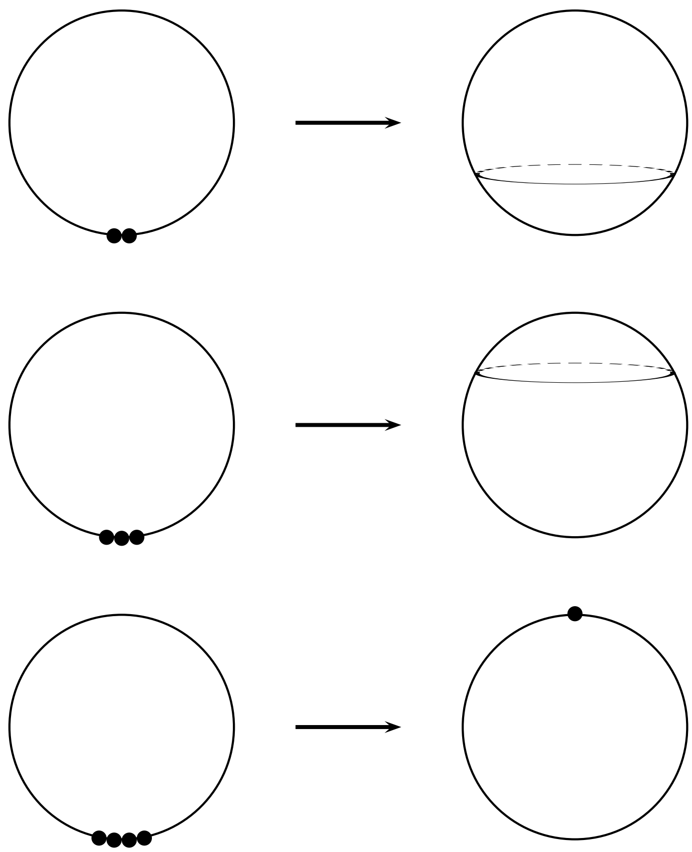


Figure 6.1: Brane dynamics on S^3: A stack of D0-branes at e can decay to a D2-brane. Putting more and more D0-branes at e the resulting brane will be localized further and further away from the group unit and eventually the decay product will be a single D0-brane at $-e$.

corresponding to a process where a stack of dim V^S branes of type λ condenses. It is usually sufficient to only evaluate a subset of these equations to determine the charge group. For $G = SU(n)$, it is proven in Appendix C that the full set of conditions (6.4) on the charges can be reduced to the cases when S is a fundamental representation ω_i. This result is probably valid for most of the other groups, too.

We can easily see that there can only be one type of charge for untwisted branes in WZW models. From the process (5.11), we know that any untwisted brane labeled by L can be obtained from a stack of $\dim V^L$ elementary branes labeled by 0. Consequently, the charge of any untwisted brane is a multiple of the charge of the 0-brane,

$$q_L \;=\; \dim(V^L)\, q_0 \quad . \tag{6.5}$$

The untwisted branes therefore contribute a subgroup of $\mathbb{Z}$ to the total charge group. In the example of $SU(n)$ we will explicitly show how this subgroup can be determined.

For twisted branes, the situation is more complicated. The analysis in the limit $k \to \infty$ suggests that charges are determined by the dimensions of the corresponding representations of the invariant subgroup G^ω. In Chapter 4, we found processes constructing any brane $\lambda = \Psi(l)$ as the condensate of a stack of $\dim(V^l)$ branes of type $\Psi(0)$. Unfortunately, we do not have a finite k analogue for most of these processes[1]. Therefore, we cannot exclude that twisted branes give rise to different types of charge. Nevertheless, we will obtain a lot of informations also on the charge group of twisted branes in the example of $SU(n)$ in the following section.

6.3 Example: Charges of branes on $SU(n)$

6.3.1 Charges of untwisted branes on $SU(n)$

For branes wrapping ordinary conjugacy classes, i.e. $\omega = \mathrm{id}$, the integers n are given by the fusion rules N. From the remarks made at the end of the last section, we have learned that the charges of untwisted branes are integer multiples of q_0 (see (6.5)). If we normalize the charge of the point-like brane by $q_0 = 1$ we arrive at

$$q_L \;=\; \dim V^L \quad . \tag{6.6}$$

These equations form a subset of the equations (6.4). We can see that the charges $q_L = \dim V^L$ solve the full set of eqs. (6.4) in the limit $k \to \infty$ where the N are just

[1]Observe that these solutions were not constant in general. Thus the corresponding perturbations involve not only the currents, but also fields that become irrelevant when k decreases.

the Clebsch-Gordan multiplicities of the simple Lie algebra $su(n)$. In this limit, the equations express that the dimension of a tensor product of $su(n)$ representations is a sum of dimensions of its irreducible subrepresentations. For finite k, however, the fusion rules N differ from the Clebsch-Gordan multiplicities of $su(n)$ so that typically the right hand side of eqs. (6.4) with $q_L = \dim V^L$ is smaller than the left hand side. The equations can only hold, if they are evaluated modulo some integer x that we need to determine. Charges then take values in the group $\mathbb{Z}_x$.

Hence, the task is to find the largest number x such that (6.4) is fulfilled modulo x. As we can generate all representations out of the fundamental ones, ω_i, $i = 1, \ldots, n-1$, we can reduce our problem to processes involving stacks of $\dim V^{\omega_i}$-branes. In other words, the general charge conservation condition is fulfilled if

$$\dim(V^{\omega_i})\, q_L \;=\; \sum_M N_{\omega_i L}{}^M\, q_M \quad \mod\; x \tag{6.7}$$

for all $i = 1, \ldots, n-1$. A rigorous proof of this statement can be found in Appendix C.

Denote by $\mathcal{J} = k \cdot \omega_1$ the generator of the simple current group $\mathbb{Z}_n$ of $\widehat{su}(n)_k$. It can be shown that it suffices to evaluate the equations (6.4) for stacks of branes labeled by the simple current $\mathcal{J}$ (see Appendix C). Thus, the charge conservation condition reduces to

$$\dim(V^{\omega_i})\, q_{\mathcal{J}} \;=\; \sum_L N_{\omega_i \mathcal{J}}{}^L\, q_L \quad \mod\; x \tag{6.8}$$

for all $i = 1, \ldots, n-1$. Taking the difference between both sides with $q_L = \dim(V^L)$ inserted, gives the following $n-1$ numbers a_i, (see (C.2))

$$\begin{aligned} a_i \;&=\; \dim(V^{\mathcal{J}})\, \dim(V^{\omega_i}) \;-\; \sum_L N_{\omega_i \mathcal{J}}{}^L\, \dim(V^L) \\ &=\; \frac{(k+1)\ldots\widehat{(k+i)}\ldots(k+n)}{(i-1)!\,(n-i)!} \end{aligned} \tag{6.9}$$

where the hat over a factor indicates that this factor is omitted. These numbers have to vanish modulo x. This means that x is given by the greatest common divisor of these numbers. It can be shown [8] (see appendix C) that $x = \gcd(a_i)$ is given by

$$x \;=\; \frac{k+n}{\gcd(k+n, \mathrm{lcm}(1, \ldots, n-1))} \quad . \tag{6.10}$$

Hence, the charge group of the untwisted branes for $X = SU(n)$ is $\mathbb{Z}_x$ with x as in formula (6.10).

6.3.2 Charges of twisted branes on $SU(n)$

Let us now take a look at branes that wrap twisted conjugacy classes. The gluing automorphism is induced by the reflection ω of the Dynkin diagram. Their action on the vertices of the Dynkin diagram induces the following map on the weight space,

$$\omega(L_1, \ldots, L_{n-1}) \;=\; (L_{n-1}, \ldots, L_1)\;, \tag{6.11}$$

where the L_i are (finite) Dynkin labels. Details on our notations and some fundamental results on the representation theory of $su(n)$ can be found in Appendix C.

As in the untwisted case we get a charge conservation condition,

$$\dim(V^S)\; q_\lambda \;=\; \sum_\mu n_{S\lambda}{}^\mu\; q_\mu \quad \text{for all} \quad \lambda \in \mathcal{B}^\omega(\widehat{su}(n)_k)\quad . \tag{6.12}$$

Again it turns out that it is sufficient to evaluate this condition for S being a fundamental representation ω_i. This is proven in Appendix C.

We would like to perform a similar analysis as for the untwisted branes, but we are faced with the problem that the integers $n_{S\lambda}{}^\mu$ are more difficult to handle than the fusion rules. Recently, however, the integers n have been expressed in terms of the fusion rules for the affine Lie algebras of the B- and C-series [60, 61]. One can hope that this result will allow to fully exploit the conditions on the charges in the future. For now, we content ourselves to evaluate the conditions only partly by using symmetry considerations. Already these limited informations will help us to derive severe constraints on the charge group.

We will see that the numbers $n_{S\,\lambda}{}^\mu$ are invariant under the action of some simple currents $\mathcal{I}$,

$$n_{\mathcal{I}S\,\lambda}{}^\mu \;=\; n_{S\,\lambda}{}^\mu \quad . \tag{6.13}$$

To derive this result we look at the explicit expressions (3.8) for the numbers $n_{S\,\lambda}{}^\mu$ and investigate what happens under the action of a simple current $\mathcal{I}$:

$$\begin{aligned} n_{\mathcal{I}S\,\lambda}{}^\mu &= \sum_{L\in\mathrm{Rep}^\omega(\hat{\mathfrak{g}})} \frac{\bar{S}^\omega_{L\mu} S^\omega_{L\lambda} S_{L\,\mathcal{I}S}}{S_{L0}} \\ &= \sum_{L\in\mathrm{Rep}^\omega(\hat{\mathfrak{g}})} e^{2\pi i Q_{\mathcal{I}}(L)}\; \frac{\bar{S}^\omega_{L\mu} S^\omega_{L\lambda} S_{LS}}{S_{L0}} \quad . \end{aligned}$$

Here $Q_{\mathcal{I}}(L)$ is the monodromy charge of L with respect to the simple current $\mathcal{I}$. If it is zero, we infer that the coefficients $n_{S\,\lambda}{}^\mu$ are invariant under the action of the simple current.

For a symmetric weight $L = \omega(L) \in \mathrm{Rep}^{\omega}(\widehat{\mathfrak{g}})$ we know that

$$\omega(\mathcal{J}^i\, L) \;=\; \mathcal{J}^{n-i}\, L \quad . \tag{6.14}$$

This implies immediately that $Q_{\mathcal{J}}(L) = Q_{\mathcal{J}^{n-1}}(L)$. If n is odd, it follows that $Q_{\mathcal{J}^i}(L) = 0$ for all $i = 1, \ldots, n-1$. If n is even, we can only deduce that $Q_{\mathcal{J}^i}(L) = 0$ for i even. We thus arrive at the result that

$$n_{\mathcal{J}^i S\,\lambda}{}^{\mu} \;=\; n_{S\,\lambda}{}^{\mu} \tag{6.15}$$

for arbitrary i if n is odd and for even i if n is even which is the precise formulation of the invariance properties of $n_{S\,\lambda}{}^{\mu}$ we anticipated in eq. (6.13).

