Correlated Defaults, Incomplete Information, and the Term Structure of Credit Spreads

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Präsident der Humboldt-Universität zu Berlin:
Prof. Dr. J. Mlynek

Dekan der Wirtschaftswissenschaftlichen Fakultät:
Prof. Dr. L. Hildebrandt

Gutachter:

1. Prof. Dr. U. Kamecke
2. Prof. Dr. H. Föllmer

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Abstract

In this dissertation, we study the term structure of credit spreads on corporate bonds in the presence of default interdependencies and informational asymmetries. Dependence of default events has a variety of sources. Issuers may be directly associated through mutual capital holdings or financial guarantees. Since all issuers' financial health is contingent on general economic conditions, issuers can also be indirectly related. Incomplete information arises from the fact that it is typically difficult for investors to observe all parameters needed to assess the credit quality of an issuer. Investors are instead forced to estimate issuers' health based on the imperfect information which is publicly available, such as accounting reports and the default status of other issuers in the market.

While public bond investors observe default incidents in the market, we suppose that they have only incomplete information on a firm's assets and/or the threshold asset level at which informed equity investors liquidate the firm. Stochastic dependence between default events is introduced through the correlated evolution of issuers' assets through time and correlated default thresholds. The former models dependence of firms on common economic factors, while the latter models direct inter-firm linkages. Throughout, we emphasize the representation of dependence via copula functions and the newly introduced conditional copulas. We characterize joint conditional default probabilities in terms of asset and threshold copula, given the incomplete information of the secondary market. We propose and characterize the default time copula as a consistent default correlation measure, which overcomes the limitations of existing covariance based measures.

As a natural tool for the characterization of credit spreads, we introduce the compensator of default. In terms of this compensator, we represent conditional default probabilities and prices of default-contingent claims as assessed by the imperfectly informed secondary bond market. We show that short spreads (spreads for maturities going to zero) are only determined by the probabilistic properties of the default event. If a default is predictable as with perfect information, short spreads are zero regardless of how the default is modeled. Zero short spreads are empirically not plausible. We construct the compensator in terms of investors' threshold prior and the conditional running minimum asset distribution. With perfect asset observation, an arrival intensity for default does not exist, though the default is completely unpredictable. Short spreads are only positive if the assets are at an historic low. With imperfect asset observation, an arrival intensity for default exists and is characterized through the compensator. The strictly positive short spreads are given by the intensity.

We exemplify our results in a setting where a firm's assets are modeled by a geometric Brownian motion and where investors' threshold prior is uniform. We show that the credit spread term structure properties implied by our model are consistent with empirical observations. In particular, through the incorporation of direct links between firms, default clusters can endogenously arise.

We apply our findings to the analysis of some problems currently being discussed by the financial industry and its regulatory authorities. We consider a portfolio of defaultable bonds and examine its aggregated default risk. In order to (partially) hedge this exposure, we propose some credit derivative structures based on the notion of a default scenario.

Keywords:
credit spread, default correlation, incomplete information, compensator

Zusammenfassung

In dieser Dissertation untersuchen wir die Terminstruktur der Kreditspreads auf Unternehmensanleihen bei korrelierten Unternehmensausfällen und asymmetrischer Information. Ausfallabhängigkeiten haben eine Reihe von Ursachen. Emittenten können durch direkte Beziehungen wie Kapitalverflechtungen oder Garantiezusagen miteinander verbunden sein. Durch ihre gemeinsame konjunkturelle Abhängigkeit stehen Emittenten aber auch in indirekten Beziehungen zueinander. Die den Bond-Investoren zur Verfügung stehenden Unternehmensinformationen sind in der Regel unvollständig. Es ist schwierig, alle Faktoren zur korrekten Bonitätseinschätzung direkt zu beobachten. Die Bonität muß vielmehr basierend auf den öffentlich zur Verfügung stehenden unvollkommenen Informationen wie Bilanzen oder dem Zustand assozierter Unternehmen geschätzt werden.

Zur Modellierung dieser Situation nehmen wir an, daß Bond-Investoren zwar Unternehmensausfälle beobachten, aber nur unvollständige Informationen über das Vermögen einer Firma und/oder deren Liquidationsschwellwert besitzen. Der Liquidationsschwellwert is der Vermögenswert, bei dem die (vollständig informierten) Eigenkapitalinvestoren die Firma liquidieren. Stochastische Abhängigkeit zwischen den Ausfallereignissen wird durch korrelierte Vermögenswerte und korrelierte Schwellwerte induziert. Ersteres modelliert die Konjunkturabhängigkeit der Firmen und Letzteres die direkten Firmenbeziehungen. Abhängigkeiten werden durch Copula-Funktionen und die neu eingeführten bedingten Copulas repräsentiert. Die bedingten gemeinsamen Ausfallwahrscheinlichkeiten werden mittels der Vermögens- und der Schwellwerte-Copula charakterisiert. Da die existierenden Kovarianz-basierten Ausfallabhängigkeitsmaße inkonsistet sind, schlagen wir zur Messung von Ausfallabhängigkeiten die Ausfallzeiten-Copula vor und charakterisieren diese.

Zur Charakterisierung der Kreditspreads führen wir den Ausfallkompensator ein. Mit diesem Kompensator werden bedingte Ausfallwahrscheinlichkeiten und Preise von Ausfall-bedingten Derivaten dargestellt, wie sie von unvollständig informierten Bond-Investoren eingeschätzt werden. Wir zeigen, daß kurzfristige Spreads (Spreads für gegen Null konvergierende Restlaufzeiten) nur durch die probabilistischen Eigenschaften der Ausfallzeit bestimmt werden. Wenn der Ausfall wie bei perfekter Information vorhersehbar ist, dann sind die kurzfristigen Spreads Null, unabhängig von der Ausfallmodellierung. Empirisch plausibel sind jedoch nur strikt positive kurzfristige Spreads. Wir konstruieren den Kompensator mit der a-priori Schwellwerteverteilung und der bedingten Verteilung des laufenden Vermögensminimums. Bei perfekter Vermögensinformation existiert keine Ankunftsintensität für den Ausfall. Die kurzfristigen

Spreads sind nur dann strikt positiv, wenn die Vermögenswerte der Firma auf einem historischen Tiefstand sind. Bei unvollständiger Vermögensbeobachtung existiert eine Intensität; diese ist durch den Kompensator gegeben. Kurzfristige Spreads sind hier strikt positiv und gegeben durch die Intensität.

Wir veranschaulichen unsere Ergebnisse in einem Modell, in dem die Vermögenswerte durch eine geometrische Brown'sche Bewegung modelliert werden und die Schwellwerte a-priori gleichverteilt sind. Wir zeigen, daß die implizierten Spread-Terminstrukturen mit empirischen Beobachtungen konsistent sind. Insbesondere können in unserem Modell die beobachteten Ausfallanhäufungen endogen entstehen.

Wir wenden unsere Erkenntnisse für die Analyse einiger zur Zeit bei Banken und deren Regulatoren diskutierten Fragen an. Wir betrachten ein Portfolio von ausfallbedrohten Bonds und untersuchen das aggregierte Ausfallrisiko. Zur (teilweisen) Absicherung dieses Risikos schlagen wir einige Kreditderivate-Strukturen vor, die auf dem Begriff des Ausfall-Szenarios basieren.

Schlagwörter:
Kreditspread, Ausfallkorrelation, Unvollständige Information, Kompensator

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Chapter 1

Introduction

Debt securities traded on today's financial markets differ in maturities and in the risk their holder is exposed to. Traditional theory distinguishes between interest rate risk and default risk. The former refers to the variability of bond prices due to changes in short term risk-free interest rates. Default risk or credit risk refers to the possibility that the issuer of a bond fails to honor the corresponding obligations, such as coupons and principal. The yield on a bond reflects the compensation which risk averse investors demand for assuming the security's risk. The difference between the yields of a default-prone bond and of an equivalent default-free one is called *credit yield spread*. The term structure of credit yield spreads relates the spread to the bond's term.

In this dissertation we study credit yield spreads in a situation where issuer defaults are not independent of each other, and where bond investors' information on the default characteristics of issuers is incomplete. Dependence of default events has a variety of sources. Issuers may be directly associated through mutual capital holdings, financial guarantees, or parent-subsidiary relationships. Since all issuers' financial health is contingent on general economic conditions, issuers can also be indirectly related. Incomplete information arises from the fact that it is typically difficult for investors to observe all parameters needed to assess the credit quality of an issuer. Investors are instead forced to estimate issuers' health based on the imperfect information which is publicly available, such as accounting reports and the default status of other issuers in the market.

Theoretical Background

Two distinct approaches to model corporate default and corporate bond prices have evolved in the theoretical literature: the structural approach and the intensity based approach.

The *structural approach* has its roots in the seminal work of Black & Scholes (1973); it has been fully developed shortly thereafter by Merton (1974). These contributions shared the fundamental insight that corporate liabilities can be considered as contingent claims on the firm's assets. A firm financed by equity and zero coupon debt experiences default if the value of the firm at debt maturity is less than the face of the bonds. Given absolute priority, bond holders receive either the principal or the assets of the firm at debt maturity, whichever is smaller. But this is the cash flow of a default-free zero coupon bond minus a put option on the firm's assets with strike equal to the face of the bonds. Given this cash flow identity, Merton (1974) elegantly valued the firm's bonds using the option pricing theory initiated by Black & Scholes (1973). The price of debt is then a function of firm leverage, asset volatility, and risk-free interest rates.

Subsequent research based on this contingent claims approach has aimed mainly at relaxing various restrictive assumptions concerning interest rates, capital structure, bankruptcy costs, and taxes. Black & Cox (1976) initiated another significant development. While in Merton's setup the firm can default at debt maturity only, in their model the firm may default at any time before the bond's maturity. This is described by defining the default event as the first time the firm value hits some lower threshold, which could be imposed exogenously by bond safety covenants. Anderson & Sundaresan (1996) recognized that the default decision is actually made by the shareholders who govern the firm. They suggest a game-theoretic model of the bankruptcy process in order to endogenize the reorganization threshold. Leland (1994) assumes that debt service payments are financed by issuing new equity. The threshold is then endogenously determined by the firm value at which the market value of equity drops to zero.

In structural models the firm value is typically assumed to follow some continuous diffusion process. This implies a fundamental property which is shared by the structural approaches: Default occurs never unexpectedly, it is a predictable event. Bond holders can observe the nearness of the assets to the default threshold, and therefore, they are warned in advance when a default is imminent. Empirical research does not support the predictability of defaults; it rather suggests that defaults come as a surprise.

The more recent *intensity based approach* to model default and corporate bond prices has been developed in a series of papers by Duffie & Singleton (1999), Artzner & Delbaen (1995), Lando (1994), and Jarrow & Turnbull (1995), to mention a few. In contrast to the structural approach, here one assumes that the default event occurs completely unexpectedly; by surprise, so to speak. The canonical example for such an unpredictable event is the first jump of a Poisson process. While in a structural model default occurs if assets do not cover due obligations, in the intensity based approach the default event is typically taken as exogenously given. This avoids the need to specify economic reasons why a firm defaults. The default time is modeled in terms of some default-arrival rate, or intensity. In general, the intensity is a stochastic process. It can be thought of as the density of the conditional default time distribution, given that the default has not yet occurred. The intensity provides a convenient parametrization of default probabilities and hence credit spreads. For example, Jarrow, Lando & Turnbull (1997) and Lando (1998) describe the intensity as a function of credit ratings in a Markov chain model. Madan & Unal (1998) model the intensity as a function of the excess return on the issuer's equity. Using the properties of the intensity, one can show that in the corporate bond valuation problem the risk of a default leads to an adjustment of interest rates: instead of using the usual riskless short rate, one discounts now with a rate given by the sum of riskless rate and intensity. In the intensity based framework, the problem of valuing a defaultable bond is therefore reduced to that of an ordinary non-defaultable bond, which is well understood. This idea applies not only to corporate debt, but as well to more complex derivative securities where default risk plays a significant role, for example options written by a default-risky issuer.

Since in the usual structural approach an intensity does not exist, a natural question is whether we can construct a model that unifies desirable properties of both: the economically insightful default modeling in the structural approach and the empirical plausibility and mathematical tractability of the intensity based approach. More precisely, we look for a model where the default event is defined in terms of some fundamental firm variable in such a way that an intensity exists and is endogenously determined by the firm's fundamentals. Duffie & Lando (2001) recently presented a first example of such a model, where they assumed that bond investors cannot observe the firm's asset value perfectly, and instead receive noisy accounting reports at discrete points in time. Here bond holders are always uncertain about the nearness of the asset value to the default threshold, and therefore, the default event is unpredictable. Duffie & Lando (2001) showed that in this case an intensity exists. They characterized the intensity in terms of the conditional distribution of assets, given the received accounting reports. In this dissertation we also assume that bond investors are incompletely informed, but in the model we are going to propose investors' access to firm information may be even more restricted.

Empirical Observations

A number of studies have investigated historical bond price and default data. They found that credit spreads as well as aggregate default rates are strongly related to general macro-economic factors such as the level of default-free interest rates, GDP growth rates, equity index returns and other business cycle indicators (see, for example, Duffee (1998) and Keenan (2000)). This finding is quite plausible: default rates are likely to increase as the economy goes into a recession. Another observation from the latest Moody's report is that there are *default clusters* around times of economic downturn, cf. Jarrow & Yu (2001). This clustering refers to infection effects and cascading defaults, where the default of a firm immediately increases the default likelihood of another firm dramatically. In its extreme form, a default directly triggers the default of another firm. Such effects can for instance be induced through mutual capital holdings, financial guarantees, or parent-subsidiary relationships. In a recession, default rates increase and so does the likelihood of observing infectious defaults. Recent evidence of the default clustering phenomenon includes the banking crises in Japan and South Korea.

These empirical observations have an important consequence: defaults of firms are *stochastically dependent*. We can distinguish two mechanisms inducing default dependence. First, the state of all firms depends on common factors related to the health of the general economy. How can this be modeled? In the structural approach, the market's valuation of the firms' assets reflects contingency on general economic factors. Therefore asset processes could be correlated through time, cf. Zhou (2001) and the industry approaches of KMV (Crosbie (1997)) and J.P. Morgan (1997). In the intensity based approach, dependence on common factors can be accommodated via correlated intensity processes, cf. Duffie & Singleton (1998) and Jarrow, Lando & Yu (2000). But such a framework has a restrictive property: defaults are conditionally independent given the intensities.

Second, the state of a particular firm depends on the default status of other firms. In the intensity based approach, one would model this by assuming the intensity to be *some* function of the default status of other firms, cf. Jarrow &

Yu (2001). But without an endogenous intensity specification, we are left with the question what the appropriate functional form is. As for the structural approach, the default status dependence is difficult to incorporate; we are not aware of models beyond the case of correlated diffusions.

The approach to be proposed in this dissertation incorporates default correlation via *both* dependence mechanisms, in such a way that structural and intensity based approaches to default are integrated in a common framework.

Outline of the Dissertation

This dissertation is organized in six chapters and one appendix.

The general framework for all subsequent chapters is laid out in Chapter 2. We propose a structural-type model of firm default, where the default event is defined as the first time a firm's asset process hits some lower threshold. Our model extends the existing approaches in two directions. First, we consider several firms whose default events are stochastically dependent. Default correlation is introduced via two mechanisms: dependence of firms on common (macro) economic factors and direct firm inter-linkages. Second, while public bond investors observe default events, they have only incomplete information on firms' default characteristics, that is on firms' default thresholds and assets.

In Chapter 3, we characterize corporate bond prices and the joint conditional distribution of defaults, given the incomplete information of the secondary market. Our approach relies on two methodological innovations. First, we emphasize the representation of dependence via copulas and the newly introduced conditional copulas. Second, we systematically employ the properties of the running minimum asset process. Having modeled directly inter-linked firms, we show that default probabilities and bond prices can jump upon defaults of other firms. That is, in our structural model default clusters can in fact endogenously arise. We clarify the structure of the association between firm defaults and propose the default time copula as a consistent default correlation measure. This copula overcomes the limitations of existing covariance based measures. It separates the complete default dependence structure from idiosyncratic default behavior. The default copula is characterized in terms of both threshold copula and running minimum asset copula. Joint default probabilities are therefore critically influenced by the properties of both copulas. This finding complements recent results of Frey & McNeil (2001).

The implications of incomplete information on the term structure of credit yield spreads on corporate debt are examined in Chapter 4. We study the relation between probabilistic properties of the default time, intensity, and spreads in a general setup. We show that the short credit spread, i.e. the spread if the maturity is going to zero, is only determined by the default time properties. If a default is predictable, then short spreads are zero; if it is unpredictable, this is not necessarily the case. In a structural approach, the information available to bond investors is closely related to the default time properties. With complete information, defaults are predictable and short spreads are zero. But with incomplete information on firms' assets and/or thresholds, defaults are unpredictable events and short spreads are non-negative. These are empirically plausible properties. Since there is a one-to-one correspondence between default time properties and the *compensator* of a firm's default indicator process, we find that the compensator is the natural tool to study credit spreads. We characterize default probabilities, defaultable security prices, and credit spreads in terms of the compensator. We prove a general representation of the compensator in our setup. If assets are perfectly observable but thresholds are not, we show that an intensity does not exist. Depending on the level of the asset value, short spreads are either zero as with complete information, or strictly positive. But if assets are only imperfectly observed or not observed at all, an intensity does in fact exist. In this case structural and intensity based approaches to default are fully integrated. The intensity is implicitly characterized through the compensator. This provides an alternative approach to the intensity result of Duffie & Lando (2001) and extends their result to the multi-firm case with correlated defaults and unobservable default thresholds. Due to default dependence, the intensities are correlated through time and they jump upon defaults of correlated firms. This justifies the models of Duffie & Singleton (1998) and Jarrow & Yu (2001), who take intensities having these properties as exogenously given.

The results of the previous two chapters were derived under quite general conditions. In Chapter 5, we exemplify our findings in a two-firm model where the market value of each firm follows a geometric Brownian motion. While bond investors cannot observe the firms' default thresholds, we assume that they observe the firm values perfectly. This is in direct contrast to the one-firm informational setup of Duffie & Lando (2001), who assume that thresholds are perfectly observable, but firm values are not. Default dependence is exclusively modeled through direct firm inter-linkages, providing an alternative to the existing approaches relying on correlated firm values. In this setup we find that, depending on the level of the firm value, incomplete information can lead to decreasing and hump shaped term structures of credit spreads. The hump shaped term structure is very similar to the one which appears in the case of complete information, where short spreads are zero.

In Chapter 6, we apply the results derived in Chapter 3 to analyze some problems which are currently discussed by the financial industry and its regulatory authorities. Consider an investor holding a portfolio of defaultable bonds. We are interested in the total default risk of her holdings and in contracts allowing a risk reduction. The aggregated portfolio default risk is most comprehensively described by the distribution of losses suffered by the investor due to bond defaults. We characterize the loss distribution in terms of the joint default distribution. Here we are able to relax some restrictive assumption made in the literature. As for partial or total risk reduction, we propose and analyze a class of flexible credit derivative contract specifications which are based on the notion of a 'default scenario'. These derivatives isolate and transfer specific aspects of the portfolio's loss risk. They can be thought of as default insurance contracts where the underlying is not only a single bond, but a whole bond portfolio.

A self-contained introduction of copula functions is given in the Appendix.

Our main results can be summarized as follows:

- We model default correlation through firms' dependence on common macro-economic factors and direct firm inter-linkages. In this approach default clusters can endogenously arise.
- We introduce the default time copula to represent the complete default dependence structure. This measure overcomes the limitations of existing covariance based default correlation measures.
- The compensator of the default indicator process provides the natural framework for studying the term structure of credit yield spreads.
- A structural model based on incomplete observation of default thresholds and assets shares with an intensity based model the property of unpredictable defaults; it is therefore empirically plausible. Whether an intensity exists depends on the extent of the available information.
- If assets are perfectly observable but default thresholds are not, the term structure of credit spreads is decreasing or hump shaped, depending on the level of the asset value.

- Default scenario based credit derivatives are appropriate instruments to isolate and transfer specific parts of a bond portfolio's aggregated default risk.

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Chapter 2

Modeling Firm Defaults

In this chapter we lay out the general framework for all subsequent chapters. We will propose a structural-type model of firm default, where the default event is defined as the first time a firm's asset process hits some lower threshold.

In the usual structural approach to default, one considers an individual firm whose assets are typically modeled as an observable continuous diffusion process. The threshold is taken to be a constant or a deterministic function of time. The framework we are going to propose extends this approach in two directions. First, we consider several firms whose default events are stochastically dependent. Second, while public bond investors observe defaults, they have only incomplete information on firms' default thresholds and assets.

Incomplete information of bond investors has been introduced by Duffie & Lando (2001) to the analysis of corporate debt. In their model, investors cannot directly observe the issuer's assets and instead receive periodic and noisy accounting reports. We extend this setup to include uncertainty on the default threshold itself: while in their one-firm model the threshold is a constant, in our multi-firm model each firms' threshold is an unobservable random quantity to the bond investors. Faced with imperfect information, investors are assumed to specify a prior on the random threshold vector.

In order to address the empirically observed characteristic default pattern mentioned in the Introduction, default correlation is introduced via two mechanisms. Dependence of firms on common (macro-) economic factors is modeled through dependence between firms' asset values, as emphasized by Zhou (2001) in a two-firm model. Direct firm inter-linkages are modeled through correlated default thresholds, as reflected by investors' threshold prior.

2.1 Firms

We consider an economy with a financial market. Uncertainty is modeled by a fixed probability space $(\Omega, \mathcal{H}, P)$, equipped with a filtration $(\mathcal{H}_t)_{t\geq 0}$ describing the information flow over time.

On the financial market investors can trade in bonds issued by several firms. The index set of all firms is denoted $I := \{1, 2, \ldots, n\}$, where $n < \infty$. We take as given some $\mathbb{R}^n$-valued stochastic process $(V^1, \ldots, V^n)$, and we denote by $(\mathcal{F}_t)_{t\geq 0}$ the filtration generated by $(V^1, \ldots, V^n)$. Here and in the sequel, any filtration will be assumed to satisfy the usual conditions of right-continuity and completeness, cf. Brémaud (1980, III.5). The process $V^i := (V^i_t)_{t\geq 0}$ is Markovian, continuous, and satisfies $V^i_0 = 0$. V^i_t is a sufficient statistic for the expected discounted future cash flows of firm i as seen from time t. We will therefore call V^i *asset process*. If, for example, we model the market value of firm i as a geometric Brownian motion, then we would take V^i to be the logarithm of the firm value. This special choice is considered in Chapter 5.

Firms are financed by equity and debt. Debt is modeled as a non-callable consol bond, paying coupons at some constant rate as long as the firm operates. We suppose that there are no taxes, implying that there is no incentive for firms to issue new debt in order to explore a tax shield. Each firm has two classes of homogeneous claimants: equity investors and bond investors. Bond investors are outside investors and have no role in the governance of the firm. In addition to other risks, they are also exposed to the risk of losses due to non-performance of the borrower. When a firm stops servicing its contractual agreed obligations, we say it defaults (or loosely, the issued bond defaults). The firm then enters financial distress and some form of corporate reorganization takes place. Assuming that there are no bankruptcy costs, this means that the remaining assets are allocated to the firm's claimants according to some scheme. For a discussion of various schemes, we refer to Franks & Torous (1989). Bankruptcy in the strict legal sense may be different from the notion of default we have chosen.

Equityholders are in charge of governing and controlling the firm. This involves in particular the decision whether and when to stop debt service payments and to liquidate the firm. We assume that the shareholders choose to default if the firm's expected future cash flows are sufficiently low. Thus we assume that there exists a random vector

$$D := (D_1, \ldots, D_n) \in \mathcal{H}_0$$

which is independent of $(\mathcal{F}_t)_{t\geq 0}$, such that the shareholders of firm i announce the default on its debt at the moment the asset value V^i falls for the first time to the threshold D_i. We let $D_i < V_0^i = 0$, implying that at time $t = 0$ all firms operate. A constant threshold is consistent with our time-homogeneous setting where the firms have issued consol bonds. Letting τ_i denote the default time of firm i, the *liquidation policy* of the shareholders is therefore given by

$$\tau_i := \inf\{t > 0 \,|\, V_t^i \leq D_i\}. \tag{2.1}$$

Thus τ_i is a random variable taking values in $(0, \infty]$. We define the vector $\tau := (\tau_1, \ldots, \tau_n)$ and denote by

$$M_t^i := \inf\{V_s^i \mid s \leq t\} \tag{2.2}$$

the running minimum of V_t^i. We then have the equality

$$\{\tau_i \leq t\} = \{M_t^i \leq D_i\}. \tag{2.3}$$

Our definition of default is consistent with the case where a firm i is insolvent because the remaining assets $V_{\tau_i}^i$ do not generate sufficient cash flow to make a due payment (flow-based bankruptcy). This corresponds to the inability of the firm to obtain new financing, for example by issuing new securities. It is also consistent with the case where the assets $V_{\tau_i}^i$ violate some minimum working-capital or positive net-worth requirement (stock-based bankruptcy). For a detailed discussion of the distinction, see Wruck (1990). We note that if the market has frictions, if there is e.g. asymmetric information, then both bankruptcy definitions may be different.

In our model, the level of the default threshold D_i of issuer i is determined by both *firm-specific* factors and by *industrial organization-related* factors. Idiosyncratic factors include for example the liability and retirement structure of the firm, safety covenants of the bond issue, or management's/shareholders' preferences. Industrial organization-related factors are determined by the particular market structure in which the firms operate. They result from the linkages of issuers through legal agreements such as parent-subsidiary relationships, financial guarantees, or mutual capital holdings[1]. To give an example, the default point of a firm being an subsidiary of another company will not only depend on its own firm-specific characteristics, but also on the type and

[1]The latter is very common in Germany's financial services industry so that one often speaks of 'Deutschland AG'. According to the Financial Times Deutschland, February 27, 2001, p. 19, the total shareholdings of insurance titan Allianz AG account for 63.5 per cent of its market capitalization (as of February 6, 2001).

extent of the linkage to the parent firm. The extent of the relation is for instance determined by the profit and cost allocation between the firms. Also the degree of dependence of the subsidiary on the human or capital resource inflow from the parent company plays an important role.

We assume that all agents are risk-neutral. This assumption eliminates the need to specify a risk premium associated with the bankruptcy of firms. If the market is incomplete, this presents a challenge in itself and is not the focus of this work.

2.2 Asymmetric Information

In our model, corporate claimants cannot only be distinguished by their role in controlling a firm, but equity and bond investors are also *asymmetrically informed*. Shareholders are assumed to have complete information. Their information flow is modeled by the filtration $(\mathcal{H}_t)_{t\geq 0}$ generated by

$$\mathcal{H}_t := \sigma(\tau \wedge t) \vee \mathcal{F}_t \vee \sigma(D), \tag{2.4}$$

where $a \wedge b := \min(a, b)$; here and in the sequel, we pass from the σ-algebras defined in (2.4) to the induced filtration $(\mathcal{H}_t)_{t\geq 0}$ without changing the notation. Bond investors' access to information is limited and modeled by a filtration $(\mathcal{G}_t)_{t\geq 0}$, satisfying

$$\sigma(\tau \wedge t) \subseteq \mathcal{G}_t \subseteq \sigma(\tau \wedge t) \vee \mathcal{F}_t \subset \mathcal{H}_t. \tag{2.5}$$

That means that bond investors observe defaults as they occur, but they may not have information on the asset processes. They are informed to a maximum extent if assets are perfectly observable. While bond investors are aware of the liquidation policy (2.1) of the shareholders, we stress that they are *not* informed about the thresholds: $\sigma(D) \not\subseteq \mathcal{G}_0$. This is consistent with the role bond investors typically play in the governance of the firm. In general, shareholders do neither disclose the relevant factors in their (debt servicing) decisions nor the stage of the decision making process to the market. It is therefore reasonable to suppose that shareholders as firm insiders are not permitted, say by insider legislation, to trade in the bond market. If this assumption were not maintained, shareholders could control the firm so as to maximize the value of their debt holdings. Also, bond transactions could reveal inside firm information, for example on the true financial health status of the firm. This is as well the motivation for assuming that equity is privately held and not traded publicly. Otherwise, bond holders could infer (private) information from transactions in a firm's stock.

The following three specific choices of $(\mathcal{G}_t)_{t\geq 0}$ are of particular interest.

Model A. Bond investors observe the defaults in the market as well as the asset value of all firms perfectly. That is,

$$\mathcal{G}_t := \mathcal{F}_t \vee \sigma(\tau \wedge t). \tag{2.6}$$

Though somewhat simplistic, this model seems in principle appropriate for public firms where one can infer information on the firm's market value from the price of its shares (for details on that procedure we refer to Crosbie (1997)). Lambrecht & Perraudin (1996) present a related model with two bondholders, where a creditor will foreclose on the firm's debt when the observed asset value falls to her individually set barrier. The value of the barrier is private information and hence strategic behavior comes into play.

Model B. As suggested by Duffie & Lando (2001) in a one-firm model, issuers' assets may not be perfectly transparent to the secondary market. In Model B, we assume that bond investors observe the defaults in the market and receive at times $t_1 < t_2 < \ldots < t_m$ a noisy accounting report $Y_{t_k} := (Y^1_{t_k}, \ldots, Y^n_{t_k})$ from all firms:

$$\mathcal{G}_t := \sigma(Y_s, s \leq t, s \in \{t_1, \ldots, t_m\}) \vee \sigma(\tau \wedge t), \tag{2.7}$$

where $Y^i_{t_k} := V^i_{t_k} + U^i_{t_k}$ for some independent noise random variable $U^i_{t_k}$. The variance of $U^i_{t_k}$ can be interpreted as a measure of the degree of accounting noise of firm i at time t. The $U^i_{t_k}$ can be serially correlated, reflecting persistence of accounting noise in time, or correlated with the asset value $V^i_{t_k}$. The significant difference to the model of Duffie & Lando (2001) is that in our model investors do not know the default threshold (in their approach it is observable). Kusuoka (1999) provides another example of imperfect observation in a one-firm model, where bond investors observe threshold, default event, and a process whose drift is modulated by the asset process.

