Efficient Hedging In Incomplete Markets Under Model Uncertainty

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Präsident der Humboldt-Universität zu Berlin:
Prof. Dr. Jürgen Mlynek

Dekan der Mathematisch-Naturwissenschaftlichen Fakultät II:
Prof. Dr. Elmar Kulke

Gutachter:

1. Prof. Dr. H. Föllmer
2. Prof. Dr. I. Karatzas
3. Prof. Dr. M. Schweitzer

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Abstract

We consider an investor who has sold a contingent claim and intends to minimize the maximal expected weighted shortfall. Here, the maximum is taken over a family of models and the minimum is taken over all admissible hedging strategies that satisfy a given cost constraint. We call the associated minimizing strategy robust-efficient. The problem to determine a robust-efficient strategy is closely related to the statistical problem of testing a composite hypothesis against a composite alternative. The hypothesis is given by the family of pricing rules and the alternative coincides with the family of models.

The mathematical centerpiece of this thesis is the solution of the statistical testing problem on a general level by means of convex duality and game-theoretical methods. The problem differs from the classical testing problem in that the power of a test is defined in terms of a strictly concave state dependent utility function rather than the identity mapping. Furthermore, our only essential assumption is that the alternative and the hypothesis are dominated, i.e., the alternative and the hypothesis need neither be parameterized nor of the form of the neighborhoods typically considered in robust statistics. Similar to the classical notion of least-favorable pairs of prior-distributions on the hypothesis respectively alternative, we introduce the pivotal notion of a least-favorable pair of elements of the hypothesis respectively alternative. The main result of our analysis on maximin-optimal tests is that the maximin-optimal test can be found among the simple-optimal test for a least-favorable pair. If the least-favorable pair is equivalent to the dominating measure, the simple optimal test is the unique maximin-optimal test. If the latter condition is not fulfilled, we approximate the maximin-optimal test by a sequence of explicitly constructed simple optimal tests.

These results clarify the general structure of the robust-efficient hedging strategy. We also show that a least-favorable pair can be decomposed into a worst-case model and a worst-case pricing rule for this model. The worst-case model has a very direct economic interpretation, whereas the worst-case pricing rule is a more mathematical auxiliary tool. If the worst-case model dominates all models, the efficient strategy associated to the fixed worst-case model is robust-efficient. For fixed model, the worst-case pricing rule yields the optimal modified claim and allows us to make some statements about its attainability.

In the second part of this thesis, we explicitly construct the robust-efficient strategy in a series of applications. For this, the task remains to determine the efficient strategy for each fixed model and a worst-case model. First, we enlarge the family of models in order to establish existence of a worst-case model. Then we derive the dynamics of the price process, the efficient strategy and the associated risk under each (fixed) model. If the model is incomplete, we adapt the dynamic programming principle to the specific dynamics of the model to compute or approximate the efficient strategy.

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Introduction

In this thesis, we consider an investor who has sold an option contract and intends to control the associated shortfall risk by means of a dynamic hedging strategy. In doing so, he is faced with an incomplete financial market as well as uncertainty regarding the underlying model. We examine hedging strategies that satisfy a given cost constraint and allow the investor to minimize the maximal expected weighted shortfall where the maximum is taken over a family of models. This problem is motivated and explained as follows:

It is well known that in a complete or incomplete financial market free of arbitrage, there is a dynamic hedging strategy that allows the investor to super-replicate the option's payoff, thereby eliminating all shortfall risk associated to the option contract. In a complete market, this strategy actually replicates the payoff exactly, hence rendering option contracts redundant. If exact replication is not possible, the market is incomplete. In this situation, the investor can still establish a *superhedging* strategy, i.e., a strategy that yields a value that is an upper bound on the option's payoff at maturity, cf. [Kar97], [KQ95], [Kra96] and [FK97].

However, the price the investor has to pay for the perfect protection provided by a superhedging strategy is unfeasibly high in incomplete markets. More precisely, the superhedge-price, i.e., the initial capital the investor must bring up to follow a superhedging strategy will typically exceed the option-premium the investor receives when entering the contract. This is true in the realistic situation where the option is not redundant and the option premium does not allow for arbitrage. For this reason, [Cvi00], [CK99], [FL99] and [FL00] examine the problem what the investor can do in an incomplete market if he invests less capital than the superhedge price of the option. Under this *cost constraint*, the investor is unable to eliminate all risk associated to the option contract: The *shortfall*, i.e., the difference between the option's payoff and the value of the investor's hedging strategy at maturity, will be positive with positive probability. The mentioned studies differ in the choice of the risk-measure that is used to quantify shortfall risk and consequentially allows to compare the performance of different strategies.

[FL00] considered risk measures ρ of the form $\rho(S) = E_P[l(S)]$ where S is the shortfall of the investor, i.e., a nonnegative random variable on a measurable space Ω, P is a fixed probability measure on Ω and $l : \mathbb{R} \to \mathbb{R}$ is a strictly convex function. They called a strategy that minimizes the associated shortfall risk among all strategies satisfying the cost-constraint *efficient*. The risk-measure underlying the analysis in [FL99] is closely related to the value at risk (VaR) of the shortfall under a given model P: The authors examine "quantile"-hedging strategies which maximize the probability that there is no shortfall. By definition, efficient respectively quantile hedging strategies depend on the choice of the underlying measure P. Hence the investor is faced with "risk risk", i.e., the risk of being exposed to a higher than calculated risk due to incorrect model assumptions. We intend to remedy this effect,

i.e., to find a version of efficient hedging strategies that are optimal for a given class $\mathcal{U}$ of possible models P.

Independently of [FL00], [Cvi00] and [CK99] examine hedging strategies that minimize the expected shortfall (i.e., they consider $l(x) = x$ in the setting of [FL00]) for fixed P in a diffusion-model with constraints on the strategies. They then formulate the problem of determining a *worst-case model* $\tilde{P}$, i.e., a model that maximizes the minimal shortfall risk over all $P \in \mathcal{U}$ for a given family of models $\mathcal{U}$. This corresponds to the maximin version of the minimax problem considered in this thesis (see below). [Cvi00] and [CK99] show that the values of these problems coincide if $\mathcal{U}$ contains all equivalent martingale measures. In this thesis we are primarily interested in finding an optimal strategy that minimizes the maximal risk. We will also examine sufficient conditions for the existence of a saddle point.

The class of risk measures considered in this thesis is given by $\rho(S) = \sup_{P\in\mathcal{U}} E_P[l(S)]$ for a given family $\mathcal{U}$ of probability measures P and a strictly convex loss function l that allows to adjust for the investor's appetite for risk. These risk measures belong to the class of *convex* measures of risk introduced by [FS00]. We say that a strategy is *robust-efficient* if it minimizes the shortfall risk $\rho(S)$ among all strategies satisfying the cost-constraint. The choice of the risk measure ρ enables the investor to measure risk with respect to a class of models $\mathcal{U}$ rather than a fixed model P, thus reducing "risk risk" if $\mathcal{U}$ is chosen properly. The only requirement we make on $\mathcal{U}$ is the existence of a dominating measure, i.e., we assume the existence of a finite measure R that dominates all models $P \in \mathcal{U}$. We show how the general results found in this thesis can be applied to different specifications of $\mathcal{U}$: Adapting concepts from robust-statistics, we consider $L^p(R)$ neighborhoods of a given model. We also examine the case where $\mathcal{U}$ has a continuous parameterization over a compact separable metric space.

Let β^* denote the value of the problem, i.e., the risk $\rho(S)$ associated to the robust-efficient strategy. The value β^* is a benchmark for the performance of alternative strategies satisfying the same cost-constraint since it is not possible to achieve a lower risk without investing more initial capital. We will frequently encounter situations where the robust-efficient strategy and the value β^* are difficult to determine but where we are able to find an approximative strategy whose performance is guaranteed to deviate no more than ϵ from the benchmark β^* for given $\epsilon > 0$. Related results can be found in Theorems 2.32, 3.14 and 6.10.

In addition to the cost constraint, hedging strategies have to satisfy certain *admissibility* conditions that preclude doubling strategies. Given a fixed model P, the standard definition of admissibility of a strategy ξ is that the value process associated to the strategy remains nonnegative at all times P-almost surely . In our setting, this requirement has to be met simultaneously for all models $P \in \mathcal{U}$. If $\mathcal{U}$ is uncountable, this condition may be very difficult to vindicate. We therefore assume that $\mathcal{U}$ is dominated. By the Halmos-Savage Theorem 3.17, this implies existence of a probability measure R equivalent to $\mathcal{U}$. The measure R can then be used as

a *reference* model in the sense that a condition holds P-almost surely simultaneously for all models $P \in \mathcal{U}$ if and only if the condition holds true R-almost surely. Thus admissibility of a strategy with respect to R is equivalent to admissibility for all $P \in \mathcal{U}$. Furthermore a strategy is a super-hedging strategy with respect to all models $P \in \mathcal{U}$ if and only if it is a super-hedging strategy with respect to R.

The case studies in Sections 5.1 and 6 will be solved be means of auxiliary problems of robust-efficient hedging where maturity is a random time. For this reason, we allow the payoff of the option F in the original formulation to occur at a random time rather than at a deterministic time.

As mentioned above, a *worst-case model* $\tilde{P} \in \mathcal{U}$ is a solution to the maximin version of the minimax problem defining robust-efficient strategies. For this, we must pass from $\mathcal{U}$ to its convex hull or, more generally, to a *proper enlargment*, cf. Definition 2.7. Here, model-uncertainty can become a source of market-incompleteness. This is due to the fact that for any two complete models P and P' which are not equivalent, a convex combination of the two is incomplete. This is for example the case if one considers a class of Black-Scholes models with different volatilities, cf. Section 5.3. We examine the intuition that given a worst-case model $\tilde{P}$, a strategy that is efficient for $\tilde{P}$ should be robust-efficient for $\mathcal{U}$. In order to turn this intuition into a precise theorem, it is sufficient that $\mathcal{U}$ is convex and $\tilde{P}$ dominates $\mathcal{U}$. However, in Section 5.3 we provide an example where the efficient strategy for a worst-case model $\tilde{P}$ is not robust-efficient.

Our approach to solve the problem of robust-efficient hedging is similar to the approach taken by [FL00] for fixed model: We first show how the problem of robust-efficient hedging for F can be reduced to the problem of super-hedging of a modified claim $\tilde{F}$. The modified claim is given by $\tilde{F} = \tilde{Z}F$ where $\tilde{Z}$ is the maximin-optimal test for an associated statistical testing problem with a concave state-dependent utility function. Whereas in the case of model-uncertainty, this testing problem has a simple alternative and composite hypothesis, we are now in a situation where both alternative and hypothesis are composite. This is the main reason why we revisit this problem which was solved by [Leu99] for simple alternative.

The mathematical centerpiece of this thesis is the solution of the statistical testing problem on a general level by means of convex duality and game-theoretical methods. The problem differs from the classical testing problem in that the power of a test is defined in terms of a strictly concave state dependent utility function rather than the identity mapping. Furthermore, our only essential assumption is that the alternative and the hypothesis are dominated, i.e., the alternative and the hypothesis need neither be parameterized nor of the form of the neighborhoods typically arising in robust statistics. Similar to the classical notion of least-favorable pairs of prior-distributions on the hypothesis respectively alternative, we introduce the pivotal notion of a least-favorable pair of elements of the hypothesis respectively alternative. In consistency with the setting arising from the problem of robust-efficient hedging, we refer to elements of the alternative as models, elements of the hypothesis are called pricing rules. Consider a *pair* consisting of a single model and a single

pricing rule. A crucial observation is that the optimal test for the "simple" problem associated to this pair can be derived easily from the level condition. Hence the optimization problem underlying the following definition can easily be formulated: We say that a model and a pricing rule are a least-favorable pair if they jointly minimize the power of the optimal test for the associated simple problem. For this, the minimum is taken over all models within a *proper* enlargement of the alternative and all pricing rules within a *proper* enlargement of the hypothesis. Essentially, an enlargement is proper if it is convex and does not change the optimization problem. We allow for some freedom concerning the choice of the proper alternative respectively hypothesis. The main result of our analysis on maximin-optimal tests is that the maximin-optimal test can be found among the simple-optimal test for a least-favorable pair. If the least-favorable pair is equivalent to the dominating measure, the simple optimal test is the unique maximin-optimal test. If the latter condition is not fulfilled, we approximate the maximin-optimal test by a sequence of simple optimal tests. We also give an example where a least-favorable pair exists but where the solution to the simple problem is not maximin-optimal.

The approach taken in this thesis allows us to derive the robust-efficient strategy under additional trading constrains on admissible strategies.

A collection of case studies for different families $\mathcal{U}$ illustrates the concepts introduced in this thesis. In these case studies, we first determine the efficient strategy and the related minimal risk β_P for any fixed model $P \in \bar{\mathcal{U}}$ within a suitably chosen proper $\bar{\mathcal{U}}$. We then show existence of a worst-case model $\tilde{P}$, i.e., a model that maximizes the value β_P over all $P \in \bar{\mathcal{U}}$. Finally, we apply theorem 3.11 to show that the $\tilde{P}$-efficient strategy is robust-efficient for $\bar{\mathcal{U}}$ respectively $\mathcal{U}$. In the setting of a Binomial tree with uncertain transition probabilities or a Black-Scholes model with uncertain drift, all models have the same unique equivalent martingale measure. For this reason we can derive the robust-efficient hedging strategy explicitly in terms of a worst-case model in these settings. One chapter is dedicated to different families with stochastic or uncertain volatility. We first provide a detailed study of a family of generalized Black-Scholes models where volatility is constant up to a random time τ. At time τ, volatility jumps to a new value $\eta(\omega)$ according to some distribution θ. Here, model-uncertainty arises from uncertainty in the distribution θ. We show that an appropriately chosen neighborhood of a given distribution θ_0 contains a worst-case distribution $\tilde{\theta}$. The efficient hedging strategy for $\tilde{\theta}$ is robust-efficient for the neighborhood, cf. Example 5.10.

We also consider a variant of the "uncertain volatility model", cf. [ALP95] and [Lyo95]: We consider a countable family of volatility paths such that we can decide at time 0 which path is actually chosen. More generally, we consider a countable family of singular models. This provides us with an example where a worst-case model $\tilde{P}$ exists but the $\tilde{P}$-efficient strategy is not robust-efficient. Instead, the superposition of all efficient strategies for any fixed model is actually robust-efficient.

A class of models where the asset price follows a "geometric Poisson process" provides a case study beyond the standard diffusion setting and is solved as follows:

If β denotes the minimal risk in a fixed model, we have $\beta = \lim_k \beta^k$ where β^k is the minimal risk that can be achieved among all strategies that are constant after the k-th jump. The associated "k-efficient" strategy can be computed directly via methods of dynamic programming. For given error-bound ϵ, we can choose k sufficiently large such that the performance of the k-efficient strategy differs from the performance of the efficient strategy only by ϵ. We then consider the case where there is uncertainty regarding the jump-intensities and show how the robust-efficient strategy can be derived via a worst-case model.

Outline

In **Chapter 1**, the problem is formulated and the notion of admissible strategies under model uncertainty is introduced. We examine super-hedging strategies in our setting and show how the problem of robust-efficient hedging for F can be reduced to the problem of super-hedging of a modified claim $\tilde{F}$. The modified claim is given by $\tilde{F} = \tilde{Z}F$ where $\tilde{Z}$ is the maximin-optimal test for an associated statistical testing problem with composite hypotheses.

We examine maximin-optimal tests more closely in **Chapter 2**. This chapter is a stand-alone discussion of hypothesis testing with strictly concave utility functions and does not rely on any definitions made in previous sections. The general setup is similar to [CK00] who considered the testing problem for the linear utility function $u(z, .) = z$. We discuss the testing problem for any two families of finite measures that are dominated by a probability measure. No further structure on the underlying probability space or on the hypotheses is assumed. Motivated by the connection of maximin-optimal tests to applications in mathematical finance, we use the following non-statistical terms: Elements of the hypothesis are referred to as "pricing rules" Q and elements of the alternative are called "models" P. A least-favorable pair $(\tilde{P}|\tilde{Q})$ is defined by the property that it minimizes the maximal power associated to the simple problems $(P|Q)$. For this, the minimum is taken over all models within a *proper* enlargement of the alternative and all pricing rules within a *proper* enlargement of the hypothesis. Essentially, an enlargement is proper if it is convex and does not change the optimization problem, cf. Definition 2.3. We also introduce the notion of a *worst-case model* and a *worst-case pricing-rule for a fixed model*. In short, a worst-case model minimizes the maximal power of the associated semi-composite problem. A worst-case pricing rule for fixed model P minimizes the maximal power associated to the simple problem of testing any pricing rule against the model P. We show in Proposition 2.27 that $\tilde{P}$ is a worst-case model and $\tilde{Q}$ is a worst-case pricing rule for $\tilde{P}$ if and only if $(\tilde{P}|\tilde{Q})$ is a least-favorable pair.

The notion of a worst-case pricing rule is closely related to the solution of the dual problem typically considered in utility-maximization problems where there is no model-uncertainty, cf. for example [Sch00] and the references given there. Essentially, the notion of a least-favorable pair and a solution to the dual problem are equivalent, cf. Proposition 2.27. The maximin-optimal test can be found among

the solutions to the simple problem of testing $\tilde{P}$ versus $\tilde{Q}$ for a least-favorable pair $(\tilde{P}|\tilde{Q})$, cf. Proposition 2.30. If the worst-case model is equivalent to R, the solution to the simple problem is unique and maximin-optimal for the composite problem, cf. Theorem 2.31. One set of sufficient conditions to find a worst-case model is that the family $\mathcal{U}$ is uniformly integrable and u is bounded: In this case, the closed convex hull of $\mathcal{G}$ in $L^1(R)$ is proper and contains a worst-case model. If the worst-case model fails to be equivalent to R, we demonstrate how one can approximate the original problem via a series of embedded problems which possess equivalent worst-case models, cf. Theorem 2.32.

In **Chapter 3**, we apply the results of Chapter 2 to the problem of efficient hedging under model-uncertainty. In Section 3.1, we define the utility function and the families of densities $(\mathcal{G}|\mathcal{H})$ in terms of the loss function, the families of "real-world" models and martingale measures as well as the contingent claim. We pass from the family of martingale measures to its closure in L^0 and from the family of models to its closed convex hull in L^1. We then rephrase the most central results of Chapter 2 directly in terms of models P, pricing rules Q and modified claims and show how the maximin-optimal modified claim of Theorem 1.7 can be derived from a least-favorable pair $(\tilde{P}|\tilde{Q})$, cf. Theorem 3.5. Our results on the existence of a least-favorable pair carry over immediately from Chapter 2. We show that the maximin and minimax values β_* respectively β^* coincide, cf. Proposition 3.10. If a worst-case model exists, the robust-efficient hedging problem has a saddle point. If, in addition, the worst-case model is equivalent to R, the robust-efficient hedging strategy is given by the $\tilde{P}$-efficient hedging strategy, cf. Theorem 3.11. In many applications it is easier to compute the P-efficient hedging strategy for a fixed model P directly, e.g. via dynamic programming, rather than via a worst-case pricing rule, cf. also Sections 5.1 and 6. For this reason we show that a worst-case model $\tilde{P}$ can be obtained by minimizing the power of P-efficient strategies over P, i.e., by solving the problem $\min_\xi E_{\tilde{P}}[l(S_\xi)] = \sup_{P\in\mathcal{U}} \min_\xi E_P[l(S_\xi)]$, cf. Lemma 3.6.

Along the same line, we derive a worst-case pricing rule $\tilde{Q}$ a posteriori from the efficient strategy in Section 3.2. Furthermore, we establish a relationship between the attainability of the maximin-optimal modified claim and equivalence of the worst-case pricing rule to the worst-case model, cf. Corollary 3.16.

In Section 3.3, we consider different ways to specify a family $\mathcal{U}$. We show in Section 3.3.1 that a suitably chosen neighborhood of a given model P_0 contains a worst-case model. In Section 3.3.2, a parameterized family $\mathcal{U} = \{P_\theta \mid \theta \in \Theta\}$ is considered. In this situation, a worst-case model is a mixture $\int P_\theta \nu(d\theta)$ for a prior-probability distribution ν on Θ. Finally, we consider a variant of the last setting where the investor assigns weights to the models P_θ, i.e., where the investor chooses a family of prior distributions. In all these specifications, the maximin-optimal claim respectively the robust-efficient strategy can be derived via a worst-case model.

Two problems of optimal hedging under model uncertainty that do not fit in the framework of Section 1 are considered: We derive the optimal hedging strategy for an extremely risk-averse investor who intends to minimize the maximum loss in

Section 3.5. The other extreme of a risk-seeking investor corresponds to the quantile hedging problem examined by [FL99] for a single model. Robust-quantile hedging strategies are developed in Section 3.6.

In the second part of this thesis we consider several case studies. **Chapter 4** is dedicated to the discussion of examples where all models are equivalent to the same equivalent martingale measure. The first example is a family of binomial trees with uncertainty regarding the transition probabilities at each node. The second example is dedicated to the study of a class of Black-Scholes models with uncertain drift. In both examples we provide explicit formulas for a worst-case model and for the efficient strategy for the worst-case model. This strategy is robust-efficient. The case where not only the probability but also the size of the return in the Binomial model is uncertain leads to a class of incomplete models. This case is considered in a one-period setting in Section 4.2.

In **Chapter 5** we study stochastic volatility models. We first consider a family of models where the volatility jumps at a random time τ to a new value η according to some unknown distribution θ. For given distribution θ, we denote the corresponding model by P_θ. We consider a class of models of the form $\mathcal{U} = \{P_\theta \mid \theta \in \Theta\}$ for some family Θ of equivalent probability distributions on $(0, \infty)$. In Section 5.1.1, we derive the efficient hedging strategy for any fixed distribution θ. With respect to the construction of the model, we distinguish the case where τ is a stopping-time w.r.t. the filtration generated by X and the case where τ is independent of X. We generalize the results obtained previously by [FL00] for a constant jump-time and derive the efficient strategy via the dynamic programming principle. We then derive a formula for the worst-case pricing rule and give an example where the worst-case pricing rule is not equivalent to P_θ, cf. Lemma 5.6 and 5.7. In Section 5.1.3, we consider a convex family Θ of equivalent distributions of volatility and show that Θ contains a worst-case distribution $\tilde{\theta}$. Since all models P_θ are equivalent, the efficient hedging strategy for $P_{\tilde{\theta}}$ derived in Section 5.1.1 is robust-efficient for $\mathcal{U} = \{P_\theta \mid \theta \in \Theta\}$.

In Section 5.2 we derive the Bellman equation for "classical" stochastic volatility models where volatility is itself modelled as a diffusion.

In Section 5.3, we consider a variant of te uncertain volatility model of [ALP95] and [Lyo95]: We consider a countable family of volatility paths such that we can decide at time 0 which path is actually chosen. More generally, we consider a *countable* family of *singular and complete* models $\{P_n \mid n \in \mathbb{N}\}$ with support $\Omega_n \in \mathcal{F}_0$. In this setting, the family of equivalent martingale measures $\mathcal{M}$ for the dominating model has a nice structure: It is given by all convex combinations of the unique equivalent martingale measures Q_n for P_n. This allows us to prove that the superposition of all efficient strategies for any fixed path is actually robust-efficient. A one-to-one correspondence between certain sequence-spaces in $[0,1]^{\mathbb{N}}$ and closures of $\mathcal{M}$ respectively $\mathcal{U}$ is derived. We then find a worst-case model of the form $\tilde{P} = P_{\tilde{n}}$ and an associated worst-case pricing rule $\tilde{Q} = Q_{\tilde{n}}$. It is easily seen that the efficient strategy for the fixed model $\tilde{P}$ is *not* robust-efficient.

In **Chapter 6**, we provide a case study where the asset price follows a "geometric Poisson process". The efficient strategy is derived as follows: Let β denote the minimal risk among all strategies $\xi \in \mathcal{A}_\alpha$, and β^k the minimal risk that can be achieved among all strategies $\xi \in \mathcal{A}_\alpha$ that are constant after the k-th jump. The associated "k-efficient" strategy can be computed directly via methods of dynamic programming. We then have $\beta = \lim_k \beta^k$ and for a given error-bound ϵ, we can choose k sufficiently large such that the performance of the k-efficient strategy differs from the performance of the efficient strategy only by ϵ, cf. Theorem 6.10. We apply the results of Section 3.2 to show that if the intrinsic value of F is nontrivial, i.e., $F(x_0) > 0$, and we invest more than the intrinsic value, then the worst-case pricing rule is not equivalent to P.

In Section 6.3 we consider the case where there is uncertainty regarding the jump-intensities. The proper enlargement $\bar{\mathcal{U}}$ is constructed as described in Section 3.3.2. We then show how the methodology of Section 6.2 can be applied to derive the efficient strategy and associated value β_P for any fixed model $P \in \bar{\mathcal{U}}$. Theorem 3.20 allows us to derive a worst-case model and the robust-efficient strategy for $\bar{\mathcal{U}}$, cf. Theorem 6.13.

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Part I

General Results

CHAPTER 1

Robust-efficient hedging: Concepts

In this chapter, we define the problem of robust-efficient hedging for a nonnegative contingent claim with random maturity. For this, the notion of admissible strategies under model uncertainty is introduced. We revisit the problem of super-hedging in our setting and show how the problem of robust-efficient hedging for F can be reduced to the problem of super-hedging of a modified claim $\tilde{F}$. The modified claim is the maximin-optimal test for an associated statistical testing problem with composite hypotheses - a problem that will be considered in detail in the following Chapter 2.

1.1. Formulation of the problem

Let a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, R)$ and an adapted positive process $X = (X_t)_{0\leq t\leq T}$ such that X is a semi-martingale under R be given. We consider a family $\mathcal{U}$ of probability measures such that R and $\mathcal{U}$ are *equivalent*, i.e., $P[A] = 0$ for all $P \in \mathcal{U}$ implies $R[A] = 0$ and vice versa. By the Halmos-Savage Theorem 3.17, such a reference measures R can be constructed for any dominated family of probability measures.

We assume that $(\mathcal{F}_t)$ is right-continuous and complete with respect to R. The set $\mathcal{M}$ of pricing rules is the set of all measures Q equivalent to R such that X is a martingale with respect to Q. We assume absence of arbitrage in the sense that $\mathcal{M}$ is nonempty.

We assume that the riskless rate of interest is zero.

A self-financing strategy with fixed initial capital $\alpha \geq 0$ is a predictable process (ξ_t) such that the resulting value process

$$V_t = \alpha + \int_0^t \xi_s dX_s, \quad t \in [0, T] \tag{1.1}$$

is R-almost surely well defined. A self-financing strategy (ξ_t) is called *admissible* if the associated value process V satisfies

$$V_t \geq 0, \quad t \in [0, T], \quad R - \text{a.s.}. \tag{1.2}$$

We denote the set of all strategies (ξ_t) which are admissible for given initial capital α by $\mathcal{A}_{\alpha,R}$. $\mathcal{A}_{\alpha,R} \subset \mathcal{A}_{\alpha,P}$ holds if P is absolutely continuous with respect to R, cf.

also [Pro90] Theorem IV.2.25. Hence

$$\mathcal{A}_{\alpha,R} = \bigcap_{P\in\mathcal{U}} \mathcal{A}_{\alpha,P}.$$

We simply write $\mathcal{A}_\alpha = \mathcal{A}_{\alpha,R}$ in the sequel. Observe also that

$$\mathcal{A}_\alpha \subset \mathcal{A}_\beta, \quad \alpha \le \beta \tag{1.3}$$

holds.

In certain applications, the dynamic programming principle allows us to solve the problem of (robust-)efficient hedging for an option with deterministic maturity T by means of auxiliary problems with *random* maturity, cf. Sections 5.1 and 6. For this reason, we generally consider contingent claims with random maturity. We examine the risk incurred by an investor who is short a $\mathcal{F}_\tau$-measurable contingent claim F which pays out $F(\omega) \ge 0$ at the $(\mathcal{F}_t)$-stopping time $\tau(\omega) \le T$. Assume that

$$\sup_{Q\in\mathcal{M}} E_Q[F] < \infty \tag{1.4}$$

holds. The following cash flows occur in our setting: Today, i.e., at time $t = 0$, the investor receives a premium p from the buyer of the option and invests the amount V_0 of initial capital in a self-financing strategy ξ. His portfolio at time $t \le \tau$ consists of ξ_t assets, -1 option and an amount of $V_t - \xi_t X_t$ invested in the money market. No cash flows occur between today and the exercise time τ of the option. At time $\tau(\omega)$, the investor closes his positions, i.e., he pays $-F(\omega)$ to the holder of the option and receives the value $V_\tau(\omega)$ of the hedging strategy.

time	cash flow
$t = 0$	$p - V_0$
$0 < t < \tau$	none
$t = \tau$	$V_\tau - F$

While today's cash flow is deterministic, the remaining liability $F - V_\tau$ at time τ is random and therefore risky. The positive part $S = (F - V_\tau)^+$ of the remaining liability is the investor's *shortfall*. We propose to measure risk associated to the shortfall $S \le F$ by the quantity

$$\rho(S) = \sup_{P\in\mathcal{U}} E_P[l(S,.)] \tag{1.5}$$

for a given *loss function* l:

1.1 Definition. A function $l : \mathbb{R} \times \Omega \to [0,\infty)$ such that $z \mapsto l(z,\omega)$ is increasing, strictly convex and continuously differentiable on $(0, F(\omega))$ $\forall\omega$ and

$$l(z,.) = 0 \quad \forall z \le 0; \quad l(z,.) \text{ bounded} \quad \forall z \ge 0 \tag{1.6}$$

$$E_P[l(F,.)] < \infty, \quad P \in \mathcal{U} \tag{1.7}$$

is called *loss function*.

The risk measure ρ defined via (1.5) belongs to the class of *convex* measures of risk introduced by [FS00]. The authors extend the concept of *coherent* measures of risk proposed by [ADEH99]. Especially, convex measures of risk allow for liquidity risk. Starting from an axiomatic description, [FS00] show that a convex measure of risk admits alternative representations in terms of an associated "penalty function" or an "acceptance set".

An important class of loss functions is given by $l(z) = (z^+)^p$, $p \geq 1$. In this case we propose to normalize ρ by taking the p-th root, i.e.,

$$\rho_p(V) := \sup_{P \in \mathcal{U}} ||V^+||_{L^p(P)}. \tag{1.8}$$

The case $p = 1$ is referred to as risk-neutral whereas $p > 1$ corresponds to a risk-averse attitude. The degree of risk-averseness increases with the parameter p. In Section 3.5, we revisit this class of risk-measures (1.8) and consider the case of extreme risk averseness as p tends to infinity.

If V_0 does not exceed the option price today *and* the option's payoff $F(\omega)$ at time $\tau(\omega)$ does not exceed the value $V_\tau(\omega)$ of the hedging strategy R-almost surely, then the investor receives only positive cash flows, i.e., he is able to make a riskless profit by selling the option. We first show that the minimum of all prices that allow for a riskless profit is given by $F_0 := \sup_{Q \in \mathcal{M}} E_Q[F]$.

1.2 Lemma. *Consider a $\mathcal{F}_\tau$-measurable contingent claim $F \geq 0$. There exists an admissible strategy $\xi \in \mathcal{A}_\alpha$ such that*

$$F \leq \alpha + \int_0^\tau \xi_s dX_s, \quad R - \textit{almost surely} \tag{1.9}$$

if and only if

$$F \in \mathcal{V}_\alpha := \left\{ V \geq 0 \,|\, V \ \mathcal{F}_\tau - \textit{measurable}, \ \sup_{Q \in \mathcal{M}} E_Q[V] \leq \alpha \right\}. \tag{1.10}$$

Proof. (1) For any admissible strategy ξ, the corresponding value process is a super-martingale for any $Q \in \mathcal{M}$.

2) Hence, if ξ satisfies (1.9), we immediately obtain $F \in \mathcal{V}_\alpha$.

3) For the reverse implication, consider any $F \in \mathcal{V}_\alpha$. Denote by Y a right-continuous version of the process

$$Y_t := \operatorname*{ess\,sup}_{Q \in \mathcal{M}} E_Q[F | \mathcal{F}_t].$$

By hypothesis,

$$Y_0 \leq \alpha. \tag{1.11}$$

(Y_t) is a super-martingale under any pricing rule $Q \in \mathcal{M}$ and $\mathcal{M}$ contains all equivalent martingale measures for R. According to the optional decomposition theorem (cf. [FK98], [Kra96]), this implies the existence of an admissible strategy

$(\xi_t) \in \mathcal{A}_{Y_0} \subset \mathcal{A}_\alpha$ and an increasing optional process C satisfying $C_0 = 0$ such that for all t

$$Y_t = Y_0 + \int_0^t \xi_s dX_s - C_t, \quad R - \text{a.s.}$$

holds. Thus, we can estimate F R-almost surely by

$$\begin{aligned} F &= E_Q[F|\,\mathcal{F}_\tau] \\ &= E_Q[Y_T|\,\mathcal{F}_\tau] \\ &\leq Y_\tau \\ &= Y_0 + \int_0^\tau \xi_s dX_s - C_\tau \\ &\leq Y_0 + \int_0^\tau \xi_s dX_s. \end{aligned}$$

Taking equation (1.11) into account we obtain validity of (1.9). □

1.3 Definition.

(i) The *superhedge price* F_0 for F is defined as

$$F_0 := \inf\{\alpha \mid \exists \xi \in \mathcal{A}_\alpha : \; F \leq \alpha + \int_0^\tau \xi_s dX_s \; R - \text{almost surely}\}. \tag{1.12}$$

It follows from Lemma 1.2 that

$$F_0 = \sup_{Q \in \mathcal{M}} E_Q[F]$$

holds and that the infimum in (1.12) is attained, i.e., there exists an admissible strategy $\xi \in \mathcal{A}_{F_0}$ with

$$F \leq F_0 + \int_0^\tau \xi_s dX_s, \quad R - \text{almost surely.}$$

Any such strategy ξ is called a *super-hedging* strategy for F.

(ii) F is called *attainable* if there exists a super-hedging strategy $\xi \in \mathcal{A}_{F_0}$ such that the associated cumulative consumption vanishes, i.e.,

$$F = F_0 + \int_0^\tau \xi_s dX_s, \quad R - \text{almost surely.}$$

The super-hedge price F_0 is the smallest amount of initial capital which allows to eliminate all shortfall risk. However, if the option is not attainable, the superhedge price allows for arbitrage. Hence the superhedge price must exceed the option-premium in a friction free market, cf. also [CSS99] and [FS99] for a more quantitative analysis. For this reason, we investigate how the investor can reduce his risk-exposure if he is not willing or able to invest the super-hedge price in a hedging strategy, i.e., if he has only an amount α of initial capital available with

$$0 < \alpha < F_0.$$

Obviously, he is not able to eliminate all risk. For any admissible strategy $\xi \in \mathcal{A}_\alpha$ there is a model $P \in \mathcal{U}$ such that $P[F > V_\tau^\xi] > 0$ holds. We examine hedging strategies that minimize the quantity

$$\sup_P E_P[l(F - \alpha - \int_0^\tau \xi_s dX_s, .)] :$$

1.4 Definition. An admissible strategy $\tilde{\xi} \in \mathcal{A}_\alpha$ is called *robust-efficient* if it solves the problem

$$\left[\begin{array}{c} \sup_P E_P[l(F - \alpha - \int_0^\tau \xi_s dX_s, .)] = \min_\xi \\ \xi \in \mathcal{A}_\alpha \end{array} \right]. \tag{1.13}$$

We denote the value of the problem by β^*:

$$\begin{aligned} \beta^* &= \min_{\xi \in \mathcal{A}_\alpha} \sup_{P \in \mathcal{U}} E_P[l(F - \alpha - \int_0^\tau \xi_s dX_s, .)] \\ &= \min_{\xi \in \mathcal{A}_\alpha} \rho(F - \alpha - \int_0^\tau \xi_s dX_s). \end{aligned}$$

We will also refer to β^* as the *robust minimal shortfall risk* due to the following consideration: Given a robust-efficient strategy $\tilde{\xi}$, we obtain from (1.13) that the expected remaining shortfall risk with respect to *any model* $P \in \mathcal{U}$ is bounded by β^*:

$$E_P[l(F - \alpha - \int_0^\tau \tilde{\xi}_s dX_s, .)] \leq \beta^*, \quad P \in \mathcal{U}.$$

This is a desirable feature from a risk-management point of view.

We will examine the solution to problem (1.13) on a general level in Chapter 3. As an auxiliary tool, we will consider strategies that are efficient with respect to a fixed model P:

1.5 Definition. Consider some fixed model P in the convex hull $co(\mathcal{U})$ of $\mathcal{U}$. A strategy is efficient for P if it solves the problem

$$\left[\begin{array}{c} E_P[l(F - \alpha - \int_0^\tau \xi_s dX_s, .)] = \min_\xi \\ \xi \in \mathcal{A}_\alpha \end{array} \right]. \tag{1.14}$$

We denote the value function for this problem by β_P.

Observe that an admissible strategy is admissible with respect to any $P \in co(U)$ since R also dominates $co(U)$. Since the class $\mathcal{A}_\alpha$ of admissible strategies is defined in terms of R rather than P, the notion of a strategy that is "efficient for P" differs from the definition of an efficient strategy under model-certainty $\mathcal{U} = \{P\}$ considered by [FL00]. Only if P is equivalent to R, these notions coincide.

We will say that a measure $\tilde{P} \in co(\mathcal{U})$ is a worst-case model if it maximizes the minimal shortfall risk for fixed model P:

$$\beta_{\tilde{P}} = \sup_{P \in co(\mathcal{U})} \beta_P =: \beta_*, \tag{1.15}$$

cf. Lemma 3.6. It is easily seen that

$$\beta_* = \beta^*$$

holds, cf. Proposition 3.10.

In view of Propositions 3.10 and 3.9, the joint solution $(\tilde{\xi}, \tilde{P})$ to problem (1.13) is a saddle-point for $\rho(.,.)$ in $\mathcal{A}_\alpha \times \bar{\mathcal{U}}^1$. However, the efficient strategy for a worst-case model $\tilde{P}$ is not necessarily robust-efficient, cf. examples 5.21 and 5.22. Only in the case where the worst-case model $\tilde{P}$ is equivalent to R we can conclude that the $\tilde{P}$-efficient strategy is robust-efficient.

To allow for model uncertainty in (1.13) yields a robust version of the efficient hedging strategy introduced by [FL00]. This is mirrored in the fact that the auxiliary testing problem which we introduce in (1.16) corresponds to a robust version of standard Neyman-Pearson tests: Instead of maximizing the power of the test for a single measure P, robust tests maximize the minimal power over a suitably chosen neighborhood of P, e.g. in the total variation distance, cf. Chapter 2 and the textbook [Hub81]. In Section 3.3.1, we modify the concept of a total-variation neighborhood to find a neighborhood that is more suitable for the purpose of efficient hedging. We consider L^p-neighborhoods of a given model P_0 to derive robust versions of P_0 efficient strategies.

The family of models is often given in a parameterized form. In this setting, we show in Section 3.3.2 under mild assumptions that a worst-case model is a mixture obtained from a "worst-case prior-probability distribution" on the parameter-space.

One can argue that the min-max criterion (1.13) is extreme in the sense that all models have the same weight in the optimization procedure, regardless of the likelihood the investor might assign to each model. We discuss in Section 3.3.3 how one can incorporate the investor's view on the likelihood of models in our approach by choosing a prior distribution on $\mathcal{U}$. We then obtain a problem that is a special case of (1.13) where a class of models $\mathcal{U}'$ is obtained by considering a class of prior distributions.

Finally, a remark on the R-almost sure validity of conditions (1.2) and (1.9) is in order: The economically sound formulation would be to require that these conditions hold P-almost surely for any $P \in \mathcal{U}$. Clearly, both formulations are equivalent since R and $\mathcal{U}$ are equivalent.

1.2. Reduction to a testing problem

We now formulate a testing problem that is closely related to (1.13).

1.6 Definition. A $\mathcal{F}_\tau$-measurable random variable $\tilde{V}$ is called *maximin-optimal* if it solves

$$\left[\begin{array}{c} \sup_{P\in\mathcal{U}} E_P[l(F-V,.)] = \min_V \\ 0 \le V \le F,\ E_Q[V] \le \alpha\ \forall Q \in \mathcal{M} \end{array}\right]. \tag{1.16}$$

We link the testing-problem to the original problem (1.13):

1.7 Theorem (Reduction to a stationary problem). *Given a maximin-optimal modified claim $\tilde{V}$ for initial capital α, the super-hedging strategy $\tilde{\xi} \in \mathcal{A}_\alpha$ for the modified claim $\tilde{V}$ is robust-efficient and the minimal risk is given by*

$$\beta^* = \sup_{P\in\mathcal{U}} E_P[l(F-\tilde{V},.)]. \tag{1.17}$$

Proof. Let $\tilde{V}$ and $\tilde{\xi}$ as in the theorem be given.

(1) It follows from the side conditions of problem (1.16) that

$$\sup_{Q\in\mathcal{M}} E_Q[\tilde{V}] \le \alpha$$

holds. Due to Definition 1.3, the super-hedging strategy $\tilde{\xi}$ satisfies the side conditions in problem (1.13), i.e., $\tilde{\xi} \in \mathcal{A}_\alpha$.

2) For optimality, consider any admissible strategy $\xi \in \mathcal{A}_\alpha$. We define V via

$$V := (\alpha + \int_0^\tau \xi_s dX_s) \wedge F \le \alpha + \int_0^\tau \xi_s dX_s.$$

Hence $V \in \mathcal{V}_\alpha$, i.e., V satisfies the side conditions of (1.16). Equation (1.6) and maximin-optimality of $\tilde{V}$ imply

$$\begin{aligned} \sup_{P\in\mathcal{U}} E_P[l(F-\alpha-\int_0^\tau \xi_s dX_s,.)] &= \sup_{P\in\mathcal{U}} E_P[l(F-V,.)] \\ &\ge \sup_{P\in\mathcal{U}} E_P[l(F-\tilde{V},.)] \\ &\ge \sup_{P\in\mathcal{U}} E_P[l(F-\alpha-\int_0^\tau \tilde{\xi}_s dX_s,.)] \end{aligned}$$

where the last inequality is due to the super-hedging property of $\tilde{\xi}$ for $\tilde{V}$ and equation (1.6). Hence $\tilde{\xi}$ is robust-efficient.

3) Considering $\xi = \tilde{\xi}$ in (2) yields

$$\begin{aligned} \sup_{P\in\mathcal{U}} E_P[l(F-\tilde{V},.)] &= \sup_{P\in\mathcal{U}} E_P[l(F-\alpha-\int_0^\tau \tilde{\xi}_s dX_s,.)] \\ &= \beta^* \end{aligned}$$

□

In fact, the converse implication of Theorem 1.7 holds as well:

1.8 Corollary. *Consider a robust-efficient strategy $\tilde{\xi}$. Then the modified claim*

$$\tilde{V} = \left(\alpha + \int_0^\tau \tilde{\xi}_s dX_s\right) \wedge F$$

is maximin-optimal.

Proof. By definition of $\tilde{V}$ and Proposition 1.2, we obtain $\tilde{V} \in \mathcal{V}_\alpha$.

For optimality, we obtain from the definition of $\tilde{V}$ and (1.6) that

$$\sup_{P \in \mathcal{U}} E_P[l(F - \tilde{V}, .)] = \beta^*$$

holds. Optimality follows via theorem 1.7:

$$\beta^* = \inf_{V \in \mathcal{V}_\alpha} \sup_{P \in \mathcal{U}} E_P[l(F - V, .)].$$

□

CHAPTER 2

Maximin-optimal tests

2.1. Introduction

In the statistical situation of hypothesis testing one tries to discriminate between two families of probability measures, the hypothesis and the alternative. The *maximin*-optimal test maximizes the minimal *power* over all tests which do not exceed a given *size*. The minimal power of a test is the minimal probability of correctly rejecting the hypothesis. The size is the maximal probability of falsely rejecting the hypothesis. Maximin-optimal tests appear as building stones in the construction of (robust-) efficient hedging strategies, cf. Theorem 1.7.

The situation considered here differs from the classical testing problem in that the power of a test is defined in terms of a strictly concave state dependent utility function rather than the identity mapping. Furthermore, our only essential assumption is that the alternative and the hypothesis are dominated, i.e., the alternative and the hypothesis need neither be parameterized nor of the form of the neighborhoods typically considered in robust statistics. Similar to the classical notion of least-favorable pairs of prior-distributions on the hypothesis respectively alternative, we introduce the pivotal notion of a least-favorable pair of elements of the hypothesis respectively alternative. In consistency with the setting arising from the problem of robust-efficient hedging, we refer to elements of the alternative as models, elements of the hypothesis are called pricing rules. Consider a *pair* consisting of a single model and a single pricing rule. A crucial observation is that the optimal test for the "simple" problem associated to this pair can be derived easily from the level condition. Hence the optimization problem underlying the following definition can easily be formulated: We say that a model and a pricing rule are a least-favorable pair if they jointly minimize the power of the optimal test for the associated simple problem. For this, the minimum is taken over all models within a *proper* enlargement of the alternative and all pricing rules within a *proper* enlargement of the hypothesis. Essentially, an enlargement is proper if it is convex and does not change the optimization problem, cf. Definition 2.3. This definition leaves some freedom concerning the choice of the proper alternative respectively hypothesis which appear as the domain of the dual problem.

The main result of our analysis on maximin-optimal tests is that the maximin-optimal test can be found among the simple-optimal test for a least-favorable pair. If the least-favorable pair is equivalent to the dominating measure, the simple optimal

test is the unique maximin-optimal test. If the latter condition is not fulfilled, we approximate the maximin-optimal test by a sequence of simple optimal tests.

A least-favorable pair can be described equivalently as a worst-case model and a worst-case pricing rule for this model or as a solution to the dual-problem, cf. Proposition 2.27. The definition of a least-favorable pair is also closely connected to the well known notion of a *pair of least-favorable prior-distributions* examined e.g. by [Leh86] and [KW67]. [KW67] introduce the dual problem by means of prior distributions on the hypothesis and the alternative. The crucial observation is that the value of this dual problem depends only on the means of both prior-distributions and the total mass of the prior distribution on the hypothesis. If the alternative and the hypothesis are (measure-)convex, the mean of the prior distribution is itself a model respectively a pricing rule. Hence we can define the dual problem directly on the convex hulls of the alternative and the hypothesis plus a new parameter $k \in (0, \infty)$. The parameter k accounts for the total mass of the prior distribution on the hypothesis. One advantage of this approach is that one does not have to impose additional measurability-assumptions.