Assuming that there is at least one twisted brane which can be assigned a charge with value 1 we immediately deduce the following condition on the unknown integer x_ω

$$\dim(V^{\mathcal{J}^i S}) \;=\; \dim(V^S) \quad \mathrm{mod} \;\; x_\omega \;\; , \tag{6.16}$$

where the values for i depend on whether n is even or odd, as formulated before.

Let us first concentrate on the case that n is odd. Using (6.16) with $S = 0, \omega_i$ we obtain

$$\dim(V^{\mathcal{J}}) \;=\; 1 \qquad\qquad \mathrm{mod} \;\; x_\omega \tag{6.17}$$

$$\dim(V^{\mathcal{J}\omega_i}) \;=\; \dim(V^{\omega_i}) \quad \mathrm{mod} \;\; x_\omega \quad . \tag{6.18}$$

The two relations combine into the following statement for the numbers a_i that were defined in eqs. (6.9) above,

$$a_i \;\;=\;\; \dim(V^{\mathcal{J}})\,\dim(V^{\omega_i}) - \dim(V^{\mathcal{J}\omega_i}) \;=\; 0 \quad \mathrm{mod} \;\; x_\omega \quad . \tag{6.19}$$

By definition, the greatest common divisor of these numbers a_i is x and hence we deduce that x_ω is a divisor of x, i.e. that the order of an element in the charge group for twisted branes cannot exceed the order of the charge subgroup from untwisted branes. It can be shown that $x_\omega = x$ does imply eqs. (6.16) but we cannot exclude that the full set of eqs. (6.12) force x_ω to be smaller than x.

If n is even, we find x_ω not so strongly restricted by the eqs. (6.16). Introducing the integers $b_0 = \dim(V^{\mathcal{J}^2}) - 1$ and $b_i = \dim(V^{\mathcal{J}^2\omega_i}) - \dim(V^{\omega_i})$ for $i = 1, \ldots, n-1$, one can show that x_ω must divide $\gcd(b_i)$. Note that $\gcd(b_i)$ is a possibly non-trivial integer multiple of x. For $SU(4)$ we still get the result that x_ω divides x but already for $SU(6)$ one finds situations where (6.16) can be fulfilled modulo $x_\omega > x$. There is some evidence that eqs. (6.16) provide enough restrictions for $n = 0 \mod 4$ to guarantee that x_ω divides x.

6.3.3 Comparison with twisted K-theory

Let us briefly summarize the results for the charge groups $C(SU(n), K)$ that we obtained in the previous two subsections, before we compare them with the twisted K-groups $K^*_H(SU(n))$. To do the comparison, we have to switch to the supersymmetric WZW model of level $K = k+g^\vee = k+n$. It contains bosonic currents of level k. In the supersymmetric model, one might have the idea that one has to perturb with the supersymmetric current of level K and apply the 'absorption of boundary spin'-principle to the whole model including bosons and fermions. It turns out that this does not give the right prescription, and one has to use the bosonic current of level k instead. For some discussion on this issue see [19, 97, 26].

The charge group that governs the dynamics of branes in a $\widehat{su}(n)_k$ WZW-model is

$$C(SU(n), K) \ = \ \mathbb{Z}_x \oplus \mathbb{Z}_y \oplus \dots \tag{6.20}$$

where x is given by (6.10) and y divides x_ω. Branes wrapping ordinary conjugacy classes can all be obtained from stacks of point-like branes. This guarantees that there is a unique way to assign charges to such branes as we have seen in our discussion leading to eqs. (6.6). Hence, untwisted branes contribute a single cyclic subgroup to the group of charges $C(SU(n), K)$. For branes wrapping twisted conjugacy classes, similar arguments do only exist for $k \to \infty$. As a consequence, we cannot exclude the existence of several independent charge assignments for twisted branes. Furthermore it is possible that other stable types of branes exist that could give rise to additional contributions in the group $C(SU(n), K)$. This uncertainty is reflected by the $\oplus \dots$ in (6.20).

The formula (6.20) is the general form for all n, so let us state what is known in addition for special values of n.

- For $n = 2$, the second summand is absent and x is given by $x = K$.
- For $n = 3$, one can show that there is only one summand from the twisted part, and $y = x_\omega = x$.
- For $n = 4$, x^ω must divide x.
- For odd n, the same is true: x_ω divides x.

For even $n \geq 6$ we can only show that x_ω divides $\gcd(b_i)$ with the integers b_i being introduced in the last paragraph of the previous subsection. In general, $\gcd(b_i)$ could be some possibly non-trivial integer multiple of x, but it is very likely that $\gcd(b_i) = x$ when $n = 0 \mod 4$.

According to the proposal of Bouwknegt and Mathai [25], our results on the group $C(SU(n), K)$ should be compared with the twisted K-groups $K^*_H(SU(n))$.

The definition of $K^*_H(X)$ uses the space of sections in a bundle over X with fiber being the algebra of compact operators on a separable Hilbert space. This space of sections can be turned into an algebra and it is known that algebras of this form are classified by elements of $H^3(X, \mathbb{Z})$. In other words, there exists some way of assigning an algebra $\mathcal{A}_H$ to any choice of $H \in H^3(X, \mathbb{Z})$. The K-group of this algebra is denoted by $K^*_H(X)$. If H vanishes, the algebra $\mathcal{A}_H$ factorizes globally into functions on X and compact operators. Hence, by Morita invariance of K-theory, $K^*_{H=0}(X)$ coincide with ordinary K-groups.

One way to calculate such K-groups makes use of Atiyah-Hirzebruch spectral sequences. These start from the de Rham cohomology groups and then proceed through a sequence of complexes whose cohomology stabilizes after a finite number of steps. The resulting cohomology provides some information on the desired K-group, though there is still some extension problem to solve. Generically, the latter may have several solutions. In any case, the problem of these computations for $K^*_H(G)$ starts earlier because almost nothing is known about the differentials that appear in the sequence of complexes. Only for the first non-trivial step, the required differential was obtained by Rosenberg in [98]. This suffices to compute the twisted K-group for $G = SU(2)$. The result is

$$K^*_H(SU(2)) \ = \ \mathbb{Z}_K \quad .$$

Here $H = K\Omega_3$ and Ω_3 is the normalized volume form of the unit sphere. This fits precisely the result from conformal field theory.

For $G = SU(3)$, Rosenberg's results still allow to show that

$$K^*_H(SU(3)) \ = \ \mathbb{Z}_r + \mathbb{Z}_r \quad ,$$

where r is known to divide K. If all the higher differentials that are not determined by the result of Rosenberg would vanish, then one would get $r = K$.

Fortunately, the twisted K-groups for $SU(n)$ have been computed recently by Hopkins, using a cell decomposition of $SU(n)$ together with the Meyer-Vietoris sequence (the result was announced in [26]). His findings show that the higher differentials are nonzero in general. The outcome of his calculations is [26]

$$K^*_H(SU(n)) \ = \ \underbrace{\mathbb{Z}_x \oplus \cdots \oplus \mathbb{Z}_x}_{2^{n-2}} \tag{6.21}$$

involving our by now well known integer x. For $n = 2, 3$ we have an exact concordance of the charge groups of maximally symmetric branes and twisted K-groups.

The constraints on the charge groups for general n (see (6.20)) are compatible with K-theory. The result (6.21) implies, however, that there are a lot more summands $\mathbb{Z}_x$ for higher n, and it is unlikely that the maximally symmetric branes can account for all these charges. In [26] it was suggested that the non-symmetric branes constructed in [44, 45] could represent the remaining K-theory classes. A confirmation from conformal field theory has not been established yet, the dynamics of these non-symmetric branes remains to be investigated.

It would be highly desirable to understand directly why the physical analysis of RG invariants, and the mathematical computation of K-groups lead to the same result. An analysis of Freed, Hopkins and Teleman shows that G-equivariant twisted K-theory of a group G is given by the fusion ring,

$$K^*_{H;G}(G) \;=\; C_0^{\mathrm{id}}(\widehat{\mathfrak{g}}_k) \;=\; \bigoplus_{L\in\mathrm{Rep}(\widehat{\mathfrak{g}}_k)} \mathbb{Z} \;\; ,$$

and this is precisely the starting point (6.3) for the construction of the charge group of untwisted branes before we divide out the charge conservation conditions (6.2).

Is there a mathematical way of understanding the conditions (6.2) as a procedure for going from equivariant to ordinary twisted K-theory? It was suggested to us by Wassermann that the techniques in [99, 100] could indeed lead to a computation of $K^*_H(SU(n))$ which resembles the CFT calculation.

Chapter 7

Summary and outlook

In this thesis we have investigated the dynamics of branes in various non-trivial backgrounds. We encountered different facets of dynamical processes: their description in non-commutative gauge theories and their appearance as boundary RG flows in two-dimensional field theories on the world-sheet. Remarkable is the K-theoretic nature of the conserved charges that rule the brane dynamics.

Starting from the knowledge of the spectrum and the geometry of maximally symmetric branes in WZW models and coset models, we have discussed in Chapter 4 the non-commutative gauge theories that govern the dynamics of these branes in the decoupling limit.

In Section 4.3 we presented the complete classification of symmetric solutions for maximally symmetric branes on simple, simply-connected compact group manifolds G that was obtained in [12]. We learned that any two configurations X and Y are connected by a solution, precisley if the corresponding representations V^X and V^Y of G^ω have the same dimension. Let us briefly recall how this result was obtained. The theory of solutions has a nice geometric interpretation in terms of vector bundles E^X of rank $\dim V^X$ that are associated to the principal G^ω-bundle G over G/G^ω. A solution describing a process between X and Y exists if and only if the two vector bundles E^X and E^Y are isomorphic. We have shown in Appendix D that for all cases the bundles E^X are trivial. Thus, two bundles E^X and E^Y are isomorphic if their rank coincides,

$$\dim V^X \;=\; \dim V^Y \quad .$$

Many properties of branes on group manifolds carry over to coset models. We showed in [10] that the effective theory on a coset brane in the decoupling limit can be derived from the non-commutative gauge theory of a brane in a group manifold

$G \times H$ by putting certain constraints on the fields (see Section 4.4). The symmetric solutions have been classified completely for untwisted coset branes. For twisted branes we were still able to find a large class of solutions. We labeled a brane configuration X by a representation V^X of the invariant subgroup $G^\omega \times H^\omega$. Two configurations X and Y that are connected by a solution coincide on the diagonal $H^\omega_{\text{diag}} \subset G^\omega \times H^\omega$,

$$V^X\big|_{H^\omega_{\text{diag}}} \;\cong\; V^Y\big|_{H^\omega_{\text{diag}}} \;.$$

As for group manifolds, we were able to reformulate the solution theory into a geometric problem. A solution exists precisely if the H-equivariant vector bundles E^X_H and E^Y_H are isomorphic (see eq. (4.54)).