Model C. Here we suppose that only the information whether a firm has defaulted or not reaches the secondary market:

$$\mathcal{G}_t := \sigma(\tau \wedge t). \tag{2.8}$$

It is clear that our Model B nests Model C prior to the first noisy observation, i.e. for all $t < t_1$. Both Model A and B are particularly realistic when the considered issuers are private firms whose equity is not traded publicly. The information policy of these corporates is usually very restrictive. In general there are only minimal informational duties imposed by law.

Bond investors are not informed about the default threshold vector D, cf. (2.5). Investors therefore form a common prior distribution G on D. We take this prior as given in form of the following two components:

(1) *Idiosyncratic prior*: marginal distributions functions G^i of D on $(-\infty, 0)$, assumed to be continuous, strictly increasing, and to admit a density g^i:

$$G^i(x) := P[D_i \leq x] = \int_{-\infty}^{x} g^i(y)dy, \quad x \leq 0. \tag{2.9}$$

(2) *Interdependence prior*: copula C^D of the vector D (see Appendix for a definition of copulas and related results).

By a key property of copulas (cf. Theorem A.2), the continuous marginal distributions G^i together with the copula C^D determine the joint threshold distribution function G uniquely via

$$G(x_1, \ldots, x_n) = C^D(G^1(x_1), \ldots, G^n(x_n)), \quad x_i \leq 0. \tag{2.10}$$

Thus, prescribing C^D and the G^i is equivalent to prescribing the prior G. We will not fix a particular G, nor do we suppose that the G^i are of the same distribution type. Separating the threshold prior corresponds to separating the threshold determinants in idiosyncratic and industrial organization-related factors, which we have suggested above. Marginal distributions G^i account for firm-specific factors whereas the copula C^D reflects the threshold dependence structure across firms.

The copula represents the dependence structure of a random vector irrespective of its multivariate distribution type. The copula C^D and the continuous G^i specify G *uniquely*. Linear correlation (covariance), in contrast, is only the natural measure of dependence for a joint elliptically (e.g. normally) distributed D. In case D is elliptical, the G^i and a covariance matrix specify G uniquely. In case of a non-elliptical D, marginals and covariances do *not uniquely* specify G. Using linear correlation in the non-elliptical case can lead to severe misinterpretations of the dependence structure.

2.3 Default Correlation

In the Introduction we have recognized that default correlation – the stochastic dependence between defaults of different firms – is essential for the analysis of corporate debt. Let us clarify how such dependence arises in the model described above. Default time correlation is induced via the definition of the default times in (2.1). We distinguish two types:

(1) *Macro-Correlation: correlation of firms' asset values* V^i.
All firms' health depend on common macro-economic factors, such as the stage of some business cycle, commodity prices, interest rates, or consumer behavior. This is reflected by the correlated evolution of firms' asset values through time. We will call this more indirect association of firms macro-correlation. This is a dynamic phenomenon since it can vary with the economy through time.

(2) *Micro-Correlation: correlation of firms' default thresholds* D_i.
Due to parent-subsidiary relationships, financial guarantees or capital holdings, firms are also directly inter-linked. Since these links are determined by the micro-structure of the market in which the firms operate, we will call this direct but more static (time-independent) association of firms micro-correlation. This phenomenon is induced through correlation of the default thresholds. The threshold dependence structure is formally represented by the copula C^D.

Thus, the resulting correlation in firm defaults is determined by both asset correlation and threshold correlation. The precise relation between these three quantities is discussed in Section 3.4 below.

Chapter 3

An Analysis of Correlated Defaults

Based on the framework laid out in the previous chapter, in this chapter we will focus on two issues. The first concerns the characterization of joint conditional default probabilities for any group of firms, given the incomplete information of the secondary market. The second concerns the measurement and representation of stochastic dependence across firm defaults.

Zhou (2001) recently derived the unconditional joint default distribution explicitly in a two-firm structural model with complete information on firms' assets and thresholds. Assets are modeled as correlated Brownian motions; direct firm inter-linkages do not exist. This has an important consequence: with macro-correlation alone default clusters do not arise. Our model differs significantly. Investors cannot observe the thresholds and may not observe assets perfectly. Moreover, firms are macro *and* micro-correlated. We show that in this setup, for the first time in a structural model, default clusters can in fact arise. Our approach to the joint conditional default distribution differs from that of Zhou (2001); it relies on two methodological innovations. First, we emphasize the representation of dependence via copulas and the newly introduced conditional copulas. Second, we systematically employ the properties of the running minimum asset process.

In existing structural approaches to correlated firm defaults, among them Zhou (2001) and Hull & White (2000*b*), default correlation is measured via *linear* correlation (covariance). Given the non-elliptical distribution of defaults, this can lead to severe misinterpretations of the default dependence because the covariance is not the natural measure of dependence any more. We clarify the structure of the association between firm defaults and propose the default time copula as a consistent default correlation measure. This copula overcomes the limitations of existing covariance based measures: it separates the complete default dependence structure from idiosyncratic default behavior. We characterize the default copula in terms of the threshold copula (the interdependence prior) and the running minimum asset copula. Joint default probabilities are therefore critically influenced by the properties of both copulas. This finding extends recent results of Frey & McNeil (2001), who found that in latent variable models with *observable* thresholds joint default probabilities are determined by the copula of the latent variables. The latent variables can in our setup be interpreted as the assets' running minima.

We first study in Section 3.1 Bayesian updating of the threshold prior given the information revealed over time to the secondary market. From observing firms' default status, bond investors can infer information about default threshold values. In Section 3.2 we introduce conditional copulas to represent the conditional threshold dependence structure. The joint conditional default distribution is established under quite general conditions (no specific assumption on information structure, asset processes and prior). Our characterization involves the (suitably transformed) conditional threshold copula, conditional threshold marginals, and the joint conditional distribution of the assets' running minima. As for the derivation of the conditional asset distribution, we provide an alternative to the approach pursued by Duffie & Lando (2001). The equivalence between the two approaches will be a prerequisite for a key insight in Chapter 4. In Section 3.3, corporate bond prices are characterized in terms of conditional default probabilities. We show how we can extract bond investors' perception of the relation between firms from bond prices observed in the market. We moreover find that default probabilities and thus bond prices jump upon the default of directly inter-linked (micro-correlated) issuers. This is characteristic for the default clustering phenomenon. The goal of Section 3.4 is to clarify the structure of the association between firm defaults in our model and to propose a consistent framework for measuring its degree. We introduce and characterize the n-dimensional default time copula as well as some pairwise copula-based default correlation measures.

A comprehensive case study, exemplifying the results derived in this chapter, is discussed in Chapter 5. In Chapter 6, the default distribution is used to study some problems related to the aggregation of correlated default risks and the analysis of credit derivatives.

3.1 Belief Updating

We begin by studying Bayesian updating of bond investors' given prior threshold belief G. That is, we will consider

$$G_t(x) := P[D \leq x \,|\, \mathcal{G}_t], \quad x \in \mathbb{R}^n_-.$$

We assume that D has a $\mathcal{G}_t$-conditional joint density g_t. For concreteness, we stick to our three information models defined in Chapter 2.

Let us define a process $(S_t)_{t \geq 0}$, taking values in $\mathbb{P}(I)$, by

$$S_t := \{i \in I \,:\, \tau_i \leq t\}, \tag{3.1}$$

where I is the index set of all bonds. Since it collects the indices of all those bonds having defaulted by time t, we will call the set S_t the *default scenario* at t. It satisfies $S_t \subseteq S_u$ for all $t \leq u$. For some $s \in \mathbb{P}(I)$ we have

$$\{S_t = s\} = \bigcap_{i \in s} \{\tau_i \leq t\} \cap \bigcap_{i \in I - s} \{\tau_i > t\}. \tag{3.2}$$

By the definition of τ_i (cf. (2.1) and (2.3)) and the continuity of the asset process V^i, we get

$$\{S_t = s\} = \bigcap_{i \in s} \{D_i = M^i_{\tau_i}\} \cap \bigcap_{i \in I - s} \{D_i < M^i_t\},$$

where $D_i < 0$ is the random default threshold and M^i_t is the running minimum of V^i_t. For $M_t := (M^1_t, \ldots, M^n_t)$, we define a set $B(M_t, \cdot) \in \mathcal{B}^n_-$ such that

$$\{D \in B(M_t, s)\} = \{S_t = s\}, \quad s \in \mathbb{P}(I).$$

Given some default scenario s is observed at time t, bond investors know that $D_i = M^i_{\tau_i}$ for all defaulted firms $i \in s$ and that $D_i < M^i_t$ for all operating firms $i \in I - s$. That means, the threshold vector D must belong to $B(M_t, s)$.

The following proposition relates the updated threshold belief to the prior.

Proposition 3.1 (A posteriori belief: Model A). *Let L denote the joint law of D. On the set $\{S_t = s\}$, the a posteriori threshold belief is represented by the distribution*

$$P[D \in A \,|\, \mathcal{G}_t] = \frac{L(A \cap B(M_t, s))}{L(B(M_t, s))}, \quad A \in \mathcal{B}^n_-.$$

PROOF. In Model A the firms' assets are perfectly observable ($M_t \in \mathcal{F}_t \subset \mathcal{G}_t$). From the structure of the σ-field $\sigma(\tau \wedge t)$[1] and from Bayes' Theorem, on the set $\{S_t = s\}$ we have

$$\begin{aligned} P[D \in A \,|\, \mathcal{G}_t] &= P[D \in A \,|\, \sigma(\tau \wedge t) \vee \mathcal{F}_t] \\ &= P[D \in A \,|\, D \in B(M_t, s),\, \mathcal{F}_t] \\ &= \frac{P[D \in A \cap B(M_t, s) \,|\, \mathcal{F}_t]}{P[D \in B(M_t, s) \,|\, \mathcal{F}_t]}. \end{aligned}$$

This implies our assertion because D is independent of $\mathcal{F}_t$. $\square$

To give an example, assume that there has been no default so far, $S_t = \emptyset$. Then the thresholds satisfy $D_i < M_t^i$ or, put another way, $D \in B(M_t, \emptyset) = (-\infty, M_t^1) \times \ldots \times (-\infty, M_t^n)$. Noting that $G(x_1, \ldots, x_n) = L((-\infty, x_1) \times \ldots \times (-\infty, x_n))$, Proposition 3.1 implies

$$G_t(x_1, \ldots, x_n) = \frac{G(x_1 \wedge M_t^1, \ldots, x_n \wedge M_t^n)}{G(M_t^1, \ldots, M_t^n)}, \quad x_i \leq 0. \tag{3.3}$$

Using the copula C^D and the marginals G^i of the vector D, by virtue of (2.10) we can also write

$$G_t(x_1, \ldots, x_n) = \frac{C^D(G^1(x_1 \wedge M_t^1), \ldots, G^n(x_n \wedge M_t^n))}{C^D(G^1(M_t^1), \ldots, G^n(M_t^n))}.$$

For future use, we note the following. By our assumptions on the prior G (cf. Chapter 2), the associated marginal distribution $G_t^i(x)$ is for a fixed $t < \tau_i$ strictly increasing on $(-\infty, M_t^i]$ and continuous on $(-\infty, 0]$. By assumption (2.9), the associated density $g_t^i(x)$ exists and satisfies $g_t^i(x) = 0$ for all $x \geq M_t^i$ on the set $\{t < \tau_i\}$.

In contrast to Model A, in Model B investors cannot observe the issuers' assets, $M_t \notin \mathcal{G}_t$, and instead receive periodic and noisy accounting reports.

Proposition 3.2 (A posteriori belief: Model B). *Fix some time t and suppose that investors have received some noisy accounting reports $Y_{t_k} := (Y_{t_k}^1, \ldots, Y_{t_k}^n)$ at times $t_k \in \mathcal{T}$, where $\mathcal{T}$ is the set of observation times. On the set $\{S_t = s\}$, the a posteriori threshold belief is then represented by the distribution*

$$P[D \in A \,|\, \mathcal{G}_t] = \frac{E[L(A \cap B(M_t, s)) \,|\, Y_u, u \leq t, u \in \mathcal{T}]}{E[L(B(M_t, s)) \,|\, Y_u, u \leq t, u \in \mathcal{T}]}, \quad A \in \mathcal{B}_-^n.$$

[1] The σ-field $\sigma(\tau \wedge t) \subseteq \mathcal{G}_t$ is generated by the events $\{\tau_i \leq u\} = \{M_u^i \leq D_i\}$ for $u \leq t$ and $i \in S_t$ as well as the atoms $\{\tau_i > t\} = \{M_t^i > D_i\}$ for $i \in I - S_t$.

Proof. In analogy to the proof of Proposition 3.1, on $\{S_t = s\}$ we get

$$P[D \in A \,|\, \mathcal{G}_t] = P[D \in A \,|\, \sigma(\tau \wedge t) \vee \sigma(Y_u, u \le t, u \in \mathcal{T})]$$
$$= \frac{P[D \in A \cap B(M_t, s) \,|\, Y_u, u \le t, u \in \mathcal{T}]}{P[D \in B(M_t, s) \,|\, Y_u, u \le t, u \in \mathcal{T}]},$$

implying our assertion because D is independent of the asset observations. □

As an example, assume again that $S_t = \emptyset$. Proposition 3.2 then yields

$$G_t(x_1, \ldots, x_n)$$
$$= \frac{\int_{\mathbb{R}^n_-} G(x_1 \wedge y_1, \ldots, x_n \wedge y_n) P[M_t \in dy \,|\, Y_u, u \le t, u \in \mathcal{T}]}{\int_{\mathbb{R}^n_-} G(y_1, \ldots, y_n) P[M_t \in dy \,|\, Y_u, u \le t, u \in \mathcal{T}]}, \tag{3.4}$$

for $x_i \le 0$. Let us consider the joint conditional asset distribution. To be specific, suppose that $\mathcal{T} := \{t\}$, i.e. at time t bond investors receive the first noisy accounting report $Y_t = V_t + U$ of all firms', where $U := (U^1, \ldots, U^n)$ is a noise vector, independent of V and D. We can think of the variance of U^i as a measure of the degree of accounting noise of firm i. For $t \ge 0$ we define

$$\varphi(x, y, t)dy := P[M_t \le x, V_t \in dy], \quad x \in \mathbb{R}^n_-, \quad y \in \mathbb{R}^n \tag{3.5}$$

i.e., $\varphi(0, \cdot, t)$ is the density of the asset vector V_t at time t. φ is available explicitly if the asset value V follows a standard n-dimensional Brownian motion with drift (cf., for example, Borodin & Salminen (1996)). By Bayes' rule and the independence of U, we have

$$P[M_t \le x \,|\, Y_t] = \int_{\mathbb{R}^n} P[M_t \le x \,|\, V_t = v] P[V_t \in dv \,|\, Y_t]$$
$$= \int_{\mathbb{R}^n} \frac{h_U(Y_t - v)}{h_Y(Y_t, t)} \varphi(x, v, t) dv,$$

where h_U (resp. $h_Y(\cdot, t)$) is the density of U (resp. Y_t). h_U is exogenously given and $h_Y(\cdot, t)$ can be obtained via convolution of h_U and asset density $\varphi(0, \cdot, t)$. Note that we can easily generalize the above calculations to multiple observation times, including for example serially correlated noise.

In Model C investors have only survivorship information:

Proposition 3.3 (A posteriori belief: Model C). *On the set $\{S_t = s\}$, the a posteriori threshold belief is represented by the distribution*

$$P[D \in A \,|\, \mathcal{G}_t] = \frac{E[L(A \cap B(M_t, s))]}{E[L(B(M_t, s))]}, \quad A \in \mathcal{B}^n_-.$$

PROOF. Use the same argument as in the proof of Proposition 3.2. $\square$

As an example, let again $S_t = \emptyset$. Proposition 3.3 then implies

$$G_t(x_1, \dots, x_n) = \frac{\int_{\mathbb{R}^n_-} G(x_1 \wedge y_1, \dots, x_n \wedge y_1) P[M_t \in dy]}{\int_{\mathbb{R}^n_-} G(y_1, \dots, y_n) P[M_t \in dy]}, \quad x_i \leq 0, \qquad (3.6)$$

where $P[M_t \leq x] = \int_{\mathbb{R}^n} \varphi(x, y, t) dy$.

3.2 The Joint Distribution of Defaults

Based on bond investors' a posteriori belief G_t, the goal of this section is to characterize the joint distribution of defaults as assessed by the incompletely informed secondary market. That is, we will establish

$$F_t(T_1, \dots, T_n) := P[\tau_1 \leq T_1, \dots, \tau_n \leq T_n \,|\, \mathcal{G}_t]. \qquad (3.7)$$

We set $\tau := (\tau_1, \dots, \tau_n)$. While we do not yet fix a particular model for bond investors' filtration $(\mathcal{G}_t)_{t \geq 0}$, we will suppose that condition (2.5) holds.

Having derived the conditional threshold distribution G_t in the previous section, we now introduce the *conditional copula process* $(C_t^D)_{t \geq 0}$, which is implicitly defined by

$$G_t(x_1, \dots, x_n) = C_t^D(G_t^1(x_1), \dots, G_t^n(x_n)), \quad x_i \leq 0.$$

The copula C_t^D is the $\mathcal{G}_t$-conditional threshold dependence structure. Let us denote by $I_t^i(u) = \inf\{x \geq 0 : \, G_t^i(x) \geq u\}$ the generalized inverse of G_t^i. If the G_t^i are continuous for all fixed times $t < \tau_i$, then we have for any $\mathcal{G}_t$-measurable random variable $u_i \in (0, 1)$ the relation

$$C_t^D(u_1, \dots, u_n) = G_t\left(I_t^1(u_1), \dots, I_t^n(u_n)\right), \qquad (3.8)$$

cf. Corollary A.3. Observe that C^D and C_t^D are not equal in general; for an example we refer to Chapter 5.

The (ordinary) copula couples marginal distributions with the joint distribution of some random vector. The copula linking the marginal survival functions with the joint survival function of some random vector will be called 'survival' copula. Ordinary copula and survival copula express in an equivalent way the dependence structure of a given random vector. For the thresholds

D, let us in analogy to $(C_t^D)_{t\geq 0}$ introduce the *survival copula process* $(\bar{C}_t^D)_{t\geq 0}$, satisfying

$$P[D_1 > x_1, \dots, D_n > x_n \,|\, \mathcal{G}_t] = \bar{C}_t^D(1 - G_t^1(x_1), \dots, 1 - G_t^n(x_n)), \qquad (3.9)$$

for any $x_i \leq 0$.

Lemma 3.4. *For each $t \geq 0$, the conditional survival threshold copula can be constructed from the conditional threshold copula via*

$$\bar{C}_t^D(u_1, \dots, u_n) = \sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1+\dots+i_n} C_t^D(v_{1i_1}, \dots, v_{ni_n}),$$

where $v_{j1} = 1 - u_j$ and $v_{j2} = 1$ and $u_i \in [0,1]$ is $\mathcal{G}_t$-measurable.

Proof. Fixing some t, we have for $u_i = 1 - G_t^i(x_i) \in [0,1]$ and any $x_i \leq 0$ the equalities

$$\begin{aligned}
&\bar{C}_t^D(1 - G_t^1(x_1), \dots, 1 - G_t^n(x_n)) \\
&\qquad = P[D_1 > x_1, \dots, D_n > x_n \,|\, \mathcal{G}_t] \\
&\qquad = \sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1+\dots+i_n} P[D_1 \leq v_{1i_1}, \dots, D_n \leq v_{ni_n} \,|\, \mathcal{G}_t] \\
&\qquad = \sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1+\dots+i_n} C_t^D(G_t^1(v_{1i_1}), \dots, G_t^n(v_{ni_n})),
\end{aligned}$$

where $v_{j1} = x_j$ and $v_{j2} = 0$. The last equality is again a consequence of the fundamental property of copulas, cf. Theorem A.2. Since $G_t^i(0) = 1$ the claim follows. $\square$

We are now in a position to state the main result of this section.

Theorem 3.5 (Joint default distribution). *The $\mathcal{G}_t$-conditional joint default probability is for $T_i > t \geq 0$ and $\max_i \tau_i > t$ given by*

$$F_t(T_1, \dots, T_n) = E\left[\bar{C}_t^D(1 - G_t^1(M_{T_1}^1), \dots, 1 - G_t^n(M_{T_n}^n)) \mid \mathcal{G}_t\right].$$

Proof. Using the equality $\{\tau_i \leq t\} = \{D_i \geq M_t^i\}$, by the law of iterated expectations we can write

$$\begin{aligned}
F_t(T_1, \dots, T_n) &= P[\tau_1 \leq T_1, \dots, \tau_n \leq T_n \,|\, \mathcal{G}_t] \\
&= P[D_1 \geq M_{T_1}^1, \dots, D_n \geq M_{T_n}^n \,|\, \mathcal{G}_t] \\
&= E\left[P[D_1 \geq M_{T_1}^1, \dots, D_n \geq M_{T_n}^n \,|\, \mathcal{G}_t \vee \mathcal{F}_T] \,|\, \mathcal{G}_t\right],
\end{aligned}$$

where we set $T \geq \max_i T_i$. Since D is independent of $(\mathcal{F}_t)_{t\geq 0}$ and $M^i_{T_i} \in \mathcal{F}_T$, our assertion now follows from (3.9). $\square$

The joint conditional default distribution describes both the individual default behavior of firms and the interrelation across the performance of firms. This distribution is the key for the solution of a variety of problems related to the measurement of dependent credit risks. These problems include the aggregation of correlated default risks, and the analysis of derivative instruments having payoffs contingent on the performance of a whole bond portfolio. These issues are the subject of Chapter 6.

The default probability of a single firm is obtained from Theorem 3.5 by setting $n = 1$, where we have to take into account copula property (2) given in Definition A.1. On the set $\{\tau_i > t\}$, we have

$$F^i_t(T) := P[\tau_i \leq T \mid \mathcal{G}_t] = E[1 - G^i_t(M^i_T) \,|\, \mathcal{G}_t], \quad t \leq T. \tag{3.10}$$

Since D_i has conditional density g^i_t, we can equivalently write

$$F^i_t(T) = \int_{-\infty}^{0} P[M^i_T \leq x \mid \mathcal{G}_t] g^i_t(x) dx, \quad t \leq T. \tag{3.11}$$

The default probability is the key ingredient needed for the valuation of a firm's bonds, cf. Section 3.3 below.

The information available to bond investors includes the default status of all firms; investors observe a default in the moment where it occurs. The mapping $t \to F^i_t(T)$ can for fixed T be subject to jumps upon the default arrival of other firms, given a sufficient degree of *micro*-correlation between the firms. To give an example, suppose that there are two firms i and j where each firm holds a substantial amount of the other's debt. If either firm defaults, the risk of the remaining firm to experience financial distress is increased substantially. Letting $\tau_j < \tau_i$, for a sufficiently large horizon T the default probability $F^i_t(T)$ is likely to jump upwards at time $t = \tau_j$. The jump corresponds to an immediate re-assessment of firm i's future performance by bond investors, given the information that the directly inter-linked firm j has defaulted. Loosely, the stronger the link, the more intense the jump. This behavior of default probabilities matches the empirically observed default clustering discussed in the Introduction.

The jump effect is a consequence of two properties of our model. The first is that firms are micro-correlated, which corresponds to dependent default thresholds. Macro-correlation does not play any role here. The second is that defaults are observable 'surprise events'. Upon any default in the market, bond investors revise their a posteriori threshold belief G_t immediately. This updating leads then to a revision of default probabilities via (3.10). Since with macro-correlation default events do not affect each other directly, we see that the model of Zhou (2001) based on correlated asset values cannot explain default clusters. Our model is a first example of a structural approach in which default clusters can arise.

Having established the default distribution for general information structures satisfying (2.5), we now consider the three information models introduced in Chapter 2. This also allows us to relate our approach to that of Duffie & Lando (2001) from a methodological point of view. Theorem 3.5 can be rewritten as

$$F_t(T_1, \ldots, T_n) = \int_{\mathbb{R}^n_-} \bar{C}_t^D(1 - G_t^1(x_1), \ldots, 1 - G_t^n(x_n))P[M_T \in dx \,|\, \mathcal{G}_t], \quad (3.12)$$

where we put $M_T := (M^1_{T_1}, \ldots, M^n_{T_n})$. The conditional distribution G_t is determined for all models by (3.3), (3.4), and (3.6), respectively. It remains to derive the joint distribution $P[M_T \leq x \,|\, \mathcal{G}_t]$ of the asset's minima for some future date, given the current information of the secondary market. For ease of exposition, for the remainder of this section let us fix some time t such that $S_t = \emptyset$. I.e., there has been no default by t and we have for all firms $D_i < M^i_t$. In Model A, where the assets V are perfectly observable, from the Markov property of V and the independence of D and V we have that

$$\begin{aligned} P[M_T \leq x \,|\, \mathcal{G}_t] &= P[M_T \leq x \,|\, D < M_t, V_t] \\ &= P[(M^1_{T_1 - t}, \ldots, M^n_{T_n - t}) \leq x - V_t] \\ &= \Psi(x - V_t, T_1 - t, \ldots, T_n - t), \end{aligned} \quad (3.13)$$

where $x \in (-\infty, M^1_t) \times \cdots \times (-\infty, M^n_t)$ and where

$$\Psi(x, T_1, \ldots, T_n) := P[M_T \leq x], \quad x \in \mathbb{R}^n_-. \quad (3.14)$$

We clearly have

$$\Psi(x, T, \ldots, T) = \int_{\mathbb{R}^n} \varphi(x, y, T)dy.$$

If assets follow a standard n-dimensional Brownian motion, Ψ is explicitly available (cf., e.g. Borodin & Salminen (1996)). Hence Model A is relatively easy to handle, which will be exploited in our case study developed in Chapter 5.

In Model B there is a noisy accounting report $Y_t = V_t + U$ available to the bond investors and by the Markov property we get

$$P[M_T \le x \,|\, \mathcal{G}_t] = P[M_T \le x \,|\, D < M_t, Y_t], \quad x \in \mathbb{R}^n_- \\ = \int_{\mathbb{R}^n} \Psi(x - z, T_1 - t, \dots, T_n - t) P[V_t \in dz \,|\, D < M_t, Y_t], \tag{3.15}$$

and it remains to calculate the conditional joint asset density, given survivorship and the observation Y_t. As an intermediate step, we have by Bayes' rule and the independence of U,

$$P[M_t > y, V_t \in dz \,|\, Y_t] = \frac{h_U(Y_t - z)\,\bar{\varphi}(y, z, t)\,dz}{h_Y(Y_t, t)},$$

where we set $\bar{\varphi}(x, y, t)dx := P[V_t \in dx, M_t > y]$, which can be obtained from φ (cf. (3.5)) via standard arguments. We can now write

$$\begin{aligned} &P[V_t \in dz \,|\, D < M_t, Y_t] \\ &= \int_{\mathbb{R}^n_-} \frac{P[M_t > y, V_t \in dz \,|\, Y_t]}{P[M_t > y \,|\, Y_t]} P[D \in dy \,|\, D < M_t, Y_t] \\ &= \int_{\mathbb{R}^n_-} \frac{h_U(Y_t - z)\,\bar{\varphi}(y, z, t)\,dz}{\int_{\mathbb{R}^n} h_U(Y_t - v)\,\bar{\varphi}(y, v, t)\,dv} P[D \in dy \,|\, D < M_t, Y_t]. \end{aligned} \tag{3.16}$$

Moreover,

$$P[D \in dy \,|\, D < M_t, Y_t] = \int_{\mathbb{R}^n_-} \frac{1_{\{y<z\}} g(y) dy}{G(z)} P[M_t \in dz \,|\, Y_t], \tag{3.17}$$

where G represents the threshold prior and g is the associated density, cf. (2.9).

For comparison, suppose for the moment that $I = \{i\}$ and that the threshold is observable, $D_i \in \mathcal{G}_t$. In this special case we would obtain

$$P[V_t^i \in dz \,|\, D_i < M_t^i, Y_t^i] = \frac{h_U(Y_t^i - z)\,\bar{\varphi}(D_i, z, t)\,dz}{\int_{D_i}^{\infty} h_U(Y_t^i - v)\,\bar{\varphi}(D_i, v, t)\,dv}. \tag{3.18}$$

In the one-firm model of Duffie & Lando (2001), investors receive a noisy asset report while the threshold is observable. In our notation, they write that (see their Section 2.2)

$$P[V_t^i \in dz \,|\, \tau_i > t, Y_t^i] = \frac{h_U(Y_t^i - z)\,\pi(-D_i, z - D_i, t)\,P[V_t^i \in dz]}{\int_{D_i}^{\infty} h_U(Y_t^i - v)\,\pi(-D_i, v - D_i, t)\,P[V_t^i \in dv]}, \tag{3.19}$$

where, taking V_t^i to be a Brownian bridge with $V_0^i = x$ and$V_t^i = y$, $\pi(x, y, t)$ is the probability that $\min\{V_s^i : 0 \le s \le t\} > 0$. We now have

$$\begin{aligned}
\pi(-D_i, z - D_i, t)\, P[V_t^i \in dz] &= P[\min_{0 \le s \le t} V_s^i > D_i \,|\, V_0^i = 0, V_t^i = z]\, P[V_t^i \in dz] \\
&= P[M_t^i > D_i, V_t^i \in dz] \\
&= \bar{\varphi}(D_i, z, t)\, dz,
\end{aligned}$$

showing the equivalence between the assets density characterization of Duffie & Lando (2001) and ours based on the running minimum asset process in the univariate case for a known default threshold. This equivalence will be central to a result derived in Chapter 4. Assuming that the asset value follows a Brownian motion and that the noise variable U is normal, Duffie & Lando (2001) provide an explicit solution for the conditional asset density (3.19) in terms of the standard normal distribution function.

In Model C investors have only survivorship information and we have

$$\begin{aligned}
P[M_T \le x \,|\, \mathcal{G}_t] &= P[M_T \le x \,|\, D < M_t], \quad x \in \mathbb{R}^n_- \\
&= \int_{\mathbb{R}^n} \Psi(x - z, T_1 - t, \ldots, T_n - t) P[V_t \in dz \,|\, D < M_t]. \quad (3.20)
\end{aligned}$$

Using Bayes' rule and (3.17), the conditional joint asset density given survivorship only can be written as

$$P[V_t \in dz \,|\, D < M_t] = \int_{\mathbb{R}^n_-} \int_A \frac{\bar{\varphi}(y, z, t)\, dz}{\bar{\Psi}(y, t, \ldots, t)} \frac{g(y)}{G(v)} \frac{\partial}{\partial v} \Psi(v, t, \ldots, t) dv dy, \quad (3.21)$$

where integration is over $A := (y_1, 0) \times \ldots \times (y_n, 0)$ and where $\bar{\Psi}(y, t, \ldots, t) := P[M_t > y]$ which can be obtained from $\Psi(y, t, \ldots, t)$ via standard arguments. We close this section by remarking that in Chapter 5, we discuss a comprehensive case study where we fix information Model A, a particular prior, and assume that assets follow a standard Brownian motion. This then allows us to derive more explicit results for F_t.

