The Twistor Equation in Lorentzian Spin Geometry

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Zusammenfassung

Gegenstand dieser Arbeit ist die Untersuchung der Twistorgleichung in der Lorentzschen Spin-Geometrie. Man betrachte zunächst allgemein eine semi-Riemannsche Spin-Mannigfaltigkeit (M^n_p, g) der Dimension $n \geq 3$ vom Index p mit fixierter Spin-Struktur. Über M^n_p existiert das Spinorbündel S, welches ein $2^{[\frac{n}{2}]}$-dimensionales komplexes Vektorbündel ist. Auf dem Spinorbündel S ist in natürlicher Weise die kovariante Spinorableitung ∇^S gegeben. Weiterhin existiert auf dem Spinorbündel die Clifford-Multiplikation $\mu : TM^n_p \otimes S \to S$.

Auf den glatten Spinorfeldern $\Gamma(S)$ wirkt nun der Twistor-Operator (auch genannt Penrose-Operator) P, welcher definiert ist durch die Superposition von Spinorableitung und Projektion auf den Kern der Clifford-Multiplikation

$$P : \Gamma(S) \xrightarrow{\nabla^S} \Gamma(T^*M \otimes S) \overset{g}{\cong} \Gamma(TM \otimes S) \xrightarrow{proj^\perp} \Gamma(ker\mu).$$

Der Twistor-Operator P ist ein konform kovarianter Differentialoperator erster Ordnung. Ein Spinorfeld φ wird Twistor-Spinor genannt, falls φ im Kern des Twistor-Operators liegt ($P\varphi = 0$). Diese Eigenschaft eines Spinorfeldes φ ist gleichbedeutend damit, dass φ die Twistorgleichung

$$\nabla^S_X + \frac{1}{n} X \cdot D\varphi = 0 \qquad \text{für alle } X \in TM^n_p$$

erfüllt. Twistor-Spinoren wurden von R. Penrose (siehe [Pen67], [PR86]) im Rahmen der allgemeinen Relativitätstheorie eingeführt. Spezielle Lösungen der Twistorgleichung auf semi-Riemannschen Spin-Mannigfaltigkeiten sind die sogenannten Killing-Spinoren, welche die stärkere Feldgleichung

$$\nabla^S_X \varphi = \lambda X \varphi \qquad \text{für alle } X \in TM^n_p$$

zur Killing-Zahl $\lambda \in \mathbb{C}$ erfüllen. In der Riemannschen Geometrie sind heutzutage eine Reihe von Strukturresultaten und Beispielen für Twistor- und Killing-Spinoren bekannt.

Wir betrachten die Twistorgleichung im Rahmen der Lorentzschen Spin-Geometrie (Index $p = 1$). Folgende geometrische Fragestellungen sind für uns von Interesse:

1. Welche Klassen von Lorentzschen Spin-Geometrien lassen Lösungen der Twistorgleichung zu?
2. Wie stehen die Eigenschaften von Twistor-Spinoren in Beziehung zu den geometrischen Strukturen, auf denen sie vorkommen?

Da die Twistorgleichung konform kovariant ist, ist es bei der Untersuchung dieser Fragestellungen sinnvoll die konformen Transformationseigenschaften von Twistor-Spinoren zu berücksichtigen. Von besonderem Interesse sind dabei solche Lösungen der Twistorgleichung, welche nicht konform äquivalent zu Killing-Spinoren sind.

Lösungen der Twistorgleichung sind für folgende Lorentzsche Spin-Geometrien wohl bekannt. Es existieren parallele Spinoren auf den pp-Mannigfaltigkeiten. Eine pp-Mannigfaltigkeit

zeichnet sich aus durch die Existenz eines parallelen lichtartigen Vektorfeldes und die Krümmungsbedingung $trace_{(3,5)(4,6)} R^\nabla \otimes R^\nabla$ an den Riemannschen Krümmungstensor (vgl. [EK62]). Weiterhin gibt es in ungerader Dimension die Klasse der Lorentzschen Einstein-Sasaki-Mannigfaltigkeiten, welche als S^1-Bündel über Kähler-Einstein-Mannigfaltigkeiten mit negativer Skalarkrümmung konstruiert werden. Sie besitzen imaginäre Killing-Spinoren (siehe [Kat99]). Darüber hinaus kennt man in gerader Dimension die Klasse der Fefferman-Räume, welche als S^1-Bündel über pseudokonvexen Spin-Mannigfaltigkeiten konstruiert werden. Die Fefferman-Räume besitzen ebenfalls Twistor-Spinoren. Diese Twistor-Spinoren haben die bemerkenswerte Eigenschaft, dass sie nicht konform äquivalent zu Killing-Spinoren sind (siehe [Baum99a]).

Es gibt nun einige charakteristische Größen, die bei der Untersuchung von Twistor-Spinoren auf Lorentzschen Spin-Mannigfaltigkeiten eine nützliche Rolle spielen. Als erstes ist dabei die Längenfunktion $|\varphi|^2$ eines Twistor-Spinors φ zu nennen. Weiterhin tritt das assoziierte Feld (Dirac-Strom) V_φ auf. Das assoziierte Feld V_φ ist nicht-trivial und konform ($L_{V_\varphi} g = \rho \cdot g$). Aus dem assoziierten Feld V_φ erzeugt man den Twist $\omega_\varphi \wedge d\omega_\varphi$, welcher eine 3-Form ist. Wir beweisen die folgenden Integrabilitätsbedingungen für Lorentzsche Spin-Mannigfaltigkeiten mit Twistor-Spinoren:

Satz — *(Abschnitt 4.1, Nr.5) Sei φ ein Twistor-Spinor mit lichtartigem Dirac-Strom V_φ, so gilt*

$$V_\varphi \lrcorner\ C \equiv 0 \qquad (C \text{ Schouten-Weyl-Tensor}),$$

$$V_\varphi \lrcorner\ W \equiv 0 \qquad (W \text{ Weyl-Tensor}).$$

Für Twistor-Spinoren φ mit trivialem Twist zeigen wir:

Satz — *(Abschnitt 4.1, Nr.14) Sei φ ein Twistor-Spinor ohne Nullstelle mit lichtartigem Dirac-Strom V_φ ohne Twist, so ist φ lokal konform äquivalent zu einem parallelen Spinor.*

In kleinen Dimensionen hat man folgende Strukturresultate für Lorentzsche Spin-Mannigfaltigkeiten mit Twistor-Spinoren. Es ist bekannt, dass in der Dimension 4 Twistor-Spinoren ohne 'Singularitäten' nur auf den pp-Mannigfaltigkeiten und den Fefferman-Räumen vorkommen (vgl. [Lew91]). In der Dimension 3 beweisen wir

Satz — *(Abschnitt 4.2, Nr.3) Sei (M^3_1, g) eine Lorentzsche Spin-3-Mannigfaltigkeit mit Twistor-Spinor φ ohne Nullstelle. Dann ist g lokal konform äquivalent zu einer pp-Metrik mit parallelem Spinor.*

In der Dimension 5 hat man die Klassifizierung

Satz — *(Abschnitt 4.4, Nr.7) Sei φ ein Twistor-Spinor ohne Nullstelle auf einer nicht konform-flachen Lorentzschen Spin-5-Mannigfaltigkeit (M^5_1, g). Falls*

(1) $|\varphi|^2 \equiv 0$, so ist $\omega_\varphi \wedge d\omega_\varphi \equiv 0$ und g ist lokal konform äquivalent zu einer pp-Metrik mit parallelem Spinor,

(2) $|\varphi|^2 \neq 0$ *und* $\omega_\varphi \wedge d\omega_\varphi \equiv 0$*, so ist g lokal konform äquivalent zu einer Produktmetrik der Form* $-dt^2 + \overline{g}$ *mit parallelem Spinor, wobei* $\overline{g}$ *eine Ricci-flache Kähler 4-Metrik ist.*

(3) $|\varphi|^2 \neq 0$ *und* $\omega_\varphi \wedge d\omega_\varphi \neq 0$*, dann ist* (M_1^5, g) *konform äquivalent zu einer Lorentz-Einstein-Sasaki-Mannigfaltigkeit mit einem imaginären Killing-Spinor.*

Von besonderem Interesse sind Lösungen der Twistorgleichung mit Nullstellen. Solche Lösungen besitzen insbesondere die Eigenschaft, dass sie nicht konform äquivalent zu Killing-Spinoren sind. Wir untersuchen die Gestalt der Nullstellenmenge von konformen Vektorfeldern und Twistor-Spinoren auf Lorentzschen Mannigfaltigkeiten. Es wird bewiesen

Satz — *(Abschnitt 3.3, Nr.2) Sei* $0 \not\equiv V$ *ein konformes Vektorfeld auf einer zusammenhängenden Lorentz-Mannigfaltigkeit* M_1^n *mit der Eigenschaft* $\nabla V(p) = 0$ *für alle* $p \in zero(V)$*. Dann existiert eine Umgebung* $U(p)$ *von p in* M_1^n *und eine lichtartige Geodäte* γ_p*, so dass*

$$zero(V) \cap U(p) \subset Image(\gamma_p) \cap U(p).$$

Für Twistor-Spinoren mit Nullstellen gilt der

Satz — *(Abschnitt 3.4, Nr.3) Die Nullstellenmenge* $zero(\varphi)$ *eines Twistor-Spinors* φ *auf einer Lorentz-Spin-Mannigfaltigkeit* (M_1^n, g) *ist abzählbare Vereinigung von isolierten Punkten und isolierten lichtartigen Geodäten.*

In den Dimensionen 3 und 4 können wir zeigen, dass Twistor-Spinoren mit isolierten Nullstellen nur auf konform-flachen Lorentzschen Spin-Mannigfaltigkeiten existieren. In den Dimensionen $n = 3, 4$ und 5 existieren Twistor-Spinoren mit Nullstellen in der Klasse der Einsteinschen Spin-Mannigfaltigkeiten nur auf den Räumen konstanten Schnittkrümmung (siehe Abschnitte 4.2, 4.3 und 4.4).

Weiterhin beschreiben wir eine Formulierung der Twistorgleichung im Kontext fast-Hermitischer symmetrischer Geometrie und normaler Cartanscher Zusammenhänge (vgl. [CSS97]). Genauer gesagt, wir formulieren die Twistorgleichung der semi-Riemannschen Spin-Geometrie als Parallelitätsgleichung des kanonischen normalen Zusammenhangs der konformen Cartan-Geometrie (siehe Abschnitt 2.3, Satz 2). Es wird dann über eine sogenannte Entwicklungsabbildung δ eine Holonomiedarstellung κ der Fundamentalgruppe π_1 einer konform-flachen Mannigfaltigkeit (M_p^n, g) in der konformen Möbiusgruppe realisiert. Die Formulierung der Twistorgleichung als Parallelitätsgleichung ermöglicht mit Hilfe der Holonomiedarstellung κ die Beschreibung von konform-flachen semi-Riemannschen Spin-Mannigfaltigkeiten mit Twistor-Spinoren (siehe Abschnitt 2.4, Satz 1).

Im fünften und letzten Teil der Arbeit behandeln wir ein Thema, welches nicht im direkten Zusammenhang zur Twistorgleichung steht. Wir diskutieren dort die Anwendung des Twistorraumes einer orientierten Lorenztschen 4-Mannigfaltigkeit in der Flächentheorie. Die Konstruktion verläuft in etwa wie folgt. Der Twistorraum $\mathcal{Z}(M_1^4)$ der Lorentz-4-Mannigfaltigkeit M_1^4 kann definiert werden als Sphärenbündel bestehend aus den lichtartigen Richtungen im Tangentialraum über M_1^4. Auf dem Twistorraum $\mathcal{Z}(M_1^4)$ sind natürliche fast-optische Strukturen

$\mathcal{O}^+$ und $\mathcal{O}^-$ gegeben. Die fast-optische Struktur $\mathcal{O}^+$ ist integrierbar ganau dann, wenn M_1^4 konform-flach ist, während $\mathcal{O}^-$ nie integrierbar ist (siehe Abschnitt 5.3, Satz 1). Man betrachte nun eine konform und raumartig immergierte Fläche N^2 in M_1^4. Die zweite Fundamentalform der immergierten Fläche N^2 zerlegt sich in

$$II = g \otimes (H_+ + H_-) + L_+ + L_-,$$

wobei H_+ und H_- die lichtartigen mittleren Krümmungsvektoren bezeichnen. Die immergierte Fläche N^2 besitzt nun einen natürlichen Gauß-Lift in den Twistorraum $\mathcal{Z}(M_1^4)$. Wir beweisen

Satz — *(Abschnitt 5.7, Nr.1) Konform immergierte Riemannsche Flächen in M_1^4 mit der Eigenschaft $H_- = 0$ korrespondieren bijektiv zu nicht vertikalen, $\mathcal{O}^-$-holomorphen komplexen Kurven im Twistorraum $\mathcal{Z}(M_1^4)$.*

Im folgenden wird dieses Resultat zur Untersuchung und Konstruktion spezieller isotrop-stationärer ($H_- = 0$) Flächen in den 4-dimensionalen Lorentzschen Raumformen $\mathbb{R}^{1,3}$, $S^{1,3}$ und $H^{1,3}$ verwendet (siehe Abschnitt 5.7).

Wie auch immer, es gibt eine optisch-geometrische Interpretation der Twistorgleichung auf Lorenztschen 4-Mannigfaltigkeiten im zugehörigen Twistorraum. Dazu bezeichne $(\mathcal{B}, \mathcal{O})$ das kanonische Linienbündel mit natürlicher fast-optischer Struktur über dem Twistoraum $(\mathcal{Z}(M_1^4), \mathcal{O}^+)$.

Satz — *(Abschnitt 5.4, Nr.1) Twistor-Spinoren auf Lorentzschen Spin-4-Mannigfaltigkeiten korrespondieren bijektiv zu linearen holomorphen Schnitten im Linenbündel $(\mathcal{B}, \mathcal{O})$ über dem Twistorraum $(\mathcal{Z}(M_1^4), \mathcal{O}^+)$.*

Contents

0 Introduction

Let us consider a semi-Riemannian spin manifold (M^n_p, g) of dimension $n \geq 3$ and index p. We denote by S the spinor bundle, which is a $2^{[\frac{n}{2}]}$-dimensional complex vector bundle over M^n_p. On the spinor bundle S, there is given in a natural way the covariant spinor derivative ∇^S. Moreover, we denote by $\mu : TM \otimes S \to S$ the Clifford multiplication. There exist two conformally covariant differential operators of first order acting on the spinor fields $\Gamma(S)$, the Dirac operator D and the twistor operator (also called Penrose operator) P. The Dirac operator is defined as the composition of the spinor derivative ∇^S with the Clifford multiplication μ

$$D : \Gamma(S) \xrightarrow{\nabla^S} \Gamma(T^*M \otimes S) \stackrel{g}{\cong} \Gamma(TM \otimes S) \xrightarrow{\mu} \Gamma(S).$$

The Dirac operator is of particular importance both in theoretical physics and mathematics. In the form that is given here its eigenvalue equation $D\varphi = \lambda\varphi$ generalizes the classical Dirac equation, which has been introduced in 1928 by P.A.M. Dirac in order to give a relativistic quantum mechanical description of a free spin-1/2 particle in the 4-dimensional Minkowski space $\mathbb{R}^{1,3}$.

Complementary to the Dirac operator, there is the twistor operator P, which is defined to be the composition of the spinor derivative ∇^S with the projection to the kernel of the Clifford multiplication

$$P : \Gamma(S) \xrightarrow{\nabla^S} \Gamma(T^*M \otimes S) \stackrel{g}{\cong} \Gamma(TM \otimes S) \xrightarrow{proj^{\perp}} \Gamma(ker\mu).$$

The elements of the kernel of the twistor operator are called twistor spinors. Equivalently, a spinor field φ is a twistor spinor if and only if it satisfies the conformally covariant twistor equation

$$\nabla^S_X \varphi + \frac{1}{n} X \cdot D\varphi = 0 \qquad \text{for all } X \in TM^n_p.$$

Twistor spinors were introduced by R. Penrose in General Relativity (see [Pen67], [PR86], [NW84]). In Riemannian geometry, the twistor equation appeared first as integrability condition for the natural almost complex structure of the twistor space of a 4-manifold (see [AHS78]). In the second half of the 80th A. Lichnerowicz started a systematic investigation of twistor spinors on Riemannian spin manifolds from the view point of conformal differential geometry. Nowadays there are a lot of structure results and examples for manifolds with twistor spinors in the Riemannian setting (see e.g. [Lic88], [Fri89], [BFGK91], [Hab90] and [KR94]).

Special solutions of the twistor equation are the so-called Killing spinors φ, which satisfy the stronger spinor field equation (Killing equation)

$$\nabla^S_X \varphi = \lambda X \varphi \qquad \text{for all } X \in TM^n_p$$

and some Killing number $\lambda \in \mathbb{C}$. Originally, the notion of Killing spinors came from mathematical physics, where Killing spinors are used in the context of supergravity and superstring theories (see e.g. [HPSW72], [DNP86], [AFOHS98]). In differential geometry, the interest in Killing spinors started with the observation of Th. Friedrich in 1980 that a special kind of Killing spinors realizes the limit case in the eigenvalue estimate of the Dirac operator on compact Riemannian spin manifolds of positive scalar curvature. In the following time, Killing spinors were intensively studied. The occuring geometric structures on Riemannian spin manifolds admitting Killing spinors are basically known today (see [Bau89], [BFGK91], [Bär93]). The description of these geometric structures is closely related to the holonomy theory of Riemannian manifolds admitting parallel spinors via the method of warped-product and cone constructions.

We investigate the twistor equation in the Lorentzian setting (index $p = 1$). The follwing geometric questions are interesting for us.

(1) Which Lorentzian geometries admit solutions of the twistor equation?

(2) How the properties of twistor spinors are related to the geometric structures, where they occur?

We list some examples of Lorentzian geometries, which admit twistor spinors. First, we mention the class of pp-manifolds (pp = plane waves and parallel rays), which is known by physicists in General Relativity for a long time (see [EK62], [Sch74]). The pp-metrics are distinguished by the existence of a parallel null vector field and the condition $trace_{(3,5)(4,6)} R^\nabla \otimes R^\nabla = 0$ for the Riemannian curvature tensor R^∇. The pp-manifolds admit parallel spinors. Secondly, in odd dimensions there are the Lorentzian Einstein-Sasaki manifolds, which are related to circle bundles over Kähler-Einstein manifolds with negative scalar curvature. The Lorentzian Einstein-Saski manifolds admit imaginary Killing spinors (comp. [Boh98], [Kat99]). In even dimensions, we have the class of Fefferman spaces with twistor spinors. The Fefferman spaces are constructed as circle bundles over strictly pseudoconvex spin manifolds (comp. [Lew91], [Bau99]).

Since twistor spinors are conformally invariant objects, it is natural to investigate their properties under conformal transformations. The following question arises.

(3) Which Lorentzian geometries admit 'true' solutions of the twistor equation in the sense that a twistor spinor φ is not conformally related to solutions of the Killing equation?

There is a first result to this question. Namely, the twistor spinors on Fefferman spaces have the property that they are not conformally equivalent to Killing spinors. Moreover, each twistor spinor with a zero is a 'true' solution of the twistor equation, since Killing spinors have no zeros. This is the reason why the investigation of the zero set of a 'twistor' is interesting in this context.

A useful technique for investigating the twistor equation and its related geometric structures is that of the associated vector field (Dirac current) V_φ to a spinor φ. For twistor spinors, the associated vector field is conformal, i.e. the local flow consists of conformal transformations. Moreover, an important characterization for the geometric structures admitting a twistor spinor φ is the twist 3-form $\omega_\varphi \wedge d\omega_\varphi$ of the associated conformal field V_φ. A further important characterization of a twistor spinor φ is its lenght function $|\varphi|^2$. We will see that in low dimensions a description of the occuring geometric structures on Lorentzian spin manifolds admitting a twistor spinor without 'singularities' can be done with the help of the corresponding twist and the lenght function.

A different approach to the twistor equation on semi-Riemannian spin manifolds can be done in the context of almost Hermitian symmetric geometry and normal Cartan connections. More explicit, the twistor equation can be reformulated as parallelity equation with respect to the normal conformal Cartan connection of conformal spin geometry. This formulation reflects the conformal covariance of the twistor equation in the most natural manner. From this conformal point of view the problem of describing the geometric structures admitting twistor spinors is related to the holonomy theory of the normal conformal Cartan connection.

The thesis is organized in the following way. In section 1, we introduce the basic notations and facts for the theory of twistor spinors in semi-Riemannian geometry. Moreover, we give a short review over results for solutions of the twistor equation in Riemannian and Lorentzian spin geometry. In section 2, we reformulate the twistor equation in the context of conformal Cartan geometry (see Theorem 2.3.2). We apply this approach to the description of conformally flat semi-Riemannan spin manifolds admitting twistor spinors (see Theorem 2.4.1). In the third section we discuss the zero set of twistors spinors on Lorentzian spin manifolds. This will be done in a more general way by investigating conformal vector fields. The main result states that the zero set of a twistor spinor on a Lorentzian spin manifold consists of isolated points and isolated lightlike geodesics (see Theorem 3.4.3). In section 4, we will discuss twistor spinors, their associated conformal fields and structure results for the underlying Lorentzian spin geometries in dependence of the characteristic properties consisting of the lenght function and the twist. In particular, twistor spinors in the low dimensions 3,4 and 5 are studied. Our main result says that in these small dimensions twistor spinors without 'singularities' occur only on the pp-manifolds, the Fefferman spaces and the Lorentz-Einstein-Sasaki manifolds (see Theorems 4.2.3, 4.3.7 and 4.4.7).

The last section is concerned with a topic, which is not directly related to the twistor equation. We will discuss an application of the Lorentzian twistor space construction to surface theory in oriented Lorentzian 4-manifolds. The idea for this application is derived from similar investigations in Riemannian twistor theory (comp. [Fri84], [ES85], [JR90]). In short, the construction works as follows. The twistor space $\mathcal{Z}(M^4_1)$ of a Lorentzian 4-space M^4_1 is defined to be the bundle of null directions in the tangent space and the twistor space $\mathcal{Z}(M^4_1)$ is in a natural way furnished with almost optical structures, which are related to almost CR-structures. A spacelike immersed surface N^2 in the Lorentzian 4-space M^4_1 admits a Gauss lift to the twistor space $\mathcal{Z}(M^4_1)$. The holomorphicity of this Gauss lift is related to certain curvature conditions expressed in the second fundamental form of the immersed surface N^2. We will prove a correspondence between holomorphic curves in the twistor space and immersed surfaces with null mean curvature vector (see Theorem 5.7.1). However, there exists an interpretation of the twistor equation in the twistor space $\mathcal{Z}(M^4_1)$ of a Lorentzian spin 4-manifold M^4_1 with respect to the natural optical geometries.

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1 Basic facts about twistor spinors

The first part of this section contains a short introduction to Clifford algebras, spin groups and spinor representations. In the second part we will introduce the twistor equation for spinor fields on semi-Riemannian spin manifolds. We will also present some general basic facts on twistor spinors. In the last two parts we discuss the development of the investigations of the twistor equation in Riemannian spin geometry and sum up what is known for the twistor equation in Lorentzian spin geometry.

In the first place our work is concerned with the investigation of the twistor equation in Lorentzian geometry. Nevertheless, the first two introductory parts treats the general case of semi-Riemannian geometry with arbitrary index. For a detailed description of Clifford algebras and spinor representations we refer to [Bau81] or [LM89]. For basic results on twistor spinors we refer to [BFGK91] and [Bau00a].

1.1 Clifford algebras and spinor representations

Let $\mathbb{R}^{p,q}$ denote the (pseudo)-Euclidean space $(\mathbb{R}^n, \langle\cdot,\cdot\rangle_{p,q})$ of dimension $n = p + q > 2$, where the scalar product $\langle\cdot,\cdot\rangle_{p,q}$ of index p is given by

$$\langle x, y\rangle_{p,q} := -\sum_{i=1}^{p} x_i y_i + \sum_{i=p+1}^{n} x_i y_i = \sum_{i=1}^{n} \varepsilon_i \cdot x_i y_i.$$

We denote by $SO(p,q)$ the special orthogonal group of signature (p,q), which acts by isometries on $\mathbb{R}^{p,q}$, and by $SO^+(p,q)$ we denote its identity component. Let D_{kl} be the $(n\times n)$-matrix in $\mathfrak{gl}(n,\mathbb{R})$, whose (k,l)-entry is 1 and all other entries are 0. Then we denote $E_{kl} := -\varepsilon_l D_{kl} + \varepsilon_k D_{lk}$. The matrices $\{E_{kl} \mid 1 \le k < l \le n\}$ form a basis of the Lie algebra $\mathfrak{so}(p,q)$.

We denote by $Cl_{p,q}$ the Clifford algebra of $\mathbb{R}^{p,q}$ and by $Cl^{\mathbb{C}}_{p,q}$ its complexification. The Clifford algebras are multiplicatively generated by the standard basis $(e_1, \dots, e_n)$ of $\mathbb{R}^n$ with the relations

$$e_i e_j + e_j e_i = -2\langle e_i, e_j\rangle_{p,q} \cdot \mathbf{1}\,.$$

In case that $n = 2m$ is even the Clifford algebra $Cl^{\mathbb{C}}_{p,q}$ is isomorphic to the algebra $\mathbb{C}(2^m)$ of complex $(2^m \times 2^m)$-matrices. In case that $n = 2m+1$ is odd $Cl^{\mathbb{C}}_{p,q}$ is isomorphic to $\mathbb{C}(2^m) \oplus \mathbb{C}(2^m)$. Let E, T, g_1 and g_2 denote the (2×2)-matrices

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad g_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \text{and} \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

We set

$$\tau(j) = \begin{cases} i & \text{if } \ j \le p \\ 1 & \text{if } \ j > p \end{cases}.$$

If $n = 2m$ then an isomorphism $\Phi_{p,q} : Cl^{\mathbb{C}}_{p,q} \longrightarrow \mathbb{C}(2^m)$ is given by

$$\begin{aligned} \Phi_{p,q}(e_{2j-1}) &= \tau(2j-1) \cdot E \otimes \dots \otimes E \otimes g_1 \otimes \underbrace{T \otimes \dots \otimes T}_{(j-1)-times} \\ \Phi_{p,q}(e_{2j}) &= \tau(2j) \cdot E \otimes \dots \otimes E \otimes g_2 \otimes \underbrace{T \otimes \dots \otimes T}_{(j-1)-times} \end{aligned}$$

for $j = 1, \cdots, m$. If $n = 2m+1$ and $q > 0$ an isomorphism $\Phi_{p,q} : Cl^{\mathbb{C}}_{p,q} \longrightarrow \mathbb{C}(2^m) \oplus \mathbb{C}(2^m)$ is given by

$$\begin{aligned} \Phi_{p,q}(e_j) &= (\Phi_{p,q-1}(e_j), \Phi_{p,q-1}(e_j)) \\ \Phi_{p,q}(e_n) &= \tau(n) \cdot (i\, T \otimes \cdots \otimes T, -i\, T \otimes \cdots \otimes T). \end{aligned}$$

The spin group is realized in the Clifford algebra $Cl_{p,q}$ as

$$Spin(p,q) = \{x_1 \cdot \ldots \cdot x_{2l} \in Cl_{p,q} : x_j \in \mathbb{R}^{p,q},\ \langle x_j, x_j \rangle = \pm 1,\ l \in \mathbb{N}\}$$

and its identity component is given by

$$\begin{aligned} Spin^+(p,q) = \{ & x_1 \cdot \ldots \cdot x_{2l} \in Cl_{p,q} : x_j \in \mathbb{R}^{p,q}, \\ & \langle x_j, x_j \rangle = \pm 1,\ \#\{x_j : \langle x_j, x_j \rangle = 1\} \text{ is even}\}. \end{aligned}$$

We can identify the Lie algebra $\mathfrak{spin}(p,q)$ with the subspace $Span\{e_k \cdot e_l \in Cl_{p,q} | k < l\}$ in $Cl_{p,q}$. The smooth group homomorphism

$$\begin{array}{rccc} \lambda : & Spin^+(p,q) & \longrightarrow & SO^+(p,q) \\ & u & \longmapsto & (x \in \mathbb{R}^{p,q} \mapsto uxu^{-1} \in \mathbb{R}^{p,q}) \end{array}$$

is a 2-fold covering map. For the differential $\lambda_* : \mathfrak{spin}(p,q) \to \mathfrak{so}(p,q)$ we have the relation $\lambda_*(e_k \cdot e_l) = 2E_{kl}$.

For calculations the following notations are sometimes useful. Let $u(\nu) := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i\,\nu \end{pmatrix}$ for $\nu \in \{\pm 1\}$. Then it holds

$$g_1 u(\nu) = i u(-\nu), \qquad g_2 u(\nu) = \nu u(-\nu) \quad \text{and} \qquad T u(v) = -\nu u(\nu).$$

The elements

$$u(\nu_1, \ldots, \nu_m) := u(\nu_1) \otimes \cdots \otimes u(\nu_m) \in \otimes_m \mathbb{C}^2, \quad \nu_i \in \{\pm 1\},$$

form a basis of $\otimes_m \mathbb{C}^2 \cong \mathbb{C}^{2^{[\frac{n}{2}]}}$.

A spinor representation $\rho_{p,q}$ on the spinor module $\Delta_{p,q} = \mathbb{C}^{2^{[\frac{n}{2}]}}$ is defined by

$$\rho_{p,q} = \begin{cases} \Phi_{p,q} & \text{if } n = 2m \\ proj_1 \circ \Phi_{p,q} & \text{if } n = 2m+1 \end{cases}.$$

It holds $\rho_{p,q}(e_1 \cdot \ldots \cdot e_{2m}) = (-1)^m i^{m+p}\, T \otimes \cdots \otimes T$. If $n = 2m$ is even then the $Spin^+(p,q)$-module $\Delta_{p,q}$ can be decomposed into two irreducible representation spaces $\Delta^+_{p,q}$ and $\Delta^-_{p,q}$, which are given by

$$\begin{aligned} \Delta^{\pm}_{p,q} &= \{v \in \Delta_{p,q} : \rho_{p,q}(e_1 \cdot \ldots \cdot e_{2m}) v = \pm i^{m+p} \cdot v\} \\ &= Span\{u(\nu_1, \cdots, \nu_m) : \textstyle\prod_{i=1}^m \nu_i = \pm 1\}. \end{aligned}$$

If $n = 2m+1$ is odd the representation $\rho_{p,q}$ is irreducible. The Clifford multiplication on the spinor module $\Delta_{p,q}$ is in even dimension $n = 2m$ defined by

$$\begin{array}{ccc} \mathbb{R}^{p,q} \times \Delta_{p,q} & \longrightarrow & \Delta_{p,q} \\ (x, v) & \longmapsto & x \cdot v := \Phi_{p,q}(x)(v) \end{array}$$

and in odd dimension $n = 2m + 1$ by

$$\begin{array}{ccc} \mathbb{R}^{p,q} \times \Delta_{p,q} & \longrightarrow & \Delta_{p,q} \\ (x, v) & \mapsto & x \cdot v := proj_1 \circ \Phi_{p,q}(x)(v). \end{array}$$

The Clifford multiplication of a spinor v by an arbitrary k-form ω is defined as

$$\omega \cdot v = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \varepsilon_{i_1} \cdot \ldots \cdot \varepsilon_{i_k} \omega(e_{i_1}, \cdots, e_{i_k}) e_{i_1} \cdot \ldots \cdot e_{i_k} \cdot v.$$

There exists a $Spin^+(p,q)$-invariant, non-degenerate (indefinite for $p \neq 0, n$) Hermitian scalar product $\langle \cdot, \cdot \rangle_\Delta$ on the spinor module $\Delta_{p,q}$, which is defined by

$$\langle v, w \rangle_\Delta := i^{p(p-1)/2} (e_1 \cdot \ldots \cdot e_p v, w), \qquad v, w \in \Delta_{p,q},$$

where $(z, z') = \sum_{i=1}^{2^{[\frac{n}{2}]}} z_i \cdot \overline{z'_i}$ is the standard Hermitian product on $\mathbb{C}^{2^{[\frac{n}{2}]}}$. The Hermitian product $\langle \cdot, \cdot \rangle_\Delta$ on $\Delta_{p,q}$ has the property

$$\langle x \cdot v, w \rangle_\Delta = (-1)^{p+1} \langle v, x \cdot w \rangle_\Delta$$

for all $x \in \mathbb{R}^n$ and $v, w \in \Delta_{p,q}$.

It is important to know that there is a canonical way to obtain a vector from a spinor. With the help of the Hermitian product $\langle \cdot, \cdot \rangle_\Delta$ one assigns to a spinor $v \in \Delta_{p,q}$ the vector

$$x_v := i^{p+1} \cdot \sum_{i=1}^{n} \varepsilon_i \langle v, e_i v \rangle_\Delta e_i \in \mathbb{R}^{p,q}.$$

We denote the mapping $v \in \Delta_{p,q} \mapsto x_v \in \mathbb{R}^{p,q}$ by ℓ. The map ℓ is $Spin^+(p,q)$-equivariant, i.e. $\ell(a \cdot v) = \lambda(a)\ell(v)$. The dual 1-form to the vector x_v is given by $\omega_v(x) = i^{p+1} \cdot \langle v, xv \rangle_\Delta$.

1.2 Twistor spinors on semi-Riemannian manifolds

Let (M^n_p, g) be a smooth space- and time-oriented semi-Riemannian manifold of dimension $n \geq 3$ and signature (p,q), where we set $q := n - p$. We denote by $SO(M)$ the set of positive space- and time-oriented orthonormal frames in the tangent space TM^n_p. That means an element $s \in SO(M)$ consists of a positive space- and time-oriented orthonormal basis $(s_1, \ldots, s_n)$ of the tangent space T_xM at some point $x \in M$. The set $SO(M)$ is in a natural way a smooth $SO^+(p,q)$-principal bundle over M^n_p. In the following we will choose a frame $(s_1, \ldots, s_n)$ in such a way that the first p vectors $(s_1, \ldots, s_p)$ are timelike.

A spin structure on a space- and time-oriented semi-Riemannian manifold (M^n_p, g) is a $Spin^+(p,q)$-reduction of the frame bundle $SO(M)$. It is common to call a semi-Riemannian manifold M^n_p spin if it admits a spin structure. For the rest of this section we assume M^n_p to be spin and we equip M^n_p with a fixed spin structure $(Spin(M), f)$, where $Spin(M)$ denotes the $Spin^+(p,q)$-principal bundle over M and f denotes the reduction of $Spin(M)$ to the frame bundle $SO(M)$. Then we have the spinor bundle

$$S = Spin(M) \times_{\rho_{p,q}} \Delta_{p,q}$$

over M^n_p, which decomposes for $n = 2m$ even to the bundle of positive and negative 'half' spinors $S = S^+ \oplus S^-$.

The Levi-Civita connection form on $SO(M)$ lifts to $Spin(M)$ and this lifted connection form gives us a canonical derivative ∇^S on the spinor bundle S, which is called the spinor derivative. A spinor field $\varphi \in \Gamma(S)$ satisfying $\nabla^S \varphi = 0$ is called parallel spinor (comp. [Wan89], [BK99] or [Bry00]). The curvature tensor R^S of ∇^S is locally given by

$$R^S(X,Y)\varphi = \frac{1}{2} \cdot \sum_{1 \leq k < l \leq n} \varepsilon_k \varepsilon_l \cdot R^\nabla(X,Y,s_k,s_l) s_k s_l \cdot \varphi$$

for $\varphi \in \Gamma(S)$ and $X, Y \in \Gamma(TM)$, where $(s_1, \ldots, s_n)$ denotes a local section in $SO(M)$ and R^∇ denotes the Riemannian curvature tensor, which is defined by

$$R^\nabla(X,Y,Z,W) = g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W), \qquad X, Y, Z, W \in \Gamma(TM).$$

The tangent vectors act on the spinor bundle S by Clifford multiplication

$$\begin{array}{rccc} \mu : & TM \otimes S & \longrightarrow & S \\ & ([q,x],[q,v]) & \longmapsto & [q, x \cdot v], \qquad q \in Spin(M) \end{array}$$

and also the k-forms act on S by Clifford multiplication $\mu : \Lambda^k M \otimes S \to S$. Furthermore, there exists a non-degenerate (possibly indefinte) Hermitian product $\langle \cdot, \cdot \rangle_S$ on the spinor bundle S defined by

$$\langle \varphi, \psi \rangle_S = \langle v, w \rangle_\Delta \qquad \text{for} \quad \varphi = [q,v], \quad \psi = [q,w] \in S.$$

The Hermitian product $\langle \cdot, \cdot \rangle_S$ satisfies the following properties:

$$\begin{array}{rcl} \langle X\varphi, \psi \rangle_S & = & (-1)^{p+1} \langle \varphi, X\psi \rangle_S \\ X \langle \varphi, \psi \rangle_S & = & \langle \nabla^S_X \varphi, \psi \rangle_S + \langle \varphi, \nabla^S_X \psi \rangle_S. \end{array}$$

There are two natural conformally covariant differential operators of first order, which act on the spinor fields $\Gamma(S)$ over the semi-Riemannian spin manifold M^n_p, the Dirac operator D and the twistor operator P. The Dirac operator is defined as the composition of the spinor derivative ∇^S with the Clifford multiplication μ:

$$D : \Gamma(S) \xrightarrow{\nabla^S} \Gamma(T^*M \otimes S) \overset{g}{\cong} \Gamma(TM \otimes S) \xrightarrow{\mu} \Gamma(S).$$

Locally, the Dirac operator is given by

$$D\varphi = \sum_{i=1}^{n} \varepsilon_i s_i \cdot \nabla^S_{s_i} \varphi \qquad \text{for } \varphi \in \Gamma(S),$$

where $s = (s_1, \cdots, s_n)$ is a local frame.

The twistor operator is the composition of the spinor derivative ∇^S with the orthogonal projection $proj^\perp$ onto the kernel of the Clifford multiplication

$$P : \Gamma(S) \xrightarrow{\nabla^S} \Gamma(T^*M \otimes S) \overset{g}{\cong} \Gamma(TM \otimes S) \xrightarrow{proj^\perp} \Gamma(\ker \mu).$$

Locally, we have

$$P\varphi = \sum_{i=1}^{n} \varepsilon_i s_i \otimes (\nabla^S_{s_i}\varphi + \frac{1}{n} s_i \cdot D\varphi).$$

We define the main object of our interest.

Definition 1.2.1. — *A spinor field $\varphi \in \Gamma(S)$ in the kernel of P is called twistor spinor.*

We denote the space of twistors spinors on a semi-Riemannian spin manifold (M^n_p, g) by $\mathcal{T}(M^n_p) := kerP$. Alternatively, we can describe twistor spinors as follows.

Proposition 1.2.2. — *(comp. [BFGK91]) For a spinor field $\varphi \in \Gamma(S)$ the following conditions are equivalent:*

(1) φ is a twistor spinor.

(2) φ satisfies the so-called twistor equation

$$\nabla^S_X\varphi + \frac{1}{n} X \cdot D\varphi = 0 \qquad \text{for all } \; X \in TM.$$

(3) $X \cdot \nabla^S_Y\varphi + Y \cdot \nabla^S_X\varphi = \frac{2}{n} g(X,Y)D\varphi \qquad$ for all $\; X, Y \in TM$.

(4) There exists a spinor $\psi \in \Gamma(S)$ such that $g(X,X)X \cdot \nabla_X\varphi = \psi$ for all $X \in TM$ with $g(X,X) \in \{\pm 1\}$.