In Chapter 5, we discussed the extrapolation of the identified processes from the decoupling limit to the stringy regime. In WZW models at a finite level k, we could indeed find an analogue of the processes mediated by a constant gauge field. We discussed the relation to the 'absorption of the boundary spin'-principle of Affleck and Ludwig.

In Section 5.1.3 we made a new proposal [11] which generalizes the 'absorption of boundary spin'-principle to coset models (see eq. (5.5)). Its wide applicability was illustrated by a detailed discussion of the consequences of the conjectured principle in unitary minimal models. Our claim is supported by a comparison with renormalization group flows that have been obtained by other methods. In the application to the three-states Potts model, we made extensively use of the geometric interpretation of boundary states which helped to organize the flows in a convenient way.

We have used the achieved understanding of brane processes to analyze possible charge groups in Chapter 6. We defined charges as quantities that are conserved under brane processes, i.e. under RG flows. For a configuration of maximally symmetric branes in the decoupling limit, the charge is given by the dimension $\dim V^X$ of the corresponding representation V^X. The object carrying 'elementary' charge corresponds to the one-dimensional trivial representation V^0.

In the stringy regime, we have explicitly evaluated the conditions on the charges for the groups $SU(n)$. Eq. (6.20) summarizes the result that was published in [8]. Following a proposal of Bouwknegt and Mathai, we compared the result with the twisted K-groups that have been recently computed by Hopkins. The results coincide exactly for the charges coming from untwisted branes, and they are compatible for charges of twisted branes.

Let us discuss at the end of this thesis a loose list of open questions that arise from the obtained results.

- Are there finite k analogues of the condensation processes for twisted branes on group manifolds which involve non-constant gauge fields?
- The results can be applied to supersymmetric coset models like the Kazama-Suzuki models. What are the charge groups in these models at finite levels, and how do they compare to equivariant K-theory?
- We discussed the relation between ordinary K-theory and tachyon condensation in Chapter 2. Is there a similar way to understand why twisted K-theory appears in curved backgrounds more generally from the nature of the involved fields?
- What is the effective theory describing the dynamics of the non-symmetric branes constructed in [44, 45]? Do their dynamics explain the twisted K-groups of $SU(n)$?

Besides these questions which are closely connected to the work in the present thesis, it would be highly interesting to investigate the dynamics of branes in more general backgrounds. The study of branes in non-compact models has just begun (see e.g. [101, 102, 103, 104]), and the certainly rich structure of their dynamics awaits its exploration.

Appendix A

Boundary conformal field theory

A short introduction to bulk CFT

Let us consider a conformally invariant field theory on a two-dimensional 'world-sheet'[1] Σ. In two dimensions, the symmetry algebra of conformal transformations is infinite-dimensional, in contrast to higher-dimensional theories. We will be mainly concerned with Euclidean conformal field theory on the complex plane Σ where conformal transformations correspond to holomorphic or anti-holomorphic transformations. Conformal invariance implies that the energy-momentum tensor is traceless. This leaves us with two independent components which we denote by $T(z)$ and $\overline{T}(\bar{z})$. They depend only on z or $\bar{z}$ and we call them left-moving or right-moving, respectively. In a two-dimensional scale-invariant quantum field theory, the energy-momentum tensor has the operator product expansion (OPE)

$$T(z)T(w) \;\sim\; \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w) \quad , \tag{A.1}$$

where $\sim$ means that we skipped non-singular parts on the r.h.s. of the formula. The real number c is called the central charge of the theory. It is an important characteristic of the theory, but it usually does not fix the theory uniquely. The appearance of the central charge destroys conformal invariance on the quantum level (conformal anomaly), e.g. the energy-momentum tensor does not transform covariantly under conformal transformations.

We can formally expand T into Laurent modes

$$T(z) \;=\; \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \;, \qquad \overline{T}(\bar{z}) \;=\; \sum_{n\in\mathbb{Z}} \bar{z}^{-n-2}\overline{L}_n \quad .$$

[1]We could call it space-time, but this would lead to confusion with space-time as target space of string theory.

From the OPE (A.1) we can derive commutation relations for the modes L_m,

$$[L_m, L_n] \;=\; (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n,0} \quad .$$

These are the commutation relation of a Virasoro algebra of central charge c.

We can do the same for the right-moving part which leads to commutation relations of generators $\bar{L}_m$ involving a central element $\bar{c}$. Thus, there are two copies of the Virasoro algebra acting in our theory. In many examples, the symmetry algebra is larger. We will assume in the following that the symmetry algebra can be splitted into chiral parts, and that the two parts are isomorphic, excluding so-called heterotic theories. The full symmetry algebra then has the form $\mathcal{W} \times \mathcal{W}$. In addition to the Virasoro generators L_m, $\mathcal{W}$ contains generators $W_m^{(i)}$ satisfying the commutation relations

$$[L_m, W_n^{(i)}] \;=\; \big((h_i-1)m-n\big)W_{m+n}^{(i)} \quad .$$

The state space $\mathcal{H}$ of the theory decomposes under the action of the symmetry algebra into a direct sum

$$\mathcal{H} \;=\; \bigoplus_{(l,\tilde{l})\in \mathrm{Spec}} \mathcal{H}^l \otimes \mathcal{H}^{\tilde{l}} \tag{A.2}$$

where $\mathcal{H}^l$ and $\mathcal{H}^{\tilde{l}}$ are irreducible highest-weight representations of the chiral symmetry algebra $\mathcal{W}$. These representations are labeled by a set $\mathrm{Rep}(\mathcal{W})$. The set Spec contains pairs of labels from $\mathrm{Rep}(\mathcal{W})$ including possible multiplicities. The smallest eigenvalue of L_0 in $\mathcal{H}^l$ is called the conformal weight h_l of the representation, the eigenspace to h_l is spanned by the ground-states in $\mathcal{H}^l$ and it carries a representation of the zero modes $W_0^{(i)}$.

In this thesis, we will only deal with rational conformal field theories where Spec is a finite set. A CFT is called unitary if the representation spaces $\mathcal{H}^l$ are Hilbert spaces admitting scalar products s.t.

$$(W_n^{(i)})^* \;=\; W_{-n}^{(i)} \quad \text{and} \quad (L_n)^* \;=\; L_{-n} \quad .$$

Unitary theories have a non-negative central charge $c \geq 0$ and conformal weights $h_l \geq 0$.

Essential informations on a rational conformal field theory can be obtained from the representation theory of the chiral algebra $\mathcal{W}$. A lot of the structure of the operator product expansion of primary fields is encoded in the fusion rules. The fusion coefficients ${N_{lm}}^k$ are non-negative integers, and they can be thought of as the analogue of the Clebsch-Gordan coefficients in finite-dimensional Lie algebras. We treat the labels $l \in \mathrm{Rep}(\mathcal{W})$ as formal objects which generate the fusion ring with the product

$$l \hat{\times} m \;=\; \sum_k {N_{lm}}^k \, k \quad .$$

The fusion product is associative, this means that

$$\sum_m {N_{jl}}^m \, {N_{mk}}^s \;=\; \sum_m {N_{jk}}^m \, {N_{ml}}^s \quad .$$

The trivial representation $0 \in \mathrm{Rep}(\mathcal{W})$ is the neutral element of fusions,

$${N_{0l}}^m \;=\; \delta_l^m \quad .$$

Among the elements of $\mathrm{Rep}(\mathcal{W})$ there are special ones called 'simple currents' $\mathcal{J}$. They have the property that their fusion with any representation $l \in \mathrm{Rep}(\mathcal{W})$ just yields again a single representation,

$$\mathcal{J} \hat{\times} l \;=\; \mathcal{J}l \quad .$$

The simple currents form an abelian group under the fusion product.

Our conformal field theory is meant to describe a closed string. In string perturbation theory, scattering amplitudes are calculated using the Polyakov expansion involving world-sheets of arbitrary genus. Therefore, we want to define the theory on all closed Riemann surfaces. Having defined the theory on the sphere, there is no freedom when we extend it to other surfaces, but there is one additional consistency condition coming from the theory on a torus, namely modular invariance. Let us consider the torus partition function

$$Z(q,\bar{q}) \;=\; \mathrm{tr}_{\mathcal{H}} \; q^{L_0 - \frac{c}{24}} \; \bar{q}^{\bar{L}_0 - \frac{c}{24}} \quad .$$

Here, $q = \exp(2\pi i \tau)$ is the exponential of the modular parameter τ of the torus. The partition function can be expressed in terms of the conformal characters

$$\chi^l(q) \;=\; \mathrm{tr}_{\mathcal{H}^l} \; q^{L_0 - \frac{c}{24}}$$

as

$$Z(q,\bar{q}) \;=\; \sum_{l,\tilde{l} \in \mathrm{Rep}(\mathcal{W})} Z_{l\tilde{l}} \; \chi^l(q) \; \chi^{\tilde{l}}(\bar{q}) \quad .$$

The torus that is parametrized by τ does not change under the so-called modular transformations $T : \tau \to \tau+1$ or $S : \tau \to -1/\tau$ which generate the symmetry group $SL(2,\mathbb{Z})$. To have a consistent CFT on the torus, we demand invariance of $Z(q,\bar{q})$ under these transformations.

In a rational CFT, modular invariance of the partition function implies that the characters themselves transform linearly under $SL(2,\mathbb{Z})$. The action of T on the characters is diagonal,

$$\chi^l(\tau+1) \;=\; \exp\bigl(2\pi i(h_l - c/24)\bigr)\,\chi^l(\tau) \quad .$$

The action of S is more complicated,

$$\chi^l(\tilde{q}) \;=\; \sum_m S_{lm}\chi^m(q)$$

where $\tilde{q} = \exp(-2\pi i/\tau)$ and S_{lm} are complex numbers forming the modular S-matrix. It is a unitary, symmetric matrix. It transforms under the action of a simple current $\mathcal{J}$ by a phase,

$$S_{\mathcal{J}l\,m} \;=\; \exp(2\pi i Q_{\mathcal{J}}(m))\, S_{lm} \quad .$$

Here, $Q_{\mathcal{J}}(l)$ is the monodromy charge of l w.r.t. $\mathcal{J}$, defined as

$$Q_{\mathcal{J}}(l) \;=\; h_{\mathcal{J}} + h_l - h_{\mathcal{J}l} \qquad \mathrm{mod}\ \ \mathbb{Z} \quad .$$

The monodromy charge is additive under the fusion product. Furthermore, for any fixed l, the map

$$\mathcal{J} \to e^{2\pi i Q_{\mathcal{J}}(l)}$$

is a group homomorphism from the simple current group into $U(1)$.