3.3 Corporate Bond Prices

The purpose of this section is to examine corporate bond prices given incomplete information and default dependence. Under some assumptions, from observed bond prices we can recover the secondary market's perception of firms' relation to each other.

In our model, the capital structure of the firms is based on consol bonds having no fixed maturity and paying out a constant coupon to the bond investors, cf. Chapter 2. We can strip the consol coupon into a continuum of zero coupon bonds with recovery being pro-rata based on the default-free market value that the strips contribute to the consol. As for the valuation of the consol bond, it is therefore enough to consider the valuation of the zero bonds.

A zero bond with maturity date T issued by firm i pays one unit of account at T if the issuer has not defaulted by time T (that is, $\tau_i > T$). If the firm defaults before the maturity of the bond ($\tau_i \leq T$), some recovery payment is made at T. Letting the independent $\mathcal{G}_{\tau_i}$-measurable random variable $\delta_i \in [0, 1]$ denote the default loss, we model this payment as a fraction $1 - \delta_i$ of the bond's face value. δ_i is independent also across firms so that recovery is a completely idiosyncratic issue. δ_i has expected value $\bar{\delta}_i$. This recovery-of-par specification is common in the industry. Other forms of recovery assumptions include the payment of some fraction of an equivalent but default-free Treasury bond (cf., for example, Madan & Unal (1998) or Jarrow & Turnbull (1995)), or the payment of some fraction of the pre-default market value of the bond (cf. Duffie & Singleton (1999) and Schönbucher (1998)). For an empirical investigation of recovery procedures we refer to Franks & Torous (1994).

Carty & Lieberman (1996) show in an empirical study based on a comprehensive Moody's data set that recovery rates vary by both instrument type and seniority. They found that subordinated bond classes are quite different in their recovery realizations, whereas the difference between secured and unsecured senior debt is not significant. For example, senior unsecured bonds have a mean recovery of 51% of face value and a standard deviation of 25%. This would suggest to model recovery rates of senior unsecured bonds as uniform on $(0, 1)$.[2] For a given debt class, it is generally straightforward to fit distributions to the available data.

We take as given some $(\mathcal{G}_t)$-adapted and locally riskless short rate process $(r_t)_{t \geq 0}$. We assume that each τ_i is independent of riskless rates. The independence assumptions on interest rates and recoveries imply that our results in Sections 3.1 and 3.2 apply as stated. Recalling that all agents are risk-neutral, the price $d(t, T)$ of a default-free zero-coupon bond maturing at T is given by

$$d(t, T) = E[e^{-\int_t^T r_s ds} \,|\, \mathcal{G}_t], \quad t \leq T.$$

A defaultable zero coupon bond with maturity T issued by firm i has at time

[2] A uniform $(0, 1)$ random variable has expected value of 0.5 and a variance of 0.28.

$t < \tau_i$ a price equal to[3]

$$p_i(t,T) = E\left[e^{-\int_t^T r_s ds}(1-\delta_i)1_{\{\tau_i \le T\}} + e^{-\int_t^T r_s ds}1_{\{\tau_i > T\}} \,|\, \mathcal{G}_t\right], \quad t \le T$$
$$= d(t,T) - d(t,T)\,\bar{\delta}_i\, F_t^i(T), \tag{3.22}$$

which is the value of a riskless zero bond less a discount for the default risk. The discount is composed of the present value of the expected default loss weighted by the default probability F_t^i, which is given by Theorem 3.5. In the previous section we have shown that the mapping $t \to F_t^i(T)$ can for a fixed T jump upon defaults of micro-correlated firms. From (3.22) it is clear that the mapping $t \to p^i(t,T)$ can exhibit an analogous pattern. These effects are characteristic for the default clustering phenomenon.

Let us consider firm i's zero bonds if the interdependence prior is $C^D = \Pi$, i.e. in case where investors perceive the default thresholds of the firms to be independent. Fixing information Model A, on the set $\{\tau_i > t\}$ we find then by Proposition 3.1 or direct calculations using Bayes' rule the a posteriori threshold distribution

$$G_t^i(x) = P[D_i \le x \,|\, \mathcal{G}_t] = \frac{G^i(x \wedge M_t^i)}{G^i(M_t^i)}, \quad x \le 0.$$

G^i is the (idiosyncratic) threshold prior. The firm's default probability $F_t^i(T)$ now satisfies

$$F_t^i(T) = E[1 - G_t^i(M_T^i) \,|\, \mathcal{G}_t] = E\left[1 - \frac{G^i(M_T^i)}{G^i(M_t^i)} \,\middle|\, \mathcal{G}_t\right], \quad t \le T, \quad \tau_i > t,$$

cf. (3.10). Letting $p_i^\Pi(t,T)$ denote the zero bond price if $C^D = \Pi$, we then have from (3.22) that

$$p_i^\Pi(t,T) = d(t,T)\left(1 - \bar{\delta}_i + \frac{\bar{\delta}_i}{G^i(M_t^i)} E[G^i(M_T^i) \,|\, \mathcal{G}_t]\right),$$

where $E[G^i(M_T^i) \,|\, \mathcal{G}_t]$ can be computed given assumptions on the asset processes, cf. Section 3.2 and Chapter 5.

Let us assume market efficiency and that corporate bond prices carry only a default risk premium (this requires particularly liquidity). Then we can recover the implied secondary market's perception of the micro-dependence structure across firms by using p^Π as a benchmark. For fixed $(t,T,\bar{\delta}_i)$, the

[3] If $A \in \mathcal{H}$, the indicator function for A is the $\{0,1\}$-valued random variable 1_A defined by $1_A(\omega) = 1$ if $\omega \in A$ and $1_A(\omega) = 0$ otherwise.

difference between the actually observed bond price of firm i on the financial market and p_i^Π reflects the micro-correlation between firm i and all other firms as assessed by the incompletely informed bond investors. A zero difference would indicate that firm i is not directly linked to other firms in the market. This provides a *test for micro-independence*. If the difference is nonzero, there is however a problem with interpretation in case $n > 2$: one cannot attribute the difference to a particular firm relation. The sign of the difference has a clear interpretation only if we know whether prices are increasing or decreasing in C^D given a particular default scenario (cf. Definition A.9).[4] Given $n = 2$ and bond prices decrease in C^D on a fixed scenario, then a positive (negative) difference would indicate that the market believes in a negative (positive) direct linkage between the two firms. This intuition will be made more precise in Chapter 5, where a concrete example is studied. An example for a negative link is when the default of one firm actually helps the remaining firm: the situation in a duopoly or in some strongly competive market where the remaining firm takes over (most of) the defaulted issuer's market share. A parent-subsidiary relationship is an example for a positive link.

3.4 Characterizing Default Correlation

The joint default distribution derived in Theorem 3.5 completely describes both the individual default behavior of firms and the interdependence across firms. The goal of this section is to clarify the structure of the association between firm defaults. We propose the default time copula for measuring the degree of default association. This copula separates the default dependence structure from idiosyncratic, i.e. marginal, default behavior. A copula-based default dependence measure overcomes the limitations of traditional covariance-based measure.

3.4.1 Default Dependence Measures

There is a broad consensus in the existing literature to measure the pairwise default correlation between two firms i and j over some time period $[0, t]$ via

[4] At first glance, we would conjecture that default probabilities are increasing and thus bond prices are decreasing in the threshold copula C^D. This is however not always the case and depends on the default scenario, as an example in Chapter 5 shows.

the *linear* correlation coefficient

$$\rho(N_t^i, N_t^j) = \frac{\mathrm{Cov}[N_t^i\, N_t^j]}{\sqrt{\mathrm{Var}[N_t^i]\ \mathrm{Var}[N_t^j]}}, \tag{3.23}$$

cf., for example, Zhou (2001), Hull & White (2000*b*), Kealhofer (1998), and Lucas (1995). $N_t^k := 1_{\{t \geq \tau_k\}}$ is the default indicator for some default time τ_k, which is not necessarily defined as in (2.1). Li (2000) considers the linear survival time correlation $\rho(\tau_i - t, \tau_j - t)$. In our model, $E[N_t^i\, N_t^j]$ and $E[N_t^k]$ are directly available from Theorem 3.5 and ρ can be easily computed.

However, measuring default correlation via $\rho(N_t^i, N_t^j)$ has severe limitations from a theoretical perspective. This then causes problems with the interpretation of the quantity ρ. The problems arise from the fact that covariance is the natural measure of dependence only for multivariate elliptically distributed random variables, cf. Embrechts, McNeil & Straumann (1999). Elliptical distributions extend the joint normal $N(\mu, \Sigma)$, i.e. the distribution with mean μ and covariance matrix Σ, such that the contours of equal density are ellipsoids. The normal distribution is thus an elliptical distribution. The essence here is that an elliptical distribution is uniquely determined by μ, Σ, and the distribution type of the admissible marginals. This fact suggests the covariance and thus linear correlation as a natural and complete dependence measure for elliptical random vectors.

There is no reason to expect that the vector (N_t^i, N_t^j) is multivariate normal or elliptical; the marginal distributions are Bernoulli with success probability $E[N_t^k]$. Thus linear correlation ρ and mean do not uniquely specify a joint distribution for (N_t^i, N_t^j). Consequently, ρ is not the natural dependence measure anymore: it cannot completely capture the true dependence in this case. At least three problems with the interpretation of ρ arise in this case. First, besides the fact that ρ measures linear dependence only, $\rho = 0$ does not imply independence between the defaults of the two firms. Second, small correlations cannot be interpreted as implying weak dependence. Third, not all linear correlations between -1 and 1 can be attained given the marginal distributions $E[N_t^k]$. Given these weaknesses, the conclusions drawn in Zhou (2001), Lucas (1995) and Standard & Poor's (1999) based on $\rho(N_t^i, N_t^j)$ should be taken with care.

As far as default interrelations are concerned, linear correlation seems not to be an appropriate association measure. A more general dependence concept, though not scalar-valued as linear correlation, is the copula. The copula represents the complete dependence structure for any random vector, regardless of its distribution. It would therefore make sense to consider the copula of the default indicator vector $(N_t^1, \ldots, N_t^n)$ or that of the default time vector τ as a measure of default correlation. As an example, we examine the copula C^τ of τ, which is implicitly defined by

$$C^\tau(P[\tau_1 \le T_1], \ldots, P[\tau_n \le T_n]) = P[\tau_1 \le T_1, \ldots, \tau_n \le T_n],$$

for all $T_i > 0$. Using our earlier notation, we have $C^\tau(F_0^1(T_1), \ldots, F_0^n(T_n)) = F_0(T_1, \ldots, T_n)$. In later sections we will also use a conditional copula C_t^τ, which can be defined analogously. Let us also introduce the copula C^M of the assets' running minimum vector M, given by

$$C^M(P[M_{T_1}^1 \le x_1], \ldots, P[M_{T_n}^n \le x_n]) = P[M_{T_1}^1 \le x_1, \ldots, M_{T_n}^n \le x_n],$$

for all $x_i \le 0$ and all $T_i > 0$. Hence the dependence structure of the vector $(M_{T_1}^1, \ldots, M_{T_n}^n)$ is equal for all choices of $(T_1, \ldots, T_n)$.

Proposition 3.6. *Define the generalized inverse of F_0^i by*

$$J^i(u) := \inf\{x \ge 0 : \ F_0^i(x) \ge u\}, \quad u \in (0,1).$$

If each $F_0^i(T)$ is continuous on $(0, \infty)$, then the copula C^τ of the default times is for all $u_i \in (0,1)$ given by

$$\begin{aligned} C^\tau(u_1, \ldots, u_n) = \int_{\mathbb{R}_-^n} & \bar{C}^D(1 - G^1(x_1), \ldots, 1 - G^n(x_n)) \\ & dC^M(P[M_{J^1(u_1)}^1 \le x_1], \ldots, P[M_{J^n(u_n)}^n \le x_n]). \end{aligned}$$

Proof. This is a consequence of Corollary A.3, which implies that

$$\begin{aligned} C^\tau(u_1, \ldots, u_n) &= F_0(J^1(u_1), \ldots, J^n(u_n)) \\ &= E\big[\bar{C}^D(1 - G^1(M_{J^1(u_1)}^1), \ldots, 1 - G^n(M_{J^n(u_n)}^n))\big]. \end{aligned}$$

Our assertion now follows from the definition of the copula C^M. □

From (3.11), the default probability $F_0^i(T)$ is continuous if the mapping $T \to P[M_T^i \le x]$ is continuous for all fixed $x \le 0$. If assets V^i follow a Brownian motion, then this is satisfied, cf. (5.12).

The copula C^τ captures the complete dependence across defaults, irrespective of their joint distribution type. C^τ measures *any* correlation between the random variables $\tau_1, \ldots, \tau_n$, whether induced by macro or micro-correlation.

Thus, the default dependence structure is a function of both asset and threshold dependence structure, represented by C^M and $\bar{C}^D$. This fact formally manifests our discussion of default correlation in Section 2.3.

Let us recall from Proposition A.7 that C^τ must satisfy

$$W(u) \leq C^\tau(u) \leq M(u), \quad u \in [0,1]^n.$$

If C^τ attains the lower bound, the defaults are perfectly negatively correlated (countermonotonicity), if it attains the upper bound the defaults are perfectly positively related (comonotonicity). If $C^\tau = \Pi$ then there is independence. A copula C_1 is said to be larger than a copula C_2 if for any $u \in [0,1]^n$ we have $C_1(u) \geq C_2(u)$ (Definition A.9). In this sense W is smaller than every copula and M is larger than every copula. This partial ordering can be used to rank the degree of dependence of default time vectors.

A pairwise dependence measure $C^\tau(u,v)$ is easily constructed by

$$C^\tau(u,v) = C^\tau(1,\ldots,1,u,1,\ldots,1,v,1,\ldots,1),$$

cf. property (2) in Definition A.1. Also, note that C^τ has the advantageous property of being invariant under monotonic transformations of the default times, cf. Propositions A.4 and A.5.

A scalar-valued measure of dependence can perhaps provide more intuition about the degree of stochastic dependence between two random default times. In view of our earlier discussion on linear correlation, the most important requirement on such a measure is that it captures the complete dependence irrespective of the underlying multivariate distribution. That is, such a measure should be defined on copula level. *Rank correlation* fulfills this requirement; it measures the degree of monotonic dependence, whereas linear correlation measures the degree of linear dependence only. For example, Spearman's rank default correlation can be defined by

$$\rho_S(\tau_i,\tau_j) = \rho(F_0^i(\tau_i), F_0^j(\tau_j)), \tag{3.24}$$

where ρ is the ordinary linear correlation and F_0^k is the default probability of firm k. Since $(F_0^i(\tau_i), F_0^i(\tau_i))$ has joint distribution C^τ (we have $F_0^k(\tau_k) \sim U(0,1)$), ρ_S is the linear correlation of the copula C^τ. Kendall's rank correlation can be used in an equivalent way to express the monotonic association among the default times (we will introduce Kendall's measure in Chapter 5).

By using the definition of linear correlation, we obtain

$$\begin{aligned}\rho_S(\tau_i, \tau_j) &= 12\ E[F_0^i(\tau_i) \cdot F_0^j(\tau_j)] - 3 \\ &= 12 \int_0^1 \int_0^1 C^\tau(u, v)\, du\, dv - 3 \\ &= 12 \int_0^1 \int_0^1 (C^\tau(u, v) - uv)\, du\, dv, \end{aligned} \tag{3.25}$$

showing that Spearman's rank correlation depends on the copula only. Also, ρ_S is seen to be a scaled version of the signed volume enclosed by the copula C^τ and the product copula $\Pi(u, v) = uv$. Thus ρ_S is a measure of the 'average distance' between the actual distribution of (τ_i, τ_j) and their distribution given independence. Besides being invariant under increasing transformations, ρ_S has the following useful properties, which are easily verified using (3.24):

(1) $-1 \leq \rho_S(\tau_i, \tau_j) \leq 1$.

(2) $\rho_S(\tau_i, \tau_j) = 1$ iff $C^\tau(u, v) = M(u, v)$.

(3) $\rho_S(\tau_i, \tau_j) = -1$ iff $C^\tau(u, v) = W(u, v)$.

(4) $\rho_S(\tau_i, \tau_j) = 0$ if $C^\tau(u, v) = \Pi(u, v)$.

For copulas of complex structure, it might be difficult to calculate ρ_S. Another measure based on distances can be defined via

$$\rho^L(\tau_i, \tau_j) = 4 \sup_{u,v \in [0,1]} |C^\tau(u, v) - uv|,$$

which is motivated by a suggestion of Schweizer & Wolff (1981). This measure satisfies $0 \leq \rho^L(\tau_i, \tau_j) \leq 1$, and $\rho^L(\tau_i, \tau_j) = 0$ iff $C^\tau(u, v) = \Pi(u, v)$ (compare property (4) above), but has the property $\rho^L(\tau_i, \tau_j) = 1$ iff $C^\tau(u, v) = M(u, v)$ or $W(u, v)$, i.e., ρ^L cannot differentiate between positive and negative dependence. If the dependence type is known a priori, then ρ^L is an easily computed measure for default correlation.

3.4.2 Dependence, Information, and Prior

In Proposition 3.6 we have derived the default time copula C^τ, which represents the default dependence structure. In order to clarify the relation between dependence, the available information, and investors' prior, we will now consider some instructive special cases.

Proposition 3.7. *Suppose the default thresholds are observable by bond investors, $D \in \mathcal{G}_0$. If each $F_0^i(T)$ is continuous on $(0, \infty)$, then the default dependence structure is given by*

$$C^\tau(u) = C^M(u), \quad u \in [0,1]^n.$$

PROOF. We consider only the case $n = 2$; the general case is obvious. From Proposition 3.6 we obtain

$$\begin{aligned} C^\tau(u,v) &= \int_{\mathbb{R}^2_-} \bar{C}^D(1_{\{x_1 < D_1\}}, 1_{\{x_2 < D_2\}}) dC^M(P[M^1_{J^1(u)} \leq x_1], P[M^2_{J^2(v)} \leq x_2]) \\ &= \int_{-\infty}^{D_1} \int_{-\infty}^{D_2} dC^M(P[M^1_{J^1(u)} \leq x_1], P[M^2_{J^2(v)} \leq x_2]) \\ &= C^M(P[\tau_1 \leq J^1(u)], P[\tau_2 \leq J^2(v)]), \end{aligned}$$

where we have used the fact that $\{\tau_i \leq t\} = \{M^i_t \leq D_i\}$. Since J^i is the inverse of F^i, we have $P[\tau_i \leq J^i(u)] = u$ and the assertion follows. □

If the thresholds are observable, the dependence structure of the default times is that of the assets' running minima. This is consistent with a result derived by Frey & McNeil (2001) for general latent variable models. For given marginal default probabilities, they showed that the copula of the latent variables completely determines the joint default probability of several issuers. The latent variables can in our setup be interpreted as the assets' running minima. By the definition of the default time copula C^τ and by Proposition 3.7, if the thresholds are observable we get

$$P[\tau_1 \leq T_1, \ldots, \tau_n \leq T_n] = C^M(P[\tau_1 \leq T_1], \ldots, P[\tau_n \leq T_n]),$$

confirming the result of Frey & McNeil (2001).

Frey & McNeil (2001) considered latent variable models with constant and observable thresholds. In our model the thresholds are unobservable. Proposition 3.6 shows that in this general case C^τ is a function of both asset *and* threshold dependence structure, represented by C^M and $\bar{C}^D$. Then joint default probabilities critically depend upon the properties of both copulas:

$$\begin{aligned} P[\tau_1 \leq T_1, \ldots, \tau_n \leq T_n] = \int_{\mathbb{R}^n_-} &\bar{C}^D(1 - G^1(x_1), \ldots, 1 - G^n(x_n)) \\ &dC^M(P[M^1_{T_1} \leq x_1], \ldots, P[M^n_{T_n} \leq x_n]). \end{aligned}$$

All else being equal, copulas C^M implying low values in all margins simultaneously will lead to more joint defaults. Likewise, copulas $\bar{C}^D$ implying high threshold values in all margins simultaneously will cause more joint defaults. This is closely connected to the concept of tail dependence of copulas, which is introduced in Definition A.10 in the Appendix.

Frey & McNeil (2001) performed a simulation study for an exchangeable Merton (1974) type setup, where defaults can occur at some fixed horizon only (cf. the Introduction). The latent variables are in this case taken to be the joint normally distributed asset values at that date. They compared two asset copulas, the Gaussian and the t-copula (cf. Appendix). While the latter exhibits both upper and lower tail dependence, the Gaussian exhibits asymptotic independence. For a given asset correlation matrix, they examined for both copulas the distribution of the total number of defaults. The results show that for the t-copula the distribution has heavier tails than that implied by the Gaussian, meaning that joint defaults are more likely with the t-copula.

In our framework both asset running minima and threshold copula play a role. Although we have not performed a simulation study yet, it is very plausible that the properties of C^M and $\bar{C}^D$ are closely related to joint defaults, similarly to what Frey & McNeil (2001) report. In our case study in Chapter 5, we assume assets to be independent and focus on the effects the threshold copula has on joint defaults. There we choose the Clayton copula family, which exhibits lower tail dependence.

In the following we are concerned with the link between investors' prior and the default dependence structure in two particular cases.

Proposition 3.8. *The following holds true:*

(1) Assume that each $F_0^i(T)$ is continuous on $(0, \infty)$. Then the default times are independent iff $C^D = C^M = \Pi$.

(2) The default times are $\mathcal{F}$-conditionally independent iff $C^D = \Pi$.

Proof. (1) We consider the only the case $n = 2$. The general case is obvious. Assume that $C^D = C^M = \Pi$. By virtue of Lemma 3.4, $\bar{C}^D = \Pi$. From Proposition 3.6 we then find

$$\begin{aligned}
C^\tau(u, v) &= \int_{-\infty}^{0} (1 - G^1(x)) dP[M^1_{J^1(u)} \leq x]) \int_{-\infty}^{0} (1 - G^2(y)) dP[M^2_{J^2(v)} \leq y]) \\
&= P[D_1 > M^1_{J^1(u)}] \cdot P[D_2 > M^2_{J^2(v)}] \\
&= u \cdot v,
\end{aligned}$$

since $\{M_t^i \leq D_i\} = \{\tau_i \leq t\}$ and $P[\tau_i \leq J^i(u)] = u$. The converse is obvious.

(2) Let $T \geq \max_i T_i$. If $C^D = \Pi$, since $M_{T_i}^i \in \mathcal{F}_T$ we obtain

$$\begin{aligned} P[\tau_1 \leq T_1, \ldots, \tau_n \leq T_n \,|\, \mathcal{F}_T] &= P[M_{T_1}^1 \leq D_1, \ldots, M_{T_n}^n \leq D_n \,|\, \mathcal{F}_T] \\ &= P[M_{T_1}^1 \leq D_1 \,|\, \mathcal{F}_T] \cdots P[M_{T_n}^n \leq D_n \,|\, \mathcal{F}_T] \\ &= P[\tau_1 \leq T_1 \,|\, \mathcal{F}_T] \cdots P[\tau_n \leq T_n \,|\, \mathcal{F}_T], \end{aligned}$$

meaning that the τ_i are conditionally independent given $\mathcal{F}_T$. The converse is straightforward. □

Statement (2) has an interesting implication. From Proposition 3.7, in a structural default model with observable thresholds the complete range of default dependence can be induced by a suitable choice of the assets' running minima copula C^M. By a complete range we mean a dependence range from countermonotonicity to comonotonicity, cf. Definition A.8. If thresholds are unobservable and independent, Proposition 3.8 suggests that the range of achievable default time correlation is in fact limited: the τ_i are conditionally independent given the asset value paths. Under incomplete information, the full range of default dependence can only be achieved by admitting correlation between the thresholds. In our case study in Chapter 5, we will show that even in the absence of asset correlation the complete range of default dependence can be induced through a suitable choice of the threshold copula C^D. Recall that by the association of thresholds we model firm dependence on the micro-level, i.e. direct firm inter-linkages.

Chapter 4

The Term Structure of Credit Spreads

While in the previous Chapter 3 the focus was on the dependence between debt issuing firms, in this chapter we will concentrate on the informational asymmetries between equity holders and bond investors in the framework laid out in Chapter 2. We will examine the implications of incomplete information on the term structure of credit yield spreads on corporate debt (the excess over risk-free rates at which corporate bond prices are quoted in public markets).

The effect of incomplete information on credit spreads has been first studied in a one-firm structural model by Duffie & Lando (2001). They assumed that bond investors know the firm's default threshold but receive only imperfect asset information in form of noisy asset reports. In this setting the default is an unpredictable event and spreads are bounded away from zero, which is empirically plausible. Duffie & Lando (2001) showed that a default-arrival intensity exists and characterized it in terms of the conditional asset distribution. This proves that a structural model with incomplete asset information is consistent with an intensity based approach, in which corporate bond prices and credit spreads are parametrized through the intensity.

In our multi-firm structural model, defaults are correlated and bond investors' information can be even more limited. Investors cannot observe firms' default thresholds. The assets of firms may be observed perfectly, imperfectly or not at all. Does this setup also imply strictly positive spreads? What spread term structure shapes are possible? What role does the intensity play, and when does it exist? Led by these questions, we study the relation between probabilistic properties of the default time, intensity, and spreads in a general setup. We show that the short credit spread, i.e. the spread if the maturity is going to zero, is only determined by the default time properties. If a default is predictable, then short spreads are zero; if it is unpredictable, this is not necessarily the case. If in the latter case the default time admits an intensity, then short spreads are equal to this intensity. Since there is a one-to-one correspondence between default time properties and the *compensator* of a firm's default indicator process, spreads are in fact determined through the compensator. We therefore find that the compensator, which always uniquely exists, is the natural tool to study corporate bond prices and credit spreads. We characterize default probabilities, defaultable security prices, and credit spreads in terms of the compensator. This characterization holds for unpredictable defaults, irrespective of the existence of an intensity.

What do these findings imply for the modeling of default in general? In a structural approach, the information available to bond investors is closely related to the default time properties. With complete information, defaults are predictable and short spreads are zero. But with incomplete information on firms' assets and/or thresholds, defaults are unpredictable events and short spreads are non-negative. This is empirically plausible. It also indicates that a structural model with incomplete information unifies structural and intensity based approach to some extent. If an intensity does in fact exist, both approaches are integrated. In this case our compensator based characterization of bond prices collapses to the well known intensity based pricing relationships.

In our multi-firm default model of Chapter 2, defaults are correlated and investors receive only incomplete information on thresholds and assets. We prove a general representation of the compensator in this setup. If assets are perfectly observable, the compensator is determined by investors' a posteriori threshold belief only; it is independent of the asset distribution. We show that in this case a default-arrival intensity does not exist. Depending on the level of the asset value, short spreads are either zero as with complete information, or strictly positive. If assets are only imperfectly observed or not observed at all, the compensator representation involves besides investors' beliefs also the asset law. But in this case an intensity does in fact exist; it is implicitly characterized through the compensator. This provides an alternative approach to the intensity result of Duffie & Lando (2001) and extends their result to the multi-firm case with correlated defaults and unobservable default thresholds.

Our intensity characterization furnishes pure intensity based approaches to correlated default, which typically take intensity processes as exogenously given. In our model, default dependence across firms has two endogenously arising effects on intensities. Intensities are correlated through time, which justifies models where this is simply assumed, for example Duffie & Singleton (1998). On the other hand, a firm's intensity is subject to jumps upon defaults of directly-linked (micro-correlated) firms. A jump corresponds to the default clustering phenomenon: it reflects an immediate re-assessment of a firm's short term performance. This justifies the approach of Jarrow & Yu (2001), who modeled default dependence by assuming some functional intensity form which accommodates this characteristic jump pattern.

4.1 Compensator, Prices, and Credit Spreads

In this section we will introduce the notion of the compensator of the default process. We will then work out the relation between compensator, probabilistic properties of the default time, default probabilities, defaultable security prices and credit yield spreads.

4.1.1 The Default Compensator

We fix a probability space $(\Omega, \mathcal{G}, P)$ equipped with a filtration $(\mathcal{G}_t)_{t\geq 0}$ satisfying the usual conditions (see, for example, Brémaud (1980, III.5)). $\mathcal{G}_t$ is taken to be the information available to the bond investors at time t. The particular structure of $\mathcal{G}_t$ is not of interest in the general discussion in this section (bond investors' information may be complete or incomplete). It will also suffice to consider a single issuer only; we will hence drop the index i.

Let us start by recalling some formal concepts. A *stopping time* with respect to the filtration $(\mathcal{G}_t)_{t\geq 0}$ is a random variable τ valued in $[0, \infty]$ such that the event $\{\tau \leq t\}$ belongs to $\mathcal{G}_t$ for all $t \geq 0$. τ is called *predictable* if there is an increasing sequence of stopping times (T_n) such that $\tau > T_n$ and $\lim_n T_n = \tau$. Intuitively, one can foretell the phenomenon associated with τ by observing a succession of 'forerunners'. We also say that (T_n) announces τ. The stopping time τ is called *totally inaccessible* if $P[\tau = T < \infty] = 0$ for all predictable times T. Here an announcing sequence does not exist. An inaccessible event is the probabilistic concept of a completely unpredictable phenomenon.