Special solutions of the twistor equation are the Killing spinors, which satisfy the spinor field equation

$$\nabla^S_X\varphi = \lambda X \cdot \varphi \qquad \text{for all } X \in TM \quad \text{and some } \lambda \in \mathbb{C} \setminus 0\,.$$

This equation is called the Killing equation and the complex number λ is called Killing number of the Killing spinor φ. The Killing number λ is real or purely imaginary. Consequently, if λ is real and non-zero the Killing spinor φ is called real Killing spinor and, if λ is purely imaginary φ is called imaginary Killing spinor. Killing spinors are exactly those twistor spinors, which satisfy the eigenvalue equation $D\varphi = -n\lambda\varphi$ for the Dirac operator (comp. [Hij86]).

There are some basic integrability conditions for twistor spinors on semi-Riemannian spin manifolds, which we want to list here. Let us denote by R the scalar curvature and by Ric the Ricci curvature of (M^n_p, g). By K we denote the Schouten tensor

$$K = \frac{1}{n-2}\left(\frac{R}{2(n-1)} g - Ric\right)$$

and by C we denote the Schouten-Weyl tensor

$$C(X,Y) = (\nabla_X K)(Y) - (\nabla_Y K)(X).$$

Furthermore, let W denote the Weyl tensor, which is the traceless part of the Riemannian curvature tensor R^∇ on (M^n_p, g).

Proposition 1.2.3. — *(comp. [BFGK91] and [Bau00a]) Let $\varphi \in \Gamma(S)$ be a twistor spinor and $X, Y, Z \in TM$. Then*

(1) $D^2\varphi = \frac{1}{4}\frac{n}{n-1}R\varphi$

(2) $\nabla^S_X D\varphi = \frac{n}{2}K(X) \cdot \varphi$

(3) $W(Y,Z) \cdot \varphi = 0$

(4) $W(Y,Z) \cdot D\varphi = nC(Y,Z) \cdot \varphi$

(5) $(\nabla_X W)(Y,Z) \cdot \varphi = X \cdot C(Y,Z) \cdot \varphi + \frac{2}{n}(X \lrcorner W(Y,Z)) \cdot D\varphi.$

If (M^n_p, g) admits a Killing spinor the Ricci and the scalar curvature of M satisfy

Proposition 1.2.4. — *(comp. [BFGK91] and [Bau00a]) Let $\varphi \in \Gamma(S)$ be a Killing spinor to the Killing number $\lambda \in \mathbb{C}$. Then*

(1) $(Ric(X) - 4\lambda^2(n-1)X) \cdot \varphi = 0$

(2) $R = 4n(n-1)\lambda^2 = const.$

From (2) in Proposition 1.2.4, it follows that the Killing number λ is real or purely imaginary, as we already mentioned above.

Proposition 1.2.5. — *(comp. [BFGK91] and [Bau00a]) Let (M^n_p, g) be an Einstein space. If $R \neq 0$ and φ is a twistor spinor on M^n_p then φ is the sum of two Killing spinors to the Killing numbers $\lambda_\pm = \pm\frac{1}{2}\sqrt{\frac{R}{n(n-1)}}$. In case that $R = 0$, either φ or $D\varphi$ is a parallel spinor.*

We remark that in even dimension, if we split a Killing spinor φ into its half spinors φ_+ and φ_- then the half spinors φ_+ and φ_- are not Killing spinors, since Clifford multiplication by a vector maps a positive half spinor to a negative half spinor and vice versa. Notice also that the half spinors to a twistor spinor are again twistor spinors.

The basic property of the twistor equation is that it is conformally covariant. In more detail, let $\tilde{g} = e^{2\sigma} \cdot g$, where $\sigma \in C^\infty(M)$, be a conformally equivalent metric to g. Then there is a natural bundle isomorphisms between the $Spin^+(p,q)$-principal fibre bundles $Spin(M,g)$ and $Spin(M,\tilde{g})$ and we obtain a natural identification

$$\begin{array}{ccc} S & \cong & \tilde{S}\ . \\ \varphi = [q,v] & \mapsto & \tilde{\varphi} = [\tilde{q},v], \qquad q \in Spin(M,g) \sim \tilde{q} \in Spin(M,\tilde{g}) \end{array}$$

The following relations for the conformally changed objects hold:

$$\begin{aligned}\nabla^{\tilde S}_{\tilde X}\tilde\varphi &= e^{-\sigma}\cdot\widetilde{\nabla^S_X\varphi}-\frac{1}{2}e^{-2\sigma}\left(X\cdot grad(e^\sigma)\cdot\varphi+g(X,grad(e^\sigma))\cdot\varphi\right)^{\sim}\\ \widetilde{D}\tilde\varphi &= e^{-\frac{n+1}{2}\sigma}\cdot\Big(D(e^{\frac{n-1}{2}\sigma}\cdot\varphi)\Big)^{\sim}\\ \tilde P\tilde\varphi &= e^{-\frac{\sigma}{2}}\cdot\Big(P(e^{-\frac{\sigma}{2}}\varphi)\Big)^{\sim},\end{aligned}$$

where $\tilde X = e^{-\sigma}X$. Obviously, $\varphi\in\Gamma(S)$ is a twistor spinor on (M^n_p,g) if and only if $e^{\frac{1}{2}\sigma}\tilde\varphi\in\Gamma(\tilde S)$ is a twistor spinor with respect to $\tilde g$ on M^n_p. We say that a twistor spinor φ on (M^n_p,g) is conformally equivalent to a Killing spinor if there exists a conformal factor $e^{2\sigma}$ such that $e^{\frac{1}{2}\sigma}\tilde\varphi$ is a Killing spinor with respect to $\tilde g = e^{2\sigma}g$ on M^n_p. In this case the Killing equation $\nabla^{\tilde S}_X(e^{\frac{1}{2}\sigma}\tilde\varphi) = \lambda X\cdot(e^{\frac{1}{2}\sigma}\tilde\varphi)$ is equivalent to (comp. [Fri89])

$$-2\lambda\varphi + grad(e^{-\sigma})\cdot\varphi = \frac{2}{n}e^{-\sigma}D\varphi.$$

We introduce a further interpretation of the twistor equation. Let us consider the covariant derivative ∇^{TC} on the doubled spinor bundle $S\oplus S$ defined by

$$\nabla^{TC}_X := \begin{pmatrix}\nabla^S_X & \frac{1}{n}X\cdot\\ -\frac{n}{2}K(X)\cdot & \nabla^S_X\end{pmatrix}.$$

Using the integrability condition (2) of Proposition 1.2.3 we obtain

Proposition 1.2.6. — *(comp. [BFGK91] and [Bau00a]) A twistor spinor $\varphi\in\Gamma(S)$ satisfies $\nabla^{TC}\begin{pmatrix}\varphi\\ D\varphi\end{pmatrix}=0$. Conversely, if $\begin{pmatrix}\varphi\\ \psi\end{pmatrix}$ is ∇^{TC}-parallel then φ is a twistor spinor and $\psi = D\varphi$.*

Proposition 1.2.6 says that the twistor equation can be interpreted as parallelity equation on the bundle $S\oplus S$ with respect to the covariant derivative ∇^{TC}. This interpretation respects the conformal covariance of the twistor equation and gives a link to conformal Cartan geometry. We will discuss this point of view of the twistor equation in detail in section 2. The curvature of ∇^{TC} on $S\oplus S$ is

$$R^{TC}(X,Y)\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = \frac{1}{2}\begin{pmatrix}W(X,Y)\cdot\varphi\\ W(X,Y)\cdot\psi - nC(X,Y)\cdot\varphi\end{pmatrix}.$$

With Proposition 1.2.6 we obtain

Proposition 1.2.7. — *(comp. [BFGK91], [Bau00a]) The dimension of the space of twistor spinors is conformally invariant and bounded by*

$$dim(ker P)\le 2^{[\frac{n}{2}]+1} = 2\cdot rang\, S =: d_n.$$

For each simply connected and conformally flat Lorentzian spin manifold the dimension of the space of twistor spinors equals d_n. The maximal dimension d_n can only occur if (M^n_p,g) is conformally flat.

A standard construction, which is very useful for the investigation of twistor spinors, is the following. To each spinor field φ we can associate the vector field $V_\varphi = \ell(\varphi)$ by the condition

$$g(V_\varphi, X) = i^{p+1}\langle\varphi, X\varphi\rangle_S \qquad \text{for all } X \in TM.$$

Locally, the associated field is given by $V_\varphi := i^{p+1}\sum_{i=1}^n \varepsilon_i\langle\varphi, s_i\varphi\rangle_\Delta s_i$. The definition of V_φ is conformally invariant, since

$$\begin{aligned} V_{e^{\frac{1}{2}\sigma}\tilde{\varphi}} &= i^{p+1}\sum_i \varepsilon_i\langle e^{\frac{1}{2}\sigma}\tilde{\varphi}, \tilde{s}_i\cdot(e^{\frac{1}{2}\sigma}\tilde{\varphi})\rangle_{\tilde{S}}\tilde{s}_i = i^{p+1}\sum_i \varepsilon_i\langle\tilde{\varphi}, \tilde{s}_i\cdot\tilde{\varphi}\rangle_{\tilde{S}}\cdot(e^\sigma\tilde{s}_i) \\ &= i^{p+1}\sum_i \varepsilon_i\langle\varphi, s_i\cdot\varphi\rangle_S s_i = V_\varphi. \end{aligned}$$

The twistor equation implies for the associated vector field of a spinor the following properties.

Proposition 1.2.8. — *(comp. [Bau00a]) Let $\varphi \in \Gamma(S)$ be a twistor spinor on M^n_p.*

(1) The vector field V_φ is a conformal field, i.e. $L_{V_\varphi}g = \frac{2}{n}div(V_\varphi)\cdot g$, and the divergence satisfies

$$div(V_\varphi) = -2\cdot b(\langle\varphi, D\varphi\rangle_S),$$

where $b(f)$ denotes the real part of f if the index p of g is odd and the imaginary part of f if the index p is even.

(2) If φ is a real (imaginary) Killing spinor and p is even (odd) then V_φ is a Killing vector field.

(3) If φ is a parallel spinor then V_φ is a parallel vector field.

(4) It holds for all $p \in zero(\varphi)$:

$$V_\varphi(p) = 0, \qquad \nabla V_\varphi(p) = 0, \qquad (divV_\varphi)(p) = 0 \qquad \textit{and} \qquad d(divV_\varphi)(p) \neq 0.$$

1.3 The twistor equation in Riemannian geometry

We give in this part a short summary concerning the twistor equation on Riemannian spin manifolds. The twistor equation in the Riemannian case has been systematically investigated, since the second half of the 80th. Nowadays a lot of structure results and examples for manifolds with twistor spinors in the Riemannian setting are known (see e.g. [Lic88], [Fri89], [Hab90], [BFGK91], [KR94] and [KR97a]). However, when we discuss the twistor equation on Lorentzian spin manifolds, we will see that methods and results for solving the twistor equation in both geometries differ.

Let (M^n, g) be a Riemannian spin manifold of dimension $n \geq 3$ with fixed spin structure $(Spin(M), f)$ and let S denote the spinor bundle over M^n. The spinor bundle S is furnished with the Hermitian product $\langle\cdot,\cdot\rangle_S$, which is here in the Riemannian case positive definite. It holds that

(1) $X\cdot\varphi = 0$ for $\varphi(p) \neq 0$ implies $X(p) = 0$ in $p \in M^n$

(2) $\langle X\varphi, \psi\rangle_S = -\langle \varphi, X\psi\rangle_S$

(3) $Re\langle X\varphi, Y\varphi\rangle = g(X,Y)|\varphi|^2$

for all spinor fields $\varphi, \psi \in \Gamma(S)$ and vector fields $X, Y \in \Gamma(TM)$.

The associated vector field to a spinor $\varphi \in \Gamma(S)$ is given by

$$V_\varphi = i \sum_i \varepsilon_i \langle \varphi, s_i\varphi\rangle_S s_i,$$

where $(s_1, \ldots, s_n)$ is a local frame. It is important to notice that V_φ may vanish even if the spinor φ is not trivial.

Let $\varphi \in \Gamma(S)$ be a twistor spinor on (M^n, g). On the space of twistor spinors $\mathcal{T}(M^n)$ exist a quadratic form C and a form Q of order four, which are defined by

$$C_\varphi := Re\langle D\varphi, \varphi\rangle \qquad \text{and}$$

$$Q_\varphi = |\varphi|^2|D\varphi|^2 - (Re\langle D\varphi, \varphi\rangle)^2 - \sum_{i=1}^{n} (Re\langle D\varphi, s_i\varphi\rangle)^2 \geq 0.$$

The functions C_φ and Q_φ are constant on M^n for every twistor spinor φ and we call them the first integrals on $\mathcal{T}(M^n)$. Moreover, C_φ and Q_φ are invariant under conformal change of the metric g. Furthermore, we define for $\varphi \in \mathcal{T}(M^n)$

$$L_\varphi := \{X \cdot \varphi : X \in TM\} \subset S$$

$$H_\varphi := dist^2(i\varphi, L_\varphi)$$

$$\eta_\varphi(X) = Im\langle X\varphi, \varphi\rangle$$

$$zero(\varphi) = \{p \in M : \varphi(p) = 0\}.$$

We have the following results for twistor spinors. First, one can state that a Riemannian 3-manifold (M^3, g) with a twistor spinor is conformally flat. A Riemannian 4-manifold with a twistor spinor is self-dual, i.e. $W_- \equiv 0$. A Riemannian manifold with Killing spinors is an Einstein space. These are direct consequences of the integrability conditions (Proposition 1.2.3 and 1.2.4). Using the solution of the Yamabe problem A. Lichnerowicz proved in 1988

Theorem 1.3.1. *— ([Lic88]) Let (M^n, g) be a compact Riemannian spin manifold. There exists a conformally equivalent metric $\tilde{g} = e^{2\sigma}g$ of constant scalar curvature on M^n such that*

$$ker P \;=\; Span\{\tilde{\varphi} \in \Gamma(\tilde{S}) : \; \tilde{\varphi} \text{ is a Killing spinor on } (M^n, \tilde{g})\}.$$

Theorem 1.3.1 means that on compact Riemannian manifolds the twistor equation can be reduced to the Killing equation. But this is not only true on compact manifolds. Moreover, it holds

Theorem 1.3.2. *— (comp. [BFGK91]) Let φ be a twistor spinor on the Riemannian spin manifold (M^n, g). The set $zero(\varphi)$ is discrete on M^n and $(M^n \setminus zero(\varphi), \tilde{g} = \frac{1}{|\varphi|^4}g)$ is an Einstein manifold with non-negative scalar curvature $\tilde{R} = \frac{4(n-1)}{n}(C_\varphi^2 + Q_\varphi)$.*

(1) In case that $\tilde{R} = 0$, the spinor $\frac{1}{|\varphi|}\tilde{\varphi}$ is parallel on $(M^n \setminus zero(\varphi), \tilde{g})$. In particular, if $zero(\varphi) \neq \emptyset$ then $\tilde{R} = 0$ and $\frac{1}{|\varphi|}\tilde{\varphi}$ is parallel on $(M^n \setminus zero(\varphi), \tilde{g})$.

(2) In case that $\tilde{R} > 0$, the set $zero(\varphi)$ is empty and the twistor spinor $\frac{1}{|\varphi|}\tilde{\varphi}$ on $(M^n, \tilde{g})$ is a sum of Killing spinors.

Theorem 1.3.2 says that a Riemannian spin manifold (M^n, g) with a twistor spinor φ, which has no zeros, is conformally equivalent to an Einstein manifold, which admits a parallel or a Killing spinor. In connection with this result we remember to the works, which describe Riemannian geometries admitting parallel or Killing spinors. First, it was M.Y. Wang, who classified the possibly holonomy groups of complete simply connected irreducible non-flat Riemannian spin manifolds, which admit parallel spinors (see ([Wan89]). In 1993, Ch. Bär obtained a description of the geometrical structure of all complete simply connected Riemannian spin manifolds admitting real Killing spinors with the help of the cone construction and Wang's holonomy classification (see [Bär93], comp. [BFGK91]). The geometry of Riemannian manifolds with imaginary Killing spinors has been described by H. Baum in [Bau89].

The next theorem gives an answer to the question when a single twistor spinor is conformally equivalent to a Killing spinor.

Theorem 1.3.3. — *([Fri89]) Let (M^n, g) be a Riemannian manifold of dimension $n \geq 3$ admitting a twistor spinor φ.*

(1) If $zero(\varphi) = \emptyset$ and $C_\varphi = Q_\varphi = 0$ then φ is conformally equivalent to a parallel spinor.

(2) The twistor spinor φ is conformally equivalent to a real Killing spinor if and only if $C_\varphi \neq 0$ and $Q_\varphi = 0$.

(3) In case that $Q_\varphi = 0$ the twistor spinor φ is conformally equivalent to an imaginary Killing spinor if and only if $C_\varphi = 0$, $H_\varphi \equiv 0$ and $\pm\frac{\eta_\varphi}{|\varphi|^4} = dk$ for some positive function k on M^n.

(4) In case that $Q_\varphi \neq 0$ the twistor spinor φ is conformally equivalent to an imaginary Killing spinor if and only if

$$C_\varphi = 0 \quad \textit{and} \quad dist^2(D\varphi, Lin_{\mathbb{R}}(i\varphi, L_\varphi)) \equiv 0.$$

Since parallel and Killing spinors have no zeros, the question is posed whether there exist twistor spinors with zeros on complete, non-compact and non-conformally flat Riemannian spin manifolds. A first result to this question is the following. The length function $u = \langle\varphi, \varphi\rangle_S$ of a twistor spinor φ on an Einstein manifold (M^n, g) satisfies the PDE

$$\nabla^2 u = -\frac{\Delta u}{n} \cdot g.$$

It is well-known that if a solution u of this PDE on an Einstein manifold has a zero then (M^n, g) is of constant sectional curvature (comp. [KR94]). Moreover, in 1994 W. Kühnel and H.-B. Rademacher proved

Theorem 1.3.4. — *([KR94]) Let φ be a twistor spinor with zero on (M^n, g) such that the associated field V_φ does not vanish. Then (M^n, g) is conformally flat.*

The proof of Theorem 1.3.4 uses the observation that a Riemannian manifold (M^n, g) with a conformal vector field V, which has a zero in $p \in M$ such that $\nabla V(p) = 0$, is conformally flat in a neigborhood of p. Moreover, since the metric g is conformally equivalent to an Einstein metric outside of the zero set, it follows from the analyticity of a Riemannian Einstein metric that (M^n, g) is everywhere conformally flat.

But later in 1996, W. Kühnel and H.-B. Rademacher gave also the first example of a twistor spinor with zero and vanishing associated vector field on a complete non-conformally flat manifold. Namely, they generated a zero of a twistor spinor in the point at infinity of the conformal completion of the Eguchi-Hanson metric in dimension 4. They also generalized this construction to conformal completions of $U(n)$-invariant Ricci-flat Kähler metrics in dimension $n = 2m$ (see [KR96], citeKR97a). All these twistor spinors with zeros have the property that they are not a sum of Killing spinors with respect to any metric in the conformal class of the completed metric.

1.4 The twistor equation in Lorentzian geometry

In the following we collect results on solutions of the twistor equation, which are characteristic for the development of the theory of twistor spinors in Lorentzian spin geometry. Thereby, the solutions of the twistor equation on Fefferman spaces, on Lorentzian Einstein-Sasaki spaces and on pp-manifolds are of particular importance for us.

Let (M_1^n, g) be a time- and space-oriented Lorentzian spin manifold of dimension $n \geq 3$ and signature $(-+++\ldots)$ with spin structure $(Spin(M), f)$ and let S denote the spinor bundle. We have on S the Hermitian product $\langle\cdot,\cdot\rangle_S$, which is indefinite with index $2^{[\frac{n}{2}]-1}$. The Hermitian product $\langle\cdot,\cdot\rangle_S$ and the Clifford multiplication have the properties:

(1) $\langle X\varphi, \psi\rangle_S = \langle\varphi, X\psi\rangle_S$

(2) $Re\langle X\varphi, Y\varphi\rangle_S = -g(X, Y)\cdot\langle\varphi,\varphi\rangle_S$

(3) $Im\langle X\varphi, \varphi\rangle_S = 0$

(4) $X\cdot\varphi = 0$ for $\varphi \neq 0$ implies $\|X\|^2 := g(X, X) = 0$ and $|\varphi|^2 := \langle\varphi,\varphi\rangle_S = 0$.

The associated vector field V_φ to a spinor $\varphi \in \Gamma(S)$, which is in the Lorentzian context also called the Dirac current, is locally defined by

$$V_\varphi := -\sum_{i=1}^{n} \varepsilon_i \langle\varphi, s_i\varphi\rangle_S s_i$$

and V_φ satisfies $g(V_\varphi, V_\varphi) \leq 0$. For $n = 2m$ and $\varphi = \varphi_+ + \varphi_- \in \Gamma(S^+ \oplus S^-)$ we have $V_\varphi = V_{\varphi_+} + V_{\varphi_-}$. Since $\langle\varphi, s_1\varphi\rangle = 0$ if and only if $\varphi = 0$, it holds the remarkable property that the zero sets of φ and V_φ coincide:

$$zero(\varphi) = zero(V_\varphi).$$

We denote the dual 1-form to V_φ by ω_φ.

Let $\varphi \in \Gamma(S)$ be a non-trivial twistor spinor on a Lorentzian spin manifold. Then the associated vector field V_φ is a non-trvial conformal vector field on (M^n_1, g) (see Proposition 1.2.8). Further important characteristic data of a twistor spinor φ are its length function $|\varphi|^2$ and the 3-form $\omega_\varphi \wedge d\omega_\varphi$, which is called the twist of the associated field (comp. section 4).

In order to give a first example of twistor spinors in Lorentzian spin geometry and to illustrate that new effects occur (compared to the Riemannian case) we consider parallel spinors in dimension 3. It is known that a Lorentzian 3-metric with a parallel spinor takes locally with respect to suitable coordinates (x, y, z) the normal form

$$g = dx \circ dy - dz \circ dz + f(y, z)dy^2,$$

where f is an arbitrary function in the coordinates (y, z) (comp. e.g. [Bry00]). It is easy to see that the metric g is flat if and only if $\frac{\partial^2 f}{(\partial z)^2} = 0$ and g is conformally flat if and only if $\frac{\partial^3 f}{(\partial z)^3} = 0$. Imposing the Einstein condition for g makes the curvature vanish identically. In particular, this normal form shows that there exist parallel spinors on Lorentzian 3-manifolds, which are neither conformally flat nor Einstein.

We start now with the presentation of the Fefferman spaces and its solutions of the twistor equation. The construction that we explain here in short has been done by H. Baum in 1999 and is interesting for us, since one can look at the twistor spinors on the Fefferman spaces as the first examples of 'true' solutions of the twistor equation. For more details we refer to the original work (see [Bau99]).

Let N^{2m+1} be a smooth oriented manifold of odd dimension $2m+1$. A CR-structure on N^{2m+1} is a pair (H, J), where H is a subbundle of codimension 1 in the tangent bundle TN and $J : H \to H$ is a complex structure on H, i.e. J satisfies $J^2 = -id$ and the integrability condition

$$[JX, Y] + [X, JY] \in \Gamma(H) \quad \text{and} \quad J([JX, Y] + [X, JY]) - [JX, JY] + [X, Y] \equiv 0.$$

Besides a CR-structur (H, J), we fix a pseudohermitian 1-form $\theta \in \Omega^1(N)$ on N^{2m+1} with $\theta|_H \equiv 0$. In case that the corresponding Levi-form $L_\theta(X, Y) := d\theta(X, JY)$ is positive definite, the space (N^{2m+1}, H, J, θ) is called a strictly pseudoconvex manifold. Then the tensor $g_\theta := L_\theta + \theta \circ \theta$ is a Riemannian metric on N^{2m+1} and there is a special metric covariant derivative ∇^W with torsion on (N^{2m+1}, g_θ), which is called the Tanaka-Webster connection (see [Tan75], [Web78]).

Now, let us assume that (N^{2m+1}, H, J, θ) is a strictly pseudoconvex spin manifold. The spin structure of (N^{2m+1}, g_θ) defines a square root of the canonical line bundle

$$\Lambda^{m+1,0}N := \{\omega \in \mathbb{C} \otimes \Lambda^{m+1}N | \ X \lrcorner \ \omega = 0 \quad \text{for all } X \in \overline{T}_{10}\},$$

where $\overline{T}_{10} \subset \mathbb{C} \otimes TN^{2m+1}$ denotes the eigenspace of J to the eigenvalue $-i$. Then we denote by (F^{2m+2}, π, N) the S^1-principal fibre bundle associated to the square root of $\Lambda^{m+1,0}N$. There exists a unique connection A^W in the S^1-bundle F^{2m+2}, which induces the Tanaka-Webster covariant derivative on the square root of $\Lambda^{m+1,0}N$. The Fefferman metric on F^{2m+1} is defined to be

$$h_\theta := \pi^* L_\theta - i\frac{8}{m+2}\pi^*\theta \circ A_\theta,$$

where $A_\theta := A^W - i\frac{R^W}{4(m+1)}\pi^*\theta$ and R^W is the Tanaka-Webster-scalar curvature. By definition, the Fefferman metric h_θ is a Lorentzian metric and the conformal class $[h_\theta]$ does not depend on the pseudohermitian form θ, but depends only on the CR-structure of N^{2m+1}. Moreover, the Fefferman metric h_θ is S^1-invariant and the S^1-fibres of (F^{2m+2}, π, N) are lightlike with respect to h_θ. The Lorentzian manifold (F^{2m+2}, h_θ) with its canonically induced spin structure is called Fefferman space of the strictly pseudoconvex spin manifold (N^{2m+1}, H, J, θ). The Fefferman metric was first discovered by C. Fefferman for the case of strictly pseudoconvex hypersurfaces $N \subset \mathbb{C}^{m+1}$ (see [Fef76]).

The Fefferman spaces admit the following twistoriel characterization.

Theorem 1.4.1. — *([Bau99]) Let (N^{2m+1}, H, J, θ) be a strictly pseudoconvex spin manifold and let (F^{2m+2}, h_θ) be its Fefferman space. Then there exist two linearly independent twistor spinors φ on (F^{2m+2}, h_θ) with the properties:*

(1) V_φ is a regular lightlike Killing field

(2) $V_\varphi \cdot \varphi = 0$

(3) $\nabla^S_{V_\varphi}\varphi = ic\varphi$, where $c \in \mathbb{R} \setminus 0$.

Conversely, let (B^{2m+2}, h) be a Lorentzian spin manifold, which admits a non-trivial twistor spinor satisfying the conditions (1), (2) and (3). Then there exists a strictly pseudoconvex spin manifold (N^{2m+1}, H, J, θ) such that (B^{2m+2}, h) is locally isometric to the Fefferman space (F, h_θ) of (N^{2m+1}, H, J, θ).

The second part of the proof of Theorem 1.4.1 is based on a geometric characterization of Fefferman spaces given by Sparling (see [Spa85], [Gra87]). Let (B^{2m+2}, h) be a Lorentzian manifold of dimension $2m + 2$. If V is a regular lightlike Killing field on (B^{2m+2}, h) such that

$$V \lrcorner W = 0, \quad V \lrcorner C = 0 \quad \textit{and} \quad K(V, V) = \textit{const} < 0,$$

then (B^{2m+2}, h) is locally isometric to the Fefferman space of a strictly pseudoconvex manifold (N^{2m+1}, H, J, θ) of dimension $2m + 1$. In the situation of Theorem 1.4.1 the associated field V_φ of a twistor spinor φ satisfies Sparling's caracterization conditions.

The interesting point of the solutions of the twistor equation on Fefferman spaces is that a Fefferman space (F^{2m+2}, h) is never conformally equivalent to an Einstein space and the twistor spinors on (F^{2m+2}, h) are not conformally equivalent to a sum of Killing spinors. In particular, the twistor spinors φ have vanishing spinor norm $|\varphi|^2 = 0$ on F^{2m+2}, but they do not have zeros, and the twist $\omega_\varphi \wedge d\omega_\varphi$ of the associated vector fields V_φ is not trivial. The twistor spinors on Fefferman spaces are the only known examples with such properties.

Next we present solutions of the Killing equation on Lorentzian Einstein-Sasaki manifolds in odd dimensions $n = 2m + 1$. One can show that a Lorentzian spin manifold (M_1^n, g) has imaginary Killing spinors to the Killing number $i\lambda$ if and only if the cone

$$C_{2\lambda}^{-}(M_1^n) := (M \times \mathbb{R},\ g_C := (2\lambda t)^2 g - dt^2)$$

over M_1^n admits parallel spinors. We describe the case of irreducible cone $(C_{2\lambda}^{-}(M_1^n), g_C)$ with parallel spinor, which leads to Lorentzian Einstein-Sasaki structures on M_1^n.

A Lorentzian Sasaki manifold is a tripel (M, g, ξ), where g is a Lorentzian metric, the vector field ξ is timelike with $g(\xi, \xi) = -1$ and the tensor $J := -\nabla\xi : TM \to TM$ satisfies

$$J^2(X) = -X - g(X, \xi)\xi \quad \text{and} \quad (\nabla_X J)(Y) = -g(X, Y)\xi + g(Y, \xi)X.$$

It is well-known that a manifold (M, g) has a Lorentzian Sasaki structure if and only if the cone $C_1^{-}(M)$ admits a pseudo-Riemannian Kähler structure. Moreover, the Einstein condition on a Lorentzian Sasaki manifold (M, g, ξ) is equivalent to the property that the cone $C_1^{-}(M)$ is in addition Ricci-flat, i.e. the cone $C_1^{-}(M)$ has holonomy in $SU(1, m)$.

There exists a twistoriel characterization of the Lorentzian Einstein-Sasaki manifolds.

Theorem 1.4.2. *— (comp. [Kat99] and [Bau00a]) Let (M^{2m+1}, g, ξ) be a simply connected, Lorentzian Einstein-Sasaki manifold. Then (M^{2m+1}, g) is a spin manifold and there exists a twistor spinor $\varphi \in \Gamma(S)$ such that*

(1) V_φ is a timelike Killing vector field with $g(V_\varphi, V_\varphi) = -1$

(2) $V_\varphi \cdot \varphi = -\varphi$

(3) $\nabla^S_{V_\varphi}\varphi = -\frac{1}{2}i\varphi$.

In particular, φ is an imaginary Killing spinor and $V_\varphi = \xi$. Conversely, let (M^{2m+1}, g) be a Lorentzian spin manifold with a twistor spinor satisfying (1), (2) and (3) then (M^{2m+1}, g, V_φ) is a Lorentzian Einstein-Sasaki manifold.

Lorentzian Einstein-Sasaki manifolds arise as S^1-fibre bundles over Kähler manifolds with negative scalar curvature. This S^1-bundle construction provides an explicit way of constructing the twistor spinors on Lorentzian Einstein-Sasaki manifolds in Theorem 1.4.2.

Finally, we want to mention some results on parallel spinors and real Killing spinors in Lorentzian spin geometry.

It exists a complete classification of the geometries of simply connected, irreducible and non-locally symmetric semi-Riemannian spin manifolds, which admit parallel spinors (see [BK99]). But the list shows that there is no irreducible Lorentzian spin manifold with parallel spinors and a classification of the possible holonomies of indecomposable non-irreducible manifolds does not exist. However, in low dimensions the local normal forms of Lorentzian metrics with parallel spinors are well studied (comp. [Bry00]).

One class of Lorentzian spin manifolds with parallel spinors are the pp-manifolds (pp = plane waves and parallel rays). A pp-manifold (M_1^n, g) is distinguished by the properties that it admits a parallel null vector field V and that the Riemannian curvature tensor satisfies the condition

$$tr_{(3,5)(4,6)} R^{\nabla} \otimes R^{\nabla} = 0.$$

A local normal form for a pp-metric is given by

$$dx_1 \circ dx_2 + f(x_2, \dots, x_n) dx_2^2 + \sum_{i=3}^{n} dx_i^2.$$

A further class of Lorentzian spin manifols with parallel spinors is the following generalization of pp-manifolds. Let (N^n, h) be a Riemannian manifold with holonomy in $SU(m)$, $Sp(m)$, G_2 or $Spin(7)$ and let $f : \mathbb{R} \times N \to \mathbb{R}$ be a smooth function. Then the Lorentzian manifold

$$M_1^n := \mathbb{R}^2 \times N, \quad g_{(t,s,x)} := -2dt \circ ds + f(s,x) ds^2 + h_x$$

has parallel spinors. The metric g is Ricci-flat if and only if the functions $f(s, \cdot) : F \to \mathbb{R}$ are harmonic for all $s \in \mathbb{R}$.

In [Bau00b] all twistor spinors on Lorentzian symmetric spaces (M_1^n, g) are explicitly described. In particular, it is proven that if (M_1^n, g) is indecomposable and non-conformally flat then each twistor spinor is parallel and $\dim \mathcal{T}(M_1^n) = \frac{1}{2} \dim S$. These Lorentzian symmetric spaces have solvable transvection group and they are special pp-manifolds.

Lorentzian spin manifolds with real Killing spinors were investigated by Ch. Bohle in [Boh00]. It is proved there that a Lorentzian spin manifold with real Killing spinors is locally isometric to a warped product of the form

$$F \times_\sigma I := (F \times I,\ g = \sigma^2 h + \varepsilon dt^2), \qquad \varepsilon = \pm 1,$$

where $\sigma > 0$ is some warping function and F is a Riemannian or Lorentzian spin manifold, which admits parallel or Killing spinors. In particular, one can see from the warped product structure that a Lorentzian spin manifold with real Killing spinor is in any case an Einstein space. There exist also classifying results on complete Lorentzian spin manifolds with real Killing spinors (comp. [Boh00] or [Bau00a]). The lenght function $|\varphi|^2 : M_1^n \to \mathbb{R}$ of a real Killing spinor φ on a complete Lorentzian spin manifold is always surjective.

2 The twistor equation in conformal Cartan geometry

In 1.2 we introduced twistor spinors on a semi-Riemannian spin manifold (M^n_p, g) as solutions of the twistor equation

$$\nabla^S_X \varphi + \frac{1}{n} X \cdot D\varphi = 0 \qquad \text{for all} \quad X \in \Gamma(TM).$$

In [PM72] the twistor equation on a space-time was set in relation to a 'twistor connection' on the vector bundle of 'local twistors'. It was H. Friedrich, who showed in [Fri77] that the 'twistor connection' on a 4-dimensional space-time is induced by the canonical normal Cartan connection of conformal geometry. In the following we will work out this fact on arbitrary semi-Riemannian spin manifolds. From the point of view that we describe here the solvability of the twistor equation is related to the holonomy theory of the normal conformal Cartan connection.

In the first part of this section we will recall the basic definitions of Cartan geometry and we will introduce the development of a flat Cartan geometry and the corresponding holonomy representation of the fundamental group of the base manifold. Then we will define almost Hermitian symmetric structures and normal Cartan connections. In 2.2 we will discuss conformal geometries of arbitrary signature, which belong to the class of Hermitian symmetric geometries. In particular, we define the standard conformal spin spaces $\hat{C}^{p,q}$. As we stated in 1.2, a spinor field $\varphi \in \Gamma(S)$ on a semi-Riemannian manifold is a 'twistor' if and only if

$$\nabla^{TC}_X \begin{pmatrix} \varphi \\ D\varphi \end{pmatrix} := \begin{pmatrix} \nabla^S_X & \frac{1}{n} X \cdot \\ -\frac{n}{2} K(X) \cdot & \nabla^S_X \end{pmatrix} \begin{pmatrix} \varphi \\ D\varphi \end{pmatrix} = 0.$$

In 2.3 we will prove that the covariant derivative ∇^{TC} is induced by the normal Cartan connection of conformal geometry. Once we have understood twistor spinors as parallel sections in $E \cong S \oplus S$, we are able to characterize conformally flat manifolds, which admit twistor spinors, with the help of a development in the standard conformal spin space $\hat{C}^{p,q}$. This will be done in the last part.

2.1 Cartan geometry, normal Cartan connections and development

First, we recall the definition of Cartan geometry and define the development map of a flat Cartan geometry. Then we give a brief summary of the construction of the canonical normal Cartan connection on an almost Hermitian symmetric structure due to [CSS97].

We start with the definition of Klein geometry, which is a generalized concept for classical geometries like Euclidean or affine geometry (comp. [Sha96]).

Definition 2.1.1. — *A Klein geometry is a pair (G, H), where G is a Lie group and H is a closed subgroup of G such that the homogenous space G/H, which is called the Klein model, is connected.*

Example. Let $Aff(n)$ denote the group of affine transformations on $\mathbb{R}^n$. Affine geometry is the pair $(Aff(n), Gl(n))$ and its standard model is the affine n-space $\mathbb{A}^n = Aff(n)/Gl(n)$.

A Cartan geometry on a C^∞-manifold M is modeled on a Klein geometry (G, H). Let $(\mathfrak{g}, \mathfrak{h})$ denote the pair of Lie algebras to (G, H).

Definition 2.1.2. — *Let M be a C^∞-manifold, (G, H) a Klein geometry and $P \to M$ an H-principal fibre bundle. A $\mathfrak{g}$-valued 1-form $\omega : TP \to \mathfrak{g}$, which satisfies the conditions*

(1) $\omega(A^*) = A$, where A^* is the fundamental vector field on P corresponding to $A \in \mathfrak{h}$

(2) $R_a^* \omega = Ad(a^{-1})\omega$

(3) $\omega_p : T_pM \to \mathfrak{g}$ is an isomorphism for each $p \in P$,

is called a Cartan connection of type (G, H). The pair (P, ω) is called a Cartan geometry on M.

The third condition of Definition 2.1.2 implies $dimP = dimG$ for a Cartan geometry (P, ω) of type (G, H) on M.

Example. Let $P := Aff(n) \xrightarrow{\pi} \mathbb{A}^n$ be the $Gl(n)$-principal fibre bundle of linear frames on the affine n-space $\mathbb{A}^n$ and denote by

$$\eta : TP \to \mathfrak{gl}(n)$$

the standard connection on P with vanishing torsion and curvature. The canonical form θ on P is defined by

$$\begin{array}{rcccl} \theta : & TP & \to & \mathbb{R}^n, & \\ & X_p & \mapsto & p^{-1}\pi_*(X_p), & \quad p \in P \end{array}$$

where a linear frame p is viewed as a linear isomorphism $p : \mathbb{R}^n \to T_{\pi(p)}\mathbb{A}^n$ as usual. The 1-form

$$\omega = \theta + \eta : TP \to \mathbb{R}^n \oplus \mathfrak{gl}(n) \cong \mathfrak{aff}(n)$$

is a Cartan connection on $P \to \mathbb{A}^n$ of affine type $(Aff(n), Gl(n))$ and (P, ω) is a Cartan geometry on $\mathbb{A}^n$.

Cartan connections induce connections in the usual sense on an enlarged bundle. Let (P, ω) be a Cartan geometry on M of type (G, H). The enlarged bundle $\bar{P} := P \times_H G \to M$ is a G-principal fibre bundle and the mapping

$$p \in P \mapsto [p, e] \in \bar{P}$$

is a natural embedding of principal fibre bundles. The image of the Cartan connection ω on P in $\bar{P}$ can be uniquely extended to a $\mathfrak{g}$-valued 1-form $\bar{\omega}$, which is a principal fibre bundle connection on $\bar{P}$. The $\mathfrak{g}$-valued 2-form

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega]$$

is the Cartan curvature of ω on P. If the curvature form Ω of (P, ω) vanishes, then the Cartan geometry (P, ω) is called flat. The curvature form $\bar{\Omega} = d\bar{\omega} + \frac{1}{2}[\bar{\omega}, \bar{\omega}]$ of the corresponding connection $\bar{\omega}$ on $\bar{P}$ also vanishes in this case.