There is a remarkable relation between the modular S-matrix and the fusion rules, namely the Verlinde formula [105]

$$N_{kl}{}^m \;=\; \sum_j \frac{S_{kj}S_{lj}\overline{S}_{mj}}{S_{0j}} \quad . \tag{A.3}$$

We can read this formula as a diagonalization of the matrix N_k having components $(N_k)_l{}^m$ with the help of the matrix S,

$$(N_k)_l{}^m \;=\; \sum_{j,j'} S_{lj}\,\bigl[\frac{S_{kj}}{S_{0j}}\delta_{j,j'}\bigr]\overline{S}_{mj'} \quad ,$$

with eigenvalues $\frac{S_{kj}}{S_{0j}}$. The maximal eigenvalue is called the quantum dimension D_l of $\mathcal{H}^l$,

$$D_l \;=\; \frac{S_{l0}}{S_{00}} \quad . \tag{A.4}$$

The quantum dimensions form a one-dimensional representation of the fusion algebra

$$D_k D_l \;=\; {N_{kl}}^m\, D_m \quad .$$

Boundary conformal field theory

Our starting point in this discussion will be a conformal field theory defined on closed Riemann surfaces. We can think of this conformal field theory as describing a background for closed strings. We now want to extend this theory to world-sheets with boundaries, i.e. we want to describe open strings in the same background. This amounts to solving a number of consistency conditions.

When we start to construct a conformal field theory on a surface with boundaries, we have to impose some continuity or gluing condition on the boundary to ensure conformal invariance. Let our world-sheet be the upper half plane. Then the energy-momentum tensor has to obey

$$T(z) \;=\; \overline{T}(\bar{z})\big|_{z=\bar{z}}$$

at the real line. The physical meaning of this condition is the absence of momentum flow across the boundary. Only in a few cases it is possible to tackle the problem of finding all solutions to the 'conformal' gluing condition above (see [43, 106] for recent progress).

Usually, one is content with the restriction to boundary theories preserving further symmetries at the boundary. We find maximally symmetric boundary conditions by imposing the gluing condition

$$W(z) \;=\; \Omega(\overline{W})(\bar{z}) \tag{A.5}$$

on the generators of the symmetry algebra $\mathcal{W}$. Here, Ω denotes an automorphism of the algebra of fields that leaves the energy-momentum tensor invariant.

A boundary CFT can be characterized by a coherent boundary state $|\lambda\rangle\!\rangle$ in $\mathcal{H}$. This is the closed string point of view: the boundary states summarize the coupling of bulk fields to the boundary. The gluing conditions (A.5) translate into conditions on the boundary state

$$\big(W_n - (-1)^h\Omega(\overline{W}_{-n})\big)\,|\lambda\rangle\!\rangle \;=\; 0 \quad \text{for all } n\in\mathbb{Z} \quad . \tag{A.6}$$

Since the modes W_n map each sector $\mathcal{H}^l$ into itself, we can solve the constraint in each sector $\mathcal{H}^l \otimes \mathcal{H}^{\tilde{l}}$ of (A.2) separately. A nontrivial and (up to normalization) unique solution can only be found if the two representations $\mathcal{H}^l$ and $\mathcal{H}^{\omega(\tilde{l})}$ are conjugate, $\omega(\tilde{l}) = l^+$. Here, $\omega : \mathrm{Rep}(\mathcal{W}) \to \mathrm{Rep}(\mathcal{W})$ denotes the permutation of representations induced by the automorphism Ω. The solutions $|l\rangle\rangle$ are called Ishibashi states. They are labeled by the finite set

$$\mathrm{Spec}^\omega \;=\; \{l|(l, \omega(l^+)) \in \mathrm{Spec}\} \quad . \tag{A.7}$$

If a sector appears more than once in $\mathcal{H}$, we have to take multiplicities into account.

As the Ishibashi states form a complete basis of solutions to the gluing constraint (A.6), we can write every boundary state $|\lambda\rangle\rangle$ as linear combination

$$|\lambda\rangle\rangle \;=\; \sum_l \frac{\psi^l_\lambda}{\sqrt{S_{0l}}} |l\rangle\rangle$$

where the constants ψ^l_λ characterize the boundary condition, and the square root of the modular S-matrix element S_{0l} has only been introduced for later convenience.

Not any choice of ψ^l_λ leads to a consistent boundary state, the solutions are constrained by two sets of conditions, the Cardy constraint [13] and the sewing conditions [107, 62]. The basic idea behind the sewing conditions is the associativity of the operator product expansion. We will not say more about these here, and concentrate on the Cardy constraint in the following.

The Cardy constraint is a consequence of world-sheet duality. Let us consider the open string partition function on a cylinder with boundary conditions λ and μ on the two ends,

$$Z_\lambda{}^\mu(q) \;=\; \mathrm{tr}_{\mathcal{H}_\lambda{}^\mu}\, q^{\frac{1}{2}(L_0 - \frac{c}{24}) + \frac{1}{2}(\overline{L}_0 - \frac{c}{24})} \;=\; \sum_{l \in \mathrm{Rep}(\mathcal{W})} n_{l\lambda}{}^\mu\, \chi^l(q) \tag{A.8}$$

where $q = e^{2\pi i \tau}$, and $n_{l\lambda}{}^\mu$ are non-negative integers. We have used here that the boundary conditions preserve the symmetry algebra $\mathcal{W}$, so that we can decompose the open string Hilbert space into representations of $\mathcal{W}$,

$$\mathcal{H}_\lambda{}^\mu \;=\; \bigoplus_{l \in \mathrm{Rep}(\mathcal{W})} n_{l\lambda}{}^\mu\, \mathcal{H}^l \quad .$$

Following Cardy, we can express the partition function in terms of boundary states. This involves an exchange of Euclidean space and time direction which in terms of the parameter τ amounts to a modular transformation $\tau \to -1/\tau$. The partition function can be written as the overlap

$$Z_\lambda{}^\mu(q) \;=\; \langle\langle \mu | \tilde{q}^{\frac{1}{2}(L_0 - \frac{c}{24}) + \frac{1}{2}(\overline{L}_0 - \frac{c}{24})} | \lambda \rangle\rangle$$

where $\tilde{q} = e^{-2\pi i/\tau}$. We can expand the boundary states in terms of Ishibashi states and perform a modular transformation to obtain

$$\begin{aligned} Z_{\lambda}{}^{\mu}(q) &= \sum_{m\in\mathrm{Spec}^{\omega}} \frac{\overline{\psi}^m_{\mu}\psi^m_{\lambda}}{S_{0m}}\ \chi^m(\tilde{q}) \\ &= \sum_{m\in\mathrm{Spec}^{\omega}} \frac{\overline{\psi}^m_{\mu}\psi^m_{\lambda}S_{ml}}{S_{0m}}\ \chi^l(q) \quad . \end{aligned} \tag{A.9}$$

Here, we used that the Ishibashi states are orthogonal and normalized by

$$\langle\langle l|\, q^{\frac{1}{2}(L_0-\frac{c}{24})+\frac{1}{2}(\overline{L}_0-\frac{c}{24})}\,|m\rangle\rangle \;=\; \delta_{lm}\,\chi^l(q) \quad .$$

The comparison of the two expressions (A.8) and (A.9) leads to the 'Cardy equation',

$$n_{l\lambda}{}^{\mu} \;=\; \sum_{m\in\mathrm{Spec}^{\omega}} \frac{\overline{\psi}^m_{\mu}\psi^m_{\lambda}S_{ml}}{S_{0m}} \quad . \tag{A.10}$$

We look for a set of elementary or fundamental boundary conditions which cannot be written as a superposition of other boundary conditions by demanding that the vacuum representation appears only once in the self-overlap of a boundary state, $n_{0\lambda}{}^{\mu} = \delta^{\mu}_{\lambda}$.

Provided that we have found a complete set $\mathcal{B}^{\omega}(\mathcal{W})$ of elementary boundary conditions, meaning that we found as many boundary states as there are Ishibashi states, one can show that the matrices $(n_l)_{\lambda}{}^{\mu}$ form a representation of the fusion algebra,

$$n_l\; n_m \;=\; \sum_{k\in\mathrm{Rep}(\mathcal{W})} N_{lm}{}^{k}\; n_k \quad .$$

Furthermore, the matrices have the properties $n_0 = \mathbf{1}$ and $n_{l^+} = n_l^T$. Such a set of matrices is called a non-negative integer valued matrix representation (NIM-rep) of the fusion rules.

In the case of a charge-conjugated modular invariant and a trivial automorphism $\omega = \mathrm{id}$, a set of fundamental boundary states has been constructed by Cardy [13]. Here, Ishibashi states and boundary states can both be labeled by $\mathrm{Rep}(\mathcal{W})$. For each sector $\mathcal{H}^l$ we find a boundary state λ_l that can be expressed as a combination of Ishibashi states by using the modular S-matrix,

$$|\lambda_l\rangle\rangle \;=\; \sum_{m\in\mathrm{Rep}(\mathcal{W})} \frac{S_{lm}}{\sqrt{S_{0m}}}|m\rangle\rangle \quad .$$

When we plug this expansion into eq. (A.10) and use the Verlinde formula (A.3), we find as a result that in this 'Cardy case' the integers ${n_{k\lambda_l}}^{\mu_m}$ are simply the fusion matrices ${N_{kl}}^m$.

Appendix B

Twisted branes and the invariant subgroup

The map Ψ

In Chapter 4, we made extensively use of a map Ψ that relates branes on a group G which are twisted by an automorphism ω to representations of the invariant subgroup $G^\omega \subset G$ in the limit $k \to \infty$,

$$\Psi : \mathrm{Rep}(G^\omega) \;\longrightarrow\; \mathcal{B}^\omega(\mathfrak{g}) \quad .$$

From the comparison of the proposed world-volume algebras of branes with the spectrum (3.6) of open strings described by the integers ${n_{J\lambda}}^\mu$, we found in Section 4.3.1 the condition (see eq. (4.12))

$${n_{J\lambda}}^\mu \;=\; \sum_{j\in\mathrm{Rep}(G^\omega)} b_{Jj}\, {N_{jl}}^m \tag{B.1}$$

for $\Psi(l) = \lambda$ and $\Psi(m) = \mu$. Here, b_{Jj} are the branching coefficients characterizing the decomposition of a representation V^J of G into representations V^j of G^ω,

$$V^J \;=\; \bigoplus_{j\in\mathrm{Rep}(G^\omega)} b_{Jj}\, V^j \quad .$$

Before we start the discussion, let us set up some notation. The embedding $\mathfrak{g}^\omega \subset \mathfrak{g}$ is characterized by a projection map $\mathcal{P} : L_w^{(\mathfrak{g})} \to L_w^{(\mathfrak{g}^\omega)}$ sending a weight in the weight lattice $L_w^{(\mathfrak{g})}$ of $\mathfrak{g}$ to a weight of $\mathfrak{g}^\omega$. Let us also introduce another projection map $\mathcal{P}_\omega$ from the weight lattice $L_w^{(\mathfrak{g})}$ to the lattice $\langle\mathcal{B}^\omega(\mathfrak{g})\rangle$ of fractional weights generated by $\mathcal{B}^\omega(\mathfrak{g})$, given by $\mathcal{P}_\omega = \frac{1}{2}(1+\omega)$ for an order 2 automorphism, or $\mathcal{P}_\omega = \frac{1}{3}(1+\omega+\omega^2)$ if ω is of order 3. On the lattice $\langle \mathcal{B}^\omega(\mathfrak{g})\rangle$, we have the action of the symmetric part W_ω of the Weyl group of $\mathfrak{g}$. This group is isomorphic to the Weyl group $W_{\mathfrak{g}^\omega}$ which acts on the weight lattice $L_w^{(\mathfrak{g}^\omega)}$.