This chapter is organized as follows: In the following Section 2.2, we formulate the problem. The optimal test for the simple problem of testing a fixed pair of a model and a pricing rule is presented. *Proper* alternatives respectively hypotheses as well as worst-case models and least-favorable pairs are defined.

In Section 2.3, we examine different notions of optimality. We show that the maxmin and minimax-values for the problem coincide. We establish sufficient conditions for the existence of a worst-case model. Existence of a worst-case model implies existence of a saddle point.

In Section 2.4, we define the dual problem (2.37). We show that there is no duality gap, cf. Lemma 2.24. As a consequence, a solution to the dual-problem is essentially equivalent to a least-favorable pair, cf. Proposition 2.27. The solution to the original problem is then constructed via a solution to the dual problem in several stages. First we consider the simple problem for fixed model and pricing rule in Lemma 2.26. Then, the semi-composite problem for fixed model is solved in Lemmata 2.28 and 2.29. Finally we turn to the solution of the full problem in Theorems 2.31 and 2.32. We state in Theorem 2.31 that the simple optimal test for a least-favorable pair is the unique maximin-optimal test in the case where the least-favorable pair is equivalent to R. For the case where the least-favorable pair is not equivalent to R, the maximin-optimal test is approximated by a series of testing problems with associated least-favorable pairs equivalent to R in Theorem 2.32.

We sketch briefly how our approach carries over to the linear case in Section 2.5: We describe the results obtained recently by [CK00] in terms of a least-favorable pair.

2.2. Definition of the problem

Consider a probability space $(\Omega, \mathcal{F}, R)$ and two families $\mathcal{G}$ and $\mathcal{H}$ of nonnegative R-integrable random variables such that

$$G_0 := \sup_{G\in\mathcal{G}} E[G] < \infty, \quad H_0 := \sup_{H\in\mathcal{H}} E[H] < \infty \tag{2.1}$$

holds. We call $\mathcal{G}$ the family of *models* G and $\mathcal{H}$ the family of *pricing rules* H.

2.1 Definition. A state-dependent utility function is a measurable function $u : [0,1] \times \Omega \to \mathbb{R}$ such that $u(.,\omega)$ is increasing, strictly concave and continuously differentiable on $(0,1)$ for all ω, and u fulfills

$$-\infty < E[G\,u(0,.)] \text{ and } E[G\,u(1,.)] < \infty, \quad G \in \mathcal{G}. \tag{2.2}$$

Our intention is to find a random-variable $\tilde{Z}$ that solves the problem

$$\left[\begin{array}{c} \inf\limits_{G\in\mathcal{G}} E[G\,u(Z,.)] = \max\limits_{Z} \\ 0 \le Z \le 1, \;\; E[HZ] \le \alpha \;\forall H \in \mathcal{H} \end{array}\right]. \tag{2.3}$$

for some constant $\alpha \in (0, H_0)$.

A test $\tilde{Z}$ that solves problem (2.3) is called *maximin*-optimal. We will also refer to this problem as $(\mathcal{G}|\mathcal{H})$ with the understanding that we derive problems of the form $(G|\mathcal{H})$ from (2.3) by setting $\mathcal{G} = \{G\}$. If $u(z,\omega) = z$, problem $(\mathcal{G}|\mathcal{H})$ is the statistical problem of testing the composite hypothesis $\mathcal{G}$ against the composite hypothesis $\mathcal{H}$. We call $E[G\,u(Z,.)]$ the *power* and $E[HZ]$ the *price* of Z (with respect to G respectively H). Problem $(\mathcal{G}|\mathcal{H})$ then corresponds to the following procedure: One fixes a maximal price one is willing to pay, i.e., the maximal probability α of falsely rejecting $\mathcal{H}$. Under this constraint one then tries to maximize the minimal power, i.e., the minimal probability of correctly rejecting $\mathcal{H}$.

For the problem of robust-efficient hedging, we define $\mathcal{G}, \mathcal{H}$ and u in equations (3.1)-(3.3), page 60. In this situation, $-E[G\,u(Z,.)]$ is the risk of the modified claim ZF under the model P with $dP/dR = G$, hence the criterion (2.3) amounts to minimizing the worst-case risk over all models. The quantity $\sup_H E[HZ]$ corresponds to the super-hedging price of ZF and $\mathcal{H}$ corresponds to the class of martingale measures, i.e., pricing rules in the standard formulation of financial market models.

Obviously, a problem of the form $(\mathcal{G}|\mathcal{H})$ with nonconstant $\alpha = \alpha(H)$ can be transformed to a problem of the form $(\mathcal{G}|\mathcal{H}')$ with constant $\alpha' = 1$ by setting $\mathcal{H}' = \{H/\alpha(H) \mid H \in \mathcal{H}\}$. As long as the function $\alpha(.)$ is bounded from below by some strictly positive constant, the family $\mathcal{H}'$ will satisfy the condition (2.1) if $\mathcal{H}$ does. Similarly, if $\alpha(H) < E[H]$ holds for some $H \in \mathcal{H}$, then $\alpha' < H_0'$ is satisfied. In this sense we may assume without loss of generality that α is constant.

We introduce the class of all tests of maximal price α:

$$\mathcal{Z}_{\alpha,\mathcal{H}} = \mathcal{Z}_\alpha := \{0 \le Z \le 1 \mid \sup_{H\in\mathcal{H}} E[HZ] \le \alpha\}. \tag{2.4}$$

We call

$$u_* = u_*(\alpha) := \max_{Z \in \mathcal{Z}_\alpha} \inf_{G \in \mathcal{G}} E[G\, u(Z, .)]$$

the *value* of the problem.

For a fixed pricing rule $H \in \bar{\mathcal{H}}$, $\mathcal{Z}_{\alpha,H}$ is the class of all tests satisfying the side condition for simple problems with pricing rule H, i.e.,

$$\mathcal{Z}_{\alpha,H} = \{0 \le Z \le 1 \mid E[HZ] \le \alpha\}. \tag{2.5}$$

Any *simple* problem $(G|H)$ can be solved straightforward, cf. [Leu99] or Section 2.4:

2.2 Example (Solution of the simple problem). Consider a utility function u as in Definition 2.1 with derivative $\partial_z u(z, \omega) = u'(z, \omega)$. In Section 2.4 we define the inverse function I of $u'(., \omega)$ on $[0, \infty)$ for each ω. We show in Corollary 2.21 (i) that the solution $\tilde{Z}$ to a simple problem $(G|H)$ is GR-almost surely unique and given by

$$\tilde{Z}_{(G|H)} := I(\tilde{k}_{(G|H)} \frac{H}{G}, .)\, \mathbf{1}_{\{G>0\}},$$

where the *critical value* $\tilde{k}_{(G|H)} \in [0, \infty)$ for $(G|H)$ is the unique solution of

(i) $E[H\tilde{Z}_{(G|H)}] = \alpha$ if $E[H\, \mathbf{1}_{\{G>0\}}] < \alpha$

(ii) $k = 0$ (i.e., $\tilde{Z}_{(G|H)} = \mathbf{1}_{\{G>0\}}$) else.

Hence

$$\max_{Z \in \mathcal{Z}_{\alpha,H}} E[Gu(Z, .)] = E[Gu(\tilde{Z}_{(G|H)}, .)].$$

The principal result of this chapter is the following: We demonstrate that under appropriate conditions, a maximin-optimal test for $(\mathcal{G}|\mathcal{H})$ is given by the solution to the simple problem $(\tilde{G}|\tilde{H})$ for a least-favorable pair $\tilde{G}, \tilde{H}$ - a notion we define subsequently. For this, we must pass to the convex hulls of $\mathcal{G}$ respectively $\mathcal{H}$. In order to establish existence of a least-favorable pair, we may have to enlarge $\mathcal{G}$ and $\mathcal{H}$ even further. Therefore we define *proper* enlargements in Definition 2.3. These are convex families $\bar{\mathcal{G}} \supset \mathcal{G}$, $\bar{\mathcal{H}} \supset \mathcal{H}$ such that the problems $(\mathcal{G}|\mathcal{H})$ and $(\bar{\mathcal{G}}|\bar{\mathcal{H}})$ are equivalent, i.e., the values of these problems coincide and a maximin-optimal test for one problem is also maximin-optimal for the other. Observe that indeed the convex hulls of $\mathcal{G}$ respectively $\mathcal{H}$ fulfill this requirement. The largest families $\bar{\mathcal{G}}_\infty$ respectively $\bar{\mathcal{H}}_\infty$ such that the problems $(\mathcal{G}|\mathcal{H})$ and $(\bar{\mathcal{G}}|\bar{\mathcal{H}})$ are equivalent are given by

$$\bar{\mathcal{H}}_\infty = \Big\{H \ge 0 \mid E[HZ] \le \alpha,\ \forall\, Z \in \mathcal{Z}_\alpha\Big\} \tag{2.6}$$

$$\bar{\mathcal{G}}_\infty = \Big\{G \ge 0 \mid E[G] \le G_0;\ E[G\, u(Z, .)] \ge \inf_{G' \in \mathcal{G}} E[G' u(Z, .)],\ \forall\, Z \in \mathcal{Z}_\alpha\Big\}. \tag{2.7}$$

It is easily seen that these sets are convex. We now provide the formal definition of proper enlargements:

2.3 Definition. A *convex* hypothesis $\bar{\mathcal{H}} \supseteq \mathcal{H}$ (respectively alternative $\bar{\mathcal{G}} \supseteq \mathcal{G}$) is called *proper*, if $\bar{\mathcal{H}} \subset \bar{\mathcal{H}}_\infty$ (respectively $\bar{\mathcal{G}} \subseteq \bar{\mathcal{G}}_\infty$) holds. I.e., the convex hull $co(\mathcal{H})$ is

the smallest proper hypothesis and $\bar{\mathcal{H}}_\infty$ is the largest proper hypothesis (respectively $co(\mathcal{G})$ and $\bar{\mathcal{G}}_\infty$).

Before we give some examples of proper enlargements in Proposition 2.7, we introduce the pivotal notion of a least-favorable pair:

2.4 Definition.

(i) Consider a model $G \in \bar{\mathcal{G}}$. We say that $\tilde{H} \in \bar{\mathcal{H}}$ is a *worst-case pricing rule* for G if it solves

$$\left[\begin{array}{c} E[G\, u(\tilde{Z}_{(G|H)}, .)] = \inf\limits_{H} \\ H \in \bar{\mathcal{H}} \end{array}\right] \tag{2.8}$$

for some proper hypothesis $\bar{\mathcal{H}}$.

(ii) $\tilde{G} \in \bar{\mathcal{G}}$ is a *worst-case model* if it solves

$$\left[\begin{array}{c} \max\limits_{Z \in \mathcal{Z}_\alpha} E[Gu(Z, .)] = \inf\limits_{G} \\ G \in \bar{\mathcal{G}} \end{array}\right] \tag{2.9}$$

for some proper alternative $\bar{\mathcal{G}}$.

(iii) $(\tilde{G}, \tilde{H}) \in \bar{\mathcal{G}} \times \bar{\mathcal{H}}$ is a *least-favorable pair* if it solves the problem

$$\left[\begin{array}{c} E[G\, u(\tilde{Z}_{(G|H)}, .)] = \inf\limits_{G,H} \\ G \in \bar{\mathcal{G}}, H \in \bar{\mathcal{H}} \end{array}\right] \tag{2.10}$$

for a proper hypothesis $\bar{\mathcal{H}}$ and a proper alternative $\bar{\mathcal{G}}$.

We will show in Proposition 2.27 that $(\tilde{G}, \tilde{H})$ is a least-favorable pair if and only if $\tilde{G}$ is a worst-case model and $\tilde{H}$ is a worst-case pricing rule for $\tilde{G}$.

By the definition of a proper alternative, we have

$$u_* = \max_{Z \in \mathcal{Z}_\alpha} \inf_{G \in \bar{\mathcal{G}}} E[Gu(Z, .)],$$

i.e., this value is independent of the choice of $\bar{\mathcal{G}}$. It follows from Proposition 2.10 below that

$$u_* = u^* = \inf_{G \in \bar{\mathcal{G}}} \max_{Z \in \mathcal{Z}_\alpha} E[Gu(Z, .)]$$

holds for any proper alternative $\bar{\mathcal{G}}$. Hence the value of problem (2.9) is independent of the choice of $\bar{\mathcal{G}}$.

We show subsequently that

$$\max_{Z \in \mathcal{Z}_\alpha} E[Gu(Z, .)] = \inf_{H \in \bar{\mathcal{H}}} E[Gu(\tilde{Z}_{(G|H)}, .)]$$

holds for any $G \in \bar{\mathcal{G}}$ and any proper hypothesis $\bar{\mathcal{H}}$, cf. equation (2.51). Hence the value of problems (2.8) and (2.10) is independent of the choice of $\bar{\mathcal{H}}$ and $\bar{\mathcal{G}}$. The motivation to allow for different proper domains $\bar{\mathcal{G}}, \bar{\mathcal{H}}$ for the above problems is

the following: These problems are viewed as auxiliary means to solve the original problem $(\mathcal{G}|\mathcal{H})$. Hence we intend to choose the smallest proper domain that contains a least-favorable pair rather than increasing the complexity of the problem by trying to find a least-favorable pair in the largest proper domain $(\bar{\mathcal{G}}_\infty|\bar{\mathcal{H}}_\infty)$. On the other hand, the smallest proper domain $(co(\mathcal{G})|co(\mathcal{H}))$ may in general not contain a least-favorable pair.

2.5 Lemma. *The hypothesis $\bar{\mathcal{H}}_\infty$ enjoys the following properties:*

(i) $\mathcal{H} \subset \bar{\mathcal{H}}_\infty$.

(ii) $\bar{\mathcal{H}}_\infty$ *is countably convex.*

(iii) $E[H] \leq H_0$ *for all* $H \in \bar{\mathcal{H}}_\infty$.

(iv) $\bar{\mathcal{H}}_\infty$ *is closed in* $L^0(R)$.

(v) $\bar{\mathcal{H}}_\infty$ *and* $\mathcal{Z}_\alpha$ *are bipolar, i.e., for nonnegative random variables* Z *and* H *we have*

$$\begin{aligned} Z \in \mathcal{Z}_\alpha \quad & \text{if and only if} \quad E[HZ] \leq \alpha, \quad \forall\, H \in \bar{\mathcal{H}}_\infty \\ H \in \bar{\mathcal{H}}_\infty \quad & \text{if and only if} \quad E[HZ] \leq \alpha, \quad \forall\, Z \in \mathcal{Z}_\alpha. \end{aligned} \tag{2.11}$$

Proof. (i) is immediate.

For a given set M, we denote by

$$co_\infty(M) := \{\sum_{n=1}^{\infty} \lambda_n m_n \mid \sum_{n=1}^{\infty} \lambda_n = 1,\ \lambda_n \in [0,1],\ m_n \in M, \quad n \in \mathbb{N}\}$$

the countably-convex hull of M. For any test Z, we have

$$\sup_{H \in \mathcal{H}} E[HZ] = \sup_{H \in co_\infty(\mathcal{H})} E[HZ]. \tag{2.12}$$

This proves (ii).

(iii) is a consequence of $0 < \alpha/H_0 \in \mathcal{Z}_\alpha$.

(iv) Consider a sequence $(H_n) \subseteq \bar{\mathcal{H}}_\infty$ with limit H in $L^0(R)$. For any $Z \in \mathcal{Z}_\alpha$ we obtain from Fatou's Lemma that

$$E[HZ] \leq \liminf_n E[H_n Z] \leq \alpha$$

holds. Hence H lies in $\bar{\mathcal{H}}_\infty$, i.e., $\bar{\mathcal{H}}_\infty$ is closed in $L^0(R)$.

(v) is immediate. □

2.6 Lemma. *The alternative $\bar{\mathcal{G}}_\infty$ enjoys the following properties:*

(i) $\mathcal{G} \subset \bar{\mathcal{G}}_\infty$.

(ii) $\bar{\mathcal{G}}_\infty$ *is countably convex.*

(iii) If u *is bounded, then* $\bar{\mathcal{G}}_\infty$ *is closed in* $L^1(R)$.

Proof. (i) is immediate.

For any test Z we have

$$\inf_{G\in\mathcal{G}} E[G\,u(Z,.)] = \inf_{G\in co_\infty(\mathcal{G})} E[G\,u(Z,.)].$$

This proves (ii).

(iii) Consider a sequence $(G_n) \subset \bar{\mathcal{G}}_\infty$ that converges to G in $L^1(R)$. Clearly, the condition $E[G] \leq G_0$ is fulfilled. Since u is bounded, we have for any $Z \in \mathcal{Z}_\alpha$:

$$\begin{aligned} E[G\,u(Z,.)] &\geq \lim_{n\uparrow\infty} E[G_n\,u(Z,.)] \\ &\geq \inf_{G'\in\mathcal{G}} E[G'u(Z,.)]. \end{aligned}$$

Hence $G \in \bar{\mathcal{G}}_\infty$.

□

For $i = 0, 1$ we denote by $\bar{\mathcal{G}}^i$ the closed convex hull of $\mathcal{G}$ in $L^i(R)$. We have

$$\bar{\mathcal{G}}^i = \Big\{G \mid \exists\ (G_n) \subset co(\mathcal{G}),\ \lim_{n\uparrow\infty} G_n = G \ \text{ in } L^i(R)\Big\}, \quad i = 0, 1 \tag{2.13}$$

where $co(\mathcal{G})$ is the convex hull of $\mathcal{G}$. We define the families $co(\mathcal{H})$, $\bar{\mathcal{H}}^1$ and $\bar{\mathcal{H}}^0$ analogously.

2.7 Proposition. *The following are proper hypotheses:*

(1) The hull $co_\infty(\mathcal{H})$ of all countable convex combinations of $\mathcal{H}$.

(2) The closed convex hull $\bar{\mathcal{H}}^p$ of $\mathcal{H}$ in $L^p(R)$, $p \geq 0$.

The following are proper alternatives:

(3) The hull $co_\infty(\mathcal{G})$ of all countable convex combinations of $G \in \mathcal{G}$.

(4) If u is bounded: the closed convex hull $\bar{\mathcal{G}}^1$ of $\mathcal{G}$ in $L^1(R)$.

(5) If u is non-positive: the closed convex hull $\bar{\mathcal{G}}^p$ of $\mathcal{G}$ in $L^p(R)$ for $p \geq 0$.

Proof. We obtain from Lemma 2.5 that

$$\mathcal{H} \subseteq co_\infty(\mathcal{H}) \subseteq \bar{\mathcal{H}}^1 \subseteq \bar{\mathcal{H}}^0 \subseteq \bar{\mathcal{H}}_\infty$$

holds. This proves (1) - (2).

Item (3) is a consequence of Lemma 2.6 (ii).

Item (4) is a consequence of Lemma 2.6 (iii).

If $u \leq 0$ holds, we can conclude that $\bar{\mathcal{G}}_\infty$ is closed in $L^0(R)$ via Fatou's Lemma. Hence

$$\mathcal{G} \subseteq co_\infty(\mathcal{G}) \subseteq \bar{\mathcal{G}}^1 \subseteq \bar{\mathcal{G}}^0 \subseteq \bar{\mathcal{G}}_\infty$$

holds. This proves item (5). □

2.3. Existence of worst-case models and saddle-points

In this Section, we examine different notions of optimality. We show that the maxmin and minimax-values for problem (2.3) coincide, cf. Proposition 2.10. As a consequence, the value of problem (2.9) is independent of the choice of $\bar{\mathcal{G}}$. We establish sufficient conditions for the existence of a worst-case model, cf. Propositions 2.14, 2.15 and 2.17. The value function $\alpha \mapsto u_*(\alpha)$, is strictly concave under the assumptions of any of these Propositions, cf. Lemma 2.18.

We introduce the function $f(Z,G) = E[G\,u(Z,.)]$ for $G \in \bar{\mathcal{G}}$ and $Z \in \mathcal{Z}_\alpha$ and the quantities

$$u_* = \sup_{Z\in\mathcal{Z}_\alpha} \inf_{G\in\bar{\mathcal{G}}} f(Z,G), \quad u^* := \inf_{G\in\bar{\mathcal{G}}} \sup_{Z\in\mathcal{Z}_\alpha} f(Z,G)$$

for some proper alternative $\bar{\mathcal{G}}$. We encounter different notions of optimality:

(a) $\tilde{Z}$ is maximin-optimal, i.e.,

$$\inf_{G\in\bar{\mathcal{G}}} f(Z,G) \le \inf_{G\in\bar{\mathcal{G}}} f(\tilde{Z},G), \quad Z \in \mathcal{Z}_\alpha.$$

(b) $\tilde{G}$ is a worst-case model, i.e.,

$$\max_{Z\in\mathcal{Z}_\alpha} f(Z,\tilde{G}) \le \max_{Z\in\mathcal{Z}_\alpha} f(Z,G), \quad G \in \bar{\mathcal{G}}.$$

(c) $\tilde{Z}$ solves problem $(G|\mathcal{H})$ for some fixed $G \in \bar{\mathcal{G}}$, i.e.,

$$f(Z,G) \le f(\tilde{Z},G), \quad Z \in \mathcal{Z}_\alpha.$$

(d) $\tilde{G}$ is optimal for fixed $\tilde{Z}$, i.e.,

$$f(\tilde{Z},\tilde{G}) \le f(\tilde{Z},G), \quad G \in \bar{\mathcal{G}}.$$

(e) The pair $(\tilde{Z},\tilde{G})$ is a saddle point if $\tilde{G}$ is optimal for fixed $\tilde{Z}$ and vice versa, i.e.,

$$f(Z,\tilde{G}) \le f(\tilde{Z},\tilde{G}) \le f(\tilde{Z},G), \quad (Z,G) \in \mathcal{Z}_\alpha \times \bar{\mathcal{G}}.$$

In the terminology of game-theory, notion (a) (respectively (b)) corresponds to a *conservative strategy* $\tilde{Z}$ (respectively $\tilde{G}$). A saddle point is also called a *noncooperative equilibrium.*

We take the following proposition from Chapter 6 Section 2 in [AE84], cf. also Theorem 1.73 in [Wit85].

2.8 Proposition. *The following conditions are equivalent:*

(i) $(\tilde{Z},\tilde{G}) \in \mathcal{Z}_\alpha \times \bar{\mathcal{G}}$ *is a saddle point.*

(ii) $u_* = u^*$ *holds,* $\tilde{Z}$ *is maximin-optimal for* $(\mathcal{G}|\mathcal{H})$ *and* $\tilde{G}$ *is a worst-case model.*

2.9 Remark. *The family* $\mathcal{Z}_\alpha$ *is* $\sigma(L^\infty, L^1)$ *compact, cf. e.g.* [Wit85]. *It follows as in* [Leu99], *Lemma 2, that for fixed* $G \in \bar{\mathcal{G}}$, *the function* $Z \mapsto E[G\,u(Z,.)]$ *is concave and lower-semicontinuous with respect to the* $\sigma(L^\infty, L^1)$*-topology on* $\mathcal{Z}_\alpha$. *The function* $G \mapsto E[G\,u(Z,.)]$ *is linear, hence convex.*

It is easily seen that the values u_* and u^* coincide:

2.10 Proposition. *$u_* = u^*$ holds and there exists a test $\tilde{Z} \in \mathcal{Z}_\alpha$ that solves problem $(\mathcal{G}|\mathcal{H})$:*

$$u_* = \inf_{G \in \mathcal{G}} E[Gu(\tilde{Z}, .)].$$

Proof. Similar to the proof of Theorem 41 in [Leu99], this follows from Remark 2.9 and Theorem 2 in [AE84], Chapter 6 Section 2. □

2.11 Corollary. *Consider a worst-case model $\tilde{G}$ and a maximin-optimal test $\tilde{Z}$. The pair $(\tilde{Z}, \tilde{G})$ is a saddle-point.*

Proof. This is an immediate consequence of Propositions 2.10, and 2.8. □

2.12 Proposition. *The solution to the semi-composite problem $(G|\mathcal{H})$ is R-almost surely unique on the event $\{G > 0\}$.*

Proof. Assume that $\tilde{Z}_1$ and $\tilde{Z}_2$ are two solutions to $(G|\mathcal{H})$ and define $Z = \frac{1}{2}\tilde{Z}_1 + \frac{1}{2}\tilde{Z}_2 \in \mathcal{Z}_\alpha$. Concavity of u implies

$$E[G\,u(Z, .)] \geq \frac{1}{2}E[G\,u(\tilde{Z}_1, .)] + \frac{1}{2}E[G\,u(\tilde{Z}_2, .)] = E[G\,u(\tilde{Z}_1, .)]$$

and this inequality is strict if $R[\{\tilde{Z}_1 \neq \tilde{Z}_2,\, G > 0\}] > 0$ holds. This would contradict optimality of $\tilde{Z}_1$. □

2.13 Theorem. *Consider a worst-case model $\tilde{G}$ such that $\tilde{G} > 0$ holds R-almost surely. Then the maximin-optimal test is the R-almost surely unique solution to the semi-composite problem $(\tilde{G}|\mathcal{H})$.*

Proof. We obtain $R-a.s.$ uniqueness of the solution $\tilde{Z}_{\tilde{G}}$ to the problem $(\tilde{G}|\mathcal{H})$ from Proposition 2.12. Due to Corollary 2.11, any maximin-optimal test $\tilde{Z}$ is a solution to problem $(\tilde{G}|\mathcal{H})$. □

We define

$$F(G) := \max_{Z \in \mathcal{Z}_\alpha} E[Gu(Z, .)]. \tag{2.14}$$

2.14 Proposition. *Assume that $u(0, .) \geq 0$ holds and that the closure $\bar{\mathcal{G}}^0$ of $co(\mathcal{G})$ in $L^0(R)$ is proper. Then $\bar{\mathcal{G}}^0$ contains a worst-case model.*

Proof. Consider a minimizing sequence (G_n) for $F(G)$ of equation (2.14). Due to Lemma 3.3 of [KS99] there exists a sequence of convex combinations $G'_n \in \mathrm{co}\{G_n, G_{n+1}, \ldots\}$ which converge to G' R-almost surely. Hence G' is an element of $\bar{\mathcal{G}}^0$. Due to Proposition 2.10 there exists a random variable $\tilde{Z}$ that solves the

problem $(G'|\mathcal{H})$, i.e.,

$$\begin{aligned} F(G') &= \max_{Z\in\mathcal{Z}_\alpha} E[G'\,u(Z,.)] \\ &= E[G'\,u(\tilde{Z},.)] \\ &= E[\lim_n G'_n\,u(\tilde{Z},.)]. \end{aligned}$$

Due to nonnegativity of u, we can apply Fatou's Lemma to obtain

$$\begin{aligned} F(G') &\le \liminf_n E[G'_n\,u(\tilde{Z},.)] \\ &= \liminf_n \sum_{k\ge n} \lambda_k E[G_k\,u(\tilde{Z},.)] \\ &\le \liminf_n \sup_{k\ge n} E[G_k\,u(\tilde{Z},.)] \\ &\le \liminf_n \sup_{k\ge n} \max_{Z\in\mathcal{Z}_\alpha} E[G_k\,u(Z,.)] \\ &= \liminf_n \sup_{k\ge n} F(G_k) \\ &= \inf_{G\in\bar{\mathcal{G}}^0} F(G) \end{aligned}$$

□

2.15 Proposition. *Assume there exist models $G_1,\dots,G_N$ such that $co(\mathcal{G}) = co(G_1,\dots,G_N)$. Then $co(\mathcal{G})$ contains a worst-case model.*

Proof. Consider the proper alternative

$$co(\mathcal{G}) = \{G_\gamma = \sum_{n=1}^N \gamma_n G_n \mid \gamma\in\mathcal{S}_N\}$$

where $\mathcal{S}_N$ is the compact N-dimensional simplex defined by

$$\mathcal{S}_N := \{\gamma = (\gamma_1,\dots\gamma_N) \mid \gamma_n\in[0,1], \sum_{n=1}^N \gamma_n = 1\}.$$

The convex function $\gamma\mapsto F(G_\gamma)$ attains its minimum at some point $\tilde{\gamma}\in\mathcal{S}_N$. Hence $G_{\tilde{\gamma}}\in co(\mathcal{G})$ is a worst-case model. □

Due to (2.1), $\mathcal{G}$ is uniformly integrable if and only if it is uniformly absolutely continuous, i.e.,

$$\sup_{G\in\mathcal{G}} E[G;A] \to 0 \quad \text{for } R[A]\to 0. \tag{2.15}$$

2.16 Lemma. *Assume $\mathcal{G}$ is uniformly integrable. Then the families $co(\mathcal{G})$, $\bar{\mathcal{G}}^1$ and $\bar{\mathcal{G}}^0$ are uniformly integrable and $\bar{\mathcal{G}}^1 = \bar{\mathcal{G}}^0$ holds.*

Proof. We first show that the convex hull $co(\mathcal{G})$ is uniformly integrable. Consider any $\epsilon > 0$. Due to equation (2.15) there exists some $\delta > 0$ such that $R[A] < \delta$

implies

$$\sup_{G\in\mathcal{G}} E[G;A] < \epsilon.$$

Hence we obtain

$$E[G';A] < \epsilon.$$

for any $G' = \sum_{n=1}^{N} \lambda_n G_n \in co(\mathcal{G})$, i.e.,

$$\sup_{G'\in co(\mathcal{G})} E[G';A] < \epsilon.$$

Since $E[G'] \leq G_0$ holds for all $G' \in co(\mathcal{G})$, the last equation implies uniform integrability of $co(\mathcal{G})$.

Similarly, it follows from equation (2.13) that $\bar{\mathcal{G}}^1$ is uniformly integrable.

Finally, uniform integrability of $\bar{\mathcal{G}}^1$ and $G \geq 0$ forall $G \in \bar{\mathcal{G}}^1$ implies $\bar{\mathcal{G}}^1 = \bar{\mathcal{G}}^0$. □

2.17 Proposition. *Assume that $\mathcal{G}$ is uniformly integrable and that u is bounded. Then $\bar{\mathcal{G}}^1$ is a proper alternative and contains a worst-case model.*

Proof. It follows from Proposition 2.7 (4) that $\bar{\mathcal{G}}^1$ is a proper alternative.

Since u is bounded, the mapping $G \mapsto E[G\,u(Z,.)]$ is continuous with respect to $L^1(R)$. Hence the mapping

$$G \mapsto \max_{Z\in\mathcal{Z}_\alpha} E[G\,u(Z,.)] = F(G)$$

is lower semi-continuous with respect to $L^1(R)$. Now consider a minimizing sequence (G_n) in $\bar{\mathcal{G}}^1$ for F. Due to convexity of F we can assume that (G_n) is convergent to G in $L^0(R)$ (for otherwise there is a sequence of convex combinations of (G_n) that is convergent in $L^0(R)$ and minimizing). Due to uniform integrability of $\bar{\mathcal{G}}^1$, the sequence is convergent in $L^1(R)$ which implies $G \in \bar{\mathcal{G}}^1$. By lower semi-continuity of F w.r.t. $L^1(R)$ we obtain

$$F(G) \leq \lim_{n\uparrow\infty} F(G_n) = \inf_{G\in\bar{\mathcal{G}}^1} F(G).$$

Hence G is a worst-case model. □

2.18 Lemma. *Under the assumptions of either Propositions 2.14, Proposition 2.15 or Proposition 2.17, the value function $\alpha \mapsto u_*(\alpha)$ is increasing and strictly concave on $(0,\tilde{\alpha})$ where we have set*

$$\tilde{\alpha} := \inf\{\alpha > 0 \mid u_*(\alpha) = u_*(H_0)\}. \tag{2.16}$$

Proof. For $0 \leq \alpha_1 < \alpha_2 < H_0$ and $0 < \lambda < 1$ we have to show that

$$u_*(\lambda\alpha_1 + (1-\lambda)\alpha_2) > \lambda u_*(\alpha_1) + (1-\lambda)u_*(\alpha_2)$$

holds.

Let $\tilde{Z}_1 \in \mathcal{Z}_{\alpha_1}$ and $\tilde{Z}_2 \in \mathcal{Z}_{\alpha_2}$ denote the maximin-optimal tests corresponding to α_1 and α_2. Due to $Z_3 := \lambda\tilde{Z}_1 + (1-\lambda)\tilde{Z}_2 \in \mathcal{Z}_{\lambda\alpha_1+(1-\lambda)\alpha_2}$ we have

$$u_*(\lambda\alpha_1 + (1-\lambda)\alpha_2) \geq \inf_{G\in\bar{\mathcal{G}}} E[Gu(Z_3,.)].$$

Under the assumptions of either Proposition 2.14, Proposition 2.15 or Proposition 2.17, we obtain existence of a model G_3 that minimizes the last expression, i.e.,

$$\inf_{G\in\bar{\mathcal{G}}} E[Gu(Z_3,.)] = E[G_3u(Z_3,.)].$$

From the strict concavity of u we obtain

$$\begin{aligned} u_*(\lambda\alpha_1 + (1-\lambda)\alpha_2) &\geq E[G_3u(Z_3,.)] \\ &\geq \lambda E[G_3u(\tilde{Z}_1,.)] + (1-\lambda)E[G_3u(\tilde{Z}_2,.)] \qquad (2.17) \\ &\geq \lambda \inf_{G\in\bar{\mathcal{G}}} E[Gu(\tilde{Z}_1,.)] + (1-\lambda)\inf_{G\in\bar{\mathcal{G}}} E[Gu(\tilde{Z}_2,.)] \qquad (2.18) \\ &= \lambda u_*(\alpha_1) + (1-\lambda)u_*(\alpha_2) \end{aligned}$$

which proves concavity of u_* on $[0, H_0]$.

Now consider $\alpha_2 < \tilde{\alpha}$ and assume we have equality everywhere in the above estimate. Equality in (2.18) implies that G_3 is a minimizing model for both $\tilde{Z}_1$ and $\tilde{Z}_2$, i.e.,

$$u_*(\alpha_i) = E[G_3u(\tilde{Z}_i,.)], \quad i = 1,2$$

Equality in (2.17) implies

$$R[\tilde{Z}_1 = \tilde{Z}_2, G_3 > 0] = R[G_3 > 0]. \qquad (2.19)$$

Putting together the last two equations we can conclude

$$u_*(\alpha_1) = u_*(\alpha_2).$$

Since u_* is increasing and concave,the last equation implies $u_*(\alpha_1) = u_*(H_0)$ in contradiction to $\alpha_1 < \tilde{\alpha}$. □

2.4. Convex duality

In this section, we define the dual problem (2.37). We show that there is no duality gap, cf. Lemma 2.24. As a consequence, a solution to the dual-problem is essentially equivalent to a least-favorable pair, cf. Proposition 2.27. The solution to the original problem is then constructed via a solution to the dual problem in several stages. First we consider the simple problem for fixed model and pricing rule in Lemma 2.26. Then, the semi-composite problem for fixed model is solved in Lemmata 2.28 and 2.29. Finally we turn to the solution of the full problem in Theorems 2.31 and 2.32. We state in Theorem 2.31 that the simple optimal test for a least-favorable pair is the unique maximin-optimal test in the case where the least-favorable pair is equivalent to R. This follows easily from the combination of the results on the semi-composite problem in this section and some more general results of Section 2.3, especially uniqueness of the optimal test. We provide an alternative proof which is closer in spirit to the convex-duality methods applied for the semi-composite

problem: We derive optimality of the worst-case model $\tilde{G}$ for the $\tilde{G}$-optimal test $\tilde{Z}_{\tilde{G}}$ directly from the fact that $\tilde{G}$ is a solution to the dual problem. For the case where the least-favorable pair is not equivalent to R, the maximin-optimal test is approximated by a series of testing problems with associated least-favorable pairs equivalent to R in Theorem 2.32.

We now outline in more detail the different stages of the dual problem. Subsequently, $\bar{\mathcal{H}}$ denotes a proper hypothesis and $\bar{\mathcal{G}}$ a proper alternative. We consider the "dual" problem of minimizing the function $g = g(G, H, k)$ defined in (2.37) over the space $\bar{\mathcal{G}} \times \bar{\mathcal{H}} \times [0, \infty)$. We shall see that the *value*

$$g^* = \inf_{(G,H,k)\in\bar{\mathcal{G}}\times\bar{\mathcal{H}}\times[0,\infty)} g(G, H, k)$$

of the dual problem coincides with the value of the primal problem u_*, cf. Lemma 2.24. As a consequence, the value of the dual problem is independent of the choice of $\bar{\mathcal{G}}, \bar{\mathcal{H}}$. We chose to allow for different domains of the dual problem (different proper $\bar{\mathcal{G}}$ respectively $\bar{\mathcal{H}}$) since the smallest proper domain that contains a solution to the dual problem may depend on the specific problem under consideration.

For fixed model G and pricing rule H, a solution $\tilde{k}$ to the "simple" dual problem $g(G, H, \tilde{k}) = \min_k g(G, H, k)$ will be called the critical value $\tilde{k} =: \tilde{k}_{(G|H)}$. The solution to the simple problem $(G|H)$ is given by

$$\tilde{Z}_{(G|H)} := I(\tilde{k}_{(G|H)}\frac{H}{G}, .)\, \mathbf{1}_{\{G>0\}}$$

and the values of the simple problem and the dual problem for fixed G, H coincide (cf. Lemma 2.26):

$$\max_{Z\in\mathcal{Z}_{\alpha,H}} E[Gu(Z, .)] = \min_{k\geq 0} g(G, H, k). \tag{2.20}$$

This is the main content of Section 2.4.1.

Second we consider in Section 2.4.2 the dual problem for fixed model G. If $(\tilde{H}, \tilde{k})$ denotes a solution to this problem, we find that the solution to the semi-composite problem $(G|\mathcal{H})$ is given by

$$I(\tilde{k}\frac{\tilde{H}}{G}, .)\, \mathbf{1}_{\{G>0\}}.$$

On the other hand, it follows from equation (2.20) that $(\tilde{H}, \tilde{k})$ solves the dual problem for fixed G if and only if $\tilde{k}$ is the critical value for $(G|\tilde{H})$ and $\tilde{H}$ is a worst-case pricing rule for G. Hence the solution to the semi-composite problem $(G|\mathcal{H})$ is given by the solution $\tilde{Z}_{(G|\tilde{H})}$ to the simple problem $(G|\tilde{H})$ for a worst-case pricing rule $\tilde{H}$. Furthermore, we obtain strong duality:

$$\max_{Z\in\mathcal{Z}_\alpha} E[G\, u(Z, .)] = \min_{(H,k)\in\bar{\mathcal{H}}\times[0,\infty)} g(G, H, k), \tag{2.21}$$

cf. Lemma 2.29.

As a last step, we consider the full dual problem in Section 2.4.3. It follows from equations (2.20) and (2.21) that $(\tilde{G}, \tilde{H}, \tilde{k})$ solves the dual problem if and only if

$\tilde{G}$ is worst-case model, $\tilde{H}$ is a worst-case pricing rule for $\tilde{G}$ and $\tilde{k} = \tilde{k}_{(\tilde{G}|\tilde{H})}$ is the critical value for $(\tilde{G}|\tilde{H})$, cf. Proposition 2.27. We obtain that every maximin-optimal test can be found among the solutions to the simple-problem $(\tilde{G}|\tilde{H})$. However, we cannot conclude in general that $\tilde{Z}_{(\tilde{G}|\tilde{H})}$ is maximin-optimal. This is only valid if $R[\tilde{G} = 0, \tilde{H} = 0] = 0$ holds.

2.4.1. Preliminary and auxiliary results

In this section, we define the inverse I of marginal utility and the convex conjugate V of u. We define the critical value $\tilde{k}_{(G|H)}$ from which we derive the solution $\tilde{Z}_{(G|H)}$ of the simple problem $(G|H)$, cf. Lemma 2.26. In Proposition 2.20, we exhibit a simple sufficient condition for maximin-optimality of $\tilde{Z}_{(G|H)}$. We show that there is no duality gap in Lemma 2.24.

We first define the inverse function of the derivative $\partial_z u(z,\omega) = u'(z,\omega)$. Observe that we do not impose Inada conditions on u. Instead, we introduce the conventions

- $\lim_{z\uparrow 1} u'(z,\omega) =: u'(1,\omega) \in [0,\infty)$,
- $\lim_{z\downarrow 0} u'(z,\omega) =: u'(0,\omega) \in (0,\infty]$.

Thus defined, $u'(.,\omega)$ is a strictly decreasing function on $[0,1]$. The inverse function $\tilde{I}$ of $u'(.,\omega)$,

$$\tilde{I}(.,\omega): \ [u'(1,\omega), u'(0,\omega)] \to [0,1]$$

is a well defined decreasing function.

We extend the domain of $\tilde{I}(.,\omega)$ to $[0,\infty]$ by setting $I(y,\omega) = 1$ for $y \in [0, u'(1,\omega))$ and $I(y,\omega) = 0$ for $y \in [u'(0,\omega), \infty]$. We thus obtain a unique function

$$I(.,\omega): \ [0,\infty) \to [0,1]. \tag{2.22}$$

that coincides with $\tilde{I}$ on the interval $[u'(1,\omega), u'(0,\omega)]$. Hence for any nonnegative random-variable B, the mapping $I(B,.) : \Omega \to [0,1]$ is a well defined test. The stochastic conjugate of u is given by

$$V(k,\omega) := \max_{0\le z\le 1} (u(z,\omega) - zk) \tag{2.23}$$

$$= \begin{cases} u(1,\omega) - k, & \text{if } k < u'(1,\omega) \\ u(\tilde{I}(k,\omega),\omega) - k\tilde{I}(k,\omega), & \text{if } u'(1,\omega) \le k \le u'(0,\omega) \\ u(0,\omega) & \text{if } u'(0,\omega) < k \end{cases} \tag{2.24}$$

$$= u(I(k,\omega),\omega) - kI(k,\omega) \tag{2.25}$$

We observe that $V(.,\omega)$ is decreasing, convex and differentiable with derivative

$$V'(k,.) := \partial_k V(k,.) = -I(k,.) \tag{2.26}$$

Hence $V(.,\omega)$ is strictly convex on the interval $(u'(1,\omega), u'(0,\omega))$. V is nonnegative if u is nonnegative.

2.19 Proposition. *For any $G \in \bar{\mathcal{G}}$ and $H \in \bar{\mathcal{H}}$ there exists a constant $\tilde{k}_{(G|H)} \in [0,\infty)$ that assumes the minimum*

$$\tilde{k}_{(G|H)} = \min\{k \geq 0 \mid E[HI(k\frac{H}{G},.)\,\mathbf{1}_{\{G>0\}}] \leq \alpha\}. \tag{2.27}$$

We call $\tilde{k}_{(G|H)}$ the critical value for $(G|H)$. With the definition

$$\tilde{Z}_{(G|H)} := I(\tilde{k}_{(G|H)}\frac{H}{G},.)\,\mathbf{1}_{\{G>0\}} \tag{2.28}$$

we have

(i) $E[H\tilde{Z}_{(G|H)}] = \alpha$ *and* $\tilde{k}_{(G|H)} > 0$ *if* $\alpha < E[H\,\mathbf{1}_{\{G>0\}}]$ *or*

(ii) $\tilde{k}_{(G|H)} = 0$ *if* $\alpha \geq E[H\,\mathbf{1}_{\{G>0\}}]$.

Proof. For any nonnegative random variable B, $I(kB,.)$ converges to 1 (respectively 0) R-almost surely as k tends to 0 (respectively ∞). Since $|HI| = HI$ is dominated by $H \in \mathcal{L}^1(R)$, it follows that $f(k) = E[H\,I(kB,.)\,\mathbf{1}_{\{G>0\}}]$ converges to $E[H\ \mathbf{1}_{\{G>0\}}]$ (respectively 0) as k tends to 0 (respectively ∞). Due to the Lebesque-criterion for dominated integration $f(k)$ is continuous. Hence for any $\alpha \in [0, E[H\ \mathbf{1}_{\{G>0\}}])$ there exists a $\tilde{k}$ with $f(\tilde{k}) = \alpha$ and $\tilde{k}_{(G|H)} = 0$ if and only if $\alpha \geq E[H\ \mathbf{1}_{\{G>0\}}]$ □

We begin our analysis of the composite problem by stating a simple sufficient condition for optimality:

2.20 Proposition. *Consider a model $\tilde{G} \in \bar{\mathcal{G}}$ and a pricing rule $\tilde{H} \in \bar{\mathcal{H}}$ such that*

$$E[H\tilde{Z}_{(\tilde{G}|\tilde{H})}] \leq \alpha \quad \forall H \in \mathcal{H} \tag{2.29}$$

$$E[\tilde{G}u(\tilde{Z}_{(\tilde{G}|\tilde{H})},.)] \leq E[Gu(\tilde{Z}_{(\tilde{G}|\tilde{H})},.)] \quad \forall G \in \mathcal{G} \tag{2.30}$$

Then the test $\tilde{Z}_{(\tilde{G}|\tilde{H})}$ of (2.28) *is maximin-optimal for $(\mathcal{G}|\mathcal{H})$ and $\tilde{G}$ is a worst-case model.*

Proof. (1) Due to the definition of V, we have for any choice of z, k and ω the estimate

$$u(z,\omega) \leq V(k,\omega) + kz\ . \tag{2.31}$$

Substituting $k = \tilde{k}_{(\tilde{G}|\tilde{H})}\frac{\tilde{H}}{\tilde{G}}\,\mathbf{1}_{\{\tilde{G}>0\}}$ yields

$$\begin{aligned} u(Z,.) &\leq \tilde{k}_{(\tilde{G}|\tilde{H})}\frac{\tilde{H}}{\tilde{G}}\,\mathbf{1}_{\{\tilde{G}>0\}}Z + V(\tilde{k}_{(\tilde{G}|\tilde{H})}\frac{\tilde{H}}{\tilde{G}}\,\mathbf{1}_{\{\tilde{G}>0\}},.) \\ &= u(I(\tilde{k}_{(\tilde{G}|\tilde{H})}\frac{\tilde{H}}{\tilde{G}}\,\mathbf{1}_{\{\tilde{G}>0\}},.),.) - \tilde{k}_{(\tilde{G}|\tilde{H})}\frac{\tilde{H}}{\tilde{G}}\,\mathbf{1}_{\{\tilde{G}>0\}}\Big(\tilde{Z}_{(\tilde{G}|\tilde{H})} - Z\Big) \end{aligned}$$

where the last equality is due to (2.25) and the definition of $\tilde{Z}_{(\tilde{G}|\tilde{H})}$. Multiplying this equation by $\tilde{G}$ implies

$$\begin{aligned} E[\tilde{G}u(Z,\,.\,)] &\leq E[\tilde{G}u(I(\tilde{k}_{(\tilde{G}|\tilde{H})}\frac{\tilde{H}}{\tilde{G}},.),\,.\,)] \\ &\quad -\tilde{k}_{(\tilde{G}|\tilde{H})}E[\tilde{H}\,\mathbf{1}_{\{\tilde{G}>0\}}\Big(\tilde{Z}_{(\tilde{G}|\tilde{H})}-Z\Big)] \\ &= E[\tilde{G}u(\tilde{Z}_{(\tilde{G}|\tilde{H})},\,.\,)]-\tilde{k}_{(\tilde{G}|\tilde{H})}E[\tilde{H}\,\mathbf{1}_{\{\tilde{G}>0\}}\Big(\tilde{Z}_{(\tilde{G}|\tilde{H})}-Z\Big)]. \end{aligned} \tag{2.32}$$

2) Assume that

$$\alpha < E[\tilde{H}\,\mathbf{1}_{\{\tilde{G}>0\}}] \tag{2.33}$$

holds. For any test $Z \in \mathcal{Z}_\alpha$ we have

$$E[\tilde{H}\,\mathbf{1}_{\{\tilde{G}>0\}}Z] \leq E[\tilde{H}Z] \leq \alpha$$

$$= E[\tilde{H}\tilde{Z}_{(\tilde{G}|\tilde{H})}] \tag{2.34}$$

$$= E[\tilde{H}\,\mathbf{1}_{\{\tilde{G}>0\}}\tilde{Z}_{(\tilde{G}|\tilde{H})}] \tag{2.35}$$

where equality in (2.34) is due to the definition of $\tilde{k}_{(\tilde{G}|\tilde{H})}$ and (2.33). Combining estimates (2.32) and (2.35) yields

$$E[\tilde{G}u(Z,\,.\,)] \leq E[\tilde{G}u(\tilde{Z}_{(\tilde{G}|\tilde{H})},\,.\,)]\ . \tag{2.36}$$

So far, we have not used any of the assumptions of the theorem. Assumption (2.29) guarantees $\tilde{Z}_{(\tilde{G}|\tilde{H})} \in \mathcal{Z}_\alpha$, together with assumption (2.30) and estimate (2.36) this implies that $(\tilde{Z}_{(\tilde{G}|\tilde{H})}, \tilde{G})$ is a saddle point. Hence $\tilde{Z}_{(\tilde{G}|\tilde{H})}$ is maximin-optimal and $\tilde{G}$ is a worst-case model, cf. Proposition 2.8.