Example. Let (G, H) be a Klein geometry. Then $G \to G/H$ is an H-principal bundle and the Maurer-Cartan form $\omega_G : TG \to \mathfrak{g}$ defined by

$$X_a \in T_aG \quad \mapsto \quad \omega_G(X_a) = L_{a^{-1}}(X_a) \quad \in T_aG \cong \mathfrak{g}$$

is a Cartan connection on G. Since by the Maurer-Cartan equation

$$\Omega_G = d\omega_G + \frac{1}{2}[\omega_G, \omega_G] = 0,$$

the Cartan geometry (G, ω_G) on G/H is flat. There exists a canonical parallel section with respect to $\bar{\omega}_G$ in the enlarged principal fibre bundle $\bar{G} := G \times_H G$:

$$\begin{array}{rccc} \bar{\alpha}: & G/H & \to & \bar{G} \, . \\ & aH & \mapsto & [a, a^{-1}] \end{array}$$

Let (P, ω) be a flat Cartan geometry of type (G, H) on a connected and simply connected C^∞-manifold M. The enlarged G-principal fibre bundle $\bar{P} = P \times_H G$ is equipped with the flat connection $\bar{\omega}$. We fix a point $\bar{p} \in \bar{P}$. Since $\bar{\omega}$ is flat and M is connected and simply connected, the parallel displacement on $\bar{P}$ defines in a unique way a G-equivariant C^∞-map

$$\bar{\delta}_{\bar{p}} : \bar{P} \to G$$

with $\bar{\delta}_{\bar{p}}(\bar{p}) = e$ and $\bar{\delta}_{\bar{p}}^* \omega_G = \bar{\omega}$. Moreover, there exists an element $a \in G$ for every C^∞-map $\bar{\delta} : \bar{P} \to G$ with the property $\bar{\delta}^* \omega_G = \bar{\omega}$ such that

$$\bar{\delta} = L_a \circ \bar{\delta}_{\bar{p}},$$

where L_a denotes the left translation by a. The natural embedding of P in $\bar{P}$ gives rise to a C^∞-map defined by

$$\begin{array}{rccc} \delta_{\bar{p}} : & P & \to & G. \\ & p & \mapsto & \bar{\delta}_{\bar{p}}([p, e]) \end{array}$$

This map $\delta_{\bar{p}}$ is H-equivariant and, since ω is the pullback of $\bar{\omega}$ on $\bar{P}$, it holds $\omega_G \circ d\delta_{\bar{p}} = \omega$. We can conclude that $\delta_{\bar{p}} : (P, \omega) \to (G, \omega_G)$ is a local isomorphism of Cartan geometries and every local isomorphism $\delta : (P, \omega) \to (G, \omega_G)$ is of the form $L_a \circ \delta_{\bar{p}}$ for some $a \in G$. The induced map

$$\delta_{\bar{p}}^M : M \to G/H$$

of $\delta_{\bar{p}}$ on the base spaces is called a development map of M in the Klein model G/H.

Let M be a connected C^∞-manifold with flat Cartan geometry (P, ω). The universal covering space $\tilde{M} \xrightarrow{\pi} M$ is equipped with the flat Cartan geometry $(\tilde{P}, \tilde{\omega}) := (\pi^* P, \pi^* \omega)$, which is the pullback of (P, ω) by π. The fundamental group $\pi_1(M)$ acts on $\tilde{M}$ by deck transformations and these deck transformations preserve the Cartan geometry $(\tilde{P}, \tilde{\omega})$. This induces a group homomorphism

$$w : \pi_1(M) \to Aut(\tilde{P}, \tilde{\omega})$$

into the automorphism group of the Cartan geometry $(\tilde{P}, \tilde{\omega})$ on $\tilde{M}$. Let $\delta : \tilde{M} \to G/H$ be any development defined as before. For each $\gamma \in \pi_1(M)$ there exists a unique $a_\gamma \in G$ such that

$$\delta \circ w(\gamma) = L_{a_\gamma} \circ \delta.$$

This unique relation gives rise to the holonomy representation of the fundamental group

$$\begin{array}{rcl} \kappa \ : \pi_1(M) & \to & G \ . \\ \gamma & \mapsto & a_\gamma \end{array}$$

The holonomy representations corresponding to different developments are conjugated in G.

We come now to the definition of almost Hermitian symmetric structures on a C^∞-manifold. One particularity of these structures is that they admit a uniquely determined normal Cartan connection. We will explain this in a short manner. The approach that we present here is due to the detailed work [CSS97] on almost Hermitian symmetric structures and normal Cartan connections. The important point for us is that conformal structures belong to the class of almost Hermitian symmetric structures and, in particular, it exists a canonical normal conformal Cartan connection. We will discuss the conformal cases explicitly in the next part.

First remember that a Lie algebra $\mathfrak{g}$ is called $|1|$-graded if

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

is a direct sum of subspaces such that

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \qquad \text{for all} \ \ i, j \in \{-1, 0, 1\}.$$

Then it is known that the map $\mathfrak{g}_0 \to \mathfrak{gl}(\mathfrak{g}_{-1})$ is the inclusion of a subalgebra and the Killing form of $\mathfrak{g}$ identifies $\mathfrak{g}_1$ as a $\mathfrak{g}_0$-module with the dual of $\mathfrak{g}_{-1}$ (see [Och70]).

Let G be a connected and semisimple Lie group with $|1|$-graded Lie algebra $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. We denote by B the closed (parabolic) subgroup of G corresponding to the Lie subalgebra $\mathfrak{b} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ in $\mathfrak{g}$. Then there are uniquely defined closed subgroups B_0 and B_1 in B with Lie algebras $\mathfrak{g}_0$ resp. $\mathfrak{g}_1$. The exponential map to $\mathfrak{g}_1$ is a diffeomorphism onto B_1 and B_1 is a normal vector subgroup in B. Moreover, the group B is the semidirect product of its subgroups B_0 and B_1 (see [Och70] and [CSS97]):

$$B = B_0 \rtimes B_1 \qquad \text{and} \qquad B_0 \cong B/B_1.$$

Let us consider now a Lie group G', which is not necessarily connected and is a covering group of G with covering map $\lambda' : G' \to G$. There exists a uniquely defined closed subgroup B' of G' such that the restriction $\lambda'|_{B'} : B' \to B \subset G$ is a group covering and $G'/B' \cong G/B$. The Lie group B' is isomorphic to the semidirect product $B'_0 \rtimes B_1$, where B'_0 is a closed subgroup of B', which covers B_0, and $B_1 = \exp \mathfrak{g}_1$ is the simply connected normal vector subgroup in B' generated by the Lie algebra $\mathfrak{g}_1$. Such a pair (G', B') is called a Hermitian symmetric geometry and the corresponding Klein model G'/B' is a Hermitian symmetric space.

Definition 2.1.3. — *Let (G, B) be a Hermitian symmetric geometry, let M be a C^∞-manifold of dimension $n = \dim \mathfrak{g}_{-1}$ and let $B(M) \to M$ be a B-principal fibre bundle on M equipped with a 1-form*

$$\theta = \theta_{-1} \oplus \theta_0 \in \Omega^1(B(M), \mathfrak{g}_{-1} \oplus \mathfrak{g}_0)$$

such that

(1) $\theta_{-1}(X) = 0$ if and only if $X \in TB(M)$ is vertical

(2) $\theta_0(A^* + B^*) = A$ for all $A \in \mathfrak{g}_0$ and $B \in \mathfrak{g}_1$

(3) $R_b^*\theta = Ad(b^{-1})\theta$ for all $b \in B$, where Ad means the restriction of the adjoint action to the vector space $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0$.

The form θ is called the soldering form and $(B(M), \theta)$ is called an almost Hermitian symmetric structure on M of type (G, B).

We use the following convention. Let P_1 and P_2 be principal fibre bundles over a base space M with structure groups G_1 resp. G_2. If $\bar{\lambda} : P_1 \to P_2$ is a principal fibre bundle morphism over the idenity of M associated to a homomorphism $\lambda : G_1 \to G_2$, which is a covering of a subgroup in G_2, then we call the bundle P_1 a G_1-reduction of P_2.

A (G, B)-structure $(B(M), \theta)$ defined as above may be understood as a 'second order structure' on M (comp. [CSS97] II.1.8.), whereas the induced B_0-principal fibre bundle $B_0(M) := B(M)/B_1$ with induced soldering form θ_{-1} is a B_0-reduction of the first order frame bundle $Gl^{(1)}(M)$ on M. There is also a way of constructing a (G, B)-structure from a B_0-reduction of $Gl^{(1)}(M)$. This inverse process is called the first prolongation of a first order structure (comp. [CSS97] II.1.). But it is not true in general that an almost Hermitian symmetric structrue $(B(M), \theta)$ is uniquely determined by its first order reduction $B_0(M)$, since $B_0(M)$ need not contain any geometric information. This happens in the case of projective geometry, where the first order reduction $B_0(M)$ is isomorphic to the linear frame bundle $Gl^{(1)}(M)$ itself (comp. [CSS97] II.1.9. or [Kob72], p. 143).

Let $(B(M), \theta)$ be an almost Hermitian symmetric structure and $B_0(M)$ its first order reduction. There exist B_0-equivariant sections $\sigma \in \Gamma(B_0(M); B(M))$. For such a section σ the induced 1-form

$$\sigma^*\theta = \sigma^*\theta_{-1} + \sigma^*\theta_0$$

is a Cartan connection and the $\mathfrak{g}_0$-valued component $\sigma^*\theta_0$ is a connection on $B_0(M)$. Moreover, there exists a uniquely defined Cartan connection $\omega^\sigma = \theta_{-1} + \theta_0 + \omega_1^\sigma$ on $B(M)$ satisfying $\omega_1^\sigma|_{T\sigma(B_0(M))} = 0$ (see [CSS97] I.3. and II.1.7.).

The curvature function $\nu \in C^\infty(B(M), \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}^* \otimes \mathfrak{g})$ of an arbitrary Cartan connection ω on $B(M)$ is defined by

$$\nu(u)(X, Y) := \Omega(\omega^{-1}(X), \omega^{-1}(Y))(u), \quad X, Y \in \mathfrak{g}_{-1},\ u \in B(M).$$

The map $ad : \mathfrak{g}_0 \to \mathfrak{gl}(\mathfrak{g}_{-1})$ is injective and therefore the $\mathfrak{g}_0$-valued component ν_0 of ν can be viewed as a function on $B(M)$ with values in $\mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_1^*$.

Definition 2.1.4. — *(1) An admissable connection*

$$\omega = \omega_{-1} + \omega_0 + \omega_1 \in \Omega^1(B(M), \mathfrak{g})$$

on an almost Hermitian symmetric structure $(B(M), \theta)$ is a Cartan connection with $\omega_{-1} = \theta_{-1}$ and $\omega_0 = \theta_0$.

(2) A normal Cartan connection $\omega \in \Omega^1(B(M), \mathfrak{g})$ is an admissable connection, which satisfies

$$tr\nu_0(X, Y) := \sum_{i=1}^{\dim \mathfrak{g}_{-1}} \nu_0(e_i, X)(Y)(e^i) = 0,$$

where e_i is a basis of $\mathfrak{g}_{-1}$ and e^i denotes the dual basis of $\mathfrak{g}_1$.

Two admissable Cartan connections ω and $\bar{\omega}$ on $(B(M), \theta)$ differ only in the $\mathfrak{g}_1$-component and it exists a deformation tensor $\Gamma \in C^\infty(B(M), \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_1)$ such that

$$\bar{\omega} - \omega = \Gamma \circ \theta_{-1}.$$

The tensor Γ on $B(M)$ is always the pullback of a tensor on the base space M (see [CSS97] I.3.10.).

It is known that on every almost Hermitian symmetric structure $(B(M), \theta) \to M$ (except some very low dimensional cases), there exists a canonical normal Cartan connection ω_N ([CSS97] II. 2. and [Kob72]). Moreover, for every B_0-equivariant section $\sigma \in \Gamma(B_0(M); B(M))$ the difference between the admissable connection ω^σ and the canonical Cartan connection ω_N is described by the deformation tensor Γ^σ and this tensor Γ^σ can be expressed by a universal formula in terms of the curvature tensor of the connection $\sigma^*\theta_0$ on $B_0(M)$. Notice that if the canonical Cartan connection ω_N on $B(M)$ is flat then the Cartan geometry $(B(M), \omega_N) \to M$ is locally isomorphic to the model $(G, \omega_G) \to G/B$.

2.2 The conformal spin spaces $C^{p,q}$ and $\hat{C}^{p,q}$

After we have defined Klein geometries in general, we present in this part the standard models of conformal (spin) geometry. We introduce these models first in a geometric manner and then give the group-theoretical description.

We denote by $\mathbb{R}^{p,q} := (\mathbb{R}^n, \langle\cdot,\cdot\rangle_{p,q}), n \geq 3$ and $p \in \{0, \dots, n\}$, $q := n - p$, the n-dimensional Euclidean space of index p with scalar product

$$\langle x, y\rangle_{p,q} := -\sum_{i=1}^{p} x_i y_i + \sum_{i=p+1}^{n} x_i y_i.$$

Remember that $Cl_{p,q}$ denotes the Clifford algebra of $\mathbb{R}^{p,q}$, $Spin(p,q)$ denotes the spin group and $\lambda : Spin(p,q) \to SO(p,q)$ is the canonical two-fold group covering.

By $\mathbb{R}^{p+1,q+1} := (\mathbb{R}^{n+2}, (\cdot,\cdot)_{p+1,q+1})$ we denote the pseudo-Euclidean space of dimension $n+2$ and index $p+1$, but this time equipped with the scalar product

$$(x,y)_{p+1,q+1} := x_0 y_{n+1} + x_{n+1} y_0 - \sum_{i=1}^{p} x_i y_i + \sum_{i=p+1}^{n} x_i y_i .$$

Notice that we use for this scalar product the round brackets $(\cdot,\cdot)_{p+1,q+1}$. Furthermore, we define the space

$$Q^{p,q} := \{[x] \in P^{n+2}(\mathbb{R}) |\; x \in \mathbb{R}^{p+1,q+1} \setminus \{0\},\; (x,x)_{p+1,q+1} = 0\},$$

which is a regular quadric in the projective space $P^{n+2}(\mathbb{R})$. The regular quadric $Q^{p,q}$ is two-fold covered by the set $\hat{Q}^{p,q}$ of time-oriented null directions in $\mathbb{R}^{p+1,q+1}$ and the space $\hat{Q}^{p,q}$ is embedded in $\mathbb{R}^{p+1,q+1}$ by

$$\begin{array}{rccc} i: & \hat{Q}^{p,q} & \hookrightarrow & \mathbb{R}^{p+1,q+1} \\ & \mathbb{R}_+ \cdot x & \mapsto & \sqrt{\dfrac{2}{\langle x,x\rangle_{n+2}}} \cdot x \end{array},$$

where $\langle \cdot,\cdot \rangle_{n+2}$ is the usual Euclidean scalar product. One can see that the space $\hat{Q}^{p,q}$ is homeomorphic to $S^p \times S^q$, since the image $i(\hat{Q}^{p,q})$ is the Carthesian product of the unit spheres in the semi-Riemannian product $\mathbb{R}^{p+1,q+1} \cong \mathbb{R}^{0,p+1} \times \mathbb{R}^{q+1,0}$. Notice that $\hat{Q}^{0,n}$ is not connected and $\hat{Q}^{1,n-1}$ is not simply connected.

We denote by $\hat{c}$ the conformal structure on $\hat{Q}^{p,q}$ that arises from the induced metric $i^*(\cdot,\cdot)_{p+1,q+1}$. This conformal structure $\hat{c}$ is flat, since the induced metric is conformally flat. The flat conformal structure $\hat{c}$ projects to a flat conformal structure c on the quadric $Q^{p,q}$. For $p \neq 0, n$ we denote the arising conformal spaces by

$$C^{p,q} := (Q^{p,q}, c) \quad \text{and} \quad \hat{C}^{p,q} := (\hat{Q}^{p,q}, \hat{c}).$$

In case that $p = 0$ we use the notation

$$\hat{C}^{0,n} = C^{0,n} := (Q^{0,n}, c),$$

since $\hat{Q}^{0,n}$ is not connected. The conformally flat space $C^{p,q}$ is the conformal compactification of $\mathbb{R}^{p,q}$ and is called the (pseudo)-Möbius sphere of dimension n and index p (comp. e.g. [Sch97]). The space $\hat{C}^{p,q}$ is a conformal spin manifold, since the normal bundle of $i(Q^{p,q})$ in $\mathbb{R}^{p+1,q+1}$ admits a globally defined orthonormal frame field (comp. 2.4). We call the model $\hat{C}^{p,q}$ the conformal spin space of dimension n and index p. The space $C^{p,q}$ is not in general conformally spin (comp. 2.4).

We want to describe the models $\hat{C}^{p,q}$ as homogenous spaces. Let us denote

$$GSO(p,q) := \begin{cases} SO(p+1,q+1) : p \neq 0, n \\ SO^+(p+1,q+1) : p = 0, n \end{cases}.$$

The group $GSO(p,q)$ acts naturally on $\hat{C}^{p,q}$ through its embedding in $\mathbb{R}^{p+1,q+1}$. This action is conformal, orientation-preserving and transitive. With respect to some point $e \in \hat{C}^{p,q}$ the isotropy group of the $GSO(p,q)$-action on $\hat{C}^{p,q}$ is the subgroup

$$BSO(p,q) := \Big\{ \begin{pmatrix} a^{-1} & s & b \\ 0 & A & r \\ 0 & 0 & a \end{pmatrix} \mid A \in SO(p,q), a \in \mathbb{R}_+, s \in (\mathbb{R}^n)^* \Big\}$$

of $GSO(p,q)$, where $r = -aAI_{p,q}s^*$ with $I_{p,q} := \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}$ and $b = -\frac{a}{2}\langle s^*, s^*\rangle_{p,q}$, and it holds $\hat{C}^{p,q} \cong GSO(p,q)/BSO(p,q)$. The kernel of the linear isotropy representation of $BSO(p,q)$ on the tangent space $T_e\hat{C}^{p,q}$ is the vector group

$$B_1(p,q) := \left\{ \begin{pmatrix} 1 & s & b \\ 0 & I & -I_{p,q}s^* \\ 0 & 0 & 1 \end{pmatrix} \in GSO(p,q) \mid s \in (\mathbb{R}^n)^*, b = -\frac{1}{2}\langle s^*, s^*\rangle_{p,q} \right\},$$

which is a normal subgroup of $BSO(p,q)$. The group $BSO(p,q)/B_1(p,q)$ is isomorphic to the conformal group $CSO(p,q) := \mathbb{R}_+ \times SO(p,q)$, which is embedded in $GSO(p,q)$ by

$$\begin{array}{rccc} i_c : & CSO(p,q) & \hookrightarrow & GSO(p,q) \ . \\ & aA & \mapsto & \begin{pmatrix} a^{-1} & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a \end{pmatrix} \end{array}$$

The Lie algebra $\mathfrak{so}(p+1,q+1)$ is semisimple and $|1|$-graded

$$\mathfrak{so}(p+1,q+1) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

where

$$\mathfrak{g}_{-1} := \left\{ \begin{pmatrix} 0 & 0 & 0 \\ l & 0 & 0 \\ 0 & -l^*I_{p,q} & 0 \end{pmatrix} \mid l \in \mathbb{R}^n \right\},$$

$$\mathfrak{g}_0 := \left\{ \begin{pmatrix} -a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a \end{pmatrix} \mid A \in \mathfrak{so}(p,q), a \in \mathbb{R} \right\},$$

$$\mathfrak{g}_1 := \left\{ \begin{pmatrix} 0 & s & 0 \\ 0 & 0 & -I_{p,q}s^* \\ 0 & 0 & 0 \end{pmatrix} \mid s \in \mathbb{R}^{n^*} \right\}.$$

The subalgebra $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ is the Lie algebra of $BSO(p,q)$, the subalgebra $\mathfrak{g}_1$ is the Lie algebra of $B_1(p,q)$ and $\mathfrak{g}_0 \cong \mathfrak{cso}(p,q)$. With the definitions in 2.1 we see that $(GSO(p,q), BSO(p,q))$ is a Hermitian symmetric geometry and $\hat{C}^{p,q} \cong GSO(p,q)/BSO(p,q)$ is a Hermitian symmetric space for every dimension n and index p. In particular, it holds that the $\mathfrak{g}_0$-modules $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ are dual to each other with respect to the Killing form on $\mathfrak{so}(p+1,q+1)$ and that

$$BSO(p,q) \cong CSO(p,q) \rtimes B_1(p,q)$$

is a semidirect product.

Since the model $\hat{C}^{p,q}$ is conformally spin, it is useful to give a spin description of the Hermitian symmetric geometry $(GSO(p,q), BSO(p,q))$. We denote

$$GSpin(p,q) := \begin{cases} Spin(p+1,q+1) : p \neq 0, n \\ Spin^+(p+1,q+1) : p = 0, n \end{cases} .$$

The group $GSpin(p,q)$ acts on $\hat{C}^{p,q}$ via the canonical group-covering $\lambda : GSpin(p,q) \to GSO(p,q)$. There exists a uniquely defined closed subgroup $BSpin(p,q)$ of $GSpin(p,q)$, which covers $BSO(p,q)$ in $GSO(p,q)$ by λ, such that

$$\hat{C}^{p,q} \cong GSpin(p,q)/BSpin(p,q).$$

The pair $(GSpin(p,q), BSpin(p,q))$ is a Hermitian symmetric geometry. There is a uniquely defined embedding

$$i_{cs} : CSpin(p,q) \hookrightarrow GSpin(p,q)$$

of the conformal spin group $CSpin(p,q) := \mathbb{R}_+ \times Spin(p,q)$ such that $\lambda \circ i_{cs} = i_c \circ \lambda$. Then we have

$$BSpin(p,q) \cong CSpin(p,q) \rtimes B_1(p,q),$$

where in this case $B_1(p,q)$ should be understood as the simply connected normal vector subgroup in $BSpin(p,q)$ corresponding to the subalgebra $\lambda_*^{-1}\mathfrak{g}_1 \subset \mathfrak{spin}(p+1,q+1)$. The differential of λ is given by

$$\begin{array}{rccc} \lambda_* : & \mathfrak{spin}(p+1,q+1) & \to & \mathfrak{so}(p+1,q+1) \ , \\ & \frac{1}{2}f_{n+1}f_i & \mapsto & \begin{pmatrix} 0 & 0 & 0 \\ e_i & 0 & 0 \\ 0 & -e_i^* I_{p,q} & 0 \end{pmatrix} \\ & \frac{1}{2}f_0 f_i & \mapsto & \begin{pmatrix} 0 & 0 & 0 \\ e_i & 0 & 0 \\ 0 & -e_i^* I_{p,q} & 0 \end{pmatrix} \\ & \frac{1}{2}f_0 f_i & \mapsto & \begin{pmatrix} 0 & -e_i^* I_{p,q} & 0 \\ 0 & 0 & e_i \\ 0 & 0 & 0 \end{pmatrix} \\ & \frac{1}{2}f_0 f_{n+1} & \mapsto & \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ & \frac{1}{2}f_i f_j & \mapsto & E_{ij} \end{array}$$

where $\{f_i : i = 0, \ldots, n+1\}$ denotes the standard basis of $\mathbb{R}^{n+2}$, $\{e_i : i = 1, \ldots, n\}$ denotes the standard basis of $\mathbb{R}^n$ and E_{ij} is the standard basis of $\mathfrak{so}(p,q)$. It holds

$$\begin{aligned} \lambda_*^{-1}(\mathfrak{g}_{-1}) &= Span\{f_{n+1}f_i : i = 1, \ldots, n\}, \\ \lambda_*^{-1}(\mathfrak{g}_0) &= Span\{f_i f_j\} \oplus \mathbb{R} f_0 f_{n+1} \cong \mathfrak{cspin}(p,q), \\ \lambda_*^{-1}(\mathfrak{g}_1) &= Span\{f_0 f_i : i = 1, \ldots, n\}. \end{aligned}$$

2.3 Twistor spinors and the normal conformal Cartan connection

It is well-known that a twistor spinor $\varphi \in \Gamma(S)$ on a semi-Riemannian spin manifold (M^n_p, g) may be interpreted as a parallel section in the two-fold spinor bundle $S \oplus S$ with respect to a certain covariant derivative ∇^{TC} (comp. [BFGK91] and 1.2). It was already proved in [Fri77] that on a space-time the 'twistor connection' is induced by the canonical normal Cartan connection of conformal geometry. We give here a prove of this fact on semi-Riemannian spin manifolds with arbitrary signature using our approach to normal Cartan connections due to [CSS97].

Let (M^n, c) be an oriented C^∞-manifold of dimension $n \geq 3$ with conformal structure c of index p, that is a class of conformally equivalent (pseudo)-Riemannian metrics with index p on M^n. The conformal structure c is equivalently described by the conformal frame bundle $CSO(M)$, which is a $CSO(p,q)$-principal fibre subbundle of the first order linear frame bundle $Gl^{(1)}(M)$. The first order conformal structure $CSO(M)$ gives rise via the canonical first prolongation to the almost Hermitian symmetric structure $(BSO(M), \theta)$ of type $(GSO(p,q), BSO(p,q))$, where the soldering form θ has no torsion, i.e.

$$T := d\theta_{-1} - [\theta_{-1}, \theta_0] = 0.$$

This implies that $BSO(M)$ is a subbundle of the second order frame bundle $Gl^{(2)}(M)$ over M and θ is the restriction of the canonical form on $Gl^{(2)}(M)$ (comp. [CSS97], I.6.2., II.1.). On the other hand, let $BSO(M)$ be a $BSO(p,q)$-subbundle of $Gl^{(2)}(M)$ over a C^∞-manifold M and θ the restriction of the canonical soldering form to $BSO(M)$. Then the induced $CSO(p,q)$-principal fibre bundle $BSO(M)/B_1(p,q)$ can be interpreted in a natural way as a subbundle of $Gl^{(1)}(M)$ (comp. [CSS97] I.3.). This leads to

Definition 2.3.1. — *(1) A conformal structure of index p on an oriented C^∞-manifold M^n of dimension $n \geq 3$ is an almost Hermitian symmetric structure $(BSO(M), \theta)$ of type $(GSO(p,q), BSO(p,q))$, whose soldering form θ has vanishing torsion.*

(2) A conformal spin structure on M^n of index p is a $BSpin(p,q)$-reduction $(BSpin(M), f)$ of a conformal structure $(BSO(M), \theta)$, i.e. it holds

$$f(l) \cdot \lambda(a) = f(l \cdot a) \qquad \text{for all } \; l \in BSpin(M) \quad \text{and} \quad a \in BSpin(p,q),$$

where $f : BSpin(M) \to BSO(M)$ is a two-fold covering map. The 1-form $f^*\theta$ is the soldering form on $BSpin(M)$.

A conformal spin structure $(BSpin(M), f)$ on M^n in the sense of Definition 2.3.1 induces a $CSpin(p,q)$-reduction of the first order conformal frame bundle $CSO(M)$:

$$CSpin(M) := BSpin(M)/B_1(p,q) \xrightarrow{f'} CSO(M),$$

where f' is induced by f.

For the rest of this section we assume (M^n, c) to be a conformal spin manifold of dimension $n \geq 3$ and index p and by $(BSpin(M), f)$ we denote a fixed conformal spin structure on M.

We want to define conformal spinor bundles on (M^n, c). Let $\Delta_{p,q} \cong \mathbb{C}^{[\frac{n}{2}]}$ denote the standard (complex) spinor module with $Spin(p,q)$-action

$$\rho : Spin(p,q) \times \Delta_{p,q} \to \Delta_{p,q}$$

and Clifford multiplication

$$\mu : \mathbb{R}^{p,q} \otimes \Delta_{p,q} \to \Delta_{p,q}.$$

Let us consider the spinor module $\Delta_{p+1,q+1}$ and the null vectors $f_0, f_{n+1} \in \mathbb{R}^{p+1,q+1}$. We denote by

$$\begin{aligned} Ann(f_0) &:= \{v \in \Delta_{p+1,q+1} \mid f_0 \cdot v = 0\} \quad \text{and} \\ Ann(f_{n+1}) &:= \{v \in \Delta_{p+1,q+1} \mid f_{n+1} \cdot v = 0\} \end{aligned}$$

the annulation spaces in $\Delta_{p+1,q+1}$ with respect to Clifford multiplication by f_0 resp. f_{n+1}. Since every spinor $v \in \Delta_{p+1,q+1}$ decomposes to $f_0 w + f_1 w$ for some $w \in \Delta_{p+1,q+1}$ and

$$x f_i = -f_i x \qquad \text{for all} \;\; x \in \mathbb{R}^{p,q} \cong Span\{f_1, \ldots, f_n\} \subset \mathbb{R}^{p+1,q+1}, \;\; i \in \{0, n+1\},$$

we can conclude that the annulation spaces $Ann(f_0)$ and $Ann(f_{n+1})$ are isomorphic to $\Delta_{p,q}$ as $Spin(p,q)$-representations:

$$\Delta_{p+1,q+1}|_{Spin(p,q)} = Ann(f_{n+1}) \oplus Ann(f_0) \cong \Delta_{p,q} \oplus \Delta_{p,q}.$$

We fix an isomorphism

$$\alpha : Ann(f_0) \xrightarrow{\sim} \Delta_{p,q}$$

of $Spin(p,q)$-representations and define a corresponding isomorphism β by

$$\begin{array}{rccc} \beta : & Ann(f_{n+1}) & \xrightarrow{\sim} & \Delta_{p,q} \\ & v & \longmapsto & -\frac{1}{n}\alpha(f_0 v) \end{array}.$$

Then it holds

$$\begin{aligned} \beta(\tfrac{1}{2} f_{n+1} \cdot x \cdot \alpha^{-1}(u)) &= \tfrac{1}{n} x \cdot u \qquad \text{for all} \;\; x \in \mathbb{R}^{p,q} \;\; \text{and} \;\; u \in \Delta_{p,q}, \\ \alpha(\tfrac{1}{2} f_0 \cdot x \cdot \beta^{-1}(u)) &= \tfrac{n}{2} x \cdot u \qquad \text{for all} \;\; x \in \mathbb{R}^{p,q} \;\; \text{and} \;\; u \in \Delta_{p,q}. \end{aligned}$$

Let

$$GSpin(M) := BSpin(M) \times_{BSpin(p,q)} GSpin(p,q)$$

denote the enlarged $GSpin(p,q)$-principal fibre bundle. We define the vector space bundles

$$E := GSpin(M) \times_{\rho} \Delta_{p+1,q+1} \qquad \text{and} \qquad T := GSpin(M) \times_{\lambda \circ \rho} \mathbb{R}^{p+1,q+1}$$

over (M^n, c). On $BSO(M)$ exists a canonical normal Cartan connection (comp. [CSS97] I.6.3., [Kob72]), which we denote by ω_{NC}. The canonical normal Cartan connection on $BSpin(M)$ is then given by

$$\omega_{NCS} := f^* \omega_{NC}.$$

Furthermore, we have the connection $\bar{\omega}_{NCS}$ on the enlarged bundle $GSpin(M)$ and this connection induces a derivative ∇^E on the conformal spinor bundle E.

It is known that in conformal geometry there exists a bijective correspondence between $CSO(p,q)$-equivariant sections in $\Gamma(CSO(M); BSO(M))$ and torsion-free connections (Weyl structures) on the conformal frame bundle $CSO(M)$. Let us fix a metric g in the conformal class c on M^n. With $(Spin(M,g), f')$ we denote the corresponding semi-Riemannian spin structure on (M,g). The usual spinor bundle on (M^n_p, g) is here denoted by

$$S^g := Spin(M,g) \times_\rho \Delta_{p,q}$$

and ∇^{S^g} denotes the spinor derivative on S^g. The Levi-Civita connection to g on the bundle of orthonormal frames $SO(M,g)$ induces a torsion free connection ω_g on the $CSpin(p,q)$-principal bundle $CSpin(M)$. But then exists a uniquely determined $CSpin(p,q)$-equivariant section $\sigma^g \in \Gamma(CSpin(M); BSpin(M))$ such that

$$\sigma^{g^*} \circ f^*\theta_0 = \omega_g,$$

where $f^*\theta_0$ is the $\mathfrak{g}_0$-valued component of the soldering form $f^*\theta$ on $BSpin(M)$ (comp. [CSS97] II.1.7.). It holds

$$\omega_{NCS} = \omega^{\sigma^g} - \Gamma^{\sigma^g} \circ f^*\theta_{-1},$$

where $\omega^{\sigma^g} = f^*\theta + \omega_1^{\sigma^g}$ is the admissible connection to the section σ^g and Γ^{σ^g} is the pullback of the Schouten tensor

$$K^g = \frac{1}{n-2}\left(\frac{R^g}{2(n-1)} - \mathrm{Ric}^g\right)$$

to $BSpin(M)$ (comp. [CSS97] I.6.4.).

With the help of the section σ^g to $g \in c$ we are able to reduce the structure group of the bundles E and T to $Spin(p,q)$:

$$\begin{array}{rcl} Spin(M,g) \times_{\rho \circ i_{cs}|_{Spin(p,q)}} \Delta_{p+1,q+1} & \cong & E, \\ {[l,v]} & \mapsto & [[\sigma^g(l), e], v] \end{array}$$

$$\begin{array}{rcl} Spin(M,g) \times_{\lambda \circ \rho \circ i_{cs}|_{Spin(p,q)}} \mathbb{R}^{p+1,q+1} & \cong & T. \\ {[l,x]} & \mapsto & [[\sigma^g(l), e], x] \end{array}$$

There are canonical sections ζ_0^g and ζ_{n+1}^g in the bundle $T \to M$ with respect to g, which are defined by

$$\zeta_0^g(\pi(l)) := [[\sigma^g(l), e], f_0] \qquad \text{and} \qquad \zeta_{n+1}^g(\pi(l)) := [[\sigma^g(l), e], f_{n+1}], \qquad l \in Spin(M,g).$$

Then we have a decomposition of the bundle E into the annulation spaces

$$\begin{array}{rcl} E & = & Ann(\zeta_{n+1}^g) \oplus Ann(\zeta_0^g), \\ {[[\sigma^g(l), e], v]} & = & [[\sigma^g(l), e], f_{n+1}w] \oplus [[\sigma^g(l), e], f_0 w] \end{array}$$

where $v = f_{n+1}w + f_0 w \in \Delta_{p+1,q+1}$, and we denote by

$$proj_i^g : E \to Ann(\zeta_i^g), \quad i \in \{0, n+1\},$$

the projections onto the annulation spaces. Moreover, there is the isomorphism

$$\begin{array}{cccc} \Phi^g : & E & \cong & S^g \oplus S^g \\ & [[\sigma^g(l), e], v] & \mapsto & [l, \beta(f_{n+1}w)] \oplus [l, \alpha(f_0 w)] \end{array} .$$

After we have defined the appropriate bundles and isomorphisms, we can explain how the twistor equation is related to the normal conformal Cartan connection ω_{NCS} on a conformal spin manifold (M^n, c). Remember that $\varphi \in \Gamma(S)$ is a twistor spinor on (M_p^n, g) if and only if (see Proposition 1.2.6)

$$\nabla_X^{TC} \begin{pmatrix} \varphi \\ D\varphi \end{pmatrix} = \begin{pmatrix} \nabla_X^{S^g} & \frac{1}{n} X \cdot \\ -\frac{n}{2} K(X) \cdot & \nabla_X^{S^g} \end{pmatrix} \begin{pmatrix} \varphi \\ D\varphi \end{pmatrix} = 0.$$

Theorem 2.3.2. — *Let $\psi \in \Gamma(E)$ be a spinor field on (M^n, c) and let g be a metric in the conformal class c on M^n. The spinor field ψ is parallel with respect to the normal conformal Cartan connection ($\nabla^E \psi = 0$) if and only if*

$$\varphi := \Phi^g \circ proj_{n+1}^g(\psi) \in ker(P^g) \qquad \textit{and} \qquad D^g \varphi = \Phi^g \circ proj_0^g(\psi).$$

Proof. It is

$$\psi = [[\sigma^g(l), e], v] + [[\sigma^g(l), e], w]$$

for some functions $v : M \to Ann(f_0)$ and $w : M \to Ann(f_{n+1})$. Let

$$\varrho^g : U \subset M \to Spin(M, g)$$

be a local section and let us denote

$$f' \circ \varrho^g(x) = (s_1(x), \ldots, s_n(x)) \in SO(M, g) \quad \text{for} \quad x \in U \quad \text{and}$$

$$\varrho^c := \sigma^g \circ \varrho^g : U \to GSpin(M).$$

Remember that $\sigma^{g^*} \circ f^* \theta_0$ is the principal connection on $CSpin(M)$, which is induced by the

Levi-Civita connection to g on $CSO(M)$. We have locally

$$\begin{aligned}
\Phi^g(\nabla^E_X\psi) &= \Phi^g[\varrho^c, d(v+w) + \rho_* \circ \omega_{NCS} \circ d\varrho^c(X)(v+w)] \\
&= \Phi^g[\varrho^c, dv + \rho_* \circ f^*\theta_0 \circ d\varrho^c(X)v] + \Phi^g[\varrho^c, dw + \rho_* \circ f^*\theta_0 \circ d\varrho^c(X)w] \\
&\quad + \Phi^g[\varrho^c, \rho_* \circ f^*\theta_{-1} \circ d\varrho^c(X)v] - \Phi^g[\varrho^c, \rho_* \circ \Gamma^{\sigma^g} \circ f^*\theta_{-1} \circ d\varrho^c(X)w] \\
&= \left(\nabla^{S^g}_X(\Phi^g \circ proj^g_{n+1}(\psi)) + \Phi^g[\varrho^c, \frac{1}{2}f_{n+1} \cdot \big(\sum_{i=1}^n \varepsilon_i g(X,s_i)f_i\big) \cdot v]\right) \\
&\quad \oplus \left(\nabla^{S^g}_X(\Phi^g \circ proj^g_0(\psi)) + \Phi^g[\varrho^c, -\frac{1}{2}f_0 \cdot \big(\sum_{i=1}^n \varepsilon_i g(K(X),s_i)f_i\big) \cdot w]\right) \\
&= \left(\nabla^{S^g}_X(\Phi^g \circ proj^g_{n+1}(\psi)) + \frac{1}{n}X \cdot \Phi^g \circ proj^g_0(\psi)\right) \\
&\quad \oplus \left(\nabla^{S^g}_X(\Phi^g \circ proj^g_0(\psi)) - \frac{n}{2}K(X) \cdot \Phi^g \circ proj^g_{n+1}(\psi)\right).
\end{aligned}$$

It follows that $\nabla^E_X\psi = 0$ if and only if

$$\nabla^{S^g}_X(\Phi^g \circ proj^g_{n+1}(\psi)) + \frac{1}{n}X \cdot \Phi^g \circ proj^g_0(\psi) = 0 \quad \text{and}$$

$$\nabla^{S^g}_X(\Phi^g \circ proj^g_0(\psi)) - \frac{n}{2}K(X) \cdot \Phi^g \circ proj^g_{n+1}(\psi) = 0.$$

With Proposition 1.2.6 we can conclude that $\nabla^E_X\psi = 0$ if and only if

$$\Phi^g \circ proj^g_{n+1}(\psi) \in ker(P^g) \qquad \text{and} \qquad D^g(\Phi^g \circ proj^g_{n+1}(\psi)) = \Phi^g \circ proj^g_0(\psi).$$

□

2.4 Twistor spinors on conformally flat manifolds

In 2.2 we introduced the conformal spin spaces $\hat{C}^{p,q}$ and described them as homogenous spaces

$$\hat{C}^{p,q} = GSO(p,q)/BSO(p,q) = GSpin(p,q)/BSpin(p,q).$$

In the following we explicitly describe the space of twistor spinors on the models $\hat{C}^{p,q}$. After we have done this, we will classify conformally flat spin manifolds admitting twistor spinors with the help of a development and its corresponding holonomy representation.