There is a canonical choice for the map Ψ in all cases except for $\mathfrak{g} = A_{2n}$, so let us exclude this special case for a moment. We are not going to write down the map Ψ explicitly, but we will rather quote some of the properties which justify to call Ψ a canonical choice in the case of $\mathfrak{g} \neq A_{2n}$:

1. Ψ can be extended to a linear map from the weight lattice $L_w^{\mathfrak{g}^\omega}$ to the lattice of fractional symmetric weights $\langle \mathcal{B}^\omega(\mathfrak{g})\rangle$
2. Ψ is isometric with respect to the standard scalar products defined via the Killing forms of $\mathfrak{g}^\omega$ and $\mathfrak{g}$
3. there is an associated isomorphism $\Psi : W_{\mathfrak{g}^\omega} \to W_\omega$ of Weyl groups s.t. $\Psi(wl) = \Psi(w)\Psi(l)$
4. $\Psi(\rho) = \rho_\omega$ where $\rho = (1,\ldots,1)$ is the Weyl vector of $\mathfrak{g}$, and ρ_ω is its fractional analog, namely the smallest fractional weight with non-zero entries.
5. $\Psi \circ \mathcal{P} = \mathcal{P}_\omega$.

Explicit expressions for Ψ along with the proof of (B.1) can be found in [12].

The special case $\mathfrak{g} = A_{2n}$

Let us turn now to the case $\mathfrak{g} = A_{2n}$. Here, we have to face the problem that there is no map Ψ with the properties stated above. Even more mysteriously, such a map exists if one replaces the invariant subalgebra $\mathfrak{g}^\omega = B_n$ by C_n, the orbit Lie algebra of $\mathfrak{g}$. From geometric considerations, however, we expect the invariant subgroup G^ω to be the relevant structure for branes close to the group unit.

This paradox can be solved when we carefully analyze what happens to the boundary labels when we perform the limit $k \to \infty$. We have to keep the labels in the right way to really describe branes sitting close to the group unit. In most of the cases, this forces us to consider brane labels close to $(0,\ldots,0)$, only in the special case $\mathfrak{g} = A_{2n}$, we instead have to keep the difference between boundary labels and

$$(0,\ldots,0,\frac{k}{4},\frac{k}{4},0,\ldots,0) \tag{B.2}$$

small. This can be inferred from an analysis of brane geometry seen by closed strings [16].

Hence, our task is to find a map from representations of the invariant subgroup to boundary labels close to the label given above. Such a map has been introduced in [73], and we will briefly review this identification here.

The invariant subgroup of $SU(2n+1)$ under the automorphism 'complex conjugation' is $SO(2n+1)$. This group is not simply connected, therefore the set $\mathrm{Rep}(G^\omega)$ of representations forms a subset of representations $\mathrm{Rep}(\mathfrak{g}^\omega)$ of the corresponding Lie algebra $\mathfrak{g}^\omega = B_n$. The easiest example is $SO(3)$ which only admits $SU(2)$ representations of integer spin.

Representations of $\mathfrak{g}^\omega = B_n$ are labeled by dominant weights $(l_1, \dots, l_n)$. The representations which may be integrated to representations of $SO(2n+1)$ are those whose last Dynkin label l_n is even. The relevant projection of the embedding $B_n \subset A_{2n}$ reads

$$\mathcal{P}\big(L_1, \dots, L_{2n}\big) \;=\; \big(L_1 + L_{2n}, L_2 + L_{2n-1}, \dots, 2(L_n + L_{n+1})\big) \quad ,$$

and one can immediately see that the last label on the r.h.s. is always even. The label for twisted boundary conditions in $SU(2n+1)$ are given by half-integer symmetric weights λ of A_{2n}. The map from admissible dominant weights of B_n to boundary labels s.t. (B.1) holds is given by [73, 12]

$$\Psi(l_1, \dots, l_n) \;=\; \Big(\frac{l_{n-1}}{2}, \dots, \frac{l_1}{2}, \frac{k}{4} - \frac{l_n}{4} - \sum_{i=1}^{n-1} \frac{l_i}{2}, \dots, \frac{l_{n-1}}{2}\Big) \quad . \tag{B.3}$$

It involves the level k explicitly and is only well-defined for even level k. When we send $k \to \infty$ and keep the representation of B_n fixed, our boundary labels are indeed close to the label (B.2) and describe branes in the vicinity of the group unit.

Subtleties with odd level

Writing down (B.3), we have solved the problem of how to relate representations of the invariant subgroup to twisted branes in the case of $\mathfrak{g} = A_{2n}$ for even level k. What should we do when the level is odd? The map Ψ given in (B.3) does not give half-integer boundary labels in that case and seems to be ill-defined. A consistent map, however, is obtained when we restrict the last Dynkin label l_n to be odd. Taking this idea seriously, we label the boundary conditions now by the set $\mathrm{Rep}(\mathfrak{g}^\omega) \setminus \mathrm{Rep}(G^\omega)$. At first, this seems odd, but note that the tensor product of two such representations is again in $\mathrm{Rep}(G^\omega)$. The definition of the world-volume algebra as sections in the bundle

$$G \times_{G^\omega} \mathrm{End}(V^l)$$

is not jeopardized: although V^l is not a proper representations of G^ω, the space $\mathrm{End}(V^l)$ does carry the structure of a G^ω-module.

The theory of solutions to the effective gauge theory on the worldvolume is considerably changed. It is still possible to relate symmetric solutions to the existence of a function K with certain covariance condition, i.e. two configurations X and Y are related by a solution if the homomorphism bundle

$$G \times_{G^\omega} \mathrm{Hom}(V^X, V^Y)$$

admits an invertible section. This problem cannot be reformulated in terms of vector bundles associated to V^X or V^Y as we did in Chapter 4, because these do not exist in the case at hand. Probably, one should try a reformulation involving projective representations and associated bundles. But at the moment, the classification of solutions remains an open problem for these theories.

To conclude, in the case $\mathfrak{g} = A_{2n}$ we obtain two very different theories for $k \to \infty$ depending on whether k is even or odd. Most of our discussion in Chapter 4 is only applicable in the case when k is even.

g-factors and characters

In this section, we want to determine the ratio of two g-factors $g_{\Psi(l)}$ and $g_{\Psi(0)}$ where l is a representation of $\mathfrak{g}^\omega$. From the definition (4.39) we find

$$\frac{g_{\Psi(l)}}{g_{\Psi(0)}} = \frac{S^\omega_{0\Psi(l)}}{S^\omega_{0\Psi(0)}} \quad .$$

The components of the matrix S^ω have the form [14]

$$S^\omega_{0\Psi(l)} \;\propto\; \sum_{w\in W_\omega} \epsilon_\omega(w)\, \exp\Big\{-\frac{2\pi i}{k+g^\vee}\big(w(\Psi(l)+\rho_\omega)\,,\,\rho\big)\Big\} \quad .$$

We first want to consider the case of $\mathfrak{g} \neq A_{2n}$, because then we can use the five nice properties of the identification map Ψ presented on page 113.

Using properties 2 and 3 we obtain

$$S^\omega_{0\Psi(l)} \;\propto\; \sum_{w\in W_{\mathfrak{g}^\omega}} \epsilon(w)\, \exp\Big\{-\frac{2\pi i}{k+g^\vee}\big(w\Psi^{-1}(\Psi(l)+\rho_\omega)\,,\,\Psi^{-1}(\rho)\big)_{\mathfrak{g}^\omega}\Big\} \quad .$$

The subscript $\mathfrak{g}^\omega$ on the scalar product should stress that this is now a scalar product in $L_w^{\mathfrak{g}^\omega}$. With the help of the linearity (property 1) and property 4 we find that the twisted S-matrix can be expressed as a usual S-matrix of $\mathfrak{g}^\omega$ at level $k + g^\vee - g^\vee_\omega$,

$$S^{\omega,k}_{0\Psi(l)} \;\propto\; S^{\mathfrak{g}^\omega,k+g^\vee-g^\vee_\omega}_{(\Psi^{-1}(\rho)-\rho_{\mathfrak{g}^\omega})\,l} \quad .$$

The ratio of the g-factors thus reduces to a ratio of S-matrices which can be written as a finite character of the representation l of $\mathfrak{g}^\omega$ (see e.g. formula (14.247) in [66]),

$$\frac{g_{\Psi(l)}}{g_{\Psi(0)}} \;=\; \chi_l\Big(-\frac{2\pi i}{k+g^\vee}\Psi^{-1}(\rho)\Big) \quad .$$

Note that this expression is still exact and not a large k approximation. It is convenient to rewrite the result in a form which can be generalized also to the case of A_{2n},

$$\frac{g_{\Psi(l)}}{g_{\Psi(0)}} \;=\; \chi_l\Big(-\frac{2\pi i}{k+g^\vee}\frac{1}{x_e}\mathcal{P}(\rho)\Big) \quad . \tag{B.4}$$

where we used $\Psi^{-1}(\rho) = \mathcal{P}(\rho)$ which is a consequence of property 5 together with the fact that the Weyl vector ρ is symmetric; the embedding index x_e is 1 for all $\mathfrak{g} \neq A_{2n}$.

Now let us turn to the case $\mathfrak{g} = A_{2n}$. We start with $n = 1$, so we consider the embedding $A_1 \subset A_2$ at embedding index $x_e = 4$. Here, the calculation of S^ω can be easily carried out since the invariant Weyl group consists only of two elements. One obtains

$$S^{\omega,k}_{0\Psi(l)} \;\propto\; S^{A_1,k+1}_{0\,l}$$

and thus

$$\frac{g_{\Psi(l)}}{g_{\Psi(0)}} \;=\; \chi^{A_1}_l\Big(-\frac{2\pi i}{k+3}\,\rho_{A_1}\Big) \;=\; \chi^{A_1}_l\Big(-\frac{2\pi i}{k+3}\cdot\frac{1}{4}\,\mathcal{P}(\rho_{A_2})\Big) \quad .$$

So we can confirm the result (B.4) also for the case $\mathfrak{g} = A_2$.