The default time of our issuer is denoted by τ. We assume that τ is a stopping time, meaning that bond investors can observe the default of the issuer. We denote by $N := (N_t := 1_{\{t\geq\tau\}})_{t\geq 0}$ the adapted default indicator process. Clearly, N is a submartingale. The Doob-Meyer decomposition theorem states that there exists a unique, increasing, and predictable process $A := (A_t)_{t\geq 0}$ with $A_0 = 0$ and such that the difference process

$$N - A \tag{4.1}$$

is a $(\mathcal{G}_t)$-martingale. The process A is called the *compensator* of the one-jump point process N with respect to $(\mathcal{G}_t)$. Note that $A_t = A_{t\wedge\tau}$: A is stopped at τ. It is well known that

$$A \text{ continuous } \Leftrightarrow \tau \text{ totally inaccessible.}$$

If A is absolutely continuous with respect to Lebesgue measure, that is if

$$A_t = \int_0^t \lambda_s ds, \quad t \geq 0, \tag{4.2}$$

for some bounded progressively measurable process $\lambda := (\lambda_t)_{t\geq 0}$, then we say that τ admits the *intensity* λ. Thus if an intensity exists τ must be an inaccessible stopping time. On the other hand, τ being inaccessible is necessary but not sufficient for an intensity to exist. If the intensity is predictable, it is essentially unique. By using the martingale property of the process $N - A$, it follows from (4.2) that on $\{\tau > t\}$ the intensity satisfies

$$\lambda_t = \lim_{h\downarrow 0} \frac{1}{h} E[N_{t+h} - N_t \,|\, \mathcal{G}_t] = \lim_{h\downarrow 0} \frac{1}{h} P[\tau_i \in (t, t+h] \,|\, \mathcal{G}_t] \quad \text{a.s.} \tag{4.3}$$

Hence λ_t can be interpreted as the conditional event arrival rate at time t, given $\mathcal{G}_t$ and that $\tau > t$. For more details we refer to Brémaud (1980, Ch. 2.3).

As a submartingale, the default indicator process N tends to rise on average. The idea of compensation via A involves the counteraction of this tendency in a predictable way such that the residual process $N - A$ follows a martingale. In view of this concept, let us consider a simple default insurance contract that stipulates a payment of 1 upon a default. We clearly have $E[dN_t] = E[dA_t] = \lambda_t dt$, the second equality being valid if τ admits an intensity λ. Thus, λ can be interpreted as the continuously paid actuarial fair variable rate premium for the default contingent insurance payment. Since the fair risk premium corresponds to risk-neutral investors having linear utility functions, in our setting this fair premium is also the premium for the default insurance contract.

4.1.2 Valuation Using the Compensator

We will now characterize default probabilities and prices of defaultable securities in terms of the compensator. We start with a general result.

Proposition 4.1. *Let the default stopping time τ be totally inaccessible. Denote the compensator by A and let, for a fixed time T, X be some bounded $\mathcal{G}_T$-measurable random variable. If the process Y defined by*

$$Y_t := E[Xe^{A_t - A_T} \,|\, \mathcal{G}_t], \quad t \leq T,$$

is continuous at τ, then on the set $\{\tau > t\}$ we have a.s. that

$$E[X(1 - N_T) \,|\, \mathcal{G}_t] = E[Xe^{A_t - A_T} \,|\, \mathcal{G}_t], \quad t \leq T.$$

Proof. Letting $K_t := E[Xe^{-A_T} \,|\, \mathcal{G}_t]$, we can write $Y_t = e^{A_t} K_t$. Noting the continuity of A, by virtue of Itô's product rule we have

$$dY_t = e^{A_t} dK_t + Y_{t-} dA_t.$$

Denote by $\Delta Z_t := Z_t - Z_{t-}$ the jump of the process Z at t. Defining $U_t := (1 - N_t) Y_t$, we find again with the aid of the product rule that

$$\begin{aligned} dU_t &= -Y_{t-} dN_t + (1 - N_{t-}) dY_t + \Delta(1 - N_t)\Delta Y_t \\ &= (1 - N_{t-}) e^{A_t} dK_t - Y_{t-} d(N_t - A_t) - N_{t-} Y_{t-} dA_t, \end{aligned} \tag{4.4}$$

where we have used our assumption that Y is continuous at τ to set $\Delta(1 - N_t)\Delta Y_t = 0$. Now integration of both sides of (4.4) yields

$$U_T - U_t = \int_t^T (1 - N_{t-}) e^{A_t} dK_t - \int_t^T Y_{t-} d(N_t - A_t) - \int_t^T Y_{t-} N_{t-} dA_t. \tag{4.5}$$

Note that $(K_t)_{0 \leq t \leq T}$ and $N - A$ are martingales. Since the integrands are bounded and predictable, the first two terms of the right hand side of (4.5) are martingales. Using the fact that A is the compensator of N, we note that

$$\begin{aligned} E\left[\int_t^T Y_{t-} N_{t-} dA_t \,\middle|\, \mathcal{G}_t\right] &= E\left[\int_t^T Y_{t-} N_{t-} dN_t \,, \middle|\, \mathcal{G}_t\right] \\ &= E[1_{\{t < \tau \leq T\}} Y_{\tau-} N_{\tau-} \,|\, \mathcal{G}_t] \\ &= 0 \end{aligned}$$

because $N_{\tau-} = 1_{\{\tau < \tau\}} = 0$. Thus, taking conditional expectation of (4.5) yields

$$U_t = Y_t(1 - N_t) = E[U_T \,|\, \mathcal{G}_t] = E[X(1 - N_T) \,|\, \mathcal{G}_t],$$

which is our assertion. $\square$

Duffie, Schroder & Skiadas (1996) establish a similar result, which is however based on the intensity of τ. While they require the compensator of the stopping time τ to be absolutely continuous (cf. (4.2)), we presume continuity only. If an intensity does in fact exist, our result is that of Duffie et al. (1996). Elliott, Jeanblanc & Yor (2000) consider a similar problem. They however distinguish between some initial filtration and the one enlarged by the random variable τ.

The following result is a direct consequence of Proposition 4.1. It characterizes defaultable security prices in terms of the (continuous) compensator.

Proposition 4.2. *Consider a security promising a payoff at time T given by a random variable $X \in \mathcal{G}_T$. In the event of a default before T the security pays nothing. If τ is totally inaccessible and the process Y defined by $Y_t := E[Xe^{-\int_t^T r_s ds + A_t - A_T} \,|\, \mathcal{G}_t]$ is continuous at τ, then the value of this security is on $\{\tau > t\}$ a.s. given by*

$$E\left[e^{-\int_t^T r_s ds} X 1_{\{\tau > T\}} \,|\, \mathcal{G}_t\right] = E\left[X e^{-\int_t^T r_s ds + A_t - A_T} \,|\, \mathcal{G}_t\right], \quad T \geq t.$$

Observe that on the right hand side the default time does not appear any more. While this feature is very similar to valuation in the intensity based model, we emphasize that the existence of an intensity is not required. Given that the default time τ is totally inaccessible, the above result implies that the default probability of an issuer can be characterized in terms of A as

$$P[\tau \leq T \,|\, \mathcal{G}_t] = 1 - E[e^{A_t - A_T} \,|\, \mathcal{G}_t], \quad \tau > t, \quad T \geq t.$$

If the risk-neutrality assumption is relaxed, the valuation of contingent claims by the no-arbitrage argument requires a change of the objective probability measure P to some equivalent measure $\tilde{P} \approx P$ under which all discounted price processes are martingales (Harrison & Kreps (1979)). To apply Proposition 4.2 in this case, we need the compensator $\tilde{A}$ of τ with respect to $\tilde{P}$. We now provide a characterization of $\tilde{A}$ in terms of the P-compensator A. Our result is analogous to that of Artzner & Delbaen (1995, Appendix A1), who examine an intensity under a change of measure. If a P-intensity does in fact exist, both results are essentially equivalent.

Proposition 4.3. *Let $\tilde{P}$ be some probability measure equivalent to P and set $Z := d\tilde{P}/dP$. Define a martingale $(Z_t)_{t \geq 0}$ by $Z_t := E[Z \,|\, \mathcal{G}_t]$ and denote by*

$K_\tau = E[Z \,|\, \mathcal{G}_{\tau-}]$ the predictable projection of Z. If τ is totally inaccessible, then the $\tilde{P}$-compensator $\tilde{A}$ can in terms of the P-compensator A be written as

$$\tilde{A}_t = \int_0^t \frac{K_s}{Z_s} dA_s, \quad t \geq 0.$$

Proof. Clearly, $\tilde{A}$ is increasing, predictable, and satisfies $\tilde{A}_0 = 0$. If the process $N - \tilde{A}$ is a $\tilde{P}$-martingale, then $\tilde{A}$ is the $\tilde{P}$-compensator of τ. For all non-negative and predictable process C we have

$$\tilde{E}[C_\tau] = E[ZC_\tau] = E[E[Z \,|\, \mathcal{G}_{\tau-}]C_\tau] = E[K_\tau C_\tau].$$

Since KC is predictable and A is the P-compensator of τ ($N - A$ is a P-martingale and thus also $(\int_0^t L_s d(N_s - A_s))_{t \geq 0}$ for all non-negative and predictable L), we get

$$E[K_\tau C_\tau] = E\left[\int_0^\infty K_t C_t dN_t\right] = E\left[\int_0^\infty K_t C_t dA_t\right].$$

By Fubini's Theorem and the definition of Z_t,

$$E\left[\int_0^\infty K_t C_t dA_t\right] = \int_0^\infty E[K_t C_t] dA_t = \int_0^\infty \tilde{E}\left[\frac{K_t C_t}{Z_t}\right] dA_t.$$

Another application of Fubini's Theorem and the definition of $\tilde{A}$ yields

$$\int_0^\infty \tilde{E}\left[\frac{K_t C_t}{Z_t}\right] dA_t = \tilde{E}\left[\int_0^\infty \frac{K_t C_t}{Z_t} dA_t\right] = \tilde{E}\left[\int_0^\infty C_t d\tilde{A}_t\right].$$

We have thus shown that for all predictable C

$$\tilde{E}[C_\tau] = \tilde{E}\left[\int_0^\infty C_t dN_t\right] = \tilde{E}\left[\int_0^\infty C_t d\tilde{A}_t\right],$$

implying that the process $N - \tilde{A}$ is a $\tilde{P}$-martingale. □

Note that if a predictable P-intensity λ of τ exists, then the $\tilde{P}$-intensity $\tilde{\lambda}$ exists as well and is predictable. Choosing a càdlàg -version of the martingale $(Z_t)_{t \geq 0}$, by virtue of Proposition 4.3 we then have

$$\tilde{\lambda}_t = \frac{K_t}{Z_{t-}} \lambda_t,$$

which is the result of Artzner & Delbaen (1995).

4.1.3 Credit Spreads

Our next goal will be to relate the properties of the compensator to credit spreads of zero coupon bonds issued by the considered firm. Let us begin by recalling some terminology. The yield-to-maturity $y(t,T)$ at time t on a zero coupon bond with maturity date T traded at $B(t,T)$ is defined by

$$y(t,T) = -\frac{\ln B(t,T)}{T-t}, \quad T > t.$$

The price of a default-free zero bond with maturity T is at time t given by

$$d(t,T) = E[e^{-\int_t^T r_s ds} \,|\, \mathcal{G}_t], \quad T \geq t.$$

A credit-risky zero bond, with zero recovery in the event of default, has a price at time $t \leq T$ equal to

$$p(t,T) = E[e^{-\int_t^T r_s ds}(1 - N_T) \,|\, \mathcal{G}_t], \quad \tau > t.$$

The term structure of defaultable bond prices is then given by the schedule of $p(t,T)$ against T. The *credit yield spread* $S(t,T)$ is the difference between the yield on a credit risky zero bond and that on a credit risk-free zero bond:

$$S(t,T) := \frac{\ln d(t,T) - \ln p(t,T)}{T-t}, \quad T > t, \quad \tau > t. \tag{4.6}$$

If defaults are independent of riskless rates, this expression becomes

$$S(t,T) = -\frac{1}{T-t} \ln P[\tau > T \,|\, \mathcal{G}_t].$$

The term structure of credit yield spreads at t is the schedule of $S(t,T)$ against the horizon T.

In order to relate credit spread properties to that of default times, we will need the following general result.

Proposition 4.4. *Let τ be predictable with respect to $(\mathcal{G}_t)$. Defining $Z_n := \{k2^{-n} \,|\, k = 0, 1, \ldots\}$ for $n \geq 1$, on the set $\{\tau > t\}$ we have $P \times dt$ a.s. that*

$$\lim_{n \uparrow \infty} \sum_{t_i \in Z_n} \frac{1}{2^{-n}} P[\tau \leq t_{i+1} \,|\, \mathcal{G}_{t_i}] \, 1_{\{t_i < t \leq t_{i+1}\}} = 0.$$

Proof. Define the non-negative supermartingale M by $M_t := 1 - N_t = 1_{\{\tau > t\}}$. To M corresponds a unique finite measure P^M on the σ-field $\mathcal{P}$ of predictable sets in $\Omega \times (0, \infty]$ such that

$$P^M[B \times (t, \infty]] = E[M_t 1_B], \quad t \geq 0, \quad B \in \mathcal{G}_t,$$

cf. Föllmer (1972). Define furthermore a measure P^λ on $\mathcal{P}$ such that

$$P^\lambda[B \times (t,T]] = (T-t)P[B], \quad 0 \le t \le T, \quad B \in \mathcal{G}_t.$$

Note that on $\mathcal{P}$ the measure P^M has support $S := \{(\omega,t) \,|\, \tau(\omega) = t\}$. Since

$$P^\lambda[S] = \int P[d\omega] \int_0^\infty dt 1_B(\omega,t) = 0,$$

the measures P^M and P^λ are singular on $\mathcal{P}$. In a general semi-martingale setting, Airault & Föllmer (1974) introduced the Radon-Nikodym density dP^M/dP^λ of the absolutely continuous part of P^M with respect to P^λ. Now we have

$$\left.\frac{dP^M}{dP^\lambda}\right|_{\mathcal{P}} = 0.$$

The general results of Airault & Föllmer (1974) imply that the predictable density dP^M/dP^λ can be identified as

$$\frac{dP^M}{dP^\lambda}(\omega,t) = \lim_{n\uparrow\infty} \sum_{t_i \in Z_n} \frac{1}{2^{-n}} E[M_{t_i} - M_{t_{i+1}} \,|\, \mathcal{G}_{t_i}](\omega)\, 1_{\{t_i < t \le t_{i+1}\}} \quad P^\lambda \text{ - a.s.}$$

Since $E[M_{t_i} - M_{t_{i+1}} \,|\, \mathcal{G}_{t_i}] = P[\tau \le t_{i+1} \,|\, \mathcal{G}_{t_i}]\, 1_{\{\tau > t_i\}}$, our claim is proved. $\square$

Let us now consider the *short credit spread*

$$\lim_{T\downarrow t} S(t,T) = \frac{\partial}{\partial T} P[\tau \le T \,|\, \mathcal{G}_t]\Big|_{T=t}, \quad t < \tau. \tag{4.7}$$

The short spread is the excess yield over the risk-free yield demanded by bond investors for assuming the default risk of the bond issuer over the infinitesimal time period $(t, t+dt]$. Its relation to the probabilistic properties of the default time is elaborated in the following result.

Theorem 4.5. *Let the default time τ be a stopping time.*

(1) If τ is totally inaccessible then $\lim_{T\downarrow t} S(t,T) \ge 0$ on $\{\tau > t\}$. Assume that τ admits moreover a right-continuous and bounded intensity λ and that the process Y defined by $Y_t = E[e^{-\int_t^T \lambda_s ds} \,|\, \mathcal{G}_t]$ is continuous at τ. Then on $\{\tau > t\}$ short spreads satisfy

$$\lim_{T\downarrow t} S(t,T) = \lim_{T\downarrow t} \frac{1}{T-t}(A_T - A_t) = \lambda_t \quad a.s.$$

(2) Let τ be predictable. Defining $Z_n := \{k2^{-n} \,|\, k = 0, 1, \ldots\}$ for $n \geq 1$, then on $\{\tau > t\}$ short spreads satisfy

$$\lim_{n\uparrow\infty} \sum_{t_i \in Z_n} S(t_i, t_{i+1})\, 1_{\{t_i < t \leq t_{i+1}\}} = 0 \quad a.s.$$

Proof. (1) That the spread is non-negative follows trivially from its definition (4.6). If τ is totally inaccessible, then we can apply Proposition 4.1 to (4.7) to see that on the set $\{\tau > t\}$ a.s.

$$\lim_{T\downarrow t} S(t,T) = -\frac{\partial}{\partial T} E[e^{A_t - A_T} \,|\, \mathcal{G}_t]\Big|_{T=t}$$

By dominated convergence and the fact that $A_t = \int_0^t \lambda_s ds$, we have

$$\begin{aligned}
\lim_{T\downarrow t} S(t,T) &= -E[\frac{\partial}{\partial T}\, e^{A_t - A_T} \,|\, \mathcal{G}_t]\Big|_{T=t} \\
&= E[\lambda_T\, e^{A_t - A_T} \,|\, \mathcal{G}_t]\Big|_{T=t} \\
&= \lambda_t,
\end{aligned}$$

which completes the proof of statement (1).

Now consider statement (2). If τ is predictable, N is predictable as well and its decomposition is trivial: the compensator of N is $A = N$ itself. A is thus not absolutely continuous. Using the definition of the spread, on $\{\tau > t\}$ we have a.s.

$$\begin{aligned}
&\lim_{n\uparrow\infty} \sum_{t_i \in Z_n} S(t_i, t_{i+1})\, 1_{\{t_i < t \leq t_{i+1}\}} \\
&\quad = -\lim_{n\uparrow\infty} \sum_{t_i \in Z_n} \frac{1}{t_{i+1} - t_i} (\ln P[\tau > t_{i+1} \,|\, \mathcal{G}_{t_i}])\, 1_{\{t_i < t \leq t_{i+1}\}} \\
&\quad = \lim_{n\uparrow\infty} \sum_{t_i \in Z_n} \frac{1}{2^{-n}} \frac{1}{P[\tau > t_{i+1} \,|\, \mathcal{G}_{t_i}]}\Big|_{t_{i+1}=t_i} P[\tau \leq t_{i+1} \,|\, \mathcal{G}_{t_i}]\, 1_{\{t_i < t \leq t_{i+1}\}} \\
&\quad = \lim_{n\uparrow\infty} \sum_{t_i \in Z_n} \frac{1}{2^{-n}} P[\tau \leq t_{i+1} \,|\, \mathcal{G}_{t_i}]\, 1_{\{t_i < t \leq t_{i+1}\}},
\end{aligned}$$

which is zero by Proposition 4.4. □

This result is quite remarkable, as it shows that the properties of the short spread derive *only* from the probabilistic properties of the default time. Since there is a one-to-one correspondence between default time and compensator properties, we can also say that the spread is determined by the compensator. How a default event is constructed plays a role only insofar as it determines the default time properties. This concerns in particular the asset process. By direct calculation (cf. (4.9) below), Duffie & Lando (2001) argue that short spreads are zero for a default being defined by first hitting of a Brownian motion to some constant boundary. Theorem 4.5 proves that this is in fact due to the default time being predictable in this case. In any default model where the default time is predictable, the short spread is zero.

Empirical studies indicate that a default time is inaccessible rather than predictable. Sarig & Warga (1989) and Helwege & Turner (1999) find that credit spreads remain in general bounded away from zero. Also, we observe jumps in bond prices at or around the bankruptcy announcement. If the default were a predictable event, prices would converge continuously to their default-contingent values; there would be no sudden drop in the value upon default.

4.2 Implications for Default Modeling

In the previous section we have worked out the relation between the default time, its compensator, its intensity, and credit yield spreads. These results have significant implications for the modeling of default, both in general and in particular for our model of Chapter 2.

On the basis of Theorem 4.5, we can distinguish the essence of the two default-modeling paradigms. In the *intensity based* approach, one starts right away by assuming that the default time is totally inaccessible and admits an intensity λ. By virtue of Theorem 4.5 (1), λ_t then constitutes the short spread at time t. By construction, short spreads are not generally zero. Also, Proposition 4.2 with $A_t = \int_0^{t\wedge\tau} \lambda_s ds$ can be applied to value defaultable securities. If the process Y defined by $Y_t := E[e^{-\int_t^T (r_s+\lambda_s)ds} \,|\, \mathcal{G}_t]$ is continuous at τ, then for $t < \tau$ we have

$$E[e^{-\int_t^T r_s ds} 1_{\{\tau>T\}} \,|\, \mathcal{G}_t] = E[e^{-\int_t^T (r_s+\lambda_s)ds} \,|\, \mathcal{G}_t], \quad t \leq T. \tag{4.8}$$

This is the value of a defaultable zero-recovery zero coupon bond with maturity T. Observe that on the right hand side the default time does not appear any more. In the intensity based framework, the problem of valuing a defaultable bond is therefore reduced to that of an ordinary non-defaultable bond, which is well understood and for which tractable methods exist.

In the intensity based approach the default time is typically taken as exogenously given. This avoids the need to specify economic reasons why a firm defaults. In the *structural* approach, the default event is defined as the first time the firm's asset process hits some lower threshold. This makes sense from an economic point of view, because the asset process is a sufficient statistic for the firm's future cash flows. The resulting probabilistic properties of the default time and thus the short spread properties vary with the available information.

In a structural approach to default the information between corporate claimants is typically symmetric: the secondary market has *complete* information. In our structural framework laid out in Chapter 2, this would mean that bond investors can observe for each firm asset value, default threshold, and default event; their information at time t would be given by $\mathcal{G}_t = \mathcal{H}_t$. In this case investors would always be certain about the distance of the firm to default, i.e. the nearness of the asset value to the default threshold. Then, for shareholders and for bond investors alike, a default would not come as a surprise. Indeed, given the continuity of the asset process, for all $t \geq 0$ we would get

$$\{\tau_i(\omega) \leq t\} = \{\inf_{0 \leq s \leq t} V_s^i(\omega) \leq D_i\} = \{\lim_{n \uparrow \infty} \inf_{0 \leq s \leq t - n^{-1}} V_s^i(\omega) \leq D_i\} \in \mathcal{G}_{t-},$$

meaning that the default times are predictable with respect to $(\mathcal{G}_t)_{t \geq 0}$. Hence complete information in a structural model with continuous asset processes has two consequences. First, since N is predictable, its decomposition is trivial. The compensator of N is $A = N$ itself and an intensity does not exist. Since A is discontinuous, Proposition 4.2 cannot be used for the valuation of defaultable securities. Second, by Theorem 4.5 (2), credit spreads go to zero with maturity going to zero. To give an example, assume that a firm's asset value follows a Brownian motion and that firms are independent. This is the classical setup proposed by Merton (1974). From (3.11) and the Markov property, in this case we have $P[\tau_i \leq T \,|\, \mathcal{G}_t] = P[M_{T-t}^i \leq D_i - V_t^i]$. Since the distribution of M_t^i is explicitly known, cf. (5.12), we can verify that on $\{\tau > t\}$ the spread satisfies

$$\lim_{T \downarrow t} S_i(t,T) = -\lim_{T \downarrow t} \frac{1}{T-t} \ln P[M_{T-t}^i > D_i - V_t^i] = 0 \quad \text{a.s.} \tag{4.9}$$

Figure 4.1 shows the term structure of credit spreads in this situation (the parameter are those of Section 5). Indeed, spreads are zero for maturities up to approximately 2 months. In a structural model with complete information and continuous asset processes, investors do not demand a default risk premium on

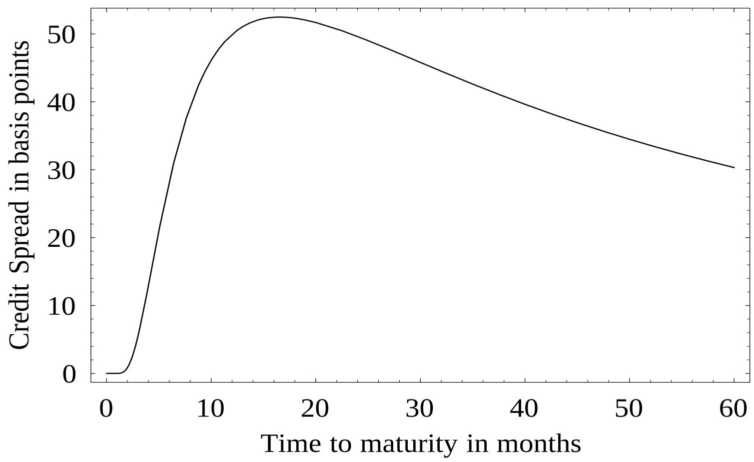

Figure 4.1: Credit spreads with completely informed bond investors.

zero coupon debt whose maturity approaches zero. But this is not supported by empirical findings.

The probabilistic properties of the default time and thus the properties of the spread change if bond investors have only *incomplete information* on the characteristics of firms. This is the situation in our structural model proposed in Chapter 2, where we suppose that $\sigma(\tau \wedge t) \subseteq \mathcal{G}_t \subseteq \sigma(\tau \wedge t) \vee \mathcal{F}_t \subset \mathcal{H}_t$. Assuming additionally that the prior satisfies $W < C^D < M$, we have that each τ_i is a totally inaccessible $(\mathcal{G}_t)$-stopping time. The assumption on the prior can be dropped in case assets are imperfectly observed. If assets are perfectly observable, it is necessary to prevent the τ_i from becoming predictable. To see this, suppose that thresholds have the same distribution and are perfectly dependent. Then the first default reveals the thresholds of all other firms, cf. (A.2). The resulting situation is equivalent to that with perfect information.

The inaccessibility of the τ_i has two consequences. First, by Theorem 4.5 (1), short credit spreads are not necessarily zero as with complete information. Second, Proposition 4.2 can be applied to value defaultable securities in our framework. As for the default probability $F_t^i(T)$, we find

$$F_t^i(T) = 1 - E[e^{A_t^i - A_T^i} \,|\, \mathcal{G}_t], \quad \tau_i > t, \quad T \geq t, \tag{4.10}$$

given that the process Y defined by $Y_t := E[e^{-A_T^i} \,|\, \mathcal{G}_t]$ is continuous at τ_i. Compare this to our earlier characterization (3.10) and (3.11) in terms of the a posteriori threshold belief of the bond investors. (4.10) can be used directly to give a new representation of the positive-recovery zero bond prices derived in Section 3.3.

To sum up, a structural model based on incomplete information shares with an intensity based model the property of inaccessible default times; it is therefore empirically plausible. *Irrespective* of the existence of an intensity, the compensator based Proposition 4.2 can be used to value defaultable securities. If an intensity does in fact exist, an application of this result leads to the familiar valuation formulas. We will see in the next section that the existence of an intensity in our model critically depends upon the extent of available information.

4.3 Characterizing the Compensator

We have demonstrated that the compensator of a default time characterizes credit spreads, and if it is continuous, default probabilities and prices of defaultable securities. In this section we will prove a representation of the compensator for our structural multi-firm default model. Though the compensator exists in any case, we will see that an intensity exists only in information Models B and C. This supports our compensator based approach to default probabilities and bond prices (Proposition 4.2), rather than an approach based on intensities only. Where it exists, the intensity is characterized through the compensator, cf. (4.2). The intensity characterization can then furnish all those pure intensity based models in which intensities are taken as exogenously given.

4.3.1 The Intensity Result of Duffie & Lando

Duffie & Lando (2001) were the first who established the existence of a default intensity in a one-firm structural model with incomplete information on the firm's assets. Specifically, they assumed that assets V follow a Brownian motion and that the default time is given by $\tau := \inf\{t : V_t \leq d\}$ for some *constant* d. Bond investors receive noisy asset reports at discrete dates t_k. Their information $(\mathcal{G}_t)_{t\geq 0}$ is defined by $\mathcal{G}_t := \sigma(\tau \wedge t) \vee \sigma((V_{t_k} + U_{t_k})1_{\{t \geq t_k\}})$ for some independent noise random variable U_{t_k}, cf. our Model B. Duffie & Lando (2001) derived the $\mathcal{G}_t$-conditional density $\xi(t, \cdot)$ of V_t in terms of the normal distribution function, cf. (3.19) and our alternative characterization (3.18). $\xi(t, x, \omega)$ is continuously differentiable at d for each (t, ω) with $\xi(t, d, \omega)$=0. They then approached the problem of deriving an intensity λ for τ via (4.3). Taking the derivative of ξ from the right, they showed that under some assumptions on the asset density $\xi(t, \cdot)$,

$$\lambda_t = \lim_{h \downarrow 0} \frac{1}{h} P[\tau \in (t, t+h] \,|\, \mathcal{G}_t] = \frac{1}{2}\sigma^2 \xi_x(t, d), \quad 0 < t \leq \tau, \tag{4.11}$$

where σ is the asset volatility. Elliott et al. (2000) provided an alternative proof of this intensity result. Observe that the intensity at t is proportional to the slope of the conditional asset density $\xi(t, d)$. The compensator A of the default time can in their model be written as

$$A_t = \frac{1}{2}\sigma^2 \int_0^{t \wedge \tau} \xi_x(s, d) ds, \quad t \geq 0. \tag{4.12}$$

Since the only source of uncertainty is the imperfectly observed firm value itself, the compensator is in this case determined by the underlying asset process only. Using the fact that the intensity does not depend on the asset's drift, Duffie & Lando (2001) extend to the case where the asset value solves the SDE $dV_t = \mu(V_t, t)dt + \sigma(V_t, t)dW_t$ for μ and σ satisfying technical conditions. Then the intensity is for $0 < t < \tau$ given by $\lambda_t = \frac{1}{2}\sigma^2(d, t)\, \xi_x(t, d)$.