Let $\omega_{GSpin(p,q)} := \omega_{-1} \oplus \omega_0 \oplus \omega_1$ be the Maurer-Cartan form of $GSpin(p,q)$. The canonical form on the model $GSpin(p,q) \to \hat{C}^{p,q}$ is

$$\theta_{GSpin(p,q)} = \omega_{-1} + \omega_0$$

and $\omega_{GSpin(p,q)}$ is the flat normal Cartan connection on the Hermitian symmetric space $(GSpin(p,q), \theta_{GSpin(p,q)}) \to \hat{C}^{p,q}$. The enlarged $GSpin(p,q)$-principal fibre bundle on $\hat{C}^{p,q}$ is given by

$$\overline{GSpin}(p,q) = GSpin(p,q) \times_{BSpin(p,q)} GSpin(p,q).$$

The bundle $\overline{GSpin}(p,q)$ is equipped with the induced flat connection $\bar{\omega}_{GSpin(p,q)}$. We have a canonical parallel section γ in $\overline{GSpin}(p,q) \to \hat{C}^{p,q}$ given by

$$\gamma(a \cdot BSpin(p,q)) = [a, a^{-1}], \qquad a \in GSpin(p,q).$$

The parallel sections in the spinor bundle $E = \overline{GSpin}(p,q) \times_{GSpin(p,q)} \Delta_{p+1,q+1}$ on $\hat{C}^{p,q}$ are given by

$$\psi_v(a \cdot BSpin(p,q)) = [[a, a^{-1}], v], \qquad a \in GSpin(p,q),$$

for arbitrary $v \in \Delta_{p+1,q+1}$. Let $g \in c$ be a conformally flat metric on $\hat{Q}^{p,q}$ and $\sigma^g : CSpin(\hat{C}^{p,q}) \to GSpin(p,q)$ the corresponding $CSpin(p,q)$-equivariant section. It is

$$\psi_v(\pi(l)) = [[\sigma^g(l), e], (\sigma^g(l))^{-1} \cdot v] \qquad \text{for } \; l \in CSpin(\hat{C}^{p,q})$$

and the parallel section ψ_v induces a twistor spinor on $(\hat{Q}^{p,q}, g)$, which is given by

$$\varphi_v(\pi(l)) := [l, \beta \circ proj^g_{n+1}((\sigma^g(l))^{-1} \cdot v)], \qquad l \in Spin(\hat{Q}^{p,q}, g).$$

Since the bundle $E \to \hat{C}^{p,q}$ is globally trivializable by parallel sections, we can conclude that the space of twistor spinors on $\hat{C}^{p,q}$ has maximal dimension $2^{[\frac{n}{2}]+1}$.

There is another way of constructing twistor spinors on $\hat{C}^{p,q}$, which we also want to describe here. In general, let $(M^{n+2,p+1}, h_1)$ be a semi-Riemannian spin manifold of dimension $n+2$ and index $p+1$ with spin structure $(Spin(M^{n+2,p+1}), f)$. Furthermore, let (F^n_p, h_2) be a semi-Riemannian manifold of dimension n and index p and let $i : F^n_p \hookrightarrow M^{n+2,p+1}$ be an isometric embedding such that the normal bundle NF^n_p of F^n_p in $M^{n+2,p+1}$ can be trivialized by a global orthonormal frame field. Under these assumptions the spin structure $(Spin(M^{n+2,p+1}), f)$ on $M^{n+2,p+1}$ induces in a natural way a spin structure on F^n_p, which we denote by $(Spin(F^n_p), \bar{f})$. Let us denote the spinor bundles on $M^{n+2,p+1}$ and F^n_p by S^M resp. S^F and let (ζ_0, ζ_{n+1}) be any global orthonormal frame field in NF^n_p. The restricted spinor bundle $S^M|_F$ on F^n_p decomposes to the annulation spaces

$$S^M|_F = Ann(\zeta_0 + \zeta_{n+1}) \oplus Ann(\zeta_0 - \zeta_{n+1})$$

(comp. 2.3). Each of these annulation spaces is isomorphic to the spinor bundle S^F on F^n_p, i.e. we have an isomorphism

$$S^M|_F \cong S^F \oplus S^F.$$

Let $\varphi \in \Gamma(S^M)$ be a spinor field on $M^{n+2,p+1}$ and

$$\varphi|_F = \varphi_1 \oplus \varphi_2 \in \Gamma(S^F \oplus S^F).$$

The spinor derivatives ∇^M and ∇^F are related by

$$\begin{aligned}\nabla^M_X\varphi|_F &= \nabla^F_X\varphi_1 \oplus \nabla^F_X\varphi_2 + [\frac{1}{2}(\nabla^M_X\zeta_0)\cdot\zeta_0\cdot\varphi - \frac{1}{2}(\nabla^M_X\zeta_{n+1})\cdot\zeta_{n+1}\cdot\varphi \\ &\quad + \frac{1}{2}g(\nabla^M_X\zeta_{n+1},\zeta_0)\cdot\zeta_{n+1}\cdot\zeta_0\cdot\varphi]\Big|_F \qquad \text{for all } \ X\in TF^n_p\end{aligned}$$

(comp. [BFGK91] for a similar formula in codimension 1).

We can apply this formula to the embedding

$$\begin{aligned} i: \quad \hat{Q}^{p,q} &\longrightarrow (\mathbb{R}^{n+2},\langle\cdot,\cdot\rangle_{p+1,q+1})\ . \\ \mathbb{R}_+\cdot x &\mapsto \sqrt{\frac{2}{\langle x,x\rangle_{n+2}}}\cdot x\end{aligned}$$

Notice that we furnish $\mathbb{R}^{n+2}$ here with the usual scalar product $\langle\cdot,\cdot\rangle_{p+1,q+1}$. There is a natural globally defined orthonormal frame field on the normal bundle of $\hat{Q}^{p,q}$ in $\mathbb{R}^{n+2}$, which is given by

$$\begin{aligned}\zeta_0(x) &:= (x_0,\dots,x_p,0,\dots,0) \quad \text{and} \\ \zeta_{n+1}(x) &:= (0,\dots,0,x_{p+1},\dots,x_{n+1}) \qquad \text{for } x\in\mathbb{R}^{n+2}.\end{aligned}$$

Let $\varphi_v(x) := x\cdot v$, $v\in\Delta_{p+1,q+1}$. The spinor field φ_v is a twistor spinor in the spinor bundle $\mathbb{R}^{n+2}\times\Delta_{p+1,q+1}$ on $(\mathbb{R}^{n+2},\langle\cdot,\cdot\rangle_{p+1,q+1})$. We denote

$$\varphi_v|_{\hat{Q}^{p,q}} = \varphi_{v_1}\oplus\varphi_{v_2}\in\Gamma(Ann(\zeta_0+\zeta_1)\oplus Ann(\zeta_0-\zeta_1)).$$

It holds

$$\begin{aligned}& g(\nabla^{\mathbb{R}^{n+2}}_X\zeta_0,\zeta_{n+1}) = 0, \\ & \nabla^{\mathbb{R}^{n+2}}_X\zeta_0 = (x_0,\dots,x_p,0,\dots,0) =: proj_T X, \\ & \nabla^{\mathbb{R}^{n+2}}_X\zeta_{n+1} = (0,\dots,0,x_{p+1},\dots,x_{n+1}) =: proj_S X\end{aligned}$$

for all $X=(x_0,\dots,x_{n+1})\in Ti(\hat{Q}^{p,q})$ and

$$(\zeta_0+\zeta_{n+1})(x)\cdot\varphi_v|_{\hat{Q}^{p,q}} = x\cdot x\cdot v|_{\hat{Q}^{p,q}} = 0 \qquad \text{for all } \ x\in i(\hat{Q}^{p,q}).$$

We obtain $\varphi_{v_2}\equiv 0$ and

$$\begin{aligned}\nabla^{\mathbb{R}^{n+2}}_X\varphi_v|_{\hat{Q}^{p,q}} &= \nabla^{\hat{Q}^{p,q}}_X\varphi_{v_1} + [\frac{1}{2}proj_T X\cdot\zeta_0\cdot\varphi_v - \frac{1}{2}proj_S X\cdot\zeta_{n+1}\cdot\varphi_v]\Big|_{\hat{Q}^{p,q}} \\ &= \nabla^{\hat{Q}^{p,q}}_X\varphi_{v_1} + [\frac{1}{2}X\cdot\zeta_0\cdot\varphi_v]\Big|_{\hat{Q}^{p,q}}\ .\end{aligned}$$

This shows that $g(X,X)X\cdot\nabla^{\hat{Q}^{p,q}}_X\varphi_{v_1}$ is independent for $X\in T\hat{Q}^{p,q}$ with $\|X\|^2=\pm1$ and we can conclude that

$$\varphi_{v_1}\in\Gamma(Ann(\zeta_0+\zeta_{n+1}))\cong\Gamma(S^{\hat{Q}^{p,q}})$$

is a twistor spinor on $\hat{Q}^{p,q}$ for any $v \in \Delta_{p+1,q+1}$.

In 2.1 we introduced in general the notations of development and holonomy representation for flat Cartan geometries. Let (M^n, c) be an oriented and conformally flat manifold with canonical normal Cartan geometry $(BSO(M), \omega)$. Let $(\tilde{M}, \tilde{c})$ denote the universal covering space of (M, c) with induced flat conformal structure $\tilde{c}$. The manifold $(\tilde{M}, \tilde{c})$ is conformally spin. Then we have a development

$$\begin{array}{cccc} \bar{\delta}: & GSpin(\tilde{M}) & \rightarrow & \overline{GSpin}(p,q) \\ & \downarrow & & \downarrow \\ \bar{\delta}: & GSO(\tilde{M}) & \rightarrow & \overline{GSO}(p,q) \\ & \downarrow & & \downarrow \\ \delta^{\tilde{M}}: & \tilde{M} & \rightarrow & \hat{C}^{p,q} \end{array}$$

and a corresponding holonomy representation

$$\kappa : \pi_1(M) \rightarrow GSO(p,q)$$

of the fundamental group of M^n into the automorphism group $GSO(p,q)$ of $(GSO(p,q), \omega_{GSO(p,q)})$. The manifold (M^n, c) is conformally spin if and only if κ admits a lift

$$\tilde{\kappa} : \pi_1(M) \rightarrow GSpin(p,q)$$

with respect to $\lambda : GSpin(p,q) \rightarrow GSO(p,q)$. The set of conformal spin structures on (M, c) corresponds bijectively to the set of lifts of the holonomy representation κ (comp. [Bau81]).

Theorem 2.4.1. — *Let (M^n, c) be a conformally flat spin manifold with development δ and holonomy representation κ and let*

$$\tilde{\kappa} : \pi_1(M) \rightarrow GSpin(p,q)$$

be a lift of κ, which defines a spin structure $f_{\tilde{\kappa}}$ on (M, c). Let $V \subset \Delta_{p+1,q+1}$ be the maximal subspace, on which the representation $\tilde{\kappa}$ acts trivially. Then every $v \in V$ gives rise to a twistor spinor φ_v on $(M, c, f_{\tilde{\kappa}})$. In particular, the space $\mathcal{T}(M^n, c, f_{\tilde{\kappa}})$ of twistor spinors on $(M, c, f_{\tilde{\kappa}})$ and the space V have the same dimension.

Proof. Let $\delta^{\tilde{M}} : (\tilde{M}, \tilde{c}) \rightarrow \hat{C}^{p,q}$ be the development, which induces the holonomy representation κ. The development $\delta^{\tilde{M}}$ lifts to a bundle morphism

$$\delta^{\tilde{M}}_* : E(\tilde{M}, \tilde{c}) \rightarrow E(\hat{C}^{p,q})$$

of the conformal spinor bundles. Let $v \in V$ be arbitrary, ψ_v the corresponding parallel section in $E(\hat{C}^{p,q})$ and let $\tilde{\varphi}_v$ be the parallel section in $E(\tilde{M}, \tilde{c})$ such that $\delta^{\tilde{M}}_*(\tilde{\varphi}_v) = \psi_v$. Since κ acts trivially on V, the parallel section $\tilde{\varphi}_v$ projects to a parallel section $\varphi_v := \pi_* \tilde{\varphi}_v$ in $E(M, c, f_{\tilde{\kappa}})$.

Conversely, let φ be a parallel section in $E(M, c, f_{\tilde{\kappa}})$ and $\tilde{\varphi}$ its lift to $E(\tilde{M}, \tilde{c})$. Then, it exists a unique spinor $v \in V$ such that $\psi_v = \delta^{\tilde{M}}_*(\tilde{\varphi})$. The fact that parallel sections in $E(M, c, f_{\tilde{\kappa}})$

correspond bijectively to twistor spinors on $(M, c, f_{\tilde{\kappa}})$ proves the theorem. $\square$

It is well-known that a twistor spinor on the Riemannian sphere $\hat{C}^{0,n} = S^n$ has at most one zero (comp. [Lic88]). This observation leads to the following corollary of Theorem 2.4.1, which is a result of W. Kühnel and H.-B. Rademacher, proved in [KR97c]:

Corollary 2.4.2. — *Let (M^n, g) be an oriented conformally flat Riemannian spin manifold with holonomy representation*

$$\kappa : \pi_1(M) \to SO^+(n+2, 1).$$

Then (M, g) admits a twistor spinor with zero if and only if the following conditions are satisfied:

(1) The holonomy representation κ fixes a point $p \in \hat{C}^{0,n} = S^n$.

(2) The linear holonomy representation κ_{*p} in $p \in \hat{C}^{0,n}$ is orthogonal:

$$\kappa_{*p} : \pi_1(M) \to SO(n).$$

(3) The linear holonomy representation κ_{*p} has a lift

$$\tilde{\kappa}_{*p} : \pi_1(M) \to Spin(n)$$

with respect to λ.

(4) The representation $\tilde{\kappa}_{*p}$ acts trivially on a non-trivial subspace V of the spinor module $\Delta_{0,n}$.

Example A. We construct the twistor spinors on the Möbius spheres $C^{p,q}$ for $1 < p < n-1$. The two-fold covering space $\hat{C}^{p,q}$ of $C^{p,q}$ is simply connected and consequently, it holds $\pi_1(C^{p,q}) = \mathbb{Z}_2$. The corresponding holonomy representation is given by

$$\begin{aligned} \kappa : \quad \mathbb{Z}_2 \quad &\to SO(p+1, q+1) \; . \\ \pm 1 \quad &\mapsto \pm I \end{aligned}$$

Let

$$\gamma := \frac{1}{2}(f_0 - f_{n+1}) \cdot f_1 \cdot \ldots \cdot f_n \cdot (f_0 + f_{n+1}) \in Spin(p+1, q+1).$$

It holds $\lambda(\gamma) = -I_{n+2}$. The holonomy representation κ admits a lift to $Spin(p+1, q+1)$ if and only if $\gamma^2 = 1$, which is equivalent to the condition

$$n = 2 \bmod 4, \quad k = 1 \bmod 2 \qquad \text{or} \qquad n = 2 \bmod 4, \quad k = 0 \bmod 2.$$

If this condition is satisfied then there are two conformal spin structures on $C^{p,q}$, which are characterized by the following two lifts of the holonomy representations

$$\begin{aligned} \tilde{\kappa}_+ : \quad \mathbb{Z}_2 \quad &\to \quad Spin(p+1, q+1), \\ \{1, -1\} \quad &\mapsto \quad \{1, \gamma\} \end{aligned}$$

$$\begin{aligned} \tilde{\kappa}_- : \quad \mathbb{Z}_2 \quad &\to \quad Spin(p+1, q+1). \\ \{1, -1\} \quad &\mapsto \quad \{1, -\gamma\} \end{aligned}$$

Moreover, the spinor module $\Delta_{p+1,q+1}$ splits into a positive and a negative part

$$\Delta_{p+1,q+1} = \Delta^+_{p+1,q+1} \oplus \Delta^-_{p+1,q+1}$$

It holds

$$\gamma \cdot \Delta^+_{p+1,q+1} \subset \Delta^-_{p+1,q+1} \quad \text{and} \quad \gamma \cdot \Delta^-_{p+1,q+1} \subset \Delta^+_{p+1,q+1}.$$

We define

$$\begin{aligned} V_+ &:= \{v + \gamma v : v \in \Delta^+_{p+1,q+1}\} \subset \Delta_{p+1,q+1} \quad \text{and} \\ V_- &:= \{v - \gamma v : v \in \Delta^+_{p+1,q+1}\} \subset \Delta_{p+1,q+1}. \end{aligned}$$

The spaces V_+ and V_- are the eigenspaces of γ to the eigenvalues 1 resp. -1, i.e. $\tilde{\kappa}_+$ acts trivially on V_+, whereas $\tilde{\kappa}_-$ acts trivially on V_-. Since $\dim_{\mathbb{C}} V_+ = \dim_{\mathbb{C}} V_- = \frac{1}{2} \dim_{\mathbb{C}} \Delta_{p+1,q+1} = 2^{[\frac{n}{2}]}$, it follows that on $(C^{p,q}, f_{\tilde{\kappa}_+})$ and $(C^{p,q}, f_{\tilde{\kappa}_-})$ there exist $2^{[\frac{n}{2}]}$ linearly independent twistor spinors. This is one half of the maximal dimension d_n.

Example B. In the Lorentzian case, when $p = 1$, it holds

$$\pi_1(\hat{C}^{1,n-1}) = \mathbb{Z} \quad \text{and} \quad H^1(\hat{C}^{1,n-1}, \mathbb{Z}) = \mathbb{Z}_2.$$

This implies that two conformal spin structures exist on $\hat{C}^{1,n-1}$. The natural one is induced by the embedding i of $\hat{C}^{1,n-1}$ in $\mathbb{R}^{n+2,p+1}$ and admits a space of twistor spinors of maximal dimension. On the other side, it can be shown that the space $\hat{C}^{1,n-1}$ furnished with the 'non-natural' conformal spin structure admits no twistor spinors.

3 Zeros of conformal vector fields and twistor spinors in Lorentzian geometry

In this section we ask for the shape of the zero set of a twistor spinor on a Lorentzian spin manifold. To solve this problem we investigate the associated conformal vector fields of twistor spinors with zeros. Twistor spinors with zeros are 'true' solutions of the twistor equation and the associated conformal fields are essential.

3.1 Some preliminary remarks on essential conformal vector fields

Let (M^n_p, g) be a semi-Riemannian manifold of dimension $n \geq 3$. A vector field $V \in \Gamma(TM)$ is called conformal if the Lie derivative of the metric g in direction of V satisfies

$$L_V g = 2\alpha \cdot g$$

for some C^∞-function α on M^n_p. In particular, Killing vector fields on (M^n_p, g) are conformal. A conformal vector field V is called essential if V is not a Killing vector field with respect to any metric $\tilde{g}$ in the conformal class $[g]$ on M^n_p. Locally, every conformal vector field V without zeros is a Killing vector field with respect to some conformally changed metric $\tilde{g}$. Hence, in order to get informations on essential conformal vector fields, it is necessary to investigate the behaviour of conformal fields and curvature properties of the underlying manifold in the near of the set $zero(V)$.

Essential conformal fields on a Riemannian manifold have been investigated by Obata, Lelong-Ferrand and Alekseevskii (see e.g. [LF71], [Oba70] and [Ale72]). In general, a conformal transformation f on (M^n, g) with fixed point $p \in M^n$ admits the following local expansion in geodesic coordinates around p (comp. [Ale72]):

$$f(x) = Ax + \langle x, \xi \rangle Ax - \frac{1}{2}\langle x, x \rangle \xi + o(x^2),$$

where $A = df(p)$, $\xi = grad(2\lambda)(p)$ and $f^* g = e^{2\lambda} g$. Using this expansion formula one can prove that if an essential conformal field V is complete, i.e. there exists a one-parameter group Φ^V_t of essential conformal transformations on (M^n, g), then (M^n, g) is globally conformal to the Euclidean space $\mathbb{R}^n$ or to the standard sphere S^n (see [Ale72], [Yos76]).

Conformal maps and conformal vector fields were also intensively studied in pseudo-Riemannian geometry, especially in General Relativity. We mention here some papers and results concerning conformal vector fields with zeros. W. Kühnel and H.-B. Rademacher investigated conformal gradient fields on pseudo-Riemannian manifolds in [KR95] and [KR97b]. A conformal gradient field with a zero is an essential conformal field. They proved that the zero set of a conformal gradient field is discrete and the manifold is conformally flat in a neighborhood of a zero. They also obtained global results for pseudo-Riemannian manifolds admitting conformal gradient fields with zeros.

In general, the zero set of a conformal vector field on a pseudo-Riemannian manifold (M^n_p, g) is neither discrete nor a submanifold. But in case that a conformal vector field V is linearizable, the connected components of $zero(V)$ are submanifolds of (M^n_p, g). A homethetic field V is always linearizable. The connected components of $zero(V)$ are then totally geodesic submanifolds and if V is not a Killing vector field they are even totally isotropic submanifolds. Several results on the question when an algebra of conformal fields on a space-time reduces to an algebra of homothetic fields or when a single conformal field is linearizable can be found in [Hal90], [HS91] and [HCB97]. For these problems the algebraic properties of the Weyl tensor W and the conformal 2-form $F = d\omega_V,\ \omega_V := g(V, \cdot)$, in a zero of the conformal vector field V play an import role.

Now, let us consider a semi-Riemannian spin manifold (M^n_p, g), which admits a twistor spinor $\varphi \in \Gamma(S)$. The associted conformal vector field V_φ has the properties:

(1) $zero(V_\varphi) \subset zero(\varphi)$ and

(2) $\nabla V_\varphi(p) = 0$ for all $p \in zero(\varphi)$.

In case that V_φ is not trivial and admits a zero in $p \in M^n_p$ the property $\nabla V_\varphi(p) = 0$ implies that V_φ is an essential conformal field. But this property also implies that V_φ is neither a gradient field nor a linearizable field.

On a Riemannian spin manifold (M^n, g) the zero set of a twistor spinor is always discrete. However, W. Kühnel and H.-B. Rademacher proved in [KR94] using the expansion formula of Alekseevskii on essential conformal fields that a Riemannian spin manifold (M^n, g) admitting a twistor spinor with zero, whose associated conformal field does not vanish, is conformally flat (see Theorem 1.3.4).

On a Lorentzian spin manifold (M^n_1, g) it holds in general

$$zero(V_\varphi) = zero(\varphi)$$

for the zero set of a twistor spinor φ and its associated conformal field V_φ. This fact is a special feature of Lorentzian spin geometry. We will investigate in the following the zero set of conformal fields on arbitrary curved Lorentzian manifolds, which satisfy the condition $\nabla V_\varphi(p) = 0$ for all $p \in zero(\varphi)$. Our main result states (see Theorem 3.3.2):

The zero set of a conformal vector field V on a Lorentzian manifold satisfying the condition $\nabla V_\varphi(p) = 0$ for all $p \in zero(\varphi)$ lies locally on a single lightlike smooth geodesic.

However, in the Lorentzian setting the only known examples of such conformal vector fields with zeros live on conformally flat spaces.

Example. Every conformal vector field on the (pseudo)-Euclidean space $\mathbb{R}^{p,q} := (\mathbb{R}^n, \langle \cdot, \cdot \rangle_{p,q})$ of index p is of the form (see [Sch97])

$$V(x) = 2\langle x, b\rangle_{p,q} x - \langle x, x\rangle_{p,q} b + \lambda x + \omega x + c,$$

where $b, c \in \mathbb{R}^{p,q}$, $\lambda \in \mathbb{R}$ and $\omega \in \mathfrak{o}(n, k)$. A conformal vector field of the form

$$W(x) = 2\langle x, b\rangle_{p,q} x - \langle x, x\rangle_{p,q} b, \quad b \neq 0,$$

is essential, since $W(0) = 0$ and $\nabla W(0) = 0$. It is

$$zero(W) = (b^{\perp} \cap L^{p,q}) \cup \{0\},$$

where $L^{p,q} := \{x \in \mathbb{R}^{p,q} | \langle x, x \rangle_{p,q} = 0,\ x \neq 0\}$ is the lightcone in $\mathbb{R}^{p,q}$. Let $\Delta_{p,q}$ be the usual complex spinor module. The twistor spinors on $\mathbb{R}^{p,q}$ are given by

$$\varphi(x) = x \cdot v + w, \quad v, w \in \Delta_{p,q},$$

and the associated conformal field to a twistor spinor $\psi(x) = x \cdot v$ with a zero in the origin is

$$V_v = 2\langle x, b_v \rangle_{p,q} x - \langle x, x \rangle_{p,q} b_v, \qquad b_v := -(-i)^{p+1} \sum_{j=1}^{n} \varepsilon_j \langle v, e_j v \rangle_{\Delta} e_j.$$

Let us consider the Lorentzian case. There are three kinds of conformal vector fields of the form

$$W_b(x) = 2\langle x, b \rangle_{1,n-1} x - \langle x, x \rangle_{1,n-1} b, \quad b \neq 0,$$

on the Minkowski space $\mathbb{R}^{1,n-1}$ corresponding to the causal character of the vector b. In case that $b = b_s$ is a spacelike vector the zero set

$$zero(W_{b_s}) = (b_s^{\perp} \cap L^{1,n-1}) \cup \{0\} \cong L^{1,n-1} \cup \{0\}$$

is not a submanifold of $\mathbb{R}^{1,n-1}$. It holds $W_{b_s}(0) = 0$, $\nabla W_{b_s}(0) = 0$ and $\nabla W_{b_s}(x) \neq 0$ for all $x \in zero(W_{b_s}) \backslash \{0\}$. In case that $b = b_t$ is a timelike vector we have

$$zero(W_{b_t}) = \{0\} \quad \text{and} \quad \nabla W_{b_t}(0) = 0.$$

In the third case, when $b = b_l$ is lightlike, the zero set is identical to the lightlike straight line $\mathbb{R} \cdot b_l$ in $\mathbb{R}^{1,n-1}$ and it holds $\nabla W_{b_l}(x) = 0$ for all $x \in zero(W_{b_l}) = \mathbb{R} \cdot b_l$. Let

$$V_v(x) = 2\langle x, b_v \rangle_{1,n-1} x - \langle x, x \rangle_{1,n-1} b_v, \quad b_v := \sum \varepsilon_j \langle v, e_j v \rangle_{\Delta} e_j,$$

be the associated conformal field to the twistor spinor $\varphi_v(x) = x \cdot v,\ v \in \Delta_{1,n-1}$, on $\mathbb{R}^{1,n-1}$. We know that the map

$$\begin{array}{rccl} \ell : & \Delta_{1,n-1} & \to & J^+ := \{x \in \mathbb{R}^{1,n-1} | \langle x, x \rangle_{1,n-1} \leq 0,\ \langle x, e_1 \rangle_{1,n-1} \geq 0\} \\ & v & \mapsto & b_v \end{array}$$

is surjective, i.e. up to a sign every conformal field V on $\mathbb{R}^{1,n-1}$ satisfying the property

$$\nabla V(p) = 0 \quad \text{for all } \ p \in zero(V)$$

is associated to a twistor spinor $\psi = x \cdot v,\ v \in \Delta_{1,n-1}$.

3.2 Some properties of the lightcones in Lorentzian geometry

We will prove in this part three propositions on elementary properties of lightcones in a Lorentzian manifold. The propositions will later enable us to prove our result on the shape of the zero set of conformal vector fields. We will use in the following some notations and facts concerning causality properties in Lorentzian geometry, which can be found in a detailed manner in [BEE96].

Let $\mathbb{R}^{1,n-1} := (\mathbb{R}^n, \langle\ ,\ \rangle_{1,n-1})$ be the n-dimensional Minkowski space. We denote by

$$L^{1,n-1} := \{x \in \mathbb{R}^{1,n-1} \mid \langle x, x\rangle_{1,n-1} = 0,\ x \neq 0\} \subset \mathbb{R}^{1,n-1}$$

the set of lightlike vectors in the Minkowski space $\mathbb{R}^{1,n-1}$ and we call $L^{1,n-1}$ the lightcone of $\mathbb{R}^{1,n-1}$. The lightcone $L^{1,n-1}$ is a submanifold in $\mathbb{R}^{1,n-1}$ of codimension 1. The tangent space $T_l L^{1,n-1}$ at every point $l \in L^{1,n-1}$ is lightlike that means the restriction of the metric $\langle\cdot,\cdot\rangle_{1,n-1}$ to $T_l L^{1,n-1}$ is degenerate. The line $\mathbb{R}\cdot l$ is the only totally lightlike subspace in $T_l L^{1,n-1}$.

Let (M_1^n, g), $n \geq 3$, be a n-dimensional Lorentzian manifold. Let L_p denote the lightcone in the tangent space $T_p M_1^n$ at $p \in M_1^n$ and let

$$\exp_p : D_p \subset T_p M_1^n \to M_1^n$$

be the exponential map in the point $p \in M_1^n$, where D_p is the maximal domain of definition, which is an open starshaped neighborhood of the origin $0 \in T_p M$. We define the lightcone $\mathcal{L}_p$ at $p \in M_1^n$ to be set

$$\mathcal{L}_p := \exp_p(D_p \cap L_p) \subset M_1^n,$$

i.e. $\mathcal{L}_p$ is exactly the set of points that can be connected with p by a smooth lightlike geodesic. In general, $\mathcal{L}_p$ is not a submanifold of M_1^n.

A convex set U in M_1^n is an open set, which has the property that for any two points $p, q \in U$ a unique C^∞-geodesic $\gamma_{pq}(t)$ exists such that

$$\gamma_{pq}(0) = p, \qquad \gamma_{pq}(1) = q \qquad \text{and} \qquad \gamma_{pq}([0,1]) \subset U.$$

We remember that every point in a Lorentzian (semi-Riemannian) manifold admits a convex neighborhood. The quadratic distance function

$$\begin{aligned} \Gamma^U : U \times U &\to \mathbb{R} \\ (p,q) &\mapsto \|\gamma'_{pq}\|^2 := g(\gamma'_{pq}(0), \gamma'_{pq}(0)) \end{aligned}$$

is a well defined and smooth function on the convex set U. On a time-oriented open set U a causal vector $0 \neq v \in TU$, $g(v,v) \leq 0$, is either future directed ($\uparrow$-vector) or past directed

(↓-vector). We define the following subsets of a time-oriented convex set $U \subset M_1^n$ to $p \in U$:

$$\begin{aligned}
I^+(p,U) &:= \{q \in U : \|\gamma'_{pq}\|^2 < 0,\ \gamma'_{pq}(0) \text{ a } \uparrow -\text{vector}\}\\
I^-(p,U) &:= \{q \in U : \|\gamma'_{pq}\|^2 < 0,\ \gamma'_{pq}(0) \text{ a } \downarrow -\text{vector}\}\\
J^+(p,U) &:= \{q \in U : \|\gamma'_{pq}\|^2 \leq 0,\ \gamma'_{pq}(0) \text{ a } \uparrow -\text{vector}\} \cup \{p\}\\
J^-(p,U) &:= \{q \in U : \|\gamma'_{pq}\|^2 \leq 0,\ \gamma'_{pq}(0) \text{ a } \downarrow -\text{vector}\} \cup \{p\}\\
\mathcal{L}_p^{U+} &:= \{q \in U : \|\gamma'_{pq}\|^2 = 0,\ \gamma'_{pq}(0) \uparrow -\text{vector}\}\\
\mathcal{L}_p^{U-} &:= \{q \in U : \|\gamma'_{pq}\|^2 = 0,\ \gamma'_{pq}(0) \downarrow -\text{vector}\}\\
\mathcal{L}_p^{U} &:= \mathcal{L}_p^{U+} \cup \mathcal{L}_p^{U-}.
\end{aligned}$$

The sets $I^+(p,U)$ and $I^-(p,U)$ are open and it holds (comp. [BEE96]):

$$\begin{aligned}
J^\pm(p,U) &= cl_U(I^\pm(p,U))\\
\mathcal{L}_p^{U\pm} &= \partial_U(I^\pm(p,U)) \setminus \{p\}\\
\mathcal{L}_p^{U} &\subset \mathcal{L}_p \cap U.
\end{aligned}$$

Notice that if $q \in I^+(p,U)$ then $J^+(q,U) \subset I^+(p,U)$ and if $q \in J^+(p,U)$ then $I^+(q,U) \subset I^+(p,U)$ (see [Pen72]). Furthermore, since U is convex, there exists an open set $V_p \subset T_pM$ such that $\exp_p : V_p \to U$ is a diffeomorphism. Then it holds

$$\mathcal{L}_p^U = \exp_p(V_p \cap L_p)$$

and $\mathcal{L}_p^U$ is a submanifold in M_1^n of codimension 1. From the Gauss lemma it follows that the induced symmetric bilinear form of g on $T\mathcal{L}_p^U$ is degenerate in every point $l \in \mathcal{L}_p^U$.

In the following we denote by $Im\gamma$ the image of a smooth curve γ in M_1^n. Here is the first of the announced propositions.

Proposition 3.2.1. — *Let $U \subset M_1^n$ be convex and $p, q \in U$, $p \neq q$. For the intersection $\mathcal{L}_{pq}^U := \mathcal{L}_p^U \cap \mathcal{L}_q^U$ of the lightcones to p and q one of the following assertions is true:*

(1) $\mathcal{L}_{pq}^U = \emptyset$,

(2) $\mathcal{L}_{pq}^U \neq \emptyset$, $\|\gamma'_{pq}\|^2 \neq 0$ *and* $\mathcal{L}_{pq}^U$ *is a $(n-2)$-dimensional spacelike submanifold of M,*

(3) $\mathcal{L}_{pq}^U \neq \emptyset$, $\|\gamma'_{pq}\|^2 = 0$ *and* $\mathcal{L}_{pq}^U = Im\gamma_{pq} \cap U$ *is a 1-dimensional, totally lightlike submanifold of M_1^n.*

Proof. Suppose that $\mathcal{L}_{pq}^U \neq \emptyset$ and $\|\gamma'_{pq}\|^2 \neq 0$. Then we have

$$\gamma'_{pl}(1) \nparallel \gamma'_{ql}(1) \qquad \text{for all } \ l \in \mathcal{L}_{pq}^U,$$

i.e. the two vectors are not parallel, which implies

$$\gamma'_{pl}(1) \notin T_l\mathcal{L}_q^U \qquad \text{for all } \ l \in \mathcal{L}_{pq}^U.$$

Hence, the submanifolds $\mathcal{L}_p^U$ and $\mathcal{L}_q^U$ in M_1^n are transversal and $\mathcal{L}_{pq}^U$ is a $(n-2)$-dimensional submanifold in M_1^n. The tangent space $T_l(\mathcal{L}_{pq}^U) = T_l(\mathcal{L}_p^U) \cap T_l(\mathcal{L}_q^U)$ is spacelike for every $l \in \mathcal{L}_{pq}^U$.

Suppose now that $\mathcal{L}_{pq}^U \neq \emptyset$ and $\|\gamma'_{pq}\|^2 = 0$. Obviously, it holds $Im\gamma_{pq} \cap U \subset \mathcal{L}_{pq}^U$ and we observe that in a convex set on a Lorentzian manifold there are never lightlike triangles, which means that if $p, q, r \in U$ and $\|\gamma'_{pq}\|^2 = \|\gamma'_{pr}\|^2 = \|\gamma'_{qr}\|^2 = 0$, then $r \in Im\gamma_{pq} \cap U$. We can conclude that $Im\gamma_{pq} \cap U = \mathcal{L}_{pq}^U$ holds. □

Proposition 3.2.2. — *Let $p \in M_1^n$. Then exists a neighborhood $U(p)$ of p contained in a convex set U with the property that*

$$\mathcal{L}_{qr}^U \neq \emptyset \qquad \textit{for all} \;\; q, r \in U(p).$$

Proof. Let U be a time-oriented convex neighborhood of $p \in M_1^n$. It is a well-known fact that in an arbitrary neighborhood $V(p)$ of p in U, there exist points $u, v \in V(p)$ such that the open set $\langle|u,v|\rangle_U := I^+(u,U) \cap I^-(v,U)$ is a neighborhood of p in $V(p)$:

$$p \in \langle|u,v|\rangle_U \subset V(p) \subset U$$

(see [Gün88] p. 15 or [Fri75]). So let $\tilde{V}(p) \subset \mathrm{int}(U)$ be a relative compact neighborhood of p and $a, b \in \tilde{V}(p)$ such that $p \in \langle|a,b|\rangle_U \subset \tilde{V}(p)$. We show that the neighborhood $U(p) := \langle|a,b|\rangle_U \subset U$ has the desired property.

First, suppose that $q, r \in \langle|a,b|\rangle_U$ and $\|\gamma'_{qr}\|^2 > 0$. We consider the geodesic γ_{qb}. It holds $\|\gamma'_{qr}\|^2 > 0$ and $\|\gamma'_{br}\|^2 < 0$. For continuity reasons it follows the existence of $t \in [0,1]$, $\tilde{b} := \gamma_{qb}(t)$, with $\|\gamma'_{\tilde{b}r}\|^2 = 0$. Furthermore, for the same reasons one can find $\hat{t} \in [0,1]$ such that $\|\gamma'_{\gamma_{r\tilde{b}}(\hat{t})q}\|^2 = 0$. It follows $\gamma_{r\tilde{b}}(\hat{t}) \in \mathcal{L}_{qr}^U$.

Suppose now that $q, r \in \langle|a,b|\rangle_U$, $\|\gamma'_{qr}\|^2 < 0$ and $\gamma'_{qr}(0)$ a $\uparrow$-vector. The set $cl_U(\langle|a,r|\rangle_U) \subset U$ is compact. Let $\gamma_q : I_q^U \to U$ be an arbitrary maximal lightlike $\uparrow$-geodesic in U with $\gamma_q(0) = q$. Since $q \in I^+(a,U)$ and $\gamma_q(t) \in J^+(q,U)$ for every $t \in I_q^U \cap \mathbb{R}_+$, it holds

$$\gamma_q(I_q^U \cap \mathbb{R}_+) \subset I^+(a,U).$$

The set $\gamma_q(I_q^U \cap \mathbb{R}_+)$ is not contained in a compact subset of U and therefore, it exists a $t \in I_q^U \cap \mathbb{R}_+$ with $\gamma_q(t) \notin J^-(r,U)$ and $\|\gamma'_{r\gamma_q(t)}\|^2 > 0$. But then also $\hat{t} \in I_q^U \cap \mathbb{R}_+$ exists such that $\|\gamma'_{r\gamma_q(\hat{t})}\|^2 = 0$, which implies $\mathcal{L}_{qr}^U \neq 0$.

Obviously, it is $\mathcal{L}_{qr}^U \neq \emptyset$ for $q, r \in \langle|a,b|\rangle_U$ with $\|\gamma'_{qr}\|^2 = 0$. □

Proposition 3.2.3. — *Let $N^1 \subset M_1^n$ be a 1-dimensional, spacelike submanifold. Then an open set $U_N \subset M_1^n$ exists with the property that for every point $r \in U_N$ there are lightlike vectors*

$$v_r \nparallel w_r \in T_rM \quad \textit{with} \quad \exp_r(v_r), \exp_r(w_r) \in N.$$

This proposition is not true in general for a lightlike, 1-dimensional submanifold $N^1 \subset M_1^n$. To prove it we need some preparation.