3) If (2.33) does not hold, we have $\tilde{k}_{(\tilde{G}|\tilde{H})} = 0$, hence we can conclude the estimate (2.36) directly from (2.32). From here proceed as in (2). □

2.21 Corollary.

(i) Consider a model $G \in \bar{\mathcal{G}}$ and a pricing rule $H \in \bar{\mathcal{H}}$. The test $\tilde{Z}_{(G|H)}$ solves the simple problem $(G|H)$.

(ii) For a model $\tilde{G}$ consider a pricing rule $\tilde{H}$ such that condition (2.29) is satisfied. The test $\tilde{Z}_{(\tilde{G}|\tilde{H})}$ solves the semi-composite problem $(\tilde{G}|\mathcal{H})$.

Proof. Item (i) respectively (ii) follows directly from Proposition 2.20 if we consider the special case $\mathcal{G} = \{G\}$, $\mathcal{H} = \{H\}$ respectively $\mathcal{G} = \{\tilde{G}\}$. □

We define the dual problem by

$$\left[\begin{array}{c} g(G,H,k) := E[GV(k\frac{H}{G},.)] + k\alpha = \min\limits_{G,H,k} \\ (G,H,k) \in \bar{\mathcal{G}} \times \bar{\mathcal{H}} \times (0,\infty) \end{array}\right]. \tag{2.37}$$

The functions $H \mapsto g(G,H,k)$ and $k \mapsto g(G,H,k)$ are convex due to convexity of V and Jensen's inequality.

We obtain from equations (2.11) and (2.31) the basic estimate

$$E[G\,u(Z,.)] \leq g(G,H,k), \quad Z \in \mathcal{Z}_\alpha, \quad (G,H,k) \in \bar{\mathcal{G}} \times \bar{\mathcal{H}} \times (0,\infty) \tag{2.38}$$

for any proper hypothesis $\bar{\mathcal{H}}$ and proper alternative $\bar{\mathcal{G}}$.

We say that there is no duality gap if

$$\sup_{Z \in \mathcal{Z}_\alpha} \inf_{G \in \bar{\mathcal{G}}} E[G\,u(Z,.)] = \inf_{(G,H,k) \in \bar{\mathcal{G}} \times \bar{\mathcal{H}} \times (0,\infty)} g(G,H,k) \tag{2.39}$$

holds. For efficient hedging and utility maximization, problems of the form (2.3) are typically considered for a singleton $\mathcal{G} = \{G\}$, cf. for example [Cvi00], [FL00], [Leu99],[KS99] and [Sch00]. Here, the typical approach is to first construct a solution $(\tilde{H}, \tilde{k})$ to the dual problem. Then one demonstrates optimality of the random variable $I(\tilde{k}\frac{\tilde{H}}{G},.)\,\mathbf{1}_{\{G>0\}}$. It then follows from (2.25) that there is no duality gap. We demonstrate first that equation (2.39) can be obtained from rather classical results in the spirit of [Wit85], cf. Lemma 2.23 and Lemma 2.24. We define

$$\mathcal{S}_m = \{\underline{\lambda} = (\lambda_1, \ldots, \lambda_m) \mid \lambda_i \geq 0, \ \sum_{i=1}^m \lambda_i = 1\}.$$

We say that $\bar{\mathcal{H}}$ is finitely generated if there exists a finite subset $\{H_1, \ldots, H_m\} \subset \bar{\mathcal{H}}$ such that

$$\bar{\mathcal{H}} = \Big\{\sum_{i=1}^m \lambda_i H_i \mid \underline{\lambda} \in \mathcal{S}_m\Big\}$$

holds. We quote Theorem 1.71 b) of [Wit85]:

2.22 Theorem. *Consider a linear space $\mathcal{L}$, $\mathcal{C} \subset \mathcal{L}$ a convex subset, $f : \mathcal{L} \to \mathbb{R}$ a concave function, $A : \mathcal{L} \to \mathbb{R}^m$ a linear function and $\underline{c} \in \mathbb{R}^m$. For $\underline{z} \in \mathbb{R}^m$ define*

$$\begin{aligned} \hat{g}(\underline{z}) &= \underline{z}\,\underline{c} + \sup_{Z \in \mathcal{C}}[f(Z) - \underline{z}A(Z)] \\ \mathcal{C}_{\leq} &= \{Z \in \mathcal{C} \mid A_i(Z) \leq c_i, \ i = i, \ldots, m\}. \end{aligned} \tag{2.40}$$

If there exists a $Z_0 \in \mathcal{C}$ such that $A_i(Z_0) < c_i$, $i = i, \ldots, m$, we have

$$\sup_{Z \in \mathcal{C}_{\leq}} f(Z) = \inf_{\underline{z} \in \mathbb{R}^m_+} \hat{g}(\underline{z}). \tag{2.41}$$

If this value is finite, there exists a $\underline{z}^* \in \mathbb{R}^m_+$ such that

$$\sup_{Z \in \mathcal{C}_{\leq}} f(Z) = \hat{g}(\underline{z}^*).$$

2.23 Lemma (Duality for the semi-composite problem). *Consider any fixed model $G \in \bar{\mathcal{G}}$ and a proper hypothesis $\bar{\mathcal{H}}$.*

(i) If $\bar{\mathcal{H}}$ is finitely generated, strong duality holds:

$$\max_{Z \in \mathcal{Z}_\alpha} E[G\,u(Z,.)] = \min_{(H,k) \in \bar{\mathcal{H}} \times [0,\infty)} g(G,H,k).$$

(ii) For any $\bar{\mathcal{H}}$, weak duality holds:

$$\max_{Z \in \mathcal{Z}_\alpha} E[G\,u(Z,.)] = \inf_{(H,k) \in \bar{\mathcal{H}} \times [0,\infty)} g(G,H,k).$$

Proof. (i): Consider a generating subset $\{H_1, \ldots, H_m\} \subset \bar{\mathcal{H}}$. We apply Theorem 2.22 to the following objects:

$$\begin{aligned}
\mathcal{L} &= L^0(\Omega, \mathcal{F}, P) \\
\mathcal{C} &= \{Z \in \mathcal{L} \mid 0 \le Z \le 1\} \\
f(Z) = f_G(Z) &= E[Gu(Z,.)] \\
A(Z) &= (E[H_1 Z], \ldots, E[H_m Z]) \\
\underline{c} &= (\alpha, \ldots, \alpha) \in \mathbb{R}^m \\
\underline{H} &= (H_1, \ldots, H_m) \\
\underline{\lambda}\,\underline{H} &= \sum_{i=1}^m \lambda_i H_i
\end{aligned}$$

Let $\hat{g}$ be given by equation (2.40). For $k \ge 0$ and $\underline{\lambda} \in \mathcal{S}_m$, we claim that

$$\hat{g}(k\underline{\lambda}) = g(G, \underline{\lambda}\,\underline{H}, k) \tag{2.42}$$

holds. By means of (2.31) with $k = k\frac{\underline{\lambda}\,\underline{H}}{G}$ we obtain the estimate

$$\begin{aligned}
\hat{g}(k\underline{\lambda}) &= k\underline{\lambda}\,\underline{c} + \sup_{Z \in \mathcal{C}}[f(Z) - k\underline{\lambda}A(Z)] \\
&= k\alpha + \sup_{Z \in \mathcal{C}}[E[Gu(Z,.)] - k\sum_{i=1}^m \lambda_i E[H_i Z]] \\
&\le k\alpha + \sup_{Z \in \mathcal{C}}\left\{E[GV(k\frac{\underline{\lambda}\,\underline{H}}{G},.) + k\underline{\lambda}\,\underline{H}Z] - kE[\underline{\lambda}\,\underline{H}Z]\right\} \\
&= k\alpha + E[GV(k\frac{\underline{\lambda}\,\underline{H}}{G},.)] \\
&= g(G, \underline{\lambda}\,\underline{H}, k).
\end{aligned}$$

For the converse estimate, we proceed as follows:

$$\begin{aligned}
\hat{g}(k\underline{\lambda}) &= k\underline{\lambda}\,\underline{c} + \sup_{Z\in\mathcal{C}}[f(Z) - k\underline{\lambda}A(Z)] \\
&= k\alpha + \sup_{Z\in\mathcal{C}}[E[Gu(Z,.)] - kE[\underline{\lambda}\,\underline{H}Z]] \\
&\geq k\alpha + E[Gu(I(k\frac{\underline{\lambda}\,\underline{H}}{G},.),.)] - kE[\underline{\lambda}\,\underline{H}I(k\frac{\underline{\lambda}\,\underline{H}}{G},.)] \\
&= g(G,\underline{\lambda}\,\underline{H},k)
\end{aligned}$$

where the last equation is due toe (2.25). Hence equation (2.42) holds.

Since we assume $\alpha > 0$, the constant $Z_0 \equiv 0$ satisfies $A_i(Z_0) < c_i = \alpha,\ i = i,\dots,m$, and the value of (2.41) is finite due to equation (2.2). Theorem 2.22 yields the existence of an optimal $\underline{z}^* \in \mathbb{R}^m_+$ that attains the infimum in (2.41). If we set $k^* = \sum_{i=1}^m z_i^*$, $\lambda_i^* = \frac{z_i^*}{k^*}$ and $H^* = \underline{\lambda}^*\underline{H}$, it is easily seen that $k^* > 0$ holds. Equation (2.42) and optimality of $\underline{z}^*$ for (2.41) implies

$$\max_{Z\in\mathcal{Z}_\alpha} E[G\,u(Z,.)] = g(G,H^*,k^*).$$

This proofs item (i).

(ii) For subsets $\mathcal{H}' \subset \bar{\mathcal{H}}$ and $\kappa > 0$ define

$$\mathcal{C}_{\mathcal{H}',\kappa} = \{0 \leq Z \leq 1|\ E[HZ] \leq \alpha\ \forall H \in \mathcal{H}',\ E[Gu(Z,.)] \geq \kappa\}\,.$$

In view of equation (2.38), it suffices to show that $g(G,H,k) \geq \kappa\ \forall (H,k) \in \bar{\mathcal{H}} \times [0,\infty)$ implies $\mathcal{C}_{\bar{\mathcal{H}},\kappa} \neq \emptyset$. Assume that $g(G,H,k) \geq \kappa$ holds. Since

$$\mathcal{C}_{\bar{\mathcal{H}},\kappa} = \bigcap_{\substack{\mathcal{H}' \subset \bar{\mathcal{H}} \\ \mathcal{H}' \text{ finite}}} \mathcal{C}_{\mathcal{H}',\kappa},$$

we will show that for finite $\mathcal{H}' \subset \bar{\mathcal{H}}'$, the sets $\mathcal{C}_{\mathcal{H}',\kappa}$ are

a) nonempty,

b) $\sigma(L^\infty, L^1)$-closed,

c) have the finite intersection property, and that

d) $\mathcal{C} = \{Z|\ 0 \leq Z \leq 1\}$ is $\sigma(L^\infty, L^1)$- compact.

Then $\mathcal{C}_{\bar{\mathcal{H}},\kappa} \neq \emptyset$ follows from Lemma I.5.6 in [DS58]. We remark that the dual space of $\mathcal{C}$ contains $L^1(P)$.

a) For every finite set $\mathcal{H}' \subset \bar{\mathcal{H}}$, we have

$$\inf\{g(G,H,k)|\ H \in \ \mathrm{co}(\mathcal{H}'),\ k > 0\} \geq \inf\{g(G,H,k)|\ H \in \bar{\mathcal{H}},\ k > 0\} \geq \kappa.$$

Due to (i), this implies

$$\max\{E[Gu(Z,.)]|\ E[HZ] \leq \alpha \forall H \in \ \mathrm{co}(\mathcal{H}')\} \geq \kappa.$$

Hence the set $\mathcal{C}_{\mathcal{H}',\kappa}$ is nonempty.

b) Consider a sequence Z_n in $\mathcal{C}_{\mathcal{H}',\kappa}$ such that $E[\Phi Z_n] \to E[\Phi Z]$ for any $\Phi \in L^1(R)$ and some $Z \in \mathcal{C}$. Then we have $E[HZ] = \lim_{n\uparrow\infty} E[HZ_n] \leq \alpha$. Due to Komlos theorem, there exists a subsequence of convex combinations $\hat{Z}_n$ that converges $R-$a.s. to Z. Due to (2.2) and concavity of u, it follows that $E[Gu(Z,.)] \geq \lim_{n\uparrow\infty} E[Gu(Z_n,.)] \geq \kappa$. This implies $Z \in \mathcal{C}_{\mathcal{H}',\kappa}$.

c) follows from a) since the intersection of finitely many $\mathcal{C}_{\mathcal{H}'_1,\kappa}, \ldots, \mathcal{C}_{\mathcal{H}'_n,\kappa}$ is given by $\mathcal{C}_{\mathcal{H}',\kappa}$ where $\mathcal{H}' = \cup_i \mathcal{H}'_i$.

d) is a well known fact which follows from Alaoglu's theorem. This completes the proof. □

2.24 Lemma (Duality for the full problem). *For any proper alternative $\bar{\mathcal{G}}$ and hypothesis $\bar{\mathcal{H}}$ we have weak duality:*

$$\max_{Z\in\mathcal{Z}_\alpha} \inf_{G\in\bar{\mathcal{G}}} E[Gu(Z,.)] = \inf_{(G,H,k)\in\bar{\mathcal{G}}\times\bar{\mathcal{H}}\times[0,\infty)} g(G,H,k).$$

Proof. We obtain from Proposition 2.10 and Lemma 2.23 (ii) the equality

$$\begin{aligned} \max_{Z\in\mathcal{Z}_\alpha} \inf_{G\in\bar{\mathcal{G}}} E[G\,u(Z,.)] &= \inf_{G\in\bar{\mathcal{G}}} \max_{Z\in\mathcal{Z}_\alpha} E[G\,u(Z,.)] \\ &= \inf_{G\in\bar{\mathcal{G}}} \inf_{(H,k)\in\bar{\mathcal{H}}\times[0,\infty)} g(G,H,k). \end{aligned}$$

□

We define

$$\begin{aligned} v(k) &= \inf_{(G,H)\in\bar{\mathcal{G}}\times\bar{\mathcal{H}}} E[GV(k\frac{H}{G},.)] \\ &= \inf_{(G,H)\in\bar{\mathcal{G}}\times\bar{\mathcal{H}}} g(G,H,k) - k\alpha. \end{aligned}$$

2.25 Corollary. *u_* is the conjugate of v, i.e.,*

$$u_*(\alpha) = \min_{k\geq 0}(v(k) + k\alpha). \tag{2.43}$$

2.26 Lemma (The simple problem revisited). *For a given model $G \in \bar{\mathcal{G}}$ and a pricing measure $H \in \bar{\mathcal{H}}$, the following statements hold true:*

1) The function $k \mapsto g(G,H,k)$ is differentiable with derivative

$$\partial_k g(G,H,k) = E[\mathbf{1}_{\{G>0\}} HV'(k\frac{H}{G},.)] + \alpha \tag{2.44}$$

$$= -E[\mathbf{1}_{\{G>0\}} HI(k\frac{H}{G},.)] + \alpha. \tag{2.45}$$

2) For a constant $\tilde{k} \geq 0$ the following statements are equivalent:

(i) $\tilde{k} = \tilde{k}_{(G|\tilde{H})}$ the critical value for $(G|H)$.

(ii) $\tilde{k}$ *minimizes* $g(G, H, .)$*, i.e.,*

$$g(G, H, \tilde{k}) = \min_{k \geq 0} g(G, H, k).$$

3) The solution to the simple problem is given by $\tilde{Z}_{(G|H)} = \mathbf{1}_{\{G>0\}} I(\tilde{k}_{G|H)} \frac{H}{G}, .)$*. with value*

$$\begin{aligned} \max_{Z \in \mathcal{Z}_{\alpha,H}} E[Gu(Z, .)] &= \min_{k \geq 0} g(G, H, k) \\ &= g(G, H, \tilde{k}_{(G|H)}). \end{aligned}$$

Proof.

(1) Due to equation (2.26), we have

$$|HV'(k\frac{H}{G}, .)| \leq H, \tag{2.46}$$

i.e., H is a dominating integrable random variable. Due to the dominated convergence theorem, we can hence interchange differentiation and integration, i.e., equation (2.44) ensues. Validity of (2.45) follows from (2.44) and (2.26).

(2) We consider two cases:

a) $\alpha < E[H\, \mathbf{1}_{\{G>0\}}]$ or

b) $\alpha \geq E[H\, \mathbf{1}_{\{G>0\}}]$.

In case a), we obtain from (2.45) and from the definition of the critical value in (2.28) equivalence of assertion (i) and

$$\partial_{\tilde{k}} g(G, H, \tilde{k}) = 0. \tag{2.47}$$

Due to the convexity of $g(G, H, .)$, the last equation is equivalent to assertion (ii).

In case b), the critical value is given by $\tilde{k}_{(G|H)} = 0$. We obtain from (2.45) that

$$\partial_k g(G, H, k) < 0 \quad \forall k > 0. \tag{2.48}$$

Hence the function $g(G, H, .)$ assumes its minimum at $\tilde{k} = 0 = \tilde{k}_{(G|H)}$.

(3) Optimality of $\tilde{Z}_{(G|H)}$ was shown in Corollary 2.21. We obtain from (2) optimality of $\tilde{k}_{(G|H)}$, i.e.,

$$\begin{aligned} \min_{k \geq 0} g(G, H, k) &= g(G, H, \tilde{k}_{(G|H)}) \\ &= E[GV(\tilde{k}_{(G|H)} \frac{H}{G}, .)] + k\alpha \end{aligned}$$

Applying equation (2.25) to the last expression yields

$$\begin{aligned} E[GV(\tilde{k}_{(G|H)} \frac{H}{G}, .)] + k\alpha &= E[Gu(I(\tilde{k}_{(G|H)} \frac{H}{G}, .), .) - \tilde{k}_{(G|H)} \frac{H}{G} I(\tilde{k}_{(G|H)} \frac{H}{G}, .)] + k\alpha \\ &= E[Gu(\tilde{Z}_{(G|H)}, .)] \\ &= \max_{Z \in \mathcal{Z}_{\alpha,H}} E[Gu(Z, .)] \end{aligned}$$

where the last equation is due to optimality of $\tilde{Z}_{(G|H)}$. This can be obtained alternatively from Lemma 2.24. □

We can now state the equivalence of a least-favorable pair and a solution to the dual problem:

2.27 Proposition. *The following statements are equivalent:*

(i) $\tilde{G}$ *is worst-case model and* $\tilde{H}$ *is a worst-case pricing rule for* $\tilde{G}$.

(ii) $(\tilde{G}, \tilde{H})$ *is a least-favorable pair .*

(iii) $(\tilde{G}, \tilde{H}, \tilde{k}_{(\tilde{G}|\tilde{H})})$ *solves the dual problem.*

Proof.

1) Since there is no duality gap for the semi-composite problem, we have for any $G \in \bar{\mathcal{G}}$

$$\max_{Z \in \mathcal{Z}_\alpha} E[G\, u(Z, .)] = \inf_{H \in \bar{\mathcal{H}}} \min_{k>0} g(G, H, k). \tag{2.49}$$

(2) Since there is no duality gap for the simple problem, we have for any $G \in \bar{\mathcal{G}},\ H \in \bar{\mathcal{H}}$ the equation

$$E[Gu(\tilde{Z}_{(G|H)}] = \min_{k>0} g(G, H, k). \tag{2.50}$$

(3) We obtain from (1) and (2) that

$$\max_{Z \in \mathcal{Z}_\alpha} E[Gu(Z, .)] = \inf_{H \in \bar{\mathcal{H}}} E[Gu(\tilde{Z}_{(G|H)}, .)] \tag{2.51}$$

holds for any $G \in \bar{\mathcal{G}}$.

We first show that (i) implies (ii): By the definition of a worst-case model, $\tilde{G}$ solves

$$\max_{Z \in \mathcal{Z}_\alpha} E[\tilde{G}\, u(Z, .)] = \min_{G \in \mathcal{G}} \max_{Z \in \mathcal{Z}_\alpha} E[G\, u(Z, .)].$$

We now apply equation (2.51) to both sides and use the fact that $\tilde{H}$ is a worst-case pricing rule for $\tilde{G}$ to obtain

$$E[\tilde{G}\, u(\tilde{Z}_{(\tilde{G}|\tilde{H})}, .)] \quad = \quad \min_{G \in \mathcal{G}} \min_{H \in \mathcal{H}} E[G\, u(\tilde{Z}_{(G|H)}, .)]$$

i.e., $(\tilde{G}, \tilde{H})$ is a least favorable pair.

Item (ii) is equivalent to (iii) due to equation (2.50) and Lemma 2.26 (2).

We now show that (iii) implies (i). Consider a triple $(\tilde{G}, \tilde{H}, \tilde{k})$ that solves the dual problem. Equation (2.49) implies that $\tilde{G}$ is a worst-case measure.

Since $(\tilde{G}, \tilde{H}, \tilde{k})$ is a solution to the dual problem, we obtain

$$\begin{aligned} \min_{k>0} g(\tilde{G}, \tilde{H}, k) &= \inf_{H \in \bar{\mathcal{H}}} \min_{k>0} \min_{G \in \bar{\mathcal{G}}} g(G, H, k) \\ &= \min_{H \in \bar{\mathcal{H}}} \min_{k>0} g(\tilde{G}, H, k). \end{aligned}$$

By means of equation (2.50) we can transform this into

$$E[\tilde{G}u(\tilde{Z}_{(\tilde{G}|\tilde{H})}] = \min_{H\in\bar{\mathcal{H}}} E[\tilde{G}u(\tilde{Z}_{(\tilde{G}|H)}],$$

i.e., $\tilde{H}$ is a worst-case pricing density for $\tilde{G}$. □

2.4.2. The semi-composite problem for fixed model

The following lemma prepares the solution of the semi-composite problem presented in the following Lemma 2.29.

2.28 Lemma (Solution to the semi-composite dual problem). *Consider a model $G \in \bar{\mathcal{G}}$.*

(i) For any fixed $k > 0$, there is a pricing measure $\tilde{H} = \tilde{H}_{(k,G)} \in \bar{\mathcal{H}}^0$ that solves

$$g(G, \tilde{H}_{(k,G)}, k) = \min_{H\in\bar{\mathcal{H}}^0} g(G, H, k) =: f_G(k).$$

(ii) The function f_G is convex and continuous on the interval $[0, \infty)$.

(iii) f_G is continuously differentiable on $(0, \infty)$ with derivative

$$f'_G(k) = E[\mathbf{1}_{\{G>0\}}\tilde{H}_{(k,G)}V'(k\frac{\tilde{H}_{(k,G)}}{G}, .)] + \alpha \tag{2.52}$$

$$= -E[\mathbf{1}_{\{G>0\}}\tilde{H}_{(k,G)}I(k\frac{\tilde{H}_{(k,G)}}{G}, .)] + \alpha \tag{2.53}$$

(iv) Consider a worst-case pricing rule $\tilde{H}$ for G. The critical value $\tilde{k}_{(G,\tilde{H})}$ solves

$$f_G(\tilde{k}_{(G,\tilde{H})}) = \min_{k\geq 0} f_G(k). \tag{2.54}$$

If

$$\alpha < \sup_{H\in\bar{\mathcal{H}}^0} E[H\,\mathbf{1}_{\{G>0\}}] \tag{2.55}$$

is satisfied, then condition (2.54) is equivalent to

$$f'_G(\tilde{k}) = 0 \tag{2.56}$$

and $\tilde{k} > 0$ holds.

Proof. (i) We first prove item (i), i.e., for any fixed $k > 0$ and $G \in \bar{\mathcal{G}}$ we find a pricing measure $\tilde{H} = \tilde{H}_{(k,G)} \in \bar{\mathcal{H}}^0$ that minimizes $g(G, ., k)$. Consider a minimizing sequence (H_n). Due to Lemma 3.3 in [KS99], there exists a sequence $H'_n \in \mathrm{co}\{H_n, H_{n+1}, \ldots\}$ of convex combinations of $(H_l)_{l\geq n}$ which converges R-almost surely to a random variable H'. Hence $H' \in \bar{\mathcal{H}}^0$. We now show that H' is a minimizer for $g(G, ., k)$. By

Jensen's inequality, $g(G,.,k)$ inherits convexity from V. Hence

$$\begin{aligned}\inf_{H\in\bar{\mathcal{H}}^0} g(G,H,k) &\leq g(G,H'_n,k)\\ &\leq \sum_{l\geq n}\lambda_l g(G,H_l,k)\\ &\leq \sup_{l\geq n} g(G,H_l,k).\end{aligned}$$

On the other hand we know that

$$\lim_{n\uparrow\infty}\sup_{l\geq n} g(G,H_l,k) = \inf_{H\in\bar{\mathcal{H}}^0} g(G,H,k),$$

hence also $g(G,H'_n,k)$ must have the same limit:

$$\lim_{n\uparrow\infty} g(G,H'_n,k) = \inf_{H\in\bar{\mathcal{H}}^0} g(G,H,k). \tag{2.57}$$

By continuity of $V(.,\omega)$ for each ω, we obtain convergence of the sequence

$$V_n := V(k\frac{H'_n}{G},.) \to V(k\frac{H'}{G},.) =: V_\infty, \quad R-\text{almost surely}.$$

We can conclude from $u(0,.)\leq V_n \leq u(1,.)$ and equation (2.2) that

$$\begin{aligned}g(G,H',k) &= E[GV_\infty]+k\alpha\\ &\leq \lim_n E[GV_n]+k\alpha\\ &= \lim_n g(G,H'_n,k)\\ &= \inf_{H\in\bar{\mathcal{H}}^0} g(G,H,k),\end{aligned}$$

where the last equality is due to (2.57). Hence

$$g(G,H',k) = \min_{H\in\bar{\mathcal{H}}^0} g(G,H,k),$$

i.e., H' is a minimizer.

(ii) (1) We first show convexity of f_G on $(0,\infty)$. With the convention

$$k\bar{\mathcal{H}}^0 := \{kH \mid H\in\bar{\mathcal{H}}^0\}$$

we obtain

$$f_G(k) = \min_{J\in k\bar{\mathcal{H}}^0} E[GV(\frac{J}{G},.)]+k\alpha.$$

Consider a pair $0\leq k<k'$. Let $J=kH\in k\bar{\mathcal{H}}^0$ respectively $J'=k'H'\in k'\bar{\mathcal{H}}^0$ denote the minimizing random variables from (i). For any $\lambda\in(0,1)$, convexity of $\bar{\mathcal{H}}^0$ yields

$$\begin{aligned}\lambda J+(1-\lambda)J' &= \lambda kH+(1-\lambda)k'H'\\ &= \{\lambda k+(1-\lambda)k'\}\{\gamma H+(1-\gamma)H'\}\\ &\in \{\lambda k+(1-\lambda)k'\}\bar{\mathcal{H}}^0\end{aligned}$$

where we have defined

$$\gamma = \frac{\lambda k}{\lambda k + (1-\lambda)k'} \in (0,1).$$

Hence we can estimate

$$\begin{aligned} f_G(\lambda k + (1-\lambda)k') &= \min_{\hat{J} \in \{\lambda k + (1-\lambda)k'\}\tilde{\mathcal{H}}^0} E[GV(\frac{\hat{J}}{G}, .)] + \{\lambda k + (1-\lambda)k'\}\alpha \\ &\leq E[GV(\frac{\lambda J + (1-\lambda)J'}{G}, .)] + \{\lambda k + (1-\lambda)k'\}\alpha \\ &\leq \lambda\{E[GV(\frac{J}{G}, .)] + k\alpha\} + (1-\lambda)\{E[GV(\frac{J'}{G}, .)] + k'\alpha\} \qquad (2.58) \\ &= \lambda f_G(k) + (1-\lambda) f_G(k'), \end{aligned}$$

where we have used convexity of V. This shows that f_G is convex on $(0,\infty)$.

(ii) (2) Continuity of f_G on $(0,\infty)$ follows from (1). To prove continuity in zero, we now demonstrate

$$\lim_{k \downarrow 0} f_G(k) = E[Gu(1,.)] = f_G(0). \qquad (2.59)$$

We obtain from (2.31) the estimate

$$E[Gu(1,.)] \leq E[GV(k\frac{H}{G}, .)] + E[\mathbf{1}_{\{G>0\}} kH]$$

which implies

$$\begin{aligned} g(G,H,k) &= E[GV(k\frac{H}{G}, .)] + k\alpha \\ &\geq E[Gu(1,.)] - kE[\mathbf{1}_{\{G>0\}} H] + k\alpha \\ &\geq E[Gu(1,.)] - k \sup_{H \in \tilde{\mathcal{H}}^0} E[H] + k\alpha \\ &= E[Gu(1,.)] + k(\alpha - H_0). \end{aligned}$$

Hence

$$\begin{aligned} \lim_{k \downarrow 0} f_G(k) &= \lim_{k \downarrow 0} \inf_{H \in \tilde{\mathcal{H}}^0} g(G,H,k) \\ &\geq \lim_{k \downarrow 0} \{E[Gu(1,.)] + k(\alpha - H_0)\} \\ &= E[Gu(1,.)]. \qquad (2.60) \end{aligned}$$

On the other hand, we can use equation (2.25) and $0 \leq I \leq 1$ to obtain

$$\begin{aligned} g(G,H,k) &= E[Gu(I(k\frac{H}{G}, .), .) - I(k\frac{H}{G}, .)] + k\alpha \\ &\leq E[Gu(1,.)] + k\alpha \end{aligned}$$

i.e.,

$$\begin{aligned}\lim_{k\downarrow 0} f_G(k) &= \lim_{k\downarrow 0} \inf_{H\in\bar{\mathcal{H}}^0} g(G,H,k)\\ &\le \lim_{k\downarrow 0}\{E[Gu(1,.)]+k\alpha\}\\ &= E[Gu(1,.)]. \qquad (2.61)\end{aligned}$$

Combination of equations (2.60) and (2.61) proves (2.59).

(iii) We show validity of item (iii) in three steps.

(1) For given k, G, consider a pricing measure $\tilde{H}\in\bar{\mathcal{H}}^0$ that minimizes $g(G,.,k)$ and some $\delta>0$. We have

$$f_G(k+\delta) = \inf_{H\in\bar{\mathcal{H}}^0} g(G,H,k+\delta) \le g(G,\tilde{H},k+\delta). \qquad (2.62)$$

Convexity and continuity of V imply

$$V(y,.)-V(z,.)\le V'(y,.)(y-z), \quad z\in[0,\infty),\ y\in(0,\infty]. \qquad (2.63)$$

We can conclude from (2.62) and (2.63) the estimate

$$\begin{aligned}\frac{1}{\delta}(f_G(k+\delta)-f_G(k)) &\le \frac{1}{\delta}(g(G,\tilde{H},k+\delta)-g(G,\tilde{H},k))\\ &= \frac{1}{\delta}\left(E[GV((k+\delta)\frac{\tilde{H}}{G},.)]-E[GV(k\frac{\tilde{H}}{G},.)]\right)+\alpha\\ &\le \frac{1}{\delta}E[G\delta\frac{\tilde{H}}{G}V'((k+\delta)\frac{\tilde{H}}{G},.)]+\alpha\\ &= E[\tilde{H}\,\mathbf{1}_{\{G>0\}}V'((k+\delta)\frac{\tilde{H}}{G},.)]+\alpha\end{aligned}$$

Since $V'(k,.)$ is increasing, we can conclude that

$$\lim_{\delta\downarrow 0} E[\tilde{H}\,\mathbf{1}_{\{G>0\}}V'((k+\delta)\frac{\tilde{H}}{G},.)] = E[\tilde{H}\,\mathbf{1}_{\{G>0\}}V'(k\frac{\tilde{H}}{G},.)]$$

holds. Hence

$$\limsup_{\delta\downarrow 0}\frac{1}{\delta}(f_G(k+\delta)-f_G(k)) \le E[\tilde{H}\,\mathbf{1}_{\{G>0\}}V'(k\frac{\tilde{H}}{G},.)]+\alpha. \qquad (2.64)$$

(2) Similarly, we obtain

$$\liminf_{\delta\downarrow 0}\frac{1}{\delta}(f_G(k)-f_G(k-\delta)) \ge E[\tilde{H}\,\mathbf{1}_{\{G>0\}}V'(k\frac{\tilde{H}}{G},.)]+\alpha. \qquad (2.65)$$

(3) From the convexity of f_G we obtain existence of the one-sided derivatives

$$f_G'(k-)\le f_G'(k+)$$

We obtain from the last estimate, (2.64) and (2.65) the estimate

$$E[\tilde{H}\,\mathbf{1}_{\{G>0\}}V'(k\frac{\tilde{H}}{G},.)]+\alpha \le f_G'(k-)\le f_G'(k+)\le E[\tilde{H}\,\mathbf{1}_{\{G>0\}}V'(k\frac{\tilde{H}}{G},.)]+\alpha.$$

Hence f_G is differentiable with derivative

$$f_G'(k) = E[\tilde{H}\,\mathbf{1}_{\{G>0\}} V'(k\frac{\tilde{H}}{G}, .)] + \alpha.$$

(iv) f_G is continuous on $[0, \infty)$ and bounded from below. Hence there exists a minimizer $\tilde{k}$ for (2.54). Consider the special case $\mathcal{G}' = \{G\}$. By the definition of f_G, $\tilde{k}$ minimizes $f_G(k)$ if and only if it solves the dual problem for $\mathcal{G}'$. Due to Lemma 2.27, $\tilde{k} = \tilde{k}_{(G|\tilde{H})}$ solves the dual problem for $\mathcal{G}'$ if $\tilde{H}$ is a worst-case pricing rule for G. Hence $\tilde{k}_{(G|\tilde{H})}$ minimizes $f_G(k)$.

Consider a test $\tilde{Z}_G \in \mathcal{Z}_\alpha$ that solves the semi-composite problem $(G|\mathcal{H})$. If condition (2.55) is satisfied, we must have

$$R[\tilde{Z}_G < 1, G > 0] > 0.$$

We hence obtain from the strict concavity of u the strict inequality

$$f_G(0) = E[Gu(1, .)] > E[Gu(\tilde{Z}_G, .)].$$

We obtain from Lemma 2.24 that there is no duality gap, i.e.,

$$\begin{aligned} E[Gu(\tilde{Z}_G, .)] &= \inf_{k\geq 0} \inf_{H\in\bar{\mathcal{H}}^0} g(G, H, k) \\ &= \min_{k\geq 0} f_G(k). \end{aligned}$$

Hence the minimizing constant $\tilde{k}$ must be strictly positive under condition (2.55). □

We now apply our results on the semi-composite dual problem to the original problem:

2.29 Lemma (Solution to the semi-composite problem). *Consider a fixed model $G \in \bar{\mathcal{G}}$ such that*

$$\alpha < \sup_{H\in\bar{\mathcal{H}}^0} E[H\,\mathbf{1}_{\{G>0\}}] =: H_{0,G}$$

holds.

(i) There exists a worst-case pricing rule $\tilde{H} \in \bar{\mathcal{H}}^0$ for G and we have strong duality:

$$\max_{Z\in\mathcal{Z}_\alpha} E[G\,u(Z, .)] = \min_{(H,k)\in\bar{\mathcal{H}}^0\times[0,\infty)} g(G, H, k). \tag{2.66}$$

(ii) The test $\tilde{Z}_{(G|\tilde{H})}$ is the GR-almost surely unique solution $\tilde{Z}$ to the semi-composite problem $(G|\bar{\mathcal{H}}^0)$.

(iii) For any worst-case pricing rule $\hat{H}$, we have

$$u'(1, .) \vee \frac{\tilde{k}_{(G|\tilde{H})}}{G}\hat{H} \wedge u'(0, .) = u'(\tilde{Z}_{(G|\tilde{H})}, .) \quad R-a.s.\ on\ \ \{G > 0\}.$$

(iv) The value function

$$u_{*,G}(\alpha) = \max_{Z \in \mathcal{Z}_\alpha} E[Gu(Z,.)]$$

for the semi-composite problem is strictly concave on $(0, H_{0,G})$. *If* $u'(1,.) = 0$ *holds, then* $u_{*,G}$ *is differentiable on* $(0, H_{0,G})$ *with derivative*

$$\partial_\alpha u_{*,G}(\alpha) = \tilde{k}_{(G|\tilde{H})} \tag{2.67}$$

where $\tilde{k}_{(G|\tilde{H})}$ *denotes the critical value for* $(G|\tilde{H})$ *at the level* α .

Proof. (i) Consider a constant $\tilde{k}$ that minimizes f_G and a pricing measure $\tilde{H}$ that minimizes $g(G,.,\tilde{k})$ as in Lemma 2.28. The pair $(\tilde{H}, \tilde{k})$ then minimizes the semi-composite dual problem

$$g(G, \tilde{H}, \tilde{k}) = \min_{k \geq 0, H \in \bar{\mathcal{H}}^0} g(G, H, k).$$

Due to Proposition 2.27, $\tilde{H}$ is a worst-case pricing rule for G. Strong duality follows from Lemma 2.23.

(ii) Consider a worst-case pricing rule $\tilde{H}$ for G. We apply Proposition 2.20 to the special case $\mathcal{G} = \{G\}$ to show optimality of the test $\tilde{Z}_{(G|\tilde{H})}$. Uniqueness of the optimal solution was shown in Proposition 2.12. It remains to show that for any $H \in \bar{\mathcal{H}}^0$ the condition

$$E[H\tilde{Z}_{(G|\tilde{H})}] \leq \alpha$$

is satisfied. For this, denote $H_\epsilon := (1-\epsilon)\tilde{H} + \epsilon H \in \bar{\mathcal{H}}^0$. Due to Proposition 2.27, $\tilde{H}$ solves

$$g(G, \tilde{H}, \tilde{k}) = \min_{H \in \bar{\mathcal{H}}^0} g(G, H, \tilde{k})$$

where $\tilde{k}$ denotes the critical value $\tilde{k}_{(G|\tilde{H})}$. Hence

$$0 \leq E[GV(\tilde{k}\frac{H_\epsilon}{G},.)] - E[GV(\tilde{k}\frac{\tilde{H}}{G},.)].$$

We apply the estimate (2.63) and equation (2.26) to obtain

$$\begin{aligned} 0 &\leq E[GV'(\tilde{k}\frac{H_\epsilon}{G},.)\tilde{k}\{\frac{H_\epsilon}{G} - \frac{\tilde{H}}{G}\}] \\ &= -\epsilon\tilde{k}E[\mathbf{1}_{\{G>0\}}\{H - \tilde{H}\}I(\tilde{k}\frac{H_\epsilon}{G},.)]. \end{aligned}$$

Due to continuity of $I(,.\omega)$ for each ω, we have convergence

$$\lim_{\epsilon \downarrow 0} I(\tilde{k}\frac{H_\epsilon}{G},.)] = I(\tilde{k}\frac{\tilde{H}}{G},.), \quad R-\text{a.s. on the set } \{G > 0\}.$$

The random variables $|H - \tilde{H}|I(\tilde{k}\frac{H_\epsilon}{G},.)$ are dominated in absolute value by the R-integrable variable $H + \tilde{H}$. We can hence conclude from the dominated convergence

theorem that

$$0 \geq E[\,\mathbf{1}_{\{G>0\}}\{H-\tilde{H}\}I(\tilde{k}\frac{\tilde{H}}{G},.)]$$

holds. By means of $\tilde{k} = \tilde{k}_{(G|\tilde{H})}$ and the definition of $\tilde{Z}_{(G|\tilde{H})}$ we can hence conclude that

$$\begin{aligned} E[H\tilde{Z}_{(G|\tilde{H})}] &\leq E[\tilde{H}\tilde{Z}_{(G|\tilde{H})}] \\ &= \alpha \end{aligned}$$

holds. This proves (ii).

(iii) From (ii) we obtain

$$I(\tilde{k}\frac{\hat{H}}{G},.) = \tilde{Z}_{(G|\tilde{H})}, \quad R-\text{a.s. on the set } \{G>0\}$$

which is equivalent to the assertion made in (iii)

(iv) Strict concavity of $u_{*,G}$ follows from Lemma 2.18.

We first show strict convexity of the convex dual $v_G(k) = f_G(k) - \alpha k$ on the interval $(0, \tilde{k}_1)$ where we have set

$$\tilde{k}_1 := \inf\{k \geq 0 \,|\, E[\,\mathbf{1}_{\{G>0\}}\tilde{H}_{(k,G)}I(k\frac{\tilde{H}_{(k,G)}}{G},.)] = 0\}. \tag{2.68}$$

For this, consider $0 \leq k \leq k' < \tilde{k}_1$. We jump into the inequality (2.58) of the proof of convexity of f_G. Due to strict convexity of $V(.,\omega)$ on the interval $(u'(1,\omega), u'(0,\omega))$, this inequality is strict if

$$R[G>0, \frac{J}{G} \in (u'(1,.), u'(0,.)), \frac{J'}{G} \in (u'(1,.), u'(0,.)), J' \neq J] > 0$$

holds. Assume the last condition is violated. We then have

$$v_G(k) = v_G(k')$$

due to $u'(1,.) = 0$ and (2.24). The last equation implies $v_G'(k) = 0$ in contradiction to

$$k < \tilde{k}_1 = \inf\{k \geq 0 \,|\, v_G'(k) = 0\}$$

by means of (2.68) and (2.53). This proves strict convexity of v_G on the interval $(0, \tilde{k}_1)$. Due to equation (2.43), $u_{*,G}$ is the convex dual of v_G. The strict convexity of v_G on $(0, \tilde{k}_1)$ implies continuous differentiability of $u_{*,G}$ on the range of v_G'. Due to Lemma 2.28 (iv), this range is $(0, H_{0,G})$. By convex duality, the derivative of $u_{*,G}$ is $u_{*,G}'(\alpha) = k(\alpha)$ where $k(\alpha)$ is given by the condition $\partial_k v_G(k) = -\alpha$. This condition is equivalent to $\partial_k f_G(k) = 0$ which is again equivalent to $k(\alpha) = \tilde{k}_{(G|\tilde{H})}$ due to Lemma 2.28 (iv). This proves equation (2.67). □

2.4.3. Solution of the full problem

We now turn to the solution of the full composite problem.

2.30 Proposition. *Consider a least-favorable pair* $(\tilde{G}, \tilde{H})$. *Any maximin-optimal test* $\tilde{Z}$ *solves the simple problem* $(\tilde{G}, \tilde{H})$ *and we have*

$$\tilde{Z} = \tilde{Z}_{(\tilde{G}|\tilde{H})} \quad R - \textit{almost surely on the event } \tilde{G} > 0. \tag{2.69}$$

Proof. Since $\tilde{G}$ is a worst-case model, it follows from Corollary 2.11 that any maximin-optimal test $\tilde{Z}$ is a solution to the semi-composite problem $(\tilde{G}|\mathcal{H})$. Proposition 2.27 implies that $\tilde{Q}$ is a worst-case pricing rule for $\tilde{P}$. Due to Proposition 2.12 and Lemma 2.29, we obtain (2.69). □

2.31 Theorem. *Consider a least-favorable pair* $(\tilde{G}, \tilde{H})$ *such that* $R[\tilde{G} = \tilde{H} = 0] = 0$ *holds. Then* $\tilde{Z}_{(\tilde{G}|\tilde{H})}$ *is the* R*-almost surely unique maximin-optimal test for the composite problem* $(\mathcal{G}|\mathcal{H})$. *We have*

$$E[\tilde{G}\, u(\tilde{Z}_{(\tilde{G}|\tilde{H})}, .)] = u_* \tag{2.70}$$

$$E[\tilde{H}\, \tilde{Z}_{(\tilde{G}|\tilde{H})}] = \alpha. \tag{2.71}$$

and the pair $(\tilde{Z}, \tilde{G})$ *is a saddle-point.*

Proof. From the definition of $\tilde{Z}_{(\tilde{G},\tilde{H})}$, we obtain validity of equation (2.71) respectively $E[\tilde{H}\tilde{Z}_{(\tilde{G}|\tilde{H})}; \tilde{G} > 0] = \alpha$. By this equation and

$$\alpha \geq E[\tilde{H}\tilde{Z}] = E[\tilde{H}\tilde{Z}; \tilde{G} = 0] + E[\tilde{H}\tilde{Z}_{(\tilde{G}|\tilde{H})}; \tilde{G} > 0]$$

we can conclude that

$$R[\tilde{Z} > 0, \tilde{H} > 0, \tilde{G} = 0] = 0$$

holds. Hence we obtain from $R[\tilde{H} = \tilde{G} = 0] = 0$ that

$$\begin{aligned} R[\tilde{Z} > 0, \tilde{G} = 0] &= R[\tilde{Z} > 0, \tilde{H} > 0, \tilde{G} = 0] + R[\tilde{Z} > 0, \tilde{H} = 0, \tilde{G} = 0] \\ &= 0 \end{aligned}$$

holds. Together with equation (2.69) we conclude

$$\tilde{Z} = \tilde{Z}_{(\tilde{G}|\tilde{H})}, \quad R - \text{almost surely},$$

i.e., $\tilde{Z}_{(\tilde{G}|\tilde{H})}$ is the unique maximin-optimal test.