Now we are left with the cases $A_{2n}, n > 1$. Here we have to consider the embedding of B_n in A_{2n} at embedding index $x_e = 2$. Using formulas from [61] one can show that

$$\frac{S^{\omega,k}_{L\Psi(l)}}{S^{\omega,k}_{L\Psi(m)}} \;=\; \frac{S^{B_n,k+2}_{\Phi(L)\,l}}{S^{B_n,k+2}_{\Phi(L)\,m}}$$

where Φ is the map

$$\Phi(L_1,\ldots,L_n,L_n,\ldots,L_1) \;=\; (L_1,\ldots,L_{n-1},2L_n+1) \quad .$$

From this we obtain

$$\frac{g_{\Psi(l)}}{g_{\Psi(0)}} \;=\; \chi^{B_n}_l\Big(-\frac{2\pi i}{k+2+g^\vee_{B_n}}[\rho_{B_n}+\Phi(0)]\Big) \quad .$$

It is not hard to see that this result coincides with the general formula (B.4) by using $g^\vee_{A_{2n}} = g^\vee_{B_n} + 2$ and

$$\rho_{B_n} + \Phi(0) = (1, \dots, 1, 2) = \frac{1}{2}\mathcal{P}(\rho_{A_{2n}}) \quad .$$

So, we have shown that the result (B.4) is valid for all simple Lie algebras $\mathfrak{g}$.

When we want to get approximate expressions for large k, we can use the expansion formula

$$\chi_l(t\cdot m) \;=\; \dim(V^l)\Big(1 + \frac{t^2}{2}\frac{C_l}{\dim \mathfrak{g}^\omega}(m,m) + \mathcal{O}(t^4)\Big)$$

which is a generalized version of formula (13.175) in [66] using the fact that the character does not depend on the direction of m in quadratic order.

Expanding our result (B.4) for the ratio of g-factors we obtain

$$\frac{g_{\Psi(l)}}{g_{\Psi(0)}} \;=\; \dim(V^l)\Big(1 - \frac{2\pi^2}{(k+g^\vee)^2}\frac{C_l}{\dim \mathfrak{g}^\omega}\frac{1}{x_e^2}(\mathcal{P}\rho,\mathcal{P}\rho) + \mathcal{O}\big(\frac{1}{k^4}\big)\Big)$$

The square $|\mathcal{P}\rho|^2$ is related to the square $|\rho|^2$ by the embedding index, and $|\rho|^2$ can be calculated using the Freudenthal-de Vries strange formula,

$$|\mathcal{P}\rho|^2 \;=\; x_e|\rho|^2 \;=\; \frac{x_e}{12}g^\vee \dim \mathfrak{g} \quad .$$

As final result we find

$$\frac{g_{\Psi(l)}}{g_{\Psi(0)}} \;=\; \dim(V^l)\Big(1 - \frac{\pi^2}{6(k+g^\vee)^2}\frac{\dim \mathfrak{g}}{\dim \mathfrak{g}^\omega}g^\vee\frac{C_l}{x_e} + \mathcal{O}\big(\frac{1}{k^4}\big)\Big) \quad .$$

Appendix C

Some results needed in Section 5.3

Some representation theory of $SU(n)$

In this appendix we will briefly review some facts in $\widehat{su}(n)$-representation theory. Details can be found e. g. in [66].

An affine weight L can be expanded in fundamental weights,

$$L = L_0\omega^0 + L_1\omega^1 + \cdots + L_{n-1}\omega^{n-1} \ .$$

The expansion coefficients are the Dynkin labels. When we consider representations at level k, the zeroth Dynkin label is fixed by the others,

$$L_0 = k - \sum_{i=1}^{n-1} L_i \ ,$$

therefore L is determined by its finite Dynkin labels $(L_1, \ldots, L_{n-1})$.

The fundamental weights are then given by

$$\omega_i = (0, \ldots, 0, \underset{i}{1}, 0, \ldots, 0) \ ,$$

the vacuum representation is $(0, \ldots, 0)$.

We are interested in integrable highest-weight representations. We find that their highest weight L has to be dominant, i. e. the Dynkin labels of L have to be non-negative integers. For a given level k there are only finitely many dominant weights restricted by

$$\sum_{i=1}^{n-1} L_i \leq k \ .$$

Instead of using Dynkin labels we can specify a weight L in terms of its partition

$$L = \{\ell_1; \ell_2; \cdots ; \ell_{n-1}\}$$

where

$$\ell_i = L_i + \cdots + L_{n-1} \ .$$

The dimension of the representation of the finite simple Lie algebra $su(n)$ belonging to the highest weight L can be easily given by the partition,

$$\dim(V^L) = \prod_{1 \leq i < j \leq n} \frac{\ell_i - \ell_j + j - i}{j - i} \ , \tag{C.1}$$

where $\ell_n = 0$.

To a partition we can associate a Young tableau, a box array of rows, such that the i-th row has length ℓ_i.

Young tableaux can be used to calculate tensor-product coefficients by a powerful algorithm called Littlewood-Richardson rule. We start with two tableaux and want to get the tensor product of the corresponding representations. In the second tableau, we fill the first row with 1's, the second row with 2's, and so on. Then we add all boxes with a 1 to the first tableau such that we produce a regular Young tableau (i. e. $\ell_i \geq \ell_j$ for $i < j$ and maximal n rows) without two 1's in the same column. Then we add the boxes marked by 2, again keeping only those that result in a regular tableau without two 2's in the same column. We continue with the 3's and so on. We have an additional restriction on the occurring tableaux. In counting from right to left and from top to bottom, the number of 1's must not be smaller than the number of 2's, the number of 2's must not be smaller than the number of 3's and so on.

The resulting tableaux then belong to the representations in the decomposition of the tensor product.

There is a procedure how to construct the fusion rules at level k from the tensor-product decomposition. For every tableau in the decomposition there are two possibilities. Either the corresponding weight lies on the boundary of an affine Weyl chamber and is ignored or it can be reflected by an appropriate shifted Weyl reflection to a dominant weight.

In the case where in the decomposition all tableaux have $\ell_1 \leq k + 1$, the only effect of truncation is to leave out the tableaux with $\ell_1 = k + 1$.

Let us consider an example that we will need for our discussion. We consider the fusion of the simple current generator $\mathcal{J} = (k, 0, \ldots , 0)$ with a fundamental weight ω_i. In the tensor product decomposition we find two representations, $\mathcal{J} + \omega_i$ and

$\mathcal{J} - \omega_1 + \omega_{i+1}$ (setting $\omega_n = 0$). The first one has $\ell_1 = k+1$ and is ignored because of the truncation at level k, the second one remains.

The dimensions of the corresponding representations of the finite Lie algebra fulfil

$$\dim(V^{\mathcal{J}})\,\dim(V^{\omega_i}) \;=\; \sum_L \tilde{N}_{\mathcal{J}\omega_i}{}^L \dim(V^L)$$

where $\tilde{N}$ denote the finite tensor-product coefficients.

When we substitute $\tilde{N}$ by the fusion rules N of the affine Lie algebra, this equation is not longer valid. In our example, the difference between both sides is then given by $\dim(V^{\mathcal{J}+\omega_i})$ which is (using (C.1))

$$a_i := \dim(V^{\mathcal{J}+\omega_i}) = \frac{(k+1)\dots(\widehat{k+i})\dots(k+n)}{(i-1)!\,(n-i)!}\ . \tag{C.2}$$

Some lemmas

We consider the affine Lie algebra $\widehat{su}(n)_k$. We denote the fundamental weights by ω_i, $i = 1,\dots,n-1$. By $\dim(V^S)$ we denote the dimension of the irreducible highest-weight representation of the horizontal subalgebra corresponding to the highest-weight vector S of $\widehat{su}(n)_k$. The charge of a brane with boundary label λ is denoted by q_λ.

Lemma 1. *Suppose*

$$\dim(V^{\omega_i})\,q_\lambda \;=\; \sum_\mu\; n_{\omega_i\lambda}{}^\mu\; q_\mu \quad \mod x \quad \forall\ i,\lambda\ \ .$$

Then

$$\dim(V^{S})\,q_\lambda \;=\; \sum_\mu\; n_{S\lambda}{}^\mu\; q_\mu \quad \mod x \quad \forall\ S,\lambda\ \ .$$

Proof. We will proof the lemma by induction over the sum of the finite Dynkin labels $\ell_1(S) = \sum_{i=1}^{n-1} S_i$. The equation obviously holds for $\ell_1(S) = 0$ and for $\ell_1(S) = 1$ (fundamental weights).

Suppose now that the assertion is valid for labels with $\ell_1 \le \ell$. For a label S with $\ell_1(S) \le \ell+1$ we denote by $i = i(S)$ the number between 0 and $n-1$ satisfying $\ell_j(S) = \ell+1$ for $1 \le j \le i$ and $\ell_j(S) \le \ell$ for $j > i$. Clearly the equation holds for weights satisfying $i = 0$. By induction we show that it holds for all i and therefore for all weights with $\ell_1 \le \ell+1$.

Let S be a weight with $\ell_1(S) = \ell + 1$. Then this weight appears once in the fusion of the weight $S' = S - \omega_{i(S)}$ with $\ell_1(S') = \ell$ and the fundamental weight $\omega_{i(S)}$. The other weights L appearing in the fusion have $i(L) < i(S)$. Assuming that the equation is valid for these L and using the fact that the matrices n_L form a representation of the fusion rules, we show that the equation holds for S,

$$\begin{aligned}
\sum_\mu n_{S\lambda}{}^\mu \, q_\mu &= \sum_\mu N_{S'\omega_i}{}^S \, n_{S\lambda}{}^\mu \, q_\mu \\
&= \sum_\mu \Big[\sum_L N_{S'\omega_i}{}^L \, n_{L\lambda}{}^\mu - \sum_{L\neq S} N_{S'\omega_i}{}^L \, n_{L\lambda}{}^\mu \Big] \, q_\mu \\
&= \sum_\mu \Big[\sum_\nu n_{\omega_i\lambda}{}^\nu \, n_{S'\nu}{}^\mu - \sum_{L\neq S} N_{S'\omega_i}{}^L \, n_{L\lambda}{}^\mu \Big] \, q_\mu \\
&\overset{\text{mod } x}{=} \sum_\nu n_{\omega_i\lambda}{}^\nu \, \dim(V^{S'}) \, q_\nu - \sum_{L\neq S} N_{S'\omega_i}{}^L \, \dim(V^L) \, q_\lambda \\
&\overset{\text{mod } x}{=} \Big[\dim(V^{S'}) \, \dim(V^{\omega_i}) - \sum_L N_{S'\omega_i}{}^L \, \dim(V^L)\Big] \, q_\lambda + \dim(V^S) \, q_\lambda \\
&= \dim(V^S) \, q_\lambda \quad .
\end{aligned}$$

This completes the proof of Lemma 1. □

We will show in the following that for untwisted branes it is sufficient to evaluate the charge conservation condition for fundamental representations ω_i and the simple current generator $\mathcal{J}$. The boundary labels are now denoted with the same capital letters as representations of $su(n)$. The annulus coefficients n are replaced by the fusion rules N. In the following we will make use of the fact that the charge q_L of an untwisted brane equals its dimension $\dim(V^L)$.