Lemma 3.2.4. *— Let $U \subset M_1^n$ be a time-oriented convex set, $r \in U$ and $a, b, c \in \mathcal{L}_r^{U-}$ points in the lightcone of the past such that*

$$\|\gamma'_{ab}\|^2, \|\gamma'_{ac}\|^2 \quad \textit{and} \quad \|\gamma'_{bc}\|^2 > 0.$$

Then there exists in every neighborhood $U(r)$ of r a point $\tilde{r} \in U(r)$ such that

$$a, b \in I^-(\tilde{r}, U) \quad \textit{and} \quad c \notin J^-(\tilde{r}, U) \cup J^+(\tilde{r}, U).$$

Proof. It is $r \in \mathcal{L}_{abc}^U := \mathcal{L}_a^U \cap \mathcal{L}_b^U \cap \mathcal{L}_c^U \neq \emptyset$. For every $\hat{r} \in \mathcal{L}_{abc}^U$ the lightlike vectors

$$\gamma'_{a\hat{r}}(1), \gamma'_{b\hat{r}}(1) \quad \text{and} \quad \gamma'_{c\hat{r}}(1) \in T_{\hat{r}}M_1^n$$

are not pairwise parallel by assumption and thus they are linearly independent. It follows

$$T_{\hat{r}}\mathcal{L}_{ab}^U = Span\{\gamma'_{a\hat{r}}(1), \gamma'_{b\hat{r}}(1)\}^\perp \not\subset (\gamma'_{c\hat{r}}(1))^\perp = T_{\hat{r}}\mathcal{L}_c^U$$

and the submanifolds $\mathcal{L}_c^U$ and $\mathcal{L}_{ab}^U$ of M_1^n are transversal. Hence, $\mathcal{L}_{abc}^U$ is a submanifold in $\mathcal{L}_{ab}^U$ of codimension 1. This implies the existence of a C^∞-curve $\alpha : (-\delta, \delta) \to \mathcal{L}_{ab}^U \subset M_1^n$ with $\alpha(0) = r$ and $\alpha'(0) \notin T_r\mathcal{L}_c^U$, i.e. the curve α intersects the lightcone $\mathcal{L}_c^U$ at the point r. Then there must be $\hat{t} \neq 0$ such that $c \notin J^-(\alpha(\hat{t}), U) \cup J^+(\alpha(\hat{t}), U)$. Since $\alpha(\hat{t}) \in \mathcal{L}_{ab}^U$, we can find a point $\tilde{r} \in I^+(\alpha(\hat{t}), U)$ in the near of $\alpha(\hat{t})$ such that $c \notin J^-(\tilde{r}, U) \cup J^+(\tilde{r}, U)$ and $a, b \in I^-(\tilde{r}, U)$. By construction the point $\tilde{r}$ can be chosen arbitrary close to r. □

We use the following notation. Let $a, b, r \in U$ points in a convex set such that

$$\#\big(\ \{a, b\} \cap I^-(r, U)\ \big) = 1$$

$$\#\big(\ \{a, b\} \cap (J^-(r, U) \cup J^+(r, U))\ \big) = 1.$$

Then we call $r \in U$ an (a, b)-separating point in U.

Lemma 3.2.5. *— Let $U \subset M_1^n$ be a time-oriented convex set, $r \in U$ and $a_1, a_2, b_1, b_2 \in I^-(r, U)$ such that*

$$\|\gamma'_{xy}\|^2 > 0 \qquad \textit{for all} \ \ x, y \in \{a_1, a_2, b_1, b_2\}, \quad x \neq y.$$

Then there exists a point $s \in U$, which separates the pairs (a_1, a_2) and (b_1, b_2) in U.

Proof. We consider the geodesic $\gamma_{a_1 r}$. There are real numbers t_{a_2}, t_{b_1} and $t_{b_2} \in (0, 1)$ such that

$$x \in \mathcal{L}^{U-}_{\gamma_{a_1 r}(t_x)} \qquad \text{for all } x \in \{a_2, b_1, b_2\}.$$

In case that one of the numbers t_{a_2}, t_{b_1} and t_{b_2} is greater then the others, it obviously exists $\hat{t} < \max\{t_{a_2}, t_{b_1}, t_{b_2}\}$ such that for $\tilde{r} := \gamma_{a_1 r}(\hat{t})$ the condition

$$\#(\ \{a_1, a_2, b_1, b_2\} \cap I^-(\tilde{r}, u)\ ) = \#(\ \{a_1, a_2, b_1, b_2\} \cap (J^-(\tilde{r}, u) \cup J^+(\tilde{r}, u)\ ) = 3$$

is satisfied. By Lemma 3.2.4 we can also find a point $\tilde{r} \in U$, which satisfies this condition for the case when $t_{a_2} = t_{b_1} = t_{b_2}$. In case that two of the numbers t_{a_2}, t_{b_1} and t_{b_2} are equal and greater then the third, Lemma 3.2.4 is all the more applicable. Altogether, we have in any case a point $\tilde{r} \in U$ such that after eventually changing the notation

$$a_1, b_1, b_2 \in I^-(\tilde{r}, U) \qquad \text{and} \qquad a_2 \notin J^-(\tilde{r}, U) \cup J^+(\tilde{r}, U).$$

With the same procedure as before applied to the points $\{a_1, b_1, b_2\}$ and the geodesic $\gamma_{a_1\tilde{r}}$, it follows the existence of a point $s \in U$ such that

$$a_1, b_1 \in I^-(s, U) \quad \text{and} \quad b_2 \notin J^-(s, U) \cup J^+(s, U) \qquad \text{or}$$

$$a_1, b_2 \in I^-(s, U) \quad \text{and} \quad b_1 \notin J^-(s, U) \cup J^+(s, U).$$

Obviously, the point s can be chosen such that $a_2 \notin J^-(s, U) \cup J^+(s, U)$ is still satisfied. □

Proof of Proposition 3.2.3. Let $U \subset M_1^n$ be a convex set and $s \in U \subset M_1^n$ such that

$$N^1 \cap I^-(s, U) \neq \emptyset.$$

Since N^1 is spacelike, there is a spacelike C^∞-curve $\alpha : (-1, 1) \to N^1 \cap I^-(s, U)$. An easy consideration shows that there are real numbers t_1, t_2, t_3 and $t_4 \in (-1, 1)$ with $t_1 < t_2 < t_3 < t_4$, such that

$$\|\gamma'_{\alpha(t_1)\alpha(t_2)}\|^2,\ \|\gamma'_{\alpha(t_3)\alpha(t_4)}\|^2 > 0 \quad \text{and}$$

$$\|\gamma'_{xy}\|^2 > 0 \qquad \text{for all } x \in \alpha([t_1, t_2]),\ y \in \alpha([t_3, t_4]).$$

From Lemma 3.2.5 it follows the existence of a point $\tilde{s} \in U$, which separates the pairs $(\alpha(t_1), \alpha(t_2))$ and $(\alpha(t_3), \alpha(t_4))$ in U. Moreover, there is a neighborhood U_N of $\tilde{s}$ such that every point $\hat{s} \in U_N$ separates these pairs in U. This shows for every $\hat{s} \in U_N$ the existence of points

$$x_{\hat{s}} \in \mathcal{L}_{\hat{s}}^{U-} \cap \alpha([t_1, t_2]) \qquad \text{and} \qquad y_{\hat{s}} \in \mathcal{L}_{\hat{s}}^{U-} \cap \alpha([t_3, t_4]).$$

Since $\|\gamma'_{x_{\hat{s}} y_{\hat{s}}}\|^2 > 0$, it follows $\gamma'_{\hat{s}x_{\hat{s}}}(0) \nparallel \gamma'_{\hat{s}y_{\hat{s}}}(0)$. □

3.3 The zero set of a conformal vector field

Let (M_1^n, g), $n \geq 3$, be a n-dimensional Lorentzian manifold and let $V \in \Gamma(TM_1^n)$ be a conformal vector field, i.e. it holds

$$L_V g = 2\alpha \cdot g$$

for some function $\alpha \in C^\infty(M_1^n)$. A conformal vector field V on a connected Lorentzian manifold M_1^n is uniquely determined by the values of

$$V(x_o), \quad \nabla V(x_o), \quad \alpha(x_o) \qquad \text{and} \qquad d\alpha(x_o)$$

at some point $x_o \in M^n_1$. In particular, if V vanishes on an open set in a connected manifold M^n_1 then $V \equiv 0$. Let $\Phi^V : A^V \subset \mathbb{R} \times M^n_1 \to M^n_1$ denote the maximal local flow of the conformal vector field V, i.e. for every point $p \in M^n_1$ the map

$$\Phi^V_t(p) = \Phi^V(t,p), \quad t \in I_p := A^V \cap (\mathbb{R} \times \{p\}),$$

is the maximal integral curve of the field V through $p \in M^n_1$. In case that the flow Φ^V is defined on an open subset $W \subset M^n_1$ for all $t \in (-\varepsilon, \varepsilon)$, $\varepsilon > 0$, the mapping

$$\Phi^V_t : W \to \Phi^V_t(W) \subset M^n_1$$

is a conformal diffeomorphism for every $t \in (-\varepsilon, \varepsilon)$. The zero set of the conformal vector field V is denoted by $zero(V)$. The property $\nabla V(p) = 0$ in $p \in zero(V)$ for a conformal vector field V implies that

$$d\Phi^V_t(p) = \mathrm{id}|_{T_pM} \qquad \text{for all } t \in I_p.$$

Lemma 3.3.1. — *Let V be a conformal vector field on a Lorentzian manifold M^n_1 and $p \in M^n_1$ a zero of V with $\nabla V(p) = 0$. For every point $q \in \mathcal{L}_p$ and every lightlike smooth geodesic $\gamma : [0,1] \to M^n_1$, $\gamma(0) = p$, $\gamma(1) = q$, it holds*

$$V(q) \parallel \gamma'(1) \in T_qM^n_1 \qquad \textit{or} \qquad V(q) = 0.$$

Proof. Let W_γ be an open neighborhood of the compact set $\gamma([0,1]) \subset M^n_1$ and $\varepsilon > 0$ such that the flow Φ^V_t on W_γ is defined for every $t \in (-\varepsilon, \varepsilon)$. Because $\Phi^V_t : W_\gamma \to M^n_1$ is a conformal transformation for $t \in (-\varepsilon, \varepsilon)$, every C^∞-curve $\gamma_t := \Phi^V_t \circ \gamma : [0,1] \to M^n_1$ is a lightlike pregeodesic in M^n_1 with $\gamma_t(0) = p$. Since

$$d\Phi^V_t(p) = \mathrm{id}|_{T_pM} \qquad \text{for all } t \in (-\varepsilon, \varepsilon),$$

it holds $\gamma'_t(0) = \gamma'(0)$ for all $t \in (-\varepsilon, \varepsilon)$, which implies the existence of a smooth function $\lambda : (-\varepsilon, \varepsilon) \to \mathbb{R}$ with

$$\Phi^V_t(q) = \gamma_t(1) = \exp_p(\lambda(t)\gamma'(0)).$$

It follows $V(q) = \frac{d}{dt}\big|_{t=0} \Phi^V_t(q) = \lambda'(0)\gamma'(1)$. □

Remark. Let us call a point $q \in M^n_1$ lightlike conjugated to $p \in M^n_1$ if there exist lightlike C^∞-geodesics

$$\gamma_i : [0,1] \to M^n_1 \quad \text{with} \quad \gamma_i(0) = p,\ \gamma_i(1) = q \qquad \text{for } \ i = 1,2,$$

such that $\gamma'_1(1) \nparallel \gamma'_2(1)$. Then we denote by $lc(p)$ the set of lightlike conjugated points to p in M^n_1. Lemma 3.3.1 implies that if $p \in M^n_1$ is a zero of a conformal vector field V on M^n_1 with $\nabla V(p) = 0$ then

$$lc(p) \subset zero(V).$$

With the results of 3.2 we can prove

Theorem 3.3.2. — *Let $0 \not\equiv V$ be a conformal vector field on a connected Lorentzian manifold M^n_1 with the property $\nabla V(p) = 0$ for all $p \in zero(V)$. Then there exists for every $p \in zero(V)$ a neighborhood $U(p) \subset M^n_1$ and a lightlike C^∞-geodesic γ_p such that*

$$zero(V) \cap U(p) \subset Im\gamma_p \cap U(p).$$

Proof. From Proposition 3.3.2 it follows the existence of a neighborhood $U(p)$ of p, which is contained in a convex set U, such that

$$\mathcal{L}^U_{qr} \neq \emptyset \qquad \text{for all } q, r \in U(p).$$

Suppose that there are points $q, r \in zero(V) \cap U(p)$ with $\|\gamma'_{qr}\|^2 \neq 0$. Then we have

$$\gamma'_{ql}(1) \not\parallel \gamma'_{rl}(1) \qquad \text{for all } l \in \mathcal{L}^U_{qr}$$

and by Lemma 3.3.1 it follows $V(l) = 0$ for all $l \in \mathcal{L}^U_{qr}$. Proposition 3.2.1 says that $\mathcal{L}^U_{qr} \subset zero(V)$ is a spacelike submanifold of M^n_1. In particular, a spacelike curve in $\mathcal{L}^U_{qr}$ exists. But then Proposition 3.2.3 together with Lemma 3.3.1 shows the existence of an open set $U_{qr} \subset zero(V)$. This is not possible, since we have assumed that M^n_1 is connected and $V \not\equiv 0$. We can conclude that

$$\|\gamma'_{qr}\|^2 = 0 \qquad \text{for all } q, r \in zero(V) \cap U(p).$$

We mentioned already that lightlike triangles do not exist in a convex subset of a Lorentzian manifold. Hence, the set $zero(V) \cap U(p)$ must be contained in the image of a single lightlike C^∞-geodesic γ_p. □

On the Minkowski space $\mathbb{R}^{1,n-1}$ we know from the explicit form of the conformal vector fields given in the beginning of this section that the zero set of a conformal vector field V with

$$\nabla V(p) = 0 \qquad \text{for all } p \in zero(V)$$

is a lightlike straight line or a single point. The statement of Theorem 3.2.2 for arbitrary curved Lorentzian manifolds is a bit weeker, since it could happen that a zero of a conformal vector field of the considered form does not lie on a lightlike geodesic, where the conformal field vanishes, but also is not isolated in the zero set.

We can prove a more global version of Theorem 3.3.2.

Theorem 3.3.3. — *Let M^n_1 be a connected Lorentzian manifold and $V \in \Gamma(TM^n_1)$ a conformal vector field. If $p, q \in zero(V)$ exist with lightlike tangent vectors $v_p \in T_pM$, $v_q \in T_q(M)$ such that*

(1) $\mathrm{rg}(d\exp_p(v_p)) = \mathrm{rg}(d\exp_q(v_q)) = n$

(2) $r := \exp_p(v_p) = \exp_q(v_q) \in M^n_1$

(3) $\frac{d}{dt}\big|_{t=1} \exp_p tv_p \not\parallel \frac{d}{dt}\big|_{t=1} \exp_q tv_q$

then the conformal vector field V vanishes identically.

PROOF. The three assumptions imply the existence of neighborhoods $V_p \subset T_pM$ of v_p and $V_q \subset T_qM$ of v_q such that

$$\mathcal{L}_1 := \exp_p(V_p \cap L_p) \quad \text{and} \quad \mathcal{L}_2 := \exp_p(V_q \cap L_q)$$

are transversal submanifolds of codimension 1 in M_1^n. Hence, $W_{pq} := \mathcal{L}_1 \cap \mathcal{L}_2$ is a spacelike submanifold of M_1^n. With the same arguments as in the proof of Theorem 3.3.2 we can conclude that $V \equiv 0$ on M_1^n. $\square$

3.4 The zero set of a twistor spinor

A twistor spinor φ on a Lorentzian spin manifold (M_1^n, g) induces the conformal vector field V_φ on M_1^n, which is locally defined by

$$V_\varphi := -\sum_i^n \varepsilon_i \langle \varphi, s_i\varphi \rangle_S s_i,$$

where $(s_1, \dots, s_n)$ is a local orthonormal frame on M_1^n. We mentioned already the following two important properties of V_φ. It holds

(1) $zero(V_\varphi) = zero(\varphi)$ and

(2) $\nabla V_\varphi(p) = 0$ for all $p \in zero(V_\varphi)$.

We want to apply the results of 3.3 to twistor spinors and its zero sets. Let us consider a smooth geodesic $\gamma(t)$ on a Lorentzian spin manifold M_1^n admitting a twistor spinor $\varphi \in \Gamma(S)$. We denote by $Im\gamma$ the image of the geodesic γ in M_1^n. Let $p \in Im\gamma$. The set $U_\gamma := \exp_p(D_p) \subset M_1^n$ is a time-orientable neighborhood of $Im\gamma$ in M_1^n. Furthermore, let $\{f_i(t) : i = 1, \dots, r\}$, $r = 2^{[\frac{n}{2}]}$, be a parallel translated basis field of S along the geodesic $\gamma(t)$, i.e.

$$\nabla^S_{\gamma'} f_i = 0 \qquad \text{for all } i \in \{1, \dots, r\}.$$

We choose on U_γ a time-orientation and define the functions

$$u_i(t) := \langle \varphi(\gamma(t)), f_i(\gamma(t)) \rangle_S \qquad \text{for } i \in \{1, \dots, r\}.$$

It holds

$$\begin{aligned}
\frac{du_i}{dt} &= \gamma'(u_i) = \langle \nabla^S_{\gamma'}\varphi, f_i \rangle_S = -\frac{1}{n}\langle \gamma' \cdot D\varphi, f_i \rangle_S, \\
\frac{d^2u_i}{dt^2} &= \gamma'\gamma'(u_i) = -\frac{1}{n}\langle \nabla^S_{\gamma'}(\gamma' \cdot D\varphi), f_i \rangle_S = -\frac{1}{2}\langle \gamma' \cdot K(\gamma')\varphi, f_i \rangle_S \\
&= -\frac{1}{2}\langle \varphi, K(\gamma') \cdot \gamma' f_i \rangle_S \qquad \text{for all } i \in \{1, \dots, r\}.
\end{aligned}$$

For the vector function $U(t) := \begin{pmatrix} u_1(t) \\ \vdots \\ u_r(t) \end{pmatrix}$ we obtain the linear differential equation system

$$U'' = -\frac{1}{2}\overline{C} \cdot U,$$

where $C(t) \in M(r, \mathbb{C})$ is the complex matrix function of the endomorphisms $s \in S_{\gamma(t)} \mapsto K(\gamma') \cdot \gamma'(t)s \in S_{\gamma(t)}$ with respect to the basis $\{f_i(t) : i = 1, \ldots, r\}$.

Lemma 3.4.1. — *Let M^n_1 be a Lorentzian spin manifold, $\varphi \in \Gamma(S)$ a twistor spinor on M^n_1, $p \in zero(\varphi)$ and $\gamma_p(t)$ a C^∞-geodesic on M^n_1 with $\gamma_p(0) = p$.*

(1) If $\gamma'_p(0) \cdot D\varphi(p) = 0$ then $Im\gamma_p \subset zero(\varphi)$.

(2) If $\gamma'_p(0) \cdot D\varphi(p) \neq 0$ then there exists a neighborhood $U(p)$ of p with

$$zero(\varphi) \cap Im\gamma_p \cap U(p) = \{p\}.$$

Proof. Let us consider the second order ODE

$$U'' = -\frac{1}{2}\overline{C} \cdot U$$

with initial conditions $U(0) = 0$ and $u_i(0) = -\frac{1}{n}\langle \gamma'(0) \cdot D\varphi(p), f_i\rangle_S$ for the functions $u_i(t) := \langle \varphi(\gamma(t)), f_i(\gamma(t))\rangle_S$ with respect to a parallel frame $\{f_i(t)\}$. If $\gamma'(0) \cdot D\varphi(p) = 0$ then $U(0) = U'(0) = 0$, which implies that $U \equiv 0$ on $Im\gamma_p$. If $\gamma'(0) \cdot D\varphi(p) \neq 0$ then $U'(0) \neq 0$ and $p \in zero(\varphi)$ is isolated on the geodesic γ_p. □

Definition 3.4.2. — *Let M^n_1 be a Lorentzian spin manifold, $\varphi \in \Gamma(S)$ a twistor spinor and*

$$\gamma_p : \{t \in \mathbb{R} : tv_p \in D_p\} \to M^n_1, \quad v_p \in T_pM,$$

a maximal geodesic such that $Im\gamma_p \subset zero(\varphi)$. Then the set $Im\gamma_p$ is called a zero set geodesic of φ.

Obviously, the image $Im\gamma$ of a maximal C^∞-geodesic γ in M^n_1 is a zero set geodesic to a twistor spinor $\varphi \not\equiv 0$ if and only if there exists a point $p \in Im\gamma \cap zero(\varphi)$ with $\gamma' \cdot D\varphi(p) = 0$.

Theorem 3.4.3. — *Let M^n_1 be a connected Lorentzian spin manifold and $0 \not\equiv \varphi \in \Gamma(S)$ a twistor spinor on M^n_1.*

(1) Every zero set geodesic to φ in M^n_1 is a totally lightlike, 1-dimensional submanifold of M^n_1.

(2) Every zero set geodesic $Im\gamma$ of φ is isolated, i.e. there exists an open set $U(\gamma)$ such that

$$zero(\varphi) \cap \overline{U(\gamma)} = Im\gamma.$$

(3) The set $zero(\varphi)$ is the countable union of isolated points and isolated zero set geodesics:

$$zero(\varphi) = \bigcup_{i \in \mathbb{N}} Im\gamma_i \cup \bigcup_{i \in \mathbb{N}} \{p_i\}.$$

PROOF. By Theorem 3.3.2, it exists a time-oriented neighborhood $U(p)$ of $p \in zero(\varphi)$ and a C^∞-geodesic γ_p with

$$zero(\varphi) \cap U(p) = zero(V_\varphi) \cap U(p) \subset Im\gamma_p,$$

where V_φ is the associated conformal field to φ. Moreover, from Lemma 3.4.1 it follows the existence of an open neighborhood $\tilde{U}(p) \subset U(p)$ of p with

$$\begin{aligned} zero(\varphi) \cap \tilde{U}(p) &= Im\gamma_p \cap \tilde{U}(p) \qquad \text{or} \\ zero(\varphi) \cap \tilde{U}(p) &= \{p\}. \end{aligned}$$

This proves that every zero set geodesic $Im\gamma$ of φ is a submanifold in M^n_1 and $Im\gamma$ is isolated. In particular, the third assertion follows with the second countability axiom.

It remains to prove that a zero set geodesic is lightlike. So let us assume that $Im\gamma$ is a zero set geodesic. It holds $\gamma' \cdot D\varphi(p) = 0$ and $D\varphi(p) \neq 0$ for $p \in Im\gamma$. This implies $\|\gamma'\|^2 = 0$. □

One should notice that Theorem 3.4.3 is not a direct consequence of Lemma 3.4.1, which is proved only by using spinor calculus, since we also used Theorem 3.3.2 for conformal vector fields.

For arbitrary vectors $X, Y \in T_pM^n_1$ the mapping

$$s \in S_p \;\mapsto\; XY \cdot s \in S_p$$

is a complex linear endomorphism. Let $w \in S_p$ be an eigenspinor of $X \cdot Y$ to the eigenvalue $c \neq 0$. We have

$$XY \cdot w = cw, \quad YX \cdot w = \frac{g(X,X)g(Y,Y)}{c} w = (-c - 2g(X,Y))w \qquad \text{and}$$

$$c = -g(X,Y) \pm \sqrt{g(X,Y)^2 - g(X,X)g(Y,Y)} \in \mathbb{C}\,.$$

If there is in addition a spinor $v \in S_p$ with $XYv = 0$, then $g(X,X)g(Y,Y) = 0$. Hence, the endomorphism $XY \in \mathrm{End}(S_p)$, $X, Y \in T_pM$, has at most the eigenvalues

$$\begin{aligned} c_+ &:= -g(X,Y) + \sqrt{g(X,Y)^2 - g(X,X)g(Y,Y)}, \\ c_- &:= -g(X,Y) - \sqrt{g(X,Y)^2 - g(X,X)g(Y,Y)}. \end{aligned}$$

Moreover, the endomorphism $X \cdot Y \in End(S_p)$ has no positive eigenvalues if and only if

$$g(X,Y)^2 - g(X,X) \cdot g(Y,Y) < 0 \quad \text{or}$$

$$\|X\|^2 \cdot \|Y\|^2 \geq 0, \quad g(X,Y) \geq 0.$$

Proposition 3.4.4. — *Let M^n_1 be a Lorentzian spin manifold with $\nabla\mathrm{Ric} = 0$, $\varphi \in \Gamma(S)$ a non-trivial twistor spinor on M^n_1, $p \in zero(\varphi)$ and $\gamma_p(t)$ a C^∞-geodesic with $\gamma_p(0) = p$ and $\gamma'_p \cdot D\varphi(p) \neq 0$. If*

$$g(\gamma'_p, K(\gamma'_p))^2 - g(\gamma'_p, \gamma'_p) \cdot g(K(\gamma'_p), K(\gamma'_p)) < 0 \quad \textit{or}$$

$$\|\gamma'_p\|^2 \cdot \|K(\gamma'_p)\|^2 \geq 0, \quad g(\gamma'_p, K(\gamma'_p)) \geq 0,$$

then $Im\gamma_p \cap zero(\varphi) = \{p\}$.

PROOF. Let $\{f_i(t) : i = 1, \dots, r\}$ be a parallel basis field along $\gamma_p(t)$. From $\nabla \mathrm{Ric} = 0$ it follows that the scalar curvature R is constant and therefore $\nabla K = 0$. Then

$$\gamma_p' \langle K(\gamma_p')\gamma_p' f_i, f_j\rangle_S = 0,$$

i.e. the matrix function $C(t) \equiv C$ is constant and it exists a parallel eigenspinor field $s(t)$ on $Im\gamma_p$ such that the function $u_s(t) := \langle \varphi(\gamma_p(t), s(t)\rangle_S \not\equiv 0$ satisfies

$$\frac{d^2 u_s(t)}{dt^2} = -\frac{1}{2}\overline{c} \cdot u_s,$$

where $c \in \mathbb{C}$ is a constant eigenvalue of C. Hence, the function u_s is of the form

$$u_s(t) = A \cdot \left(e^{\sqrt{-\frac{1}{2}\overline{c}} \cdot t} - e^{-\sqrt{-\frac{1}{2}\overline{c}} \cdot t} \right), \quad A \neq 0.$$

If $u_s(t) = 0$ for $t \neq 0$ then $\sqrt{-\frac{1}{2}\overline{c}} \in i\mathbb{R}$ and thus $c > 0$. But $K(\gamma_p')\gamma_p'$ has no positive eigenvalues and therefore, such a $t \neq 0$ does not exist. □

Proposition 3.4.5. — *Let M^n_1 be a Lorentzian Einstein spin manifold, $0 \not\equiv \varphi \in \Gamma(S)$ a twistor spinor on M^n_1, $p \in zero(\varphi)$ and $\gamma_p(t)$ a C^∞-geodesic with $\gamma_p(0) = p$ and $\gamma_p' \cdot D\varphi(p) \neq 0$.*

(1) If $g(\gamma_p', \gamma_p') \cdot R \leq 0$ then $zero(\varphi) \cap Im\gamma_p = \{p\}$.

(2) If $g(\gamma_p', \gamma_p') \cdot R > 0$ then

$$zero(\varphi) \cap Im\gamma_p = \{\gamma_p(\frac{n \cdot \pi}{d}) | n \in \mathbb{N}\}, \qquad d = \sqrt{\frac{R \cdot g(\gamma_p', \gamma_p')}{4n(n-1)}},$$

i.e. the zero set is periodic on $Im\gamma_p$.

PROOF. On an Einstein space the Schouten tensor K equals $\frac{-R}{2n(n-1)}\mathrm{id}$ and therefore

$$K(\gamma_p')\gamma_p' = \frac{R \cdot g(\gamma_p', \gamma_p')}{2n(n-1)}\mathrm{id}_S.$$

The first assertion is a special case of Theorem 3.4.4. If $g(\gamma_p', \gamma_p') \cdot R > 0$ then the solution of

$$U'' = -\frac{R \cdot g(\gamma_p', \gamma_p')}{4n(n-1)} U, \quad U = \begin{pmatrix} u_1(t) \\ \vdots \\ u_r(t) \end{pmatrix} \in \mathbb{C}^r,$$

is given by

$$U = \sin(dt) \cdot U'(0), \quad d = \sqrt{\frac{R \cdot g(\gamma_p', \gamma_p')}{4n(n-1)}}.$$

This proves the second assertion. □

There is an easy consequence of Proposition 3.4.5. Let M^n_1 be a Lorentzian Einstein spin manifold admitting a twistor spinor $\varphi \not\equiv 0$ and let $p, q \in zero(\varphi)$ be zeros, which do not lie on a common zero set geodesic of φ. By Proposition 3.4.5 there are no isolated zeros in the lightcones $\mathcal{L}_p$ and $\mathcal{L}_q$. Moreover, zero set geodesics can not intersect. It follows that the intersection $\mathcal{L}_p \cap \mathcal{L}_q$ of the lightcones to p and q must be empty:

$$\mathcal{L}_p \cap \mathcal{L}_q = \emptyset.$$

Example. The even-dimensional pseudosphere

$$S^{1,2n-1} := \{x \in \mathbb{R}^{1,2n} | \langle x, x\rangle_{1,2n} = 1\}$$

is a totally umbilic hypersurface in $\mathbb{R}^{1,2n}$. The spin structure on $\mathbb{R}^{1,2n}$ induces via the canonical embedding a spin structure on $S^{1,2n-1}$. The zero set of a twistor spinor φ on $\mathbb{R}^{1,2n}$ is empty, a single point or a lightlike straight line. The restriction $\varphi|_{S^{1,2n-1}}$ to the totally umbilic hypersurface $S^{1,2n-1}$ of a twistor spinor φ on $\mathbb{R}^{1,2n}$ is again a twistor spinor (comp. e.g. [Bau00b]) and the zero set $zero(\varphi|_{S^{1,2n-1}}) = zero(\varphi) \cap S^{1,2n-1}$ is also empty, a single point or a lightlike geodesic (that is a straight line in $S^{1,2n-1} \subset \mathbb{R}^{1,2n}$). We consider now a twistor spinor $\varphi|_{S^{1,2n-1}}$ on $S^{1,2n-1}$ with $e_2 \in zero(\varphi|_{S^{1,1}})$ and the spacelike geodesic $\gamma(t) = \cos(t)e_2 + \sin(t)e_3$. It is $d = \sqrt{\frac{R \cdot g(\gamma',\gamma')}{4n(n-1)}} = \frac{1}{2}$ and in fact we have

$$\gamma(t) \in zero(\varphi|_{S^{1,2n-1}}) \quad \text{if and only if} \quad t = 2\pi \cdot n = \frac{\pi \cdot n}{d}.$$

This is in accordance with Proposition 3.4.5. Let us consider now the universal covering

$$\pi : \tilde{S}^{1,1} \to S^{1,1}$$

with induced metric $g_{\tilde{S}^{1,1}}$ and induced spin structure. The space $\tilde{S}^{1,1}$ is geodesically complete and conformally flat. Every twistor spinor $\tilde{\varphi}$ on $\tilde{S}^{1,1}$ is induced by a twistor spinor $\varphi|_{S^{1,1}}$ on $S^{1,1}$ via the condition

$$\pi_*(\tilde{\varphi}) = \varphi|_{S^{1,1}}.$$

If $\varphi|_{S^{1,1}}$ on $S^{1,1}$ admits a zero or a zero set geodesic, then $\tilde{\varphi}$ on $\tilde{S}^{1,1}$ admits infinitely many zeros or zero set geodesics. The product $\mathbb{R} \times \tilde{S}^{1,1}$ with metric $dt \oplus g_{\tilde{S}^{1,1}}$ is a geodesically complete and conformally flat Lorentzian spin manifold of dimension 3. The space of twistor spinors on $\mathbb{R} \times \tilde{S}^{1,1}$ has maximal dimension 4. Every twistor spinor $\tilde{\varphi}$ on $\tilde{S}^{1,1}$ can be extended to a twistor spinor ψ on $\mathbb{R} \times \tilde{S}^{1,1}$ such that $\psi|_{\{0\}\times\tilde{S}^{1,1}} = \tilde{\varphi}$ (comp. [BFGK91]). The space $\mathbb{R} \times \tilde{S}^{1,1}$ is an example of a geodesically complete Lorentzian spin manifold of dimension 3 that admits twistor spinors with infinitely many zero set geodesics.

4 Further investigations of the twistor equation in Lorentzian spin geometry

In 1.4 we presented in short the important facts and results for twistor spinors in Lorentzian spin geometry, which have been worked out in the last years. Building upon these works, we will go on here to investigate the twistor equation in Lorentzian spin geometry. The essential ingredients to our investigations are the characteristic data of a twistor spinor φ consisting of the length function $\langle\varphi,\varphi\rangle_S$, the associated conformal field V_φ, its corresponding twist $\omega_\varphi \wedge d\omega_\varphi \in \Omega^3(M)$ and the spinor field $V_\varphi \cdot \varphi$.

In the first part of this section, we establish some results on twistor spinors φ in arbitrary dimension for the case when the twist $\omega_\varphi \wedge d\omega_\varphi$ of the associated field vanishes and the lenght function $|\varphi|^2$ has no singularities. In the following parts we discuss the twistor equation in the low dimensions $n = 3, 4$ and 5. It turns out that we can describe the Lorentzian metrics in dimension 3 and 5, which admit twistor spinors φ without 'singularities'. The classification of half spinors without 'singularities' in dimension 4 has been done already by J. Lewandowski in [Lew91]. Nevertheless, we will recall this classification result and will extend the theory in dimension 4. Moreover, in dimension $n = 3, 4$ and 5 we can prove some statements on twistor spinors with zeros.

4.1 Twistor spinors in arbitrary dimension

We start with some calculations in the spinor module $\Delta_{1,n-1}$ concerning the relation between spinors and its associated vectors.

Lemma 4.1.1. — *Let $0 \neq v \in \Delta_{1,n-1}$, $n \geq 3$, be a spinor and let $x_v = -\sum_{i=1}^n \varepsilon_i \langle v, e_i v\rangle_\Delta e_i \in \mathbb{R}^{1,n-1}$ be its associated vector.*

(1) The set $\ell(\Delta_{1,n-1})$ of associated vectors coincides with the set of future-directed causal vectors

$$J^+ = \{x \in \mathbb{R}^{1,n-1} : \langle x,x\rangle_{1,n-1} \leq 0 \text{ and } \langle x, e_1\rangle_{1,n-1} \leq 0\}.$$

(2) If $\|x_v\|^2 = 0$ then $x_v \cdot v = 0$ and $\langle v,v\rangle_\Delta = 0$.

(3) If $x_v \cdot v = \delta v$ for some $\delta \in \mathbb{C}$ then $\delta = \langle v,v\rangle_\Delta \in \mathbb{R}$ and $-\langle x_v, x_v\rangle_{1,n-1} = \langle v,v\rangle_\Delta^2$.

(4) If $y \cdot v = \delta v$ for some $0 \neq y \in \mathbb{R}^{1,n-1}$ and $\delta \in \mathbb{R}$ then the vectors y and x_v are parallel.

Proof. (1) Let $v_1 = a \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \Delta_{1,n-1} \cong \bigotimes_{[\frac{n}{2}]} \mathbb{C}^2$ with $(a,a) = 1$. It is

$$ig_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \qquad g_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix} \qquad \text{and} \qquad T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix}$$

(comp. 1.1), which yields $\langle v_1, e_1 v_1\rangle_\Delta = +1$ and

$$\langle v_1, e_i v_1\rangle_\Delta = 0 \qquad \text{for all } \; i \in \{2, \ldots, n\}.$$

It follows that $\ell(v_1) = e_1$. Now, let $v_2 = a \otimes \begin{pmatrix} 1 \\ -i \end{pmatrix}$. Then

$$ig_1 \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} -1 \\ -i \end{pmatrix}, \qquad g_2 \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix} \qquad \text{and} \qquad T \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} -1 \\ i \end{pmatrix},$$

which implies that $\ell(v_2) \in \mathbb{R}(e_1 + e_2)$. The spin group $Spin^+(1, n-1)$ acts transitively on the future-directed timelike unit vectors and also on the future-directed lightcone. Moreover, it holds $\langle v, e_1 v\rangle_\Delta \geq 0$ and $\ell(\nu \cdot sv) = |\nu|^2 \lambda(s) \circ \ell(v)$ for all $v \in \Delta_{1,n-1}$, $s \in Spin^+(1, n-1)$ and $\nu \in \mathbb{C}$. These properties show that for all $x \in J^+$ there exist $s \in Spin^+(1, n-1)$ and $\nu \in \mathbb{C}$ such that $x = \ell(\nu \cdot sv_1)$ or $x = \ell(\nu \cdot sv_2)$.

(2) It is sufficent to prove that $x_v \cdot v = 0$ for the case that $x_v = e_1 + e_2$. To show this we calculate the inverse image $\ell^{-1}(\mathbb{R}(e_1 + e_2))$. Let

$$v = \sum_{(\varepsilon_1,\dots,\varepsilon_m)\in\{\pm 1\}^m} a_{\varepsilon_1,\dots,\varepsilon_m} \cdot u(\varepsilon_1, \dots, \varepsilon_m), \qquad a_{\varepsilon_1,\dots,\varepsilon_m} \in \mathbb{C},$$

where $m = \left[\frac{n}{2}\right]$, be an arbitrary spinor represented in the standard basis of $\Delta_{1,n-1}$ (comp. 1.1). It holds

$$e_1 \cdot v = -\sum a_{\varepsilon_1,\dots,\varepsilon_m} \cdot u(\varepsilon_1, \dots, -\varepsilon_m) \qquad \text{and}$$

$$e_2 \cdot v = \sum \varepsilon_m \cdot a_{\varepsilon_1,\dots,\varepsilon_m} u(\varepsilon_1, \dots, -\varepsilon_m).$$

We obtain

$$\langle v, e_1 v\rangle_\Delta = \sum |a_{\varepsilon_1,\dots,\varepsilon_m}|^2 \qquad \text{and} \qquad \langle v, e_2 v\rangle_\Delta = \sum -\varepsilon_m \cdot |a_{\varepsilon_1,\dots,\varepsilon_m}|^2.$$

Obviously, it is $x_v \in \mathbb{R}(e_1 + e_2)$ if and only if $\langle v, e_1 v\rangle = -\langle v, e_2 v\rangle$. The latter condition is equivalent to

$$a_{\varepsilon_1,\dots,\varepsilon_{m-1},-1} = 0 \qquad \text{for all } (\varepsilon_1, \dots, \varepsilon_{m-1}) \in \{\pm 1\}^{m-1}.$$

We can conclude that a spinor v with $\ell(v) \in \mathbb{R}(e_1 + e_2)$ has the form

$$v = a \otimes \begin{pmatrix} 1 \\ -i \end{pmatrix}, \qquad a \in \bigotimes_{m-1} \mathbb{C}^2.$$

Obviously, it holds $\langle v, v\rangle_\Delta = 0$ and

$$(e_1 + e_2) \cdot v = a \otimes \left((ig_1 + g_2) \begin{pmatrix} 1 \\ -i \end{pmatrix}\right) = 0.$$

To prove the formulas in (3), it is sufficient to handle the cases when $x_v = e_1$ and $x_v = e_1 + e_2$. The case when $x_v = e_1 + e_2$ is already proved. So let us assume that $x_v = e_1$. Then

$$x_v \cdot v = \pm v \qquad \text{and} \qquad 1 = \langle v, e_1 v\rangle = \pm\langle v, v\rangle,$$

which implies that $\delta = \langle v, v\rangle$ and $-\langle x_v, x_v\rangle = \langle v, v\rangle^2$. With the equivariance property $\ell(\nu \cdot sv) = |\nu|^2 \lambda(s) \circ \ell(v)$ we can conclude that the formulas in (3) hold, whenever $x_v \cdot v = \delta v$ for some $\delta \in \mathbb{C}$.