From Propositions 2.24 and 2.27 we obtain

$$\begin{aligned} u_* &= \inf_{(G,H,k) \in \bar{\mathcal{G}} \times \bar{\mathcal{H}} \times [0,\infty)} g(G, H, k) \\ &= \inf_{(G,H) \in \bar{\mathcal{G}} \times \bar{\mathcal{H}}} E[G\, u(\tilde{Z}_{(G,H)}, .)] \\ &= E[\tilde{G}\, u(\tilde{Z}_{(\tilde{G},\tilde{H})}, .)], \end{aligned}$$

i.e., equation (2.70) holds.

Due to Corollary 2.11, the pair $(\tilde{Z}, \tilde{G})$ is a saddle-point. $\square$

We will give an alternative proof of Theorem 2.31 that uses directly the fact that $(\tilde{G}, \tilde{H}, \tilde{k}_{(\tilde{G}|\tilde{H})})$ is a solution to the dual problem on pages 23 - 56.

If we find a least favorable pair $(\tilde{G}, \tilde{H})$ such that $\tilde{G} > 0$ holds R-almost surely, the test $\tilde{Z}_{(\tilde{G}|\tilde{H})}$ is maximin-optimal. We now consider the case where one finds a worst-case model that is not equivalent to R. In this case, the maximin-optimal claim can be approximated by a sequence of solutions $\tilde{Z}_n$ to embedded problems $\bar{\mathcal{G}}_n \subset \bar{\mathcal{G}}^1$ such that each $\bar{\mathcal{G}}_n$ contains a worst-case model $\tilde{G}_n > 0$ R-almost surely, cf. Theorem 2.32. For this, we remark that we can assume without loss of generality that

$$1 \in co_\infty(\mathcal{G}) \tag{2.72}$$

holds. This is due the following reasoning: By the Halmos-Savage theorem 3.17, we find a sequence $(G_n) \subseteq \mathcal{G}$ such that the families $\mathcal{G}$ and (G_n) are equivalent, i.e., $E[G_n; A] = 0 \forall n \in \mathbb{N}$ implies $E[G; A] = 0$ $\forall G \in \mathcal{G}$, $A \in \mathcal{F}$. We set

$$G_0 = \sum_{n=0}^{\infty} \frac{1}{2^n} G_n.$$

and introduce

$$\begin{aligned}
\frac{dR'}{dR} &= \frac{G_0}{E[G_0]} \\
\mathcal{G}' &= \left\{ G \frac{1}{G_0} \mathbf{1}_{\{G_0>0\}} \mid G \in \mathcal{G} \right\} \\
\mathcal{H}' &= \left\{ H \frac{E[G_0]}{G_0} \mathbf{1}_{\{G_0>0\}} \mid H \in \mathcal{H} \right\} \\
u' &:= E[G_0] u.
\end{aligned}$$

Then the density $G_0' := 1$ R'-almost surely is an element of $co_\infty(\mathcal{G}')$. Given a maximin-optimal test $\tilde{Z}'$ for the problem defined in terms of $\{R'; (\mathcal{G}'|\mathcal{H}'); u'; \alpha\}$, the test $\tilde{Z} := \tilde{Z}' \mathbf{1}_{\{G_0>0\}}$ is maximin-optimal for the problem $\{R; (\mathcal{G}|\mathcal{H}); u; \alpha\}$ due to

$$\sup_{H \in \mathcal{H}} E[H\tilde{Z}] = \sup_{H' \in \mathcal{H}'} E'[H'\tilde{Z}'] \leq \alpha$$

and

$$\inf_{G \in \mathcal{G}} E[Gu(\tilde{Z}, .)] = \inf_{G' \in \mathcal{G}'} E'[G'u'(\tilde{Z}', .)] = u_*.$$

In this sense, the problems $\{R; (\mathcal{G}|\mathcal{H}); u; \alpha\}$ and $\{R'; (\mathcal{G}'|\mathcal{H}'); u'; \alpha\}$ are equivalent. Thus we can assume validity of (2.72) without loss of generality in the following theorem:

2.32 Theorem. *We assume u is bounded, $\mathcal{G}$ is uniformly integrable and validity of (2.72). Then the family*

$$\bar{\mathcal{G}}_n := \left\{ (1 - \frac{1}{n}) G + \frac{1}{n} \mid G \in \bar{\mathcal{G}}^1 \right\} \tag{2.73}$$

is convex and closed in $L^1(R)$ and we have:

(i) For every n, there exists a least-favorable pair $(\tilde{G}_n, \tilde{H}_n) \in \bar{\mathcal{G}}_n \times \bar{\mathcal{H}}^0$ for the problem $(\bar{\mathcal{G}}_n|\mathcal{H})$ and the R-almost surely unique solution to this problem is given by $\tilde{Z}_n = \tilde{Z}_{(\tilde{G}_n|\tilde{H}_n)}$ of equation (2.28).

(ii) $\lim_{n\uparrow\infty} u_{*,n} = u_*$ where $u_{*,n}$ denotes the value of the problem $(\bar{\mathcal{G}}_n|\mathcal{H})$.

(iii) For the performance of $\tilde{Z}_n$ for the original problem $(\mathcal{G}|\mathcal{H})$, we have the estimate

$$u_{*,n} - \frac{c}{n-1} \leq \inf_{G\in\mathcal{G}} E[G\, u(\tilde{Z}_n, .)] \leq u_* \leq u_{*,n}$$

where we can replace the constant $c \geq 0$ by $c_n = E[u(\tilde{Z}_n, .)] - u_{*,n}$.

(iv) There exists a sequence of convex combinations $Z_n = \sum_{k\geq n} \lambda_k \tilde{Z}_k$ which converges to a test $Z^* \in \mathcal{Z}_\alpha$ R-almost surely. Any such limit Z^* is maximin-optimal for the original problem $(\mathcal{G}|\mathcal{H})$.

Proof. (0) Clearly, $\bar{\mathcal{G}}_n$ inherits convexity and closedness in $\mathcal{L}^1(R)$ from $\bar{\mathcal{G}}^1$.

(i) It follows from Proposition 2.17 and Lemma 2.29 that $\bar{\mathcal{G}}_n \times \bar{\mathcal{H}}^0$ contains a worst case model $\tilde{G}_n$ and worst-case pricing rule $\tilde{H}_n$, i.e., a least favorable pair. $\tilde{G}_n \geq \frac{1}{n}$ holds R-almost surely due to the definition of $\bar{\mathcal{G}}_n$. Theorem 2.31 implies that $\tilde{Z}_{(\tilde{G}_n|\tilde{H}_n)}$ is the R-almost surely unique maximin-optimal test for problem $(\bar{\mathcal{G}}_n|\mathcal{H})$. This proves item (i).

Item (ii) is a consequence of item (iii).

(iii) We obtain from (2.73)

$$\begin{aligned} u_{*,n} &= \inf_{G\in\bar{\mathcal{G}}_n} E[G\, u(\tilde{Z}_n, .)] \\ &= \inf_{G\in\bar{\mathcal{G}}^1} \Big\{ E[(1-\frac{1}{n}) G\, u(\tilde{Z}_n, .)] \Big\} + E[\frac{1}{n} u(\tilde{Z}_n, .)] \end{aligned}$$

or equivalently

$$\frac{n}{n-1}\Big(u_{*,n} - E[\frac{1}{n} u(\tilde{Z}_n, .)]\Big) = \inf_{G\in\bar{\mathcal{G}}^1} E[G\, u(\tilde{Z}_n, .)].$$

We can estimate

$$\begin{aligned} \frac{n}{n-1}\Big(u_{*,n} - E[\frac{1}{n} u(\tilde{Z}_n, .)]\Big) &= u_{*,n} - \frac{1}{n-1}\Big(E[u(\tilde{Z}_n, .)] - u_{*,n}\Big) \\ &\geq u_{*,n} - \frac{1}{n-1}\Big(E[u(1, .)] - u_*\Big) \\ &= u_{*,n} - \frac{c}{n-1} \end{aligned}$$

where we have set $c := E[u(1, .)] - u_* \geq 0$. Due to $\tilde{Z}_n \in \mathcal{Z}_\alpha$ and $\bar{\mathcal{G}}_n \subseteq \bar{\mathcal{G}}^1$ we have

$$\inf_{G\in\mathcal{G}} E[G\, u(\tilde{Z}_n, .)] \leq u_* \leq u_{*,n}.$$

This proves (iii).

(iv) Due to lemma 3.3 in [KS99], there exists a sequence of convex combinations $Z_n = \sum_{k\geq n} \lambda_k \tilde{Z}_k$ which converges to a limit Z^* R-almost surely. Fatous Lemma implies $Z^* \in \mathcal{Z}_\alpha$. It remains to show optimality. Since u is bounded, continuous and concave, we can estimate

$$\begin{aligned}
\inf_{G\in\mathcal{G}} E[G\,u(Z^*,.)] &= \inf_{G\in\mathcal{G}} E[G\,u(\lim_n Z_n,.)] \\
&= \inf_{G\in\mathcal{G}} \lim_n E[G\,u(Z_n,.)] \\
&\geq \inf_{G\in\mathcal{G}} \lim_n \sum_{k\geq n} \lambda_k E[G\,u(\tilde{Z}_k,.)] \\
&\geq \inf_{G\in\mathcal{G}} \lim_n \inf_{k\geq n} E[G\,u(\tilde{Z}_k,.)] \\
&\geq \limsup_n \inf_{k\geq n} \inf_{G\in\mathcal{G}} E[G\,u(\tilde{Z}_k,.)]
\end{aligned}$$

We know from item (iii) that

$$\begin{aligned}
\limsup_n \inf_{k\geq n} \inf_{G\in\mathcal{G}} E[G\,u(\tilde{Z}_k,.)] &\geq \limsup_n \inf_{k\geq n} \left\{ u_{*,k} - \frac{c}{k-1} \right\} \\
&\geq \limsup_n \inf_{k\geq n} u_{*,k} - \liminf_n \inf_{k\geq n} \frac{c}{k-1}
\end{aligned}$$

holds. Since the sequence $(u_{*,n})_{n\in\mathbb{N}}$ is convergent to u_* and due to $c \geq 0$, we have

$$\begin{aligned}
\limsup_n \inf_{k\geq n} u_{*,k} - \liminf_n \inf_{k\geq n} \frac{c}{k-1} &= u_* \\
&= \max_{Z\in\mathcal{Z}_\alpha} \inf_{G\in\mathcal{G}} E[G\,u(Z,.)].
\end{aligned}$$

Hence

$$\inf_{G\in\mathcal{G}} E[G\,u(Z^*,.)] \geq \max_{Z\in\mathcal{Z}_\alpha} \inf_{G\in\mathcal{G}} E[G\,u(Z,.)]$$

holds. This proves maximin-optimality of Z^* due to $Z^* \in \mathcal{Z}_\alpha$. □

We remark that for the sequence of worst-case models $\tilde{G}_n$ for $\bar{\mathcal{G}}_n$ of Theorem 2.32, there exists a sequence of convex combinations G'_n that converges R-almost surely and in in $L^1(R)$ to a model $G' \in \bar{\mathcal{G}}$. It follows as in Proposition 2.17 that the limit G' is a worst-case model.

We now give an ALTERNATIVE PROOF of Theorem 2.31 via convex duality in the case where

$$G\Big(u(1,.) - u(0,.)\Big) \in L^1(R), \quad G \in \mathcal{G} \tag{2.74}$$

holds. This proof is closer in spirit to the convex-duality methods applied for the semi-composite problem: We derive optimality of the worst-case model $\tilde{G}$ for the $\tilde{G}$-optimal test $\tilde{Z}_{\tilde{G}}$ directly from the fact that $\tilde{G}$ is a solution to the dual problem.

We show maximin-optimality of $\tilde{Z}_{(\tilde{G}|\tilde{H})}$ by applying Proposition 2.20. Validity of condition (2.29) follows from Lemma 2.29. It remains to demonstrate optimality of

$\tilde{G}$ for $\tilde{Z}_{\tilde{G}}$, i.e., condition (2.30). For this, we must show that

$$E[(G-\tilde{G})u(I(\tilde{k}\frac{\tilde{H}}{\tilde{G}},.),.)] \geq 0$$

holds for any $G \in \bar{\mathcal{G}}$

Now consider some $G \in \bar{\mathcal{G}}$ and let $\tilde{G}, \tilde{H}$ be given as in the theorem. We know from Proposition 2.27 that $\tilde{G}, \tilde{H}, \tilde{k} := \tilde{k}_{(\tilde{G}|\tilde{H})}$ solves the dual problem. For any $\epsilon > 0$ we have $G_\epsilon = (1-\epsilon)\tilde{G} + \epsilon G \in \bar{\mathcal{G}}$, hence due to optimality of $\tilde{G}$:

$$\begin{aligned}
(2.75) \qquad 0 &\leq E[G_\epsilon V(\tilde{k}\frac{\tilde{H}}{G_\epsilon},.) - \tilde{G}\, V(\tilde{k}\frac{\tilde{H}}{\tilde{G}},.)] \\
&= E[G_\epsilon \Big(u(I(\tilde{k}\frac{\tilde{H}}{G_\epsilon},.),.) - \tilde{k}\frac{\tilde{H}}{G_\epsilon} I(\tilde{k}\frac{\tilde{H}}{G_\epsilon},.)\Big)] \\
&\quad -E[\tilde{G}\Big(u(I(\tilde{k}\frac{\tilde{H}}{\tilde{G}},.),.) - \tilde{k}\frac{\tilde{H}}{\tilde{G}} I(\tilde{k}\frac{\tilde{H}}{\tilde{G}},.)\Big)] \\
&= E[G_\epsilon \Big(u(I(\tilde{k}\frac{\tilde{H}}{G_\epsilon},.),.) - \tilde{k}\frac{\tilde{H}}{G_\epsilon} I(\tilde{k}\frac{\tilde{H}}{G_\epsilon},.)\Big)] \\
&\quad -E[\tilde{G}\Big(u(I(\tilde{k}\frac{\tilde{H}}{\tilde{G}},.),.) - \tilde{k}\frac{\tilde{H}}{\tilde{G}} I(\tilde{k}\frac{\tilde{H}}{\tilde{G}},.)\Big)] \\
(2.76) \qquad &= \epsilon E[(G-\tilde{G})\,\mathbf{1}_{\{\tilde{G}>0\}}\, u(I(\tilde{k}\frac{\tilde{H}}{\tilde{G}},.),.)] \\
(2.77) \qquad &\quad +E[G_\epsilon \Big(u(I(\tilde{k}\frac{\tilde{H}}{G_\epsilon},.),.) - \mathbf{1}_{\{\tilde{G}>0\}} u(I(\tilde{k}\frac{\tilde{H}}{\tilde{G}},.),.)\Big)] \\
(2.78) \qquad &\quad +\tilde{k}E[\tilde{H}\Big(\mathbf{1}_{\{\tilde{G}>0\}} I(\tilde{k}\frac{\tilde{H}}{\tilde{G}},.) - \mathbf{1}_{\{G_\epsilon>0\}} I(\tilde{k}\frac{\tilde{H}}{G_\epsilon},.)\Big)] \\
&=: F_1(G_\epsilon) + F_2(G_\epsilon) + F_3(G_\epsilon)
\end{aligned}$$

where F_1, F_2 and F_3 are given in (2.76), (2.77) and (2.78). For any $\epsilon > 0$, we obtain from (2.75) the estimate

$$\begin{aligned}
E[(G-\tilde{G})\, u(\tilde{Z}_{(\tilde{G}|\tilde{H})},.)] &= E[(G-\tilde{G})\,\mathbf{1}_{\{\tilde{G}>0\}}\, u(I(\tilde{k}\frac{\tilde{H}}{\tilde{G}},.),.)] + E[G\, u(0,.)\,\mathbf{1}_{\{\tilde{G}=0\}}] \\
&= \frac{1}{\epsilon} F_1(G_\epsilon) + E[G\, u(0,.)\,\mathbf{1}_{\{\tilde{G}=0\}}] \\
(2.79) \qquad &\geq \frac{1}{\epsilon}\Big(-F_2(G_\epsilon) - F_3(G_\epsilon)\Big) + E[G\, u(0,.)\,\mathbf{1}_{\{\tilde{G}=0\}}].
\end{aligned}$$

We now demonstrate that the expression on the right side of (2.79) is nonnegative for sufficiently small ϵ. Then condition (2.30) follows from (2.79).

For ease of exposition, we assume $u'(0,.) = \infty$ and $u'(1,.) = 0$ holds, but this is not essential.

In this case, we have

$$u(y,.) - u(z,.) \geq u'(y,.)(y-z), \quad z \in [0,1],\ y \in (0,1] \tag{2.80}$$

and

$$u'(I(k,.),.) = k, \quad I(k,.) > 0 \quad \forall k \geq 0.$$

Hence

$$\begin{aligned}
-F_2(G_\epsilon) &= (1-\epsilon)E[\tilde{G}\left(u(I(\tilde{k}\frac{\tilde{H}}{\tilde{G}},.),.) - u(I(\tilde{k}\frac{\tilde{H}}{G_\epsilon},.),.)\right)] \\
&\quad +\epsilon E[G\left(u(I(\tilde{k}\frac{\tilde{H}}{\tilde{G}},.),.)\,\mathbf{1}_{\{\tilde{G}>0\}} - u(I(\tilde{k}\frac{\tilde{H}}{G_\epsilon},.),.)\right)] && (2.81)\\
&\geq (1-\epsilon)E[\tilde{G}\tilde{k}\frac{\tilde{H}}{\tilde{G}}\Big(I(\tilde{k}\frac{\tilde{H}}{\tilde{G}},.) - I(\tilde{k}\frac{\tilde{H}}{G_\epsilon},.)\Big)] \\
&\quad +\epsilon F_4(G_\epsilon) \\
&= (1-\epsilon)E[\tilde{k}\tilde{H}\,\mathbf{1}_{\{\tilde{G}>0\}}\Big(I(\tilde{k}\frac{\tilde{H}}{\tilde{G}},.) - I(\tilde{k}\frac{\tilde{H}}{G_\epsilon},.)\Big)] && (2.82)\\
&\quad +\epsilon F_4(G_\epsilon) \\
&=: (1-\epsilon)F_3'(G_\epsilon) + \epsilon F_4(G_\epsilon) && (2.83)
\end{aligned}$$

where F_4 and F_3' are defined via (2.81) and (2.82). Since the difference

$$F_3'(G_\epsilon) - F_3(G_\epsilon) = \tilde{k}E[\tilde{H}\,\mathbf{1}_{\{G_\epsilon>0\}}\,\mathbf{1}_{\{\tilde{G}=0\}}I(\tilde{k}\frac{\tilde{H}}{\epsilon G},.)]$$

is nonnegative we obtain from (2.83) the estimate

$$\begin{aligned}
\frac{1}{\epsilon}\Big(-F_2(G_\epsilon) - F_3(G_\epsilon)\Big) &\geq \frac{1}{\epsilon}\Big(F_3'(G_\epsilon) - F_3(G_\epsilon) - \epsilon F_3'(G_\epsilon) + \epsilon F_4(G_\epsilon)\Big) \\
&\geq -F_3'(G_\epsilon) + F_4(G_\epsilon). && (2.84)
\end{aligned}$$

Now we can finish up by letting ϵ tend to zero:

$$\begin{aligned}
F_3'(G_\epsilon) &= E[\tilde{k}\tilde{H}\,\mathbf{1}_{\{\tilde{G}>0\}}\Big(I(\tilde{k}\frac{\tilde{H}}{\tilde{G}},.) - I(\tilde{k}\frac{\tilde{H}}{G_\epsilon},.)\Big)] \\
&\to 0 && (2.85)
\end{aligned}$$

as ϵ tends to zero by continuity of $y \mapsto I(y,.)$ and $I \leq 1$.

$$\begin{aligned}
F_4(G_\epsilon) &= E[G\left(u(I(\tilde{k}\frac{\tilde{H}}{\tilde{G}},.)\,\mathbf{1}_{\{\tilde{G}>0\}},.) - u(I(\tilde{k}\frac{\tilde{H}}{G_\epsilon},.),.)\right)] \\
&= E[G\,\mathbf{1}_{\{\tilde{G}>0\}}\left(u(I(\tilde{k}\frac{\tilde{H}}{\tilde{G}},.),.) - u(I(\tilde{k}\frac{\tilde{H}}{G_\epsilon},.),.)\right)] && (2.86)\\
&\quad -E[G\,\mathbf{1}_{\{\tilde{G}=0\}}\,u(I(\tilde{k}\frac{\tilde{H}}{\epsilon G},.),.)] && (2.87)
\end{aligned}$$

We have

$$G\,\mathbf{1}_{\{\tilde{G}>0\}}\left|u(I(\tilde{k}\frac{\tilde{H}}{\tilde{G}},.),.) - u(I(\tilde{k}\frac{\tilde{H}}{G_\epsilon},.),.)\right| \leq G\Big(u(1,.) - u(0,.)\Big).$$

Due to continuity of I and (2.74), the expression in (2.86) converges to zero as ϵ approaches the origin. By the same arguments, we can can compute the limit for the expression in (2.87) to obtain

$$F_4(G_\epsilon) \to -E[G\,\mathbf{1}_{\{\tilde{G}=0,\tilde{H}>0\}}\,u(0,.)] - E[G\,\mathbf{1}_{\{\tilde{G}=0,\tilde{H}=0\}}\,u(1,.)], \quad \epsilon \downarrow 0. \tag{2.88}$$

Combining (2.79), (2.84), (2.85) and (2.88) we arrive at

$$\begin{aligned} E[(G-\tilde{G})\,u(\tilde{Z}_{(\tilde{G}|\tilde{H})},.)] &\geq E[G\,\mathbf{1}_{\{\tilde{G}=0,\tilde{H}=0\}}\Big(u(0,.)-u(1,.)\Big)] \\ &= 0 \end{aligned}$$

This proves equation (2.30) □

2.5. Linear case

[CK00] examine the statistical testing problem

$$\left[\begin{array}{c} \inf\limits_{G\in\mathcal{G}} E[GZ] = \max\limits_{Z} \\ 0 \leq Z \leq 1, \ \sup\limits_{H\in\mathcal{H}} E[HZ] \leq \alpha \end{array}\right] \tag{2.89}$$

by means of convex duality and nonlinear optimization methods under the following assumptions:

(i) $\mathcal{G} \cap \mathcal{H} = \emptyset$,

(ii) $E[G] = E[H] = 1$, $G \in \mathcal{G}$, $H \in \mathcal{H}$.

(iii) $\mathcal{G}$ is convex and closed under R-a.s. convergence.

The authors consider the following dual problem:

$$\left[\begin{array}{ccc} g(G,H,k) := E[(G-kH)^+] + \alpha k & = & \inf\limits_{G,H,k} \\ (G,H,k) \in \bar{\mathcal{G}} \times \bar{\mathcal{H}}_\infty \times (0,\infty) & & \end{array}\right]. \tag{2.90}$$

One important difference to the case of a strict concave utility function u is that in the linear case, a solution to the simple problem (G,H) is no longer unique. The Neyman-Pearson Lemma provides a solution $\tilde{Z}_{(G|H)}$ to the simple problem (G,H) in terms of the acceptance set and the critical value. [CK00] prove existence of a solution $(\tilde{G},\tilde{H},\tilde{k})$ to the dual problem and find that every solution $\tilde{Z}$ to problem (2.89) solves the simple problem $(\tilde{G}|\tilde{H})$ with critical value $\tilde{k}$.

With the obvious definition of a least-favorable pair in the linear case, it is straightforward to prove the exact analogue of Proposition 2.27 in the linear case since there is "no duality gap". Using this Proposition we can rephrase Theorem 4.1 of [CK00] as follows:

2.33 Theorem. *Consider a least-favorable pair $\tilde{G},\tilde{H}$. Any maximin-optimal test $\tilde{Z}$ for $(\mathcal{G}|\mathcal{H})$ can be found among the optimal tests for the simple problem $(\tilde{G}|\tilde{H})$.*

This theorem links the duality result of [CK00] to the more classical results on least-favorable prior distributions in [Wit85] and [KW67]: [KW67] introduce the dual problem on the class of all prior distributions on the hypothesis and the alternative. The crucial observation is that the value of this dual problem depends only on the means of both prior-distributions and the total mass of the prior distribution on the hypothesis. If the alternative and the hypothesis are measure-convex, the mean of the (normalized) prior distribution is itself a model respectively a pricing rule. Hence one can define the dual problem directly on all models and pricing rules plus a new parameter $k \in (0, \infty)$. The parameter k accounts for the total mass of the prior distribution on the hypothesis. One then obtains problem (2.90). In our formulation, the parameter k vanishes: For given model G and pricing rule H, the optimal choice for k is the critical value for $(G|H)$, cf. Proposition 2.27.

CHAPTER 3

Robust-efficient hedging: Solutions

In this chapter, we apply the results of Chapter 2 to the problem of efficient hedging under model-uncertainty. In **Section 3.1**, we define the utility function and the families of densities $(\mathcal{G}|\mathcal{H})$ in terms of the loss function, the families of "real-world" models and martingale measures as well as the contingent claim. We pass from the family of martingale measures to its closure in L^0 and from the family of models to its closed convex hull in L^1. We then rephrase the most central results of Chapter 2 directly in terms of models P, pricing rules Q and modified claims and show how the maximin-optimal modified claim of Theorem 1.7 can be derived from a least-favorable pair $(\tilde{P}|\tilde{Q})$, cf. Theorem 3.5. We show that the maximin and minimax values β_* respectively β^* coincide, cf. Proposition 3.10. Under the additional condition that the worst-case model is equivalent to R, the robust-efficient hedging problem has a saddle point, cf. Theorem 3.11. Especially, the robust-efficient hedging strategy in this situation is given by the $\tilde{P}$-efficient hedging strategy. For fixed model P, it is in many applications easier to compute the P-efficient hedging strategy for a fixed model P directly, e.g. via dynamic programming, rather than via a worst-case pricing rule, cf. also sections 5.1 and 6. For this reason we show that a worst-case model $\tilde{P}$ can be obtained by minimizing the power of P-efficient strategies over P, i.e., by solving the problem $\min_\xi E_{\tilde{P}}[l(S_\xi)] = \sup_{P\in\mathcal{U}} \min_\xi E_P[l(S_\xi)]$, cf. Lemma 3.6.

Our results on the existence of a least-favorable pair carry over immediately from Chapter 2: We obtain existence of a worst-case pricing rule in the closure of $\mathcal{M}$ in L^0. The closed convex hull of $\mathcal{U}$ in L^1 contains a worst-case model if F is bounded and $\mathcal{U}$ is uniformly integrable, cf. Proposition 3.13. If F is not bounded, the existence of a worst-case model is no longer guaranteed. Hence we show how the maximin-optimal claim can be approximated by optimal tests associated to bounded $F_n := F \wedge n$ by taking the limit $n \uparrow \infty$, cf. Theorem 3.14.

We derive a worst-case pricing rule $\tilde{Q}$ a posteriori from the efficient strategy in **Section 3.2**. Furthermore, we establish a relationship between the attainability of the maximin-optimal modified claim and equivalence of the worst-case pricing rule to the worst-case model, cf. Corollary 3.16.

In **Section 3.3**, we consider different ways to specify a family $\mathcal{U}$. We show in Section 3.3.1 that a suitably chosen neighborhood of a given model P_0 contains a worst-case model. In Section 3.3.2, a parameterized family $\mathcal{U} = \{P_\theta \mid \theta \in \Theta\}$ is considered. In this situation, a worst-case model is a mixture $\int P_\theta \tilde{\nu}(d\theta)$ for a worst-case prior-probability distribution $\tilde{\nu}$ on Θ. Finally we consider in Section 3.3.3 a variant where

the investor assigns weights to the models P_θ, i.e., where the investor chooses a family of prior distributions.

We also examine two problems of optimal hedging under model uncertainty that do not fit in the framework of Section 1. We derive the optimal hedging strategy for an extremely risk-averse investor who intends to minimize the maximum loss in **Section 3.5**. The other extreme of a risk-seeking investor corresponds to the quantile hedging problem examined by [FL99] for a single model. Robust quantile hedging strategies are developed in **Section 3.6**.

3.1. Worst-case measures

In our setting, tests Z correspond to the ratio V/F for some modified claim V. The corresponding σ-field is given by $\mathcal{F} = \mathcal{F}_\tau$. We introduce

$$\mathcal{D} = \{\frac{dQ}{dR}|_{\mathcal{F}_\tau} \mid Q \in \mathcal{M}\}$$

and denote by $\bar{\mathcal{D}}^0$ the closure of $\mathcal{D}$ in $L^0(R)$.

We consider problem (2.3) with the definitions

$$\mathcal{G} = \{\frac{dP}{dR}|_{\mathcal{F}_\tau} \mid P \in \mathcal{U}\} \tag{3.1}$$

$$\mathcal{H} = \{FD \mid D \in \mathcal{D}\} \tag{3.2}$$

$$u(z,\omega) = -l(F(\omega)(1-z),\omega). \tag{3.3}$$

As before, let $\bar{\mathcal{G}}^1$ denote the closed convex hull of $\mathcal{G}$ in $L^1(R)$. Due to convexity of $\mathcal{D}$, the closed convex hull $\bar{\mathcal{H}}^0$ of $\mathcal{H}$ in $L^0(R)$ is given by

$$\bar{\mathcal{H}}^0 = \{FD \mid D \in \bar{\mathcal{D}}^0\}.$$

We define

$$\begin{aligned} \bar{\mathcal{M}}^0 &:= \{DR \mid D \in \bar{\mathcal{D}}^0\}, \\ \bar{\mathcal{U}}^1 &:= \{GR \mid G \in \bar{\mathcal{G}}^1\}. \end{aligned}$$

Obviously, the function u defined in (3.3) is a state dependent utility function in the sense of Definition 2.1. Observe that u is non-positive.

3.1 Proposition. *A test $\tilde{Z}$ is maximin-optimal for problem $(\mathcal{G}|\mathcal{H})$ if and only if $\tilde{V} = \tilde{Z}F$ is maximin-optimal for problem (1.16). The values of the problems are related by $\beta^* = -u_*$.*

Proof. The first assertion follows immediately from definitions (3.1)-(3.3). The equality $\beta^* = -u_*$ is a consequence of these definitions and Theorem 1.7. □

We can now apply the results of Chapter 2 to determine the maximin-optimal test $\tilde{Z}$ respectively the optimal modified claim $\tilde{V} = \tilde{Z}F$, confer especially Theorems 2.31 and 2.32. The robust-efficient strategy for F is then given by the super-hedging strategy for $\tilde{Z}F$ and the robust minimal shortfall risk is given by $\beta^* = -u_*$, cf.

Theorem 1.7. For sake of intuition, we paraphrase the optimal modified claim, worst-case pricing rules and worst-case models directly in terms of measures $P \in \bar{\mathcal{U}}^1$ and $Q \in \bar{\mathcal{M}}^0$.

We define

$$I_l(k,\omega) = \sup\{z \geq 0 \mid l'(z,\omega) \leq k\}.$$

with the convention $\sup(\emptyset) = 0$. Observe that the set on the righthand side is nonempty for any $k \geq l'(0,.)$. I_l is strictly increasing on $[l'(0,.), l'(F,.)]$ since l is strictly convex on $[0, F(\omega)]$.

Let the inverse function I of u' be defined as in (2.22), page 34.

Due to

$$\begin{aligned} u'(z,.) &= \partial_z\{c - l(F(1-z),.)\} \\ &= Fl'(F(1-z),.) \end{aligned} \tag{3.4}$$

the inverse functions for u' and l' are related by

$$I(kF(\omega),\omega) = \left(1 - \frac{I_l(k,\omega)}{F(\omega)}\right)^+. \tag{3.5}$$

For given $(P,Q) \in \bar{\mathcal{U}}^1 \times \bar{\mathcal{M}}^0$ we define $(G,H) \in \bar{\mathcal{G}}^1 \times \bar{\mathcal{H}}^0$ via

$$G = \frac{dP}{dR}|_{\mathcal{F}_\tau} \in \bar{\mathcal{G}}^1, \tag{3.6}$$

$$H = F\frac{dQ}{dR}|_{\mathcal{F}_\tau} \in \bar{\mathcal{H}}^0. \tag{3.7}$$

and the critical value $\tilde{k}_{(P|Q)}$ for $(P|Q)$ via

$$\tilde{k}_{(P|Q)} := \tilde{k}_{(G|H)} \tag{3.8}$$

where $\tilde{k}_{(G|H)}$ was defined in 2.19.

With the definition

$$\tilde{V}_{(P|Q)} = \left(F - I_l(\tilde{k}_{(P|Q)}\frac{dQ}{dP}|_{\mathcal{F}_\tau},.)\right)^+ \mathbf{1}_{\{\frac{dP}{dR}|_{\mathcal{F}_\tau}>0\}}. \tag{3.9}$$

we obtain from equations (3.5) - (3.8) and (2.28) validity of

$$\tilde{V}_{(P|Q)} = \tilde{Z}_{(G|H)}F. \tag{3.10}$$

3.2 Lemma. *The critical value $\tilde{k}_{(P|Q)}$ defined via (3.8) satisfies*

(i) $E_Q[\tilde{V}_{(P|Q)}] = \alpha$ and $\tilde{k}_{(P|Q)} > 0$ if $\alpha < E_Q[F\,\mathbf{1}_{\{\frac{dP}{dR}>0\}}]$ or

(ii) $\tilde{k}_{(P|Q)} = 0$ if $\alpha \geq E_Q[F\,\mathbf{1}_{\{\frac{dP}{dR}>0\}}]$.

These conditions determine $\tilde{k}_{(P|Q)}$ uniquely.

Proof. This follows immediately from Proposition 2.19 and (3.10). □

3.3 Lemma. *The modified claim $\tilde{V}_{(P|Q)}$ given by* (3.9) *is the P-almost surely unique solution to the problem*

$$(P|\,Q) \qquad \left[\begin{array}{c} E_P[l(F-V,.)] = \min\limits_V \\ V \in \mathcal{F}_\tau,\ \ 0 \le V,\ \ E_Q[V] \le \alpha \end{array}\right]$$

Proof. We define G and H via (3.6) respectively (3.7). From equation (3.10) we obtain $\tilde{V}_{(P|Q)} = \tilde{Z}_{(G|H)}F$. By definition, $\tilde{Z}_{(G|H)}$ is optimal for the problem $(G|H)$. Hence the assertion is a special case of Proposition 3.1 for $\mathcal{U} = \{P\}$ and $\mathcal{M} = \{Q\}$. □

We can transfer the notion of least-favorable densities $(G, H) \in \bar{\mathcal{G}}^1 \times \bar{\mathcal{H}}^0$ to the corresponding measures $P \in \bar{\mathcal{U}}^1$, $Q \in \bar{\mathcal{M}}^0$ via equations (3.6) and (3.7). For example, we say that $(\tilde{P}|\tilde{Q})$ is a least-favorable pair if $(\tilde{G}|\tilde{H})$ defined via (3.6) and (3.7) is a least-favorable pair in the sense of Definition 2.4. We then have the following

3.4 Lemma.

(i) Consider a model $P \in \bar{\mathcal{U}}^1$. $\tilde{Q} \in \bar{\mathcal{M}}^0$ is a worst-case pricing rule *for P if it solves the problem*

$$\left[\begin{array}{c} E_P[l(F-\tilde{V}_{(P|Q)},.)] = \max\limits_Q \\ Q \in \bar{\mathcal{M}}^0 \end{array}\right].$$

(ii) $\tilde{P} \in \bar{\mathcal{U}}^1$ is a worst-case model *if it solves the problem*

$$\left[\begin{array}{c} \min\limits_{V \in \mathcal{V}_\alpha} E_P[l(F-V,.)] = \max\limits_P \\ P \in \bar{\mathcal{U}}^1 \end{array}\right].$$

(iii) $(\tilde{P}, \tilde{Q}) \in \bar{\mathcal{U}}^1 \times \bar{\mathcal{M}}^0$ is a least-favourable pair *if it solves the problem*

$$\left[\begin{array}{c} E_P[l(F-\tilde{V}_{(P|Q)},.)] = \max\limits_{P,Q} \\ P \in \bar{\mathcal{U}}^1, Q \in \bar{\mathcal{M}}^0 \end{array}\right].$$

The **Proof** is immediate from the definitions made in this section. □

We remark that

$$E_P[l(F-\tilde{V}_{(P|Q)},.)] = E_P[l(I_l(\tilde{k}_{(P|Q)}\frac{dQ}{dP},.) \wedge F,.)]$$

holds.

We now state the analogue of Theorem 2.31 in the context of robust-efficient hedging:

3.5 Theorem. *Consider a least-favorable pair $\tilde{P}, \tilde{Q}$ such that*

$$R[\frac{d\tilde{P}}{dR} > 0 \text{ or } \frac{d\tilde{Q}}{dR} > 0] = 1$$

holds. Then $\tilde{V}_{(\tilde{P}|\tilde{Q})}$ *is the* R*-almost surely unique maximin-optimal modified claim and*

$$E_{\tilde{P}}[l(F-\tilde{V}_{(\tilde{P}|\tilde{Q})},.)] = \beta^* \tag{3.11}$$

$$E_{\tilde{Q}}[\tilde{V}_{(\tilde{P}|\tilde{Q})}] = \alpha \tag{3.12}$$

holds. The pair $(\tilde{V}_{(\tilde{P}|\tilde{Q})},\tilde{P})$ *is a saddle-point, i.e.,*

$$E_P[l(F-\tilde{V}_{(\tilde{P}|\tilde{Q})},.)] \leq E_{\tilde{P}}[l(F-\tilde{V}_{(\tilde{P}|\tilde{Q})},.)] \leq E_{\tilde{P}}[l(F-V,.)], \quad P\in\bar{\mathcal{U}}^1, V\in\mathcal{V}_\alpha \tag{3.13}$$

Proof. This follows immediately from Theorem 2.31. □

There are situations where it is easier to compute the efficient hedging strategy for a fixed model P directly, e.g. via dynamic programming, rather than finding a worst-case pricing rule for P. Hence we examine strategies that are efficient with respect to a fixed model P. We recall that the minimal shortfall risk for fixed model P is denoted by

$$\beta_P = \min_{\xi\in\mathcal{A}_\alpha} E_P[l(F-\alpha-\int_0^\tau \xi_s dX_s,.)].$$

cf. Definition 1.5.

3.6 Lemma. *A measure* $\tilde{P}\in\bar{\mathcal{U}}^1$ *is a worst-case model if and only if it solves*

$$\beta_{\tilde{P}} = \max_{P\in\bar{\mathcal{U}}^1}\beta_P. \tag{3.14}$$

The assertion follows immediately from Theorem 1.7 if all models are equivalent: Consider the special case $\mathcal{U}=\{P\}$. For the general case, we need the following proposition:

3.7 Proposition. *Consider a model* $P\in\bar{\mathcal{U}}^1$ *and a test* $\tilde{V}$ *that solves the semi-composite problem* $(P|\mathcal{M})$*. Then the super-hedging strategy* $\tilde{\xi}$ *for the modified claim* $\tilde{V}$ *is efficient for* P *and*

$$\begin{aligned}\beta_P &= E_P[l(F-\alpha-\int_0^\tau \tilde{\xi}_s dX_s,.)]\\ &= E_P[l(F-\tilde{V},.)]\end{aligned} \tag{3.15}$$

holds.

Proof. Observe that we cannot paraphrase this proposition as a special case of theorem 1.7 with $\mathcal{U}=\{P\}$ if P is not equivalent to R. Nevertheless, the proof proceeds exactly as in 1.7. We subsequently iterate the proof of theorem 1.7 where we replace $\sup_{P\in\mathcal{U}}$ by P.

Let $\tilde{V}$ and $\tilde{\xi}$ as in the proposition be given.

(1) It follows from Definition 1.3 and the side conditions of problem $(P|\mathcal{M})$ that

$$\sup_{Q\in\mathcal{M}} E_Q[\tilde{V}] \leq \alpha$$

Hence $\tilde{\xi}$ satisfies the side conditions in problem (1.13).

2) For optimality, consider any admissible strategy $\xi \in \mathcal{A}_\alpha$. We define V via

$$V := (\alpha + \int_0^\tau \xi_s dX_s) \wedge F \leq \alpha + \int_0^\tau \xi_s dX_s.$$

Hence V is $\mathcal{F}_\tau$-measurable and $V \in \mathcal{V}_\alpha$. This implies

$$\sup_{Q \in \mathcal{M}} E_Q[V] \leq \alpha,$$

i.e., V satisfies the side conditions of (1.16). Equation (1.6) and optimality of $\tilde{V}$ imply

$$\begin{aligned} E_P[l(F - \alpha - \int_0^\tau \xi_s dX_s, .)] &\geq E_P[l(F - V, .)] \\ &\geq E_P[l(F - \tilde{V}, .)] \\ &= E_P[l(F - \alpha - \int_0^\tau \tilde{\xi}_s dX_s, .)] \end{aligned}$$

where the last inequality is due to the super-hedging property of $\tilde{\xi}$ for $\tilde{V}$ and equation (1.6). Hence $\tilde{\xi}$ is efficient for P.

3) Considering $\xi = \tilde{\xi}$ in (2) yields

$$\beta_P = E_P[l(F - \tilde{V}, .)].$$

□

Again, the converse implication holds as well:

3.8 Corollary. *Consider an efficient strategy $\tilde{\xi}$ for the model $P \in \bar{\mathcal{U}}^1$. Then the modified claim*

$$\tilde{V} = \left(\alpha + \int_0^\tau \tilde{\xi}_s dX_s\right) \wedge F$$

is the P-almost surely unique solution dominated by F to the problem $(P|\mathcal{M})$.

Proof. Similarly to the last proof, the proof proceeds exactly as in 1.8 where we replace $\sup_P$ by P. Uniqueness of the solution follows from Proposition 2.12. □

Proof of Lemma 3.6. By Lemma 3.4, a worst-case model is given by the problem

$$\left[\begin{array}{c} \min_{V \in \mathcal{V}_\alpha} E_P[l(F - V, .)] = \min_P \\ P \in \bar{\mathcal{U}}^1 \end{array} \right].$$

Due to Proposition 3.7, we have

$$\min_{V \in \mathcal{V}_\alpha} E_P[l(F - V, .)] = \beta_P$$

which completes the proof. □

With the definition

$$\beta(\xi, P) := E_P[l(F - \alpha - \int_0^\tau \xi_s dX_s, .)],$$

Proposition 2.8 can be rephrased as follows (cf. equation (1.15) for the definition of β_*):

3.9 Proposition. *The following conditions are equivalent:*

(i) $(\tilde{\xi}, \tilde{P}) \in \mathcal{A}_\alpha \times \bar{\mathcal{U}}^0$ *is a saddle point for* $\beta(.,.)$

(ii) $\beta_* = \beta^*$ *holds,* $\tilde{\xi}$ *is robust-efficient and* $\tilde{P}$ *is a worst-case model.*

3.10 Proposition. $\beta_* = \beta^*$ *holds and there is a robust-efficient strategy* $\tilde{\xi} \in \mathcal{A}_\alpha$.

Proof. We know from Proposition 3.1 (respectively 3.7) that $\beta^* = -u_*$ (respectively $\beta_* = -u^*$) holds. We conclude from Proposition 2.10 that $\beta^* = \beta_*$ holds and the existence of a maximin-optimal modified claim $\tilde{V}$. The superhedging strategy for $\tilde{V}$ is robust-efficient due to Theorem 1.7. □

The following theorem allows us to derive a robust-efficient strategy from a worst-case model.

3.11 Theorem. *Consider a worst-case model* $\tilde{P} \in \bar{\mathcal{U}}^1$ *equivalent to* R *and an* $\tilde{P}$*-efficient strategy* $\tilde{\xi}$ *for the fixed model* $\tilde{P}$.

The strategy $\tilde{\xi}$ *is robust-efficient and the pair* $(\tilde{\xi}, \tilde{P})$ *is a saddle-point, i.e.,*

$$\beta(\tilde{\xi}, P) \leq \beta(\tilde{\xi}, \tilde{P}) \leq \beta(\xi, \tilde{P}), \quad \xi \in \mathcal{A}_\alpha, \quad P \in \bar{\mathcal{U}}^1. \tag{3.16}$$

Proof. By the definition of $\tilde{\xi}$, the second part of the saddle-point-equation holds:

$$\beta(\tilde{\xi}, \tilde{P}) \leq \beta(\xi, \tilde{P}).$$

Due to Corollary 3.8, the modified claim

$$\tilde{V} := \Big(\alpha + \int_0^\tau \tilde{\xi}_s dX_s\Big) \wedge F$$

is optimal for the semi-composite problem $(\tilde{P}|\mathcal{M})$. Since $\tilde{P}$ is a worst-case model equivalent to R, the pair $(\tilde{V}, \tilde{P})$ is a saddle point, cf. Theorem 2.13 and Corollary 2.11. Hence

$$E_{\tilde{P}}[l(F - \tilde{V}, .)] \geq E_P[l(F - \tilde{V}, .)]$$

holds. Thus we have

$$\begin{aligned}
\beta(\tilde{\xi}, \tilde{P}) &= E_{\tilde{P}}[l(F - \tilde{V}, .)] \\
&\geq E_P[l(F - \tilde{V}, .)] \\
&= E_P[l(F - \alpha - \int_0^\tau \tilde{\xi}_s dX_s, .)] \\
&= \beta(\tilde{\xi}, P)
\end{aligned}$$

which proves equation (3.16).