4.3.2 Perfect Observation of Issuers' Assets

Let us now consider the framework of Chapter 2. We start with information Model A, where issuers' assets are perfectly observable, but where investors have no information on the default thresholds of the firms at all. In Subsection 4.3.3 below, we will extend to situations where information on assets is imperfect (Models B and C). We suppose that

$$\mathcal{G}_t := \mathcal{F}_t \vee \sigma(\tau \wedge t).$$

If we assume additionally that the prior satisfies $W < C^D < M$, then each τ_i is a totally inaccessible $(\mathcal{G}_t)$-stopping time. Since the default process N^i is not predictable, its continuous compensator A^i is non-trivial. Our goal for the remainder of this subsection is to characterize A^i. To this end, let us assume that there is some auxiliary filtration $(\mathcal{G}_t^i)_{t \geq 0}$ such that $(\mathcal{G}_t)_{t \geq 0}$ is the progressive enlargement of $(\mathcal{G}_t^i)_{t \geq 0}$ with the random variable τ_i. That is, $\mathcal{G}_t^i$ contains the asset information up to time t and survivorship information on all bonds except the i-th up to time t. We thus set

$$\mathcal{G}_t^i := \mathcal{F}_t \vee \sigma(\tau_j \wedge t, j \in I - \{i\}),$$

so that $\mathcal{G}_t = \mathcal{G}_t^i \vee \sigma(\tau_i \wedge t)$. Now we assume that D_i has a regular $\mathcal{G}_t^i$-conditional distribution function H_t^i. We also assume that H_t^i admits a density h_t^i. H_t^i is related to the a posteriori threshold distribution G_t^i derived in Proposition 3.1 via

$$\begin{aligned} G_t^i(x) &= P[D_i \le x \,|\, \mathcal{G}_t] \\ &= N_t^i 1_{\{M^i_{\tau_i} \le x\}} + (1 - N_t^i)(1_{\{M^i_t \le x\}} + \frac{H_t^i(x)}{H_t^i(M_t^i)} 1_{\{M^i_t > x\}}). \end{aligned}$$

Using the fact that $H_t^i(M_t^i) = P[\tau_i > t \,|\, \mathcal{G}_t^i]$, we have also

$$\begin{aligned} F_t^i(T) &= P[\tau_i \le T \,|\, \mathcal{G}_t] \\ &= N_t^i + (1 - N_t^i) E[H_t^i(M_t^i) - H_T^i(M_T^i) \,|\, \mathcal{G}_t^i]. \end{aligned}$$

H_t^i can be calculated from the given threshold prior G by applying a similar argument as in Proposition 3.1. To give an example, fix some time t and suppose that $S_t = \emptyset$, i.e. no firm has defaulted by t. By Bayes' Theorem and the independence of D and $\mathcal{F}_t$ we get

$$H_t^i(x) = \frac{G(M_t^1, \ldots, M_t^{i-1}, x, M_t^{i+1}, \ldots, M_t^n)}{G(M_t^1, \ldots, M_t^{i-1}, 0, M_t^{i+1}, \ldots, M_t^n)}.$$

In order to prove our main result, the following lemma is needed. Its proof is analogous to the proof of Lemma 2.4 in Rutkowski (2000).

Lemma 4.6. *For all bounded and $(\mathcal{G}_t^i)$-adapted processes Z we have*

$$E[1_{\{t < \tau_i \le s\}} Z_{\tau_i} \,|\, \mathcal{G}_t^i] = E\left[- \int_t^s Z_u dH_u^i(M_u^i) \,\middle|\, \mathcal{G}_t^i \right], \quad s \ge t.$$

Note that the process $(H_t^i(M_t^i))_{t \ge 0}$, where $H_t^i(M_t^i) = P[D_i < M_t^i \,|\, \mathcal{G}_t^i] = P[\tau_i > t \,|\, \mathcal{G}_t^i]$, is a $(\mathcal{G}_t^i)$-supermartingale. Thus there is a unique increasing and predictable process K^i called the $(\mathcal{G}_t^i)$-compensator of $H^i(M^i)$ such that $K^i + H^i(M^i)$ is a $(\mathcal{G}_t^i)$-martingale. We are now ready to characterize the default compensator given incomplete information of the bond investors on firms' default thresholds as well as correlation between firm defaults.

Theorem 4.7. *In information model A, the $(\mathcal{G}_t)$-compensator A^i is given by*

$$A_t^i = \int_0^{t \wedge \tau_i} \frac{dK_s^i}{H_{s-}^i(M_s^i)}, \qquad t \ge 0.$$

Proof. Clearly, A^i is increasing, predictable, and satisfies $A_0^i = 0$. It remains to show that the process $N^i - A^i$ is a martingale. We thus check that

$$E[A_s^i - A_t^i \,|\, \mathcal{G}_t] = E[N_s^i - N_t^i \,|\, \mathcal{G}_t], \quad s \ge t.$$

We start with the right hand side:

$$
\begin{aligned}
E[N_s^i - N_t^i \,|\, \mathcal{G}_t] &= E[N_s^i - N_t^i \,|\, \mathcal{G}_t^i \vee \sigma(\tau_i \wedge t)] \\
&= \frac{(1-N_t^i)}{P[\tau_i > t \,|\, \mathcal{G}_t^i]} E[(N_s^i - N_t^i)(1-N_t^i) \,|\, \mathcal{G}_t^i] \\
&= \frac{(1-N_t^i)}{H_t^i(M_t^i)} E[N_s^i - N_t^i \,|\, \mathcal{G}_t^i] \\
&= \frac{(1-N_t^i)}{H_t^i(M_t^i)} E[H_t^i(M_t^i) - H_s^i(M_s^i) \,|\, \mathcal{G}_t^i] \qquad (4.13) \\
&= \frac{(1-N_t^i)}{H_t^i(M_t^i)} E[K_s^i - K_t^i \,|\, \mathcal{G}_t^i]. \qquad (4.14)
\end{aligned}
$$

On the set $\{\tau_i \leq t\}$ the left hand side can be written as

$$
\begin{aligned}
E[A_s^i - A_t^i \,|\, \mathcal{G}_t] &= (1-N_t^i) E\left[\int_t^{s\wedge\tau_i} \frac{dK_s^i}{H_{s-}^i(M_s^i)} \,\middle|\, \mathcal{G}_t^i \vee \sigma(\tau_i \wedge t)\right] \\
&= \frac{(1-N_t^i)}{H_t^i(M_t^i)} E\left[\int_t^{s\wedge\tau_i} \frac{dK_s^i}{H_{s-}^i(M_s^i)} \,\middle|\, \mathcal{G}_t^i\right]. \qquad (4.15)
\end{aligned}
$$

Now we calculate

$$
\begin{aligned}
& E\left[\int_t^{s\wedge\tau_i} \frac{dK_s^i}{H_{s-}^i(M_s^i)} \,\middle|\, \mathcal{G}_t^i\right] \\
&= E\left[(1-N_s^i)\int_t^{s} \frac{dK_s^i}{H_{s-}^i(M_s^i)} + (N_s^i - N_t^i)\int_t^{s\wedge\tau_i} \frac{dK_s^i}{H_{s-}^i(M_s^i)} \,\middle|\, \mathcal{G}_t^i\right] \\
&= E\left[H_s^i(M_s^i)\int_t^{s} \frac{dK_s^i}{H_{s-}^i(M_s^i)} + (N_s^i - N_t^i)\int_t^{s\wedge\tau_i} \frac{dK_s^i}{H_{s-}^i(M_s^i)} \,\middle|\, \mathcal{G}_t^i\right] \\
&= E\left[H_s^i(M_s^i)(A_s^i - A_t^i) + \int_t^{s}\int_t^{u\wedge\tau_i} \frac{dK_v^i}{H_{v-}^i(M_v^i)} dK_u^i \,\middle|\, \mathcal{G}_t^i\right] \\
&= E\left[H_s^i(M_s^i)(A_s^i - A_t^i) - \int_t^{s} A_u^i dH_u^i(M_u^i) + A_t^i[H_s^i(M_s^i) - H_t^i(M_t^i)] \,\middle|\, \mathcal{G}_t^i\right],
\end{aligned}
$$

where for the third equality we have used Lemma 4.6. Noting that K^i is a process of bounded variation, by virtue of the product formula we get

$$
\begin{aligned}
\int_t^s A_u^i dH_u^i(M_u^i) &= A_s^i H_s^i(M_s^i) - A_t^i H_t^i(M_t^i) - \int_t^s H_{u-}^i(M_u^i) dA_u^i \\
&= A_s^i H_s^i(M_s^i) - A_t^i H_t^i(M_t^i) - K_s^i + K_t^i.
\end{aligned}
$$

Collecting and canceling terms yields the equality

$$E[A_s^i - A_t^i \,|\, \mathcal{G}_t] = \frac{(1-N_t^i)}{H_t^i(M_t^i)} E[K_s^i - K_t^i \,|\, \mathcal{G}_t^i],$$

which completes the proof. □

The law of the asset process does not enter the characterization of A^i, i.e. the compensator does not depend on the choice of the underlying asset process. Rather, A^i is determined by the a posteriori threshold beliefs of the bond investors. Compare this to the structure of the compensator (4.12) in the model of Duffie & Lando (2001). We should bear in mind that default dependence enters the distribution H_t^i via the conditioning information $\mathcal{G}_t^i$, which contains the default status of all firms. Thus, A^i as firm i's default compensator reflects also the default correlation with all other firms. If the thresholds are independent ($C^D = \Pi$), we have that $H_t^i(x) = P[D_i \leq x] = G^i(x)$ for all t, where G^i is the prior. Hence $H_t^i(M_t^i) = G^i(M_t^i)$ is continuous and decreasing in t. Therefore $K_t^i = 1 - G^i(M_t^i)$ and $dK_t^i = -g^i(M_t^i)dM_t^i$, using the fact that the quadratic variation of M^i is zero. Now the compensator simplifies to

$$A_t^i = -\int_0^{t\wedge\tau_i} \frac{g^i(M_s^i)}{G^i(M_s^i)} dM_s^i, \qquad t \geq 0.$$

With threshold independence the compensator A^i is equivalent to that in the one-firm case where $I = \{i\}$.

But what can we say about the intensity of τ_i? Because the Lebesgue measure of the set

$$\{t \geq 0 : V_t^i = M_t^i\}$$

is zero, the continuous process M^i is not absolutely continuous with respect to Lebesgue measure. It follows that the compensator A^i is not absolutely continuous with respect to Lebesgue measure either. Thus an intensity does not exist if bond investors' information is given by Model A and asset follow some continuous process.

We can also use the approach of Duffie & Lando (2001) to explain why an intensity does not exist. Although their informational setup is in direct contrast to ours, the formal problem of establishing an intensity is seen to be very similar to ours. Approaching the problem like Duffie & Lando (2001) via (4.3) and substituting the default probability from (3.11), we conjecture that the process Λ^i defined by

$$\Lambda_t^i := \lim_{h \downarrow 0} \frac{1}{h} F_t^i(t+h) = \lim_{h \downarrow 0} \frac{1}{h} \int_{-\infty}^{V_t^i} P[M_{t+h}^i \leq x \,|\, \mathcal{G}_t] g_t^i(x) dx$$

is for $t < \tau$ the default intensity in our case. If assets are assumed to follow a standard Brownian motion, $P[M_{t+h}^i \leq x \,|\, \mathcal{G}_t] = P[M_h^i \leq x - V_t^i]$ is available explicitly, cf. (5.12). Calculating the limit under the condition that $g_t^i(x)$ is smooth at $x = V_t^i$ on the set $\{V_t^i = M_t^i\}$, we obtain

$$\Lambda_t^i = -\frac{1}{2}\sigma_i^2 \frac{\partial}{\partial x} g_t^i(V_t^i), \quad 0 \leq t < \tau_i, \tag{4.16}$$

where we take the derivative of the $\mathcal{G}_t$-conditional density g_t^i of D_i to be from the left. While this looks very similar to the intensity (4.11) of Duffie & Lando (2001), Λ^i involves the threshold density $g_t^i(V_t^i)$ at the asset value V_t^i rather than the asset density $\xi(d,t)$ at the constant threshold d. However, (4.16) is only valid if $g_t^i(x)$ is smooth at $x = V_t^i$ on the set $\{V_t^i = M_t^i\}$. If this does not hold, $\Lambda_t^i = \infty$ for all t. The density $g_t^i(x)$ is the result of Bayesian updating; its smoothness is not natural in our setting. Given asset values are observable, on $\{\tau_i > t\}$ investors know that $D_i < M_t^i$, and hence $g_t^i(x) = 0$ for $x \geq M_t^i$. Thus, with $g_t^i(x) > 0$ for $x < M_t^i$, the density is not even continuous at $x = V_t^i$, cf. our discussion after Proposition (3.1). Hence in our setting we have $\Lambda_t^i = \infty$ for all t, and an intensity does not exist.

What can we say about the resulting credit spread term structure properties? By Theorem 4.5 (1), the spread $S_i(t,T)$ remains non-negative for $T \to t$. But we can say even more: on the set $\{V_t^i = M_t^i\}$, when the asset value is 'at a new low', $S_i(t,T) > 0$ for $T \to t$. On the other hand, on $\{V_t^i > M_t^i\}$, we have $S_i(t,T) = 0$ for $T \to t$, despite the fact that τ_i is inaccessible. This means that, depending on the firm's current asset value, the credit spread term structure with incomplete information (Model A) can be very similar to the one which appears in the case of complete information. This property corresponds to the fact that, whenever assets $V_t^i > M_t^i$ and $\tau_i > t$, bondholders are, loosely spoken, in a relieved position which is similar to that with complete information. Having incomplete information $\mathcal{G}_t$ allows to deduce that $D_i < M_t^i$; hence the firm cannot default immediately if the assets V^i follow a continuous process. This results in zero short term default probabilities and thus zero spreads. If in contrast $V_t^i = M_t^i$, then the firm can default immediately and short term default probabilities are strictly positive.

Credit spreads in Model A are further exemplified in our case study in Chapter 5. We will show that, depending on the current asset value, incomplete information and default dependence can lead to *decreasing and hump shaped* term structures of credit spreads.

4.3.3 Imperfect Observation of Issuers' Assets

In this subsection we examine the default compensator when issuers' assets can only be imperfectly observed, as is the case in information Models B and C. Specifically, we suppose that bond investors' filtration $(\mathcal{G}_t)$ satisfies

$$\sigma(\tau \wedge t) \subseteq \mathcal{G}_t \subset \sigma(\tau \wedge t) \vee \mathcal{F}_t,$$

implying that each τ_i is totally inaccessible.

The following general result will allow us to exploit the compensator characterization for Model A also in the current situation.

Proposition 4.8. *Let τ be some $(\mathcal{G}_t)$-stopping time with $(\mathcal{G}_t)$-compensator A (we again drop the index i). Let $(\hat{\mathcal{G}}_t)_{t\geq 0}$ be a filtration satisfying $\sigma(\tau \wedge t) \subseteq \hat{\mathcal{G}}_t \subseteq \mathcal{G}_t$ and let $\hat{A}$ denote an increasing right-continuous version of the process*

$$\hat{A}_t := E[A_t \,|\, \hat{\mathcal{G}}_t], \qquad t \geq 0.$$

If $\hat{A}$ is $(\hat{\mathcal{G}}_t)$-predictable, then $\hat{A}$ is the $(\hat{\mathcal{G}}_t)$-compensator of τ.

PROOF. $\hat{A}$ is increasing, predictable, and satisfies $\hat{A}_0 = 0$. For $s \geq t$ we have

$$E[N_s - \hat{A}_s \,|\, \hat{\mathcal{G}}_t] = E[N_s - E[\hat{A}_s \,|\, \hat{\mathcal{G}}_s] \,|\, \hat{\mathcal{G}}_t] = E[N_s - A_s \,|\, \hat{\mathcal{G}}_t].$$

Since $N - A$ is a $(\mathcal{G}_t)$-martingale, we get

$$E[N_s - A_s \,|\, \hat{\mathcal{G}}_t] = E[E[N_s - A_s \,|\, \mathcal{G}_t] \,|\, \hat{\mathcal{G}}_t] = E[N_t - A_t \,|\, \hat{\mathcal{G}}_t] = N_t - \hat{A}_t,$$

showing that the process $N - \hat{A}$ is a $(\hat{\mathcal{G}}_t)$-martingale. $\square$

We are now ready to give a characterization of the default compensator given incomplete information on firms' default thresholds *and* assets. This representation applies for both information Models B and C.

Theorem 4.9. *Let A^i denote an increasing right-continuous version of the process*

$$A^i_t = E\left[\int_0^{t\wedge\tau_i} \frac{dK^i_s}{H^i_{s-}(M^i_s)} \,\Big|\, \mathcal{G}_t\right], \qquad t \geq 0.$$

If A^i is $(\mathcal{G}_t)$-predictable, then A^i is the $(\mathcal{G}_t)$-compensator of τ_i under our current informational assumptions.

Proof. Since the current filtration $(\mathcal{G}_t)$ is smaller than the information filtration in Model A where assets are observable, the statement follows by virtue of Proposition 4.8 from the compensator in Model A, which is characterized in Theorem 4.7. $\square$

If assets are not perfectly observable, the compensator depends on the law of the asset process. Whether an intensity λ_i exists is not obvious from the compensator characterization given in Theorem 4.9. Let us look at the intensity from a different perspective. Define the processes $(m^i(t,x))_{t\geq 0}$ by

$$m^i(t,x) := \lim_{h\downarrow 0} \frac{1}{h} P[M^i_{t+h} \leq x \,|\, \mathcal{G}_t, D_i = x] \quad \text{on} \quad \{M^i_t > D_i\},$$

for $x \leq 0$. For Models B and C, the $\mathcal{G}_t$-conditional distribution of M^i_{t+h} is derived in (3.15) and (3.20). If assets follow a Brownian motion, then $m^i(t,x)$ is well-defined and can be computed explicitly, cf. Proposition 4.11 below. Assume that the threshold D_i admits a $\mathcal{G}_t$-conditional density g^i_t, i.e.

$$g^i_t(x)dx = P[D_i \in dx \,|\, \mathcal{G}_t], \qquad x \leq 0.$$

The g^i_t can for information models B and C be obtained from Propositions 3.2 and 3.3.

Proposition 4.10. *Assume that the processes $(m^i(t,x))_{t\geq 0}$ and $(g^i_t(x))_{t\geq 0}$ are well-defined and satisfy the following conditions:*

(1) For almost every ω and all t, $m^i(t,x,\omega)$ and $g^i_t(x,\omega)$ are bounded.

(2) For all fixed $x \leq 0$, the processes $(m^i(t,x))_{t\geq 0}$ and $(g^i_t(x))_{t\geq 0}$ are progressively measurable.

Assume also that for almost every ω, for all (t,x,h) $|P[M^i_{t+h} \leq x \,|\, \mathcal{G}_t, D_i = x](\omega)|$ has an integrable upper bound. Then τ_i admits on the set $\{\tau_i > t\}$ the $(\mathcal{G}_t)$-intensity

$$\lambda^i_t = \int_{-\infty}^0 m^i(t,x)\, g^i_t(x)\, dx.$$

Proof. From (4.3), on $\{\tau_i > t\}$ we have a.s.

$$\begin{aligned}
\lambda^i_t &= \lim_{h\downarrow 0} \frac{1}{h} P[\tau_i \leq t+h \,|\, \mathcal{G}_t] \\
&= \lim_{h\downarrow 0} \frac{1}{h} P[M^i_{t+h} \leq D_i \,|\, \mathcal{G}_t] \\
&= \lim_{h\downarrow 0} \frac{1}{h} \int_{-\infty}^0 P[M^i_{t+h} \leq x \,|\, \mathcal{G}_t, D_i = x]\, P[D_i \in dx \,|\, \mathcal{G}_t],
\end{aligned}$$

so that the claim follows by dominated convergence. $\square$

This result shows that we can write for the compensator $A^i_t = \int_0^{\tau_i \wedge t} \lambda^i_s ds$. We hence have that

$$E\left[\int_0^{t\wedge\tau_i} \frac{dK^i_s}{H^i_{s-}(M^i_s)} \,\Big|\, \mathcal{G}_t\right] = \int_0^{\tau_i \wedge t} \int_{-\infty}^0 m^i(s,x)\, g^i_s(x)\, dx\, ds.$$

The intensity λ^i_t can be viewed as the $\mathcal{G}_t$-conditional density of τ_i on the set $\{\tau_i > t\}$. In analogy to the distribution F^i_t of τ_i, the the mapping $t \rightarrow \lambda^i_t$ experiences a jump at other firms' default times τ_j given a sufficient degree of micro-correlation between the firms. Macro-correlation does not play any role here. At an intuitive level, a jump corresponds to an immediate re-assessment of firm i's expected future performance by the secondary market. Technically, this is a consequence of conditioning on $\mathcal{G}_t$, which includes the default status of all firms, and the inaccessible nature of defaults. The jumps in λ^i are induced via the conditional density g^i_t, which is updated whenever unpredictable defaults occur (cf. Section 3.2).

The jump behavior of intensities is very natural in the context of direct links (micro-correlations) between firms. In a recent contribution, Jarrow & Yu (2001) modeled this behavior in a pure intensity based approach by *assuming* a particular functional form of the intensity that accommodates the jumps. They supposed that there are two types of firms: 'primary' firms whose default depends only on general macro-factors and 'secondary' firms whose intensity depends additionally upon the default status of the primary firms. In case $n = 2$, they model the intensity γ^2 of the secondary firm 2 as

$$\gamma^2_t := a + b1_{\{t \geq \tau_1\}},$$

for some $a > 0$ and b such that $\gamma^2_t \geq 0$ for all t. Jarrow & Yu (2001) offer no specification of a and b. With this assumption in place, the usual intensity based pricing formulas can be shown to hold. Our approach is fundamentally different: instead of exogenously assuming a functional form of the intensity that accommodates the natural jump behavior, our model generates the jumps *endogenously*, in the manner described above.

There is another point related to default correlation in the intensity based approach which is worth emphasizing. Duffie & Singleton (1998) and Jarrow et al. (2000) suggest to model dependence of firms on common (macro-) economic factors by letting intensity processes γ^i be correlated. Taking the γ^i as exogenously given, they define some stopping time $\tilde{\tau}_i$ having intensity γ^i by

$$\tilde{\tau}_i := \inf\{t \geq 0 : \int_0^t \gamma^i_s ds \geq e_i\},$$

where e_i is some independent exponentially distributed random variable. The e_i are also *mutually independent*. The $\tilde{\tau}_i$ are interpreted as default times. But $\tilde{\tau}_i$ is equal to the 'true' default time τ_i only in distribution, cf. Elliott et al. (2000). In such an approach the $\tilde{\tau}_i$ are however conditionally independent given the γ^i. Thus the range of achievable default dependence is limited. Even perfect correlation in intensities does not lead to perfect dependent $\tilde{\tau}_i$. In our approach, the default time dependence is *not* constrained, as was discussed in Subsection 3.4.2. We showed in Proposition 3.8 that only if we assume the thresholds to be independent, then defaults are conditionally independent given the asset value paths. With correlated thresholds, the full range of default correlation can be attained; the associated intensities are determined through Proposition 4.10.

In order to give an example and to relate our result to that of Duffie & Lando (2001), we consider the instructive one-firm case.

Proposition 4.11. *Let $I = \{i\}$ and suppose that assets V^i follow a Brownian motion with drift $\mu_i \in \mathbb{R}$ and volatility $\sigma_i > 0$:*

$$dV_t^i = \mu_i dt + \sigma_i dW_t^i,$$

where W^i is a standard Brownian motion. Assume, for $t < \tau_i$, that V_t^i admits a conditional density $a^i(t, x, \cdot, \omega)$ given $\mathcal{G}_t$ and $M_t^i > x$ on $[x, \infty)$, which satisfies the following conditions:

(1) For each (t, x, ω), we have $a^i(t, x, \cdot, \omega) = 0$ on $(-\infty, x]$ and $a^i(t, x, \cdot, \omega)$ is continuously differentiable on (x, ∞) and differentiable from the right at x. For almost every ω, the derivative $|a_z^i(s, x, z, \omega)|$ is bounded on sets of the form $\{(s, x, z) :\ 0 \le s \le t,\ -\infty < x < 0,\ x \le z < \infty\}$.

(2) For all fixed $x \le 0$ and $z \ge x$, the process $(a_z^i(t, x, z))_{t \ge 0}$ is progressively measurable.

Then τ_i admits on the set $\{\tau_i > t\}$ the $(\mathcal{G}_t)$-intensity

$$\lambda_t^i = \frac{1}{2}\sigma_i^2 \int_{-\infty}^{0} a_z^i(t, x, x)\, g_t^i(x)\, dx.$$

PROOF. We show that $m^i(t, x) = \frac{1}{2}\sigma_i^2 a_z^i(t, x, x)$ for all $x \le 0$. The result follows then from Proposition 4.10. On the set $\{M_t^i > D_i\}$, from the Markov property of Brownian motion and by substituting $y = x - z/\sigma_i\sqrt{h}$ we obtain

$$\begin{aligned} m^i(t,x) &= \lim_{h\downarrow 0} \frac{1}{h} P[M^i_{t+h} \le x \,|\, \mathcal{G}_t, D_i = x] \\ &= \lim_{h\downarrow 0} \frac{1}{h} \int_x^\infty P[M^i_h \le x - z] a^i(t,x,z)\, dz \\ &= \sigma_i \lim_{h\downarrow 0} \int_{-\infty}^0 P[M^i_h \le y\sigma_i\sqrt{h}] \frac{1}{\sqrt{h}} a^i(t,x,x-y\sigma_i\sqrt{h}) dy. \end{aligned}$$

Since the distribution of M^i is explicitly known in the Brownian case, with (5.12) we find

$$\begin{aligned} &\lim_{h\downarrow 0} P[M^i_h \le y\sigma_i\sqrt{h}] \\ &\quad = \lim_{h\downarrow 0} \left(1 - \Phi\left(\frac{\mu_i\sqrt{h}}{\sigma_i} - y\right) + \exp\left(\frac{2\mu_i y\sqrt{h}}{\sigma_i}\right)\Phi\left(y + \frac{\mu_i\sqrt{h}}{\sigma_i}\right)\right) \\ &\quad = 2\Phi(y), \end{aligned}$$

where Φ is the standard normal distribution function. Now, since $a^i(t,x,x) = 0$, we obtain

$$\lim_{h\downarrow 0} \frac{1}{\sqrt{h}} a^i(t,x,x-y\sigma_i\sqrt{h}) = y\sigma_i a^i_z(t,x,x),$$

where the derivative is taken from the right. If dominated convergence can be applied,

$$\begin{aligned} m^i(t,x) &= 2\sigma_i^2 a^i_z(t,x,x) \int_{-\infty}^0 \Phi(y)\, y\, dy \\ &= \frac{1}{2}\sigma_i^2 a^i_z(t,x,x), \end{aligned}$$

as desired. To justify dominated convergence, note that for $h < 1$

$$|P[M^i_h \le y\sigma_i\sqrt{h}]| \le M(y) := 1 - \Phi(-y) + \exp(2|\mu_i|y\sigma_i^{-1})\Phi(y - |\mu_i|\sigma_i^{-1})$$

and $M(y)$ goes to zero exponentially fast as $y \to -\infty$. Since $a^i_z(t,x,z)$ is bounded, we have for some constant B that

$$|P[M^i_h \le y\sigma_i\sqrt{h}] \frac{1}{\sqrt{h}} a^i(t,x,x-y\sigma_i\sqrt{h})| < M(y)\, B,$$

providing an integrable upper bound for all $h < 1$. $\square$

For Model B, the conditional asset density $a^i(t, x, \cdot)$ is established in (3.18). But this density was shown to be equivalent with the conditional asset density derived by Duffie & Lando (2001) in their framework. Since Duffie & Lando (2001) calculated the density explicitly in terms of the normal distribution function, we can verify that $a^i(t, x, \cdot)$ satisfies the conditions of Proposition 4.11. As for Model C, $a^i(t, x, \cdot)$ can be easily calculated from the multivariate asset density (3.21). In case of Brownian motion, $a^i(t, x, \cdot)$ is explicitly available, using for example the results in Borodin & Salminen (1996). We can then verify that the conditions of Proposition 4.11 are satisfied.

Now let us additionally assume that the default threshold is known to the bond investors, $D_i \in \mathcal{G}_0$. Then we get from Proposition 4.11 for the $(\mathcal{G}_t)$-intensity

$$\lambda_t^i = \frac{1}{2}\, \sigma_i^2\, a_z^i(t, D_i, D_i) \quad \text{on} \quad \{\tau_i > t\}.$$

This is the result (4.11) of Duffie & Lando (2001) in case of Brownian motion if $a^i(t, x, x) = \xi(t, x)$ for all $t \geq 0$ and $x \geq D_i$ on $\{\tau_i > t\}$. The equality of the densities has been verified in Section 3.2.

Chapter 5

A Two-Firm Model with Log-Normal Assets

The results in the previous two Chapters 3 and 4 could be established without specific assumptions on the law of the underlying asset processes and the prior threshold beliefs of the bond investors. In order to illustrate and exemplify our findings, in this section we will examine default distribution and credit spreads given the following assumptions:

(1) there are two firms;

(2) the market value of each firm follows a geometric Brownian motion;

(3) bond investors observe the firm values perfectly (information Model A);

(4) the idiosyncratic threshold prior is uniform and the interrelation prior is represented by the parametric Clayton copula family.

We will also provide methods to calibrate the threshold copula to historical data as well as to market bond price data.

Though a special case of our general framework, it will prove insightful to study this particular setup for two reasons. The first concerns the modeling of default dependence. The Brownian motions driving the firm values are chosen to be independent. Default correlation is exclusively caused by default threshold dependence. We show that any degree of default correlation can be induced through a suitable choice of the threshold copula. This setup provides a first example of a two-firm structural default model, in which default correlation is not modeled via asset correlation.

The second second concerns the resulting credit spreads. In the one-firm model of Duffie & Lando (2001), bond investors cannot observe assets directly while the constant default threshold is known. The credit spread term structure is shown to be hump shaped with strictly positive short spread. We reverse the informational setup of Duffie & Lando (2001). The resulting credit spread term structures are hump shaped or decreasing, depending on the firms' market value. Although the default time is unpredictable, the short spread in the hump shaped term structure is zero as with complete information.

5.1 Assets, Information, and Prior

We consider a bond market where zero coupon bonds of two firms, labeled 1 and 2, are traded. Let us suppose that the firms are positively related. For example, each firm could hold a substantial part of the other one's debt. Following the tradition in structural default modeling, we assume that the total market value Z^i of firm i follows a geometric Brownian motion. That is, Z^i obeys the stochastic differential equation

$$dZ_t^i = Z_t^i(m_i dt + \sigma_i dW_t^i), \tag{5.1}$$

where $m_i \in \mathbb{R}$ and $\sigma_i > 0$ are constant drift and volatility parameters, respectively. (W^1, W^2) is a two-dimensional standard Brownian motion. This implies that the firm values itself are not correlated. This is plausible if the firms are from quite different industries, so that their earnings/cash flows and hence firm values are not directly related. Macro-correlation is hence not considered; we will focus instead on direct linkages between firm defaults.

(5.1) has unique solution

$$Z_t^i = Z_0^i \, e^{B_t^i},$$

where $Z_0^i > 0$ is the initial value of firm i and B^i is a Brownian motion with drift $\mu_i = m_i - \frac{1}{2}\sigma_i^2$, i.e.,

$$B_t^i = \mu_i t + \sigma_i W_t^i. \tag{5.2}$$

The incomplete information available to the secondary market is modeled by the filtration $(\mathcal{G}_t)_{t \geq 0}$, which satisfies

$$\mathcal{G}_t := \mathcal{F}_t \vee \sigma(\tau_1 \wedge t) \vee \sigma(\tau_2 \wedge t),$$

where $(\mathcal{F}_t)_{t\geq 0}$ is the filtration generated by the firm value process (Z^1, Z^2). While bond investors observe defaults as well as firm values perfectly, they do not know the thresholds at which the shareholders liquidate the firm. This is corresponds to our information Model A, which is the converse of that in Duffie & Lando (2001).