(4) Again, it is sufficient to consider the cases $y = e_1$ and $y = e_1 + e_2$. First, we assume that $e_1 \cdot v = \pm v$. Then

$$\langle v, e_i v\rangle_\Delta = \langle e_1 v, e_i e_1 v\rangle_\Delta = -\langle v, e_i v\rangle_\Delta \qquad \text{for all } \; i \in \{2, \dots, n\}.$$

This shows that $x_v \in \mathbb{R}(e_1)$. If $(e_1 + e_2) \cdot v = 0$. Then $\langle v, e_1 v\rangle_\Delta = -\langle v, e_2 v\rangle_\Delta$ and this already implies $x_v \in \mathbb{R}(e_1 + e_2)$. □

Remember that the Clifford multiplication of a spinor v by an arbitrary k-form ω is defined as

$$\omega \cdot v = \sum_{1 \le i_1 < \cdots < i_k \le n} \varepsilon_{i_1} \cdot \ldots \cdot \varepsilon_{i_k} \omega(e_{i_1}, \cdots, e_{i_k}) e_{i_1} \cdot \ldots \cdot e_{i_k} \cdot v.$$

Lemma 4.1.2. — *Let $0 \neq v \in \Delta_{1,n-1}$ be a spinor such that $\|x_v\|^2 = 0$ and let $\eta \in \Lambda^2$ be a 2-form with $\eta \cdot v = 0$ then $x_v \lrcorner\ \eta = 0$.*

Proof. Since $\|x_v\|^2 = 0$, we may assume without loss of generality that $x_v = e_1 + e_2$. Moreover, in this case we know from the proof of Lemma 4.1.1 that the spinor v has the form

$$v = a \otimes \begin{pmatrix} 1 \\ -i \end{pmatrix}, \qquad a \in \bigotimes_{i=1}^{m-1} \mathbb{C}^2.$$

Since $(e_1 + e_2) \cdot v = 0$, we have

$$\begin{aligned} 0 = \eta \cdot v \quad &= \quad -\eta(e_1, e_2) e_1 e_2 \cdot v + \frac{1}{2} \cdot \sum_{j=3}^{n} \eta(e_1 + e_2, e_j)(-e_1 + e_2) e_j \cdot v \\ &\quad + \sum_{3 \le i < j \le n} \eta(e_i, e_j) e_i e_j \cdot v. \end{aligned}$$

It holds

$$\begin{aligned} (-e_1 + e_2) e_j \cdot v \quad &= \quad -2(e_j \cdot a) \otimes \begin{pmatrix} 1 \\ i \end{pmatrix} \\ e_1 e_2 \cdot v \quad &= \quad -v \\ e_i e_j \cdot v \quad &= \quad (e_i e_j \cdot a) \otimes \begin{pmatrix} 1 \\ -i \end{pmatrix}. \end{aligned}$$

Since $Re\left(\sum_{3 \le i < j \le n} \eta(e_i, e_j) e_i e_j \cdot a, a\right) = 0$ (positive definite scalar product), there exists no $0 \neq \delta \in \mathbb{R}$ such that

$$\sum_{3 \le i < j \le n} \eta(e_i, e_j) e_i e_j \cdot a = \delta a.$$

We can conclude that $\eta(e_1, e_2) = 0$ and $\eta(e_1 + e_2, e_j) = 0$ for all $j \in \{3, \cdots, n\}$, which implies that $(e_1 + e_2) \lrcorner\ \eta = 0$. □

We introduce a second indefinite $Spin^+(1,n-1)$-invariant Hermitian product $\langle\cdot,\cdot\rangle_2$ on the spinor module $\Delta_{1,n-1}$, which is defined by

$$\langle v,w\rangle_2 := i^{\frac{n(n-1)}{2}}(e_2\cdot\ldots\cdot e_n v,w)$$

for $v,w\in\Delta_{1,n-1}$ (see [Kat99]). With respect to the Hermitian product $\langle\cdot,\cdot\rangle_2$ we define an associated vector

$$t_v := i\cdot\sum_{i=1}^{n}\varepsilon_i\langle v,e_iv\rangle_2 e_i\in\mathbb{R}^{1,n-1}$$

to $v\in\Delta_{1,n-1}$. The dual 1-form of t_v is given by $\omega_{t_v}(x)=i\langle v,xv\rangle_2$. It is a straight forward calculation to see that the Hermitian product $\langle\cdot,\cdot\rangle_2$ and the associated vectors t_v have the following properties:

Lemma 4.1.3. — *(1)* $\langle x\cdot v,w\rangle_2=(-1)^{n+1}\langle v,x\cdot w\rangle_2$ *for all* $x\in\mathbb{R}^{1,n-1}$.

(2) The product $\langle\cdot,\cdot\rangle_2$ *is indefinite with index* $2^{[\frac{n}{2}]-1}$.

(3) If $n=2m+1$ *is odd then* $\langle v,w\rangle_2=(-1)^{m+1}\langle v,w\rangle_\Delta$.

(4) If $n=2m$ *is even then*

$$\begin{aligned}\langle v,w\rangle_2 &= (-1)^m i\cdot\langle v_+-v_-,w_++w_-\rangle_\Delta \qquad \textit{and}\\ \langle v,v\rangle_2 &= (-1)^{m+1}\cdot 2\,Im\langle v_+,v_-\rangle_\Delta,\end{aligned}$$

where $v=v_++v_-$ *and* $w=w_++w_-$ *are the decompositions into the half spinors.*

(5) If $n=2m$ *then* $t_v=(-1)^m(x_{v_+}-x_{v_-})$ *for* $v=v_++v_-$. *The map* $v\mapsto t_v$ *is surjective onto* $\mathbb{R}^{1,n-1}$. *It may happen that* $t_v=0$ *for* $v\neq 0$.

Now, let (M_1^n,g) be a smooth space- and time-oriented Lorentzian spin manifold of dimension n with spin structure $(Spin(M),f)$. The spinor bundle

$$S=Spin(M)\times_{\rho_{1,n-1}}\Delta_{1,n-1}$$

is furnished with the Hermitian product $\langle\cdot,\cdot\rangle_S$. Moreover, we have the Hermitian product $\langle\cdot,\cdot\rangle_2$ on S, which is induced by the $Spin^+(1,n-1)$-invariant product $\langle\cdot,\cdot\rangle_2$ on the spinor module $\Delta_{1,n-1}$. The Hermitian products satisfy the properties:

$$\begin{aligned}\langle X\varphi,\psi\rangle_S &= \langle\varphi,X\psi\rangle_S\\ Re\langle X\varphi,Y\varphi\rangle_S &= -g(X,Y)\cdot Re\langle\varphi,\varphi\rangle_S=-g(X,Y)\cdot|\varphi|^2\\ Im\langle X\varphi,\varphi\rangle_S &= 0\\ \langle X\varphi,\psi\rangle_2 &= (-1)^{n+1}\cdot\langle\varphi,X\psi\rangle_2\\ \langle\varphi,\psi\rangle_2 &= (-1)^m i\cdot\langle\varphi_+-\varphi_-,\psi_++\psi_-\rangle_S\\ \langle\varphi,\varphi\rangle_2 &= (-1)^{m+1}\cdot 2Im\langle\varphi_+,\varphi_-\rangle_S.\end{aligned}$$

To each spinor field we can associate the vector field (Dirac current) V_φ via the condition $g(V_\varphi, X) = -\langle\varphi, X\varphi\rangle_S$. Remember that $V_\varphi = V_{\varphi_+} + V_{\varphi_-}$ for $\varphi = \varphi_+ + \varphi_- \in \Gamma(S^+ \oplus S^-)$. Moreover, in even dimensions we have the vector field W_φ to $\varphi \in \Gamma(S)$, which is defined by

$$g(W_\varphi, X) = i\langle\varphi, X\varphi\rangle_2.$$

It holds $W_\varphi = (-1)^m(V_{\varphi_+} - V_{\varphi_-})$. So it may happen that W_φ vanishes even if $\varphi \in \Gamma(S)$ is not trivial.

We will need the following little useful lemma for twistor spinors on Lorentzian spin manifolds.

Lemma 4.1.4. — *Let φ be a twistor spinor on (M^n_1, g) and $a : M \to \mathbb{C}$ a complex function. If $a \cdot \varphi \in ker(P)$ is a twistor spinor then $a \equiv const.$*

Proof. Assume that $a \cdot \varphi$ is a twistor spinor. Let $(s_1, \ldots, s_n)$ be an orthonormal basis with $g(s_1, s_1) = -1$ in an arbitrary point $p \in M^n_1 \setminus zero(\varphi)$. From Proposition 1.2.2 it follows that $-s_1(a)s_1 \cdot \varphi = s_i(a)s_i \cdot \varphi$ for all $i = 2, \ldots, n$. But this implies that $-s_1(a)s_1 = s_i(a)s_i$ for all $i = 2, \ldots, n$. Obviously, the latter condition is possible only if $X(a) = 0$ for all $X \in T_pM$. Since the set $M^n_1 \setminus zero(\varphi)$ is dense in M^n_1, we can conclude that a is constant on M^n_1. □

We stated in Proposition 1.2.3 basic integrability conditions for semi-Riemannian spin manifolds admitting twistor spinor. Here is a further integrability condition in case of a Lorentzian spin manifold.

Proposition 4.1.5. — *Let $\varphi \in \Gamma(S)$ be a twistor spinor with lightlike Dirac current V_φ on (M^n_1, g). Then $V_\varphi \lrcorner W \equiv 0$.*

Proof. From Proposition 1.2.3 we know that $W(\eta) \cdot \varphi = 0$ for all 2-forms η on M^n_1 and from Lemma 4.1.2 we know that $V_\varphi \lrcorner W(\eta) = 0$ for all $\eta \in \Lambda^2 M^n_1$, which means that $W(X, Y, V_\varphi, Z) = 0$ for all X, Y and $Z \in TM$. □

Remark. It is also well-known that $V_\varphi \lrcorner C \equiv 0$, in case that φ is a twistor spinor on (M^n_1, g) (comp. [Bau99]).

Proposition 4.1.6. — *Let $\varphi \in ker(P)$ be a twistor spinor on (M^n_1, g) with $\|V_\varphi\|^2 \equiv 0$ and $zero(\varphi) = \emptyset$. Then the integral curves of V_φ are lightlike pregeodesics.*

Proof. Let $p \in M^n_1$ be arbitrary and let $s = (s_1 \ldots, s_n)$ be a local frame, which is parallel in $p \in M^n_1$. We calculate in $p \in M^n_1$:

$$\begin{aligned}
\nabla_{V_\varphi} V_\varphi &= \nabla_{V_\varphi}(-\textstyle\sum_i \varepsilon_i \langle\varphi, s_i \cdot \varphi\rangle s_i) \\
&= -\textstyle\sum_i \varepsilon_i \langle\nabla_{V_\varphi}\varphi, s_i \cdot \varphi\rangle s_i - \sum_i \varepsilon_i \langle\varphi, s_i \cdot \nabla_{V_\varphi}\varphi\rangle s_i \\
&= \tfrac{1}{n}\textstyle\sum_i \varepsilon_i \langle V_\varphi \cdot D\varphi, s_i \cdot \varphi\rangle s_i + \tfrac{1}{n}\sum_i \varepsilon_i \langle\varphi, s_i V_\varphi \cdot D\varphi\rangle s_i \\
&= -\tfrac{2}{n}\langle D\varphi, \varphi\rangle \cdot \textstyle\sum_i \varepsilon_i g(V_\varphi, s_i)s_i - \tfrac{2}{n}\langle\varphi, D\varphi\rangle \cdot \sum_i \varepsilon_i g(V_\varphi, s_i)s_i \\
&= -\tfrac{4}{n} Re\langle D\varphi, \varphi\rangle V_\varphi.
\end{aligned}$$

This shows that the integral curves of V_φ are pregeodesics. $\square$

On the conformally covariant kernel $\mathcal{T}(M^n_1) = ker(P)$ of the twistor operator P on a connected Lorentzian spin manifold (M^n_1, g), there exist a quadratic form C and a form A of order four defined by

$$C_\varphi := Im\langle D\varphi, \varphi\rangle$$

$$A_\varphi := |\varphi|^2|D\varphi|^2 + \sum_{i=1}^{n} \epsilon_i \cdot (Re\langle D\varphi\ , s_i\varphi\rangle)^2 .$$

(comp. 1.3 and [BFGK91]). Let K_φ denote the real subspace of S defined by

$$K_\varphi := Span_{\mathbb{R}}\{X\cdot\varphi |\ X \in TM\} = TM\cdot\varphi.$$

In case that $|\varphi|^2 \neq 0$ the bilinear form $Re\langle\cdot,\cdot\rangle_S$ on K_φ is non-degenerate and we have a unique decomposition of $D\varphi$ into $D\varphi = \psi_\perp + \psi$ such that $\psi \in K_\varphi$ and $Re\langle\psi, \psi_\perp\rangle = 0$. Then it holds $A_\varphi = |\varphi|^2|\psi_\perp|^2$. We set

$$T_\varphi := -n \cdot grad|\varphi|^2 = 2\cdot\sum_{i=1}^{n} \epsilon_i \cdot Re\langle D\varphi, s_i\cdot\varphi\rangle s_i.$$

Proposition 4.1.7. — *Let $\varphi \in \Gamma(S)$ be a twistor spinor on (M^n_1, g). Then*

(1) C_φ and A_φ are constant on M^n_1.

(2) The constants C_φ and A_φ are conformally invariant, i.e. if $\tilde g = e^{2\sigma}g$ is a conformally equivalent metric to g on M^n_1 then $C_{e^{\sigma/2}\tilde\varphi} = C_\varphi$ and $A_{e^{\sigma/2}\tilde\varphi} = A_\varphi$.

(3) $\left|iC_\varphi\cdot\varphi + |\varphi|^2 D\varphi + \frac{1}{2}T_\varphi\cdot\varphi\right|^2 = |\varphi|^2(A_\varphi - C^2_\varphi)$.

(4) If φ is a Killing spinor to the real Killing number λ then

$$C_\varphi = 0 \qquad \text{and} \qquad A_\varphi = n^2\lambda^2\cdot\left(|\varphi|^4 + g(V_\varphi, V_\varphi)\right).$$

(5) If φ is an imaginary Killing spinor then

$$C_\varphi = -nIm\lambda\cdot|\varphi|^2 \qquad \text{and} \qquad A_\varphi = -n^2\lambda^2\cdot|\varphi|^4.$$

PROOF. (1) It holds for all $X \in TM$

$$\begin{aligned}\nabla_X C_\varphi &= Im\langle\nabla_X D\varphi, \varphi\rangle + Im\langle D\varphi, \nabla_X\varphi\rangle \\ &= Im\langle\frac{n}{2}L(X)\cdot\varphi, \varphi\rangle - Im\langle D\varphi, \frac{1}{n}X\cdot D\varphi\rangle = 0,\end{aligned}$$

$$\begin{aligned}
\nabla_X A_\varphi &= 2Re\langle\nabla_X\varphi,\varphi\rangle|D\varphi|^2 + 2|\varphi|^2 Re\langle\nabla_X D\varphi, D\varphi\rangle \\
&\quad +2\cdot\sum_{i=1}^n \epsilon_i\cdot Re\langle\nabla_X D\varphi, s_i\cdot\varphi\rangle\cdot Re\langle D\varphi, s_i\cdot\varphi\rangle \\
&\quad +2\cdot\sum_{i=1}^n \epsilon_i\cdot Re\langle D\varphi, s_i\cdot\nabla_X\varphi\rangle\cdot Re\langle D\varphi, s_i\cdot\varphi\rangle \\
&= -\frac{2}{n}Re\langle D\varphi, X\cdot\varphi\rangle|D\varphi|^2 + n|\varphi|^2 Re\langle L(X)\cdot\varphi, D\varphi\rangle \\
&\quad +n\sum_{i=1}^n \epsilon_i\cdot Re\langle L(X)\cdot\varphi, s_i\cdot\varphi\rangle\cdot Re\langle D\varphi, s_i\cdot\varphi\rangle \\
&\quad -\frac{2}{n}\sum_{i=1}^n \epsilon_i\cdot Re\langle D\varphi, s_i X\cdot D\varphi\rangle\cdot Re\langle D\varphi, s_i\cdot\varphi\rangle \\
&= -\frac{2}{n}Re\langle D\varphi, X\cdot\varphi\rangle|D\varphi|^2 + n|\varphi|^2 Re\langle L(X)\cdot\varphi, D\varphi\rangle \\
&\quad -n|\varphi|^2 Re\langle D\varphi, L(X)\cdot\varphi\rangle + \frac{2}{n}|D\varphi|^2 Re\langle D\varphi, X\cdot\varphi\rangle = 0\ .
\end{aligned}$$

(2) We have

$$\tilde{D}(e^{\sigma/2}\tilde{\varphi}) = e^{-\sigma/2}(\widetilde{D\varphi} + \frac{n}{2}\widetilde{grad}(e^\sigma)\cdot\tilde{\varphi}) =: \rho_\perp + \rho,$$

where $\rho \in K_{e^{\sigma/2}\tilde{\varphi}} = K_\varphi$ and $Re\langle\rho_\perp,\rho\rangle_{\tilde{S}} = 0$. It follows

$$\begin{aligned}
Im\langle\tilde{D}(e^{\sigma/2}\varphi), e^{\sigma/2}\varphi\rangle_{\tilde{S}} &= Im\langle e^{-\sigma/2}(\widetilde{D\varphi} + \frac{n}{2}\widetilde{grad}(e^\sigma)\cdot\tilde{\varphi}), e^{\frac{1}{2}\sigma}\tilde{\varphi}\rangle_{\tilde{S}} \\
&= Im\langle\widetilde{D\varphi},\tilde{\varphi}\rangle_{\tilde{S}} = Im\langle D\varphi,\varphi\rangle_S.
\end{aligned}$$

In case that $|\varphi|^2 \not\equiv 0$ we choose $p\in M$ with $|\varphi(p)|^2 \neq 0$. Since $K_{e^{\sigma/2}\tilde{\varphi}} = K_\varphi$, it holds

$$\rho_\perp(p) = e^{-\sigma/2}\cdot\tilde{\psi}_\perp(p)$$

and then

$$\begin{aligned}
A_{e^{\sigma/2}\tilde{\varphi}} &= A_{e^{\sigma/2}\tilde{\varphi}}(p) = |e^{\sigma/2}\tilde{\varphi}(p)|^2_{\tilde{S}}\cdot|\rho_\perp(p)|^2_{\tilde{S}} \\
&= e^\sigma|\varphi(p)|^2\cdot e^{-\sigma}|\psi_\perp(p)|^2 = A_\varphi.
\end{aligned}$$

In case that $|\varphi|^2\equiv 0$, it holds $A_\varphi = A_{e^{\sigma/2}\tilde{\varphi}} = 0$, since $0 = \nabla_X Re\langle\varphi,\varphi\rangle = -\frac{2}{n}Re\langle X\cdot D\varphi,\varphi\rangle$.

(3) We calculate that

$$\begin{aligned}\left|iC_\varphi\varphi+|\varphi|^2D\varphi+\frac{1}{2}T_\varphi\cdot\varphi\right|^2 &= C_\varphi^2|\varphi|^2+|\varphi|^4|D\varphi|^2-|\varphi|^2\sum_{\alpha=1}^n\epsilon_\alpha(Re\langle D\varphi,s_\alpha\varphi\rangle)^2\\ &\quad -2C_\varphi^2|\varphi|^2+|\varphi|^2Re\langle D\varphi,T_\varphi\cdot\varphi\rangle\\ &= -C_\varphi^2|\varphi|^2+|\varphi|^4|D\varphi|^2+2|\varphi|^2\sum_{\alpha=1}^n\epsilon_\alpha(Re\langle D\varphi,s_\alpha\varphi\rangle)^2\\ &\quad -|\varphi|^2\sum_{\alpha=1}^n\epsilon_\alpha(Re\langle D\varphi,s_\alpha\varphi\rangle)^2\\ &= |\varphi|^2(A_\varphi-C_\varphi^2).\end{aligned}$$

The formulas in (4) and (5) for C_φ and A_φ in case that φ is a Killing spinor to the Killing number λ follow easily with $D\varphi=-n\lambda\cdot\varphi$. □

Remark. The property that a twistor spinor φ is conformally equivalent to a parallel spinor is equivalent to the existence of a function σ on M_1^n such that

$$grad(e^{-\sigma})\cdot\varphi=\frac{2}{n}e^{-\sigma}D\varphi$$

(see 1.2). In case that $|\varphi|^2\neq 0$ this condition is equivalent to

$$grad|\varphi|^2\cdot\varphi=\frac{2}{n}|\varphi|^2D\varphi.$$

If φ is parallel then C_φ and A_φ are zero. On the other hand, if $C_\varphi=A_\varphi=0$ then

$$0=\left||\varphi|^2D\varphi+\frac{1}{2}T_\varphi\cdot\varphi\right|^2=\left||\varphi|^2D\varphi-\frac{n}{2}grad|\varphi|^2\cdot\varphi\right|^2,$$

which does not imply that $grad|\varphi|^2\cdot\varphi=\frac{2}{n}|\varphi|^2D\varphi$, i.e. it does not follow that φ has to be conformally equivalent to a parallel spinor. For example, a real Killing spinor φ on a complete Lorentzian spin manifold M_1^n has the property, that its lenght function $u=\langle\varphi,\varphi\rangle:M_1^n\to\mathbb{R}$ is surjective (see 1.4). Then locally in any point of the level set $\{u=0\}$ the real Killing spinor φ can not be conformally equivalent to a parallel spinor. But outside of the level set $\{u=0\}$ the real Killing spinor may be conformally equivalent to a parallel spinor (at least in dimensions 3 and 5; see 4.2, 4.4 and Proposition 4.1.16), which implies $A_\varphi=C_\varphi=0$.

Remember that in the Riemannian case the converse conclusion is true. A twistor spinor φ without zeros and $Q_\varphi=C_\varphi=0$ is conformally equivalent to a parallel spinor (see Theorem 1.3.3). This is useful to know, since then a twistor spinor with a zero has to be conformally equivalent to a parallel spinor outside of the zero set. In the Lorentzian case this argument does not count when we are looking for twistor spinors with zeros.

Proposition 4.1.8. — *Let (M_1^n,g) be a Lorentzian spin manifold, which admits a twistor spinor $\varphi\in\Gamma(S)$ with $|\varphi|^4\equiv 1$. Then (M_1^n,g) is an Einstein manifold with scalar curvature*

$$R=-\frac{4(n-1)}{n}\cdot A_\varphi.$$

PROOF. It holds $Re\langle D\varphi, X\varphi\rangle = 0$ for all $X \in TM$, since $|\varphi|^2 = const$. Then we have

$$\begin{aligned} -\frac{n}{2}g(K(X),Y)|\varphi|^2 &= Re\langle \nabla_X D\varphi, Y\cdot\varphi\rangle = X Re\langle D\varphi, Y\cdot\varphi\rangle - Re\langle D\varphi, \nabla_X(Y\cdot\varphi)\rangle \\ &= -Re\langle D\varphi, Y\cdot\nabla_X\varphi\rangle = \frac{1}{n}Re\langle D\varphi, YX\cdot D\varphi\rangle \\ &= -\frac{1}{n}g(X,Y)|D\varphi|^2. \end{aligned}$$

It follows for the Schouten tensor $K(X) = \frac{2}{n^2}\frac{|D\varphi|^2}{|\varphi|^2}\cdot X$, i.e. (M_1^n, g) is Einstein and

$$R = -\frac{4(n-1)}{n}\frac{|D\varphi|^2}{|\varphi|^2} = -\frac{4(n-1)}{n}A_\varphi.$$

□

REMARK. A Lorentzian spin manifold admitting a real Killing spinor is always an Einstein space (see [Boh00] and 1.4). In case that φ is an imaginary Killing spinor on (M_1^n, g) the lenght function $|\varphi|^2$ is constant by Proposition 4.1.7 (5). We can conclude with Proposition 4.1.8 that if (M_1^n, g) is not Einstein, but φ is a Killing spinor on M_1^n, then the Killing number λ is imaginary and $|\varphi|^2 \equiv 0$.

Corollary 4.1.9. — *Let (M_1^n, g) be a Lorentzian spin manifold with twistor spinor φ and let $N_\varphi := \{x \in M : |\varphi(x)|^2 = 0\}$. Then $(M^n\backslash N_\varphi, \frac{1}{|\varphi|^4}\cdot g)$ is an Einstein manifold with scalar curvature $\tilde{R} = -\frac{4(n-1)}{n}\cdot A_\varphi$.*

PROOF. The set N_φ is closed in M_1^n and $|e^{\sigma/2}\tilde{\varphi}|^2_{\tilde{S}} \equiv const \neq 0$ on $M_1^n\backslash N_\varphi$. Now, we can apply the previous Proposition 4.1.8. □

REMARK. (1) In connection with Proposition 1.2.5, the Corollary 4.1.9 reduces the twistor equation for spinor fields with $A_\varphi \neq 0$ and nowhere vanishing spinor norm to the Killing equation. Hence, in order to find solutions of the twistor equation, which are not conformally related to Killing spinor, one should investigate twistor spinor with vanishing spinor norm.

(2) It is known that on even-dimensional Fefferman spaces there exist twistor spinors. These twistor spinors have vanishing lenght. It turns out that a Fefferman space is (locally) never conformally equivalent to an Einstein space and the occuring twistor spinors are neither conformally equivalent to Killing spinors nor to a sum of Killing spinors (comp. 1.4, 4.3 and [Bau99]).

The Dirac current V_φ of a twistor spinor φ is a very useful object for the investigation of twistor spinors in Lorentzian spin geometry. An important characteristic property of V_φ is its twist (comp. [Lew91]). The twist of a vector field $V \in \Gamma(TM)$ is defined to be the 3-form

$$\omega_V \wedge d\omega_V,$$

where ω_V denotes the dual 1-form to V.

Lemma 4.1.10. — *Let V be a vector field on (M_1^n, g) without zeros. The following conditions are equivalent:*

(1) The twist of V vanishes, i.e. $\omega_V \wedge d\omega_V \equiv 0$.

(2) There exist locally functions f and λ such that $V = \lambda \cdot grad(f)$.

(3) The vector field V is hypersurface orthogonal, i.e. the smooth distribution $V^\perp \subset TM$ of codimension 1 is integrable.

The Lemma 4.1.10 is a direct consequence of the Frobenius' Theorem.

Lemma 4.1.11. — *Let V be a conformal vector field without zeros and with the property $\|V\|^2 \equiv 0$ or $\|V\|^2 \neq 0$ on (M_1^n, g). If $\omega_V \wedge d\omega_V \equiv 0$ then there exists locally a function σ on M_1^n such that V is parallel with respect to the metric $\tilde{g} = e^{2\sigma} \cdot g$.*

Proof. Assume that $\|V\|^2 \neq 0$ on M_1^n. We choose $\tilde{g} = \frac{1}{|g(V,V)|} \cdot g$. Then $\tilde{g}(V,V) = \pm 1$ and V is a Killing vector field with respect to $\tilde{g}$. Since V is also hypersurface orthogonal, we can find locally coordinates $(x_1, \ldots, x_n)$ such that $V = \frac{\partial}{\partial x_1}$ and

$$V^\perp = Span\{\frac{\partial}{\partial x_i} : i = 2, \ldots, n\}.$$

Locally in the coordinates $(x_1, \ldots, x_n)$, it holds $\widetilde{grad}(x_1) = \pm \frac{\partial}{\partial x_1} = \pm V$. Then we have

$$\pm \tilde{g}(\tilde{\nabla}_X V, Y) = \widetilde{Hess}(x_1)(X, Y) = \frac{1}{2} L_V \tilde{g}(X, Y) = 0$$

for all $X, Y \in TM_1^n$, i.e. V is parallel.

In case that $\|V\|^2 \equiv 0$, we can find locally a conformally changed metric $\tilde{g}$ such that $V = \widetilde{grad}(f)$ for some local function f on M_1^n (Lemma 4.1.10 (3)). Obviously, a conformal lightlike gradient field V satisfies

$$\tilde{\nabla}_X V = \frac{\widetilde{div}(V)}{n} \cdot X \qquad \text{and}$$

$$0 = X\tilde{g}(V,V) = \frac{2 \cdot \widetilde{div}(V)}{n} \cdot \tilde{g}(X, V)$$

for all $X \in TM$, where $\widetilde{div}(V) := tr(\tilde{\nabla} V)$ denotes the divergence of V with respect to $\tilde{g}$. These formulas imply that $\widetilde{div}(V) = 0$ and $\tilde{\nabla}_X V = 0$. □

Lemma 4.1.12. — *Let φ be a twistor spinor with $V_\varphi \cdot \varphi = \delta \cdot \varphi$ for some real function δ on (M_1^n, g). Then the spinor φ is parallel if and only if V_φ is parallel.*

Proof. It is clear that if φ is parallel then V_φ is parallel. On the other hand, assume that V_φ is parallel and $V_\varphi \cdot \varphi = \delta \cdot \varphi$ for some real smooth function δ. Then the function $\delta = \pm\sqrt{-g(V_\varphi, V_\varphi)}$ has to be constant. Consider the subbundle

$$R_\delta := \{\psi \in S : \; V_\varphi \cdot \psi = \delta \cdot \psi\} \subset S.$$

Locally on a suitable subset $U \subset M$, let us choose a spinor frame field of R_δ, which we denote by $\{\psi_i^U\}_{i=1,\dots,r}$, where $r = 2^{[\frac{n}{2}]}$. We have

$$\nabla_X^S(V_\varphi \cdot \varphi) = V_\varphi \cdot \nabla_X^S \varphi = \delta \cdot \nabla_X^S \varphi,$$

which implies that $\nabla_X^S \varphi = \sum_{i=1}^r a_i(X) \cdot \psi_i^U$ for some functions $a_i : TU \subset TM \to \mathbb{R}$. Moreover, we may assume without loss of generality that there exists a local orthonormal frame s on U such that $V_\varphi = s_1$ or $V_\varphi = s_1 + s_2$. Since φ is a twistor spinor, there exists a unique spinor field α such that

$$\alpha = g(X,X)X \cdot \nabla_X \varphi = g(X,X) \cdot \sum_i a_i(X) X \cdot \psi_i^U$$

for all $X \in TM$ with $\|X\|^2 = \pm 1$. It follows

$$\sum_i a_i(s_1) s_1 \psi_i^U = -\sum_i a_i(s_j) s_j \psi_i^U \qquad \text{for all } j \in \{2, \dots, n\}.$$

In case that $V_\varphi = s_1$ we have $\delta \neq 0$ and

$$V_\varphi \left(\sum_i a_i(s_1) s_1 \psi_i^U \right) = \delta \cdot \sum_i a_i(s_1) s_1 \psi_i^U \in R_\delta,$$

$$V_\varphi \left(\sum_i a_i(s_j) s_j \psi_i^U \right) = -\delta \cdot \sum_i a_i(s_j) s_j \psi_i^U \notin R_\delta$$

for $j = 2, \dots, n$. In case that $V_\varphi = s_1 + s_2$ we have $\delta = 0$,

$$V_\varphi \left(\sum_i a_i(s_1) s_1 \psi_i^U \right) = 2 \cdot \sum a_i(s_1) \psi_i^U \quad \text{and} \quad V_\varphi \left(\sum_i a_i(s_j) s_j \psi_i^U \right) = 0$$

for all $j \in \{3, \dots, n\}$. This implies in both cases $\sum_i a_i(s_1) s_1 \psi_i^U = 0$. Since $\{e_j \psi_i^U\}_{i=1,\dots,r}$ is a set of linearly independent spinors fields for all $j \in \{1, \dots, n\}$, we can conclude $a_i(s_j) = 0$ for all $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, n\}$. Hence, $\nabla_X^S \varphi = 0$ for all $X \in TM$. □

Proposition 4.1.13. — *Let φ be a twistor spinor without zeros such that $|\varphi|^2 \neq 0$ or $|\varphi|^2 \equiv 0$, $\omega_\varphi \wedge d\omega_\varphi \equiv 0$ and $V_\varphi \cdot \varphi = \delta \cdot \varphi$ for some real function δ on (M_1^n, g). Then φ is locally conformally equivalent to a parallel spinor.*

Proof. From Lemma 4.1.1 (3) we know that $\|V\|^2 \neq 0$ or $\|V\|^2 \equiv 0$. Then we can apply Lemma 4.1.11 and Lemma 4.1.12 to complete the proof. □

Moreover, with Lemma 4.1.1 (2) we obtain

Corollary 4.1.14. — *Let φ be a twistor spinor without zeros such that V_φ is lightlike and non-twisting. Then the twistor spinor φ is locally conformally equivalent to a parallel spinor.*

We want to prove further consequences of Proposition 4.1.13 concerning Killing spinors. First, we prove that a real Killing spinor has always vanishing twist, i.e. $\omega_\varphi \wedge d\omega_\varphi \equiv 0$. Let $\varphi \in \Gamma(S)$ be an arbitrary spinor field without zeros and let V_φ be its associated vector field. It holds $\omega_\varphi(X) = -\langle \varphi, X\varphi \rangle_S$ and

$$\begin{aligned} d\omega_\varphi(X,Y) &= -X\langle \varphi, Y\varphi\rangle + Y\langle \varphi, X\varphi\rangle + \langle \varphi, [X,Y]\varphi\rangle \\ &= -\langle \nabla_X\varphi, Y\cdot\varphi\rangle - \langle \varphi, Y\cdot\nabla_X\varphi\rangle + \langle \nabla_Y\varphi, X\cdot\varphi\rangle + \langle \varphi, X\cdot\nabla_Y\varphi\rangle \\ &= -2Re\langle \nabla_X\varphi, Y\cdot\varphi\rangle + 2Re\langle \nabla_Y\varphi, X\cdot\varphi\rangle. \end{aligned}$$

Then we have for the twist

$$\begin{aligned} \omega_\varphi \wedge d\omega_\varphi(X,Y,Z) = \; & \langle \varphi, X\varphi\rangle \cdot [Re\langle \nabla_Z\varphi, Y\varphi\rangle - Re\langle \nabla_Y\varphi, Z\varphi\rangle] \\ & +\langle \varphi, Y\varphi\rangle \cdot [Re\langle \nabla_X\varphi, Z\varphi\rangle - Re\langle \nabla_Z\varphi, X\varphi\rangle] \\ & +\langle \varphi, Z\varphi\rangle \cdot [Re\langle \nabla_Y\varphi, X\varphi\rangle - Re\langle \nabla_X\varphi, Y\varphi\rangle]. \end{aligned}$$

Moreover, in even dimensions we calculate for the conformal field $W_\varphi = (-1)^m(V_{\varphi_+} - V_{\varphi_-})$ that

$$d\omega_{W_\varphi}(X,Y) = -2Im\langle \nabla_X\varphi, Y\varphi\rangle_2 + 2Im\langle \nabla_Y\varphi, X\varphi\rangle_2 \quad \text{and}$$

$$\begin{aligned} \omega_{W_\varphi} \wedge d\omega_{W_\varphi}(X,Y,Z) = \; & i\cdot\langle \varphi, X\varphi\rangle_2 \cdot [Im\langle \nabla_Z\varphi, Y\varphi\rangle_2 - Im\langle \nabla_Y\varphi, Z\varphi\rangle_2] \\ & +i\cdot\langle \varphi, Y\varphi\rangle_2 \cdot [Im\langle \nabla_X\varphi, Z\varphi\rangle_2 - Im\langle \nabla_Z\varphi, X\varphi\rangle_2] \\ & +i\cdot\langle \varphi, Z\varphi\rangle_2 \cdot [Im\langle \nabla_Y\varphi, X\varphi\rangle_2 - Im\langle \nabla_X\varphi, Y\varphi\rangle_2]. \end{aligned}$$

Lemma 4.1.15. — *Let φ be a Killing spinor on (M^n_1, g) to the Killing number λ.*

(1) If λ is real then $\omega_\varphi \wedge d\omega_\varphi \equiv 0$.

(2) If $n = 2m$ and λ is imaginary then $\omega_{W_\varphi} \wedge d\omega_{W_\varphi} \equiv 0$, i.e. W_φ is conformal and non-twisting.

Obviously, the Lemma 4.1.15 follows from the above formulas for the twist of the associated vector fields by inserting the Killing equation.

Proposition 4.1.16. — *Let φ be a real Killing spinor and $V_\varphi \cdot \varphi = \delta \cdot \varphi$ for some real function $\delta \neq 0$ on (M^n_1, g). Then φ is locally conformally equivalent to a parallel spinor.*

The proof uses Proposition 4.1.13 and Lemma 4.1.15. Notice that if φ is a real Killing spinor on (M^n_1, g) with $V_\varphi \cdot \varphi$ for some real function δ then $zero(\delta) = zero|\varphi|^2$ is a hypersurface in M^n_1, since $grad|\varphi|^2 = -2\lambda V_\varphi \neq 0$. Around a point of this hypersurface a real Killing spinor is not conformally equivalent to a parallel spinor.

Remark. (1) We mentioned already that a twistor spinor φ is conformally equivalent to a parallel spinor if and only if $grad(e^{-\sigma}) \cdot \varphi = \frac{2}{n} e^{-\sigma} D\varphi$ for some function σ. In case that φ is a real Killing spinor this condition is equivalent to

$$grad(e^{-\sigma}) \cdot \varphi = -2\lambda e^{-\sigma} \cdot \varphi.$$

It is clear, that we can not find a solution for σ in this equation, if $V_\varphi \cdot \varphi \not\equiv \delta \cdot \varphi$ for any real function δ. On the other hand, if $V_\varphi \cdot \varphi = \delta \cdot \varphi$ for $\delta \neq 0$ then we showed above that this equation admits locally a solution σ.

(2) We will see later that in the small dimension $n = 3$ and 5 the condition $V_\varphi \cdot \varphi = \delta\varphi$ is no restriction to a spinor field $\varphi \in \Gamma(S)$.

(3) Similar to Proposition 4.1.13, one can prove that a twistor spinor φ with

$$\|W_\varphi\|^2 > 0, \qquad \omega_{W_\varphi} \wedge d\omega_{W_\varphi} \equiv 0 \qquad \text{and} \qquad W_\varphi \cdot \varphi = i\delta\varphi$$

for some real function δ is locally conformally equivalent to a parallel spinor. In particular, if φ is an imaginary Killing spinor with $|\varphi|^2 \equiv 0$ and $W_\varphi \cdot \varphi = i\delta\varphi$ for some $\delta \neq 0$ then φ is conformally equivalent to a parallel spinor. Obviously, an imaginary Killing spinor φ with $|\varphi|^2 \neq 0$ can not satisfy $W_\varphi \cdot \varphi = i\delta\varphi$ for any real function δ, since $A_\varphi \neq 0$ and hence, φ is not conformally equivalent to a parallel spinor.

4.2 Twistor equation in dimension 3

Let (M^3_1, g) be a 3-dimensional Lorentzian manifold. The Riemannian curvature tensor R^∇ decomposes to $R^\nabla = g * K$, where $*$ denotes the Kulkarni-Nomizu product and the Schouten tensor satisfies $K = \frac{R}{4}g - Ric$. Therefore, the curvature of M^3_1 is completely determined by the Ricci curvature. The Weyl tensor W, which is the traceless part of R^∇, vanishes identically. The Lorentzian 3-manifold M^3_1 is Einstein if and only if M^3_1 has constant sectional curvature and M^3_1 is conformally flat if and only if the Schouten-Weyl tensor C vanishes.