Equation (3.16) implies that $\tilde{\xi}$ is robust-efficient. □

3.12 Proposition (Minimal Risk for fixed model). *For a fixed model $P \in \bar{\mathcal{U}}^1$ consider a worst-case pricing rule $\tilde{Q}$.*

(i) The minimal risk under the model P is given by

$$\beta_P = E_P[l(I_l(\tilde{k}_{(P|\tilde{Q})}\frac{d\tilde{Q}}{dP}, .) \wedge F, .)]. \tag{3.17}$$

(ii) The value function $\alpha \mapsto \beta_P(\alpha)$ is decreasing and strictly convex on $(0, F_{0,P})$ where we have set

$$F_{0,P} = \sup_{Q \in \mathcal{M}} E_Q[F; \{\frac{dP}{dR}|_{\mathcal{F}_\tau} > 0\}].$$

The derivative with respect to the capital constraint α is

$$\partial_\alpha \beta_P(\alpha) = -\tilde{k}_{(P|\tilde{Q})}. \tag{3.18}$$

Proof. Item (i) is a consequence of equation (3.9) and Proposition 3.7.

(ii) Due to Lemma 2.29 (iv), we have strict convexity of $\beta_P(.)$ on $(0, F_{0,P})$ and

$$\partial_\alpha u_{*,P}(\alpha) = \tilde{k}_{(P|\tilde{Q})}.$$

We know from Theorem 1.7 that

$$\partial_\alpha \beta_P(\alpha) = -\partial_\alpha u_{*,P}(\alpha)$$

holds. □

For to the existence of a worst-case model, we refer to Propositions 2.15, 2.17 and Theorem 2.32. As an example, we rephrase Proposition 2.17 for the purpose of robust-efficient hedging:

3.13 Proposition. *Assume that $\mathcal{G}$ is uniformly integrable and that $l(F, .)$ is bounded. Then $\bar{\mathcal{G}}^1$ contains a worst-case model.*

Proof. This follows immediately from Proposition 2.17. □

For the applicability of Theorem 2.32, it remains to relax the assumption of boundedness of u which is equivalent to boundedness of $l(F, .)$. Due to Definition 1.1, $l(F, .)$ is bounded if F is bounded. If F is not bounded, the existence of a worst-case model is no longer guaranteed. Hence we now show how the maximin-optimal claim $\tilde{V}$ and the value β^* can be obtained from the testing problems associated to bounded $F_n := F \wedge n$ by taking the limit $n \uparrow \infty$. We proof a more general theorem:

3.14 Theorem. *Consider a sequence of contingent claims $F_n \leq F$ with*

$$\lim_{n \uparrow \infty} F_n = F \quad R - \textit{almost-surely}.$$

Let $\tilde{V}_n$ denote the maximin-optimal modified claim for F_n and β_n^* the associated value, i.e.,

$$\beta_n^* \quad = \quad \sup_{P \in \mathcal{U}} E_P[l(F_n - \tilde{V}_n, .)].$$

The following statements hold:

(i) $\lim_{n \uparrow \infty} \beta_n^* = \beta^*$.

(ii) There exists a sequence of convex combinations $V_n = \sum_{k \geq n} \lambda_k \tilde{V}_k$ which converges to a random variable $\tilde{V}$ R-almost surely. Any such limit $\tilde{V}$ is maximin-optimal for the original problem (1.16).

(iii) Consider the case where the payoff of the contingent claim F_n occurs at time τ_n for an increasing series of stopping times with limit $\tau_n \to \tau$. Let $\tilde{\xi}^n$ denote a robust-efficient strategy for (F_n, τ_n) and set $\tilde{\xi}_t^n = \tilde{\xi}_{\tau_n}^n$ for $t \geq \tau_n$. Assume the limit

$$\lim_{n \uparrow \infty} (\tilde{\xi}_s^n)_{0 \leq s \leq T} = (\tilde{\xi}_s)_{0 \leq s \leq T}$$

exists R-almost surely and $|\tilde{\xi}^n| \leq \zeta$ for some integrable process ζ. Then $(\tilde{\xi}_s)$ is robust-efficient for the claim F.

Proof. We may assume without loss of generality that

$$\tilde{V}_n \leq F_n$$

holds. We introduce the sequence

$$Y_n := F_n - \tilde{V}_n.$$

Applying lemma 3.3 in [KS99], there exists a sequence of convex combinations (Y_n') which converges R-almost surely to a random variable Y', i.e.,

$$Y_n' = \sum_{k \geq n} \lambda_k Y_k \to Y', \quad n \uparrow \infty \quad R - \text{almost surely}.$$

We can conclude from equation (1.4) and $0 \leq Y_n' \leq F$ that Y' takes values in $[0, \infty)$ only. The sequence $\sum_{k \geq n} \lambda_k F_k$ converges to F R-almost surely. Hence

$$\sum_{k \geq n} \lambda_k \tilde{V}_k = \sum_{k \geq n} \lambda_k F_k - Y_n'$$

converges R almost surely. The limit

$$\tilde{V} := \lim_{n \uparrow \infty} \sum_{k \geq n} \lambda_k \tilde{V}_k$$

is nonnegative and bounded by F. $\tilde{V}_k \in \mathcal{V}_\alpha$ and $\tilde{V}_k \geq 0$ yields

$$\tilde{V} \in \mathcal{V}_\alpha. \tag{3.19}$$

By construction, we have

$$\lim_{n\uparrow\infty} Y'_n = F - \tilde{V}, \quad R- \text{ almost surely.} \tag{3.20}$$

From (3.19) and (3.20) we can conclude

$$\begin{aligned}
\beta^* &= \min_{V\in\mathcal{V}_\alpha} \sup_{P\in\mathcal{U}} E_P[l(F-V,.)] \\
&\leq \sup_{P\in\mathcal{U}} E_P[l(F-\tilde{V},.)] \\
&= \sup_{P\in\mathcal{U}} E_P[l(\lim_n Y'_n,.)] \\
&\leq \sup_{P\in\mathcal{U}} \limsup_n E_P[l(Y'_n,.)] \\
&\leq \sup_{P\in\mathcal{U}} \limsup_n \sum_{k\geq n} \lambda_k E_P[l(Y_k,.)] \\
&\leq \sup_{P\in\mathcal{U}} \limsup_n \sup_{k\geq n} E_P[l(Y_k,.)] \\
&\leq \limsup_n \sup_{P\in\mathcal{U}} \sup_{k\geq n} E_P[l(Y_k,.)] \\
&= \limsup_n \sup_{k\geq n} \sup_{P\in\mathcal{U}} E_P[l(F-\tilde{V}_k,.)] \\
&= \limsup_n \sup_{k\geq n} \beta^*_k.
\end{aligned} \tag{3.21}$$

On the other hand we obtain from $F_n \leq F$ the estimate

$$\beta^*_n \leq \min_{V\in\mathcal{V}_\alpha} \sup_{P\in\mathcal{U}} E_P[l((F-V,.)] = \beta^*.$$

Hence in the above chain of estimates we have equality everywhere. This proves item (i). Equation (3.21) and (3.19) yield optimality of $\tilde{V}$ for the original problem, i.e., assertion (ii).

(iii): We know from corollary (1.8) that the modified claim

$$\tilde{V}_n = \alpha + \int_0^{\tau_n} \tilde{\xi}^n_s dX_s$$

is maximin-optimal for F_n. By dominated convergence for stochastic integrals, the sequence $(\tilde{V}_n)$ converges R-almost surely to

$$\tilde{V} = \alpha + \int_0^{\tau} \tilde{\xi}_s dX_s,$$

cf. [Pro90] Theorem IV.2.32. Due to (ii), the claim $\tilde{V}$ is maximin-optimal for F. By definition, $\tilde{\xi}$ is a super-hedging strategy for $\tilde{V}$. We can conclude from Theorem 1.7 that $\tilde{\xi}$ is robust-efficient. □

3.2. Singular part of worst-case pricing rules

In this short section, we derive a worst-case pricing rule $\tilde{Q}$ from the efficient strategy, cf. Proposition 3.15. Furthermore, we establish a relationship between the attainability of the maximin-optimal modified claim and equivalence of the worst-case pricing rule to the worst-case model, cf. Corollary 3.16.

3.15 Proposition. *Any robust-efficient strategy $\tilde{\xi}$, least-favorable pair $(\tilde{P}|\tilde{Q})$ and $\tilde{V}$ the maximin-optimal claim are related as follows:*

(i) $(\alpha + \int_0^\tau \tilde{\xi}_s dX_s) \wedge F = \tilde{V} \quad \tilde{P} -$ *almost-surely.*

(ii) $l'(F - \tilde{V}, .) = l'(0, .) \vee \tilde{k}_{(\tilde{P}|\tilde{Q})} \frac{d\tilde{Q}}{d\tilde{P}}|_{\mathcal{F}_\tau} \wedge l'(F, .) \quad \tilde{P}$*-almost surely.*

(iii) $\left\{\frac{d\tilde{Q}}{d\tilde{P}}|_{\mathcal{F}_\tau} = 0, F > 0\right\} = \left\{\tilde{V} = F, F > 0\right\} \quad \tilde{P} -$ *almost-surely if* $l'(0, .) = 0$ *holds.*

Proof. (i) Due to Corollary 2.11, the maximin-optimal claim $\tilde{V}$ is optimal for the semi-composite problem for $\tilde{P}$. Due to Corollary 1.8 and Corollary 2.11, the modified claim

$$V := (\alpha + \int_0^\tau \tilde{\xi}_s dX_s) \wedge F.$$

is optimal for the semi-composite problem for $\tilde{P}$. Proposition 2.12 implies $V = \tilde{V}$ $\tilde{P}$-almost surely, i.e., item (i).

(ii) Follows from 2.29 (iii) and equations (3.4)-(3.10).

(iii) If $l'(0, .) = 0$ holds, we can conclude from (ii) that

$$\left\{\tilde{V} = F\right\} = \left\{\frac{d\tilde{Q}}{d\tilde{P}}|_{\mathcal{F}_\tau} = 0 \text{ or } F = 0\right\}$$

holds $\tilde{P}$-almost surely. □

3.16 Corollary. *Again, we consider a robust-efficient strategy $\tilde{\xi}$, a least-favorable pair $(\tilde{P}|\tilde{Q})$ and $\tilde{V}$ the maximin-optimal claim. We assume that $l'(0, .) = 0$ holds.*

(i) $\tilde{P}\Big[\alpha + \int_0^\tau \tilde{\xi}_s dX_s \geq F, F > 0\Big] > 0$ *implies* $\tilde{P}\Big[\frac{d\tilde{Q}}{d\tilde{P}} = 0\Big] > 0.$

(ii) If $F > 0$ and $\frac{d\tilde{Q}}{d\tilde{P}} > 0$ holds $\tilde{P}$-almost surely, the modified claim $\tilde{V}$ is attainable under $\tilde{P}$.

Proof. (i) We can estimate with the help of Proposition 3.15 (ii) and (i) the probability $\tilde{P}\Big[\frac{d\tilde{Q}}{d\tilde{P}} = 0\Big]$ as follows:

$$\begin{aligned}
\tilde{P}\Big[\frac{d\tilde{Q}}{d\tilde{P}} = 0\Big] &\geq \tilde{P}\Big[\frac{d\tilde{Q}}{d\tilde{P}}|_{\mathcal{F}_\tau} = 0, F > 0\Big] \\
&= \tilde{P}\Big[\tilde{V} = F, F > 0\Big] \\
&= \tilde{P}\Big[\alpha + \int_0^\tau \tilde{\xi}_s dX_s \geq F, F > 0\Big] \\
&> 0
\end{aligned}$$

(ii) By definition, we have

$$\alpha + \int_0^\tau \tilde{\xi}_s dX_s \geq \tilde{V} \quad R - \text{almost-surely.}$$

In order to show attainability of $\tilde{V}$, it remains to show that

$$\tilde{P}\Big[\alpha + \int_0^\tau \tilde{\xi}_s dX_s > \tilde{V}\Big] = 0$$

holds. By means of Proposition 3.15 (i) and (ii), we can conclude

$$\begin{aligned}
\tilde{P}\Big[\alpha + \int_0^\tau \tilde{\xi}_s dX_s > \tilde{V}\Big] &= \tilde{P}\Big[\alpha + \int_0^\tau \tilde{\xi}_s dX_s > \tilde{V}, F > 0\Big] \\
&= \tilde{P}\Big[\tilde{V} = F, F > 0\Big] \\
&= \tilde{P}\Big[\frac{d\tilde{Q}}{d\tilde{P}}|_{\mathcal{F}_\tau} = 0, F > 0\Big] \\
&= 0.
\end{aligned}$$

□

3.3. Specification of the family of models

We subsequently consider different approaches to specify a family of models $\mathcal{U}$ and examine sufficient conditions under which we can establish existence of a worst-case model $\tilde{P}$. In Section 3.3.1 we adapt concepts of robust statistics to define L^p-neighborhoods of a given model P_0. In Section 3.3.2, a parameterized family $\mathcal{U} = \{P_\theta \mid \theta \in \Theta\}$ is considered. In this situation, a worst-case model is a mixture $\int P_\theta \tilde{\nu}(d\theta)$ for a worst-case prior-probability distribution $\tilde{\nu}$ on Θ. Finally we consider in Section 3.3.3 a variant where the investor assigns weights to the models P_θ, i.e., where the investor chooses a family of prior distributions.

For the existence of an equivalent reference measure R, we first recall the Halmos-Savage Theorem (cf. e.g. [Leh86]):

3.17 Theorem (Halmos Savage). *A family $\mathcal{U}$ of probability measures is dominated if and only if $\mathcal{U}$ has a countable equivalent subset, i.e., there is a sequence $(P_n) \subseteq \mathcal{U}$ such that $P_n[A] = 0$ for all $n \in \mathbb{N}$ implies $P[A] = 0$ for all $P \in \mathcal{U}$.*

We introduce the notations

$$\begin{aligned} m_1(\Omega) &= \{ \text{ all probability measures } P \text{ on } (\Omega, \mathcal{F})\} \\ m_{1,R}(\Omega) &= \{ \text{ all probability measures } P \text{ on } (\Omega, \mathcal{F}) \text{ absolutely continuous w.r.t. } R\} \\ m'_{1,R}(\Omega) &= \{\frac{dP}{dR} \mid P \in m_{1,R}(\Omega)\} \\ &= \{G \in L^1(R, \Omega, \mathcal{F}) \mid G \geq 0,\ E[G] = 1\} \end{aligned}$$

with the convention $m_1 = m_1(\Omega)$ if there is no confusion about the underlying probability space Ω.

The total-variation distance d of two probability measures P, P' on $(\Omega, \mathcal{F})$ is given by

$$\begin{aligned} d(P, P') &= 2 \sup_{A \in \mathcal{F}} |P[A] - P'[A]| \\ &= \int |\frac{dP}{d\mu} - \frac{dP'}{d\mu}| d\mu \end{aligned}$$

where μ is a dominating measure, e.g., $\mu = P + P'$.

3.3.1. Neighborhood of a given model

For a given model P_0 we consider neighborhoods of P_0 of the form

$$\mathcal{U}_p(P_0, R, \epsilon) := \{ P \in m_{1,R} \mid ||\frac{dP}{dR} - \frac{dP_0}{dR}||_{L^p(R)} \leq 2\epsilon\}$$

for a reference model R dominating P_0 and $p \in [1, \infty]$ - see below for a motivation of these neighborhoods. For $p > 1$, we can establish existence of a worst-case model:

3.18 Theorem. *Consider a model P_0 and a reference model R such that $dP_0/dR \in L^p(R)$ holds for some $p > 1$. If $l(F, .)$ is bounded, the convex family $\mathcal{U}_p(P_0, R, \epsilon)$ contains a worst-case model.*

Proof. Obviously, R is equivalent to $\mathcal{U}_p(P_0, R, \epsilon)$. We define

$$G_0 := \frac{dP_0}{dR}$$

and

$$\mathcal{G} = \{\frac{dP}{dR} \mid P \in \mathcal{U}_p(P_0, R, \epsilon)\}.$$

With q such that $1/p + 1/q = 1$, we obtain from Hölders inequality

$$\begin{aligned} E[G;A] &\leq E[|G-G_0|;A] + E[|G_0|;A] \\ &\leq ||G-G_0||_{L^p(R)} R[A]^{\frac{1}{q}} + ||G_0||_{L^p(R)} R[A]^{\frac{1}{q}} \\ &\leq R[A]^{\frac{1}{q}} \Big(2\epsilon + ||G_0||_{L^p(R)}\Big) \end{aligned}$$

for any $G \in \mathcal{G}$ and $A \in \mathcal{F}$. Hence condition (2.15) holds, i.e., $\mathcal{G}$ is uniformly integrable. By Fatou's Lemma, $\mathcal{G}$ is closed in $L^1(R)$. Proposition 3.13 implies existence of a worst-case model in $\bar{\mathcal{G}}^1 = \mathcal{G}$. □

The definition of $\mathcal{U}_p(P_0, R, \epsilon)$ is motivated as follows: Given a simple testing problem $(P_0|Q_0)$, a typical approach of robust statistics is to consider total variation neighborhoods $\mathcal{U}$ respectively $\mathcal{M}$ of P_0 respectively Q_0:

$$\mathcal{U}(P_0, \epsilon) = \{P \in m_1 \mid |P_0[A] - P[A]| \leq \epsilon\} \tag{3.22}$$

where m_1 denotes the class of all probability measures on $(\Omega, \mathcal{F})$. For the problem of efficient hedging, we now face the problem that we cannot choose $\mathcal{M}$ in a fashion similar to (3.22): Instead, $\mathcal{M}$ must consist of all pricing rules Q such that the superhedge price in the family $\mathcal{U}(P_0, \epsilon)$ is given by the supremum of all prices $E_Q[F]$, $Q \in \mathcal{M}$, cf. Lemma 1.2. If the family $\mathcal{U}$ is not dominated, it is not clear how this family $\mathcal{M}$ should be constructed. However, if we replace m_1 in (3.22) by the class $m_{1,R}$ of all probability measures absolutely continuous with respect to a given reference measure R, it is sufficient to consider $\mathcal{M}$ the set of all measures Q equivalent to R such that X is a martingale with respect to Q, cf. Lemma 1.2. In order for this "dominated" approach to be reasonable, we have to make sure that the dominating model R assigns nonzero-probability to all events of interest. For example, if P_0 is a Black-Scholes model with constant volatility, one might consider a model R with stochastic volatility and jumps in the asset price such that the event that no jump occurs and volatility is constant has positive probability under R. Then the family $\mathcal{U}(P_0, R, \epsilon)$ of (3.22) where we replace m_1 by $m_{1,R}$ contains any model P with jumps and stochastic volatility that is "close to P_0". It is easily seen that

$$\mathcal{U}_p(P_0, R, \epsilon) \subseteq \mathcal{U}_1(P_0, R, \epsilon) = \mathcal{U}(P_0, \epsilon) \cap m_{1,R}$$

holds.

The $\tilde{P}$-efficient strategy is robust-efficient if the worst-case model $\tilde{P}$ is equivalent to R. If the worst-case model $\tilde{P}$ is not equivalent to R, we can apply Theorem 2.32 to determine the robust-efficient strategy. For this, observe that if $R \notin \mathcal{U}_p(P_0, R, \epsilon)$ holds, we can choose some reference model R' equivalent to R such that $R' \in \mathcal{U}_p(P_0, R, \epsilon)$ holds. Hence the family $\mathcal{G}$ of all model-densities dP/dR satisfies the assumptions on the family $\mathcal{G}$ of Theorem 2.32, i.e., it is uniformly integrable and we can assume that $1 \in \mathcal{G}$ holds.

3.3.2. Parameterized family

Given a parameterized family of models

$$\mathcal{U} = \{P_\theta \mid \theta \in \Theta\}, \tag{3.23}$$

we show that under some appropriate conditions, the class

$$\bar{\mathcal{U}} = \left\{ P_\nu \mid P_\nu[A] = \int_\Theta P_\theta[A]\nu(d\theta),\ \nu \in m_1(\Theta) \right\} \tag{3.24}$$

is proper and contains a worst-case model $P_{\tilde{\nu}}$. These conditions are formulated in terms of the densities

$$G_\theta := \frac{dP_\theta}{dR}, \quad \theta \in \Theta$$

for a dominating measure R: Essentially, we have to establish product-measurability of the mapping $(\omega, \theta) \mapsto G_\theta(\omega)$ as well as uniform-integrability of the family $\mathcal{G} = \{G_\theta \mid \theta \in \Theta\}$, cf. Theorems 3.19 and 3.20.

For this reason, we also review a sufficient condition for the existence of a dominating measure R when $\mathcal{U}$ is of the form (3.23), cf. Theorem 3.21. As a Corollary, we realize that a family of Black-Scholes models parameterized via an interval of volatilities is not continuously parameterizable, cf. Corollary 3.22.

3.19 Theorem. *Consider a parameterized family of probability densities $\mathcal{G} = \{G_\theta \mid \theta \in \Theta\} \subset m'_{1,R}(\Omega)$ for a compact separable metric space Θ such that the mapping $\theta \mapsto G_\theta(\omega)$ is continuous for every $\omega \in \Omega$. Let $\mathcal{B}_\Theta$ denote the Borel-sigma field over Θ. The following statements hold:*

(i) The mapping $(\omega, \theta) \mapsto G_\theta(\omega)$ is measurable with respect to the product-sigma field $\mathcal{F} \times \mathcal{B}_\Theta$.

(ii) $G_\nu(\omega) := \int_\Theta G_\theta(\omega)\nu(d\theta)$ defines a probability density $G_\nu \in m'_{1,R}(\Omega)$ for every $\nu \in m_1(\Theta)$.

(iii) $\bar{\mathcal{G}} := \{G_\nu = \int_\Theta G_\theta \nu(d\theta) \mid \nu \in m_1(\Theta)\}$ is a proper alternative in the sense of Definition 2.3. $\bar{\mathcal{G}}$ is closed in $L^0(R)$.

Proof. Item (i) follows from Theorem 1.149 of [Wit85].

(ii) is a consequence of (i) and Fubini's Lemma.

(iii) In order to show that $\bar{\mathcal{G}}$ is proper, we have to demonstrate that $E[G_\nu] \leq 1$ and

$$E[G_\nu\, l(F - V, .)] \leq \sup_{\theta \in \Theta} E[G_\theta l(F - V, .)], \quad V \in \mathcal{V}_\alpha$$

holds for any $\nu \in m_1'(\Theta)$. Now consider some $\nu \in m_1'(\Theta)$. Item (ii) implies $E[G_\nu] = 1$. For $V \in \mathcal{V}_\alpha$, we obtain via Fubini's Lemma:

$$\begin{aligned} E[G_\nu\, l(F-V,.)] &= \int_\Omega R(d\omega) \int_\Theta \nu(d\theta) G_\theta(\omega)\, l(F(\omega - V(\omega), \omega) \\ &= \int_\Theta \nu(d\theta) \int_\Omega R(d\omega) G_\theta(\omega)\, l(F(\omega - V(\omega), \omega) \\ &\leq \sup_{\theta\in\Theta} \int_\Omega R(d\omega) G_\theta(\omega)\, l(F(\omega - V(\omega), \omega) \\ &= \sup_{\theta\in\Theta} E[G_\theta l(F-V,.)]. \end{aligned}$$

Hence $\bar{\mathcal{G}}$ is proper.

Consider a sequence $(G_{\nu_n})_{n\in\mathbb{N}} \subset \bar{\mathcal{G}}$ converging to G in $L^0(R)$. Since Θ is compact, there is a subsequence ν_{n_k} that converges weakly to $\nu \in m_1(\Theta)$. Continuity of $\theta \mapsto G_\theta(\omega)$ implies

$$\begin{aligned} \lim_k G_{\nu_{n_k}}(\omega) &= \lim_k \int_\Theta G_\theta(\omega) \nu_{n_k}(d\theta) \\ &= \int_\Theta G_\theta(\omega) \nu(d\theta) \\ &= G_\nu(\omega) \end{aligned}$$

Hence $G = G_\nu \in \bar{\mathcal{G}}$, i.e., $\bar{\mathcal{G}}$ is closed in $L^0(R)$. □

Given $\nu \in m_1(\Theta)$, let $\tilde{\xi}^\nu$ denote the efficient strategy for the fixed model P_ν. Clearly, the minimal risk β_ν in the model P_ν is given by

$$\beta_\nu = \int_\Theta E_\theta[l(F - \alpha - \int_0^\tau \tilde{\xi}_s^\nu dX_s ds, .)] \nu(d\theta).$$

3.20 Theorem. *In the situation of Theorem 3.19, assume $l(F,.)$ is bounded and*

$$E[\sup_{\theta\in\Theta} G_\theta] < \infty \tag{3.25}$$

holds. Then there is a solution $\tilde{\nu}$ to

$$\beta_{\tilde{\nu}} = \max_{\nu\in m_1(\Theta)} \beta_\nu \tag{3.26}$$

and $P_{\tilde{\nu}}$ is a worst-case model for $\mathcal{U}$. If, in addition, all models P_θ, $\theta \in \Theta$ are equivalent, then the efficient strategy $\tilde{\xi}^{\tilde{\nu}}$ for $P_{\tilde{\nu}}$ is robust-efficient for $\mathcal{U}$.

Proof. By definition, a worst-case model is given by condition (3.26). Due to Theorem 3.19 and condition (3.25), $\bar{\mathcal{G}}$ is uniformly integrable, convex and closed in $L^1(R)$. Hence existence of a worst-case model follows from Proposition 3.13.

Due to Theorem 3.11, the efficient strategy for the worst-case model $P_{\tilde{\nu}}$ is robust-efficient if all models P_θ, $\theta \in \Theta$ are equivalent. □

Given a metric space Θ, we say that the parameterization (3.23) is continuous if the mapping $\theta \mapsto P_\theta$ is continuous w.r.t. the metric on Θ and the total-variation metric on $m_1(\Omega)$. We cite Corollary 1.146 of [Wit85]:

3.21 Theorem. *Let Θ be a separable metric space and $\mathcal{U} = \{P_\theta \mid \theta \in \Theta\} \subset m_1(\Omega)$ a continuously parameterized family of models. Then $\mathcal{U}$ is dominated.*

We say that a probability measure P_σ on $\mathcal{C}[0,T]$ equipped with the Borel sigma-field is a Black-Scholes model with drift m and volatility σ if the dynamics of the coordinate process (X_t) under P_σ are given by

$$\frac{dX_t}{X_t} = \sigma dW_t + m dt \tag{3.27}$$

where W is a Brownian motion under P_σ.

3.22 Corollary. *For $\sigma \in [\sigma_*, \sigma^*]$, let P_σ denote a Black-Scholes model with constant drift m and volatility σ. Then*

(i) The family $\mathcal{U} = \{P_\sigma \mid \sigma \in [\sigma_*, \sigma^*]\}$ is not dominated.

(ii) The parameterization $\sigma \mapsto P_\sigma$ is not continuous.

Proof. Since all models $P \in \mathcal{U}$ are singular, it is easily seen that $\mathcal{U}$ does not possess a countable equivalent subset. Hence Item (i) is a direct consequence of the Halmos-Savage Theorem 3.17.

Item (ii) follows from Item (i) and Theorem 3.21. □

In a situation as in Corollary 3.22 where there is no dominating reference measure, one has to find alternative ways to define admissibility and superhedging strategies simultaneously for all models. One such alternative is to use a strictly pathwise Ito-calculus as in [Foe81]. This is especially applicable in the situation of Corollary 3.22, cf. also [Foe00].

3.3.3. Bayesian measures of risk

For ease of exposition, we restrict the analysis in this Section to countably many models. For uncountably many models, it is straightforward to apply the results of the previous Section 3.3.2 to a subset of all prior-distributions.

Given a countable family of models $\mathcal{U}' = \{P_1, P_2, \ldots\}$, we assume that the investor does not know which model P_n is the true model, but that he has a view on the likelihood γ_n for the model P_n being the true model. In this case the rational objective for the investor is to minimize the risk associated to the prior-distribution

$\gamma = (\gamma_n)_{n\in\mathbb{N}}$, i.e., he intends to solve the problem

$$\left[\begin{array}{c} \sum_{n=1}^{\infty} \gamma_n E_n[l(F-\alpha-\int_0^\tau \xi_s dX_s,.)] = \min_\xi \\ \xi \in \mathcal{A}_{\alpha,} \end{array}\right] \tag{3.28}$$

where γ fulfills

$$0 < \gamma_n < 1, \quad \sum_{n=1}^{\infty} \gamma_n = 1.$$

Now

$$\sum_{n=1}^{\infty} \gamma_n E_{P_n}[S] =: E_\gamma[S]$$

is simply the expectation of S with respect to the mixture model $P_\gamma := \gamma_1 P_1 + \gamma_2 P_2 + \ldots$. Hence problem (3.28) is a special case of (1.13) with $\mathcal{U} = \{P_\gamma\}$, i.e., this setting corresponds to the standard problem without model-uncertainty.

The investor will in general be uncertain about the correct prior distribution, i.e., he may prefer to choose a family of prior distributions Γ. For example, Γ can be a neighborhood of γ as in Section 3.3.1. Minimizing the maximal expected shortfall for any prior distributions $\gamma \in \Gamma$ brings us back to a problem of the form (1.13) with

$$\mathcal{U} = \{\sum_{n=1}^{\infty} \gamma_n P_n \mid \gamma \in \Gamma\}. \tag{3.29}$$

In this situation, the investor can choose the family Γ such that there is a lower bound $\delta_n > 0$ for the probability of each model:

$$\gamma_n \geq \delta_n, \quad n \in \mathbb{N},\ \gamma \in \Gamma, \tag{3.30}$$

This way, the investor can be certain to have a minimal exposure δ_n to each model P_n under any mixture $\gamma \in \Gamma$. Under condition (3.30), the $\tilde{\gamma}$-efficient strategy for a worst-case model $\tilde{\gamma}$ is robust-efficient , cf. Theorem 3.11.

It is straightforward to generalize the above reasoning to a non-countable family $\mathcal{U}$ of the form (3.23). One is then lead to a family of prior distributions $\Gamma \subset m_1(\Theta)$. The right measurability properties and existence of a worst-case measure are then established as in Theorem 3.19 and Theorem 3.20.

3.4. Market frictions

So far, we have used a classical definition of admissible strategies assuming friction free markets. More general concepts take market-frictions like short-sales constraints or transaction costs into account. We subsequently sketch shortly how the robust efficient strategy in this situation can be derived from a maximin-optimal test for (1.16) where one has to replace the pricing rules $\mathcal{M}$ by a new family $\mathcal{M}^c$ reflecting the specific constraints. The main contribution of this thesis is the solution of the testing problem (1.16) independently of the structure of pricing rules, cf. Chapter 2 and Section 3.1. This allows to "solve" the problem of robust-efficient hedging under

general constraints on admissible strategies. To illustrate this point, we subsequently consider shortly the case of short-sales constraints and transaction costs.

3.4.1. Short-sales constraints

Let $\mathcal{A}_\alpha^c$ denote the class of all strategies in $\mathcal{A}_\alpha$ satisfying the short-sales constraint $\xi \geq 0 \quad R-$ almost surely.

The following analogue of the super-hedging Lemma 1.2 in the case of short-sales constraints follows from the analysis of [FK97]:

3.23 Lemma. *Consider a nonnegative contingent claim V. There exists a strategy $\xi \in \mathcal{A}_\alpha^c$ with*

$$V \leq \alpha + \int_0^T \xi_s dX_s, \quad R - \textit{almost surely}$$

if and only if

$$\sup_{Q \in \mathcal{M}^c} E_Q[V] \leq \alpha$$

holds where $\mathcal{M}^c$ is the class of all probability measures Q equivalent to R such that X is a local super-martingale with respect to Q- assuming this class is nonempty.

3.24 Definition.

(i) A *robust efficient strategy under short-sales constraints* is a solution to problem (1.13) where we replace $\mathcal{A}_\alpha$ by $\mathcal{A}_\alpha^c$.

(ii) A *maximin-optimal modified claim $\tilde{V}$ for $(\mathcal{U}|\mathcal{M}^c)$* is a solution to problem (1.16) where we replace $\mathcal{M}$ by $\mathcal{M}^c$.

3.25 Theorem. *Consider a maximin-optimal modified claim $\tilde{V}$ for $(\mathcal{U}|\mathcal{M}^c)$ and initial capital α. Then there exists a super-hedging strategy $\tilde{\xi} \in \mathcal{A}_\alpha^c$ for the claim $\tilde{V}$ and this strategy is robust-efficient under short-sales constraints.*

The **Proof** proceeds analogous to the proof of Theorem 1.7 where one replaces the use of Lemma 1.2 by Lemma 3.23. □

As in Section 3.1, we can now apply Theorem 2.31 or Theorem 2.32 to determine the maximin-optimal modified claim $\tilde{V}$ for $(\mathcal{U}|\mathcal{M}^c)$. Especially, Theorems 3.5 and 3.11 hold with the appropriate substitution of $\mathcal{M}$ by $\mathcal{M}^c$.

3.4.2. Transaction costs

Consider a class $\mathcal{U}$ such that every $P \in \mathcal{U}$ yields a diffusion-model with proportional transaction costs as in [CK96]. Here, the value-process is given by a two-dimensional process (X, Y) where X_t (resp. Y_t) denotes the amount of money held in the bank account (resp. stock) at time t. A contingent claim corresponds to a two-dimensional random-variable (C_0, C_1) where C_0 denotes the amount of cash and C_1 denotes the amount of stock that has to be delivered at the exercise time T. Assume that the

holder of the option prefers to be payed out the net-worth $F := C_0 + C_1$ in cash rather than obtaining "delivery of C_1 stock at price $-C_0$". For the problem of efficient hedging the writer of the option wants to minimize the expected weighted shortfall. We can define the shortfall by $(F - V_{T+})^+$ where V_{T+} denotes the amount of cash available to the investor after liquidating all stock at time T. For fixed $P \in \mathcal{U}$ consider the class $\mathcal{D}_P$ of auxiliary martingales defined in Section 3 of [CK96]. The authors show that

$$\mathcal{C}_\alpha(\mathcal{M}) = \{V \geq 0 \,|\, \sup_{(Z_0,Z_1)\in\mathcal{D}_P} E_P[Z_0(T)V] \leq \alpha \ \forall P \in \mathcal{U}\}$$

corresponds to the set of all net-values $V = V_{T+}$ that can be attained by an admissible strategy that starts with α in cash and without any initial holdings of stock. Hence the problem of efficient hedging can be reduced to problem (1.16) with

$$\mathcal{M} = \{Z_0(T)P \,|\, (Z_0, Z_1) \in \mathcal{D}_P, \ P \in \mathcal{U}\}.$$

This has been worked out in more detail in [Kam00] for a singleton $\mathcal{U} = \{P\}$.

3.5. Minimizing the maximum loss

In this section, we derive the optimal hedging strategy for an extremely risk-averse investor who measures risk by the maximum loss.

Consider first a moderately risk-averse investor who measures risk by the L^p-norm[1]:

$$\begin{aligned} \rho_p(S) &= \left(\sup_{P\in\mathcal{U}} E_P[(S^+)^p]\right)^{\frac{1}{p}} \\ &= \sup_{P\in\mathcal{U}} ||S^+||_{L^p(P)} \end{aligned} \tag{3.31}$$

where $\mathcal{U}$ is a class of models dominated by R as in Chapter 1. As the degree of risk aversion p increases to infinity, we obtain in the limit the maximum loss:

$$\begin{aligned} \rho_\infty(S) &:= ||(S)^+||_{L^\infty(R)} \\ &= \inf\{c \in \mathbb{R} \,|\, (S)^+ \leq c \ R\text{-almost surely}\}. \end{aligned}$$

Another way to look at the maximum loss is to fix a single model R and to define $\mathcal{U}' = \{P \,|\, P \text{ equivalent to } R\}$. Then we obtain

$$\rho_\infty(S) = \sup_{P\in\mathcal{U}'} E_P[S^+]$$

In other words, the maximum loss is a coherent measure of risk in the terminology of [ADEH99]. The subsequent analysis complements the results of [Leu99], Section 4.1.3. In a complete market setting, the author demonstrated that the optimal modified claim $\tilde{F}_p$ corresponding to ρ_p converges to $\tilde{F}_\infty$ defined below as p tends to infinity. We consider the limit $\lim_{p\uparrow\infty} \rho_p = \rho_\infty$ as a risk measure by its own right and

[1]The normalization by taking the p-th root does not affect the robust-efficient hedging strategy or worst-case models.

derive the optimal modified claim $\tilde{F}_\infty$ in a general semi-martingale setting directly from the optimality criterion

$$\left[\begin{array}{c} \rho_\infty(F - \alpha - \int_0^\tau \xi_s dX_s) = \min_\xi \\ \xi \in \mathcal{A}_\alpha \end{array}\right]. \tag{3.32}$$

For this, let $\tilde{c}$ denote a solution to

$$\sup_{Q \in \mathcal{M}} E_Q[(F - c)^+] = \alpha, \tag{3.33}$$

i.e.,

$$\tilde{F}_\infty := (F - \tilde{c})^+ \tag{3.34}$$

is a modified claim with superhedge price α which is obtained by a constant shift of F. The existence of a solution $\tilde{c}$ to (3.33) follows as in Proposition 2.19.

3.26 Proposition. *Let $\tilde{c}$ be given by equation* (3.33)*. The super-hedging strategy for the modified claim $\tilde{F}_\infty = (F - \tilde{c})^+$ minimizes the maximum loss and the minimal maximum loss is $\tilde{c}$.*

Proof. We have to show that the prescribed strategy solves problem (3.32). It follows as in Theorem 1.7 that a solution to problem (3.32) is given by a super-hedging strategy for a modified claim F' which solves the problem

$$\left[\begin{array}{c} \rho_\infty(F - F') = \min_{F'} \\ F' \in \mathcal{V}_\alpha \end{array}\right] \tag{3.35}$$

and that the values of these two problems coincide. By definition of $\tilde{F}_\infty$ we have for any $F' \in \mathcal{V}_\alpha$:

$$\begin{aligned} \rho_\infty(F - F') &= \inf\{c \in \mathbb{R} \mid (F - F')^+ \leq c \ R\text{-almost surely}\} \\ &\geq \tilde{c}. \end{aligned}$$

On the other hand, this bound is achieved by $\tilde{F}_\infty$:

$$\rho_\infty(F - \tilde{F}_\infty) = \tilde{c}.$$

Since $\tilde{F}_\infty$ satisfies the side condition $\tilde{F}_\infty \in \mathcal{V}_\alpha$, it solves problem (3.35). □

From equation (3.33) we obtain the following corollary:

3.27 Corollary. *Given a level c, the minimal initial capital $\alpha(c)$ required to establish an admissible strategy that limits the maximum loss to the level c is given by*

$$\alpha(c) = \sup_{Q \in \mathcal{M}} E_Q[(F - c)^+].$$

3.6. Robust quantile hedging

In this section, we shortly show how a robust version of the quantile hedging strategy examined by [FL99] can be found via a least-favorable pair, cf. Theorems 3.28 and 3.30. We consider the situation of an investor who intends to maximize the minimal success ratio given initial capital α:

$$\left[\begin{array}{c} \inf_P E_P[F^{-1}\Big(\alpha + \int_0^\tau \xi_s dX_s\Big);\ F>0] = \max_\xi \\ \xi \in \mathcal{A}_\alpha \end{array}\right]. \tag{3.36}$$

The corresponding testing problem is given by (2.89) where Z runs through all $\mathcal{F}_\tau$-measurable tests:

$$\left[\begin{array}{c} \inf_{G\in\mathcal{G}} E[GZ] = \max_Z \\ 0 \le Z \le 1,\ \sup_{H\in\mathcal{H}} E[HZ] \le \alpha \end{array}\right] \tag{3.37}$$

where we have set

$$\begin{aligned} \mathcal{F} := \mathcal{F}_\tau\mathcal{G} &= \{\frac{dP}{dR}|_{\mathcal{F}_\tau}\, \mathbf{1}_{\{F>0\}} \mid P \in \mathcal{U}\} \\ \mathcal{H} &= \{F\frac{dQ}{dR}|_{\mathcal{F}_\tau} \mid Q \in \mathcal{M}\}. \end{aligned}$$

It is straightforward to prove the following analogue to Theorem 1.7:

3.28 Theorem. *Given a maximin-optimal test $\tilde{Z}$ for problem (3.37), the super-hedging strategy $\tilde{\xi}$ for the modified claim $\tilde{Z}\,F$ solves (3.36). The values of the problems coincide and the success ratio associated to $\tilde{\xi}$ is given by $\tilde{Z}$.*

We can now apply the general results of Section 2.5. As above, we denote by $\bar{\mathcal{G}}^1$ the closed convex hull of $\mathcal{G}$ in $L^1(R)$ and by $\bar{\mathcal{H}}^0$ the closed convex hull of $\mathcal{H}$ in $L^0(R)$. These are proper due to Proposition 2.7, i.e., we may replace $\mathcal{G}, \mathcal{H}$ by $\bar{\mathcal{G}}^1, \bar{\mathcal{H}}^0$ in problem (3.37).

The simple problem $(G|H)$ is given by

$$\left[\begin{array}{c} E[G\,Z] = \max_Z \\ 0 \le Z \le 1,\ E[HZ] \le \alpha \end{array}\right]. \tag{3.38}$$

We define

$$\begin{aligned} \tilde{k}_{(G|H)} &:= \inf\{k \ge 0 \mid E[H\,\mathbf{1}_{\{G>kH\}}] \le \alpha\} \text{ the critical value for } (G|H) \\ \gamma_{(G|H)} &:= \begin{cases} \dfrac{\alpha - E[H\,\mathbf{1}_{\{G>\tilde{k}_{(G|H)}H\}}]}{E[H\,\mathbf{1}_{\{G=\tilde{k}_{(G|H)}H\}}]} & \text{if} \quad E[H\,\mathbf{1}_{\{G=\tilde{k}_{(G|H)}H\}}] > 0 \\ 0 & \text{else} \end{cases} \\ \tilde{Z}_{(G|H)} &:= \mathbf{1}_{\{G>\tilde{k}_{(G|H)}H\}} + \gamma_{(G|H)}\,\mathbf{1}_{\{G=\tilde{k}_{(G|H)}H\}} \end{aligned}$$

The following lemma is well known, cf. for example theorem 2.7 in [Wit85].

3.29 Lemma. *Consider* $G \in \bar{\mathcal{G}}^1$, $H \in \bar{\mathcal{H}}^0$ *and* $0 < \alpha < [H;\ G > 0]$. *Then*

(i) $\tilde{Z}_{(G|H)}$ *solves the simple problem (*3.38*).*

(ii) A test $\tilde{Z}$ *solves the simple problem* $(G|H)$ *if and only if it is of the form*
$$\tilde{Z} = \begin{cases} 1 & \text{if} \quad G > \tilde{k}_{(G|H)}\, H \\ 0 & \text{if} \quad G < \tilde{k}_{(G|H)}\, H \end{cases} \qquad R - \text{almost surely and } E[H\tilde{Z}] = \alpha.$$

Obviously, we shall say that $(\tilde{G}|\tilde{H})$ is a least-favorable pair if it solves the problem
$$\begin{bmatrix} E[G\,\tilde{Z}_{(G|H)}] = \inf\limits_{G,H} \\ G \in \bar{\mathcal{G}}^1, H \in \bar{\mathcal{H}}^0 \end{bmatrix},$$
cf. also Definition 2.4.

We now reduce the problem of finding a maximin-optimal test to finding a least-favorable pair:

3.30 Theorem. *Assume that* $\mathcal{G}$ *is uniformly integrable.*

(i) There exists a least-favorable pair $(\tilde{G}|\tilde{H}) \in \bar{\mathcal{G}}^1 \times \bar{\mathcal{H}}^0$ *and any maximin-optimal test* $\tilde{Z}$ *for (*3.37*) can be found among the optimal tests for the simple problem* $(\tilde{G}|\tilde{H})$.

(ii) If, in addition, $R[\tilde{G} = \tilde{H},\ F > 0] = 0$ *holds, then the solution to the simple problem* $(\tilde{G}|\tilde{H})$ *is* R*-almost surely unique on* $\{F > 0\}$ *and maximin-optimal.*

Proof. First observe that $\bar{\mathcal{G}}^1$ is closed in $L^0(R)$, cf. also Lemma 2.16. Hence item (i) is a direct consequence of Theorem 2.33 and Theorem 4.1 of [CK00].

Item (ii) follows from the definition of $\tilde{Z}_{(G|H)}$ and Lemma 3.29. □

Part II

Applications

Introduction

In the following case studies, we consider different situations of complete and incomplete markets with model-uncertainty:

- A binomial tree with uncertain transition probabilities, Section 4.1.
- A binomial tree with uncertain return, Section 4.2.
- A Black-Scholes model with uncertain drift, Section 4.3.
- A Black-Scholes model where volatility jumps to a new value according to some unknown distribution, Section 5.1.
- A countable family of singular models - this setup can be applied to the "uncertain volatility model", Section 5.3.
- A "geometric Poisson process" with uncertain intensities, Chapter 6.

For each family, we first determine the efficient strategy and the related minimal risk β_P for any fixed model $P \in \bar{\mathcal{U}}$ within a suitably chosen proper $\bar{\mathcal{U}}$. While this is rather immediate in the complete market setting of Chapter 4, it is a nontrivial task in the incomplete markets of chapters 5 and 6. In these situations, we apply the dynamic programming principle to derive the efficient strategy for fixed model. We then establish existence of a worst-case model $\tilde{P}$, i.e., a model that maximizes the value β_P over all $P \in \bar{\mathcal{U}}$. Finally, we apply theorem 3.5 or 3.11 to show that the $\tilde{P}$-efficient strategy is robust-efficient for $\bar{\mathcal{U}}$ respectively $\mathcal{U}$.

Section 5.3 is an interesting case study in that the above approach does not allow us to derive the robust-efficient strategy in this setting. Instead, we derive the robust-efficient strategy more directly. We then establish existence of a least-favorable pair. This yields an example where a worst-case model $\tilde{P}$ exists but the $\tilde{P}$-efficient strategy is not robust-efficient.

CHAPTER 4

Complete markets with model-uncertainty

In this chapter, we derive robust versions of efficient strategies for different families of complete models. In Section 4.1, we consider the case of a binomial model where there is uncertainty regarding the transition probabilities at each node.

In Section 4.3 we examine the case where the drift of a Black-Scholes model P_m is not known with certainty but lies within a given interval. As an application of Theorem 3.19, we find a mixture worst-case model and the robust-efficient strategy. For this, we derive the dynamics of the price process and the efficient strategy under any mixed model $P_\nu = \int_{m_*}^{m^*} P_m \nu(dm)$, cf. Theorem 4.2.