Lemma 2. *Suppose*

$$q_{\mathcal{J}} \, q_{\omega_i} = \sum_L N_{\mathcal{J}\omega_i}{}^L \, q_L \quad \text{mod } x \quad \forall \; i \quad . \tag{C.3}$$

Then

$$q_S \, q_{\omega_i} = \sum_L N_{S\omega_i}{}^L \, q_L \quad \text{mod } x \quad \forall \; i, S \quad . \tag{C.4}$$

Proof. Let us first remark that the equation certainly holds for $\ell_1(S) < k$, because then the fusion matrices N coincide with the finite tensor-product coefficients. We are now going to proof the statement:

For all $i = 1, \ldots, n-1$ and $\ell = 0, \ldots, k-1$ the following is true:

A

$$\sum_L N_{S\omega_j}{}^L \, q_L \overset{\text{mod } x}{=} q_S \, q_{\omega_j} \qquad \forall \; j = 1, \ldots, n-1$$
$$\forall \; S \text{ with } \ell_1(S) = k \; , \; \ell_j(S) \leq \ell + 1 \text{ for } j \geq 2$$
$$\ell_j(M) \leq \ell \text{ for } j \geq i+1$$

B

$$\dim(V^L) \overset{\text{mod } x}{=} 0 \quad \forall \; L \quad \text{with} \quad \ell_1(L) = k+1 \; , \; \ell_j(L) \leq \ell + 2 \text{ for } j \geq 2$$
$$\ell_j(L) \leq \ell + 1 \text{ for } j \geq i+1 \quad .$$

We proof this proposition by induction over ℓ and i. We start with $\ell = 0, i = 1$. Part A is fulfilled because of (C.3). For part B consider a weight L with $\ell_1(L) = k+1, \ell_2(L) \leq 1$. Then $L = \mathcal{J} + \omega_j$ for some j. This is just the truncated weight in the fusion of $\mathcal{J}$ and ω_j, therefore

$$\dim(V^{\mathcal{J}+\omega_j}) \;=\; q_{\omega_j} \, q_{\mathcal{J}} - \sum_L N_{\omega_j \mathcal{J}}{}^L \, q_L \overset{\text{mod } x}{=} 0 \; .$$

We note that the statements A and B for $\ell, i = n-1$ are equivalent to the statements for $\ell+1, i = 1$. For the induction process we only have to show the step $(\ell, i) \Rightarrow (\ell, i+1)$.

Assume that $A_{\ell,i}$ and $B_{\ell,i}$ are valid. Let M be a label with $\ell_1(M) = k, \ell_2(M) \leq \ell + 1$. The fusion of M and ω_i differs from the finite tensor-product decomposition just by representations L with $\ell_1(L) = k+1, \ell_2(L) \leq \ell + 2$ and $\ell_{i+1}(L) \leq \ell + 1$. From $B_{\ell,i}$ we know that their dimensions vanish modulo x and hence

$$\sum_L N_{M\omega_i}{}^L \, q_L \overset{\text{mod } x}{=} q_M \, q_{\omega_i} \quad \text{for } \; \ell_1(M) = k \, , \quad \ell_2(M) \leq \ell + 1 \; . \tag{C.5}$$

Now we will proof $A_{\ell,i+1}$. Let M be a label with $\ell_1(M) = k$ and $\ell_2(M) = \cdots = \ell_{i+1}(M) = \ell + 1$, $\ell_j(M) \leq \ell$ for $j \geq i+2$. We then define

$$M' \;=\; \{k; \underbrace{\ell; \cdots ; \ell}_{i-\text{times}}; \ell_{i+2}; \cdots \} \; .$$

M occurs once in the fusion of M' and ω_i, all the other labels occurring in the fusion fulfil the requirements of $A_{\ell,i}$. Hence

$$\begin{aligned}
\sum_L N_{M\omega_j}{}^L\, q_L &= \sum_L N_{M'\omega_i}{}^M\, N_{M\omega_j}{}^L\, q_L \\
&= \sum_L \Big[\sum_R N_{M'\omega_i}{}^R N_{R\omega_j}{}^L - \sum_{R\neq M} N_{M'\omega_i}{}^R N_{R\omega_j}{}^L \Big]\, q_L \\
&\overset{\text{mod } x}{=} \sum_{R,L} N_{M'\omega_j}{}^R N_{R\omega_i}{}^L\, q_L - \sum_{R\neq M} N_{M'\omega_i}{}^R\, q_R\, q_{\omega_j} \\
&\overset{\text{mod } x}{=} \sum_R N_{M'\omega_j}{}^R\, q_R\, q_{\omega_i} - \sum_R N_{M'\omega_i}{}^R\, q_{\omega_j}\, q_R + N_{M'\omega_i}{}^M\, q_M\, q_{\omega_j} \\
&\overset{\text{mod } x}{=} q_{M'}\, q_{\omega_j}\, q_{\omega_i} - q_{M'}\, q_{\omega_i}\, q_{\omega_j} + q_M\, q_{\omega_j} \\
&= q_M\, q_{\omega_j} \quad .
\end{aligned}$$

Now we have to show $B_{\ell,i+1}$. Let L_0 be a label of the form

$$L_0 \;=\; \{k+1; \underbrace{\ell+2; \cdots; \ell+2}_{i\text{-times}}; \underbrace{\ell+1; \cdots; \ell+1}_{(j-i-1)\text{-times}}; \ell_{j+1}(L_0); \cdots; \ell_{n-1}(L_0)\}$$

with $\ell_{j+1}(L_0) \leq \ell$ and define

$$L' = \{k; \underbrace{\ell+1; \cdots; \ell+1}_{i\text{-times}}; \underbrace{\ell; \cdots; \ell}_{(j-i-1)\text{-times}}; \ell_{j+1}(L_0); \cdots; \ell_{n-1}(L_0)\} \; .$$

Then L_0 appears once in the finite tensor product of L' and ω_j. It belongs to the representations that are truncated by going over to the fusion rules of the affine Lie algebra. For the other truncated representations L we know from $B_{\ell,i}$ that $\dim(V^L) = 0 \mod x$. But since $A_{\ell,i+1}$ is applicable to $M = L'$ we get $\dim(V^{L_0}) = 0 \mod x$. This completes the proof. □

Evaluation of $\gcd(a_i)$

Lemma 3. *Let the numbers a_i be defined as in (6.9). Then their greatest common divisor is given by*

$$x \;:=\; \gcd(a_i) \;=\; \frac{k+n}{\gcd(k+n, \operatorname{lcm}(1,\ldots,n-1))} \quad .$$

Proof. We are only going to give a sketch of the proof. Let us rewrite the numbers a_i by introducing

$$b(n-1) = \frac{(n-1)!}{\mathrm{lcm}(1,\ldots,n-1)}$$

as

$$a_i = \frac{(k+1)\ldots\widehat{(k+i)}\ldots(k+n-1)}{b(n-1)}\binom{n-1}{i-1}\frac{k+n}{\mathrm{lcm}(1,\ldots,n-1)}.$$

An important observation is that the first factor in a_i is always an integer. As also the binomial coefficient is an integer, we can see that x is a divisor of all a_i.

It remains to show that it is already the greatest common divisor. Let p be a prime number. We determine the maximum y and the corresponding i such that $p^y|(k+i)$. Then one can show that $p \not| \frac{a_i}{x}$. □

Appendix D

Proof of bundle triviality

In this appendix we will present a proof that all complexified vector bundles $G \times_{G^\omega} V_R^{\mathbb{C}}$ over the base manifold G/G^ω associated to representations V_R of G^ω are trivial. Here, G is any simple simply-connected Lie group and G^ω the subgroup invariant under a diagram automorphism. All possible cases are summarized in Table (4.1).

Before we start with the actual proof, let us note that representations V_R which arise by restricting representations of G to G^ω always lead to trivial bundles. We will use this extensively to proof the triviality of all other bundles.

In the considerations below, we will meet the reduced complex K-groups $\tilde{K}_{\mathbb{C}}(M)$ of vector bundles on a manifold M. They can be defined as a set of stable equivalence classes of vector bundles where two bundles E, F are stably equivalent if they can be made isomorphic by adding trivial bundles to them[1].

The proof falls into five parts. We will first present these five propositions and then enter the detailed discussion of the single steps.

Proposition 1. *Consider the reduced K-ring $\tilde{K}_{\mathbb{C}}(G/G^\omega)$ of complex vector bundles over the base manifold G/G^ω. The map $\mathcal{K} : V_R \to [G \times_{G^\omega} V_R^{\mathbb{C}}] \in \tilde{K}_{\mathbb{C}}(G/G^\omega)$ which sends a representation V_R of G^ω to the stable equivalence class of its associated complexified vector bundle, is a ring homomorphism from the representation ring $\langle Rep(G^\omega)\rangle$ to $\tilde{K}_{\mathbb{C}}(G/G^\omega)$.*

Proposition 2. *The representation ring of G^ω is a polynomial ring on the fundamental representations, $\langle Rep(G^\omega)\rangle = \mathbb{Z}[V_{\omega_1}, \dots, V_{\omega_r}]$, $r = \operatorname{rank} G^\omega$.*

Therefore any element of $\tilde{K}_{\mathbb{C}}(G/G^\omega)$ can be expressed as a polynomial in the stable equivalence classes of $G \times_{G^\omega} V_{\omega_i}^{\mathbb{C}}$.

[1] We introduced topological K-theory already in Chapter 2 on page 16 as set of pairs (E, F) of vector bundles E, F modulo certain equivalence relations. The reduced K-group is obtained by restricting to bundles of equal rank.

Proposition 3. *All complexified vector bundles associated to fundamental representations of G^ω are mapped to the zero element in $\tilde{K}_{\mathbb{C}}(G/G^\omega)$, i.e. all these bundles are stably equivalent to trivial bundles.*

From the previous remark it follows then that all bundles $G \times_{G^\omega} V_R^{\mathbb{C}}$ are stably equivalent to trivial bundles.