In lack of complete information, investors form a prior on the default thresholds. The prior is separated into an idiosyncratic component and an interrelation component. We assume that bond investors' idiosyncratic prior is uniform. This corresponds to uninformed investors not having any specific knowledge on the barriers. Reflected by the same idiosyncratic prior for either firm, the secondary market believes that both firms have similar individual default characteristics. Since the firms' market value follows a strictly positive geometric Brownian motion process, the prior has support $(0, Z_0^i)$. Denoting the barrier by $\hat{D}_i$, the idiosyncratic prior is then represented by the distribution function $\hat{G}^i(x) := P[\hat{D}_i \leq x] = x/Z_0^i$. We set $\hat{D} := (\hat{D}_1, \hat{D}_2)$.

We assume that bondholders' interrelation threshold prior is modeled by the Clayton copula family

$$C^{\hat{D}}(u, v; \theta) := (u^{-\theta} + v^{-\theta} - 1)^{-\frac{1}{\theta}}, \quad (u, v) \in [0, 1]^2, \quad \theta > 0. \tag{5.3}$$

The Clayton family belongs to the class of Archimedean copulas. For a complete account we refer to Lindskog (2000), who also shows that this family exhibits lower tail dependence (cf. Definition A.10). This is relevant to the joint default behavior implied by this family, cf. Subsection 3.4.2. The parameter θ controls the degree of dependence between the random variables $\hat{D}_1$ and $\hat{D}_2$. Corresponding to the positive relation in the default characteristics of the two firms, we choose $\theta > 0$. This implies that $C^{\hat{D}}$ displays positive dependence (cf. Propositions A.6 and A.7):

$$\Pi(u, v) < C^{\hat{D}}(u, v; \theta) \leq M(u, v), \quad (u, v) \in [0, 1]^2.$$

Specifically, $\lim_{\theta\to\infty} C^{\hat{D}}(u, v; \theta) = M(u, v)$, reflecting perfect positive dependence, and $\lim_{\theta\to 0} C^{\hat{D}}(u, v; \theta) = \Pi(u, v)$, corresponding to independence.

In order to assess the degree of dependence associated with $C^{\hat{D}}(u, v; \theta)$ for a particular θ, we can consider Kendall's rank correlation ρ_K. The coefficient ρ_K measures the degree of monotonic dependence and is defined by

$$\rho_K(\hat{D}) := P[(\hat{D}_1' - \hat{D}_1'')(\hat{D}_2' - \hat{D}_2'') > 0] - P[(\hat{D}_1' - \hat{D}_1'')(\hat{D}_2' - \hat{D}_2'') < 0].$$

$(\hat{D}_1', \hat{D}_2')$ and $(\hat{D}_1'', \hat{D}_2'')$ are two independent pairs of random variables from the joint distribution $\hat{G}$ of $\hat{D}$. Compare ρ_K to Spearman's rank correlation

ρ_S, defined in (3.24). ρ_K and ρ_S are equivalent in that they carry the same information. We prefer the quantity that is easier to calculate in the case at hand. Since the $\hat{D}_i$ are continuous, we have $\rho_K(\hat{D}) \in [-1, 1]$, $\rho_K(\hat{D}) = 0$ if $C^{\hat{D}} = \Pi$, $\rho_K(\hat{D}) = 1$ iff $C^{\hat{D}} = M$ and $\rho_K(\hat{D}) = -1$ iff $C^{\hat{D}} = W$, cf. Nelsen (1999). Kendall's rank correlation and the copula are related via

$$\rho_K(\hat{D}) = 4 \int_0^1 \int_0^1 C^{\hat{D}}(u, v; \theta) dC^{\hat{D}}(u, v; \theta) - 1,$$

showing that ρ_K is determined by the copula only. ρ_K is thus invariant under strictly increasing transformations of the random variables. For the Clayton family we obtain in particular

$$\rho_K(\hat{D}) = \frac{\theta}{\theta + 2}. \tag{5.4}$$

The parameter θ can be used to calibrate the copula to historical default data with empirical rank correlation ρ_K. If this sort of data is not available, $C^{\hat{D}}$ can also be calibrated to historical bond price data, cf. Section 5.4 below.

From now on we set

$$V_t^i := B_t^i, \quad t \geq 0, \quad i = 1, 2.$$

That is, our 'asset process' V^i in the sense of Chapter 2 is the log-firm value B^i. We thus introduce the threshold transformation

$$D_i := \ln \hat{D}_i - \ln Z_0^i. \tag{5.5}$$

Since $V_0^i = 0$ by construction, the prior has now support $(-\infty, 0)$. We now characterize the distribution function G of $D := (D_1, D_2)$.

Proposition 5.1. *The (transformed) threshold prior is represented by the unique joint distribution function*

$$G(x, y; \theta) = (e^{-\theta x} + e^{-\theta y} - 1)^{-1/\theta}, \quad (x, y) \in \mathbb{R}_-^2.$$

Proof. For fixed $Z^i > 0$, D_i is a strictly increasing transformation of $\hat{D}_i$. By the invariance property established in Proposition A.4, the copula C^D of the transformed threshold vector D remains unchanged:[1]

$$C^D(u, v; \theta) = C^{\hat{D}}(u, v; \theta), \quad (u, v) \in [0, 1]^2.$$

[1]If we had worked with linear correlation as dependence measure, such a simple transformation procedure would not apply here, since linear correlation is *not* invariant under increasing nonlinear transformations of the underlying random variables.

It follows directly from (5.5) that the idiosyncratic prior with respect to D_i is represented by the marginal distribution function

$$G^i(x) = P[D_i \le x] = \hat{G}^i(Z_0^i\, e^x) = e^x, \quad x \in \mathbb{R}_-,$$

with density function $g^i(x) = e^x$. Since the G^i are continuous, G is now uniquely determined by

$$G(x, y, \theta) = C^D(G^1(x), G^2(y)), \quad (x, y) \in \mathbb{R}_-^2,$$

from which the conclusion follows. □

Let us note that the invariance of the copula implies that $\rho_K(\hat{D}) = \rho_K(D)$.

5.2 Belief Updating

Without loss of generality we set $\tau_1 < \tau_2$, i.e., Firm 1 defaults before Firm 2. Define for $t \ge 0$

$$K_t(\theta) := G(M_t^1, M_t^2; \theta) = (e^{-\theta M_t^1} + e^{-\theta M_t^2} - 1)^{-1/\theta}.$$

The $\mathcal{G}_t$-conditional joint law of D is found by specializing in Proposition 3.1:

Corollary 5.2. *Under the current assumptions, the a posteriori belief of the bond investors is represented by the $\mathcal{G}_t$-conditional distribution*

$$G_t(x, y; \theta) = \frac{1}{K_t(\theta)}(e^{-\theta x} + e^{-\theta y} - 1)^{-1/\theta} \quad \text{on} \quad \{0 \le t < \tau_1\},$$

where $x \in (-\infty, M_t^1]$ and $y \in (-\infty, M_t^2]$, and

$$G_t(x, y; \theta) = \left[\frac{e^{-\theta d_1} + e^{-\theta y} - 1}{e^{-\theta d_1} + e^{-\theta M_t^2} - 1}\right]^{-(1/\theta + 1)} \quad \text{on} \quad \{\tau_1 \le t < \tau_2\},$$

for $x \in [d_1 := M_{\tau_1}^1, 0]$ and $y \in (-\infty, M_t^2]$.

Proof. At time $t \in [0, \tau_1)$ the bond holders are on the set (cf. (3.2))

$$\{t < \tau_1\} \cap \{t < \tau_2\} = \{D_1 < M_t^1\} \cap \{D_2 < M_t^2\},$$

where $M_t^i \in \mathcal{G}_t$. Using Bayes' rule, for $x \in (-\infty, M_t^1]$ and $y \in (-\infty, M_t^2]$,

$$G_t(x, y; \theta) = P[D_1 \le x, D_2 \le y \,|\, D_1 < M_t^1, D_2 < M_t^2] = \frac{G(x, y; \theta)}{K_t(\theta)}.$$

Similarly, if $t \in [\tau_1, \tau_2)$, they are on the set $\{D_1 = d_1\} \cap \{D_2 < M_t^2\}$. Since

$$P[D_2 \leq y \,|\, D_1 = d_1] = \frac{1}{g^1(d_1)} \frac{\partial}{\partial x} G(x, y; \theta)\bigg|_{x=d_1}$$

we see that for $x \in [d_1, 0]$ and $y \in (-\infty, M_t^2]$,

$$\begin{aligned} G_t(x, y; \theta) &= P[D_1 \leq x, D_2 \leq y \,|\, D_1 = d_1, D_2 < M_t^2] \\ &= \frac{P[D_2 \leq y \,|\, D_1 = d_1]}{P[D_2 \leq M_t^2 \,|\, D_1 = d_1]} \\ &= \left[\frac{G(d_1, y; \theta)}{G(d_1, M_t^2; \theta)}\right]^{1+\theta}, \end{aligned}$$

which completes the proof in view of Proposition 5.1. □

To illustrate the a posteriori beliefs, we consider the $\mathcal{G}_t$-conditional density g_t^2 of D_2. When both firms still operate, we have on $\{-\infty < y < M_t^2\}$

$$g_t^2(y; \theta) = \frac{1}{K_t(\theta)} (e^{-\theta M_t^1} + e^{-\theta y} - 1)^{-(1+\theta)/\theta} e^{-\theta y}, \tag{5.6}$$

and, if Firm 1 has already defaulted,

$$g_t^2(y; \theta) = \frac{(e^{-\theta d_1} + e^{-\theta y} - 1)^{-(1+2\theta)/\theta}}{(e^{-\theta d_1} + e^{-\theta M_t^2} - 1)^{-(1+\theta)/\theta}} e^{-\theta y}(1 + \theta). \tag{5.7}$$

The parameter for computations are fixed in Table 5.1. Figure 5.1 (Figure 5.2)

	Mean Asset Return	Asset Volatility	Running Minimum
Firm 1	$\mu_1 = 0.015$	$\sigma_1 = 0.06$	$M_t^1 = -0.2$
Firm 2	$\mu_2 = 0.01$	$\sigma_2 = 0.05$	$M_t^2 = -0.1$

Table 5.1: Base case parameter

graphs this density for varying degrees of monotonic dependence between D_1 and D_2 on the set $\{0 \leq t < \tau_1\}$ (resp. on the set $\{\tau_1 \leq t < \tau_2\}$). We put $D_1 = -0.2$.

Clearly, given information $\mathcal{G}_t$, there is no mass on $[M_t^2, 0]$. To understand the observed pattern, let us first consider two special cases. Threshold independence corresponds to $C^D = \Pi$ (or $\theta \to 0$) and implies that $g_t^2(y; \theta) = \exp(y - M_t^2)$. Threshold comonotonicity requires $C^D = M$ (or $\theta \to \infty$). Then $D_2 = T(D_1)$ with $T = I^2 \circ G^1$, where I^i is the inverse of G^i (cf. A.2). Since

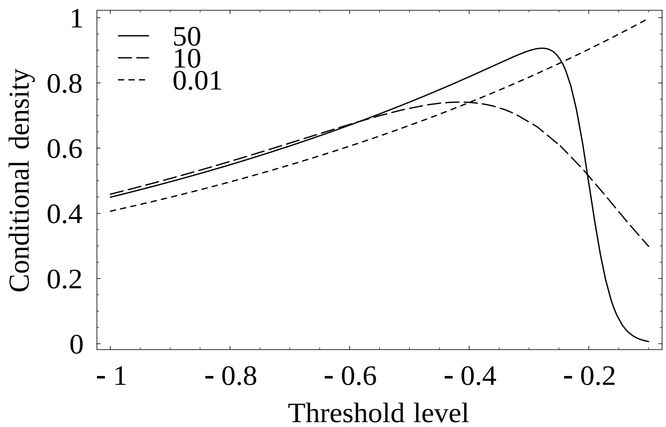

Figure 5.1: Conditional threshold density Firm 2, varying θ (both firms operate).

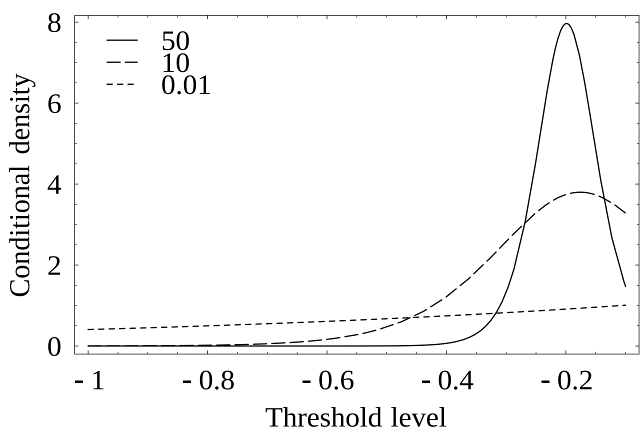

Figure 5.2: Conditional threshold density Firm 2, varying θ (Firm 1 has defaulted).

by assumption $G^1 = G^2$, we have that $D_1 = D_2$: both thresholds coincide in this extreme case.

If both firms still operate (Figure 5.1), comonotonicity implies that $D_i < M_t^1 \wedge M_t^2 = M_t^1 = -0.2$. Thus in this extreme case $g_t^2(y;\theta) = 0$ on $[M_t^1, M_t^2]$. By increasing the parameter θ we increase the degree of positive dependence, implying a shift in the density towards the extreme comonotonicity case. $\theta = 50$ corresponds to a rank correlation of $\rho_K(D) = 0.962$. If Firm 1 has defaulted (Figure 5.2), comonotone thresholds imply $D_1 = D_2$. This means that D_2 is known with certainty before the default of Firm 2. Note that this is always the case with perfect information. Indeed, with $\theta \to \infty$ the density degenerates since there is mass one on the point $D_2 = d_1$: we have $G_t^2(y;\theta) = 1_{\{y \geq d_1\}}$. With increasing positive dependence we hence observe a concentration of mass around the known threshold d_1 of Firm 1.

Let us consider the conditional copula C_t^D associated with the conditional distribution G_t, cf. (3.8). The inverse functions I_t^i of the marginals G_t^i derived in Corollary 5.2 are for $t < \tau_1$ given by

$$I_t^i(u) = -\frac{1}{\theta} \ln((uK_t(\theta))^{-\theta} - e^{-\theta M_t^j} + 1), \quad u \in [0,1].$$

From (3.8) we then find

$$\begin{aligned} C_t^D(u,v;\theta) &= G_t(I_t^1(u), I_t^2(v);\theta) \\ &= \frac{1}{K_t(\theta)}((uK_t(\theta))^{-\theta} + (vK_t(\theta))^{-\theta} - (K_t(\theta))^{-\theta})^{-1/\theta} \\ &= (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta} \\ &= C^D(u,v;\theta) \quad \forall t \geq 0, \end{aligned} \tag{5.8}$$

so that the threshold copula is invariant under conditioning on $\{t < \tau_1\}$. We emphasize that this is not always the case in our setting. To give an example, suppose that the thresholds are comonotone, $C^D(u,v;\theta) = u \wedge v$. Then on $\{\tau_1 \leq t < \tau_2\}$, the copula C_t^D must satisfy

$$G_t(x,y;\theta \to \infty) = 1_{\{x \geq d_1, y \geq d_1\}} = C_t^D(1_{\{x \geq d_1\}}, 1_{\{y \geq d_1\}}; \theta \to \infty),$$

implying that $C_t^D(u,v;\theta \to \infty) = uv$ for all $t \geq 0$.

5.3 Default Distribution

We now turn to the joint default distribution when both firms still operate. We will need the density $\psi_i(\cdot,t)$ of the running minimum M_t^i of the Brownian motion V_t^i with drift μ_i and volatility σ_i (cf. Borodin & Salminen (1996)):

$$\begin{aligned} \psi_i(x,t) &= \frac{1}{\sigma_i\sqrt{t}}\phi\left(\frac{-x+\mu_i t}{\sigma_i\sqrt{t}}\right) \\ &\quad + \exp\left(\frac{2\mu_i x}{\sigma_i^2}\right)\left[\frac{2\mu_i}{\sigma_i^2}\Phi\left(\frac{x+\mu_i t}{\sigma_i\sqrt{t}}\right) + \frac{1}{\sigma_i\sqrt{t}}\phi\left(\frac{x+\mu_i t}{\sigma_i\sqrt{t}}\right)\right], \end{aligned} \tag{5.9}$$

where Φ (resp. ϕ) is the standard normal distribution (resp. density) function.

Corollary 5.3. *Under the current assumptions, the $\mathcal{G}_t$-conditional joint distribution of the default times is for $t < T_i$ on the set $\{t < \tau_1\}$ given by*

$$F_t(T_1, T_2; \theta) = \frac{1}{K_t(\theta)} \int_{-\infty}^{M_t^2} \int_{-\infty}^{M_t^1} (G(x, y) - G(x, M_t^2) - G(M_t^1, y) + 1) \\ \times \psi_1(x - V_t^1, T_1 - t)\, \psi_2(y - V_t^2, T_2 - t)\, dx\, dy.$$

Proof. The basis for the derivation in the Brownian case is Theorem 3.5 for $n = 2$. In Section 3.2, we also specialized to the case where the information is given by Model A. Using the fact that V^1 and V^2 are independent, the default distribution can be written as

$$F_t(T_1, T_2; \theta) = \int_{-\infty}^{M_t^2} \int_{-\infty}^{M_t^1} \bar{C}_t^D(1 - G_t^1(x), 1 - G_t^2(y)) \\ \times \psi_1(x - V_t^1, T_1 - t)\, \psi_2(y - V_t^2, T_2 - t)\, dx\, dy.$$

The conditional survival threshold copula $\bar{C}_t^D$ is given by (3.8). In the bivariate case it simplifies to

$$\bar{C}_t^D(u, v; \theta) = C_t^D(1 - u, 1 - v; \theta) + u + v - 1.$$

From (5.8), $C_t^D = C^D$ and by using (5.3) we have that

$$\bar{C}_t^D(u, v; \theta) = ((1 - u)^{-\theta} + (1 - v)^{-\theta} - 1)^{-1/\theta} + u + v - 1, \quad \forall t \geq 0. \tag{5.10}$$

Substituting the marginals G_t^i, after simplification we get

$$\bar{C}_t^D(1 - G_t^1(x), 1 - G_t^2(y) = G_t(x, y) - G_t^1(x) - G_t^2(y) + 1$$

and the result follows from Corollary 5.2. $\square$

From the preceding result we obtain the $\mathcal{G}_t$-conditional default probability $F_t^i(T; \theta)$ of firm i for the horizon $T > t$. For $\tau_i > t$ we have

$$\begin{aligned} F_t^i(T; \theta) &= \int_{-\infty}^{M_t^i} (1 - G_t^i(x; \theta)) \psi_i(x - V_t^i, T - t) dx \qquad (5.11) \\ &= \int_{-\infty}^{M_t^i} g_t^i(x; \theta) \Psi_i(x - V_t^i, T - t) dx, \end{aligned}$$

where Ψ_i is the distribution function of M_t^i:

$$\Psi_i(x,t) := P[M_t^i \le x] = \int_{-\infty}^{x} \psi_i(y,t)dy, \qquad x \le 0$$
$$= 1 - \Phi\left(\frac{-x+\mu_i t}{\sigma_i\sqrt{t}}\right) + \exp\left(\frac{2\mu_i x}{\sigma_i^2}\right)\Phi\left(\frac{x+\mu_i t}{\sigma_i\sqrt{t}}\right). \qquad (5.12)$$

For $V_t^2 = -0.05$, Figure 5.3 displays $F_t^2(T;\theta)$ as a function of θ for various time horizons T when both firms still operate. Somewhat surprising at a first glance, $F_t^2(T;\theta)$ decreases in the degree of association θ (one can easily check that G_t^i is increasing in θ, cf. Corollary 5.2). This effect rests on the fact that investors observe the default status of all firms. The intuition here is that, given a positive degree of *monotonic* association between the firms, the information that Firm 1 has 'survived' up to time t signals a 'good health' of Firm 2. The stronger the association, the more convincing is the fact that Firm 1 still operates, the higher bond investors rate Firm 2 and the lower $F_t^2(T;\theta)$.

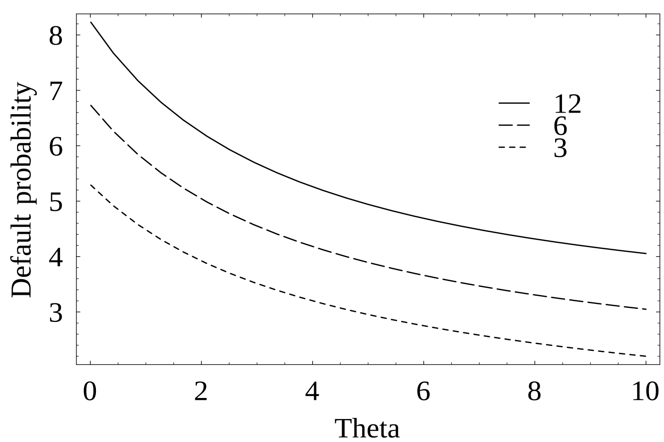

Figure 5.3: Conditional default probability Firm 2, varying horizon T (both firms operate).

Figure 5.4 shows $F_t^2(T;\theta)$ on $\{\tau_1 \le t < \tau_2\}$ as a function of θ for various time horizons T. We see that if Firm 1 has already defaulted, the default probability of Firm 2 is increasing in θ (again from Corollary 5.2, G_t^i is decreasing in θ). Here the intuition is as follows: under positive association between the firms, a default of Firm 1 lets investors conceive a bad state of Firm 2. The stronger the dependence, the lower is Firm 2's rating, and the higher its default probability for a given horizon. The effect of a given change of θ on $F_t^2(T;\theta)$

is for all horizons higher if Firm 1 has defaulted, compared to the case where it still operates.

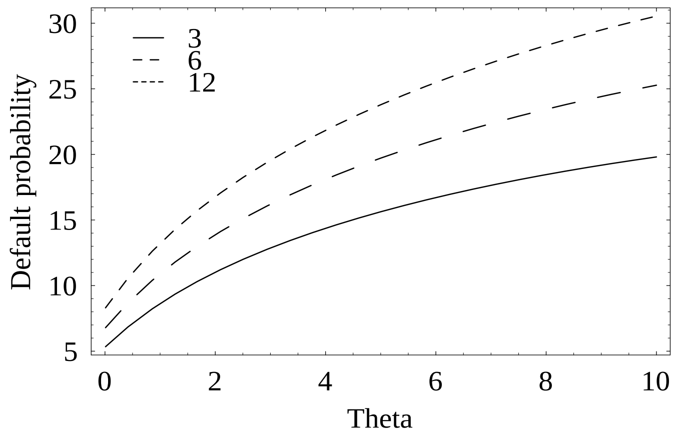

Figure 5.4: Conditional default probability Firm 2, varying horizon T (Firm 1 has defaulted).

The term structure of Firm 2's default probabilities $F_t^2(T;\theta)$ on the set $\{\tau_1 \leq t < \tau_2\}$ is plotted in Figure 5.5. We observe that the stipulated degree of dependence has a significant effect on the default probability level. This level increases with longer horizons.

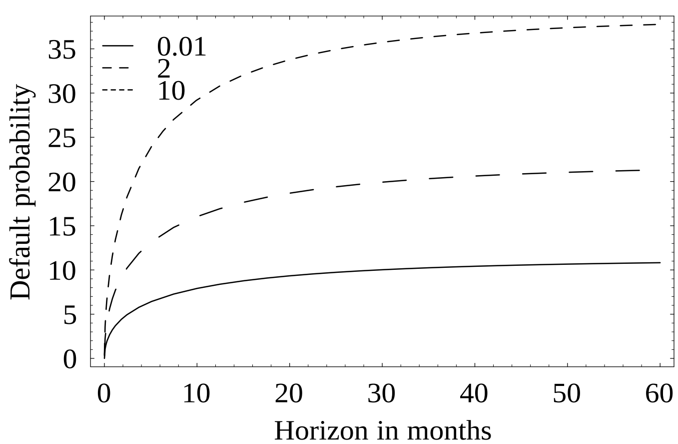

Figure 5.5: Term structure of conditional default probabilities for Firm 2, varying θ (Firm 1 has defaulted).

In our general analysis in Section 3.2, we have seen that default probabilities in our model can exhibit a distinguishing jump behavior: the default probability of a firm can for a fixed horizon exhibit jumps upon the default of correlated bonds. For a time horizon of $T = 12$ months, Figure 5.6 displays this effect for Firm 2. It shows $F_t^2(T;\theta)$ for a fixed t when both firms still operate and when Firm 1 has defaulted. The difference between the curves for a given θ is the jump that would $F_t^2(T;\theta)$ experience if $t = \tau_1$. We observe that the jump size is increasing in the degree of association θ between the two firms. Clearly, if there is no relation between the firms, the default status of Firm 1 does not affect $F_t^2(T;\theta)$.

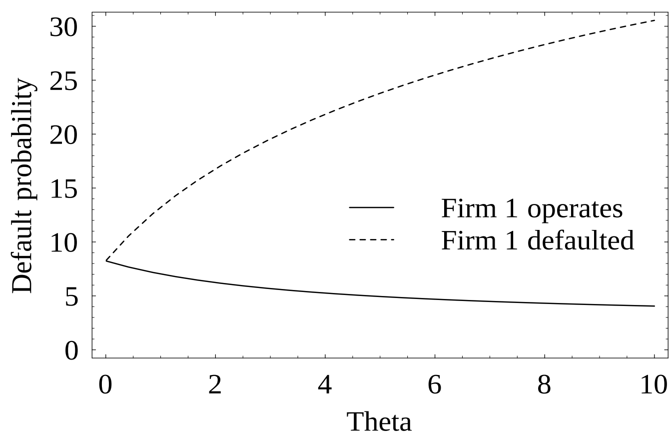

Figure 5.6: Conditional default probabilities of Firm 2 when both firms operate and when Firm 1 has defaulted.

Note that the jump effect is critically dependent on the time horizon T, as shown in Figure 5.7. There we have plotted the difference between the two curves in Figure 5.6 for two fixed values of θ. Fairly intuitive, with $T \to 0$ the jump effect vanishes and the jumps size is increasing in T.

The default correlation between the two firms can be measured by the copula C^τ of (τ_1, τ_2). Since $F_0^i(T)$ is continuous on $(0,\infty)$, we can apply Proposition 3.6 together with (5.10) to see that

$$C^\tau(u,v;\theta) = \int_{-\infty}^{0}\int_{-\infty}^{0} (G(x,y;\theta) + u + v - 1)\,\psi_1(x, J^1(u))\,\psi_2(y, J^2(v))\,dx\,dy,$$

where J^i is the inverse of F_0^i. The prior G is given by Proposition 5.1. Spearman's rank default time correlation $\rho_S(\tau_1,\tau_2;\theta)$, defined in (3.24), can now be computed via (3.25). Figure 5.8 plots ρ_S as a function of the threshold

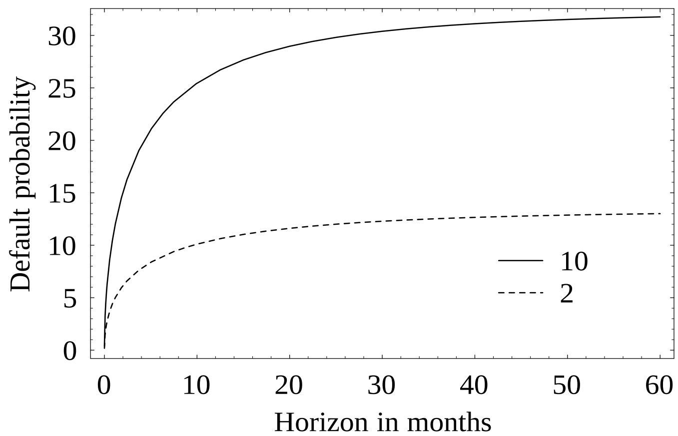

Figure 5.7: Difference between conditional default probabilities of Firm 2 when both firms operate and when Firm 1 has defaulted, varying θ.

association parameter θ. We see that any desired degree of default correlation can be induced by a suitable choice of the Clayton-copula parameter θ. Even with independent assets, the range of achievable correlation degrees is not restricted. It can be scaled from independence through comonotonicity.

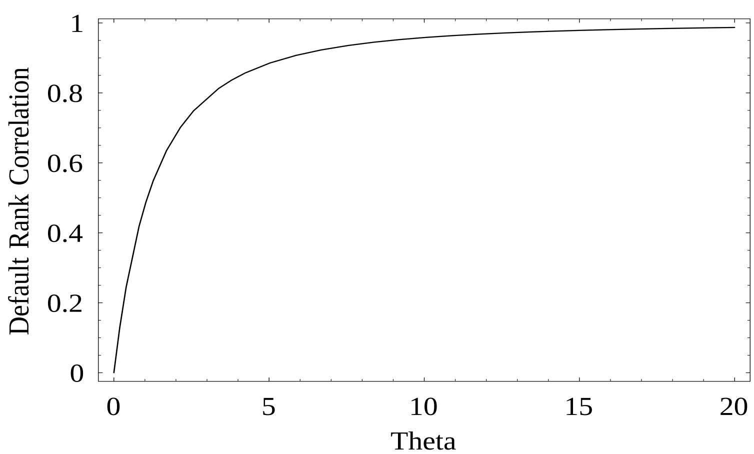

Figure 5.8: Spearman's rank default time correlation as a function of θ.