Both above studies are easily solvable since the convex hull of the given model-family (i.e., the minimal *proper alternative*) is equivalent to the original unique martingale measure. Hence one need not leave the complete market setting. Typically, this is not the case when there is uncertainty about a complete model. For example, any convex combination of two complete models that do not possess the same equivalent martingale measure is incomplete. This is illustrated in the Section 4.2 where we consider the simple setting of a one-period binomial model with uncertainty regarding the size of the return. For a similar situation of a Black-Scholes model with uncertainty regarding the volatility we refer to Section 5.3.

4.1. Binomial tree with uncertain transition-probabilities

Given constants $0 < p_- < p_+ < 1$ and $u > 1$, we define a family $\mathcal{U} = \{P_\theta\}$ of N-period Binomial-models on the space $\Omega = \{u^{-1}, u\}^N$ as follows: Let X_n denote the value of the underlying asset at period $n = 0, \ldots, N$. The set

$$\begin{aligned} \Theta &:= [p_-, p_+]^N \\ &= \{\theta = (\theta_1, \ldots, \theta_N) \mid \theta_n \in [p_-, p_+],\ n = 1, \ldots, N\} \end{aligned}$$

contains vectors of transition probabilities θ with the definition

$$P_\theta\Big[\frac{X_{n+1}}{X_n} = u\Big] = \theta_{n+1},$$

$$P_\theta\Big[\frac{X_{n+1}}{X_n} = u^{-1}\Big] = 1 - \theta_{n+1} \tag{4.1}$$

$$\mathcal{U} = \{P_\theta \mid \theta \in \Theta\}. \tag{4.2}$$

Each model P_θ is equivalent to the same unique martingale measure $\tilde{Q}$. Hence, for fixed model P_θ, the P_θ-efficient strategy and associated minimal shortfall risk β_θ can

be computed explicitly - only the critical value $\tilde{k}_\theta$ has to be computed. From β_θ, we can easily determine a worst-case model and the robust-efficient strategy:

4.1 Theorem. *There exists a parameter $\tilde{\theta}$ that solves*

$$\beta_{\tilde{\theta}} = \max_{\theta \in \Theta} \beta_\theta.$$

The associated model $P_{\tilde{\theta}}$ is a worst-case model. The perfect hedging strategy for the modified claim $\tilde{F}_{\tilde{\theta}}$ is robust-efficient and the mini-maximal shortfall risk is given by $\beta^* = \beta_{\tilde{\theta}}$.

Proof. Clearly, the concave function $\theta \mapsto \beta_\theta$ attains its maximum on the compact set Θ. The class $\mathcal{U}$ defined via (4.2) is convex. Hence $P_{\tilde{\theta}}$ is a worst-case model. The remaining assertions follows from Theorem 3.5. □

Subsequently, we provide more detailed formulas for β_θ: The unique equivalent martingale measure $\tilde{Q}$ is given by

$$\begin{aligned} \frac{d\tilde{Q}}{dP_\theta}(\omega) &= \prod_{n=1}^{N} \left[\frac{q}{\theta_n} \mathbf{1}_{\{u\}}(\omega_n) + \frac{1-q}{1-\theta_n} \mathbf{1}_{\{u^{-1}\}}(\omega_n) \right] \\ q &:= \frac{u-1}{u^2-1} \end{aligned} \tag{4.3}$$

where we use the convention $\omega = (\omega_1, \ldots, \omega_N) \in \{u^{-1}, u\}^N$.

We define

$$\begin{aligned} g_\theta(x_1, \ldots, x_N) &= \prod_{n=1}^{N} m_n \left(\frac{x_n}{x_{n-1}} - u\right) + \frac{q}{\theta_n} \\ m_n &= \left[\frac{1-q}{1-\theta_n} - \frac{q}{\theta_n} \right] (u^{-1} - u)^{-1} \end{aligned}$$

which yields

$$\frac{d\tilde{Q}}{dP_\theta} = g_\theta(X_1, \ldots, X_N). \tag{4.4}$$

Let E^* denote expectation under $\tilde{Q}$. The unique arbitrage-free premium F_0 for any claim F is

$$F_0 = E^*[F]$$

Consider some $\alpha < F_0$ and a strictly convex loss function l as in the last section. Let $\tilde{k}_\theta$ denote a solution to

$$\alpha = E^*[F - I_l(\tilde{k}_\theta g_\theta(X_1, \ldots, X_N), .)].$$

For fixed $\theta \in \Theta$, the efficient hedging strategy for F in the model θ is given by the perfect hedging strategy for

$$\tilde{F}_\theta = F - I_l(\tilde{k}_\theta g_\theta(X_1, \ldots, X_N), .)$$

and the minimal shortfall risk under θ is

$$\beta_\theta = E_\theta[l(I_l(\tilde{k}_\theta g_\theta(X_1,\dots,X_N),.),.)]. \tag{4.5}$$

For example if $l(z,.) = z^2$, then

$$\begin{aligned} \tilde{F}_\theta &= F - \frac{1}{2}\tilde{k}_\theta g_\theta(X_1,\dots,X_N) \wedge F \\ \beta_\theta &= E_\theta[\Big(\frac{1}{2}\tilde{k}_\theta g_\theta(X_1,\dots,X_N) \wedge F\Big)^2]. \end{aligned}$$

4.2. Uncertain size of return

We shortly discuss what happens if we allow for uncertainty regarding the size u_n of the return X_n/X_{n-1} in the binomial model. This situation differs considerably from the setting with uncertain probabilities. For sake of simplicity, we consider a one-step tree $N = 1$ and only four possible returns. Given two possible up-moves $1 < u_0 < u_1$, $X_0 = 1$ and some fixed p we define

$$\begin{aligned} \Omega &= \{u_0, u_0^{-1}, u_1, u_1^{-1}\} \\ &= \{\omega_1, \omega_2, \omega_3, \omega_4\} \\ X(\omega) &= \omega \\ P_i\Big[X = \omega\Big] &= p\,\delta_{\{u_i\}}(\omega) + (1-p)\delta_{\{u_i^{-1}\}}(\omega) \end{aligned}$$

P_i is a one-step binomial model that assigns probability p to u_i respectively $1-p$ to u_i^{-1}. We set

$$\mathcal{U} = \{P_0, P_1\}.$$

These two models are singular. Opposed to the setting of Section 5.3, we cannot decide at time 0 which model is the true model.

The investor has to decide which fraction γ of the available capital α he intends to invest in the stock. The amount invested in the bond is then given by $(1-\gamma)\alpha$. The value at time 1 is $\gamma\alpha X + (1-\gamma)\alpha$. Hence the robust-efficient strategy is given by the solution to

$$\left[\begin{array}{c} \max\limits_{j\in\{0,1\}} E_j[l((F - \gamma\alpha X - (1-\gamma)\alpha)^+)] = \min\limits_{\gamma} \\ \gamma \in \mathcal{A}_\alpha \end{array}\right] \tag{4.6}$$

where

$$\begin{aligned} \mathcal{A}_\alpha &= \{\gamma \in \mathbb{R} \mid \gamma\alpha X(\omega) + (1-\gamma)\alpha \geq 0 \quad \forall \omega \in \Omega\} \\ &= \left[\frac{1}{1-\alpha}, \frac{1}{1-\alpha^{-1}}\right]. \end{aligned}$$

This problem is straightforward to solve without any worst-case - measure considerations. It does not yield a worst-case model.

In order to determine a worst-case model, we have to pass to the convex hull

$$\bar{\mathcal{U}} = \{P_\lambda = \lambda P_1 + (1-\lambda)P_0 \mid \lambda \in [0,1]\}.$$

Any model $P_\lambda \in \bar{\mathcal{U}} \setminus \mathcal{U}$ is an incomplete quadrinomial model. The new parameter λ determines the probabilities assigned to the nodes:

$$P_\lambda[X = u_1] = \lambda p, \quad P_\lambda[X = u_0] = (1-\lambda)p.$$

Any P_λ with $\lambda > 0$ can be chosen as a reference model. In this setting, the class $\mathcal{M}$ of pricing rules is given by

$$\begin{aligned} \mathcal{Q} &:= \{q = (q_1, q_2, q_3, q_4) \mid \sum_{n=1}^{4} q_n \omega_n = 1,\ q_4 = 1 - q_1 - q_2 - q_3\} \\ \mathcal{M} &= \{Q_q \mid Q_q[X = \omega_i] = q_i,\ q \in \mathcal{Q}\} \end{aligned} \tag{4.7}$$

We have two alternatives to compute the optimal test for fixed P_λ. We can either determine a worst-case pricing rule $\tilde{Q} = Q_{\tilde{q}}$. To this end we have to solve a two-dimensional optimization problem on $\mathcal{Q}$. Alternatively, we can compute the efficient strategy $\tilde{\gamma}^\lambda$ for fixed P_λ directly:

$$\left[\begin{array}{c} E_\lambda[l((F - \gamma\alpha X - (1-\gamma)\alpha)^+)] = \min\limits_{\gamma} \\ \gamma \in \mathcal{A}_\alpha \end{array} \right].$$

This is a one-dimensional problem. As usual, we denote the value of this problem by β_λ. A worst-case-model is then given by

$$\beta_{\tilde{\lambda}} = \max_{\lambda \in [0,1]} \beta_\lambda$$

and the strategy $\tilde{\gamma}^{\tilde{\lambda}}$ is robust-efficient, i.e., a solution to (4.6) provided $\tilde{\lambda} \in (0,1)$ holds. Due to concavity of the function $\lambda \mapsto \beta_\lambda$, we can expect that in general $\tilde{\lambda} \in (0,1)$ should hold. However, we may have good reason to restrict the analysis a priori to mixtures P_λ for $\lambda \in [\delta, 1-\delta]$, cf. also Section 3.3.3. In this case, the worst-case model is automatically equivalent to the reference model R.

4.3. Black-Scholes models with uncertain drift

In this section, we examine a family of Black Scholes models P_m with uncertain drift m, i.e., $\mathcal{U} = \{P_m \mid m \in [m_*, m^*]\}$ where $[m_*, m^*]$ represents an interval of drift parameters m. As an application of Theorem 3.19, we find a worst-case model of the form $\int_{m_*}^{m^*} P_m \tilde{\nu}(dm)$ and the robust-efficient strategy. For this, we derive the dynamics of X under any mixture $P_\nu = \int_{m_*}^{m^*} P_m \nu(dm)$, cf. Theorem 4.2.

Let Q denote the unique measure on the measurable space $(\mathcal{C}[0,T], \mathcal{B}^{[0,T]})$ such that the coordinate process (X_t):

$$\begin{aligned} \Omega \times [0,T] &\to \mathcal{R} \\ X_t(\omega) &= \omega(t) \end{aligned}$$

satisfies

$$\frac{dX_t}{X_t} = \sigma dW_t$$

for some constant $\sigma > 0$ and a Brownian motion W. I.e., X is a geometric Brownian motion with volatility σ and zero drift:

$$X_t = X_0 \exp(\sigma W_t - \frac{1}{2}\sigma^2 t).$$

Given constant drift parameter $0 < m$ we define a model P_m with density G^m via

$$G^m = \frac{dP_m}{dQ} = \exp\left(\frac{m}{\sigma}W_T - \frac{1}{2}\frac{m^2}{\sigma^2}T\right). \tag{4.8}$$

By the Girsanov transformation, the process

$$\tilde{W}_t^m = W_t - \frac{m}{\sigma}t$$

is a Brownian motion under P_m. As a consequence, X is a geometric Brownian motion with drift m under P_m:

$$\frac{dX_t}{X_t} = \sigma dW_t^m + m dt,$$

respectively

$$X_t = X_0 \exp\left(\sigma W_t^m + (m - \frac{1}{2}\sigma^2)t\right).$$

In order to find a worst-case model for the parameterized family

$$\mathcal{U} = \{P_m \mid m \in [m_*, m^*]\}$$

we have to pass to the proper enlargement

$$\bar{\mathcal{U}} = \left\{P_\nu := \int_{m_*}^{m^*} P_m \nu(dm) \mid \nu \in m_1([m_*, m^*])\right\},$$

cf. Theorem 3.19.

We introduce the new drift process m^ν,

$$m_t^\nu := \frac{1}{G_t^\nu}\int_{m_*}^{m^*} m G_t^m \nu(dm)$$

where we have set

$$\begin{aligned} G_t^m &= \frac{dP_m}{dQ}|_{\mathcal{F}_t} = E_Q[G^m \,|\mathcal{F}_t] \\ G_t^\nu &= \frac{dP_\nu}{dQ}|_{\mathcal{F}_t} \\ &= \int_{m_*}^{m^*} G_t^m \nu(dm). \end{aligned}$$

The dynamics of X under $P_\nu \in \bar{\mathcal{U}}$ are given by

$$\frac{dX_t}{X_t} = \sigma dW_t^\nu + m_t^\nu dt \tag{4.9}$$

where W^ν is a Brownian motion under P_ν. This is an immediate consequence of the following Theorem 4.2:

4.2 Theorem. *We have*

$$\frac{dP_\nu}{dQ} = \exp\left(\int_0^T \frac{m_t^\nu}{\sigma} dW_t - \frac{1}{2}\int_0^T \frac{m_t^\nu}{\sigma^2} dt\right). \tag{4.10}$$

The process

$$W^\nu = W_t - \int_0^t \frac{m_s^\nu}{\sigma} ds$$

is a Brownian motion under P_ν.

Proof. We obtain from equation (4.8) the representation

$$G_t^m = 1 + \int_0^t \frac{m}{\sigma} D_s^m dW_s. \tag{4.11}$$

For given prior distribution $\nu \in m_1([m_*, m^*])$ we can conclude from (4.11) the representation

$$\begin{aligned} G_t^\nu &= 1 + \int_{m_*}^{m^*} \int_0^t \frac{m}{\sigma} G_s^m dW_s \nu(dm) \\ &= 1 + \int_0^t \int_{m_*}^{m^*} \frac{m}{\sigma} G_s^m \nu(dm) dW_s \\ &= 1 + \frac{1}{\sigma}\int_0^t m_s^\nu G_s^\nu dW_s \end{aligned}$$

respectively

$$G_T^\nu = \exp\left(\int_0^T \frac{m_t^\nu}{\sigma} dW_t - \frac{1}{2}\int_0^T \frac{m_t^\nu}{\sigma^2} dt\right) \tag{4.12}$$

which proves equation (4.10). Hence P_ν is obtained from Q by means of a Girsanov-transformation with drift m^ν. As a consequence, W^ν is a Brownian motion under P_ν. □

Now consider a contingent claim F, a state-dependent loss function l and initial capital α with

$$\alpha < E_Q[F].$$

For $\nu \in m_1([m_*, m^*])$, we consider the modified claim

$$\tilde{V}_\nu = \left(F - I_l(\tilde{k}_\nu (G^\nu)^{-1}, .)\right)^+$$

where the critical value $\tilde{k}_\nu \in (0, \infty)$ is determined uniquely from the capital constraint

$$E_Q[\tilde{V}_\nu] = \alpha.$$

It follows immediately from Lemma 3.3 that the modified claim $\tilde{V}_\nu$ is optimal for the simple testing problem $(P_\nu|Q)$, i.e., the perfect hedging strategy for $\tilde{V}_\nu$ is P_ν-efficient, cf. Theorem 1.7. The minimal shortfall-risk β_ν under the model P_ν is given

by

$$\beta_\nu = \int_{m_*}^{m^*} E_m[l(I_l(\tilde{k}_\nu (G^\nu)^{-1}, .) \wedge F, .)]\nu(dm), \tag{4.13}$$

cf. also Proposition 3.12.

4.3 Theorem. *Assume $l(F, .)$ is bounded. There is a probability measure $\tilde{\nu} \in m_1([m_*, m^*])$ that solves*

$$\beta_{\tilde{\nu}} = \max_{\nu \in m_1([m_*, m^*])} \beta_\nu. \tag{4.14}$$

The perfect hedging strategy for the modified claim $\tilde{V}_{\tilde{\nu}}$ is robust efficient for the families $\mathcal{U}$ and $\bar{\mathcal{U}}$ and the robust-minimal shortfall risk is given by $\beta^* = \beta_{\tilde{\nu}}$.

Proof. $\mathcal{U}$ is of the form (3.23) with $\Theta = [m_*, m^*]$. The parameterization fulfills the assumptions of Theorem 3.19, i.e., Θ is compact, the mapping $m \mapsto G_m(\omega)$ is continuous for every ω and condition (3.25) holds. Hence the class $\bar{\mathcal{U}}$ contains a worst-case model $P_{\tilde{\nu}}$ if $l(F, .)$ is bounded. Clearly, $\tilde{\nu}$ is given by condition (4.14).

Since all models $P_\nu \in \bar{\mathcal{U}}$ are equivalent, the $P_{\tilde{\nu}}$-efficient strategy $\tilde{\xi}$ is robust-efficient for $\bar{\mathcal{U}}$, cf. Theorem 3.11. As discussed above, $\tilde{\xi}$ is given by the perfect hedging strategy for $\tilde{V}_{\tilde{\nu}}$. Since $\bar{\mathcal{U}}$ is a proper enlargement of $\mathcal{U}$, $\tilde{\xi}$ is also robust-efficient for $\mathcal{U}$. □

The optimization problem (4.14) is a rather complex task. For this reason we examine the special case where one considers only two drift-parameters m_1, m_0 with corresponding models $P_i = P_{m_i}$. In this case the class $m_1(\{0, 1\})$ is simply the interval $[0, 1]$ and we have

$$\bar{\mathcal{U}} = \{P_\nu := \nu P_1 + (1 - \nu)P_0 \mid \nu \in [0, 1]\}. \tag{4.15}$$

With

$$\rho(\nu, x) = \frac{1}{\nu c_1 x^{\frac{m_1}{\sigma^2}} + (1 - \nu)c_0 x^{\frac{m_0}{\sigma^2}}}$$

we can substitute

$$(G^\nu)^{-1} = \frac{dQ}{dP_\nu} = \rho(\nu, X_T).$$

In order to simplify the notation, we assume that l is not state-dependent. We introduce the functions

$$\begin{aligned} s(\nu, x) &:= I_l(\tilde{k}_\nu \rho(\nu, x)) \\ \tilde{k}'_\nu &:= -\frac{E_Q[\,\mathbf{1}_{\{s(\nu, X_T) < F\}} \tilde{k}_\nu \frac{\rho_\nu(\nu, X_T)}{l''(I_l(\tilde{k}_\nu \rho(\nu, X_T)))}]}{E_Q[\,\mathbf{1}_{\{s(\nu, X_T) < F\}} \frac{\rho(\nu, X_T)}{l''(I_l(\tilde{k}_\nu \rho(\nu, X_T)))}]} \\ \rho'(\nu, X_T) &= \rho(\nu, X_T)^2 \left[c_0 X_T^{\frac{m_0}{\sigma^2}} - c_1 X_T^{\frac{m_1}{\sigma^2}}\right]. \end{aligned} \tag{4.16}$$

4.4 Proposition. *For fixed model $P_\nu \in \bar{\mathcal{U}}$ of* (4.15)*, the minimal risk β_ν is given by*

$$\beta_\nu = \nu E_1[l(s(\nu, X_T) \wedge F)] + (1-\nu) E_0[l(s(\nu, X_T) \wedge F)]. \tag{4.17}$$

(i) The mapping $\nu \mapsto \beta_\nu$ is differentiable with derivative

$$\begin{aligned} \partial_\nu \beta_\nu &= E_1[\mathbf{1}_{\{s(\nu,X_T)<F\}} l(s(\nu, X_T))] - E_0[\mathbf{1}_{\{s(\nu,X_T)<F\}} l(s(\nu, X_T))] \\ &\quad + E_\nu[\mathbf{1}_{\{s(\nu,X_T)<F\}} l'(s(\nu, X_T)) \frac{\tilde{k}_\nu \rho'(\nu, x) + \tilde{k}'_\nu \rho(\nu, x)}{l''(I_l(\tilde{k}_\nu \rho(\nu, x)))}]. \end{aligned}$$

(ii) Let $\tilde{\nu}$ be determined via either

$$\partial_\nu \beta_\nu = 0, \quad \nu \in (0,1) \tag{4.18}$$

or

$$\beta_\nu = \max_{\nu=1,0} \beta_\nu$$

if (4.18) *has no solution. Then $P_{\tilde{\nu}}$ is a worst-case model for $\bar{\mathcal{U}}$ of equation* (4.15)*.*

Proof. Equation (4.17) follows from (4.13).

Ad (i): (1) The function

$$(\nu, k) \mapsto f(\nu, k) = E_Q[(F - I_l(k\rho(\nu, X_T)))^+]$$

is differentiable with partial derivatives

$$\begin{aligned} f_\nu(\nu, k) &= -E_Q[\mathbf{1}_{\{F - I_l(k\rho(\nu,X_T))>0\}} k \frac{\rho'(\nu, X_T)}{l''(I_l(k\rho(\nu, X_T)))}] \\ f_k(\nu, k) &= -E_Q[\mathbf{1}_{\{F - I_l(k\rho(\nu,X_T))>0\}} \frac{\rho(\nu, X_T)}{l''(I_l(k\rho(\nu, X_T)))}] \end{aligned}$$

From the condition

$$f(\nu, \tilde{k}_\nu) = \alpha$$

we obtain the derivative

$$\tilde{k}'_\nu := \partial_\nu \tilde{k}_\nu = -\frac{f_\nu(\nu, \tilde{k}_\nu)}{f_k(\nu, \tilde{k}_\nu)},$$

i.e., the mapping $\nu \mapsto \tilde{k}_\nu$ is differentiable and its derivative is given by equation (4.16).

(2) The mapping $\nu \mapsto s(\nu, x)$ is differentiable with derivative

$$s_\nu(\nu, X_T) = \frac{\tilde{k}_\nu \rho'(\nu, x) + \tilde{k}'_\nu \rho(\nu, x)}{l''(I_l(\tilde{k}_\nu \rho(\nu, x)))}.$$

(3) We conclude from (1) and (2) differentiability of $\nu \mapsto \beta_\nu$ and

$$\begin{aligned}
\partial_\nu \beta_\nu &= E_1[\,\mathbf{1}_{\{s(\nu,X_T)<F\}}(l(s(\nu,X_T)) + \nu\partial_\nu l(s(\nu,X_T)))] \\
&\quad -E_0[\,\mathbf{1}_{\{s(\nu,X_T)<F\}}(l(s(\nu,X_T)) + \nu\partial_\nu l(s(\nu,X_T)))] \\
&= E_1[\,\mathbf{1}_{\{s(\nu,X_T)<F\}} l(s(\nu,X_T))] - E_0[\,\mathbf{1}_{\{s(\nu,X_T)<F\}} l(s(\nu,X_T))] \\
&\quad +E_\nu[\,\mathbf{1}_{\{s(\nu,X_T)<F\}} \partial_\nu l(s(\nu,X_T))] \\
&= E_1[\,\mathbf{1}_{\{s(\nu,X_T)<F\}} l(s(\nu,X_T))] - E_0[\,\mathbf{1}_{\{s(\nu,X_T)<F\}} l(s(\nu,X_T))] \\
&\quad +E_\nu[\,\mathbf{1}_{\{s(\nu,X_T)<F\}} l'(s(\nu,X_T)) s_\nu(\nu,X_T)].
\end{aligned}$$

This proves (i).

Ad (ii): Due to concavity of the mapping $\nu \mapsto \beta_\nu$, a solution $\tilde{\nu}$ to (4.18) fulfills

$$\beta_{\tilde{\nu}} = \max_{\nu\in[0,1]} \beta_\nu.$$

If there is no solution to (4.18), the maximum is attained either at $\nu = 1$ or $\nu = 0$.
□

CHAPTER 5

Stochastic volatility

In this chapter, we consider three different settings of models with stochastic volatility. In Section 5.1, we consider a family of models where volatility jumps at a random time τ to a new value η according to some unknown distribution θ. We apply the dynamic programming principle to construct the efficient strategy for fixed model. We then derive existence of a worst-case model $\tilde{P}$ and show that the $\tilde{P}$-efficient strategy is robust-efficient.

In Section 5.2, we give the Bellman equation for the efficient strategy in a single "classical" stochastic volatility model.

Section 5.3 is motivated by the uncertain volatility model of [ALP95] and [Lyo95]: We consider a countable family of volatility paths such that we can decide at time 0 which path is actually chosen. This provides us with an example where a worst-case model $\tilde{P}$ exists but the $\tilde{P}$-efficient strategy is not robust-efficient.

5.1. Volatility jump model

We consider a family of models where the volatility jumps at a random time τ to a new value η according to some unknown distribution θ. For given distribution θ, we denote the corresponding model by P_θ. In Section 5.1.1, we derive the efficient hedging strategy for any fixed distribution θ. We distinguish the case where τ is a stopping-time w.r.t. the filtration generated by X and the case where τ is independent of X. We first generalize the results obtained previously by [FL00] for a constant jump-time and derive the efficient strategy via the dynamic programming principle. We then derive a formula for the worst-case pricing rule and give an example where the worst-case pricing rule is not equivalent to P_θ, cf. Lemma 5.6 and 5.7.

In Section 5.1.3, we consider a convex family Θ of equivalent distributions of volatility and show that Θ contains a worst-case distribution $\tilde{\theta}$. Since all models P_θ are equivalent, the efficient hedging strategy for $P_{\tilde{\theta}}$ derived in Section 5.1.1 is robust-efficient for $\mathcal{U} = \{P_\theta \mid \theta \in \Theta\}$.

5.1.1. Efficient hedging for fixed model θ

We start with an explicit construction of the model $(\Omega, (\mathcal{F}_t), P_\theta)$. Consider $\Omega^0 = \mathcal{C}[0,T]$ with P^0 such that the coordinate process X on Ω^0 is a geometric Brownian

motion with drift m and constant volatility σ_0:

$$X_t = X_0 \exp\left(\sigma_0 W_t + (m - \frac{1}{2}\sigma_0^2)t\right)$$

where W is a Brownian motion under P^0. Let $(\mathcal{F}_t^0)$ denote the natural filtration generated by X. With respect to the construction of the volatility-jump model, we consider two cases:

(i) Let τ be a $(\mathcal{F}_t^0)$-stopping time, e.g., $\tau = \inf\{t \mid X_t \notin B\}$ for some neighborhood B of X_0. Let P^ϑ denote the probability measure on Ω^0 such that X is a geometric Brownian motion with piecewise constant volatility process (σ_t) given by

$$\sigma_t = \sigma_0 \, \mathbf{1}_{[0,\tau)}(t) + \vartheta \, \mathbf{1}_{[\tau,T]}(t).$$

Under P^ϑ, the values of volatility are deterministic, but the time of the jump is random. We define the model $P = P_\theta$ on $\Omega := \Omega^0 \times (0,\infty)$ via

$$P_\theta(d\omega^0, d\vartheta) := \theta(d\vartheta) P^\vartheta(d\omega^0). \tag{5.1}$$

We denote the distribution of τ under P by ρ.

(ii) Alternatively, we construct τ independently of X. Let $P^{\vartheta,s}$ denote the probability measure on Ω^0 such that X is a geometric Brownian motion with piecewise constant deterministic volatility process (σ_t) given by

$$\sigma_t = \sigma_0 \, \mathbf{1}_{[0,s)}(t) + \vartheta \, \mathbf{1}_{[s,T]}(t).$$

Given a probability distribution ρ on $(0,\infty)$ we define $\Omega = \Omega^0 \times (0,\infty) \times (0,\infty)$ and a probability measure P_θ on Ω via

$$P_\theta(d\omega^0, d\vartheta, ds) := \rho(ds)\theta(d\vartheta)P^{\vartheta,s}(d\omega^0).$$

In both cases, we consider the filtration $(\mathcal{F}_t)$ on Ω given by the right-continuous filtration generated by (X_t) and (σ_t). We can assume that $\theta(\sigma_0) = 0$ holds. Then τ is a $(\mathcal{F}_t)$-stopping time with distribution ρ . Under P_θ, volatility is a stochastic process σ_t of the form

$$\sigma_t(\omega) = \sigma_0 \, \mathbf{1}_{[0,\tau(\omega))}(t) + \eta(\omega) \, \mathbf{1}_{[t_0,\tau(\omega)]}(t)$$

for a random variable η independent $X_{[0,\tau]}$ with distribution θ.

Let $\mathcal{A}_\alpha(\theta)$ denote the class of admissible strategies in the model $(\Omega, (\mathcal{F}_t), P_\theta)$ as defined in Section 1.1. Clearly, $\mathcal{A}_\alpha(\theta) = \mathcal{A}_\alpha(\theta')$ holds if θ and θ' are equivalent. Since we are going to consider equivalent distributions θ only, we drop the dependence on θ and denote $\mathcal{A}_\alpha = \mathcal{A}_\alpha(\theta)$. In the following, E_θ denotes expectation under P_θ and E_0 denotes expectation under P_0.

We consider an European contingent claim $F = f(X_T)$ and a non-state dependent loss function l. Given a constant $\alpha > 0$ strictly less than the super-hedging price of

the option, an efficient hedging strategy is a solution to the problem

$$\left[\begin{array}{c} E_\theta[l(F-\alpha-\int_0^T \xi_s dX_s)] = \min_\xi \\ \xi \in \mathcal{A}_\alpha \end{array}\right]. \tag{5.2}$$

For given time t of the volatility jump, asset value x, new volatility ϑ and capital α' we define the auxiliary problem

$$\left[\begin{array}{c} E_\theta[l(F-\alpha'-\int_t^T \xi_s dX_s)|\, X_\tau = x, \tau = t, \eta = \vartheta] = \min_\xi \\ \xi = (\xi_s)_{[t\le s\le T]} \in \mathcal{A}_{\alpha',t,\vartheta} \end{array}\right]. \tag{5.3}$$

Here, $\mathcal{A}_{\alpha',t,\vartheta}$ denotes the class of all admissible strategies starting at time t with initial capital α' in the standard Black-Scholes model with maturity $T-t$, volatility ϑ, current asset price x. We could as well drop the subscript θ to stress the fact that this problem does not depend on θ. The solution $\tilde{\xi}$ to problem (5.3) is the efficient hedging strategy for the claim F in this model. This strategy is discussed thoroughly in [FL00].

Let $g^\vartheta(x,t)$ denote the unique arbitrage free price of F in this model, i.e., the Black-Scholes price. We define

$$g(x,t) := \inf\{c \,|\, g^\vartheta(x,t) \le c \;\; \theta - \text{almost-surely }\}.$$

We set $||\eta||_\infty = \inf\{\vartheta \,|\, \theta([0,\vartheta]) = 1\}$. If F is a call option, it is easily seen that

$$g(x,t) = g^{||\eta||_\infty}(x,t)$$

holds with $g^\infty(x,t) = x$.

Let $\beta^\vartheta(\alpha',x,t)$ denote the value function of problem (5.3), i.e.,

$$\beta^\vartheta(\alpha',x,t) := E[l(F-\alpha'-\int_t^T \tilde{\xi}_s dX_s)|\, X_\tau = x, \tau = t, \eta = \vartheta]$$

where $\tilde{\xi} = \tilde{\xi}(\alpha',x,t)$ denotes a solution to problem (5.3). On $t = T$ we extend β^ϑ to

$$\beta^\vartheta(\alpha',x,T) := l(f(x)-\alpha'). \tag{5.4}$$

We set $\hat{\tau} := \tau \wedge T$. Consider the function

$$\bar{\beta}_\theta(\alpha',x,t) := \int \beta^\vartheta(\alpha',x,t)\theta(d\vartheta). \tag{5.5}$$

We define a state-dependent loss function $L_\theta(\alpha',\omega)$ via

$$\begin{aligned} \bar{l}_\theta(\alpha',x,t) &:= \bar{\beta}_\theta(g(x,t)-\alpha',x,t) \\ L_\theta(\alpha',.) &:= \bar{l}_\theta(\alpha',X_{\hat{\tau}}(.),\hat{\tau}(.)). \end{aligned} \tag{5.6}$$

From now on we assume that

$$\beta_\theta(\alpha',x,t) < \infty \tag{5.7}$$

holds for all α',x,t.

5.1 Lemma. *L_θ is a loss function for the contingent claim $g(X_{\hat\tau}, \hat\tau)$ in the sense of Definition 1.1. Especially, $l_\theta(., x, t)$ is increasing and strictly convex on the interval $[0, g(x,t)]$ with derivative*

$$\partial_{\alpha'} \bar l_\theta(\alpha', x, t) = \int [\partial_{\alpha'} \beta^\vartheta(g(x,t) - \alpha', x, t)]\theta(d\vartheta).$$

We have

$$\lim_{\alpha' \downarrow 0} \partial_{\alpha'} \bar l_\theta(\alpha', x, t) = 0 \tag{5.8}$$

$$\lim_{\alpha' \uparrow g(x,t)} \partial_{\alpha'} \bar l_\theta(\alpha', x, t) = \infty \tag{5.9}$$

Proof. Up to equation (5.9), this is a direct consequence of Lemma 8.1 of [FL00] with $u(\alpha', x) = l_\theta(g(x,t)) - l_\theta(g(x,t) - \alpha', x, t)$ and (5.7).

Theorem 21, equation (6.21) and Remark 6 of [Leu99] imply

$$\lim_{\alpha' \downarrow 0} \partial_{\alpha'} \beta^\vartheta(\alpha', x, t) = -\infty$$

which proves equation (5.9). □

Let P_1 denote the projection of P_θ on $\mathcal{F}_{\tau-}$ with associated expectation operator E_1. Consider the second auxiliary problem

$$\left[\begin{array}{c} E_1[L_\theta(g(X_{\hat\tau}, \hat\tau) - \alpha - \int_0^{\hat\tau} \xi_s dX_s, X_{\hat\tau}, .)] = \min_\xi \\ \xi \in \mathcal{A}_\alpha \end{array} \right] \tag{5.10}$$

Problem (5.10) can be reduced to a statistical testing problem via Theorem 1.7. This testing problem may be solved using the methodology of Chapter 2 respectively Chapter 3.

If τ is constructed as in case (i) above, e.g. of the form $\tau = \inf\{t \mid X_t \notin B\}$, we have $P_1 = P_0$. Especially, the equivalent martingale measure for problem (5.10) is unique. In this situation, the optimal test can be computed explicitly, cf. Lemma 3.3.

In case (ii) where τ is independent of X, we have $P_1(d\omega^0, ds) = \rho(ds) P^{\sigma_0, s}(d\omega^0)$. Hence the equivalent martingale measure is not unique. In this case, one has to find a worst-case pricing rule in order to solve the semi-composite testing problem associated to (5.10).

We link auxiliary problems (5.3), (5.10) and the original problem (5.2):

5.2 Theorem. *Let $\tilde\xi^1$ be a solution to (5.10). Then the P_θ-efficient strategy is given by the following procedure:*

(i) On $[0, \hat\tau]$: Use $\tilde\xi^1$

(ii) On $(\hat{\tau}, T]$: Use the efficient hedging strategy given by problem (5.3) with new volatility $\vartheta = \eta$, capital $\alpha' = \alpha + \int_0^{\hat{\tau}} \tilde{\xi}_s dX_s$, asset price $x = X_{\hat{\tau}}$ and maturity $T - \hat{\tau}$.

Proof. It is easily seen that the strategy $\tilde{\xi}$ defined by (i) and (ii) is admissible: It is predictable and it satisfies

$$\alpha + \int_0^t \tilde{\xi}_s dX_s \geq 0, \quad 0 \leq t \leq T \quad P_\theta - \text{almost surely.}$$

For proof of optimality, consider any admissible strategy $\xi \in \mathcal{A}_\alpha$. The associated value at time $\hat{\tau}$ is given by

$$V_{\hat{\tau}} = \alpha + \int_0^{\hat{\tau}} \xi_s dX_s.$$

We obtain from the definition of β^ϑ validity of

$$E_\theta[l(F - V_{\hat{\tau}} - \int_{\hat{\tau}}^T \xi_s dX_s) | \, X_{\hat{\tau}}, \hat{\tau}, \eta] \geq \beta^\eta(\alpha + \int_0^{\hat{\tau}} \xi_s dX_s, X_{\hat{\tau}}, \hat{\tau})$$

which implies

$$\begin{aligned}
E_\theta[l(F - \alpha - \int_0^T \xi_s dX_s)] &= E_\theta[l(F - V_{\hat{\tau}} - \int_{\hat{\tau}}^T \xi_s dX_s)] \\
&= E_\theta[E[l(F - V_{\hat{\tau}} - \int_{\hat{\tau}}^T \xi_s dX_s) | \, X_{\hat{\tau}}, \hat{\tau}, \eta] \,] \\
&\geq E_\theta[\beta^\eta(\alpha + \int_0^{\hat{\tau}} \xi_s dX_s, X_{\hat{\tau}}, \hat{\tau})] \\
&= E_1[\bar{\beta}_\theta(\alpha + \int_0^{\hat{\tau}} \xi_s dX_s, X_{\hat{\tau}}, \hat{\tau})] \\
&\geq E_1[\bar{\beta}_\theta(\alpha + \int_0^{\hat{\tau}} \tilde{\xi}_s^1 dX_s, X_{\hat{\tau}}, \hat{\tau})],
\end{aligned}$$

the last inequality holds by definition of $\tilde{\xi}^1$. Hence $E_1[\bar{\beta}_\theta(\alpha + \int_0^{\hat{\tau}} \tilde{\xi}_s^1 dX_s, X_{\hat{\tau}}, \hat{\tau})]$ is a lower bound for the risk associated to any strategy if the initial cost is bounded by α. If we replace ξ by $\tilde{\xi}$ in the above calculation, it follows that $\tilde{\xi}$ actually achieves this bound:

$$E_\theta[l(F - \alpha + \int_0^T \tilde{\xi}_s dX_s)] = E_1[\bar{\beta}_\theta(\alpha + \int_0^{\hat{\tau}} \tilde{\xi}_s^1 dX_s, X_{\hat{\tau}}, \hat{\tau})]$$

□

5.3 Corollary. *Let $\tilde{\xi}^1 = \tilde{\xi}^1(\theta)$ be a solution to* (5.10)*. Then the minimal shortfall risk with respect to the model P_θ is given by*

$$\beta_\theta = E_1[\bar{\beta}_\theta(\alpha + \int_0^{\hat{\tau}} \tilde{\xi}_s^1(\theta) dX_s, X_{\hat{\tau}}, \hat{\tau})].$$

5.4 Remark (Process of jump times). *It is straightforward to iterate the above reasoning to an increasing sequence of random times of volatility jumps, e.g., where volatility jumps occur at the jump-times of a Poisson process independent of W. This can be solved in analogy to Corollary 6.3. Observe that in this case, the number of volatility jumps prior to maturity is unbounded.*

5.1.2. Worst-case pricing rule

In this section, we derive a formula for the worst-case pricing rule from the efficient strategy and provide an example where the worst-case pricing rule is not equivalent to the model, cf. Lemma 5.6 and 5.7.

We consider the special case (5.1) where we have $P_1 = P_0$ with unique equivalent martingale measure Q_0 where:

$$\frac{dQ_0}{dP_0} = c_1 X_{\hat{\tau}}^{-\frac{m}{\sigma_0^2}}$$

for some constant c_1. In this case, Problem (5.10) is the problem of efficient hedging for the contingent claim $g(X_{\hat{\tau}}, \hat{\tau})$ given initial capital α and loss function L_θ in the complete market P_0. Let E_0^* denote expectation with respect to Q_0. Equation (3.9) and Proposition 3.15 imply that the value at time τ of a solution $\tilde{\xi}^1$ to (5.10) and the optimal modified claim $\tilde{F}_1$ are given by

$$(\alpha + \int_0^{\hat{\tau}} \tilde{\xi}_t^1 dX_t) \wedge g(X_{\hat{\tau}}, \hat{\tau}) = \tilde{F}_1 = \Big(g(X_{\hat{\tau}}, \hat{\tau}) - I_{\bar{l}}(k_1 X_{\hat{\tau}}^{1-\gamma(\sigma)}, X_{\hat{\tau}}, \hat{\tau})\Big)^+, \tag{5.11}$$

where we have defined $\gamma(\vartheta) = \frac{m}{\vartheta^2}$, $I_{\bar{l}}$ denotes the inverse function for $\bar{l}'_\theta$ defined via (5.6) and the critical value k_1 is determined by the condition

$$E_0^*[\Big(g(X_{\hat{\tau}}, \hat{\tau}) - I_{\bar{l}}(k_1 X_{\hat{\tau}}^{1-\gamma(\sigma)}, X_{\hat{\tau}}, \hat{\tau})\Big)^+] = \alpha. \tag{5.12}$$

Due to (5.9), we have

$$\tilde{F}_1 = g(X_{\hat{\tau}}, \hat{\tau}) - I_{\bar{l}}(k_1 X_{\hat{\tau}}^{1-\gamma(\sigma)}, X_{\hat{\tau}}, \hat{\tau}) > 0 \quad P - \text{almost-surely}. \tag{5.13}$$

The optimal modified claim for problem (5.3) for $\tau = t$, $X_t = x$, new volatility $\eta = \vartheta$ and capital α' is of the form

$$\tilde{F}_{(x,t,\vartheta,\alpha')} = \Big(F - I_l(k_2(x,t,\vartheta,\alpha') X_T^{1-\gamma(\vartheta)})\Big)^+. \tag{5.14}$$

Here, the critical value $k_2(x,t,\vartheta,\alpha')$ is given by the condition

$$E^*[\tilde{F}_{(x,t,\vartheta,\alpha')} | \, X_\tau = x, \tau = t, \eta = \vartheta] = \alpha'$$

where E^* denotes expectation with respect to the unique equivalent martingale measure given the information available at time τ. Observe that we obtain the derivative of $\bar{l}_\theta$ directly from k_2 via equations (5.6) and (3.18):

$$\begin{aligned}\partial_{\alpha'}\bar{l}_\theta(\alpha',x,t) &= \int\{\partial_{\alpha'}\beta^\vartheta(g(x,t)-\alpha',x,t)\}\theta(d\vartheta)\\ &= \int\frac{k_2(x,t,\vartheta,\alpha')}{c_2(x,t,\vartheta)}d\vartheta\\ &=: \bar{k}_2(x,t,\alpha')\end{aligned}$$

where c_2 is the normalizing factor

$$\frac{1}{c_2(x,t,\vartheta)} = E[X_T^{-\gamma(\vartheta)}|\, X_\tau = x, \tau = t, \eta = \vartheta].$$

This proves the following

5.5 Lemma. *$I_{\bar{l}}(\bar{k}_2(x,t,\alpha'),x,t) = \alpha'$ holds for all $\alpha' \in [0,g(x,t)]$. The graph of $I_{\bar{l}}(.,x,t)$ is given by*

$$\{(k, I_{\bar{l}}(k,x,t)) \mid k \geq 0\} = \{(\bar{k}_2(x,t,\alpha'),\alpha') \mid \alpha' \in [0,g(x,t)]\}.$$

This lemma is very handy for practical purposes: After computation of $k_2(x,t,\vartheta,\alpha')$ for a grid of parameters (x,t,ϑ,α'), we do not need to differentiate the interpolation of $\bar{l}_\theta$ on this grid to compute $I_{\bar{l}}$. Instead, we can obtain $I_{\bar{l}}(.,x,t)$ directly from the interpolation of the points $\{(\bar{k}_2(x,t,\alpha'),\alpha') \mid \alpha' \in \text{ grid }\}$. This interpolation of $I_{\bar{l}}$ can then be used to compute k_1 according to equation (5.12). This reduces the required computation time and increases the precision for numerical algorithms.

5.6 Lemma (Formula for worst-case pricing rules). *Assume that $l'(0) = 0$ holds. For any worst-case pricing rule $\tilde{Q}$ there is a constant κ such that*

$$\kappa\frac{d\tilde{Q}}{dP} \wedge l'(F) = k_2\Big[X_{\hat{\tau}},\hat{\tau},\eta,X_{\hat{\tau}} - I_{\bar{l}}(k_1X_{\hat{\tau}}^{1-\gamma(\sigma)},X_{\hat{\tau}},\hat{\tau},\eta)\Big]X_T^{1-\gamma(\eta)} \wedge l'(F)$$

holds P_θ-almost surely.

Proof. Proposition 3.15 (ii) implies

$$l'(F-\tilde{V}) = \tilde{k}_{(P|\tilde{Q})}\frac{d\tilde{Q}}{dP} \wedge l'(F) \quad P-\text{almost surely}$$

where we can replace the optimal modified claim $\tilde{V}$ via (5.14) and α' in (5.14) via (5.13)

$$\alpha' = \tilde{F}_1 = g(X_{\hat{\tau}},\hat{\tau}) - I_{\bar{l}}(k_1X_{\hat{\tau}}^{1-\gamma(\sigma)},X_{\hat{\tau}},\hat{\tau})$$

to obtain

$$l'(F-\tilde{V}) = k_2\Big[x,t,\vartheta,g(X_{\hat{\tau}},\hat{\tau}) - I_{\bar{l}}(k_1X_{\hat{\tau}}^{1-\gamma(\sigma)},X_{\hat{\tau}},\hat{\tau})\Big]X_T^{1-\gamma(\vartheta)} \wedge l'(F)$$

□

We now show that the worst-case pricing rule is in general not an equivalent measure. We consider the special case where $\tau = s$ is deterministic, θ is equivalent to the Lebesque-measure and $F = (X_T - K)^+$.

5.7 Lemma. *Consider the situation where $\tau = s$ is deterministic, $\frac{d\theta}{d\lambda}(\vartheta) > 0\ \forall \vartheta > 0$ and $F = (X_T - K)^+$. Then for $0 < \alpha < F_0$, any worst-case pricing rule $\tilde{Q}$ is* not *equivalent to P, i.e.,*

$$P[\frac{d\tilde{Q}}{dP} = 0] > 0. \tag{5.15}$$

Proof. The Black-Scholes price $g^\vartheta(x,s)$ for the call F at time s given volatility ϑ and asset price $X_s = x$ is increasing in ϑ with

$$\lim_{\vartheta \downarrow 0} g^\vartheta(x,s) = (x-K)^+ \quad \lim_{\vartheta \uparrow \infty} g^\vartheta(x,s) = x. \tag{5.16}$$

Hence problem (5.10) is the problem of efficient hedging for the contingent claim

$$g(x,s) = x$$

given initial capital α and loss function L_θ.