Before we start to investigate the twistor equation on a Lorentzian spin 3-manifold, we discuss the standard model. Let $\mathbb{R}^{2,1} := (\mathbb{R}^3, \langle\cdot,\cdot\rangle_{2,1})$ be the 3-dimensional Minkowski space with signature $(+--)$ and let (e_1, e_2, e_3) denote the standard basis in $\mathbb{R}^{2,1}$. Notice that this convention for the signature is different then before. The reason for this choice of the signature is the following. The Clifford algebra $C_{2,1}$ of $\mathbb{R}^{2,1}$ is isomorphic to $\mathbb{R}(2) \oplus \mathbb{R}(2)$ (comp. [LM89]), where $\mathbb{R}(2)$ denotes the real (2×2)-matrices. We obtain a real Clifford representation $\Phi : C_{2,1} \to \mathbb{R}(2)$ given by

$$\Phi(e_1) := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \Phi(e_2) := \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad \Phi(e_3) := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The spin group $Spin^+(2,1)$ is mapped by Φ isomorphically to $Sl(2,\mathbb{R})$. Then we have the real spinor module $\Delta^{\mathbb{R}}_{2,1} = \mathbb{R}^2$ with a natural real Clifford action on it given by Φ. The usual complex spinor module $\Delta_{2,1}$ is equal to the complexification $\mathbb{C} \otimes \Delta^{\mathbb{R}}_{2,1}$. Notice that the Clifford algebra $C_{1,2}$ to the space $\mathbb{R}^{1,2}$ with signature $(-++)$ admits no real representation.

The $Spin^+(2,1)$-invariant Hermitian product on $\Delta_{2,1}$ is given by

$$\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \rangle_\Delta := (e_1 \cdot \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}) = -i(b\bar{c} - a\bar{d}).$$

Obviously, for $0 \neq v, w \in \Delta^{\mathbb{R}}_{2,1}$, it holds $\langle v, w\rangle_\Delta = 0$ if and only if v and w are parallel. The associated vectors to the real spinors are given by the mapping

$$\begin{array}{rccc} \ell : & \Delta^{\mathbb{R}}_{2,1} & \to & \mathbb{R}^{2,1}. \\ & v = \begin{pmatrix} a \\ b \end{pmatrix} & \mapsto & \begin{pmatrix} a^2+b^2 \\ b^2-a^2 \\ -2ab \end{pmatrix} \end{array}$$

Let L^+ denote the cone of future-directed lightlike vectors in $\mathbb{R}^{2,1}$. It holds

$$\|\ell(v)\|^2 = 0 \qquad \text{and} \qquad \ell(v) = \ell(w) \text{ if and only if } v = \pm w$$

for all real spinors $v, w \in \Delta^{\mathbb{R}}_{2,1}$. Therefore, the map $\ell : \Delta^{\mathbb{R}}_{2,1}\backslash 0 \to L^+$ is a two-fold covering of the lightcone $L^+ \cong \mathbb{R} \times S^1$. Moreover, it holds

$$\ell(v) \cdot v = 0 \qquad \text{for all } v \in \Delta^{\mathbb{R}}_{2,1}$$

and the 1-dimensional lightlike subspace $\mathbb{R} \cdot \ell(v)$ of $\mathbb{R}^{2,1}$ consists of all vectors, which annihilate the spinor v.

Let us consider the complex spinor module $\Delta_{2,1}$. It holds

$$\langle v + iw, e_i(v + iw)\rangle_\Delta = \langle v, e_i v\rangle_\Delta + \langle w, e_i w\rangle_\Delta$$

and therefore,

$$\ell(v + iw) = \ell(v) + \ell(w) \qquad \text{for all } v, w \in \Delta^{\mathbb{R}}_{2,1}.$$

This shows $\ell(\Delta_{2,1}) = J^+$. If $x \cdot v = 0$ for $0 \neq x \in \mathbb{R}^{2,1}$ and $0 \neq v \in \Delta_{2,1}$ then

$$v = c \cdot v_{\mathbb{R}} \qquad \text{and} \qquad x \in \mathbb{R} \cdot \ell(v_{\mathbb{R}})$$

for some $c \in \mathbb{C}$ and $v_{\mathbb{R}} \in \Delta^{\mathbb{R}}_{2,1}$. Moreover, it holds $\ell(v) \cdot v = i\langle v, v\rangle_\Delta v$ for all $v \in \Delta_{2,1}$.

Let (M^3_1, g) be a time- and space-oriented Lorentzian spin manifold of dimension 3 with signature $(+--)$. By the discussion above, it is clear that there exists a real spinor bundle $S^{\mathbb{R}}$ over M^3_1 such that the usual spinor bundle S is isomorphic to the complexification $\mathbb{C} \otimes S^{\mathbb{R}}$. In other words, S admits a real structure and we can take the real and imaginary parts of the spinor bundle:

$$ReS \cong ImS \cong S^{\mathbb{R}}.$$

Let $V_\varphi = \ell(\varphi)$ be the associated vector field to a spinor field $\varphi \in \Gamma(S)$. We have $\|V_\varphi\|^2 \equiv 0$ and $V_\varphi \cdot \varphi = 0$ for all real spinors $\varphi \in \Gamma(S^{\mathbb{R}})$ and $V_\varphi = V_{Re\varphi} + V_{Im\varphi}$ for all complex spinor fields $\varphi \in \Gamma(S)$.

We will now investigate the twistor equation on a 3-dimensional Lorentzian spin manifold (M^3_1, g). So, let $\varphi \in \Gamma(S)$ be a complex twistor spinor. Since

$$(\nabla^S \varphi)_{Re,Im} = \nabla^S \varphi_{Re,Im} \qquad \text{and} \qquad (X \cdot \varphi)_{Re,Im} = X \cdot \varphi_{Re,Im}$$

for all $X \in TM$, it is clear that the real and imaginary parts of a twistor spinor φ are real solutions of the twistor equation. Notice once again that we have chosen the signature $(+--)$ for the Lorentzian spin manifold (M_1^3, g) and also that a real Killing spinor on (M_1^3, g) is an imaginary Killing spinor on $(M_1^3, -g)$ and vice versa.

From the general integrability conditions for a Lorentzian spin manifold with twistor spinor φ (see Proposition 1.2.3), we know that

$$C(Y,Z) \cdot \varphi = 0 \qquad \text{for all } Y, Z \in TM$$

and if $\varphi(p) \neq 0$ in $p \in M$ then

$$C(Y,Z) \in \mathbb{R} \cdot V_\varphi(p) \qquad \text{for all } Y, Z \in T_pM.$$

Let $\mathcal{T}(M_1^3) = ker(P)$ denote the space of twistor spinors on M_1^3.

Proposition 4.2.1. — *Let (M_1^3, g) be a Lorentzian spin 3-manifold. If $dim\mathcal{T}(M_1^3) > 1$ then M_1^3 is conformally flat.*

PROOF. Suppose that $dim\mathcal{T}(M_1^3) > 1$ and let $\varphi_1, \varphi_2 \in \Gamma(S^{\mathbb{R}})$ be two linearly independent real twistor spinors. Then we know from Lemma 4.1.4 that the open set $\tilde{M} := M_1^3 \setminus \{x \in M_1^3 : \varphi_1(x) \| \varphi_2(x)\}$ is dense in M. Moreover,

$$C(Y,Z) \cdot \varphi_1 = C(Y,Z) \cdot \varphi_2 = 0 \qquad \text{for all } Y, Z \in T\tilde{M}$$

and V_{φ_1}, V_{φ_2} are not parallel on $\tilde{M}$. But then we can conclude $C \equiv 0$ on $\tilde{M}$ and, since $\tilde{M}$ is dense in M_1^3, we know that C vanishes everywhere on M_1^3. □

REMARK. If $\varphi \in \Gamma(S)$ is a complex twistor spinor such that φ_{Re} and $\varphi_{Im} \in \Gamma(S^{\mathbb{R}})$ are linearly independent, then M_1^3 is conformally flat. For example, if M_1^3 admits an imaginary Killing spinor then M_1^3 has constant sectional curvature (comp. 1.4).

For the relation between real twistor spinors and conformal null vector fields we can prove the following result.

Theorem 4.2.2. — *Let (M_1^3, g) be a Lorentzian spin 3-manifold.*

(1) If $\varphi \in \Gamma(S^{\mathbb{R}})$ is a twistor spinor then V_φ is a conformal null vector field.

(2) Let V be a conformal null vector field on (M_1^3, g) with $V(p) \neq 0$ in $p \in M$. There exists locally around $p \in M_1^3$ up to a sign a unique real twistor spinor φ with $V_\varphi = V$.

PROOF. The first assertion is known from the general theory. So, let V be a conformal null vector field with $V(p) \neq 0$ in $p \in M$. Then there exists a neighborhood $U(p)$ of p in M_1^3 and a smooth spinor field φ on $U(p)$ with $V_\varphi = V$, which is unique up to a sign. We calculate

$$\begin{aligned}
\lambda \cdot g(X,Y) &= (L_V g)(X,Y) \\
&= i\langle \varphi, \nabla_X Y \cdot \varphi\rangle - iX\langle \varphi, Y\varphi\rangle + i\langle \varphi, \nabla_Y X \cdot \varphi\rangle - iY\langle \varphi, X\varphi\rangle \\
&= -2i\langle \varphi, Y \cdot \nabla^S_X \varphi + X \cdot \nabla^S_Y \varphi\rangle \\
&= -4ig(X,Y) \cdot \langle \varphi, X \cdot \nabla^S_X \varphi\rangle - 2i\langle \varphi, XY(X \cdot \nabla^S_X \varphi - Y \cdot \nabla^S_Y \varphi)\rangle
\end{aligned}$$

for all $X, Y \in TM$ with $\|X\|^2 = \|Y\|^2 = 1$. It follows

$$i\langle\varphi, XY(X\cdot\nabla^S_X\varphi - Y\cdot\nabla^S_Y\varphi)\rangle = t\cdot g(X,Y)$$

for some function t. But then the function t must be zero. Hence, we have

$$(L_V g)(X,Y) \;=\; -4ig(X,Y)\langle\varphi, X\cdot\nabla^S_X\varphi\rangle = -4ig(X,Y)\langle\varphi, Y\cdot\nabla^S_Y\varphi\rangle$$

and consequently, $\langle\varphi, X\cdot\nabla^S_X\varphi - Y\cdot\nabla^S_Y\varphi\rangle = 0$. It follows that the real spinor $X\cdot\nabla^S_X\varphi - Y\cdot\nabla^S_Y\varphi$ is parallel to the spinors φ and $YX\cdot\varphi$ for all X, Y with $\|X\|^2 = \|Y\|^2 = 1$. This is only possible if

$$X\cdot\nabla^S_X\varphi - Y\cdot\nabla^S_Y\varphi = 0 \qquad \text{for all} \;\; X, Y \text{ with } \|X\|^2 = \|Y\|^2 = 1,$$

which implies that φ is a twistor spinor. □

Remark. There is no general correspondence between complex twistor spinors and causal (here: spacelike or lightlike) conformal vector fields on M^3_1. This is clear from the following observation. A Lorentz metric of the form

$$g_f := \sum_{i\leq j} f_{ij}(x_2,x_3)dx^i \circ dx^j, \quad f_{11} > 0,$$

posses the spacelike Killing field $\frac{\partial}{\partial x_1}$. The metric g_f is not conformally flat in general. But a twistor spinor φ with $V_\varphi = \frac{\partial}{\partial x_1}$ is not a real twistor spinor and therefore, if g_f admits such a twistor spinor φ, the metric g_f would be conformally flat. We can conclude that in general there belongs no twistor spinor to $\frac{\partial}{\partial x_1}$.

We are now going to classify Lorentzian metrics in dimension 3, which admit twistor spinors without zeros.

Theorem 4.2.3. — *Let (M^3_1, g) be a Lorentzian spin 3-manifold with real twistor spinor $\varphi \in \Gamma(S^{\mathbb{R}})$. There exists for every point $p \notin zero(\varphi)$ an open neighborhood $U(p)$ and a function $\sigma : U(P) \to \mathbb{R}$ such that $e^{\sigma/2}\tilde{\varphi} \in \Gamma(\tilde{S}^{\mathbb{R}}|_{U(P)})$ is parallel with respect to $\tilde{g} = e^{2\sigma}\cdot g$.*

Proof. Let φ be a real twistor spinor. It holds $g(V_\varphi, V_\varphi) = 0$ and

$$(L_{V_\varphi} g)(X,Y) = g(\nabla_X V_\varphi, Y) + g(X, \nabla_Y V_\varphi) = h\cdot g(X,Y)$$

for some function h on M^3_1. Furthermore, we have

$$d\omega_\varphi(X,Y) \;=\; g(\nabla_X V_\varphi, Y) - g(\nabla_Y V_\varphi, X) = h\cdot g(X,Y) - 2g(X, \nabla_Y V_\varphi)$$

and $d\omega_\varphi(V_\varphi, X) = h\cdot g(V_\varphi, X)$. Let us choose $X, Y \in T_xM$ for an arbitrary $x \in M$ such that $Span\{X, Y, V_\varphi(x)\} = T_xM$ and $g(X, V_\varphi) = 0$. Then

$$\omega_\varphi \wedge d\omega_\varphi(Y, X, V_\varphi) = \omega_\varphi(Y)\cdot d\omega_\varphi(X, V_\varphi) = h\cdot\omega_\varphi(Y)\cdot g(V_\varphi, X) = 0,$$

i.e. $\omega_\varphi \wedge d\omega_\varphi \equiv 0$ on M. In case that $\varphi(p) \neq 0$, we can apply Proposition 4.1.13, since $V_\varphi\cdot\varphi = 0$. □

Example. Let us consider $\mathbb{R}^{2,1}$ with the twistor spinor

$$\varphi = x \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_3 \\ -x_1 - x_2 \end{pmatrix}.$$

We define $\sigma := -ln(x+y)$ on $\mathbb{R}^{2,1} \setminus \{x \in \mathbb{R}^{2,1} : x_1 + x_2 = 0\}$. Notice that $zero(\varphi) \subset \{x \in \mathbb{R}^{2,1} : x_1 + x_2 = 0\}$. It is

$$grad(\sigma) = \tfrac{1}{x_1+x_2}\left(\tfrac{\partial}{\partial x_2} - \tfrac{\partial}{\partial x_1}\right) \qquad \text{and} \qquad D\varphi = \begin{pmatrix} -3 \\ 0 \end{pmatrix}.$$

It follows

$$-\frac{3}{2} grad(\sigma) \cdot \varphi = \frac{3}{2} \frac{grad(e^{-\sigma})}{e^{-\sigma}} \cdot \varphi = D\varphi,$$

which implies that $\varphi = x \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is parallel on $\mathbb{R}^{2,1} \setminus \{x \in \mathbb{R}^{2,1} : x_1 + x_2 = 0\}$ with respect to

$$\tilde{g} = e^{2\sigma} \langle \cdot, \cdot \rangle_{2,1} = \frac{1}{(x_1 + x_2)^2} (dx_1^2 - dx_2^2 - dx_2^2).$$

Notice that $\tilde{g}$ is again the flat Minkowski metric in dimension 3.

It is well-known that a normal form of a Lorentzian 3-metric with parallel spinor is given by

$$g = dx \circ dy - dz \circ dz + f(y,z) dy^2,$$

where f is an arbitrary function in the coordinates (y,z) (comp. [Bry00]). Notice that the metric g is flat if and only if $\frac{\partial^2 f}{(\partial z)^2} = 0$ and g is conformally flat if and only if $\frac{\partial^3 f}{(\partial z)^3} = 0$. A normal form of a metric with real Killing spinor is given by

$$g = -4e^{2x} dy \circ (B(x,y)dx + dz) - dx^2,$$

where B is an arbitrary function in (x,y).

In case that $\varphi \in \Gamma(S)$ is a complex twistor spinor on M_1^3 with $\varphi(p) \neq 0$ for $p \in M_1^3$, we can conclude that there exist coordinates (x,y,z) around p and a function σ such that

$$g = e^{2\sigma}(dx \cdot dy - dz \circ dz + f(y,z)dy^2)$$

and the real part $Re\varphi$ or the imaginary part $Im\varphi$ is conformally equivalent to a parallel spinor.

For the rest of this part, we want to discuss the zeros of a twistor spinor. We have to confess just in the beginning that we are not able to solve the twistor equation completely in this case.

Proposition 4.2.4. — *Let (M_1^3, g) be a Lorentzian spin 3-manifold.*

(1) If M_1^3 admits a twistor spinor φ with an isolated zero then M_1^3 is conformally flat.

(2) Let φ be a real twistor spinor on M_1^3. Then the set $zero(\varphi)$ consists of isolated lightlike geodesics.

Proof. In general, it is known that if $\gamma_p(t)$ is a geodesic through $p \in zero(\varphi)$ on M^3_1 and $\gamma_p'(0) \cdot D\varphi(p) = 0$ then $Image(\varphi_p) \subset zero(\varphi)$ (see Lemma 3.4.1). Assume that p is an isolated zero. It follows that $D\varphi(p) \neq 0$ is not annihilated by any tangent vector. This means that $D\varphi$ is not a real spinor field. But then it holds $dim\mathcal{T}(M^3_1) > 1$, and we can conclude that M^3_1 is conformally flat. Since every real spinor ψ is annihilated by V_ψ, Lemma 3.4.1 implies also the second assertion of the proposition. □

Lemma 4.2.5. — *Let $\varphi \in \Gamma(S^{\mathbb{R}})$ be a real twistor spinor on M^3_1 and let V_φ be its associated conformal null field. If there exists an 1-dimensional subbundle $T \subset TM^3_1$ such that $V_\varphi \in \Gamma(T)$ then φ admits no zero.*

Proof. Suppose that $V_\varphi \not\equiv 0$ is a section in a 1-dimensional subbundle $T \subset TM$. Then we can choose locally coordinates (x, y, z) in an arbitrary point $p \in M^3_1$ such that $V_\varphi = f\frac{\partial}{\partial x}$ for some function f. The metric g has locally the form

$$g = Adx \circ dy + Bdx \circ dz + Cdy \circ dy + Ddy \circ dz + Edz \circ dz.$$

Without loss of generality we may assume that $A(p) \neq 0$. The field V_φ is also a conformal field with respect to the conformally changed metric

$$\tilde{g} = \frac{1}{A} \cdot g = dx \circ dy + \tilde{B}dx \circ dz + \tilde{C}dy \circ dy + \tilde{D}dy \circ dz + \tilde{E}dz \circ dz.$$

The property that V_φ is conformal is equivalent to

$$\begin{aligned}
(L_{V_\varphi}\tilde{g})(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) &= 0 \\
(L_{V_\varphi}\tilde{g})(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) &= \frac{1}{2}\frac{\partial f}{\partial x} = \frac{1}{2}\lambda \\
(L_{V_\varphi}\tilde{g})(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}) &= \frac{1}{2}(\frac{\partial \tilde{B}}{\partial x}f + \tilde{B}\frac{\partial f}{\partial x}) = \frac{1}{2}\lambda\tilde{B} \\
(L_{V_\varphi}\tilde{g})(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}) &= \frac{\partial \tilde{C}}{\partial x} \cdot f + \frac{\partial f}{\partial y} = \lambda\tilde{C} \\
(L_{V_\varphi}\tilde{g})(\frac{\partial}{\partial y}, \frac{\partial}{\partial z}) &= \frac{1}{2}(\frac{\partial \tilde{D}}{\partial x}f + \tilde{B}\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}) = \frac{1}{2}\lambda\tilde{D} \\
(L_{V_\varphi}\tilde{g})(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}) &= \frac{\partial \tilde{E}}{\partial x}f + \tilde{B}\frac{\partial f}{\partial z} = \lambda\tilde{E}.
\end{aligned}$$

Suppose now that V_φ vanishes in p, i.e. $f(p) = 0$. It holds

$$\frac{\partial f}{\partial y} - \tilde{C}\frac{\partial f}{\partial x} + \frac{\partial \tilde{C}}{\partial x} \cdot f = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial z} + \tilde{B}\frac{\partial f}{\partial y} - \tilde{D}\frac{\partial f}{\partial x} + \frac{\partial \tilde{D}}{\partial x} \cdot f = 0.$$

These equations show that the function f vanishes on the integral curves to

$$\frac{\partial}{\partial y} - \tilde{C}\frac{\partial}{\partial x} \qquad \text{and} \qquad \frac{\partial}{\partial z} + \tilde{B}\frac{\partial}{\partial y} - \tilde{D}\frac{\partial}{\partial x}$$

through $p \in zero(V_\varphi)$. But the set $zero(V_\varphi)$ consists of isolated points or 1-dimensional submanifolds (see Theorem 3.4.3). This leads to a contradiction and we conclude that $zero(\varphi) = zero(V_\varphi) = \emptyset$. □

Proposition 4.2.6. — *Let φ be a twistor spinor on M^3_1 with a zero in $p \in M^3_1$ then $C(p) = 0$.*

Proof. Suppose that $C(p) \neq 0$. It holds

$$g(C(X,Y), C(Z,W)) = 0 \qquad \text{for all } X, Y, Z \quad \text{and} \quad W \in T(M^3_1 \setminus zero(\varphi)).$$

This implies that $g(C(X,Y), C(Z,W)) = 0$ on M^3_1. Hence, C defines a lightlike distribution T in a neighborhood of $p \in M^3_1$, where C does not vanish. Since C is smooth, the distribution T is smooth and $V_\varphi \in \Gamma(T)$. But then the field V_φ has no zero by Lemma 4.2.5. Hence, φ has no zero in p, which is a contradiction. □

The standard example of a twistor spinor with zero is given by $\varphi_v = x \cdot v$, $v \in \Delta_{2,1}$, on the flat Minkowski space $\mathbb{R}^{2,1}$.

Remark. (1) We sketch the construction of a twistor spinor with zeros on a non-conformally flat manifold. For this, let us consider a Lorentzian 3-metric g of the form

$$g = dx \circ dy - dz \circ dz + f(y,z) dy \circ dy$$

on $\mathbb{R}^3$ with

$$\emptyset \neq supp(f) \subset C_x := \{(x,y,z) \in \mathbb{R}^3 : y^2 + z^2 \leq 1\}.$$

The metric g has the property that it is flat on $R^3 \setminus C_x$ and non-conformally flat on the cylinder C_x. Moreover, the metric g admits a parallel spinor ψ.

Consider now the flat metric $\langle \cdot, \cdot \rangle_{2,1}$ on $\mathbb{R}^{2,1}$ with a real twistor spinor of the form $\varphi_v = x \cdot v$, $v \in \Delta^{\mathbb{R}}_{3,1}$. The twistor spinor φ_v has zeros and outside of the zero set it is (locally) conformally equivalent to a parallel spinor. It is not difficult to show that there exists an open subset $U \subset \mathbb{R}^{2,1}$ with $zero(\varphi_v) \cap U \neq \emptyset$ and a conformally changed metric $\tilde{h} = e^{2\sigma} \langle \cdot, \cdot \rangle_{2,1}$ on U such that it is possible to glue on $(U, \tilde{h})$ a non-conformally flat cylinder C with parallel spinor ψ of the above described form in the way that the twistor spinors φ_v and ψ fit smoothly together on $U \cup C$. The resulting twistor spinor on $U \cup C$ admits zeros and lives on a non-conformally flat manifold. Of course, this construction of a twistor spinor with zero has the property that the manifold is still conformally flat in a neigborhood of the zero set.

(2) We are not able to construct a Lorentzian metric admitting a twistor spinor with a zero such that the Lorentzian metric is not conformally flat in the near of the zero. It is not clear whether such a twistor spinor with zero exists or not. In particular, we do not have a complete classification of Lorentzian 3-metrics, which admit twistor spinors.

4.3 Twistor equation in dimension 4

Let $\mathbb{R}^{3,1} := (\mathbb{R}^4, \langle \cdot, \cdot \rangle_{3,1})$ be the 4-dimensional Minkowski space with signature $(+ - --)$! The Clifford algebra $C_{3,1}$ of $\mathbb{R}^{3,1}$ is isomorphic to $\mathbb{R}(4)$ and consequently, we have a real representation $\Phi : C_{3,1} \to \mathbb{R}(4)$ given by

$$\Phi(e_1) = \begin{pmatrix} 0 & L \\ -L & 0 \end{pmatrix}, \qquad \Phi(e_2) = \begin{pmatrix} 0 & L \\ L & 0 \end{pmatrix},$$

$$\Phi(e_3) = \begin{pmatrix} -L & 0 \\ 0 & L \end{pmatrix}, \qquad \Phi(e_4) = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix},$$

where $K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. In particular, we obtain a real representation of $Spin^+(3,1)$ on the real spinor module $\Delta^{\mathbb{R}}_{3,1} \cong \mathbb{R}^4$. Moreover, the spin group $Spin^+(3,1)$ is isomorphic to $Sl(2,\mathbb{C})$. Hence, there is a $Spin^+(3,1)$-equivariant complex structure defined by

$$J^{\Delta} := e_1 e_2 e_3 e_4$$

on $\Delta^{\mathbb{R}}_{3,1}$ and the spinor module $\Delta^{\mathbb{R}}_{3,1}$ is isomorphic to $\mathbb{C}^2$. Notice that J^{Δ} is not $C_{3,1}$-equivariant. The usual complex spinor module $\Delta_{3,1}$ is equal to the complexification $\mathbb{C}\otimes\Delta^{\mathbb{R}}_{3,1}$ and it decomposes to

$$\Delta_{3,1} = \Delta^+_{3,1} \oplus \Delta^-_{3,1} = \Delta^{\mathbb{R}}_{3,1} \oplus i\Delta^{\mathbb{R}}_{3,1}.$$

The real $Spin^+(3,1)$-representation spaces $\Delta^{\mathbb{R}}_{3,1}$, $\Delta^+_{3,1}$ and $\Delta^-_{3,1}$ are isomorphic:

$$\begin{array}{ccccc} \Delta^{\mathbb{R}}_{3,1} & \cong & \Delta^+_{3,1} & \cong & \Delta^-_{3,1}. \\ v & \leftrightarrow & v + iJ^{\Delta}v & \leftrightarrow & v - iJ^{\Delta}v \end{array}$$

Notice that $C_{3,1}$ acts on $\Delta^{\mathbb{R}}_{3,1}$, but not on $\Delta^+_{3,1}$ or $\Delta^-_{3,1}$.

The $Spin^+(3,1)$-invariant Hermitian product $\langle\cdot,\cdot\rangle_{\Delta}$ on $\Delta_{3,1}$ is defined by

$$\langle \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix}, \begin{pmatrix} c_1 \\ c_2 \\ d_1 \\ d_2 \end{pmatrix} \rangle_{\Delta} = -i(b_1\bar{c}_1 - b_2\bar{c}_2 - a_1\bar{d}_1 + a_2\bar{d}_2).$$

The associated vector to a real spinor is given by the map

$$\begin{array}{cccc} \ell : & \Delta^{\mathbb{R}}_{3,1} & \to & \mathbb{R}^{3,1} \quad . \\ & v = \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} & \mapsto & \ell(v) = \begin{pmatrix} |a_1 + ia_2|^2 + |b_1 + ib_2|^2 \\ |a_1 + ia_2|^2 - |b_1 + ib_2|^2 \\ 2Re[(a_1 + ia_2)(b_1 - ib_2)] \\ -2Im[(a_1 + ia_2)(b_1 - ib_2)] \end{pmatrix} \end{array}$$

It holds $\ell(v) = \ell(\exp(tJ^{\Delta}) \cdot v)$ and $\ell(v) \cdot v = 0$ for all $v \in \Delta^{\mathbb{R}}_{3,1}$ and $t \in \mathbb{R}$, i.e. the map $\ell : \Delta^{\mathbb{R}}_{3,1} \setminus 0 \to L^+$ to the lightcone is a S^1-fibration and we have the identification

$$\Delta^{\mathbb{R}}_{3,1} \setminus 0/S^1 \cong L^+.$$

Alternatively, we can identify $(\mathbb{R}^4, \langle\cdot,\cdot\rangle_{3,1})$ with $(H_2(\mathbb{C}), \det)$, the space of Hermitian symmetric (2×2)-matrices. The action of $Spin^+(3,1)$ on $\mathbb{R}^{3,1} \cong H_2(\mathbb{C})$ is then given by

$$A \cdot x = AxA^*, \qquad A \in Spin^+(3,1),\ x \in H_2(\mathbb{C}).$$

Moreover, it holds $\ell(v) = 2vv^* \in H_2(\mathbb{C})$ (comp. [Bry00]).

Let (M^4_1, g) be a Lorentzian 4-manifold. The Riemannian curvature tensor R^{∇} decomposes to $R^{\nabla} = W \oplus g * K$, where $K = \frac{1}{2}(\frac{R}{6}g - Ric)$ is the Shouten tensor. The Weyl tensor W is the tracefree part of the Riemannian curvature tensor R^{∇}. The Lorentzian manifold M^4_1 is conformally flat if and only if $W \equiv 0$.

Now, let (M^4_1, g) be a time-oriented Lorentzian spin 4-manifold with metric tensor g of signature $(+---)$. Then we have the real spinor bundle $S^{\mathbb{R}}$, which is furnished with the complex structure J induced by J^{Δ} on $\Delta^{\mathbb{R}}_{3,1}$, and the usual complex spinor bundle $S = \mathbb{C} \otimes S^{\mathbb{R}}$. Furthermore, we have the associated vector field $V_\varphi = \ell(\varphi)$ to a spinor field $\varphi \in \Gamma(S)$. If $\varphi \in \Gamma(S^{\mathbb{R}})$ then $V_\varphi \cdot \varphi = 0$. For a complex spinor φ, it holds

$$V_\varphi = V_{Re\varphi} + V_{Im\varphi}.$$

The image $\ell(S)$ coincides with the set of positive oriented causal (here: spacelike or lightlike) vectors in TM^4_1.

Let $\varphi \in \Gamma(S^{\mathbb{R}})$ be a real twistor spinor on (M^4_1, g) then the associated vector field V_φ is null and it annihilates the Weyl tensor

$$V_\varphi \lrcorner W = 0,$$

(comp. [Bau99] and Proposition 4.1.5), i.e. M^4_1 is pointwise of Petrov type N or conformally flat. In case that $W_p \neq 0$ for $p \in M^4_1$ the vector $V_\varphi(p)$ is called a 4-principal null vector in TM^4_1 (comp. [ON95] and [Bes87]). It is also well-known that $W_p = 0$ if $\varphi(p) = 0$ in $p \in M$. As in dimension 3, if $\varphi \in \Gamma(S)$ is a complex twistor spinor then $Re\varphi$ and $Im\varphi \in \Gamma(S^{\mathbb{R}})$ are real twistor spinors on M^4_1. Moreover, if $\varphi \in \Gamma(S^{\mathbb{R}})$ is a real twistor spinor on M^4_1 then $J\varphi \in \Gamma(S^{\mathbb{R}})$ is also a real twistor spinor and $\varphi, J\varphi$ are $\mathbb{C}$-linearly independent. Let $\mathcal{T}(M^4_1)$ denote the space of complex twistor spinors on M^4_1.

Theorem 4.3.1. — *Let (M^4_1, g) be a Lorentzian spin 4-manifold.*

(1) The dimension of $\mathcal{T}(M^4_1)$ is always even.

(2) If $dim\mathcal{T}(M^4_1) > 2$ then M^4_1 is conformally flat.

Proof. The first assertion follows, since S^+ and S^- are isomorphic as real vector bundles. It remains to prove the second assertion. So let α_1, α_2 and $\alpha_3 \in \mathcal{T}(M^4_1)$ be linear independent complex twistor spinors. Then we can find linearly independent real twistor spinors β_1, β_2 and $\beta_3 \in \Gamma(S^{\mathbb{R}})$ on M^4_1. Let us denote

$$Z := zero(\beta_1) \cup zero(\beta_2) \cup zero(\beta_3).$$

The set $M^4_1 \setminus Z$ is dense in M^4_1. Now suppose that $W(p) \neq 0$ for some $p \in M$. There exists a neighborhood $U(p)$ of p with $W|_{U(p)} \neq 0$ and the vectors $V_{\beta_1}(x), V_{\beta_2}(x)$ and $V_{\beta_3}(x)$ are 4-principal null vectors for all $x \in U(p) \setminus Z$. Hence, these vectors are parallel and this implies

$$\beta_2 = (a + Jb)\beta_1 \qquad \text{and} \qquad \beta_3 = (c + Jd)\beta_1$$

for some real functions a, b, c and d on $U(p) \setminus Z$. But since β_1, β_2 and β_3 are twistor spinors, it follows from a slightly different version of Lemma 4.1.4 that the functions a, b, c and d are constant and this implies

$$\beta_2, \beta_3 \in Span_{\mathbb{C}}\{\beta_1, J\beta_1\}.$$

This is a contradiction and we can conclude that $W \equiv 0$ on M^4_1. $\square$

Remark. (1) If M^4_1 is simply connected then $\dim \mathcal{T}(M^4_1) = 0, 2, 8$.

(2) Let $\varphi \in \Gamma(S)$ be an imaginary Killing spinor on M^4_1. Then one can show that $\{Re\varphi, J(Re\varphi), Im\varphi, J(Im\varphi)\}$ is a set of $\mathbb{C}$-linearly independent twistor spinors, which implies that M^4_1 is conformally flat. Moreover, since M^4_1 is Einstein, it follows that M^4_1 has constant sectional curvature (comp. 1.4). Similar, if $\varphi \in \Gamma(S)$ is a twistor spinor with $\langle\varphi, \varphi\rangle \neq 0$ on M^4_1, again we can show that the set $\{Re\varphi, Im\varphi, J(Re\varphi), J(Im\varphi)\}$ is $\mathbb{C}$-linearly independent and M^4_1 has to be be conformally flat.

(3) If $\varphi \in \Gamma(S^{\mathbb{R}})$ is a real Killing spinor then $J\varphi \in \Gamma(S^{\mathbb{R}})$ is not a Killing spinor in general, since $JX \neq XJ$ for some $X \in TM$.

Corollary 4.3.2. — *Let M^4_1 be Ricci-flat and non-flat. If φ is a twistor spinor on M^4_1 then φ is parallel.*

Proof. Let φ be a twistor spinor on M^4_1. Then

$$\dim_{\mathbb{C}} \mathcal{T}(M^4_1) = 2 \qquad \text{and} \qquad \mathcal{T}(M^4_1) = Span_{\mathbb{C}}\{\varphi, J\varphi\}.$$

Suppose that $D\varphi \not\equiv 0$ then the spinor field $D\varphi$ is parallel and $D\varphi = (a+bJ)\varphi$ for some constants $a, b \in \mathbb{C}$. It is

$$0 = \nabla^S_X D\varphi = (a + bJ)\nabla^S_X \varphi \qquad \text{for all} \quad X \in TM,$$

i.e. $\nabla^S_X \varphi = 0$ for all $X \in TM$, which is a contradiction to $D\varphi \not\equiv 0$. We conclude that $D\varphi \equiv 0$ and φ is parallel. $\square$

Corollary 4.3.3. — *Let M^4_1 be a Lorentzian Einstein spin 4-space. If M^4_1 admits a twistor spinor φ with $zero(\varphi) \neq \emptyset$ then M^4_1 has constant sectional curvature.*

Proof. Since M^4_1 is Einstein and φ has a zero, the spinor $D\varphi$ is a non-trivial twistor spinor with $D\varphi(p) \neq 0$ for all $p \in zero(\varphi)$. This shows that the twistor spinors $\varphi, J\varphi$ and $D\varphi$ are linearly independent. Therefore, M^4_1 is conformally flat. $\square$

We want to recall now the well-known description of Lorentzian spin 4-manifolds admitting twistor spinors without zeros in case that the twist of the associated vector fields has no singularities. The main classification result is Theorem 4.3.7 below, which has been proved by J. Lewandowski in 1991 (see [Lew91]). The occuring classes of Lorentzian metrics are the pp-manifolds and the Fefferman spaces (comp. 1.4). We shortly characterize these two classes of Lorentzian metrics in dimension 4. We start with the standard pp-manifolds.

Proposition 4.3.4. — *(comp. [EK62], [Lew91] and [Bau00a]) Each 4-dimensional Lorentzian spin manifold admitting parallel spinors is locally isometric to a standard pp-manifold $(\mathbb{R}^4, g_f)$, where g_f has the form*

$$g_f := -2dx_1 \circ dx_2 + f(x_2, x_3, x_4)dx_2^2 + dx_3^2 + d_4^2.$$

Conversely, every simply connected standard pp-manifold admits parallel spinors.

Remark. The class of pp-manifolds has been considered also in [Sch74]. It is proved there that a Lorentzian 4-metric with totally isotropic (i.e. lightlike) Ricci tensor admitting a lightlike parallel vector field is a standard pp-metric of the form given in Proposition 4.3.4.

Before we characterize the class of Fefferman 4-metrics, we establish the following lemma, which helps to make the situtation more clear.

Lemma 4.3.5. — *Let V be a conformal null vector field on (M^4_1, g).*

(1) The field V is geodesic, i.e. the integral curves of V are pregeodesics.

(2) The field V is shear-free, i.e. $L_V I^{[g]} = 0$, where $I^{[g]}$ is the almost complex structure on the screen space $V^\perp/V \cdot \mathbb{R}$, which is induced by the conformal class $[g]$.

(3) If the field V is twisting, i.e. $\omega_V \wedge d\omega_V \neq 0$, then the rotation $d\omega_V$ is non-degenerate on the screen space $V^\perp/V \cdot \mathbb{R}$.

Proof. It holds for all $X \in V^\perp$

$$g(\nabla_V V, X) = -g(V, \nabla_V X) = -g(V, [V, X]) = -L_V g(V, X) = \lambda g(V, X) = 0,$$

which implies that $\nabla_V V = \mu \cdot V$ for some real function μ and the integral curves to V are pregeodesics. This together with the fact, that V is conformal, also implies that V is shear-free.

Assume now that $d\omega_V$ is degenerate on $V^\perp/V \cdot \mathbb{R}$. We choose for arbitrary $p \in M^4_1$ vectors $W, X, Y \in T_pM^4_1$ such that $Span\{X, Y, V\} = V^\perp$ and $Span\{W, X, Y, V\} = TM^4_1$. Then we have $\omega_V \wedge d\omega_V(X, Y, V) = 0$ and

$$\omega_V \wedge d\omega_V(W, V, X) = r \cdot d\omega_V(V, X) = r \cdot g([V, X], V) = 0 = \omega_V \wedge d\omega_V(W, V, Y)$$

for some real function r. Moreover, since $d\omega_V(X, Y) = 0$, it holds $\omega_V \wedge d\omega_V(W, X, Y) = 0$, i.e. V has no twist. □

Proposition 4.3.6. — *(comp. [Spa85], [NP00]) A Lorentzian metric g on a 4-dimensional manifold M^4 is a Fefferman metric if and only if the following conditions are satisfied.*

(1) There exists a 4-principal null vector field V on (M^4_1, g). In particular, the metric g has Petrov type N.

(2) The field V is a conformal null vector field, which is geodesic, shear-free and twisting, i.e. $\omega_V \wedge d\omega_V \neq 0$.

Remark. A Fefferman space is (locally) never conformally equivalent to an Einstein space.

Here is the classification result due to J. Lewandowski.

Theorem 4.3.7. — *(comp. [Lew91] and [Bau00a]) Let $\varphi \in \Gamma(S^{\mathbb{R}})$ be a real twistor spinor without zeros on a spacetime (M^4_1, g).*

(1) If $\omega_\varphi \wedge d\omega_\varphi \equiv 0$ then (M^4_1, g) is locally conformally equivalent to a pp-manifold and the spinor φ is locally conformally equivalent to a parallel spinor.

(2) If $\omega_\varphi \wedge d\omega_\varphi \neq 0$ then (M^4_1, g) is locally conformally equivalent to a Fefferman space. On the other hand, there exist locally solutions of the twistor equation on each 4-dimensional Fefferman space.