Proposition 4. *Two stably equivalent complex vector bundles of rank d over a base manifold of dimension n fulfilling $2d \geq n$ are isomorphic.*

Proposition 5. *All representations V_R that are not a restriction of a representation of G obey the inequality*

$$2 \cdot \dim V_R \geq \dim G/G^\omega \quad . \tag{D.1}$$

We had seen that all bundles are stably equivalent to trivial bundles, from the last two propositions we can thus conclude that all bundles are trivial. This ends the main line of argumentation. Note that it was important that we considered complexified vector bundles, otherwise there would appear a much stronger inequality in proposition 4 which in many cases could not be fulfilled any more.

Let us now take a closer look at the single propositions. The first proposition follows from the fact that the bundles associated to the tensor product of two representations $V_R \otimes V_{R'}$ is the tensor product of the associated bundles,

$$G \times_{G^\omega} (V_R \otimes V_{R'}) \;\simeq\; (G \times_{G^\omega} V_R) \;\otimes_{G/G^\omega}\; (G \times_{G^\omega} V_{R'}) \quad .$$

Proposition 2 is a structure theorem which can be found e.g. in [108, Theorem 23.24].

The third proposition is much more technical. We have to check its validity case by case. We know that restrictions of representations of G give rise to trivial bundles and thus to the zero element in $\tilde{K}_{\mathbb{C}}(G/G^\omega)$. Studying the appearance of fundamental representations in the decomposition of G-representations, we deduce in a inductive way that all fundamental representations of G^ω correspond to bundles of trivial stable equivalence class. Let us discuss the way it works in an example.

$B_3 \subset D_4$: The fundamental representations of B_3 have the dimensions 7, 8 and 21. The first fundamental representation of D_4 is 8-dimensional and decomposes as $8 \to 7 + 1$. The corresponding bundle is trivial, as well as the bundle associated to the trivial representation 1, hence the bundle associated to the 7-dimensional representation is stably equivalent to a trivial bundle. The next fundamental representation of D_4 decomposes as $28 \to 21 + 7$ so that we find by an analogous argument as above that the bundle associated to the fundamental 21 of B_3 is stably equivalent to a trivial bundle. The remaining 8-dimensional fundamental representation of B_3 is the restriction of one of the 8-dimensional representations of D_4 and thus gives rise to a trivial bundle.

In a similar way we will prove proposition 3 for all cases at the end of this chapter. But before we do so, we want to discuss the last two propositions. After we have shown that the fundamental and thus all representations give rise to bundles which are stably equivalent to trivial bundles, we want to show that they are actually really trivial. Proposition 4 is a theorem that can be found e.g. in [109, Theorem 9.1.5] which tells us that 'stbaly equivalent' and 'isomorphic' have the same meaning if the rank of the bundles is sufficiently high. The last proposition 5 ensures that all our bundles indeed fulfil this requirement, hence they are trivial. To prove this proposition we determine for all cases the lowest-dimensional representation not arising as a restriction of a G-representation. We show then that it obeys the inequality (D.1).

As an example take again $B_3 \subset D_4$. The lowest dimensional non-trivial representation has dimension 7 which is not a restriction of a representation of D_4. The base manifold has dimension $\dim D_4 - \dim B_3 = 28 - 21 = 7$. As $2 \cdot 7 = 14 \geq 7$, the inequality holds.

We will now show the validity of this proposition for all cases together with proposition 3 in a case-by-case study.

- $B_n \subset A_{2n}$ $\quad [SO(2n+1) \subset SU(2n+1)]$:
  Decomposition of fundamental representations of A_{2n} (Dynkin label notation):

$$\begin{array}{ccccc} (0\,.\,.\,\underset{i}{1}\,.\,.\,.\,.\,.\,.\,.\,.\,.\,0) & \longrightarrow & (0\,.\,.\,\underset{i}{1}\,.\,0) & & i < n \\ (0\,.\,.\,.\,.\,.\,\underset{n}{1}\,.\,.\,.\,.\,.\,.\,0) & \longrightarrow & (0\,.\,.\,.\,.\,.\,2) & & \end{array}$$

  All fundamental weights of $SO(2n+1)$ are restrictions of fundamental representations of $SU(2n+1)$. $\Rightarrow$ Proposition 3

  The lowest dimensional representation not obtained from a restriction is the representation $(20\ldots0)$ of B_n with dimension $2n^2+3n$. The base manifold has dimension $\dim SU(2n+1) - \dim SO(2n+1) = 2n^2+3n$, so inequality (D.1) holds. $\Rightarrow$ Proposition 5

- $C_n \subset A_{2n-1}$ $\quad [Sp(2n) \subset SU(2n)]$:

Decomposition of fundamental weights of A_{2n-1}:

$$
\begin{array}{lcl}
(1\ldots\ldots\ldots\ldots.0) & \longrightarrow & (1\ldots\ldots..0) \\
(01\ldots\ldots\ldots\ldots0) & \longrightarrow & (01\ldots\ldots.0)\oplus(0\ldots\ldots..0) \\
(001\ldots\ldots\ldots.0) & \longrightarrow & (001\ldots\ldots0)\oplus(1\ldots\ldots..0) \\
(0\ldots..\underset{i}{1}\ldots\ldots.0) & \longrightarrow & (0\ldots..\underset{i}{1}..0)\oplus(0\ldots\underset{i-2}{1}\ldots0)\oplus\ldots \\
(i\leq n) & & \ldots\oplus\begin{cases}(1\ldots\ldots..0) & i \text{ odd}\\ (0\ldots\ldots..0) & i \text{ even}\end{cases}
\end{array}
$$

Proceeding inductively, we see that all bundles associated to fundamental representations of C_n belon to the zero element in $\tilde{K}_{\mathbb{C}}(G/G^{\omega})$. $\Rightarrow$ Proposition 3

The lowest dimensional representation that cannot obtained from a restriction is $(010\ldots0)$ and has dimension $2n^2-n-1$. The base manifold has dimension $\dim SU(2n)-\dim Sp(2n)=2n^2-n-1$. $\Rightarrow$ Proposition 5

- $F_4\subset E_6$:
  Decomposition of representations of E_6:

$$
\begin{array}{lcl}
(000010) & \longrightarrow & (0001)\oplus(0000) \\
(000001) & \longrightarrow & (1000)\oplus(0001) \\
(000100) & \longrightarrow & (0010)\oplus(1000)\oplus(0001)
\end{array}
$$

So we see that the three fundamental representations (1000), (0010), (0001) of F_4 give rise to bundles of vanishing K-class. From the tensor product

$$(0001)\otimes(1000) \;=\; (0001)\oplus(0010)\oplus(1001)$$

we can deduce that the same is valid for (1001).

Now we look at the decomposition

$$(001000) \longrightarrow (0100)\oplus(1001)\oplus(0010)\oplus(0010)\oplus(1000)$$

and we find a vanishing K-class also for the fourth fundamental representation (0100). $\Rightarrow$ Proposition 3

The lowest dimensional representation of F_4 is (0001) and has dimension 26. The dimension of the base manifold is $\dim E_6-\dim F_4=26$. $\Rightarrow$ Proposition 5

- $G_2 \subset D_4$:
  Decompositions of representations of D_4:

$$\begin{array}{rcl} (1000) & \longrightarrow & (01) \oplus (0,0) \\ (0100) & \longrightarrow & (10) \oplus (01) \oplus (01) \end{array}$$

$\Rightarrow$ Proposition 3

The lowest dimensional non-trivial representation of G_2 is (01) and has dimension 7. The base manifold has dimension $\dim D_4 - \dim G_2 = 28 - 14 = 14$. $\Rightarrow$ Proposition 5

- $B_{n-1} \subset D_n \quad [Spin(2n-1) \subset Spin(2n)]$:
  Decomposition of fundamental representations of D_n:

$$\begin{array}{ccccc} (10 \ldots\ldots. 0) & \longrightarrow & (10 \ldots\ldots. 0) & \oplus & (0 \ldots\ldots.. 0) \\ (01 \ldots\ldots. 0) & \longrightarrow & (01 \ldots\ldots. 0) & \oplus & (10 \ldots\ldots. 0) \\ (0 \ldots.. 1 \,.\,. 0) & \longrightarrow & (0 \ldots.. 1 \,.\,. 0) & \oplus & (0 \ldots. 1 \ldots 0) \\ i \;\; (i < n) & & i & & i-1 \end{array}$$

$$\left.\begin{array}{c} (0 \ldots\ldots. 10) \searrow \\ (0 \ldots\ldots. 01) \nearrow \end{array}\right. \; (0 \ldots\ldots.. 1)$$

$\Rightarrow$ Proposition 3

The lowest dimensional non-trivial representation of B_{n-1} is $(10 \ldots 0)$ and has dimension $2n - 1$. The dimension of the base manifold is $\dim Spin(2n) - \dim Spin(2n-1) = 2n - 1$. $\Rightarrow$ Proposition 5

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Acknowledgements

It is difficult to overstate my gratitude to my Ph.D. supervisor, Volker Schomerus. With his enthusiasm, his inspiration, and his great efforts to explain things clearly and simply, he helped to make physics fun for me. He managed to strike the perfect balance between providing direction and encouraging independence. It was a great pleasure to work with him.

I have benefited so much from the discussions and the joint work with Thomas Quella. I enjoyed the time when we learned the basics of representation theory together, and I loved our disputes on notations and conventions. Many thanks to him also for the careful reading of the manuscript.

I am grateful to the referees of this thesis, Albrecht Klemm and Stefan Theisen, for their support. I have learned a lot in the lectures of Albrecht at Humboldt university. I would also like to thank my former diploma thesis advisor, Wilfried Buchmüller, from DESY Hamburg for his encouragement and advice.

Thanks are due also to my student colleagues who helped to run our student seminar. I rejoiced in all our physics and non-physics discussions. Special thanks go to Emanuel Scheidegger who guided us through our project on K-theory. I wish to thank all the nice and helpful people at the institute, the scientists, the secretaries, the librarians and Christina Pappa. They all contributed to an enjoyable working atmosphere. Adrian Butscher deserves a special mention for his help with the english language in the present document.

I would like to acknowledge the financial support from the Studienstiftung des Deutschen Volkes during the last one and a half years.

I want to thank my friends for all the emotional support, entertainment and caring they provided. My wonderful friends from the 'OE Physik' and the 'Fachschaftsrat' in Hamburg made my years of study a marvelous time, thank you for that!

I am grateful to the Golm area for providing such a pleasurable working environment. When I will leave the institute, I will miss the fresh air, the grunting and quacking of wild pigs and frogs on my walks around the institute, the twittering and chirping of birds and crickets, and the faint green light of the little glowworms that accompanied my way home when I worked until late.

Lastly, and most importantly, I wish to thank my family and my girl-friend for their support, their patience and so much love.

Lebenslauf

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Hiermit erkläre ich, die vorliegende Arbeit selbstständig ohne fremde Hilfe verfasst zu haben und nur die angegebene Literatur und Hilfsmittel verwendet zu haben.

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4. Juli 2002