We intend to demonstrate that

$$P_\theta[\tilde{F} \geq (X_T - K)^+,\ X_T > K] > 0 \tag{5.17}$$

holds for any $\alpha > 0$. Intuitively, this is due to the fact the strategy $\tilde{\xi}^1$ aims to replicate the superhedge price $g(x,s) = x$ even though we only need a very small amount of capital α' in order to replicate $(X_T - K)^+$ on the interval $[s,T]$ if volatility is low. Since θ is equivalent to the Lebesque-measure, for any initial capital α' there is a positive probability that we need less than α' in order to cover $(X_T - K)^+$ if $X_s \leq K$ holds, cf. (5.16). We will now turn this heuristic reasoning into more rigorous statements:

The crucial fact for our analysis is that

$$\{\tilde{F} \geq (X_T - K)^+\} = \{\tilde{F}_1 \geq g^\vartheta(X_s, s)\}$$

holds. We obtain from equation (5.11)

$$\tilde{F}_1 = f_1(X_s)$$

where

$$f_1(x) = \left(x - I_{\bar{l}}(k_1 x^{1-\gamma(\sigma)}, x, s)\right)^+.$$

The function f_1 is strictly positive due to (5.13). By continuity of f_1, we find

$$\epsilon := \frac{f_1(K)}{2} > 0$$

and the set

$$J := \{x \mid f_1(x) > \epsilon,\ x \leq K\}$$

has strictly positive mass under the Lebesque measure λ. Due to $K \geq f_1(K) > \epsilon$, we can find $\vartheta^* > 0$ such that

$$g^{\vartheta^*}(K,s) = \epsilon$$

holds. Since $g^\vartheta(x,s)$ is increasing in ϑ and increasing in x, we can conclude from the last equation that

$$g^\vartheta(x,s) \le \epsilon \quad \forall \vartheta \le \vartheta^*,\ \forall x \le K.$$

By construction, we obtain

$$\{(x,\vartheta)|\ x \in J,\ \vartheta \le \vartheta^*\} \subset \{(x,\vartheta)|\ f_1(x) \ge g^\vartheta(x,s)\}$$

and the set on the left-hand side has strictly positive mass under the Lebesque-measure λ^2 on $[0,\infty) \times [0,\infty)$.

Since the measure $P_\theta[X_s \in dx, \eta \in d\vartheta]$ is equivalent to λ^2, we obtain

$$P_\theta[\tilde{F} \ge (X_T - K)^+,\ X_T > K] \ge P_\theta[X_s \in J,\ \eta \le \vartheta^*,\ X_T > K] > 0.$$

The last estimate and Corollary 3.16 (i) imply (5.15). □

5.1.3. Uncertain distribution of volatility

We now turn to robust-efficient strategies for a family of volatility-distributions Θ.

5.8 Theorem. *Consider a convex family Θ of equivalent distributions θ. Let $\tilde{\theta}$ denote a solution to*

$$\beta_{\tilde{\theta}} = \max_{\theta \in \Theta} \beta_\theta \tag{5.18}$$

where β_θ was defined in Corollary 5.3. Then $\tilde{P} = P_{\tilde{\theta}}$ is a worst-case model and the $\tilde{\theta}$-efficient hedging strategy described in Theorem 5.2 is robust-efficient.

Proof. Due to 3.6, a solution $\tilde{\theta}$ to(5.18) yields a worst-case model. Due to Theorem 3.11 the $\tilde{\theta}$-efficient strategy is robust-efficient. □

It remains to establish existence of a worst-case model. Subsequently, we identify distributions θ and their densities $\theta(\vartheta) = d\theta/d\lambda(\vartheta)$. Let $m_1'(S)$ denote the class of all probability densities on $S \subseteq (0,\infty)$.

5.9 Theorem. *Assume $l(F)$ is bounded and the convex class $\Theta \subset m_1'(0,\infty)$ is uniformly absolutely continuous and closed in $L^1(\theta_0)$ for some dominating $\theta_0 \in \Theta$. Then Θ contains a worst-case density $\tilde{\theta}$.*

Proof. Consider the dominating model $P_{\theta_0} =: R$ for $\mathcal{U} = \{P_\theta \mid \theta \in \Theta\}$. Under the assumptions of the Theorem, the family $\mathcal{G} = \{\frac{dP_\theta}{dR} \mid \theta \in \Theta\}$ is convex, uniformly integrable and closed in $L^1(R)$. Hence the assertion follows from Proposition 3.13. □

5.10 Example. This example corresponds to the total variation neighborhood of a given distribution θ_0 where all densities in the neighborhood are restricted to stay within given bounds.

Let there be given a probability density θ_0 with support $S \subset [\vartheta_*, \vartheta^*]$ for some $0 < \vartheta_* < \vartheta^* < \infty$ such that $\inf_{\vartheta \in S} \theta_0(\vartheta) > 0$ holds. We consider a lower bound θ_* such that

$$\theta_* \leq \inf_{\vartheta \in S} \theta_0(\vartheta)$$

holds. For any $p > 1$, $\epsilon > 0$, the class

$$\Theta_p(\theta_0, \epsilon) := \left\{ \theta \in m_1'(S) \mid \theta_* \leq \theta(\vartheta) \forall \vartheta \in S; \quad \int |\theta_0(\vartheta) - \theta(\vartheta)|^p d\vartheta \leq \epsilon^p \right\}$$

satisfies the assumptions of Theorem 5.8 and Theorem 5.9: Clearly, all $\theta \in \Theta(\theta_0, \epsilon)$ are equivalent, $\Theta_p(\theta_0, \epsilon)$ is convex and uniformly integrable and closed in $L^1(\theta_0)$.

5.11 Example. Let $\theta_{(e,v)}$ denote the density of the log-normal distribution with mean e and variance v with respect to the Lebesque measure. We consider

$$\Theta = \{\theta_{(e,v)} \mid e \in [e_-, e_+],\ v \in [v_-, v_+]\}$$

for given positive constants e_-, e_+, v_-, v_+. It is straightforward to apply the results of Section 3.3.2 to this parameterized family of models - here, one finds a worst-case *mixture* model $\tilde{\theta}$ of the form

$$\tilde{\theta} = \int_{[e_-,e_+]\times[v_-,v_+]} \theta_{(e,v)} \nu(de, dv),$$

cf. Theorem 3.20.

5.2. Bellman equation for classical SV-models

We consider a stochastic volatility model of the form

$$dX_t = X_t R_t dt + X_t f(Y_t) dW_t^1 \tag{5.19}$$

$$dY_t = \alpha_t(m_t - Y_t)dt + g(Y_t, t)(\rho_t dW_t^1 + \sqrt{1 - \rho_t^2} dW_t^2) \tag{5.20}$$

$$dV_t = \pi_t[R_t dt + f(Y_t) dW_t^1] \tag{5.21}$$

where (W^1, W^2) is a Brownian motion in $\mathbb{R}^2$. The parameter m_t is the long term mean of the volatility-driving process Y and α_t determines its speed of mean-reversion. The processes R_t, α_t, m_t and $-1 < \rho_t < 1$ are assumed to be deterministic. We assume that f is positive and invertible. The Heston-model $f(y) = \sqrt{y}$, $g(y,t) = \sigma_t \sqrt{y}$ is popular with practitioners who price equity options, cf. e.g. [BFF+00]. The dynamics (5.21) of the value-process V are a consequence of equation (1.1) where we replace the number of assets ξ_t by the value $\pi_t = \xi_t X_t$.

We consider a European contingent claim $F = F(X_T)$ and a loss function $l(z, .) = l(z)$. Let $\beta(t, x, y, v)$ denote the minimal expected risk at time t given $X_t = x$, $Y_t = y$ and initial capital $V_t = v$, i.e.,

$$\beta(t, x, y, v) = \min_{\pi \in \mathcal{A}_{v,t}} E_P[l(F - v - \int_t^T \frac{\pi_s}{X_s} dX_s) \mid X_t = x, Y_t = y]$$

where $\mathcal{A}_{v,t}$ denotes the set of all strategies admissible strategies π from time t on with $V_t = v$.

We apply the methodology of [FS93] to derive the Hamilton-Jacobi-Bellman (HJB) equation for β. The process $Z_t = (X_t, Y_t, V_t)$ is a controlled Markov process in $\mathbb{R}^3$ where π_t is the control applied at time t. We have

$$dZ_t = h(t, Z_t, \pi_t)dt + \sigma(t, Z_t, \pi_t)dW_t$$

where we have set

$$\begin{aligned} h(t,x,y,v,\pi) &= \begin{pmatrix} xR_t \\ \alpha_t(m_t - y) \\ \pi R_t \end{pmatrix} \\ \sigma(t,x,y,v,\pi) &= \begin{pmatrix} xf(y) & 0 \\ g(y,t)\rho_t & g(y,t)\sqrt{1-\rho_t^2} \\ \pi f(y) & 0 \end{pmatrix}. \end{aligned} \tag{5.22}$$

$$(a_{ij}) := \sigma\sigma' = \begin{pmatrix} x^2 f^2(y) & xf(y)g(y,t)\rho_t & xf^2(y)\pi \\ xf(y)g(y,t)\rho_t & g^2(y,t) & \pi f(y)g(y,t)\rho_t \\ xf^2(y)\pi & \pi f(y)g(y,t)\rho_t & \pi^2 f^2(y) \end{pmatrix} \tag{5.23}$$

To abbreviate notation, let

$$\mathcal{L}_{x,y}\beta = R_t x\beta_x + \alpha_t(m_t - y)\beta_y + \frac{1}{2}f(y)^2x^2\beta_{xx} + \rho_t f(y)g(y,t)x\beta_{xy} + \frac{1}{2}g(y,t)^2\beta_{yy}$$

denote the infinitesimal generator $\mathcal{L}_{x,y}$ for (X, Y).

We obtain from equations (5.22), (5.23) and [FS93] Chapter IV equations (3.2)-(3.4) the HJB equation for β:

$$\beta_t + \mathcal{L}_{x,y}\beta + \inf_\pi \left\{ \pi[R_t\beta_v + f(y)^2x\beta_{xv} + \rho_t g(y,t)f(y)\beta_{yv}] + \frac{1}{2}\pi^2 f(y)^2\beta_{vv} \right\} = 0$$

with terminal condition

$$\beta(T, x, y, v) = l(F(x) - v). \tag{5.24}$$

The infimum is assumed by

$$\tilde{\pi}(t,x,y,v) := -\frac{R_t\beta_v + f(y)^2x\beta_{xv} + \rho_t g(y,t)f(y)\beta_{yv}}{f(y)^2\beta_{vv}} \tag{5.25}$$

which yields the HJB equation

$$\beta_t + \mathcal{L}_{x,y}\beta - \frac{(R_t\beta_v + f(y)^2x\beta_{xv} + \rho_t g(y,t)f(y)\beta_{yv})^2}{2f(y)^2\beta_{vv}} = 0. \tag{5.26}$$

5.12 Conjecture. *Let $\beta = \beta(t,x,y,v)$ be a solution to (5.26) with terminal condition (5.24). Then the strategy*

$$\tilde{\xi}_t = \tilde{\pi}(t, X_t, Y_t, V_t)$$

defined via (5.25) is efficient.

Whereas time t, asset price X_t and the available capital V_t are observed immediately, the quantity Y_t has to be computed from the observed instantaneous volatility $\sigma_t = f(Y_t)$ via $Y_t = f^{-1}(\sigma_t)$.

The typical approach to establish Conjecture 5.12 is to apply a verification theorem, see e.g. [FS93]. However, the main problem is to solve the HJB equation. [JS00] consider the asymptotic case of "fast-mean reverting" stochastic volatility $\alpha \to \infty$ and give first- and second-order correction terms for the efficient strategy for $\alpha < \infty$. One advantage of this approach is that one does not need to solve equation (5.26) explicitly. Instead, correction terms are obtained by singular perturbation analysis.

5.3. Singular models

We consider a *countable* family of *singular and complete* models $\{P_n | \ n \in \mathbb{N}\}$, i.e., the equivalent martingale measure Q_n for each P_n is unique and there exist events $\Omega_k \in \mathcal{F}$ such that

$$P_n[\Omega_k] = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{else.} \end{cases}$$

We shall assume that

$$\Omega_n \in \mathcal{F}_0, \quad n \in \mathbb{N} \tag{5.27}$$

holds. This setup corresponds to a special case of the "uncertain volatility model" examined e.g. by [ALP95] and [Lyo95]: We consider a countable family of deterministic volatility-paths such that we can decide immediately which volatility path is the true one. This is e.g. the case if we restrict the analysis to constant volatilities and a right-continuous filtration. Intuitively, the "optimal" hedging strategy in this setting is to simply follow the efficient strategy in the complete model P_{n_0} which is revealed at time 0. However, this approach does not tell us what the maximal risk is if a different model would happen to be the true one. Furthermore, this strategy is not robust-efficient. Hence this setup is interesting not only for theoretical purposes but also to determine the maximal risk that can occur over different volatility-paths (i.e., models). We shall demonstrate that the superposition of all efficient strategies for any fixed path is actually robust-efficient, cf. Proposition 5.14. This is a rather immediate consequence of the structure of the class of all equivalent martingale measures considered in Proposition 5.13.

We then derive existence of a worst-case model and worst-case pricing rule. For this, we first define a bijection between the convex hull of $\mathcal{U}$ respectively $\mathcal{M}$ and the sequence space $\mathcal{S}$ of equation (5.32). We then relate point-wise (respectively L^1) convergence on $\mathcal{S}$ to L^0 (respectively L^1)-convergence of the densities dP/dR. This will allow us to establish existence of a worst-case model and pricing rule.

We will then see that in this setting, the efficient strategy for the worst-case model is typically *not* robust-efficient, cf. Examples 5.21 and 5.22.

We consider the dominating measure

$$R := \sum_{n=1}^{\infty} 2^{-n} P_n. \tag{5.28}$$

As usual, E denotes the expectation operator with respect to R. The decomposing events can be chosen as

$$\Omega_n = \Big\{ \frac{dP_n}{dR} > 0 \Big\}$$

and we have

$$1 = \sum_{n=1}^{\infty} \mathbf{1}_{\Omega_n} \quad R - \text{almost-surely}. \tag{5.29}$$

We examine robust-efficient hedging strategies for the class

$$\mathcal{U} = \{P_n |\ n \in \mathbb{N}\}$$

or - equivalently - for the class $co_\infty(\mathcal{U})$ of all countable convex combinations

$$co_\infty(\mathcal{U}) := \Big\{ \sum_{n=1}^{\infty} \lambda_n P_n \mid \lambda_n \in [0,1],\ \sum_{n=1}^{\infty} \lambda_n = 1 \Big\}.$$

Any point in $co_\infty(\mathcal{U}) \setminus \{P_n |\ n \in \mathbb{N}\}$ yields an incomplete financial market. This is due to fact that the equivalent martingale measure is no longer unique: Consider e.g. the case $P = \lambda P_1 + (1-\lambda) P_2$ for $\lambda > 0$. Any $\gamma \in (0,1)$ yields a martingale measure Q^γ equivalent to P via

$$Q^\gamma = \gamma Q_1 + (1-\gamma) Q_2.$$

We show in the next proposition that every martingale measure equivalent to P is of this form provided condition (5.27) holds. In general, Proposition 5.13 does not hold if condition (5.27) is violated. This can be seen e.g. in the quadrinomial setting of Section 4.2, cf. equation (4.7).

5.13 Proposition. *The class $\mathcal{M}$ of martingale measures Q equivalent to R is given by*

$$\mathcal{M} = \Big\{ \sum_{n=1}^{\infty} \lambda_n Q_n \mid \lambda_n \in (0,1),\ \sum_{n=1}^{\infty} \lambda_n = 1 \Big\}.$$

Proof. The inclusion "$\supseteq$" is immediate. We therefore only show the inclusion "$\subseteq$".

(1) For given $Q \in \mathcal{M}$ we obtain from the equivalency of Q and R that

$$\lambda_n := Q[\Omega_n] \in (0,1) \quad n \in \mathbb{N}$$

holds. Thus

$$\hat{Q}[A] := Q[A|\Omega_n] = \frac{Q[A \cap \Omega_n]}{Q[\Omega_n]}$$

defines a probability measure equivalent to P_n and we have the representation

$$Q = \sum_{n=1}^{\infty} \lambda_n \hat{Q}_n.$$

It remains to show that X is a martingale under $\hat{Q}_n$: Uniqueness of the equivalent martingale measure for P_n then implies $\hat{Q}_n = Q_n$.

2) With $D^n = d\hat{Q}/dR$, $D = dQ/dR$ and $D_s^n = E[D^n \mid F_s]$ respectively $D_s = E[D \mid F_s]$, we obtain

$$\begin{aligned} E_Q[X_t|\mathcal{F}_s] &= (D_s)^{-1} E\left[DX_t|\mathcal{F}_s\right] \\ &= (D_s)^{-1} E\Big[\sum_{n=1}^{\infty} \lambda_n D^n X_t|\mathcal{F}_s\Big] \\ &= (D_s)^{-1} \sum_{n=1}^{\infty} \lambda_n E\left[D^n X_t|\mathcal{F}_s\right] \\ &= (D_s)^{-1} \sum_{n=1}^{\infty} \lambda_n D_s^n E_{\hat{Q}_n}\left[X_t|\mathcal{F}_s\right]. \end{aligned} \tag{5.30}$$

Furthermore, we obtain from (5.27) the equation

$$D_s \, \mathbf{1}_{\Omega_n} = \lambda_n D_s^n. \tag{5.31}$$

Now we compute for any $A \in \mathcal{F}_s$ the expectation

$$\begin{aligned} \lambda_n E_{\hat{Q}_n}[X_s; A] &= \lambda_n E_{\hat{Q}_n}[X_s; A \cap \Omega_n] \\ &= E_Q[X_s; A \cap \Omega_n] \\ &= E_Q[E_Q[X_t|\mathcal{F}_s]; A \cap \Omega_n] \end{aligned}$$

Here we can substitute $E_Q[X_t|\mathcal{F}_s]$ according to equation (5.30) and (5.31) to conclude

$$\begin{aligned} E_Q[E_Q[X_t|\mathcal{F}_s]; A \cap \Omega_n] &= E_Q[E_{\hat{Q}_n}[X_t|\mathcal{F}_s]; A \cap \Omega_n] \\ &= \lambda_n E_{\hat{Q}_n}[E_{\hat{Q}_n}[X_t|\mathcal{F}_s]; A] \end{aligned}$$

i.e., X is a martingale under $\hat{Q}_n$. □

The structure of the class of equivalent martingale measures allows us to derive the robust-efficient strategy respectively the maximin-optimal modified claim immediately:

5.14 Proposition. *Let $\tilde{V}^n$ denote the optimal modified claim in the complete model P_n given initial capital α, with the understanding that $\tilde{V}^n = F$ holds whenever $\alpha \leq E_n^*[F]$. The modified claim*

$$\tilde{V} := \sum_{n=1}^{\infty} \tilde{V}_n \, \mathbf{1}_{\Omega_n}$$

is optimal for any $P \in \mathcal{U}$ and maximin-optimal.

Proof. First observe that $\tilde{V} \in \mathcal{V}_\alpha$ holds due to Proposition 5.13.

Now consider any claim $V \in \mathcal{V}_\alpha$. Then $E_n^*[V] \leq \alpha$ holds for any $n \in \mathbb{N}$, i.e., V satisfies the side condition for the simple problem $(P_n|Q_n)$. We can hence conclude from optimality of $\tilde{V}_n$ for the simple problem $(P_n|Q_n)$ that

$$E_n[l(F - \tilde{V}^n, .)] \quad \leq \quad E_n[l(F - V, .)]$$

holds. Hence $\tilde{V}$ is "uniformly most powerful", i.e., optimal for any $P_n \in \mathcal{U}$. As a trivial consequence, $\tilde{V}$ is maximin-optimal. □

In the reminder of this section, we derive existence of a worst-case model and worst-case pricing rule. For this, we first define a bijection between the convex hull of $\mathcal{U}$ respectively $\mathcal{M}$ and the sequence space $\mathcal{S}$ of equation (5.32). We then relate pointwise (respectively L-1) convergence on $\mathcal{S}$ to L^0 (respectively L^1)-convergence of the densities dP/dR. This will allow us to establish existence of a worst-case model and pricing rule.

We define

$$\begin{aligned} \bar{\mathcal{G}} &= \{\frac{dP}{dR} \mid P \in co_\infty(\mathcal{U})\} \\ \mathcal{D} &= \{\frac{dQ}{dR} \mid Q \in \mathcal{M}\} \end{aligned}$$

and denote by $\bar{\mathcal{G}}^i$ respectively $\bar{\mathcal{D}}^i$ the closure of $\bar{\mathcal{G}}$ respectively $\mathcal{D}$ in $L^i(R)$ for $i = 0, 1$. Then we have, by definition,

$$\begin{aligned} \bar{\mathcal{U}}^1 &= \{P \ll R \mid \frac{dP}{dR} \in \bar{\mathcal{G}}^1\} \\ \bar{\mathcal{U}}^0 &= \{P \ll R \mid \frac{dP}{dR} \in \bar{\mathcal{G}}^0\} \\ \bar{\mathcal{M}}^1 &= \{Q \ll R \mid \frac{dQ}{dR} \in \bar{\mathcal{D}}^1\} \\ \bar{\mathcal{M}}^0 &= \{Q \ll R \mid \frac{dQ}{dR} \in \bar{\mathcal{D}}^0\} \end{aligned}$$

We introduce the sequence-spaces

$$\begin{aligned} \bar{\mathcal{S}}^0 &:= \Big\{\lambda = (\lambda_n)_{n\in\mathbb{N}} \mid \sum_{n=1}^{\infty} \lambda_n \leq 1,\ \lambda_n \in [0,1],\ n \in \mathbb{N}\Big\} \\ \bar{\mathcal{S}}^1 &:= \Big\{\lambda \in \bar{\mathcal{S}}^0 \mid \sum_{n=1}^{\infty} \lambda_n = 1\Big\} \\ \mathcal{S} &:= \Big\{\lambda \in \bar{\mathcal{S}}^1 \mid \lambda_n \in (0,1),\ n \in \mathbb{N}\Big\}. \end{aligned} \tag{5.32}$$

We consider different topologies on the sequence space $\bar{\mathcal{S}}^0$:

(i) The topology of convergence in l_1:

$$\lim_{k\uparrow\infty} \lambda^k = \lambda \quad \text{in } l_1 \quad \text{if} \quad \lim_{k\uparrow\infty} \sum_{n=1}^{\infty} |\lambda_n^k - \lambda_n| = 0.$$

(ii) The topology of point-wise convergence l_0:

$$\lim_{k\uparrow\infty} \lambda^k = \lambda \quad \text{in } l_0 \quad \text{if} \quad \lim_{k\uparrow\infty} |\lambda_n^k - \lambda_n| = 0 \quad \forall n \in \mathbb{N}.$$

5.15 Lemma. *$\bar{\mathcal{S}}^1$ is the closure of $\mathcal{S}$ under l_1 and $\bar{\mathcal{S}}^0$ is the closure of $\mathcal{S}$ in l_0.*

5.16 Proposition. *We consider a sequence (D^k) in $\mathcal{D}$. There exists a unique sequence (γ^k) in $\mathcal{S}$ such that*

$$D^k = \sum_{n=1}^{\infty} \gamma_n^k \frac{dQ_n}{dR} \tag{5.33}$$

holds and we have

(i) (D^k) is Cauchy in $L^1(R)$ if and only if (γ^k) is Cauchy in l_1.

(ii) (D^k) is Cauchy in $L^0(R)$ if and only if (γ^k) is Cauchy in l_0.

Proof. Existence of the representation (5.33) follows from Proposition 5.13, uniqueness via

$$\gamma_n^k = E[D^k; \Omega_n].$$

(i) Due to (5.33), we have

$$\begin{aligned} ||D^k - D^l||_{L^1(R)} &= E[|D^k - D^l|] \\ &= \sum_{n=1}^{\infty} E[|D^k - D^l|; \Omega_n] \\ &= \sum_{n=1}^{\infty} E[|\gamma_n^k - \gamma_n^l| \frac{dQ_n}{dR}] \\ &= \sum_{n=1}^{\infty} |\gamma_n^k - \gamma_n^l| \end{aligned}$$

which implies (i).

(ii) First assume (γ^k) is not Cauchy in l_0. Then there is an index n such that $(\gamma_n^k)_{k\in\mathbb{N}}$ is not Cauchy, i.e., there exists $\epsilon > 0$ such that for all $K \in \mathbb{N}$ there exist $k, l > K$ with

$$|\gamma_n^k - \gamma_n^l| > \epsilon.$$

This implies

$$R[|D^k - D^l| > \epsilon] \geq R[\Omega_n] = \frac{1}{2^n}$$

for infinitely many k and l. Hence (D^k) is not Cauchy in $L^0(R)$.

Now we assume (γ^k) is Cauchy in l_0. Given some $\epsilon > 0$ and $\delta > 0$, let N be large enough such that

$$\sum_{n=N+1}^{\infty} \frac{1}{2^n} < \delta$$

holds. There exists K such that for all $n = 1, 2, \ldots, N$ we have

$$|\gamma_n^k - \gamma_n^l| \leq \epsilon, \quad k, l > K.$$

Hence we can estimate for all $k, l > K$:

$$\begin{aligned} R[|D^k - D^l| > \epsilon] &= \sum_{\substack{n=1 \\ |\gamma_n^k - \gamma_n^l| > \epsilon}}^{\infty} \frac{1}{2^n} \\ &= \sum_{\substack{n=N+1 \\ |\gamma_n^k - \gamma_n^l| > \epsilon}}^{\infty} \frac{1}{2^n} \\ &\leq \delta. \end{aligned}$$

Since δ is arbitrary, (D^k) is Cauchy in $L^0(R)$. □

With

$$P_\lambda = \sum_{n=1}^{\infty} \lambda_n P_n, \quad \lambda \in \bar{\mathcal{S}}^0 \quad \text{and} \tag{5.34}$$

$$Q_\gamma = \sum_{n=1}^{\infty} \gamma_n Q_n, \quad \gamma \in \bar{\mathcal{S}}^0 \tag{5.35}$$

we have the following bijection between $\bar{\mathcal{S}}^1$ and $co_\infty(\mathcal{U})$ respectively between $\mathcal{S}$ and $\mathcal{M}$:

$$\begin{aligned} co_\infty(\mathcal{U}) &= \{P_\lambda \mid \lambda \in \bar{\mathcal{S}}^1\} \\ \mathcal{M} &= \{Q_\gamma \mid \gamma \in \mathcal{S}\} \end{aligned}$$

and

$$\begin{aligned} \bar{\mathcal{S}} &= \{(P[\Omega_n])_{n\in\mathbb{N}} \mid P \in co_\infty(\mathcal{U})\} \\ \mathcal{S} &= \{(Q[\Omega_n])_{n\in\mathbb{N}} \mid Q \in \mathcal{M}\}. \end{aligned}$$

5.17 Proposition. *The following representations hold true:*

$$co_\infty(\mathcal{U}) = \bar{\mathcal{U}}^1 \tag{5.36}$$

$$\bar{\mathcal{U}}^0 = \{P_\lambda \mid \lambda \in \bar{\mathcal{S}}^0\} \tag{5.37}$$

$$\bar{\mathcal{M}}^1 = \{Q_\gamma \mid \gamma \in \bar{\mathcal{S}}^1\} \tag{5.38}$$

$$\bar{\mathcal{M}}^0 = \{Q_\gamma \mid \gamma \in \bar{\mathcal{S}}^0\}. \tag{5.39}$$

Proof. For $\bar{\mathcal{M}}^i$, the assertion follows immediately from Lemma 5.15 and Proposition 5.16. With respect to $\bar{\mathcal{U}}^i$, we remark that one can replace $\mathcal{D}$ by $\bar{\mathcal{G}}$ in Proposition 5.16. □

Due to the last Proposition, we have

$$\begin{aligned} Q_n &\in \bar{\mathcal{M}}^1 \setminus \mathcal{M}, \quad n \in \mathbb{N} \\ 0 &\in \bar{\mathcal{M}}^0 \setminus \bar{\mathcal{M}}^1. \end{aligned}$$

For $\lambda, \gamma \in \mathcal{S}$ we denote by $E_\lambda[\,.\,]$ respectively $E^*_\gamma[\,.\,]$ expectation under P_λ respectively Q_γ. For extremal points $P_\lambda = P_n$ for some n we denote by $E_n[\,.\,]$ respectively $E^*_n[\,.\,]$ expectation in the complete model P_n respectively its associated complete risk-neutral model Q_n.

5.18 Lemma. *The superhedge price for an Option $V \geq 0$ is given by*

$$\sup_{Q \in \mathcal{M}} E_Q[V] = \max_{n \in \mathbb{N}} E^*_n[V].$$

There exists some $\tilde{n} \in \mathbb{N}$ that attains the maximum.

Proof. We only consider the non-trivial case $R[V > 0] > 0$. We denote

$$\sup_{Q \in \mathcal{M}} E_Q[V] =: c^*.$$

Due to Propositions 2.7 and 5.17, we have

$$c^* = \sup_{Q \in \bar{\mathcal{M}}^0} E_Q[V] \tag{5.40}$$

$$= \sup_{\gamma \in \bar{\mathcal{S}}^0} E^*_\gamma[V]. \tag{5.41}$$

Due to Proposition 5.16 and Lemma 3.3 of [KS99], there exists a maximizing parameter $\tilde{\gamma} \in \bar{\mathcal{S}}^0$ such that

$$c^* = E^*_{\tilde{\gamma}}[V]$$

holds.

If $\tilde{\gamma} \in \bar{\mathcal{S}}^0 \setminus \bar{\mathcal{S}}^1$, there exists some $1 > \delta > 0$ such that

$$\sum_{n=1}^{\infty} \tilde{\gamma}_n = 1 - \delta$$

holds. With

$$\gamma'_n := (\tilde{\gamma}_n + \frac{\delta}{2^n})$$

we have $0 \leq \gamma'_n \leq 1$ and

$$\sum_{n=1}^{\infty} \gamma'_n = 1.$$

Hence $(\gamma'_n) \in \bar{\mathcal{S}}^0$ holds and we obtain from (5.41)

$$\begin{aligned} c^* &\geq E^*_{\gamma'}[V] \\ &= \sum_{n=1}^{\infty} \gamma'_n E^*_n[V] \\ &= \sum_{n=1}^{\infty} \tilde{\gamma}_n E_n[V] + \sum_{n=1}^{\infty} \delta E_n[V] \\ &> c^* \end{aligned}$$

due to $R[V > 0] > 0$, a contradiction. Hence we have $\tilde{\gamma} \in \bar{\mathcal{S}}^1$.

We now demonstrate that $\tilde{\gamma}_n > 0$ holds for only one index $n = \tilde{n} \in \mathbb{N}$. Assume to the contrary existence of two indices k and l such that $\tilde{\gamma}_k > 0$ and $\tilde{\gamma}_l > 0$ holds. If $E^*_k[V] = E^*_l[V]$ holds, we can replace $\tilde{\gamma}$ by γ with $\gamma_l = 0$ via (5.43) and still obtain $E^*_{\tilde{\gamma}}[V] = E^*_{\gamma}[V]$. Now assume that $E^*_k[V] \neq E^*_l[V]$ holds, e.g.

$$E^*_l[V] < E^*_k[V]. \tag{5.42}$$

We define

$$\gamma_n := \begin{cases} \tilde{\gamma}_k + \tilde{\gamma}_l\,, & n = k \\ 0\,, & n = l \\ \tilde{\gamma}_n\,, & \text{else} \end{cases} \tag{5.43}$$

Observe that $(\gamma_n) \in \bar{\mathcal{S}}^0$ holds. We obtain from equation (5.42) the estimate

$$\begin{aligned} c^* &= E^*_{\tilde{\gamma}}[V] \\ &= \sum_{\substack{n=1 \\ n\neq l}}^{\infty} \tilde{\gamma}_n E^*_n[V] \; + \tilde{\gamma}_l E^*_l[V] \\ &< \sum_{\substack{n=1 \\ n\neq l}}^{\infty} \tilde{\gamma}_n E^*_n[V] \; + \tilde{\gamma}_l E^*_k[V] \\ &= \sum_{n=1}^{\infty} \gamma_n E^*_n[V] \\ &= E^*_{\gamma}[V]. \end{aligned}$$

Due to $(\gamma_n) \in \bar{\mathcal{S}}^1$, this contradicts equation (5.41). Hence $\tilde{\gamma}_n > 0$ holds only for one index $\tilde{n}$. Due to $(\tilde{\gamma}_n) \in \bar{\mathcal{S}}^1$, we conclude $\tilde{\gamma}_{\tilde{n}} = 1$ and

$$c^* = E^*_{\tilde{\gamma}}[V] = E^*_{\tilde{n}}[V].$$

Clearly, we have due to equation (5.41)

$$c^* \geq \sup_{n\in\mathbb{N}} E^*_n[V].$$

Putting the last two equations together, we obtain

$$c^* = \max_{n\in\mathbb{N}} E^*_n[V].$$

□

5.19 Lemma. *The maximal shortfall risk for given modified claim V is given by*

$$\sup_{P\in\bar{\mathcal{U}}^1} E_P[l(F-V,.)] = \max_{n\in\mathbb{N}} E_n[l(F-V,.)].$$

There exists a maximizing index $\tilde{n}\in\mathbb{N}$.

Proof. With $V' = l(F-V,.)$, we are in the situation of Lemma 5.18: Instead of maximizing the value $E^*_\gamma[V']$ over $\gamma\in\bar{\mathcal{S}}^1$, we now maximize the value $E_\gamma[V']$ over $\gamma\in\bar{\mathcal{S}}^1$. Hence the proof proceeds exactly as in Lemma 5.18. □

5.20 Theorem. *(i) Let $\tilde{V}^n$ denote the optimal modified claim in the complete model P_n given initial capital α, with the understanding that $\tilde{V}^n = F$ holds whenever $\alpha\le E^*_n[F]$. The modified claim*

$$\tilde{V} := \sum_{n=1}^{\infty} \tilde{V}_n\,\mathbf{1}_{\Omega_n}$$

is optimal for any $P\in\bar{\mathcal{U}}^1$ and maximin-optimal.

(ii) Let β_n denote the minimal shortfall risk in the complete model P_n, i.e., $\beta_n = E_n[l(F-\tilde{V}^n,.)]$. Then the robust minimal risk for $\bar{\mathcal{U}}^1$ is given by

$$\beta^* = \max_{n\in\mathbb{N}} \beta_n$$

and there exists a maximizing index $\tilde{n}$.

(iii) For $\tilde{V}$ defined via (i) and $\tilde{n}$ defined via (ii), the pair $(\tilde{V}, P_{\tilde{n}})$ is a saddle point, i.e.,

$$E_P[l(F-\tilde{V},.)] \le E_{\tilde{n}}[l(F-\tilde{V})] \le E_{\tilde{n}}[l(F-V)]$$

holds for all $P\in\bar{\mathcal{U}}^1$ and all $V\in\mathcal{V}_\alpha$.

(iv) $(P_{\tilde{n}}|Q_{\tilde{n}})$ is a least-favorable pair.

(v) Let $\tilde{\xi}^n = (\tilde{\xi}^n_s)$ denote the efficient hedging strategy for F in the complete model P_n given initial capital α.
The strategy $\tilde{\xi}$ defined via

$$\tilde{\xi}_s := \sum_{n=1}^{\infty} \tilde{\xi}^n_s\,\mathbf{1}_{\Omega_n}$$

is robust-efficient for $\bar{\mathcal{U}}^1$ given initial capital α.

Proof. Item (i) follows from Proposition 5.14.

Item (ii) is a direct consequence of (i) and Lemma 5.19.

Item (iii) follows from (i) and (ii).

(iv) $P_{\tilde{n}}$ is a worst-case model due to (iii). Now clearly, $Q_{\tilde{n}}$ is a worst-case pricing rule for $P_{\tilde{n}}$. Hence $(P_{\tilde{n}}|Q_{\tilde{n}})$ is a least-favorable pair.

(v) We first show that the strategy $\tilde{\xi}$ is in $\mathcal{A}_\alpha$: Since $\Omega_n \in \mathcal{F}_0$ holds for each n, the strategy $\tilde{\xi}$ is predictable. The following equations hold R-almost surely:

$$\begin{aligned}\alpha + \int_0^\tau \tilde{\xi}_s dX_s &= \sum_{n=1}^\infty \left(\alpha + \int_0^\tau \tilde{\xi}_s dX_s)\right) \mathbf{1}_{\Omega_n} \\ &= \sum_{n=1}^\infty \left(\alpha\, \mathbf{1}_{\Omega_n} + \int_0^\tau \tilde{\xi}_s^n\, \mathbf{1}_{\Omega_n} dX_s)\right)\end{aligned}$$

Each summand $\alpha\, \mathbf{1}_{\Omega_n} + \int_0^\tau \tilde{\xi}_s^n\, \mathbf{1}_{\Omega_n} dX_s$ remains nonnegative P_n-almost surely. Since each summand vanishes on $\Omega \setminus \Omega_n$, each summand remains nonnegative R-almost surely. Hence the sum remains nonnegative R-almost surely. Hence $\tilde{\xi} \in \mathcal{A}_\alpha$.

Optimality follows from (i) due to

$$\begin{aligned}\tilde{V} &= \sum_{n=1}^\infty \tilde{V}^n\, \mathbf{1}_{\Omega_n} \\ &= \sum_{n=1}^\infty \left((\alpha \wedge E_n^*[F]) + \int_0^\tau \tilde{\xi}_s^n dX_s\right) \mathbf{1}_{\Omega_n} \\ &\leq \alpha + \int_0^\tau \sum_{n=1}^\infty \tilde{\xi}_s^n\, \mathbf{1}_{\Omega_n} dX_s \\ &= \alpha + \int_0^\tau \tilde{\xi}_s dX_s\end{aligned}$$

R-almost surely. □

We now construct an example of a family $\mathcal{U}$ that contains a worst-case model $\tilde{P}$ but where the $\tilde{P}$-efficient strategy is not robust-efficient, i.e., where the optimal modified claim $\tilde{V}$ for the semi-composite problem $(\tilde{P}|\mathcal{M})$ is not maximin-optimal.

5.21 Example. For given constant drift m and extremal volatilities $\sigma_* < \sigma^*$, we consider the countable family of Black-Scholes models P_σ (cf. (3.27)) with volatility $\sigma \in [\sigma_*, \sigma^*] \cap \mathbb{Q}$. I.e., the family of models $\mathcal{U}$ is of the form

$$\mathcal{U} = \{P_n |\ n \in \mathbb{N}\} = \{P_\sigma \mid \sigma \in [\sigma_*, \sigma^*] \cap \mathbb{Q}\}.$$

The filtration $(\mathcal{F}_t)$ is given by the right-continuous filtration generated by (X_t) and a reference model R as in equation (5.28). Then $\{dP_n/dR\} \in \mathcal{F}_0$ holds, i.e., condition (5.27) is fulfilled. We denote by E_σ the associated expectation operator for a given constant volatility σ. Consider F a European call option with strike K and a loss function of the form $l(z) = (z^+)^p$. Let $\beta_\sigma(\alpha)$ denote the minimal shortfall risk under volatility $\sigma \in [\sigma_*, \sigma^*] \cap \mathbb{Q}$ given initial capital α. Due to

$$\beta_\sigma(0) = E_\sigma[(X_T - K)^p;\ X_T > K],$$

it can then easily be seen that

$$\lim_{\sigma \to \tilde{\sigma}} \beta_\sigma(0) = \beta_{\tilde{\sigma}}(0) \tag{5.44}$$

holds. Due to Theorem 5.20 (iii), there exists a worst-case model $\tilde{\sigma} \in [\sigma_*, \sigma^*] \cap \mathbb{Q}$. We have

$$\epsilon := \beta_{\tilde{\sigma}}(0) - \beta_{\tilde{\sigma}}(\alpha) > 0. \tag{5.45}$$

Due to equation (5.44), there exists a model $\sigma \in [\sigma_*, \sigma^*] \cap \mathbb{Q}$ such that

$$|\beta_\sigma(0) - \beta_{\tilde{\sigma}}(0)| < \epsilon \tag{5.46}$$

holds. Let $\tilde{V}$ denote the optimal modified claim for the semi-composite problem $(P_{\tilde{\sigma}}|\mathcal{M})$. We obtain from Lemma 5.18 that $\tilde{V}$ is simply the optimal modified claim for the simple problem $(P_{\tilde{\sigma}}|Q_{\tilde{\sigma}})$ where $Q_{\tilde{\sigma}}$ denotes the unique equivalent martingale measure for $P_{\tilde{\sigma}}$. Alternatively, this follows from Theorem 5.20. Especially, we obtain that

$$E_{\tilde{\sigma}}[l(F - \tilde{V})] = \beta_{\tilde{\sigma}}(\alpha) \tag{5.47}$$

holds. By definition, $\tilde{V}$ vanishes $\mathcal{P}_\sigma$-almost surely. We can conclude

$$\begin{aligned} E_\sigma[l(F - \tilde{V})] &= E_\sigma[l(F)] \\ &= \beta_0(0) \\ &> \beta_{\tilde{\sigma}}(\alpha) \\ &= E_{\tilde{\sigma}}[l(F - \tilde{V})] \end{aligned} \tag{5.48}$$

where the strict estimate (5.48) is due to (5.45) and (5.46), the last equation is due to (5.47). Hence $(\tilde{V}, P_{\tilde{\sigma}})$ is not a saddle point. Especially, $\tilde{V}$ is not maximin-optimal.

To summarize, we have found that $\tilde{V}$ is a solution to the semi-composite problem $(P_{\tilde{\sigma}}|\mathcal{M})$ where $P_{\tilde{\sigma}}$ is a worst-case model, but $\tilde{V}$ is not maximin-optimal.

The above mechanism does not rely on the fact that we have infinitely many models:

5.22 Example. With the notation from example 5.21, we consider as a special case the family $\{P_1, P_2\} = \{P_{\tilde{\sigma}}, P_\sigma\}$ of Black-Scholes models with constant volatility σ respectively $\tilde{\sigma}$ and $\mathcal{U}$ its convex hull. It then follows from example 5.21 that $\tilde{V}$ is a solution to the semi-composite problem $(P_{\tilde{\sigma}}|\mathcal{M})$ where $P_{\tilde{\sigma}}$ is a worst-case model for $\mathcal{U}$, but $\tilde{V}$ is not maximin-optimal. Equivalently, the efficient strategy for $\tilde{P}$ is not robust-efficient, cf. Proposition 3.7 and Corollary 3.8.

Clearly, $Q_{\tilde{\sigma}}$ is a worst-case model for $P_{\tilde{\sigma}}$. Hence we can also conclude that $(P_{\tilde{\sigma}}|Q_{\tilde{\sigma}})$ is a least-favorable pair and $\tilde{V}$ is optimal for the simple problem $(P_{\tilde{\sigma}}|Q_{\tilde{\sigma}})$ but not maximin-optimal.

CHAPTER 6

Geometric Poisson process

In this chapter, we examine (robust-) efficient strategies in the situation where the asset price follows a geometric Poisson process. We first derive the efficient strategy for a fixed model. By dynamic programming, the efficient strategy between jumps can be derived from the value function. But the value function itself is unknown. Hence we develop the auxiliary notion of k-efficient strategies - these can be computed explicitly. We approximate efficient strategies and the value function by means of k-efficient strategies.

We then examine the case where jump-intensities are uncertain. It follows from the analysis in Section 3.3.2 that we need to consider all mixtures of models in order to find a worst-case model. Under such mixtures models, jump-intensities are no longer constant but predictable intensity processes instead. Thus we derive the efficient strategy for fixed model with predictable intensities. The robust-efficient strategy is then given by the efficient strategy for a worst-case (mixture-)model.

We now outline the content of this chapter in more detail. Consider a model P such that the (discounted) price process is given by

$$X_t = x_0 e^{aN_t^+ - bN_t^-} \tag{6.1}$$

for two independent poisson processes N^+, N^- with intensities λ^+, λ^- and constants $a, b > 0$. We say that X is a *geometric Poisson process*.

The model is incomplete. Especially, it can be shown that the superhedge-price for a European call-option with payoff $(X_T - K)^+$ is given by x_0 for any maturity T and strike K. Subsequently, we examine efficient hedging strategies for a given non-state dependent loss function $l : \mathbb{R} \to \mathbb{R}_+$ and an European option whose payoff can be written as $F(X_T)$ such that

$$E[l(F(X_T))] < \infty \tag{6.2}$$

holds. By dynamic programming we shall see that the efficient strategy between jump-times can be derived easily from the *value function* β where $\beta(v, x, T - t)$ denotes the minimal risk at time t given initial capital $V_t = v$ and asset price $X_t = x$, cf. equation (6.10). We give the Bellman equation for β which can - in principle - be solved numerically, cf. equations (6.12) and (6.7). As an alternative, we provide a conceptually different approach to determine β: We show that $\beta = \lim_k \beta^k$ holds where β^k is the minimal risk that can be achieved among all strategies that are constant after the k-th jump and satisfy the capital constraint. We call the associated minimizing strategy "k-efficient". For $k = 0$, the 0-efficient strategy can

be computed directly from the optimality constraint (6.14). For $k \geq 1$, a k-efficient strategy is $k-1$-efficient after the first jump - given the new stock-price, capital and time to maturity. Hence, by iteration, the k-efficient strategy and the associated minimal risk β^k can be determined for any $k \in \mathbb{N}$, cf. equation (6.17). If we choose k sufficiently large, the performance of the k-efficient strategy itself is close to the performance of the efficient strategy, cf. Theorem 6.10 and the error-estimate (6.20).

In Section 6.3, we consider the case where there is uncertainty regarding the intensities. For two intensities λ^+, λ^- within some given bounds $[\lambda_*, \lambda^*]$ we denote by P_{λ^+,λ^-} the associated model and consider the family $\mathcal{U} = \{P_{\lambda^+,\lambda^-} \mid (\lambda^+, \lambda^-) \in [\lambda_*, \lambda^*]^2\}$. As described in Section 3.3.2, a proper enlargement containing a worst-case model is given by all mixtures $P_\nu = \int_{[\lambda_*,\lambda^*]^2} P_{\lambda^+,\lambda^-}\nu(d\lambda^+, d\lambda^-)$. We thus examine the dynamics of X under any P_ν, cf. Proposition 6.14. We then show how k-efficient hedging strategies can be applied to derive the efficient strategy and associated value β_P for any fixed model $P \in \bar{\mathcal{U}}$. Theorem 3.20 allows us to derive a worst-case model and the robust-efficient strategy for $\bar{\mathcal{U}}$, cf. Theorem 6.13.