5.4 Credit Spreads

The credit yield spread for zero bonds issued by Firm i is defined in (4.6). Assuming zero recovery and a constant risk-free interest rate $r > 0$, we have

$$\begin{aligned} S_i(t,T) &= -\frac{\ln P[\tau_i > T \,|\, \mathcal{G}_t]}{T-t}, \quad T > t, \quad \tau_i > t \\ &= -\frac{1}{T-t} \ln \int_{-\infty}^{M_t^i} G_t^i(x;\theta)\psi_i(x - V_t^i, T-t)dx. \end{aligned} \tag{5.13}$$

In Section 3.3, we have described how we can recover information from observed bond prices on the market's perception of the relation between firms. We used as a benchmark the price of a bond given no micro-correlation, $p_i^\Pi(t,T)$. Equivalently, we can apply a similar procedure on the level of credit spreads, using the spread $S_i^\Pi(t,T)$ given no micro-correlation as a benchmark. If $C^D = \Pi$, Corollary 5.2 yields

$$G_t^i(x;\theta \to 0) = \frac{G^i(x;\theta \to 0)}{G^i(M_t^i;\theta \to 0)} = e^{x - M_t^i}, \quad x \le M_t^i.$$

By virtue of (5.13), $S_i^\Pi(t,T)$ is for $i = 1,2$ on the set $\{0 \le t < \tau_1\}$ and for $i = 2$ on $\{\tau_1 \le t < \tau_2\}$ given by

$$S_i^\Pi(t,T) = -\frac{1}{T-t} \ln \int_{-\infty}^{M_t^i} e^{x - M_t^i} \psi_i(x - V_t^i, T-t)dx.$$

In the current setting, the benchmark spread $S_i^\Pi(t,T)$ can be viewed as the idiosyncratic component of the default risk premium. This is the premium which is due to the specific individual default risk of a firm if there is neither micro-dependence between the firms nor macro-correlation. The observed difference $\Delta_i(t,T) := S_i(t,T) - S_i^\Pi(t,T)$ reflects the micro-dependence structure across the two firms as perceived by the bond investors. Let us suppose that both firms still operate. Fix a pair (t,T). Then $F_t^i(T)$ is decreasing in θ (cf. Figure 5.3) and $S_i(t,T)$ is increasing in θ. It follows that $S_i^\Pi(t,T)$ is the lowest possible spread under our current assumptions. Thus, if $\Delta_i(t,T) = 0$ we have that Firm i is completely independent from Firm j. If $\Delta_i(t,T) > 0$, the parameter θ is positive and implicitly given by

$$e^{(t-T)(\Delta_i(t,T) + S_i^\Pi(t,T))} = \frac{1}{K_t(\theta)} \int_{-\infty}^{M_t^i} (e^{-\theta x} + e^{-\theta M_t^j} - 1)^{-1/\theta} \psi_i(x - V_t^i, T-t)dx,$$

where we have used (5.13) and Corollary 5.2. This gives a method of calibrating the threshold copula C^D to market data. Alternatively, given historical default data is available, we may calibrate C^D via (5.4) using the estimated Kendall rank correlation ρ_K.

Letting $\tau_1 \leq t < \tau_2$, Figure 5.9 graphs the term structure of credit yield spreads $S_2(t,T;\theta)$ for varying degrees of threshold dependence. We set $V_t^2 = M_t^2 = -0.1$: the firm value is currently 'at a new low'. Under this choice

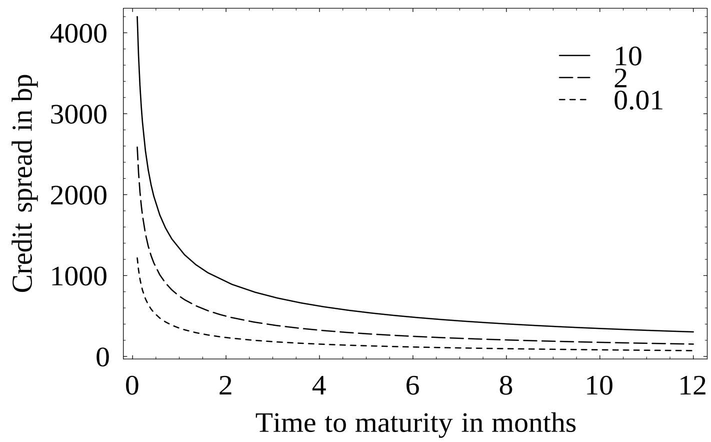

Figure 5.9: Term structure of credit spreads for Firm 2, varying θ (Firm 1 has defaulted).

of the current asset value the term structure is decreasing and the spread is strictly positive for maturities near zero. Compare this to the case of complete information displayed in Figure 4.1. Consistent with the behavior of default probabilities $F_t^2(T;\theta)$ on the set $\{\tau_1 \leq t < \tau_2\}$, the degree of positive threshold dependence θ has a significant effect on the absolute level of the spread. This effect is pronounced for short maturities. The lower curve ($\theta = 0.01$) corresponds to the idiosyncratic spread $S_2^{\Pi}(t,T)$. For example, for a time to maturity of 2 months the spread that can be attributed to a positive correlation of $\theta = 10$ (upper curve) amounts to approximately 700 basis points. The downward-sloping shape of the term structure can be intuitively explained as follows. Since the firm value $V_t^2 = M_t^2$ is at its current running minimum, at time t the bond is fairly risky. In the short run negative shocks on the asset value can quickly lead to a default before time $T > t$. This risk increases with positive dependence. Due to the positive drift in the firm value, there is room for improvement over time and only less potential to worsen. As a result, the spread decreases with the horizon T.

In Figure 5.9, we have set $V_t^2 = M_t^2$. However, as soon as we increase the current asset value V_t^2 above the level of the minimum to date M_t^2, we witness a downward shift in the spread curve towards a hump shaped term structure, cf. Figure 5.10, where $\theta = 2$. If $V_t^2 > M_t^2$, the term structure shape with incomplete information approaches that of complete information. With complete information short spreads are zero, cf. Figure 4.1. Clearly, as the default threshold must be below M_t^2 if the firm still operates and the firm value cannot jump, higher firm values correspond to zero short-term default probabilities and thus zero spreads. This effect is most interesting. Depending on the level of the firm value, information Model A implies two different sorts of term structures: decreasing *and* hump shaped. These shape properties seem very similar to those obtained by Merton (1974) in his pioneering work. In his model there is complete information. Merton finds that if the risklessly discounted face value of the zero bonds financing a firm besides equity is larger than the firm value, the term structure is decreasing; otherwise it is hump shaped. Thus the decreasing term structure corresponds to a very risky firm.

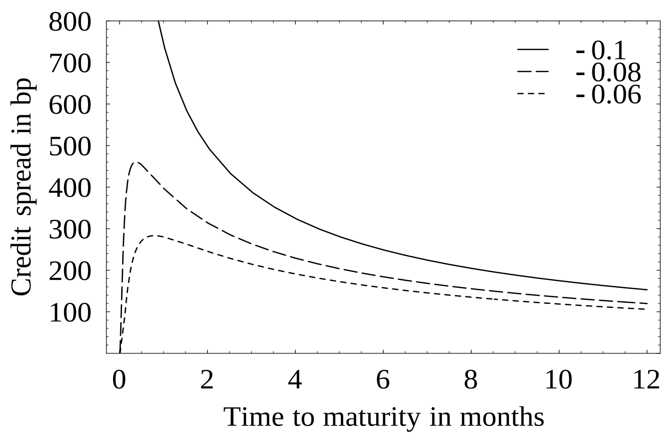

Figure 5.10: Term structure of credit spreads for Firm 2, varying current asset value V_t^2 (Firm 1 has defaulted).

As Sarig & Warga (1989) report, a hump shaped term structure is typically observed for junk quality. For $\theta = 2$, the shift of the term structure from hump shaped to monotone decreasing can be considered as an increase in risk in the short run, as we decrease the firm value towards its running minimum. In this sense the decreasing pattern is a characteristic for high risk junk quality. This interpretation is consistent with the interpretation of a decreasing shape offered by Merton (1974) in his model. Now in the extreme case, if $M_t^2 = -0.1$ and $V_t^2 = 0$, the term structure approaches that for complete information in the above junk quality case. Then correlation does hardly matter any more. In essence, the effect of incomplete information and default dependence becomes less important the more the current asset value is away from its minimum to date.

In the incomplete information model of Duffie & Lando (2001), the spread term structure is always hump shaped with strictly positive short spreads. In our setting of incomplete information, the term structure has two possible shapes. Although the default time is unpredictable, short spreads are only strictly positive if the asset value is at its running minimum. This shows that incomplete information does only in certain cases result in empirically desirable spread properties.

5.5 On Estimating the Prior

In this section we briefly comment on the estimation of the threshold prior. There are two sub-problems:

- estimation of the marginal distributions $\hat{G}^i$;
- estimation of the copula $C^{\hat{D}}$.

Generally, estimation is plagued by the lack of historical default data. The only database we are aware of is maintained by San Francisco based KMV Corporation, which provides commercial structural-based risk measurement services to financial institutions. The dataset comprises over 2000 default incidents and includes also records on the default threshold. We could then pursue some bucketing approach, in which the firms in the database divided into industrial, regional etc. groups. After some suitable normalization of the firm data, estimation of the marginal threshold distribution of each group is straightforward. Alternatively, as done in this chapter, we may simply assume a uniform marginal threshold law.

Estimation of the buckets' copula is not trivial. Given some parametric copula family, the parameter can be fitted given an estimate of the rank correlation coefficient matrix. Estimation of rank correlation is well understood; see, for example, Gibbons (1988). The relation between copula and (Kendall's) rank correlation was discussed in Section 5.1. Alternatively, we can calibrate the parameter to market bond prices, as described in Section 5.4. Other parametric estimation methods like maximum likelihood are discussed in Durrleman, Nikeghbali & Roncalli (2001). However, in all these parametric approaches we are left with the question what the 'right' copula family is. This is an area where further research is required. In practice, this question can perhaps be answered in a more pragmatic way. One simply chooses the family which is most easily calculated in higher dimensions.

A non-parametric estimation approach can be based on the *empirical* copula, cf. Nelsen (1999). For a sample $\{(D_1^u, D_2^u)\}_{u=1}^U$, the empirical copula $C_e^{\hat{D}}$ is the non-parametric estimate

$$C_e^{\hat{D}}\left(\frac{u_1}{U}, \frac{u_2}{U}\right) := \frac{1}{U}\sum_{u=1}^{U} 1_{\{D_1^{u_1} \le D_1^{(u_1)}, D_2^{u_2} \le D_2^{(u_2)}\}},$$

where $\{D_i^{(u)}\}_{u=1}^U$ is the order statistic of $\{D_i^u\}_{u=1}^U$ and $1 \le u_1, u_2 \le U$. Now $C_e^{\hat{D}}$ converges to the true $C^{\hat{D}}$ (e.g. uniformly). To choose a copula from a given subset that fits the data best, one can consider the distance based on the discrete L^p-norm between each proposed copula and the empirical copula. Such an approach is discussed by Durrleman et al. (2001).

Chapter 6

Bond Portfolios and Scenario-Based Credit Derivatives

In this chapter, we will apply the results derived in Chapter 3 to analyze some problems which are currently discussed by the financial industry and its regulatory authorities. We will focus primarily on two questions: (1) What is the credit risk associated with a given portfolio of corporate bonds? and (2) How can we hedge or insure against that risk?

The first question concerns the *aggregation* of correlated credit risks. This is an essential issue for a financial institution holding a large number of defaultable positions. From an institution's perspective, taking risks is the raison d'être of a bank: the institution is rewarded for bearing the risk. The goal is to lock in a premium while keeping the total amount of risk under control. Consequently, all risks assumed must be aggregated, priced and efficiently managed. A crucial point here is to measure and to evaluate the total amount of risk the institution has been taken, recognizing that individual portfolio positions' defaults may not be independent. Default correlation among names can have two opposite effects. If it is negative, risk is reduced through diversification. If it is positive, risk is increased through concentration. Default concentrations require special emphasis as they can lead to excessive cascading defaults in the portfolio. Even the occurrence of such extreme scenarios must not endanger the bank's existence. Since the financial distress of a bank has severe negative spillover effects to the entire financial system, regulatory authorities are empowered to enforce risk measurement controls.

Having measured the aggregated risk, the exposure is to be hedged in order to balance the total amount of risk. We are looking for contracts allowing a separation of credit risk from the underlying bond asset, and a transfer of that risk to another capital market participant. Such an instrument can then be used to buy and sell credit risks.

Using the results derived in Chapter 3, in this chapter we consider the issues raised above. We will begin in Section 6.1 by discussing measures for the default risk exposure of a corporate bond portfolio. In Section 6.2, we will establish the conditional distribution of portfolio losses. This distribution can be considered as the most comprehensive risk measure available. Our earlier results allow us to relax some assumptions made in the literature. These concern the structure of the underlying portfolio (homogeneity of issuers) and the extent of dependence between portfolio positions. In Section 6.3, we will propose some flexible credit derivative contracts based on the notion of a *default scenario*. These derivatives can be thought of as default insurance contracts where the underlying is not only a single bond, but a whole bond portfolio. While accounting for the portfolio nature of credit in particular, they allow to hedge some pre-specified part of the portfolio's risk profile. The default distribution will be the key to the analysis of these structures. In Section 6.4, we will examine a collateralized debt obligation, being the ultimate instrument to transfer the investor's total credit portfolio risk to the capital market. The loss distribution will play a central role here.

6.1 Characterizing Aggregated Credit Risk

We now discuss measures of the credit risk associated with a portfolio of defaultable bonds. The setup for this chapter is that introduced in Chapter 2. We now focus not on the secondary market in general, but on an incompletely informed individual bond holder. We consider an investor, say some financial institution or high net-worth individual, holding a fixed portfolio of n zero coupon bonds[1] issued by n different firms. The index set of the bonds in the portfolio is I. The loss in case of default of bond i is given by an independent $\mathcal{G}_{\tau_i}$-measurable random variable $\delta_i \in [0,1]$, independent also across firms. δ_i has expected value $\bar{\delta}_i$. The value of the portfolio at time t is $\sum_{i \in J} p_i(t, T_i)$, where prices p_i are given by (3.22).

[1]The assumption that the obligations are zero coupon bonds is not essential for the following analysis. It is merely chosen in order to be consistent with the firms' consol-based capital structure fixed in Chapter 2.

Credit risk is defined as the risk of losses due to changes in the credit quality of issuers. Supposing that the investor intends to hold the bonds until their maturity, in our setting credit losses are caused exclusively by defaults. Fixing some time t, the default losses prior to some horizon $T > t$ are given by the random variable

$$L_T := \sum_{i \in I} \delta_i 1_{\{\tau_i \le T\}}.$$

The time T-credit risk associated with the portfolio corresponds to the possibility that $L_T > 0$. The distribution $P[L_T \in \cdot \,|\, \mathcal{G}_t]$ represents the *complete* conditional portfolio credit risk profile with respect to the time horizon T. The typical time horizon for a risk analysis is one year. This is based on the time necessary to liquidate a portfolio of credit-risky positions. In contrast, when considering market risks one usually fixes a horizon of the order of 10 days.

To begin with, we consider some easily computed statistics of this distribution, summarizing aspects of the loss risk. Letting $\bar{\delta}_i = E[\delta_i]$, we find

$$E[L_T \,|\, \mathcal{G}_t] = \sum_{i \in I} \bar{\delta}_i F_t^i(T).$$

The loss variance indicates some shape properties of the loss distribution. We obtain

$$\begin{aligned}
\mathrm{Var}\ [L_T \,|\, \mathcal{G}_t] &= \sum_{i \in I} \mathrm{Var}\ [\delta_i 1_{\{\tau_i \le T\}} \,|\, \mathcal{G}_t] + \sum_{i,j \in I} \mathrm{Cov}\ [\delta_i 1_{\{\tau_i \le T\}}, \delta_j 1_{\{\tau_j \le T\}} \,|\, \mathcal{G}_t] \\
&= \sum_{i \in I} F_t^i(T)[E[\bar{\delta}_i^2] - \bar{\delta}_i^2 F_t^i(T)] + \sum_{i \neq j} \bar{\delta}_i \bar{\delta}_j [F_t^{ij}(T) - F_t^i(T) F_t^j(T)],
\end{aligned}$$

where $F_t^{ij}(T) := P[\tau_i \le T, \tau_j \le T \,|\, \mathcal{G}_t]$ is given by Theorem 3.5. We observe that the loss variance is increasing in the degree of positive conditional default correlation, a measure of which is the difference

$$F_t^{ij}(T) - F_t^i(T) F_t^j(T) = C_t^\tau(F_t^i(T), F_t^j(T)) - F_t^i(T) F_t^j(T),$$

cf. Section 3.4. Default correlation (larger default copulas C_t^τ) imposes a shift in mass towards the tail of the loss distribution: higher losses become more likely and the degree of diversification decreases. Moreover, the loss distribution will also be highly skewed. Intuitively, each single bond represents an asymmetric risk. There is a small probability of a large loss while gains are capped. Given this non-normality of losses, mean and variance do not complete specify the distribution of losses. Thus Var $[L_T \,|\, \mathcal{G}_t]$ can only give some sense of the loss uncertainty – as a risk measure it is clearly inappropriate. Classical mean-variance based portfolio optimization is therefore not consistent with a portfolio of defaultable positions.

For market risks, the Value-at-Risk measure plays an important role. We can analogously define the conditional Credit Value-at-Risk CVaR as the quantile function of the loss distribution. For some given level of confidence $\alpha \in (0,1)$ and time horizon T, we let

$$\mathrm{CVaR}_{t,T}(\alpha) := \inf\{z \geq 0 : \, P[L_T \leq z \,|\, \mathcal{G}_t] \geq \alpha\}, \quad T > t.$$

Thus $P[L_T \leq \mathrm{CVaR}_{t,T}(\alpha) \,|\, \mathcal{G}_t] = \alpha$ and $\mathrm{CVaR}_{t,T}(\alpha)$ is the loss that will only be exceeded with probability $1-\alpha$. Then the difference

$$\mathrm{CVaR}_{t,T}(\alpha) - E[L_T \,|\, \mathcal{G}_t]$$

is called (conditional) economic capital for time horizon T and the level α. The economic capital is the equity capital that must be held against the portfolio in order to cover its unexpected losses. The level α is imposed by the target solvency rate the investors wishes to maintain. For example, for a triple A target portfolio rating the confidence level is $\alpha = 0.9998$. From the viewpoint of a financial institution, the return of each individual bond in the portfolio must cover both its expected loss and the costs for its marginal economic capital contribution. A risk-averse bank should additionally demand a premium for taking over the bond's default risk at all.

Both variance and CVaR share the deficiency of not being a coherent risk measure in the sense of Artzner, Delbaen, Eber & Heath (1999). In general, variance is not monotone and CVaR typically fails to be sub-additive.[2] Defining capital reserves based on CVaR may thus cause problems when allocating aggregated economic capital to individual desks. Another disadvantage of CVaR is that it does not take care of the probability of losses exceeding the CVaR itself. A remedy is the expected tail loss measure defined by

$$\mathrm{TL}_{t,T}(u) := E[L_T \,|\, L_T > u, \, \mathcal{G}_t], \quad T > t,$$

which is the conditional expected loss given that it exceeds a level u. Since $\mathrm{TL}_{t,T}(u)$ is a coherent risk measure if u is some quantile of the loss distribution, economic capital should be based on TL instead of CVaR.

[2] Embrechts et al. (1999) showed that VaR is subbadditive if losses are elliptically distributed. They also demonstrated that portfolio optimization for some target return based on the variance is in the elliptical world equivalent to optimization based on a positive homogeneous and translation-invariant risk measure. That is, the variance-optimal portfolio minimizes VaR in the elliptical world.

6.2 Loss Distribution

A prerequisite for the computation of the risk measures $\mathrm{CVaR}_{t,T}$ and $\mathrm{TL}_{t,T}$ and hence economic capital is the conditional distribution of default losses L_T for the horizon T. With our assumption of independent recovery rates, for $x \geq 0$ and $T > t$ we obtain

$$P[L_T \leq x \,|\, \mathcal{G}_t] = \sum_{s \in \mathbb{P}(I)} P[\textstyle\sum_{i \in s} \delta_i \leq x] P[S_T = s \,|\, \mathcal{G}_t], \tag{6.1}$$

where $S_T \in \mathbb{P}(I)$ is the default scenario at time T, cf. (3.1). In other words, the random variable S_T is the index set of bonds having defaulted by time T. For any given set $s \in \mathbb{P}(I)$, the probability $P[S_T = s \,|\, \mathcal{G}_t]$ is given by Theorem 3.5. The distribution of $\sum_{i \in s} \delta_i$ is easily obtained via convolution of the laws of the δ_i. However, even for a moderate number $n = |I|$ of portfolio positions the computational burden for (6.1) is quite high. By assuming that the default losses δ_i are identically distributed across firms, we can reduce these efforts considerably. For $x \geq 0$ and $T > t$ we can write in this case

$$P[L_T \leq x \,|\, \mathcal{G}_t] = \sum_{k=0}^{m} P[\textstyle\sum_{i=0}^{k} \delta_i \leq x] P[N_T = k \,|\, \mathcal{G}_t], \tag{6.2}$$

where $\delta_0 := 0$. The random variable $N_T := |S_T| \in \mathbb{N}$ is the number of defaults in the portfolio by time T. As we will see, the iid-recovery assumption is quite realistic in certain applications. It thus remains to determine the distribution of N_T. In the zero-recovery case where $\delta_i = 1$, we simply have $P[L_T = k \,|\, \mathcal{G}_t] = P[N_T = k \,|\, \mathcal{G}_t]$.

If the defaults are independent and the firms have equal default probability $F_t^i(T)$, then N_T has the binomial distribution:

$$P[N_T = k \,|\, \mathcal{G}_t] = \binom{n}{k} F_t^i(T)^k (1 - F_t^i(T))^{n-k},$$

with $\binom{n}{k} = n!/k!(n-k)!$. Clearly $E[L_T \,|\, \mathcal{G}_t] = nF_t^i(T)$ and $\mathrm{Var}\,[L_T \,|\, \mathcal{G}_t] = nF_t^i(T)(1 - F_t^i(T))$.

To analyze and rate a symmetric zero-recovery bond portfolio with default correlation, Moody's Investors Services (1999) suggests the so-called Binomial Expansion Technique. The goal is to replace the original correlated portfolio (having n positions) with some comparison portfolio of $d \leq n$ *independent* bonds with equal default probability p such that the expected loss is equal to that of the original portfolio. The unconditional distribution of the default number in the comparison portfolio is then binomial with parameters p and d. The number d of the bonds in the comparison portfolio is called diversity score. It is easily seen that expected defaults and variance increase as d decreases. d is determined based on the assumption that bonds in the same industry are related while issuers in different industry sectors are independent. This procedure is very crude. It ignores among other things the dependence of all firms on common economic factors (macro-correlation). Though pragmatic, this technique does not account for default correlation in a sensible way.

Rather than replacing the original portfolio, Davis & Lo (2000) extended Moody's approach to incorporate correlation via an infection mechanism: a default may trigger off defaults of other issuers. While keeping the symmetry and industry-independence assumption, they obtain a closed-form default number distribution that reduces to the binomial when there is no infection. Duffie & Garleanu (2001) maintain symmetry and introduce default correlation through dependence of all bonds' default intensity on a common state process. This results in conditional independent defaults, given the path of the state process. Based on this assumption, they derive the distribution of the number of defaulting bonds in analytical form.

The following proposition establishes the law of N_T in terms of the joint default probabilities derived Theorem 3.5. Given this earlier result, we do not need to assume symmetry of issuers, industry independence, or conditional independence. Closed form solutions are however unlikely to arise.

Proposition 6.1. *The $\mathcal{G}_t$-conditional distribution of the number of defaulted bonds at time $T > t$ can be written as*

$$P[N_T = k \,|\, \mathcal{G}_t] = \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} \sum_{\{s_1,\dots,s_i\} \subset I} P[\tau_{s_1} \le T, \dots, \tau_{s_i} \le T \,|\, \mathcal{G}_t].$$

Proof. We define for $i \in I$ the event $A_i := \{\tau_i \le T\}$. For $\{i_1, \dots, i_k\} \subset I$ we denote the complement by $\{i_{k+1}, \dots, i_n\}$. We have

$$\begin{aligned} 1_{\{N_T = k\}} &= \sum_{\{i_1,\dots,i_k\} \subset I} 1_{A_{i_1}} \cdots 1_{A_{i_k}} (1 - 1_{A_{i_{k+1}}}) \cdots (1 - 1_{A_{i_n}}) \\ &= \sum_{\{i_1,\dots,i_k\} \subset I} 1_{A_{i_1}} \cdots 1_{A_{i_k}} \sum_{i=0}^{n-k} (-1)^i \sum_{\{j_1,\dots,j_i\} \subset \{i_{k+1},\dots,i_n\}} 1_{A_{j_1}} \cdots 1_{A_{j_i}}. \end{aligned}$$

There are $\binom{k+i}{k}$ possibilities to decompose a set $\{s_1, \dots, s_{k+i}\} \subset I$ in disjoint sets $\{i_1, \ldots, i_k\}$ and $\{j_1, \ldots, j_i\}$. Therefore,

$$
\begin{aligned}
1_{\{N_T=k\}} &= \sum_{i=0}^{n-k} (-1)^i \sum_{\{s_1,\ldots,s_{k+i}\}\subset I} \binom{k+i}{k} 1_{A_{s_1}} \cdots 1_{A_{s_{k+i}}} \\
&= \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} \sum_{\{s_1,\ldots,s_i\}\subset I} 1_{A_{s_1}} \cdots 1_{A_{s_i}},
\end{aligned}
$$

from which the statement follows by taking conditional expectation. □

6.3 Scenario-Based Credit Derivatives

Having measured the investor's aggregated portfolio credit risk exposure, in this section we will discuss means to mitigate this exposure. Specifically, we will propose and analyze a class of credit derivative contracts which are based on the notion of a default scenario.

A *credit derivative* is a bilateral financial contract involving the separation of the credit risk associated with a reference asset and the transfer of that risk between the two parties. Put another way, the payoff to a credit derivative depends on the credit quality of the underlying asset. This makes it possible to hedge default risk, or to assume a certain credit risk exposure. By disaggregating risk, these contracts contribute to efficiency gains through a process of market completion.

There is a wide variety of credit derivatives contract specifications. One of the most popular types is the credit default swap. Here the party seeking default protection for some reference asset (a *single* corporate bond say) pays a periodic fee in return for a payment by the protection seller, which is made contingent upon the underlying asset experiencing a credit event. Typical credit events are payment default, bankruptcy filing, or credit rating change. Credit swaps can thus be viewed as bond default insurance. Duffie (1999) and Hull & White (2000*a*) review the pricing of such instruments.

The effectiveness of these plain vanilla credit swaps is however limited when the aggregated exposure of several correlated positions (a bond *portfolio* say) is to be hedged. Though each bond could be insured by a single default swap, this would not account for hedge cost reducing diversification effects within the portfolio. Moreover, this maximum security insurance leaves no credit risk at all and hence no default premium margin. We hence look for a single transaction which takes care of a pre-specified part of the portfolio's risk profile.

In the following we will propose a class of flexible credit derivative specifications meeting the requirements above. The key notion we rely on is the *default scenario* S_t at date t, cf. (3.1). The $\mathcal{G}_t$-measurable random variable $S_t \in \mathbb{P}(I)$ is the index set of all bonds having defaulted by time t. From (6.1), for each pair (t, T) the portfolio credit risk is completely described by arrival probability $P[S_T = s \,|\, \mathcal{G}_t]$ and loss $\sum_{i \in s} \delta_i$ for each possible default scenario $s \in \mathbb{P}(I)$.

Definition 6.2 (Scenario-Based Credit Derivative). *The rationale of a scenario-based credit derivative is to generate cash flows upon the occurrence of pre-specified default scenarios. Precisely, it is specified by a tuple (T, I, Q, Y), consisting of*

- *a maturity date T,*
- *a reference obligation index set I,*
- *a set of settlement default scenarios $Q \subseteq \mathbb{P}(I)$, and*
- *payoff specifications given by the $\mathcal{G}_T$-measurable $\mathbb{R}^{|Q|}$-valued random variable $Y := (Y_s)_{s \in Q}$.*

The extension to a time-varying but deterministic list $(Q(t))_{t \in [0,T]}$ is straightforward. We can imagine the following cash flow definitions of this derivative.

All scenarios count. The cash flow at maturity T is defined as

$$\sum_{s \in Q} Y_s \cdot 1_{\{s \subset S_T\}}, \quad Q \subseteq \mathbb{P}(I), \tag{6.3}$$

such that the credit derivative would pay upon arrival of any scenario fixed in Q before T, irrespective of the default status of the remaining bonds. If the settlement scenarios are not mutually disjoint, multiple payments for the same event can occur. We can set, for example, Y_s equal to the default loss in the bond portfolio given scenario s:

$$Y_s := \sum_{i \in s} \delta_i, \quad s \in \mathbb{P}(I), \tag{6.4}$$

which is $\mathcal{G}_{\tau_s}$-measurable for

$$\tau_s := \max_{i \in s} \tau_i, \quad s \in \mathbb{P}(I). \tag{6.5}$$

Under risk-neutrality the value of the derivative at time $t \leq T$ can be written as

$$E[e^{-\int_t^T r_s ds} \sum_{s \in Q} Y_s \cdot 1_{\{s \subset S_T\}} \,|\, \mathcal{G}_t] = d(t,T) \sum_{s \in Q} \sum_{i \in s} \bar{\delta}_i P[\textstyle\bigcap_{i \in s} \{\tau_i \leq T\} \,|\, \mathcal{G}_t], \quad (6.6)$$

where $P[\bigcap_{i \in s} \{\tau_i \leq T\} \,|\, \mathcal{G}_t]$ is directly given by Theorem 3.5.

Scenario at maturity counts. Here the payout at T is also contingent on the default status of the bonds not included in a settlement scenario. The derivative pays only if a settlement scenario is exactly matched at T:

$$\sum_{s \in Q} Y_s \cdot 1_{\{S_T = s\}}, \quad Q \subseteq \mathbb{P}(I). \quad (6.7)$$

Note that $\{S_T = s_1\} \cap \{S_T = s_2\} = \emptyset$ for any different $s_1, s_2 \in \mathbb{P}(I)$:

$$\{S_T = s\} = \bigcap_{i \in s} \{\tau_i \leq T\} \ \cap \bigcap_{i \in I - s} \{\tau_i > T\}. \quad (6.8)$$

With a payoff as in (6.4), we find the derivative's value at time $t \leq T$:

$$E[e^{-\int_t^T r_s ds} \sum_{s \in Q} Y_s \cdot 1_{\{S_T = s\}} \,|\, \mathcal{G}_t] = d(t,T) \sum_{s \in Q} \sum_{i \in s} \bar{\delta}_i P[S_T = s \,|\, \mathcal{G}_t]. \quad (6.9)$$

For any given set $s \in \mathbb{P}(I)$, the probability $P[S_T = s \,|\, \mathcal{G}_t]$ is in view of (6.8) easily calculated from Theorem 3.5 via standard methods.