6.1. Efficient hedging

The filtration we consider is the natural filtration for the process (X_t) defined in (6.1). The total number $N := N_t^+ + N_t^-$ of jumps up to time t is again a poisson process with intensity $\lambda := \lambda^+ + \lambda^-$. We denote by τ_n the time of the nth jump, i.e.,

$$\tau_n = \inf\{t \geq 0 \mid N_t^+ + N_t^- = n\}.$$

Clearly, we have

$$\tau_1 = \tau_1^+ \wedge \tau_1^-$$

where τ_n^+ (respectively τ_n^-) denotes the time of the n-th jump up (respectively down). In the sequel, we will frequently consider the times

$$\hat{\tau}_n := \tau_n \wedge T. \tag{6.3}$$

The process X can be described equivalently as a marked point process $((\tau_n)_{n\geq 1}, (Y_n)_{n\geq 1})$ with (simple) point process τ and mark $Y_n = X_{\tau_n}/X_{\tau_{n-1}} \in \{e^a, e^{-b}\}$. The probability that the next jump is upward is given by

$$P[X_{\tau_1} = x_0 e^a] = P[\tau_1^+ < \tau_1^-] = \frac{\lambda^+}{\lambda^+ + \lambda^-}. \tag{6.4}$$

We claim that

$$E[f(X_{\tau_1}, \tau_1)] = \int_0^\infty \Big\{\lambda^+ f(x_0 e^a, t) + \lambda^- f(x_0 e^{-b}, t)\Big\} e^{-(\lambda^+ + \lambda^-)t} dt \tag{6.5}$$

holds for $f \geq 0$. Indeed:

$$
\begin{aligned}
E[f(X_{\tau_1}, \tau_1)] &= E[f(x_0 e^a, \tau_1^+),\ \tau_1^+ < \tau_1^-] + E[f(x_0 e^{-b}, \tau_1^-),\ \tau_1^- < \tau_1^+] \\
&= E[f(x_0 e^a, \tau_1^+)\ P[\tau_1^+ < \tau_1^- \,|\, \tau_1^+]\,] + E[f(x_0 e^{-b}, \tau_1^-)\ P[\tau_1^- < \tau_1^+ \,|\, \tau_1^-]\,] \\
&= \int_0^\infty f(x_0 e^a, t)\ P[t < \tau_1^-] P[\tau_1^+ \in dt] \\
&\quad + \int_0^\infty f(x_0 e^{-b}, t)\ P[t < \tau_1^+] P[\tau_1^- \in dt] \qquad (6.6) \\
&= \int_0^\infty f(x_0 e^a, t)\ e^{-\lambda^- t} \lambda^+ e^{-\lambda^+ t} dt \\
&\quad + \int_0^\infty f(x_0 e^{-b}, t)\ e^{-\lambda^+ t} \lambda^- e^{-\lambda^- t} dt,
\end{aligned}
$$

i.e., equation (6.5) holds.

In order to derive an P-efficient hedging strategy via the methodology of Part I of this thesis, we would have to derive a worst-case pricing rule. We can conclude from Corollary 3.16 that if the intrinsic value of F is nontrivial, i.e., $F(x_0) > 0$, and we invest more than the intrinsic value, then the worst-case pricing rule is not equivalent to P:

6.1 Corollary. *If $0 < F(x_0) < \alpha$ holds, the worst-case pricing rule for F and α is not equivalent to P.*

Proof. Let $\tilde{\xi} \in \mathcal{A}_\alpha$ denote the efficient strategy for F given initial capital α. On the event $\tau_1 > T$, we have $\alpha + \int_0^T \tilde{\xi}_t dX_t = \alpha$ and $F(X_T) = F(x_0) > 0$. Hence

$$
\begin{aligned}
P[\alpha + \int_0^T \tilde{\xi}_t dX_t \geq F(X_T), F(X_T) > 0] &\geq P[\tau_1 \geq T] \\
&> 0.
\end{aligned}
$$

Now the assertion follows from Corollary 3.16. □

Due to the complexity of the set of equivalent martingale measures, we prefer to derive the efficient strategy more directly via dynamic programming methods rather than via a worst-case pricing rule. One can then derive the structure of the worst-case pricing rule a posteriori from Proposition 3.15 (i) and (ii).

Let $\beta(v, x, T - t)$ denote the minimal risk at time t given initial capital $V_t = v$ and asset price $X_t = x$, i.e., $\beta(\alpha, x_0, T) = \beta^*$ and

$$
\beta(v, x, 0) = l(F(x) - v). \qquad (6.7)
$$

The well known principle of dynamic programming is the cornerstone of the analysis in this chapter:

6.2 Lemma (Dynamic Programming Principle). *For any stopping time $\tau \leq T$ we have*

$$\beta^* = \min_{\xi \in \mathcal{A}_\alpha} E[\beta(\alpha + \int_0^\tau \xi_s dX_s, X_\tau, T - \tau)] \tag{6.8}$$

and the following strategy is efficient:

(i) On $[0, \tau]$: Use the strategy (ξ_t^*) that solves (6.8). This strategy is called efficient until time τ.

(ii) On $(\tau, T]$: Establish an efficient hedging strategy for F with time to maturity $T - \tau$, current asset price X_τ and initial capital $V_\tau = \alpha + \int_0^\tau \xi_s^* dX_s$.

Proof. The strategy $\tilde{\xi}$ prescribed by (i) and (ii) is admissible by definition.

For proof of optimality, consider any admissible strategy $\xi \in \mathcal{A}_\alpha$ with associated value process V. We obtain from the definition of β validity of

$$E[l(F - V_\tau - \int_\tau^T \xi_s dX_s) | \, X_\tau, \tau] \geq \beta(\alpha + \int_0^\tau \xi_s dX_s, X_\tau, T - \tau)$$

which implies

$$\begin{aligned} E[l(F - \alpha - \int_0^T \xi_s dX_s)] &= E[l(F - V_\tau - \int_\tau^T \xi_s dX_s)] \\ &= E[E[l(F - V_\tau - \int_\tau^T \xi_s dX_s) | \, X_\tau, \tau] \,] \\ &\geq E[\beta(\alpha + \int_0^\tau \xi_s dX_s, X_\tau, T - \tau)] \end{aligned}$$

Hence

$$\beta^* \geq \min_{\xi \in \mathcal{A}_\alpha} E[\beta(\alpha + \int_0^\tau \xi_s dX_s, X_\tau, T - \tau)]$$

holds. If we repeat the above calculation for the strategy prescribed by (i) and (ii), it is easily seen that this strategy actually achieves this bound. □

6.3 Corollary. *A strategy ξ with associated value process V is efficient if it solves the problem*

$$\left[\begin{array}{c} E[\beta(V_{\hat{\tau}_{n-1}} + \xi_{\hat{\tau}_n}(X_{\hat{\tau}_n} - X_{\hat{\tau}_{n-1}}), X_{\hat{\tau}_n}, T - \hat{\tau}_n) \mid V_{\hat{\tau}_{n-1}}, X_{\hat{\tau}_{n-1}}, \hat{\tau}_{n-1}] = \min\limits_{\xi} \\ \xi \in \mathcal{A}_\alpha \end{array} \right] \tag{6.9}$$

for each $n \in \mathbb{N}$ and the series of jump-times $\hat{\tau}_n$ defined in (6.3).

Proof. This follows immediately from the iterative application of Lemma 6.2 to the jump-times $\hat{\tau}_n$, $n \in \mathbb{N}$. □

The following auxiliary problem is the simple problem of minimizing a real-valued function over an interval:

$$\left[\begin{array}{rcl} & \lambda^+\beta(v+\zeta x(e^a-1), xe^a, T-t) & \\ + & \lambda^-\beta(v+\zeta x(e^{-b}-1), xe^{-b}, T-t) & = \min\limits_{\zeta} \\ & \zeta \in \left[-\dfrac{v}{x(e^a-1)}, \dfrac{v}{x(1-e^{-b})}\right] & \end{array}\right] \tag{6.10}$$

for each $t \in [0,T]$. We use the convention $\tau_0 := 0$. Due to Corollary 6.3 and equation 6.5, we can reduce the problem of efficient hedging to problem (6.10):

6.4 Theorem. *Let $\tilde{\zeta}(v,x,t) \in \mathbb{R}$ denote a solution to problem (6.10).*

The following strategy $(\tilde{\xi}_t)$ is efficient: If the last jump has occurred at time $\tau_{n-1} < T$ for some $n \in \mathbb{N}$, then use until the next jump τ_n the deterministic strategy $\tilde{\xi}_t = \tilde{\zeta}(v,x,t-\tau_{n-1})$ given $x := X_{\tau_{n-1}}$ the asset price after the last jump and $v := \tilde{V}_{\tau_{n-1}}$ the value of the portfolio after the last jump, i.e.,

$$\tilde{\xi}_t = \sum_{n=1}^{\infty} \tilde{\zeta}(\tilde{V}_{\hat{\tau}_{n-1}}, X_{\hat{\tau}_{n-1}}, t-\hat{\tau}_{n-1})\, \mathbf{1}_{(\hat{\tau}_{n-1},\hat{\tau}_n]}(t). \tag{6.11}$$

Proof. (1) We first consider admissibility of the strategy: The strategy $(\tilde{\xi}_t)_{0\le t\le T}$ is predictable due to equation (6.11).

By definition, $\tilde{V}_{\tau_0} = \alpha$ is nonnegative. Given some $t \in (0,T]$ and state $\omega \in \Omega$, there exists $n \in \mathbb{N}$ such that $\tau_{n-1}(\omega) < t \le \tau_n(\omega)$ holds. We define x and $v \ge 0$ as in the theorem.

The value associated to the strategy at time t is

$$\begin{aligned} \tilde{V}_t(\omega) &= v + \left(\int_{\tau_{n-1}}^{\tau_n} \tilde{\zeta}(v,x,s-\tau_{n-1})dX_s\right)(\omega) \\ &= \begin{cases} v & \text{if } t < \tau_n(\omega) \\ v+\tilde{\zeta}(v,x,t-\tau_{n-1})x(e^a-1) & \text{if } t=\tau_n(\omega) \text{ and } \frac{X_{\tau_n}(\omega)}{x} = e^a \\ v+\tilde{\zeta}(v,x,t-\tau_{n-1})x(e^{-b}-1) & \text{if } t=\tau_n(\omega) \text{ and } \frac{X_{\tau_n}(\omega)}{x} = e^{-b} \end{cases} \end{aligned}$$

This is nonnegative if and only if

$$\tilde{\zeta}(v,x,t-\tau_{n-1}) \in \left[-\frac{v}{x(e^a-1)}, \frac{v}{x(1-e^{-b})}\right]$$

holds. Hence $\tilde{\xi}$ is admissible by induction.

(2) We next show that the strategy $\tilde{\xi}$ solves problem (6.9) for each $n \in \mathbb{N}$. For $n \in \mathbb{N}$, $x := X_{\tau_{n-1}}$ and $v := \tilde{V}_{\tau_{n-1}}$, any admissible strategy $\xi \in \mathcal{A}_\alpha$ satisfies the side-condition of problem (6.10):

$$\xi_t \in \left[-\frac{v}{x(e^a-1)}, \frac{v}{x(1-e^{-b})}\right], \quad t \in (\hat{\tau}_{n-1}, \hat{\tau}_n],$$

cf. (1). Hence we have by equation (6.5) and definition of $\tilde{\zeta}$:

$$\begin{aligned}
& E[\beta(v+\xi_{\hat{\tau}_n}(X_{\hat{\tau}_n}-x), X_{\hat{\tau}_n}, T-\hat{\tau}_n) \mid V_{\hat{\tau}_{n-1}}=v, X_{\hat{\tau}_{n-1}}=x, \hat{\tau}_{n-1}] \\
= & \int_{\hat{\tau}_{n-1}}^{T} e^{-(\lambda^+ + \lambda^-)t}\Big\{\lambda^+\beta(v+\xi_t x(e^a - 1), xe^a, T-t) \\
& +\lambda^-\beta(v+\xi_t x(e^{-b}-1), xe^{-b}, T-t)\Big\}dt \\
& +e^{-(\lambda^+ + \lambda^-)(T-\hat{\tau}_{n-1})} l(F(x)-v) \\
\geq & \int_{\hat{\tau}_{n-1}}^{T} e^{-(\lambda^+ + \lambda^-)t}\Big\{\lambda^+\beta(v+\tilde{\zeta}(v,x,t)x(e^a - 1), xe^a, T-t) \\
& +\lambda^-\beta(v+\tilde{\zeta}(v,x,t)x(e^{-b}-1), xe^{-b}, T-t)\Big\}dt \\
& +e^{-(\lambda^+ + \lambda^-)(T-\hat{\tau}_{n-1})} l(F(x)-v) \\
= & E[\beta(v+\tilde{\xi}_{\hat{\tau}_n}(X_{\hat{\tau}_n}-x), X_{\hat{\tau}_n}, T-\hat{\tau}_n) \mid V_{\hat{\tau}_{n-1}}=v, X_{\hat{\tau}_{n-1}}=x, \hat{\tau}_{n-1}].
\end{aligned}$$

Thus, $\tilde{\xi}$ solves (6.9). □

We now derive the Bellman equation for β. For given strategy ξ, the controlled Markov process $(\int_0^t \xi_s dX_s, X_t)$ has generator

$$\begin{aligned}
(\Lambda^\xi f)(v,x) &:= \lim_{t\downarrow 0}(E_{v,x}[f(V_t^\xi, X_t)] - f(v,x)) \\
&= \lambda^+ f(v+\xi x(e^a-1), xe^a) \\
&\quad +\lambda^- f(v+\xi x(e^{-b}-1), xe^{-b}) \\
&\quad -(\lambda^+ + \lambda^-) f(v,x)
\end{aligned}$$

The Bellman equation for β is given by

$$\partial_t \beta(v,x,T-t) + \min_\xi (\Lambda^\xi \beta)(v,x,T-t) = 0$$

respectively

$$\begin{aligned}
0 &= \partial_t \beta(v,x,T-t) + \min_\xi \{\lambda^+\beta(v+\xi x(e^a-1), xe^a, T-t) \\
&\quad +\lambda^-\beta(v+\xi x(e^{-b}-1), xe^{-b}, T-t)\} \\
&\quad -(\lambda^+ + \lambda^-)\beta(v,x,T-t)
\end{aligned} \tag{6.12}$$

where the minimum has to be taken over the interval in (6.10). The terminal condition is given by equation (6.7). Observe that the Bellman equation implies formula (6.10) for the optimal choice of ξ_t.

6.2. k-efficient hedging

In order to find a more comfortable approximation of β and efficient strategies, we introduce k-efficient strategies:

6.5 Definition. We denote by $\mathcal{A}_\alpha^k$ the class of all admissible strategies $\xi \in \mathcal{A}_\alpha$ that are constant from time τ_k on, i.e.,

$$\mathcal{A}_\alpha^k = \{\xi \in \mathcal{A}_\alpha \mid \xi_t = \xi_{\hat{\tau}_k+}, \quad t \in (\hat{\tau}_k, T]\}.$$

A strategy is called *k-efficient*, if it is most efficient of all strategies that are constant from time τ_k on, i.e., if it solves the problem

$$\left[\begin{array}{c} E[l(F(X_T) - \alpha - \int_0^T \xi_s dX_s)] = \min\limits_{\xi} \\ \xi \in \mathcal{A}_\alpha^k \end{array}\right]. \tag{6.13}$$

We denote the value of this problem by $\beta^k(\alpha, x_0, T)$.

6.6 Proposition. *A strategy that is constant from time τ_k on is admissible if and only if it is predictable and satisfies*

(i) $\xi_s \in [-\frac{V_{\tau_{n-1}}}{X_{\tau_{n-1}}(e^a - 1)}, \frac{V_{\tau_{n-1}}}{X_{\tau_{n-1}}(1 - e^{-b})}]$ *for* $\tau_{n-1} < s \leq \tau_n$, $n = 1, \ldots, k$ *and*

(ii) $\xi_{\tau_k+} \in [0, V_{\tau_k}/X_{\tau_k}]$ *where* ξ_{τ_k+} *denotes the constant value of the strategy from* τ_k *on.*

Proof. It follows as in the proof of Theorem 6.4 that (i) is necessary and sufficient for admissibility up to time τ_k.

Clearly, (ii) is a sufficient condition for admissibility. It is also necessary: Given $\tau_k < T$, there is a non-trivial probability that the asset price X_T attains arbitrary large or small positive values. Since the strategy is to be held constant, it is therefore not possible to go short the asset (i.e., to choose $\xi_{\tau_k+} < 0$) or to borrow against the asset (i.e., to choose $\xi_{\tau_k+} > V_{\tau_k}/X_{\tau_k}$) if V_T is to remain nonnegative. □

We introduce the abbreviation

$$\begin{aligned} p_{m,n,T} &:= P[N_T^+ = m, N_T^- = n] \\ &= \frac{1}{m!n!} e^{-(\lambda^+ + \lambda^-)T} (\lambda^+ T)^m (\lambda^- T)^n. \end{aligned}$$

6.7 Proposition. *The 0-efficient minimal risk β^0 is given by*

$$\beta^0(\alpha, x_0, T) = \min_{\zeta \in [0, \alpha/x_0]} \sum_{m,n=0}^{\infty} l\big((F(x_0 e^{am-bn}) - \alpha - \zeta x_0 (e^{am-bn} - 1))\big) p_{m,n,T}. \tag{6.14}$$

and the minimizing value $\tilde{\zeta}^0(\alpha, x_0, T) \in [0, \alpha/x_0]$ is the optimal constant value $\tilde{\xi}^0$ for the 0-efficient strategy for given initial capital α, asset price x_0 and time to maturity T.

Proof. Consider any strategy ξ that is constant from time 0 on, i.e., $\xi_t = \xi_0$ for $t \in [0, T]$. We can compute the expectation

(6.15)
$$E[l(F(X_T) - \alpha - \int_0^T \xi dX_s)] = \sum_{m,n=0}^{\infty} l\big((F(x_0 e^{am-bn}) - \alpha - \xi x_0(e^{am-bn} - 1))\big) p_{m,n,s}.$$

Due to Proposition 6.6 (ii), the value given by equation (6.14) is a lower bound for the expression given in (6.15). The strategy prescribed in the Proposition actually achieves this bound and it is admissible and constant from time 0 on. Hence it is 0-efficient. □

If we replace β by β^k, problem (6.10) reads

$$\left[\begin{array}{rcl} & \lambda^+\beta^k(v + \zeta x(e^a - 1), xe^a, T - t) & \\ + & \lambda^-\beta^k(v + \zeta x(e^{-b} - 1), xe^{-b}, T - t) & = \min\limits_{\zeta} \\ & \zeta \in \left[-\dfrac{v}{x(e^a - 1)}, \dfrac{v}{x(1 - e^{-b})}\right] & \end{array}\right] \tag{6.16}$$

6.8 Theorem. *Let $\tilde{\zeta}^{k+1}(v, x, t)$ denote a solution to problem (6.16). The following strategy is $(k+1)$-efficient:*

(i) Up to the first jump time τ_1, use $\tilde{\xi}_t := \tilde{\zeta}^{k+1}(\alpha, x_0, t)$.

(ii) After time τ_1, establish a k-efficient strategy given the asset price after the last jump X_{τ_1}, time to maturity $T - \tau_1$ and $\tilde{V}_{\tau_1}$ the value of the portfolio after the last jump.

The associated minimal risk is given by

$$\begin{aligned} \beta^{k+1}(\alpha, x_0, T) \;=\; & \int_0^T e^{-(\lambda^+ + \lambda^-)t}\Big\{\lambda^+\beta^k(\alpha + \tilde{\zeta}^{k+1}(\alpha, x_0, t)x_0(e^a - 1), x_0 e^a, T - t) \\ & + \lambda^-\beta^k(\alpha + \tilde{\zeta}^{k+1}(\alpha, x_0, t)x_0(e^{-b} - 1), x_0 e^{-b}, T - t)\Big\} dt \\ & + e^{-(\lambda^+ + \lambda^-)T} l(F(x_0) - \alpha). \end{aligned} \tag{6.17}$$

for given initial capital α, asset price x_0 and time to maturity T.

Proof.

(1) It follows as in Lemma 6.2 that the following strategy is $(k+1)$-efficient:

(i) On $[0, \hat{\tau}_1]$: Use the strategy (ξ_t^*) that solves

$$\min_{\xi \in \mathcal{A}_\alpha} E[\beta^k(\alpha + \int_0^{\hat{\tau}_1} \xi_s dX_s, X_{\hat{\tau}_1}, T - \hat{\tau}_1)] \tag{6.18}$$

(ii) On $(\hat{\tau}_1, T]$: Establish a k-efficient hedging strategy for F with time to maturity $T - \hat{\tau}$ given asset price $X_{\hat{\tau}}$ and initial capital $V_{\hat{\tau}} = \alpha + \int_0^{\hat{\tau}} \xi_s^* dX_s$.

(2) It follows as in Theorem 6.4 that the strategy

$$\tilde{\xi}_t := \tilde{\zeta}^{k+1}(\alpha, x_0, t), \quad 0 \le t \le \hat{\tau}_1$$

is a solution to problem (6.18).

(3) From (1) and (2) we conclude that the strategy prescribed in the theorem is $(k+1)$-efficient. As a consequence, equation (6.17) holds. □

6.9 Corollary. *The following strategy is k-efficient:*

$$\tilde{\xi}_t^k = \sum_{n=0}^{k-1} \tilde{\zeta}^{k-n}(\tilde{V}_{\hat{\tau}_n}, X_{\hat{\tau}_n}, t - \hat{\tau}_n)\, \mathbf{1}_{(\hat{\tau}_n, \hat{\tau}_{n+1}]}(t) + \tilde{\zeta}^0(\tilde{V}_{\hat{\tau}_k}, X_{\hat{\tau}_k}, t - \hat{\tau}_k)\, \mathbf{1}_{(\hat{\tau}_k, T]}(t) \tag{6.19}$$

where $\tilde{\zeta}^k$ and $\tilde{\zeta}^0$ were defined in Theorem 6.8 and Proposition 6.7, respectively.

Proof.

(1) Due to Theorem 6.8 and Proposition 6.7, the assertion holds true for $k = 1$ and $k = 0$.

(2) Now assume the assertion is true for some $k - 1 \in \mathbb{N}$. We show that it holds for k: By definition, the strategy $\tilde{\xi}^k$ satisfies the condition (i) of Theorem 6.8. Since the assertion is true for $k - 1$, the strategy satisfies the condition (ii) of Theorem 6.8, too. Hence the strategy $\tilde{\xi}^k$ is (k)-efficient. □

The following theorem (in combination with Theorem 6.8) is the fundamental result of this chapter:

6.10 Theorem. *For any initial capital α, asset price x_0 and time to maturity T, we have*

(i) $\lim_{k\uparrow\infty} \beta^k(\alpha, x_0, T) = \beta(\alpha, x_0, T) = \beta^*$ *and the error can be estimated by*

$$0 \le \beta^k(\alpha, x_0, T) - \beta(\alpha, x_0, T) \le E[l(F(x_0 e^{aN_T^+ - bN_T^-})), \tau_{k+1} \le T]. \tag{6.20}$$

(ii) $\liminf_{k\uparrow\infty} \tilde{V}^k = \tilde{V}$ *holds P-almost surely where we have set*

$$\begin{aligned} \tilde{V}^k &= \Big(\alpha + \int_0^T \tilde{\xi}_s^k dX_s\Big) \wedge F \\ \tilde{V} &= \Big(\alpha + \int_0^T \tilde{\xi}_s dX_s\Big) \wedge F \end{aligned}$$

for an efficient strategy $\tilde{\xi}$ and k-efficient strategies $\tilde{\xi}^k$.

In order to prove this theorem, we consider for $k = 0, 1, \ldots$ a second notion of k-optimal strategies, namely solutions to the problem

$$\hat{\beta}^k(\alpha, x_0, T) := \min_{\xi \in \mathcal{A}_\alpha} E[l(F(X_T) - \alpha - \int_0^T \xi_s dX_s),\ \tau_{k+1} \ge T] \tag{6.21}$$

where $E[Y, \tau_{k+1} \geq T]$ is shorthand for $E[Y\, \mathbf{1}_{\{\tau_{k+1} \geq T\}}]$.

6.11 Proposition. *Let $\hat{\zeta}^{k+1}(v, x, t)$ denote a solution to problem (6.16) where we replace β by $\hat{\beta}$. The dynamic programming equations for $\hat{\beta}$ are*

$$\begin{aligned}
\hat{\beta}^0(\alpha, x_0, T) &= e^{-(\lambda^+ + \lambda^-)T} l(F(x_0) - \alpha) \\
\hat{\beta}^{k+1}(\alpha, x_0, T) &= \int_0^T e^{-(\lambda^+ + \lambda^-)t} \Big\{ \lambda^+ \hat{\beta}^k(\alpha + \hat{\zeta}^{k+1}(\alpha, x_0, t) x_0 (e^a - 1), x_0 e^a, T - t) \\
&\quad + \lambda^- \hat{\beta}^k(\alpha + \hat{\zeta}^{k+1}(\alpha, x_0, t) x_0 (e^{-b} - 1), x_0 e^{-b}, T - t) \Big\} dt \\
&\quad + e^{-(\lambda^+ + \lambda^-)T} l(F(x_0) - \alpha).
\end{aligned}$$

Proof. The first equation is immediate. The proof of the second equation is analogous to the proof of Theorem (6.8). □

6.12 Proposition. *For any $k = 0, 1, \ldots$, initial capital α, asset price x_0 and time to maturity T, we have*

$$\hat{\beta}^k(\alpha, x_0, T) \leq \beta(\alpha, x_0, T) \tag{6.22}$$

$$\beta(\alpha, x_0, T) \leq \beta^k(\alpha, x_0, T) \tag{6.23}$$

$$\beta^k(\alpha, x_0, T) \leq \hat{\beta}^k(\alpha, x_0, T) + E[l(F(x_0 e^{aN_T^+ - bN_T^-})), \tau_{k+1} \leq T]. \tag{6.24}$$

Proof. Only the last equation needs proof. We denote $N_T := aN_T^+ - bN_T^-$.

(1) For $k = 0$, we obtain from equation (6.14)

$$\begin{aligned}
\beta^0(\alpha, x_0, T) &= \hat{\beta}^0(\alpha, x_0, T) \\
&\quad + \min_{\zeta \in [0, \alpha/x_0]} \sum_{m,n=1}^{\infty} l\big((F(x_0 e^{am-bn}) - \alpha - \zeta x_0 (e^{am-bn} - 1))\big) p_{m,n,T} \\
&\leq \hat{\beta}^0(\alpha, x_0, T) \\
&\quad + \sum_{m,n=1}^{\infty} l\big((F(x_0 e^{am-bn})\big) p_{m,n,T} \\
&= \hat{\beta}^0(\alpha, x_0, T) + E[l(F(x_0 e^{N_T})), \tau_1 \leq T].
\end{aligned}$$

(2) Now assume equation (6.24) holds for some $k \geq 0$. With the notation

$$I(v, x) := \left[-\frac{v}{x(e^a - 1)}, \frac{v}{x(1 - e^{-b})} \right]$$

we obtain from equation (6.17) and (6.24) for k the estimate

$$\begin{aligned}
\beta^{k+1}(\alpha,x_0,T) &= \int_0^T e^{-(\lambda^++\lambda^-)t} \min_{\zeta\in I(\alpha,x_0)} \Big\{\lambda^+\beta^k(\alpha+\zeta(\alpha,x_0,t)x_0(e^a-1),x_0e^a,T-t) \\
&\quad +\lambda^-\beta^k(\alpha+\zeta(\alpha,x_0,t)x_0(e^{-b}-1),x_0e^{-b},T-t)\Big\}dt \\
&\quad +e^{-(\lambda^++\lambda^-)T}l(F(x_0)-\alpha) \\
&\le \int_0^T e^{-(\lambda^++\lambda^-)t} \min_{\zeta\in I(\alpha,x_0)} \Big\{\lambda^+\hat\beta^k(\alpha+\zeta(\alpha,x_0,t)x_0(e^a-1),x_0e^a,T-t) \\
&\quad +\lambda^-\hat\beta^k(\alpha+\zeta(\alpha,x_0,t)x_0(e^{-b}-1),x_0e^{-b},T-t)\Big\}dt \\
&\quad +e^{-(\lambda^++\lambda^-)T}l(F(x_0)-\alpha) \\
&\quad +\int_0^T e^{-(\lambda^++\lambda^-)t}\Big\{\lambda^+E[l(F(x_0e^ae^{N_{T-t}})),\tau_{k+1}\le T-t] \\
&\quad +\lambda^-E[l(F(x_0e^{-b}e^{N_{T-t}})),\tau_{k+1}\le T-t]\Big\}dt \\
&= \hat\beta^{k+1}(\alpha,x_0,T)+E[l(F(x_0e^{N_T})),\tau_{k+2}\le T]
\end{aligned}$$

where the last equality is due to Proposition 6.11. □

We now turn to the **Proof** of Theorem 6.10.

The limit

$$P[\sup_{n\ge k}\left\{l(F(x_0e^{aN_T^+-bN_T^-}))\,\mathbf{1}_{\{\tau_{n+1}\le T\}}\right\}>\epsilon]\le P[\tau_{k+1}\le T]\to 0,\quad k\uparrow\infty$$

vanishes for any $\epsilon\ge 0$. Together with equation (6.2), this implies

$$\lim_{k\uparrow\infty}E[l(F(x_0e^{aN_T^+-bN_T^-})),\tau_{k+1}\le T]=0.$$

We can conclude from this equation and equations (6.22) -(6.24) that both

$$\begin{aligned}
\beta(\alpha,x_0,T) &= \lim_{k\uparrow\infty}\beta^k(\alpha,x_0,T), \\
&= \lim_{k\uparrow\infty}\hat\beta^k(\alpha,x_0,T)
\end{aligned}$$

hold. Estimate (6.20) is a consequence of equations (6.22) -(6.24).

It remains to prove (ii). Due to Corollary 3.8, $\tilde V$ is the P-almost surely unique optimal modified claim $\le F$ for the associated stationary problem $(P|\mathcal{M})$ (1.16). It follows from Fatou's Lemma that

$$V':=\liminf_{k\uparrow\infty}\tilde V^k$$

satisfies the side condition

$$\sup_{Q\in\mathcal{M}}E_Q[V']\le\alpha$$

for this problem. The value of this problem is $\beta(\alpha, x_0, T)$. Hence we obtain

$$\begin{aligned}\beta(\alpha, x_0, T) &\leq E[l(F - V')] \\ &\leq \lim_{k\uparrow\infty} E[l(F - \tilde{V}^k)] \\ &= \beta(\alpha, x_0, T)\end{aligned}$$

where the last equation is due to (i). This proves optimality of V' for $(P|\mathcal{M})$ which implies $V' = \tilde{V}$ since both claims are dominated by F. □

6.3. Uncertain intensities

In this section we consider the case where there is uncertainty regarding the intensities λ^+, λ^-. Let P_{λ^+,λ^-} denote the associated probability measure and consider $\mathcal{U} = \{P_{\lambda^+,\lambda^-} \mid (\lambda^+, \lambda^-) \in [\lambda_*, \lambda^*]^2\}$. As described in Section 3.3.2, a proper enlargement $\bar{\mathcal{U}}$ containing a worst-case model is given by all mixtures $P_\nu = \int_{[\lambda_*,\lambda^*]^2} P_{\lambda^+,\lambda^-}\nu(d\lambda^+, \lambda^-)$. Under P_ν, the driving processes N^+ and N^- are Poisson-processes with predictable intensity process, cf. Proposition 6.14. We then show how the approach of the last section can be applied to derive the efficient strategy and associated value β_P for any fixed model $P \in \bar{\mathcal{U}}$. Theorem 3.20 allows us to derive a worst-case model and the robust-efficient strategy for $\bar{\mathcal{U}}$, cf. Theorem 6.13.

We start by a more concrete construction of the underlying probability space. In accordance with the notation in [Jac99], let $(W, \mathcal{H})$ denote the canonical counting process path space up to maturity T and P_λ the measure on $(W, \mathcal{H})$ such that the coordinate process is a homogenous Poisson process with intensity λ under P_λ. We set

$$\begin{aligned}\Omega &:= W \times W = \{(\omega^+, \omega^-) \in W \times W\} \\ P_{\lambda^+,\lambda^-} &:= P_{\lambda^+} \times P_{\lambda^-} \\ N_t^+(\omega^+, \omega^-) &:= \omega_t^+ \\ N_t^-(\omega^+, \omega^-) &:= \omega_t^-.\end{aligned}$$

N^+ and N^- are independent homogenous Poisson processes with intensity λ^+ respectively λ^- under P_{λ^+,λ^-}.

Consider

$$\mathcal{U} = \left\{P_{\lambda^+,\lambda^-} \mid (\lambda^+, \lambda^-) \in [\lambda_*, \lambda^*]^2\right\}.$$

The reference measure $R := P_{1,1}$ is equivalent to $\mathcal{U}$ with densities

$$G_{\lambda^+,\lambda^-} := \frac{dP_{\lambda^+,\lambda^-}}{dP_{1,1}} = e^{-(\lambda^+ + \lambda^- - 2)T}(\lambda^+)^{N_T^+}(\lambda^-)^{N_T^-}.$$

We now show that the robust-efficient strategy can be derived via a worst-case mixture-model as in Section 3.3.2. For this, we must pass from $\mathcal{U}$ to the proper

enlargement $\bar{\mathcal{U}}$ defined via

$$\begin{aligned} P_\nu &:= \int_{[\lambda_*,\lambda^*]^2} P_{\lambda^+,\lambda^-}\nu(d\lambda^+, d\lambda^-), \quad \nu \in m_1([\lambda_*, \lambda^*]^2) \\ \bar{\mathcal{U}} &:= \{P_\nu \mid \nu \in m_1([\lambda_*, \lambda^*]^2)\} \end{aligned}$$

Theorem 3.20 yields existence of a worst-case model in $\bar{\mathcal{U}}$:

6.13 Theorem. *Let β_ν denote the minimal risk for fixed model $P_\nu \in \bar{\mathcal{U}}$ and assume $l(F, .)$ is bounded. Then there is a solution $\tilde{\nu}$ to*

$$\beta_{\tilde{\nu}} = \max_{\nu \in m_1([\lambda_*,\lambda^*]^2)} \beta_\nu.$$

The associated model $P_{\tilde{\nu}}$ is a worst-case model and the efficient strategy $\tilde{\xi}^{\tilde{\nu}}$ for $P_{\tilde{\nu}}$ is robust-efficient for $\mathcal{U}$.

Proof. Clearly, the family $\mathcal{G} = \{G_{\lambda^+,\lambda^-} \mid (\lambda^+, \lambda^-) \in [\lambda_*, \lambda^*]^2\}$ satisfies the assumptions of Theorem 3.19 and Theorem 3.20. Hence the assertion follows from Theorem 3.20. □

In the reminder of this section, we generalize the methods of Section 6.1 to derive the efficient strategy and minimal risk for fixed model P_ν. Unfortunately, N^+ and N^- are no longer Poisson-processes with constant intensities under P_ν:

6.14 Proposition. *Under the measure P_ν, N^+ and N^- are conditionally independent Poisson processes with predictable intensity processes $(\lambda_t^{+,\nu})_{0\le t\le T}$ (respectively $(\lambda_t^{-,\nu})_{0\le t\le T}$) given by*

$$\lambda_t^{+,\nu} = \frac{\int e^{-(\lambda^+-1)t}(\lambda^+)^{N_{t-}^+ +1}\nu^+(d\lambda^+)}{\int e^{-(\lambda^+-1)t}(\lambda^+)^{N_{t-}^+}\nu^+(d\lambda^+)}. \tag{6.25}$$

Here, ν^+ denotes the one-dimensional margin-distribution of ν on the coordinate λ^+.

Proof. This is the content of Example 3.7.1 in [Jac99]. Under P_ν, N^+ (respectively N^-) is also called a *Cox process.* □

Due to (6.25), the expectation $E_\nu[f(X_{\tau_1}, \tau_1)]$ cannot be expressed in a formula as simple as (6.5). However, we obtain as in (6.5) - (6.6) the expression

$$\begin{aligned} E_\nu[f(X_{\tau_1}, \tau_1)] &= \int_0^\infty f(x_0e^a, t)\ P_\nu[t < \tau_1^- \mid \tau_1^+ = t]P_\nu[\tau_1^+ \in dt] \\ &\quad + \int_0^\infty f(x_0e^{-b}, t)\ P_\nu[t < \tau_1^+ \mid \tau_1^- = t]P_\nu[\tau_1^- \in dt]. \end{aligned} \tag{6.26}$$

We have

$$P_\nu[\tau_1 \le t] = \int_0^t P_\nu[t < \tau_1^- \mid \tau_1^+ = t]P_\nu[\tau_1^+ \in dt] + \int_0^t P_\nu[t < \tau_1^+ \mid \tau_1^- = t]P_\nu[\tau_1^- \in dt],$$

i.e., the measure $\mu^{-,+,\nu}$ given by

$$\mu^{-,+,\nu}[0,t] := \int_0^t P_\nu[t < \tau_1^- \mid \tau_1^+ = t] P_\nu[\tau_1^+ \in dt]$$

is absolutely continuous with respect to the measure $P_\nu[\tau_1 \in dt]$. Let

$$g^{-,+,\nu}(t) := \frac{d\mu^{-,+,\nu}}{dP_\nu[\tau_1 \in \ .]} \tag{6.27}$$

denote the associated Radon-Nikodym derivative. We obtain from equation (6.26) the expression

$$E_\nu[f(X_{\tau_1}, \tau_1)] = \int_0^\infty \Big[g^{-,+,\nu}(t) f(x_0 e^a, t) + g^{+,-,\nu}(t) f(x_0 e^{-b}, t) \Big] P_\nu[\tau_1 \in dt]. \tag{6.28}$$

We consider the auxiliary problem

$$\left[\begin{array}{ccc} & g^{-,+,\nu}(t)\beta_\nu(\alpha + \zeta x_0(e^a - 1), x_0 e^a, T-t) & \\ + & g^{+,-,\nu}(t)\beta_\nu(\alpha + \zeta x_0(e^{-b} - 1), x_0 e^{-b}, T-t) & = \min\limits_{\zeta} \\ & \zeta \in \left[-\dfrac{\alpha}{x_0(e^a - 1)}, \dfrac{\alpha}{x_0(1 - e^{-b})} \right] & \end{array} \right] \tag{6.29}$$

where $\beta_\nu(v, x, T-t)$ denotes the the minimal risk in the model P_ν at time t given initial capital $V_t = v$ and asset price $X_t = x$, i.e., $\beta_\nu(\alpha, x_0, T) = \beta_\nu$. Clearly, the dynamic programming principle Lemma 6.2 holds for β_ν. Thus we obtain the following analogue of Theorem 6.4:

6.15 Theorem. *For given initial capital α, asset price x_0 and $t \in [0,T)$, let $\tilde{\zeta}^\nu(t)$ denote a solution to problem (6.29).*

The following strategy $(\tilde{\xi}_t^\nu)$ is efficient for P_ν: Until the first jump at time τ_1, use

$$\tilde{\xi}_t^\nu = \tilde{\zeta}^\nu(t), \quad t \le \tau_1.$$

Then establish a P_ν-efficient strategy from time τ_1 on.

The **Proof** proceeds analogous to Theorem 6.4 by means of the dynamic programming principle. We only replace the use of equation (6.5) by equation (6.28). □

As in Section 6.1, we now face the problem that the value function β_ν is unknown. Again, we approximate β_ν by means of k-efficient strategies. For this, let β_ν^k denote the risk of a k-efficient strategy under P_ν, i.e., the minimal risk under P_ν among all strategies that are constant from time τ_k on.

6.16 Theorem. *Let $\tilde{\zeta}^{\nu,k+1}(t)$ denote a solution to problem (6.29) where we replace β_ν by β_ν^k. The following strategy is $(k+1)$-efficient for P_ν:*

(i) Up to the first jump time τ_1, use $\tilde{\xi}_t^{\nu,k+1} := \tilde{\zeta}^{\nu,k+1}(t)$.

(ii) After time τ_1, establish a k-efficient strategy given the asset price after the last jump X_{τ_1}, time to maturity $T - \tau_1$ and $\tilde{V}_{\tau_1}$ the value of the portfolio after the last jump.

The **Proof** proceeds as in Theorem 6.15 by means of the dynamic programming principle and equation (6.28). □

Clearly, the 0-efficient minimal risk β_ν^0 is given by

$$\beta_\nu^0(\alpha, x_0, T) = \min_{\zeta \in [0, \alpha/x_0]} E_\nu[l\big((F(X_T) - \alpha - \zeta(X_T - x_0)\big)]. \tag{6.30}$$

Hence the target-function β_ν^k for $\tilde{\zeta}^{\nu,k+1}$ can be computed inductively for any $k \in \mathbb{N}$. In the limit, we obtain the minimal-risk β_ν of Theorem 6.13:

6.17 Theorem. *For any initial capital α, asset price x_0 and time to maturity T, we have*

(i) $\lim_{k\uparrow\infty} \beta_\nu^k(\alpha, x_0, T) = \beta_\nu(\alpha, x_0, T)$ *and the error can be estimated by*

$$0 \le \beta_\nu^k(\alpha, x_0, T) - \beta_\nu(\alpha, x_0, T) \le E_\nu[l(F(x_0 e^{aN_T^+ - bN_T^-})), \tau_{k+1} \le T].$$

(ii) $\liminf_{k\uparrow\infty} \tilde{V}_\nu^k = \tilde{V}_\nu$ *holds $P_{1,1}$-almost surely where we have set*

$$\begin{aligned} \tilde{V}_\nu^k &= \Big(\alpha + \int_0^T \tilde{\xi}_s^{k,\nu} dX_s\Big) \wedge F \\ \tilde{V}_\nu &= \Big(\alpha + \int_0^T \tilde{\xi}_s^{\nu} dX_s\Big) \wedge F. \end{aligned}$$

The **Proof** proceeds as in Theorem 6.10 due to

$$\lim_{k\uparrow\infty} P_\nu[\tau_k \le T] = 0.$$

□

Index of notation

$(\Omega, \mathcal{F})$	Measurable space
$\omega \in \Omega$	State, scenario
$(\mathcal{F}_t)$	Filtration
P	Model, i.e., a probability distribution on $(\Omega, \mathcal{F})$, page 13
R	Reference model, page 13
E	Expectation w.r.t. R, page 13
$\mathcal{U}$	Family of models, page 13
$co(\mathcal{U})$	Convex hull of $\mathcal{U}$, page 17
$co_\infty(\mathcal{U})$	Family of all countably convex combinations in $\mathcal{U}$, page 26
$\bar{\mathcal{U}}^1$	Closed convex hull of $\mathcal{U}$ in $L^1(R)$, page 60
Θ	Parameter space, page 73
θ	parameter, page 73
ν	prior distribution on Θ, page 73
Q	Martingale measure equivalent to R, page 13
$\mathcal{M}$	Family of all martingale measures Q, page 13
$\bar{\mathcal{M}}^0$	Closure of $\mathcal{M}$ in $L^0(R)$, page 60
F	Contingent claim, page 14
F_0	Superhedge price pf F, page 16
τ	Stopping time, e.g. the maturity of the option, page 14
(X_t)	Price process of the underlying, page 13
(ξ_t)	Dynamic hedging strategy, page 13
α	Initial capital, page 13
$\mathcal{A}_\alpha$	Class of all admissible hedging strategies for given initial capital α, page 13
ρ	Convex measure of risk, page 14
l	Loss function, page 14
I_l	Inverse of l', page 61
$(\tilde{\xi}_t)$	Robust-efficient strategy, page 17
β^*	Minimal robust shortfall risk, minimax value, page 17
β_*	Corresponding maximin value, page 18
β_P	Minimal shortfall risk for fixed model P, page 17

β_ν	Minimal shortfall risk for fixed mixture model P_ν, page 74
β^k	Minimal risk for k-efficient strategies for constant intensities, page 125
β^k_ν	Minimal risk for k-efficient strategies in the mixture model P_ν, page 132
V	Modified claim, page 19
$\mathcal{V}_\alpha$	All modified claims with superhedge price no greater than α, page 15
$\tilde{V}_{(P\|M)}$	Optimal modified claim for the simple problem $(P\|M)$, page 61
$\tilde{k}_{(P\|M)}$	Critical value for the simple problem $(P\|M)$, page 61
$\mathcal{F}$	Sigma-algebra, page 23 and 60
Z	$\mathcal{F}$-measurable test, page 23
$\mathcal{Z}_\alpha$	All tests of maximal size (price) α, page 23
$\mathcal{Z}_{\alpha,H}$	All tests of maximal size (price) α under H, page 24
$\tilde{Z}_{(G\|H)}$	Optimal test for the simple problem $(G\|H)$, page 24
$\tilde{k}_{(G\|H)}$	Critical value for the simple problem $(G\|H)$, page 24
u	State dependent utility function, pages 23 and 60
u_*	Value of the testing problem, maximin value, page 23
u^*	Corresponding minimax value, page 28
I	Inverse of u', page 34
$\mathcal{G}$	Alternative, page 23 and 60
G	Model-density, page 23
$\bar{\mathcal{G}}$	Proper enlargement of $\mathcal{G}$, page 24
$\bar{\mathcal{G}}^1$	Closed convex hull of $\mathcal{G}$ in $L^1(R)$, page 27 and 60
$\bar{\mathcal{G}}^0$	Closed convex hull of $\mathcal{G}$ in $L^0(R)$, page 27
$\mathcal{H}$	Hypothesis, page 23 and 60
H	Pricing-density, page 23
$\bar{\mathcal{H}}$	Proper enlargement of $\mathcal{H}$, page 24
$\bar{\mathcal{H}}^1$	Closed convex hull of $\mathcal{H}$ in $L^1(R)$, page 27
$\bar{\mathcal{H}}^0$	Closed convex hull of $\mathcal{H}$ in $L^0(R)$, page 27 and 60
g	Dual function, page 37
V	Stochastic conjugate, page 34
$m_1(\Omega)$	All probability measures on Ω, page 71
$m_{1,R}(\Omega)$	All probability measures on Ω absolutely continuous w.r.t. R, page 71

$m'_{1,R}(\Omega)$ All densities dP/dR for $P \in m_{1,R}(\Omega)$, page 71

$\mathcal{C}[0,T]$ The space of all continuous functions $[0,T] \to \mathbb{R}$.

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Ich versichere, daß ich diese Dissertation auf der Grundlage der im Literaturverzeichnis angegebenen Quellen selbständig und ohne unerlaubte Hilfe angefertigt habe.

Ich besitze weder einen Doktorgrad im Fach Mathematik noch habe ich mich an anderer Stelle um einen Doktorgrad beworben.

Die Promotionsordnung der Mathematisch-Naturwissenschaftlichen Fakultät II laut Beschluß des Fakultätsrats vom 19. Februar 1996 ist mir bekannt.

Berlin, den 19. Oktober 2001

Michael Kirch