First-to-occur scenarios. Now the payoff to the derivative is linked to the first settlement scenario to arrive before T:

$$\sum_{s \in Q} Y_s \cdot 1_{\{\tau = \tau_s \leq T\}}, \quad Q \subseteq \mathbb{P}(I), \quad (6.10)$$

with τ_s as in (6.5) and where τ is the first arrival time of any settlement scenario:

$$\tau := \min_{s \in Q} \tau_s = \min_{s \in Q} \max_{i \in s} \tau_i. \quad (6.11)$$

If the settlement scenarios are not mutually disjoint, several settlement scenarios may arrive simultaneously and multiple payments for a single default event can occur.[3] Noting that

$$\begin{aligned} \{\min_{s \in Q} \tau_s = \tau_s \leq T\} &= \{\tau_s \leq T,\, \tau_u \geq \tau_s \,(u \in Q - s)\} \\ &= \bigcap_{i \in s} \{\tau_i \leq T\} \ \cap \bigcap_{u \in Q - s} \bigcup_{j \in u} \bigcap_{i \in s} \{\tau_i \leq \tau_j\}, \end{aligned}$$

[3]If this is in fact not desired, a correction payoff deduction taking care of double payments is easily envisaged.

for fixed t and $s \in Q$, we define the conditional density

$$\begin{aligned} f_t^s(x)dx &:= P[\tau_s \in dx,\, \tau_u \geq x\,(u \in Q - s) \,|\, \mathcal{G}_t], \quad x > t \\ &= -\frac{\partial}{\partial T_s} P[\tau_u \geq T_u\,(u \in Q) \,|\, \mathcal{G}_t]\Big|_{T_u = x\,(u\in Q)} dx \\ &= \frac{\partial}{\partial T_s} P[\textstyle\bigcup_{u\in Q} A_u(T_u) \,|\, \mathcal{G}_t]\Big|_{T_u = x\,(u\in Q)} dx, \end{aligned}$$

where $A_u(T_u) := \bigcap_{i\in u}\{\tau_i \leq T_u\}$. Since $P[A_u(T_u) \,|\, \mathcal{G}_t]$ is directly available from Theorem 3.5, we can obtain $P[\bigcup_{u\in Q} A_u(T_u) \,|\, \mathcal{G}_t]$ from that via standard inclusion-exclusion type arguments. Assuming that $f_t^s(x)$ exists for all $s \in \mathbb{P}(I)$ and $x > t \geq 0$, we can write for the price of the scenario-linked derivative

$$E[e^{-\int_t^T r_s ds} \sum_{s\in Q} Y_s\, 1_{\{\tau = \tau_s \leq T\}} | \,\mathcal{G}_t] = d(t,T) \sum_{s\in Q}\sum_{i\in s} \bar{\delta}_i \int_t^T f_t^s(x)dx. \qquad (6.12)$$

The more standardized *first-to-default* structure is obtained as a special case of a first-to-occur scenario structure by setting

$$Q := \{s \in \mathbb{P}(I) :\ |s| = 1\}.$$

Thus the derivative pays upon the first default in the portfolio before the contract's maturity. Valuation of this structure has been analyzed by Duffie (1998) and Kijima (2000) in a pure intensity based framework for exogenously given intensities. Kijima (2000) assumes additionally that defaults are conditionally independent. In our scenario-based framework, the market value of a first-to-default contract at time $t < T$ collapses from (6.12) to

$$d(t,T) \sum_{i\in I} \bar{\delta}_i \int_t^T \frac{\partial}{\partial T_i} P[\textstyle\bigcup_{j\in I}\{\tau_j \leq T_j\} \,|\, \mathcal{G}_t]\Big|_{T_j = x\,(j\in I)} dx.$$

The popularity of this structure relies on the fact that it provides the investor (the seller of default protection) with an enhanced yield compared to any individual obligation in the basket I. This is because $P[\bigcup_{i\in I}\{\tau_i \leq T\}] \geq P[\tau_i \leq T]$ for all $i \in I$. This feature is relevant to investors who are constrained to investments over a certain credit quality level on stand-alone basis. The protection effect and the yield enhancement effect of a first-to-default structure decrease with increasing default correlation. It therefore makes sense to compose a reference basket with bonds of equal face, equal seniority, and similar but uncorrelated high credit quality. If, for example, the credit quality were too different, then the bond with the highest default probability would dominate the pricing of the basket. Now the assumption of iid losses δ with expected value $\bar{\delta}$ is reasonable. In this case the derivative has a cash flow at time T of $\delta \cdot 1_{\{N_T \geq 1\}}$ and we obtain for the contract's market value

$$d(t,T)\,\bar{\delta}(1 - P[N_T = 0 \,|\, \mathcal{G}_t]) = d(t,T)\,\bar{\delta}\,P[\textstyle\bigcup_{i \in I}\{\tau_i \leq T\} \,|\, \mathcal{G}_t],$$

where $P[N_T = i \,|\, \mathcal{G}_t]$ is directly calculated in Proposition 6.1. We can easily generalize to a kth-to-default structure having a cash flow of

$$\delta \sum_{i=1}^{k} i \cdot 1_{\{N_T = i\}} + \delta \cdot k \cdot 1_{\{N_T > k\}}$$

and hence a value of

$$d(t,T)\bar{\delta}\Big(\sum_{i=0}^{k}(i-k)\,P[N_T = i \,|\, \mathcal{G}_t] + k\Big).$$

The buyer of a scenario-based credit derivative is insured against the losses in the reference bond portfolio due to arrivals of all those default scenarios fixed in the list Q. The cash flow triggering mechanism can be varied, cf. (6.3), (6.7), and (6.10). These structures share an inherent hedging leverage: one contract covers a collection of names at an attractive cost, because it incoporates default correlation effects. The settlement scenario list Q can for example contain critical crash scenarios identified by some stress testing procedure. By including a scenario 'default of security issuer before maturity', we can account for the realistic situation where the issuer is itself subject to default. This additional risk should reduce the cost of insurance, if it not questions the contract at all.[4]

A default scenario-based credit derivative can for example be structured as a swap transaction. A swap with maturity date T would stipulate that counterparty

A pays an annuity at a constant swap rate R at fixed dates $T_1 < T_2 < \ldots < T_K = T$ (fee leg);

[4]This issue of counterparty credit risk is in fact a focal point in regulatory authorities' discussion on credit risk management and mitigation. The effectiveness of credit derivatives in reducing default loss risk forms the basis for regulatory capital relief when insuring positions against default. The degree of relief has clearly to account for the credit risk of the protection provider as well as its correlation with the bonds to be default-insured. For a discussion of recent advances in the regulatory treatment of credit, including the issues raised here, we refer to Basel Committee on Banking Supervision (1999) and International Swap and Derivatives Association (2000).

B pays the default loss at T according to the stipulated scheme (contingent leg), [5]

to the opposite party. The scenario swap involves thus the transfer of the default risk associated with counterparty A's bond portfolio to party B. Party A's relationship to the bond issuers is being maintained. Party B assumes the particular default risk profile of the bond portfolio without having to invest capital in it. This construction allows party B to access the bond market in order to gain diversified corporate default exposure. If neither counterparty has a contractual relationship to the underlying portfolio, a synthetic investment opportunity is created.[6]

The swap rate R has to be determined at inception of the contract. The fair rate R must compensate the protection seller B for assuming the risk of having to pay upon a settlement scenario arrival. R is thus such that the market value of the swap is zero at inception ($t = 0$ say). Equating the value of fee and contingent leg, we find that the swap rate for payoff scheme (6.3) must satisfy

$$R = \frac{d(0,T) \sum_{s \in Q} \sum_{i \in s} \bar{\delta}_i P[\bigcap_{i \in s} \{\tau_i \leq T\}]}{\sum_{j=1}^{K} d(0,T_j)},$$

where we have used the fact that the swap rate is paid until T, irrespective of settlement scenario arrivals. Likewise, for payoff scheme (6.7) we have

$$R = \frac{d(0,T) \sum_{s \in Q} \sum_{i \in s} \bar{\delta}_i P[S_T = s]}{\sum_{j=1}^{K} d(0,T_j)}.$$

In scheme (6.10), the swap rate is paid until time $\min(\tau, T)$, where τ is the first arrival time of any settlement scenario, cf. (6.11). Now the value of the fee leg is found to be

$$\begin{aligned} E[\textstyle\sum_{T_j \leq T} e^{-\int_0^{T_j} r_s ds} R\, 1_{\{\tau \geq T_j\}}] &= R \sum_{T_j \leq T} d(0,T_j) P[\min_{s \in Q} \max_{i \in s} \tau_i \geq T_j] \\ &= R \sum_{T_j \leq T} d(0,T_j)(1 - P[\textstyle\bigcup_{s \in Q} \bigcap_{i \in s} \{\tau_i \leq T_j\}]), \end{aligned}$$

[5]The contract can in principle call for physical settlement, where A would sell the portfolio of obligations to B at par, or for cash settlement, where A retains the defaulted bonds and collects the difference between pre-default and post-default bond value. The post-default bond value is usually determined by a dealer poll.

[6]Here the difference to a 'classic' insurance contract, where the insurance buyer must own the object to be insured, becomes evident.

with $P[\bigcup_{s\in Q}\bigcap_{i\in s}\{\tau_i \leq T_j\}]$ being determined by Theorem 3.5. We suppose that the first-to-occur scenario contract does not require the payment of the accrued swap rate by party A in case $\tau \leq T$. If it does indeed, we can easily model an adjustment in the swap rate by reducing B's loss payment by the accrued annuity at τ. The effect on R will however be negligible for reasonable small rate date periods and default probabilities. Since the value of the contingent swap leg is now given by (6.12), the swap rate satisfies

$$R = \frac{d(0,T)\sum_{s\in Q}\sum_{i\in s}\bar{\delta}_i\int_t^T f_t^s(x)dx}{\sum_{T_j\leq T} d(0,T_j)(1 - P[\bigcup_{s\in Q}\bigcap_{i\in s}\{\tau_i \leq T_j\}])}.$$

So far we have considered contracts focusing on occurrences of default events. Having the portfolio's loss distribution at hand, we can also consider structures with payoffs contingent on the actual default loss suffered by the portfolio investor. We can think of an European-type structure paying

$$(L_T - B)1_{\{L_T\geq B\}} = (L_T - B)^+ \tag{6.13}$$

at the maturity T of the contract. B is some constant loss threshold, for example $B = \mathrm{CVaR}_{t,T}(0.9998)$. Such a contract would be similar to that of a basket or index option with strike B, where the underlying has price process (L_t). We have for the value of this credit derivative

$$d(t,T)\Big(\int_B^\infty xP[L_T \in dx \,|\, \mathcal{G}_t] - BP[L_T > B \,|\, \mathcal{G}_t]\Big),$$

which is easily evaluated using (6.1) or (6.2).

6.4 Collateralized Debt Obligations

A collateralized debt obligation (CDO) is a structured fixed income security with cash flows linked to the performance of a reference pool of debt instruments (e.g. corporate or sovereign bonds or loans). It involves the transfer of ownership of the reference assets to a special purpose vehicle (SPV). The SPV issues prioritized tranches of securities that are collateralized by the obligations in the pool. In a 'synthetic' CDO transaction, the issuer retains the assets on its balance sheet but transfers the credit risk exposure to the SPV by means of credit derivatives. Issuers of *balance-sheet* CDOs in form of collateralized loan obligations are typically banks aiming at restructuring the credit risk profile of their loan portfolio and releasing risk-based capital. *Arbitrage*

CDOs in form of collateralized bond obligations are often issued by insurance companies and money managers whose goal is to lock in term funding and to earn incremental fee income. For more structural details we refer to Tierney & Punjabi (1999). Applying our previous results, in this section we examine the risk and valuation of a basic CDO structure.

Suppose our investor wishes to structure a CDO in order to distribute the credit risk of the bond portfolio I to the capital market. The CDO is of the cash flow type, i.e. the collateral is not subjected to active trading by the CDO manager. The maturity of the CDO is denoted by T; for simplicity we let $T_i = T$ for all $i \in I$. At inception of the CDO ($t = 0$ say), the SPV acquires the investor's collateral pool for a market value of $z_0 := \sum_{i \in I} p_i(0, T)$. The principal payment n is due at T. This acquisition is financed by issuing two CDO tranches: senior coupon bonds with a total face value of f, and a subordinated equity position with face $m - f$. To these tranches the SPV allocates any income (principal, interest and recoveries) received from the collateral pool. Management fees are neglected. The bonds promise to pay a quarterly coupon at rate c at dates T_j, $j = 1, \ldots, K$. Given sufficient cash flow is available, the senior bonds are repaid their principal at $T_K = T$. Equity investors collect any residual receipts at T, with no guaranteed coupon or principal repayment.

The rationale of the CDO is to repackage the credit risk of the collateral pool. The senior tranche concentrates only 'good' risks (low default probabilities, low default correlation) so that the bonds have a relatively high credit quality. This risk separation is achieved by having the equity tranche absorbing losses up to some extent. The equity position hence acts as a credit enhancement facility for the senior bond investors. The return to the CDO investors is solely determined by the *joint* default performance of the collateral bonds.

To assess the credit quality of the tranches, we must evaluate the losses that can potentially suffered by the CDO investors. The quality of the senior tranche then fixes its coupon rate c. Given some target bond rating, we must determine whether the loss probability is consistent with this rating. In other words, we need to check whether the size of the equity tranche is sufficient to support the bonds. We now address these issues. The difference $m - z_0$ gained at inception of the CDO will be deposited in a reserve account, which accrues interest at the riskless rate. From this account the bond coupon is paid. Its post-coupon value $R(T_k, f, c)$ at coupon date T_k is given by

$$R(T_k, f, c) = \Big(\frac{R(T_{k-1})}{d(T_{k-1}, T_k)} - c \cdot f\Big)^+, \quad k = 1, \ldots, K,$$

where $R(0, f, c) = m - z_0$. If, at some coupon date, the initial reserve $R(0, f, c)$ is not sufficient to pay the interest $c \cdot f$, unpaid coupon is accrued at the riskless rate.[7] At maturity $T = T_K$ there is a total accrued coupon of

$$U(T, f, c) = \sum_{k=1}^{K} \frac{1}{d(T_k, T)} (c \cdot f - R(T_k, f, c))^+.$$

In order to maintain a high rating for the senior bonds, the CDO is typically structured such that $U(T, f, c) = 0$. At time T, the promised cash flow to the bond investors is comprised of principal f, last coupon $c \cdot f$ and any accrued interest. The cash flow from the collateral at T is $m - L_T$, where $L_T = \sum_{i \in I} \delta_i \cdot 1_{\{\tau_i \leq T\}}$ is the random loss in collateral principal at T. Defining the $\mathcal{G}_T$-measurable random variable

$$B(T, f, c) := m + R(T, f, c) - U(T, f, c) - f,$$

it follows that the loss experienced by the bond investors at T is given by

$$(L_T - B(T, f, c))^+.$$

This resembles the payoff of a call option on the collateral loss with (stochastic) strike $B(T, f, c)$. Note that $B(T, f, c)$ depends on riskless interest rates only, but not on collateral defaults. In view of (6.13), bond investors' loss and thus the market value of the CDO-bonds is easily evaluated given the independence of interest rates and defaults. Knowing that Var (L_T) is increasing in the degree of default correlation, the market value of the CDO-bond tranche decreases with default correlation. This is very similar to the manner in which an option benefits from an increase in the volatility of the underlying.

Given some $\alpha \in (0, 1)$ and coupon rate c imposed by the target credit rating of the CDO-bonds, at inception of the CDO we choose the face value $f \in (0, m)$ of the bonds such that

$$P[L_T > B(T, f, c)] \leq \alpha.$$

The higher f, the lower the remaining face value $m - f$ of the equity tranche and the fewer subordination is available to the bond issue. The degree of

[7] Alternatively, one may stipulate that unpaid coupon is accrued at its own rate c.

subordination corresponds to the degree of loss protection provided by the equity position. Given some fixed pair (f, c), the probability $P[L_T > x]$ for $x \in [B(T, f, c), n]$ and the conditional loss expectation $E[L_T \mid L_T > B(T, f, c)]$ are easily computed measures of the riskiness of the bond issue.

After the bonds have been paid in full, any residual cash flow goes to the equity issue, which has neither a guaranteed coupon nor guaranteed principal repayment. The return to equity investors is maximal if no collateral bond defaults. Consequently, already the first default will reduce equity investors' gain from the structure. At T their loss (relative to no defaults) is given by

$$\min(L_T, \bar{B}(T, f, c)) = \bar{B}(T, f, c) - (\bar{B}(T, f, c) - L_T)^+,$$

where $\bar{B}(T, f, c) := B(T, f, c) \wedge (m - f)$. We have $\bar{B}(T, f, c) = B(T, f, c)$ if $U(T, f, c) \geq 0$, i.e. if at least one scheduled coupon has been missed. Thus the loss position of the equity issue resembles that of a loan $\bar{B}(T, f, c)$ and short put on the collateral losses with strike $\bar{B}(T, f, c)$. In effect, the equity investors take a leveraged position in the collateral pool. Their payoff resembles that of a straight put position on the collateral loss:

$$(B(T, f, c) - L_T)^+.$$

Since the put value increases in Var (L_T), the value of the equity tranche is increasing in the degree of default correlation. This means that, in contrast to the bond investors, equity investors actually *benefit* from default correlation. Because the total collateral value is not effected by the degree of diversification, the effects of default correlation in both tranches must offset each other. Having the loss distribution at hand, we can assess the risk of the equity issue by calculating the risk measures $E[L_T \mid L_T \leq \bar{B}(T, f, c)]$, and $P[L_T \leq x \mid L_T \leq \bar{B}(T, f, c)]$ for given $x \in [0, \bar{B}(T, f, c)]$.

There are some reasons suggesting that the equity tranche should be retained by the CDO-issuer, cf. Duffie & Garleanu (2001). In perfect capital markets, CDOs would not exist. Their existence in practice hinges critically on the existence of market imperfections, such as capital requirements and illiquidity of collateral positions. Illiquidity can have several sources, including adverse selection and moral hazard. Asymmetric information between seller and buyer of a collateral bond may lead to a lower transaction price, since the buyer is likely to demand some lemon's premium (compared to the price that would obtain given symmetric information). Though this effect cannot be eliminated completely through securitization of the collateral via a CDO, it can be reduced if the issuer retains a subordinated tranche. This tranche concentrates the collateral subject to adverse selection effects. The lemon's premium on the remaining collateral is then reduced to a relatively low level.

Moral hazard refers primarily to an incentive reduction for the CDO issuer in the selection of high-quality collateral and the costly enforcement of borrower interests after issuance. These potential incentive problems may lead to reductions in the valuation of collateral. They speak in fact against the creation of the CDO. These effects may be mitigated if the issuer demonstrates sufficient commitment by retaining a subordinated position. Then the issuer suffers losses resulting from poor monitoring or asset quality first. In sum, higher valuations can be achieved and the reduction of the effects of imperfect capital markets via the CDO may be larger than the effects of moral hazard, if the structure embeds prioritized tranches and the junior tranche is retained by the issuer.

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Appendix A

Copulas in a Nutshell

This Appendix is devoted to a brief review of copula functions, where we will provide all those results that are used in this thesis. For more details we refer to Nelsen (1999), Joe (1997), Embrechts et al. (1999), and Lindskog (2000). Throughout, we fix an $\mathbb{R}^n$-valued random vector $X := (X_1, \ldots, X_n)$ with marginal distribution functions $G_1, \ldots, G_n$ and joint distribution function G. We will denote by $\mathrm{Ran}H$ the range of a function H.

Let us start with a definition.

Definition A.1. *A copula is the distribution function of a random vector with all marginal distributions being uniform on* $[0,1]$. *Equivalently, a copula is a function* $C : [0,1]^n \to [0,1]$ *with the following properties:*

(1) $C(u_1, \ldots, u_n)$ *is increasing in each* u_i.

(2) $C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i$ *for all* $1 \le i \le n$ *and* $u_i \in [0,1]$.

(3) For all $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_n) \in [0,1]^n$ *with* $a_i \le b_i$ *for all* $1 \le i \le n$,

$$P[a_1 \le X_1 \le b_1, \ldots, a_n \le X_n \le b_n] = \sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1+\cdots+i_n} C(u_{1i_1}, \ldots, u_{ni_n}) \ge 0,$$

where $u_{j1} = a_j$ *and* $u_{j2} = b_j$.

The following result shows that the copula is the function that 'couples' the joint distribution function with its marginal distribution functions:

Theorem A.2 (Sklar (1996)). *There exists a copula C such that*

$$G(x) = C(G_1(x_1), \ldots, G_n(x_n)), \quad x \in \mathbb{R}^n.$$

If the G_i are all continuous, C is unique; otherwise, C is uniquely determined on $RanG_1 \times \ldots \times RanG_n$.

Define the generalized inverse of G_i by

$$G_i^{(-1)}(u) := \inf\{x \in \mathbb{R} \,:\, G_i(x) \geq u\}, \quad u \in [0,1].$$

Of course, if G_i is strictly increasing, then $G_i^{(-1)}$ is the ordinary inverse. The subsequent result is a corollary to Theorem A.2. It provides a useful method for constructing copulas from joint and marginal distributions (for general details and examples see, e.g., Joe (1997)).

Corollary A.3. *For continuous G_i, we have*

$$C(u) = G(G_1^{(-1)}(u_1), \ldots, G_n^{(-1)}(u_n)), \quad u \in [0,1]^n.$$

For example, letting Φ^2 (resp. Φ) denote the bivariate (resp. univariate) normal cumulative distribution function, the bivariate normal copula is given by

$$\begin{aligned} C_\rho^N(u,v) &= \Phi_\rho^2(\Phi^{-1}(u), \Phi^{-1}(v)) \\ &= \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(\frac{2\rho xy - x^2 - y^2}{2(1-\rho^2)}\right) dx\, dy \end{aligned}$$

where $\rho \in [0,1]$ is the linear correlation coefficient. Denoting by t_ν the univariate standard t-distribution function with ν degrees of freedom, the bivariate t-copula can be written as

$$C_{\nu,\rho}^t(u,v) = \int_{-\infty}^{t_\nu^{-1}(u)} \int_{-\infty}^{t_\nu^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \left(1 + \frac{x^2 - 2\rho xy + y^2}{\nu(1-\rho^2)}\right)^{-(\nu+2)/2} dx\, dy.$$

A simple closed-form copula family is the Frank family:

$$C_\theta^F(u,v) = -\frac{1}{\theta} \ln\left(1 + \frac{(e^{-\theta u} - 1)(e^{-\theta v} - 1)}{e^{-\theta} - 1}\right),$$

where the parameter $\theta \in \mathbb{R}-\{0\}$ controls the degree of dependence (see below).

Theorem A.2 suggests to interpret a copula as the *complete dependence structure* of X (if C is not unique we speak of a possible dependence structure).

We now prove that a copula is invariant under increasing transformations of the random variables, which undermines this interpretation.

Proposition A.4. *Let $X_1, \dots, X_n$ be continuous random variables with copula C. If each T_i is strictly increasing on $RanX_i$, then the transformed vector $T(X) := (T_1(X_1), \dots, T_n(X_n))$ has copula C also.*

Proof. We denote by G_i and G_i^T the distribution function of X_i and $T_i(X_i)$, respectively. Letting C^T denote the copula of $T(X)$, using Theorem A.2 and the fact that each $T_i(X_i)$ is a continuous random variable, we have

$$\begin{aligned} C^T(G_1^T(x_1), \dots, G_n^T(x_n)) &= P(T_1(X_1) \le x_1, \dots, T_n(X_n) \le x_n) \\ &= P(X_1 \le T_1^{-1}(x_1), \dots, X_n \le T_n^{-1}(x_n)) \\ &= C(G_1(T_1^{-1}(x_1)), \dots, G_n(T_n^{-1}(x_n))) \\ &= C(G_1^T(x_1), \dots, G_n^T(x_n)), \end{aligned}$$

where the last line is due to the relation $G_i(T_i^{-1}(x)) = P(X_i \le T_i^{-1}(x)) = P(T_i(X_i) \le x) = G_i^T(x)$ for all $x \in \mathbb{R}$. $\square$

For strictly decreasing transformations we have the following:

Proposition A.5. *Let $X_1, \dots, X_n$ be continuous random variables with copula C. If each T_i is strictly decreasing on $RanX_i$, then $T(X)$ has copula C^T given by*

$$C^T(u_1, \dots, u_n) = \sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1+\cdots+i_n} C(v_{1i_1}, \dots, v_{ni_n}) \tag{A.1}$$

where $v_{j1} = 1 - u_j$ and $v_{j2} = 1$.

Proof. Using the notation of Proposition A.4, it follows that

$$\begin{aligned} C^T(G_1^T(x_1), \dots, G_n^T(x_n)) &= P(T_1(X_1) \le x_1, \dots, T_n(X_n) \le x_n) \\ &= P(X_1 > T_1^{-1}(x_1), \dots, X_n > T_n^{-1}(x_n)) \\ &= \sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1+\cdots+i_n} P(X_1 \le y_{1i_1}, \dots, X_n \le y_{ni_n}) \\ &= \sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1+\cdots+i_n} C(G_1(y_{1i_1}), \dots, G_n(y_{ni_n})) \end{aligned}$$

where $y_{j1} = T_j^{-1}(x_j)$ and $y_{j2} = \infty$ (the last equality is again a consequence of Theorem A.2). Now the result follows by noting that

$$G_i(T_i^{-1}(x)) = P(X_i \leq T_i^{-1}(x)) = P(T_i(X_i) > x) = 1 - G_i^T(x)$$

for all $x \in \mathbb{R}$. □

The next statement is easy to verify.

Proposition A.6. *The continuous random variables $X_1, \ldots, X_n$ with copula C are mutually independent iff $C = \Pi$, where $\Pi(u_1, \ldots, u_n) := u_1 \cdots u_n$.*

As a joint distribution, C satisfies a version of the Fréchet-bounds inequality (cf. Nelsen (1999) for a proof):

Proposition A.7. *Define the functions*

$$M(u) := \min(u_1, \ldots, u_n)$$
$$W(u) := \max(u_1 + \ldots + u_n - n + 1, 0), \quad u \in [0,1]^n.$$

If C is a copula, we have

$$W(u) \leq C(u) \leq M(u), \quad u \in [0,1]^n.$$

To understand the bounds M and W let us consider the instructive bivariate case $n = 2$. Here both bounds are themselves copulas since, for $U \sim U[0,1]$,

$$M(u,v) = P[U \leq u, U \leq v]$$
$$W(u,v) = P[U \leq u, 1 - U \leq v],$$

so that M and W are the bivariate distributions functions of the random vectors (U, U) and $(U, 1-U)$, respectively. Thus the support of M is the diagonal between $[0,0]$ and $[1,1]$, that of W is the diagonal between $[0,1]$ and $[1,0]$. If X_1 and X_2 are continuous, we have

$$C = M \quad \Leftrightarrow \quad X_2 = T(X_1), \quad \text{a.s. }, \; T = G_2^{-1} \circ G_1 \text{ increasing} \tag{A.2}$$
$$C = W \quad \Leftrightarrow \quad X_2 = T(X_1), \quad \text{a.s. }, \; T = G_2^{-1} \circ (1 - G_1) \text{ decreasing}, \tag{A.3}$$

cf. Embrechts et al. (1999). Thus the copula M describes perfect positive dependence and W describes perfect negative dependence. This motivates the following terminology.

Definition A.8. *If (X_1, X_2) has copula M, then X_1 and X_2 are called* comonotonic*; if it has copula W, they are called* countermonotonic*.*

The upper bound M is also a copula for all $n > 2$, so that statement (A.2) carries over to the general multivariate case. Since for $n > 2$, W does not satisfy property (3) of Definition A.1, the lower bound W fails to be a copula for $n > 2$. However, for all $n > 2$ and any $u \in [0,1]^n$ there exists a copula C such that $C(u) = W(u)$, cf. Nelsen (1999).

Proposition A.7 suggests a partial ordering on the set of copulas:

Definition A.9. *If C_1 and C_2 are copulas, we say that C_1 is smaller than C_2 (we write $C_1 \prec C_2$) if $C_1(u) \leq C_2(u)$ for all $u \in [0,1]^n$.*

Thus, for any copula C we have $W \prec C \prec M$.

Let us finally introduce the concept of tail dependence of copulas. In the bivariate case, tail dependence concerns the amount of dependence in the lower or upper quadrant tail of a bivariate distribution.

Definition A.10. *Let the continuous random vector (X_1, X_2) have copula C. Define the coefficient of lower tail dependence by*

$$\chi^L := \lim_{u \downarrow 0} \frac{C(u,u)}{u}.$$

If $\chi^L \in (0,1]$ then C is said to exhibit lower tail dependence. If $\chi^L = 0$ we speak of asymptotic independence in the lower tail. Define the coefficient of upper tail dependence by

$$\chi^U := \lim_{u \downarrow 0} \frac{1 - 2u + C(u,u)}{u}.$$

If $\chi^U \in (0,1]$ then C is said to exhibit upper tail dependence. If $\chi^U = 0$ we speak of asymptotic independence in the upper tail.

For example, the Gaussian copula exhibits asymptotic independence in both tails and the t-copula exhibits lower and upper tail dependence. With the quantile function $G_i^{(-1)}$ we can write

$$\chi^L = \lim_{u \downarrow 0} P[X_2 \leq G_2^{(-1)}(u) \,|\, X_1 \leq G_1^{(-1)}(u)],$$

given the limit exists. Similarly,

$$\chi^U = \lim_{u \downarrow 0} P[X_2 > G_2^{(-1)}(u) \,|\, X_1 > G_1^{(-1)}(u)],$$

showing the meaning of tail dependence.

Selbständigkeitserklärung

Hiermit erkläre ich, die vorliegende Arbeit selbständig ohne fremde Hilfe verfaßt zu haben und nur die angegebene Literatur und Hilfsmittel verwendet zu haben.

Kay Giesecke
01.06.2001