Originally, Theorem 4.3.7 was stated for complex half spinors. We have formulated it here for real spinors. Notice also that by Lemma 4.3.5 the assumption that V_φ is twisting implies already that the rotation $d\omega_\varphi$ is non-degenerate. So there is no further natural distinction in the class of Fefferman 4-metrics.

Remark. (1) For every twistor spinor φ on a Feffermann space (M^4_1, g) holds $\langle \varphi, \varphi \rangle = 0$ and φ is neither conformally equivalent to a Killing spinor nor to a sum of Killing spinors.

(2) If φ is a real Killing spinor on a spacetime (M^4_1, g) with signature $(+---)$ then the twist $\omega_\varphi \wedge d\omega_\varphi$ must vanish identically, since there are no Killing spinors on Fefferman spaces. Moreover, Theorem 4.3.7 says that φ is locally conformally equivalent to a parallel spinor. This means that an imaginary Killing spinor in signature $(-+++)$ (!) is always locally conformally equivalent to a parallel spinor.

In dimension 3, we have seen that conformal null vector fields and real twistor spinors correspond to each other. Here, we have

Theorem 4.3.8. *— (comp. [Lew91]) Let (M^4_1, g) be a spacetime with a conformal 4-principal null vector field V without zeros. If $\omega_V \wedge d\omega_V \equiv 0$ or $\omega_V \wedge d\omega_V \neq 0$ everywhere on (M^n_1, g) then there exists locally a twistor spinor $\varphi \in \Gamma(S^{\mathbb{R}})$ on (M^4_1, g) such that $V = V_\varphi$.*

Proof. In case that the twist $\omega_V \wedge d\omega_V \equiv 0$ is identically zero the field V is locally parallel with respect to a conformally equivalent metric $\tilde{g}$ (Lemma 4.1.11). Then, it holds

$$V \lrcorner \, R^{\tilde{\nabla}} = 0 \qquad \text{and} \qquad V \lrcorner \, \widetilde{W} = 0$$

and this implies $V \lrcorner \, \tilde{g} * \widetilde{K} = 0$. From this property, it follows that the Ricci tensor $\widetilde{Ric}$ is totally isotropic, i.e. $\widetilde{Ric}$ maps only to lightlike tangent vectors. We can conclude that $(M^4_1, \tilde{g})$ is locally a pp-manifold (comp. [Sch74]) and there exists locally a twistor spinor φ such that $V_\varphi = V$ (Proposition 4.3.4).

In case that the twist vanishes nowhere on M^4_1, we know by the characterization in Proposition 4.3.6 that (M^4_1, g) is locally conformally equivalent to a Fefferman space. Again, we can conclude that there exists locally a twistor spinor φ with $V_\varphi = V$. □

We want to remark that there are no classification results for twistor spinors φ on a spacetime (M^4_1, g) in case that the twist $\omega_\varphi \wedge d\omega_\varphi$ is not identically zero, but vanishes somewhere on M^4_1. The only known examples of twistor spinors with zeros live on manifolds, which are conformally flat in a neighborhood of the zero set.

Proposition 4.3.9. — *Let (M^4_1, g) be a Lorentzian spin 4-manifold.*

(1) Let φ be a real twistor spinor on M^4_1 . Then the set $zero(\varphi)$ consists of isolated lightlike geodesics.

(2) If M^4_1 admits a twistor spinor φ with an isolated zero then M^4_1 is conformally flat.

Proof. (1) If φ is a real twistor spinor and $p \in zero(\varphi)$ then $D\varphi(p) \neq 0$ is real and annihilated by $V_{D\varphi}(p)$, which implies that $p \in M^n_1$ lies on a zero set geodesic.

(2) If φ is a twistor spinor with isolated zero then φ is a complex spinor and the set $\{Re\varphi, J(Re\varphi), Im\varphi, J(Im\varphi)\}$ consists of linearly independent twistor spinors. □

4.4 Twistor equation in dimension 5

The basic observation in 5-dimensional Lorentzian spin geometry is the existence of a quaternionic structure on the spinor module. So let $\mathbb{H}$ denote the quaternionic numbers and let i, j, k denote the imaginary units in $\mathbb{H}$. The space $\mathbb{H}^2$ is identified with $\mathbb{C}^4$ by

$$\begin{array}{ccc} \mathbb{H}^2 & \cong & \mathbb{C}^4, \\ \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} & \mapsto & \begin{pmatrix} z_1 \\ \bar{w}_1 \\ z_2 \\ \bar{w}_2 \end{pmatrix} \end{array}$$

where $a_\mu = z_\mu + w_\mu \cdot j$ for $\mu = 1, 2$. Associated to this identification we have the embedding

$$\begin{array}{cccc} i_{\mathbb{C}}: & \mathbb{H}(2) & \hookrightarrow & \mathbb{C}(4) \ . \\ & (a_{\mu v})_{\mu,v=1,2} & \mapsto & \begin{pmatrix} z_{\mu v} & -w_{\mu v} \\ \bar{w}_{\mu v} & \bar{z}_{\mu v} \end{pmatrix}_{\mu,v=1,2} \end{array}$$

Let $\mathbb{R}^{1,4} := (\mathbb{R}^5, \langle\cdot,\cdot\rangle_{1,4})$ be the 5-dimensional Minkowski space with the usual signature $(- + + + +)$! It is convenient to identify the Minkowski space $\mathbb{R}^{1,4}$ with the set $\{\begin{pmatrix} a & -b \\ \bar{b} & -a \end{pmatrix} \in \mathbb{H}(2) : a \in \mathbb{R} \ , b \in \mathbb{H}\}$ furnished with the scalar product

$$\langle \begin{pmatrix} a & -b \\ \bar{b} & -a \end{pmatrix}, \begin{pmatrix} c & -d \\ \bar{d} & -c \end{pmatrix} \rangle_{1,4} = -ac + Re(b\bar{d}).$$

The quadratic form r to $\langle\cdot,\cdot\rangle_{1,4}$ on $\mathbb{H}(2)$ is given by

$$r \begin{pmatrix} a & -b \\ \bar{b} & -a \end{pmatrix} = \det \begin{pmatrix} a & -b \\ \bar{b} & -a \end{pmatrix} = -a^2 + |b|^2.$$

The Clifford algebra $C_{1,4}$ of $\mathbb{R}^{1,4}$ is isomorphic to $\mathbb{H}(2) \oplus \mathbb{H}(2)$ (comp. [LM89], [Bry00]). An explicit quaternionic Clifford representation $\Phi : C_{1,4} \to \mathbb{H}(2)$ is given by

$$\Phi(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Phi(e_2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \Phi(e_3) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

$$\Phi(e_4) = \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix} \qquad \text{and} \qquad \Phi(e_5) = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}.$$

The spin group $Spin^+(1,4)$ is isomorphic to $Sp(1,1)$ and is realized in $\mathbb{H}(2)$ as the set of matrices $A \in \mathbb{H}(2)$ that satisfy $A^*QA = Q$, where $Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and A^* is the conjugate transpose to A.

The complex spinor modul $\Delta_{1,4} \cong \mathbb{C}^4$ admits a $C_{1,4}$-equivariant quaternionic structure, where the actions of $C_{1,4}$ and $Spin^+(1,4)$ on $\Delta_{1,4}$ are given by $i_{\mathbb{C}} \circ \Phi$. For simplicity, we identify $\Delta_{1,4}$ with $\mathbb{H}^2$. The action of $Spin^+(1,4)$ on the Minkowski 5-space $\mathbb{R}^{1,4} \subset \mathbb{H}(2)$ is given by

$$\begin{array}{rcl} Spin^+(1,4) \times \mathbb{R}^{1,4} & \to & \mathbb{R}^{1,4} \ . \\ (A,x) & \mapsto & AxA^{-1} \in \mathbb{H}(2) \end{array}$$

On $\Delta_{1,4}$ we have the $Spin^+(1,4)$-invariant product

$$\langle v,w\rangle_{\Delta_{\mathbb{H}}} := v^*Qw, \qquad v,w \in \Delta_{1,4} \cong \mathbb{H}^2,$$

which takes values in $\mathbb{H}$, and we use the notation $\mathbb{H}^{1,1} \cong (\Delta_{1,4}, \langle\cdot,\cdot\rangle_{\Delta_{\mathbb{H}}})$. Moreover, we have the quadratic form

$$q\begin{pmatrix} a \\ b \end{pmatrix} := \langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix}\rangle_{\Delta_{\mathbb{H}}} = |a|^2 - |b|^2 \qquad \text{for} \ \ v = \begin{pmatrix} a \\ b \end{pmatrix} \in \Delta_{1,4}$$

and it holds $\langle xv,w\rangle_{\Delta_{\mathbb{H}}} = \langle v,xw\rangle_{\Delta_{\mathbb{H}}}$, since $x^*Q = Qx$ for all $x \in \mathbb{R}^{1,4}$. Notice that the real part of the product $\langle\cdot,\cdot\rangle_{\Delta_{\mathbb{H}}}$ is equal to the real part of the usual Hermitian product $\langle\cdot,\cdot\rangle_{\Delta}$ on the complex spinor modul $\Delta_{1,4}$.

The associated vector x_v to $v \in \Delta_{1,4}$ is defined by the map

$$\begin{array}{rrcl} \ell : & \Delta_{1,4} & \to & \mathbb{R}^{1,4} \ . \\ & v = \begin{pmatrix} a \\ b \end{pmatrix} & \mapsto & 2vv^*Q - q(v)\cdot I_2 = \begin{pmatrix} |a|^2+|b|^2 & -2a\bar{b} \\ 2b\bar{a} & -(|a|^2+|b|^2) \end{pmatrix} \end{array}$$

It holds

$$\begin{aligned} &\ell(v) = -\sum_i \varepsilon_i \langle v, e_i v\rangle_{\Delta_{\mathbb{H}}} e_i, \\ &\|\ell(v)\|^2 = -(|a|^2-|b|^2)^2 = -q(v) \\ &\ell(v)\cdot v = q(v)\cdot v \qquad \text{and} \\ &\ell(v) = \ell(v\cdot p) \qquad \text{for all} \ \ p \in S^3 \subset \mathbb{H}. \end{aligned}$$

The orbits of the $Spin^+(1,4)$-action on $\Delta_{1,4} \setminus 0$ consists of the level sets of the length function $q : \Delta_{1,4} \setminus 0 \to \mathbb{R}$ (comp. [Bry00]). So, for every spinor $v \in \Delta_{1,4} \setminus 0$ an $A \in Spin^+(1,4)$ exists such that

$$\begin{aligned} &Av = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{if} \quad q(v) = 0, \\ &Av = \begin{pmatrix} r \\ 0 \end{pmatrix} \qquad \text{if} \quad q(v) = r^2 \in \mathbb{R} \quad \text{or} \\ &Av = \begin{pmatrix} 0 \\ r \end{pmatrix} \qquad \text{if} \quad q(v) = -r^2 \in \mathbb{R}. \end{aligned}$$

Let $L(\mathbb{H}^{1,1}) = q^{-1}(0)$ denote the lightcone in $\mathbb{H}^{1,1}$. Every spinor $v \in L(\mathbb{H}^{1,1})$ can be written as $v = r\begin{pmatrix} 1 \\ a \end{pmatrix}\check{a}$ for some uniquely determined $r \in \mathbb{R}_+$ and $a, \check{a} \in S^3 \subset \mathbb{H}$ and we have the identifications

$$L(\mathbb{H}^{1,1}) \cong (\mathbb{H} \setminus 0) \times S^3 \cong \mathbb{R}_+ \times S^3 \times S^3.$$

The mapping

$$\begin{array}{rcccl} \ell|_L: & L(\mathbb{H}^{1,1}) & \to & L^+(\mathbb{R}^{1,4}) & , \\ & v = r\begin{pmatrix} 1 \\ a \end{pmatrix}\check{a} & \mapsto & \ell(v) = r^2\begin{pmatrix} 1 \\ a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix}, & \qquad a = a_{11} + a_{12}i + a_{21}j + a_{22}k \in S^3, \end{array}$$

is surjective. Moreover, this map is a S^3-fibration over the lightcone $L^+(\mathbb{R}^{1,4}) \cong \mathbb{R}_+ \times S^3$. It holds

$$\ell\begin{pmatrix} 1 \\ 1 \end{pmatrix} = e_1 + e_2, \qquad \ell\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \ell\begin{pmatrix} 0 \\ 1 \end{pmatrix} = e_1,$$

$$\ell^{-1}(e_1 + e_2) = \{\begin{pmatrix} a \\ a \end{pmatrix} \mid a \in S^3\} \cong S^3 \qquad \text{and}$$

$$\ell^{-1}(e_1) = \{\begin{pmatrix} a \\ 0 \end{pmatrix} \mid a \in S^3\} \cup \{\begin{pmatrix} 0 \\ a \end{pmatrix} \mid a \in S^3\}.$$

Notice that $\dim_{\mathbb{C}}(v \cdot \mathbb{H}) = 2$ for all $0 \neq v \in \mathbb{H}^2$. The spinors v and $v \cdot j$ are $\mathbb{C}$-linear independent.

We search now for the annihilators of a spinor in the 2-forms. Let

$$\omega = \sum_{i<j} \varepsilon_i \varepsilon_j \omega_{ij} e_i^* \wedge e_j^* \in \Lambda^2$$

be a 2-form and let $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in L(\mathbb{H}^{1,1})$. Then

$$\begin{array}{rcl} \omega \cdot v & = & -\omega_{12}\begin{pmatrix} -1 \\ -1 \end{pmatrix} - \omega_{13}\begin{pmatrix} i \\ -i \end{pmatrix} - \omega_{14}\begin{pmatrix} j \\ -j \end{pmatrix} - \omega_{15}\begin{pmatrix} k \\ -k \end{pmatrix} + \omega_{23}\begin{pmatrix} -i \\ i \end{pmatrix} \\ & & +\omega_{24}\begin{pmatrix} -j \\ j \end{pmatrix} + \omega_{25}\begin{pmatrix} -k \\ k \end{pmatrix} + \omega_{34}\begin{pmatrix} k \\ k \end{pmatrix} + \omega_{35}\begin{pmatrix} -j \\ -j \end{pmatrix} + \omega_{45}\begin{pmatrix} i \\ i \end{pmatrix}. \end{array}$$

It follows that $\omega \cdot v = 0$ if and only if

$$\omega_{12} = \omega_{34} = \omega_{35} = \omega_{45} = 0 \qquad \text{and} \qquad \omega_{13} + \omega_{23} = \omega_{14} + \omega_{24} = \omega_{15} + \omega_{25} = 0,$$

which is equivalent to say that ω has the form

$$\omega = (-e_1^* + e_2^*) \wedge \sum_{i=3}^{5} a_i e_i^* \qquad \text{for some } \; a_i \in \mathbb{R}.$$

In case that $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, it holds $\omega \cdot v = 0$ if and only if

$$\omega_{12} = \omega_{13} = \omega_{14} = \omega_{15} = 0 \qquad \text{and} \qquad \omega_{23} - \omega_{45} = \omega_{25} - \omega_{34} = \omega_{24} + \omega_{35} = 0,$$

which is equivalent to

$$\omega \;=\; a \cdot (e_2^* \wedge e_3^* + e_4^* \wedge e_5^*) + b \cdot (e_2^* \wedge e_5^* + e_3^* \wedge e_4^*) + c \cdot (e_2^* \wedge e_4^* - e_3^* \wedge e_5^*)$$

for some $a, b, c \in \mathbb{R}$. Moreover, it holds $\omega \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$ if and only if

$$\omega \;=\; a \cdot (e_2^* \wedge e_3^* - e_4^* \wedge e_5^*) + b \cdot (e_2^* \wedge e_5^* - e_3^* \wedge e_4^*) + c \cdot (e_2^* \wedge e_4^* + e_3^* \wedge e_5^*).$$

for some $a, b, c \in \mathbb{R}$. These expressions for the annihilators in the 2-forms show

Lemma 4.4.1. — *Let $\omega \in \Lambda^2$ be a 2-form and $0 \neq v, u \in \Delta_{1,4}$.*

(1) If $\omega \cdot v = 0$ then $\ell(v) \lrcorner\ \omega = 0$.

(2) If $\omega \cdot v = \omega \cdot u = 0$ and v, u are $\mathbb{H}$-linear independent then $\omega = 0$.

Let (M^5_1, g) be a time-oriented Lorentzian spin 5-manifold with metric tensor g of usual signature $(- + + + +)$. Let S denote the complex spinor bundle over M^5_1. The spinor bundle admits a quaternionic structure, i.e. there exists an $\mathbb{H}$-multiplication from the right on S. In particular, we have a multiplication of a spinor $\varphi \in S$ with the imaginary units i, j, k. Moreover, let $L(S)$ denote the bundle of lightcones in S with respect to the natural product $\langle \cdot, \cdot \rangle_{S_{\mathbb{H}}}$ on S.

To a spinor field $\varphi \in \Gamma(S)$ we have the associated vector field $V_\varphi = \ell(\varphi)$ and, it holds

$$V_\varphi \cdot \varphi = \langle \varphi, \varphi \rangle \cdot \varphi \qquad \text{for all } \varphi \in \Gamma(S).$$

In particular, $V_\varphi \cdot \varphi = 0$ for all $\varphi \in \Gamma(L(S))$.

Let φ be an arbitrary twistor spinor on M^5_1. Then $W(\eta) \cdot \varphi = 0$ for all $\eta \in \Lambda^2 M^5_1$. Moreover, from Lemma 4.4.1 it follows that $V_\varphi \lrcorner\ W \equiv 0$ on M^5_1 even if V_φ is not lightlike (comp. Proposition 4.1.5). As in dimension $n = 3$ and 4, we are able to give the upper bound of the dimension of the space $\mathcal{T}(M^5_1)$, when M^5_1 is not conformally flat.

Proposition 4.4.2. — *Let (M^5_1, g) be a Lorentzian spin 5-manifold.*

(1) Let $\varphi \in \Gamma(S)$ be a twistor spinor and let $a : M^5_1 \to \mathbb{H}$ be a quaternionic function. Then $\varphi \cdot a \in \Gamma(S)$ is a twistor spinor if and only if a is constant.

(2) The dimension of $\mathcal{T}(M^5_1)$ is even.

(3) If $\dim \mathcal{T}(M^5_1) > 2$ then M^5_1 is conformally flat.

Proof. (1) Let φ be a twistor spinor and let $a : M^5_1 \to \mathbb{H}$ be a quaternionic function. The spinor $\varphi \cdot a \in \Gamma(S)$ is a twistor spinor if and only if

$$g(X, X) X \cdot \varphi X(a) = \frac{1}{n} \sum_i \varepsilon_i \cdot s_i \varphi s_i(a) \qquad \text{for all } \ X \in TM^5_1 \text{ with } \|X\|^2 = \pm 1,$$

where $(s_1, \ldots, s_5)$ denotes a local frame. Notice that the right hand side does not depend on X and that

$$Span\{X \cdot \varphi : X \in TM^5_1,\ \varphi \neq 0\} = S|_{M^5_1 \setminus zero(\varphi)}.$$

This shows that the above condition is equivalent to $X(a) = 0$ for all $X \in TM$, i.e. a is constant.

(2) Let $\{\varphi_l :\ l = 1, \ldots, s\}$ be a set of $\mathbb{H}$-linear independent twistor spinors such that $\mathcal{T}(M^5_1) = Span_{\mathbb{H}}\{\varphi_l\}$. Then the set of twistor spinors $\{\varphi_l, \varphi_l \cdot j :\ l = 1, \ldots, s\}$ is $\mathbb{C}$-linear independent and

$$\mathcal{T}(M^5_1) = Span_{\mathbb{C}}\{\varphi_\ell, \varphi_\ell \cdot j\}.$$

(3) Assume that $\dim_{\mathbb{C}} \mathcal{T}(M_1^5) > 2$. Then there exist twistor spinors $\varphi, \psi \in \Gamma(S)$ such that the set

$$\tilde{M} := \{x \in M : \nexists a \in \mathbb{H} \text{ with } \varphi(x) = \psi(x) \cdot a\}$$

is dense in M. It holds

$$W(\eta) \cdot \varphi = W(\eta) \cdot \psi = 0 \qquad \text{for all } \eta \in \Lambda^2 M$$

and with Lemma 4.4.1, it follows $W \equiv 0$ on $\tilde{M}$. We can conclude that $W \equiv 0$ on M_1^5. □

Remark. If $\varphi \in \Gamma(S)$ is a Killing spinor then $\varphi \cdot j \in \Gamma(S)$ is also a Killing spinor to the same Killing number.

Corollary 4.4.3. — *Let (M_1^5, g) be a Lorentzian Einstein spin 5-manifold.*

(1) If $\varphi \in \mathcal{T}(M_1^5)$ then $D\varphi = \varphi \cdot a$ for some $a \in \mathbb{H}$ or M_1^5 has constant sectional curvature.

(2) If $\varphi \in \mathcal{T}(M_1^5)$ and zero$(\varphi) \neq 0$ then M_1^5 has constant sectional curvature.

(3) If M_1^5 is non-flat and Ricci-flat then every twistor spinor on M_1^5 is parallel.

Proof. (1) Since M_1^5 is Einstein and φ is a twistor spinor, $D\varphi$ is a twistor spinor. In case that M_1^5 has not constant sectional curvature, it must be $D\varphi \in Span_{\mathbb{H}}\{\varphi\}$.

(2) If $\varphi \in \mathcal{T}(M_1^5)$ has a zero then $D\varphi \notin Span_{\mathbb{H}}\{\varphi\}$. Hence, M_1^5 has constant sectional curvature.

(3) It holds $D\varphi = \varphi \cdot a$ for some $a \in \mathbb{H}$. Suppose that $\nabla_X^S \varphi \neq 0$. Then $D\varphi = \varphi \cdot a$ is parallel, which is obviously impossible. □

Remark. All together we have proved that in dimension $n = 3, 4$ and 5 a twistor spinor with zero on a Lorentzian Einstein space exists only if the space has constant sectional curvature. We remember that in the Riemannian setting this statement is true in every dimension.

We give now a classification of Lorentzian 5-metrics, which admit twistor spinors without 'singularities'. We start with the parallel spinors. It is well-known that if (M_1^5, g) admits a parallel spinor then the following two types of Lorentzian 5-metrics can occur (comp. [Bry00]). For the first, if $\varphi \in \Gamma(S)$ is parallel and $\langle \varphi, \varphi \rangle \neq 0$ then g has locally the form $g = -dt^2 + \bar{g}$, where $\bar{g}$ is a Ricci-flat Kähler metric on a Riemannian 4-manifold. The associated vector field to φ is $\frac{\partial}{\partial t}$, which is a timelike unit field. Secondly, if φ is parallel and $\langle \varphi, \varphi \rangle \equiv 0$ then there exist locally coordinates

$$(x, s, r) : U \to \mathbb{R} \times Im\mathbb{H} \times \mathbb{R},$$

in which the metric g has the form

$$g = d\bar{s} \circ ds - 2dr \circ dx - (1 + 2f(x, s))dx^2,$$

where f is an arbitrary function of (x, s). These are the pp-metrics (comp. 1.4). The associated vector field V_φ is lightlike in this case. Conversely, on every metric of the two given classes a parallel spinor exists. By Proposition 4.1.13 we obtain

Proposition 4.4.4. — *Let $\varphi \in \Gamma(S)$ be a twistor spinor on (M^5_1, g) without zeros such that $|\varphi|^2 \equiv 0$ or $|\varphi|^2 \neq 0$ and with vanishing twist $\omega_\varphi \wedge d\omega_\varphi \equiv 0$. Then g is locally conformally equivalent to*

(1) a Riemannian product of the form $-dt^2 + \bar{g}$, where $\bar{g}$ is a Ricci-flat Kähler 4-metric, or

(2) a pp-metric of the form $d\bar{s} \circ ds - 2dr \circ dx - (1 + 2f(x, s))dx^2$.

We remember to Theorem 1.4.2 (comp. [Kat99]), which says that a simply connected Lorentzian Einstein-Sasaki 5-manifold admits an imaginary Killing spinor. On the other hand, let $\varphi \in \Gamma(S)$ be an imaginary Killing spinor with $\langle\varphi, \varphi\rangle \neq 0$. Then, it holds

$$-g(V_\varphi, V_\varphi) = \langle\varphi, \varphi\rangle_S^2 = const, \qquad V_\varphi = \langle\varphi, \varphi\rangle_S \cdot \varphi \qquad \text{and} \qquad \nabla^S_{V_\varphi}\varphi = i\lambda\varphi, \quad \lambda \in \mathbb{R},$$

and (M^5_1, g) is a Lorentzian Einstein-Sasaki 5-manifold.

Proposition 4.4.5. — *Let $\varphi \in \Gamma(S)$ be a twistor spinor with $\langle\varphi, \varphi\rangle \neq 0$ on a non-conformally flat Lorentzian spin 5-manifold (M^5_1, g). There occur exactly two cases:*

(1) The twist of V_φ satisfies $\omega_\varphi \wedge d\omega_\varphi \neq 0$ on M^5_1, the spinor field $\frac{1}{|\varphi|} \cdot \tilde{\varphi} \in \Gamma(\tilde{S})$ is an imaginary Killing spinor with respect to the metric $\tilde{g} = \frac{1}{|\varphi|^4} \cdot g$ on M^5_1 and $(M^5_1, \tilde{g})$ is an Einstein-Sasaki 5-manifold.

(2) The twist of V_φ satisfies $\omega_\varphi \wedge d\omega_\varphi \equiv 0$ and the spinor $\frac{1}{|\varphi|} \cdot \tilde{\varphi}$ is parallel on $(M^5_1, \tilde{g} = \frac{1}{|\varphi|^4} \cdot g)$.

Proof. By Proposition 1.2.5 and Corollary 4.4.3 we know that only one of the following cases can occur. The spinor $\frac{1}{|\varphi|} \cdot \tilde{\varphi} \in \Gamma(\tilde{S})$ is with respect to $(M^5_1, \frac{1}{|\varphi|^4} \cdot g)$

(a) a parallel spinor,

(b) a Killing spinor or

(c) a sum of Killing spinors to different Killing numbers λ_+ and λ_-.

We consider the case (c). Suppose $\frac{1}{|\varphi|} \cdot \tilde{\varphi} = \psi_+ + \psi_-$ is a sum of non-trivial Killing spinors. Then the spinor fields $\psi_+, \psi_+ \cdot j, \psi_-$ and $\psi_- \cdot j$ are $\mathbb{C}$-linearly independent Killing spinors. That means $dim\mathcal{T}(M^5_1) > 2$, which is in contradiction to the assumptions.

Hence, $\frac{1}{|\varphi|} \cdot \tilde{\varphi}$ is parallel or a Killing spinor. But the spinor norm of a real Killing spinor is never constant. We can conclude that $\frac{1}{|\varphi|} \cdot \tilde{\varphi}$ is parallel and then $\omega_\varphi \wedge d\omega_\varphi \equiv 0$ or $\frac{1}{|\varphi|} \cdot \tilde{\varphi}$ is an imaginary Killing spinor on an Einstein-Sasaki manifold, which implies that $\omega_\varphi \wedge d\omega_\varphi \neq 0$.□

It remains the case of Lorentzian 5-metrics admitting a twistor spinor φ with $\langle\varphi, \varphi\rangle \equiv 0$ and $\omega_\varphi \wedge d\omega_\varphi \neq 0$. In even dimensions we have such solutions of the twistor equation on the

Fefferman spaces. But here we can prove

Lemma 4.4.6. — *If $\varphi \in \Gamma(S)$ is a twistor spinor without zeros on (M^5_1, g) such that $\langle \varphi, \varphi \rangle \equiv 0$ then the twist $\omega_\varphi \wedge d\omega_\varphi$ vanishes.*

Proof. Let $\varphi \in \Gamma(L(S))$ be a twistor spinor without zeros. Since $Spin^+(1,4)$ acts transitively on $L(\mathbb{H}^{1,1})$, we can find locally for every point $p \in M^5_1$ a section $\hat{s} : U(p) \subset M^5_1 \to Spin(M^5_1)$ such that the twistor spinor φ is locally given as $\varphi = [\hat{s}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}]$. The map $s := f \circ \hat{s} : U(p) \to SO(M^5_1)$ is a local frame on M^5_1. We denote by θ the dual frame to s. Furthermore, let ω denote the Levi-Civita connection form on the frame bundle $SO(M^5_1)$. We have

$$\omega \circ ds = \sum_{i<j} \omega_{ij} \cdot E_{ij},$$

where E_{ij} is the standard basis of the Lie algebra $\mathfrak{so}(1,4)$ and $\omega_{ij} := \varepsilon_i \varepsilon_j g(\nabla s_i, s_j)$ are the local connection components. We calculate for the twist with respect to the frame s

$$\begin{aligned} \omega_\varphi \wedge d\omega_\varphi \quad = \quad & (-\omega_{23}(s_2) + \omega_{13}(s_2) + \omega_{13}(s_1) - \omega_{23}(s_1)) \cdot \theta_1 \wedge \theta_2 \wedge \theta_3 \\ & + (-\omega_{24}(s_2) + \omega_{14}(s_2) + \omega_{14}(s_1) - \omega_{24}(s_1)) \cdot \theta_1 \wedge \theta_2 \wedge \theta_4. \end{aligned}$$

And now we calculate the conditions on the connection components ω_{ij}, which arise from the twistor equation for φ. The calculation is straight forward and at the end we have the following linear conditions on the components ω_{ij}:

$$\begin{aligned}
&\omega_{12}(s_3) = -\omega_{34}(s_4) = -\omega_{35}(s_5) = \omega_{23}(s_2) - \omega_{13}(s_2) = \omega_{13}(s_1) - \omega_{23}(s_1), \\
&\omega_{34}(s_3) = \omega_{12}(s_4) = -\omega_{45}(s_5) = \omega_{24}(s_2) - \omega_{14}(s_2) = \omega_{14}(s_1) - \omega_{24}(s_1), \\
&\omega_{45}(s_1) = \omega_{45}(s_2) = \omega_{45}(s_3) = \omega_{35}(s_4) = \omega_{34}(s_5) = 0, \\
&\omega_{15}(s_4) - \omega_{25}(s_4) = \omega_{14}(s_5) - \omega_{24}(s_5) = 0, \\
&\omega_{35}(s_3) = \omega_{45}(s_4) = \omega_{12}(s_5) = \omega_{25}(s_2) - \omega_{15}(s_2) = \omega_{15}(s_1) - \omega_{25}(s_1), \\
&\omega_{23}(s_3) - \omega_{13}(s_3) = \omega_{24}(s_4) - \omega_{14}(s_4) = \omega_{25}(s_5) - \omega_{15}(s_5) = -\omega_{12}(s_2) = -\omega_{12}(s_1), \\
&\omega_{24}(s_3) - \omega_{14}(s_3) = \omega_{13}(s_4) - \omega_{23}(s_4) = -\omega_{34}(s_1) = -\omega_{34}(s_2) = 0 \quad \text{and} \\
&\omega_{15}(s_3) - \omega_{25}(s_3) = \omega_{23}(s_5) - \omega_{13}(s_5) = \omega_{35}(s_2) = \omega_{35}(s_1) = 0.
\end{aligned}$$

From the first two lines one can see that these conditions imply already that the twist $\omega_\varphi \wedge d\omega_\varphi$ of the associated field V_φ vanishes identically. □

Altogether, we have

Theorem 4.4.7. — *Let $\varphi \in \Gamma(S)$ be a twistor spinor without zeros such that $|\varphi|^2 \equiv 0$ or $|\varphi|^2 \neq 0$ on a non-conformally flat Lorentzian 5-manifold (M^5_1, g). Then*

(1) the twist $\omega_\varphi \wedge d\omega_\varphi$ vanishes identically on M^5_1, φ is locally conformally equivalent to a parallel spinor and the metric g has up to a conformal factor a local form given as in Proposition 4.4.4 or

(2) the twist satisfies $\omega_\varphi \wedge d\omega_\varphi \neq 0$ *everywhere on* M_1^5*, the spinor* φ *is conformally equivalent to an imaginary Killing spinor and* (M_1^5, g) *is conformally equivalent to an Einstein-Sasaki manifold.*

We remark that there is no complete description of Lorentzian 5-metrics admitting twistor spinors φ with zeros or singularities in the spinor norm $|\varphi|^2$.

5 Twistoriel construction of spacelike surfaces in Lorenztian 4-manifolds

This section is concerned with the investigation of spacelike surfaces in Lorentzian 4-manifolds with the aid of a so-called twistor construction in 4-dimensional Lorentzian geometry. The idea to this comes from Riemannian twistor theory and its well-known application to the theory of surfaces in Riemannian 4-spaces. The topic on surface theory, that we present here, is not directly related to the twistor equation. However, the twistor space of a Lorentzian 4-manifold provides an interpretation of the twistor equation in terms of optical geometry (see 5.4).

5.1 Some preliminary remarks on twistor theory

We start with a short description of the Riemannian twistor construction. Let (M^4, g) be an oriented 4-dimensional Riemannian manifold. The tangent space at every point of the manifold M^4 is isometric to the Euclidean 4-space $\mathbb{R}^4$. The set of complex structures on $\mathbb{R}^4$ can be identified with the homogenous space $GL(4,\mathbb{R})/GL(2,\mathbb{C})$. There are two kinds of orthogonal complex structures on $\mathbb{R}^4$:

$$A_+ := \{J \in SO(4)|\ \ \exists\, A \in SO(4) \text{ with } J = A^{-1} \begin{pmatrix} J_o & 0 \\ 0 & J_o \end{pmatrix} A\} \quad \text{and}$$
$$A_- := \{J \in SO(4)|\ \ \exists\, A \in SO(4) \text{ with } J = A^{-1} \begin{pmatrix} J_o & 0 \\ 0 & -J_o \end{pmatrix} A\},$$

where $J_o := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathbb{R}(2)$. Both sets A_+ and A_- of orthogonal complex structures are naturally identified with the homogenous space $SO(4)/U(2)$, which is the 2-sphere S^2. We choose the set A_- and define the twistor space $\mathcal{A}_-(M^4)$ of M^4 due to Atiyah/Hitchin/Singer [AHS78] as the associated fibre bundle

$$\mathcal{A}_-(M^4) := SO(M^4) \times_{SO(4)} A_- = SO(M^4) \times_{SO(4)} S^2$$

over M^4 consisting of orthogonal complex structures on TM^4. Thereby, $SO(M^4)$ denotes the bundle of positive oriented orthonormal frames on M^4. The twistor space $\mathcal{A}_-(M^4)$ admits two natural almost complex structures J_1 and J_2. The almost complex structure J_2 is never integrable. The almost complex structure J_1 is integrable if and only if the Riemannian 4-manifold M^4 is self-dual. In [AHS78] it is pointed out that the twistor equation on a Riemannian spin manifold appears as integrability condition for the almost complex structure J_1. In particular, twistor spinors may be interpreted as holomorphic sections in the canonical line bundle $\mathcal{B}$ over the twistor space $\mathcal{A}_-(M^4)$.

Let us consider a conformal immersion $f : N^2 \to M^4$ of a Riemannian surface N^2 into the Riemannian 4-space M^4. Every oriented 2-plane V^2 in the oriented Euclidean four-space $\mathbb{R}^4$ gives naturally rise to an orthogonal complex structure on $\mathbb{R}^4$, which is the rotation around the angle $\frac{\pi}{2}$ in positive direction on V and negative direction on $V^\perp$ and which is an element of the set A_-. The image $df(T_nN)$ of the tangent space at every point $n \in N$ is an oriented 2-plane in the Euclidean vector space $T_{f(n)}M$. Hence, the 2-plane $df(T_nN)$ in $T_{f(n)}M^4$ corresponds uniquely to an element J_n in the fibre of the twistor space $\mathcal{A}_-(M^4)$ over the point $f(n)$. This gives rise to a natural lift of the immersion f to the twistor space $\mathcal{A}_-(M^4)$:

$$\begin{array}{rccc} \gamma_f : & N & \to & \mathcal{A}_-(M^4) \ , \\ & n & \mapsto & J_n \end{array} \qquad \begin{array}{ccc} & & \mathcal{A}_-(M^4) \\ & \overset{\gamma_f}{\nearrow} & \downarrow \\ f : N & \to & M^4 \end{array} .$$

J. Eells and S. Salamon studied this 'Gauss' lift in [ES85] and proved the following relation for minimal surfaces.

Theorem. *Conformally immersed minimal surfaces in a Riemannian 4-space M^4 correspond bijectively to non-vertical J_2-holomorphic curves in the twistor space $\mathcal{A}_-(M^4)$ over M^4.*

In particular, the lift of a conformally immersed surface to the twistor space is horizontal if and only if the immersed surface is superminimal (comp. [Fri84]). This means that in the case of a self-dual Riemannian 4-space M^4, superminimal surfaces can be constructed by horizontal holomorphic curves in the twistor space, which is a complex manifold. Those constructions of superminimal surfaces have been done by Th. Friedrich in [Fri84] and [Fri97]. By the way, a well-known result is obtained, which says that a superminimal surface in the Euclidean 4-space has locally the form

$$\begin{array}{ccc} \mathbb{C} & \to & \mathbb{C} \times \mathbb{C} \cong \mathbb{R}^4 \ , \\ z & \mapsto & (z, f(z)) \end{array}$$

where f is a holomorphic function. Using the twistor space $P^3(\mathbb{C})$ of the sphere S^4, R. Bryant proved in [Bry82] the global result that every Riemannian surface admits a conformal superminimal immersion into S^4. The twistor method finds also applications in semi-Riemannian geometry. G.R. Jensen and M. Rigoli constructed in [JR90] the reflector space of a neutral 4-space and applied this construction to the theory of immersed neutral surfaces.

The idea of twistor theory in the context of Lorentzian geometry is as follows. The twistor space $\mathcal{Z}(M_1^4)$ of an oriented Lorentzian 4-manifold M_1^4 is defined to be the bundle of null directions in the tangent space TM over M_1^4 (comp. [Nur96])

$$\mathcal{Z}(M_1^4) := SO(M_1^4) \times_{SO(1,3)} Q,$$

where Q is the space of null directions in the Minkowski space $\mathbb{R}^{1,3}$. Instead of almost complex structures, the twistor space $\mathcal{Z}(M_1^4)$ admits natural almost optical structures, which are related to CR-structures (see 5.2).

Let us consider a conformal immersion $f : N^2 \to M_1^4$ of a Riemannian surface N^2 into an oriented Lorentzian 4-space M_1^4. In every point $f(n) \in M_1^4$ of the immersed surface, there exists an ordered pair of normal null directions on the tangent space $df(T_nN)$ to the surface in M_1^4. By choosing one of these normal null directions, we obtain a natural lift of the immersion f to the bundle of null directions $\mathcal{Z}(M_1^4)$

$$\begin{array}{ccc} & & \mathcal{Z}(M_1^4) \\ & \overset{\gamma_f}{\nearrow} & \downarrow \\ f : N^2 & \to & M_1^4 \end{array} .$$

Similar as in 4-dimensional Riemannian geometry, spacelike surfaces in M_1^4 with special assumptions on the extrinsic geometry can be characterized and constructed by holomorphic curves in the almost optical manifold $\mathcal{Z}(M_1^4)$.

5.2 Optical geometry and CR-geometry

We recall in this part some basic facts about CR-geometry, optical geometry and the relation between them. Optical geometry was first introduced in relativistic theories by A. Trautman (see [Tra85], [Nur96]).

We start with the definition of CR-structures (comp. [Jac90]):

Definition 5.2.1. — *Let M^{2n-1} be a $(2n-1)$-dimensional C^∞-manifold.*

(1) A pair $(\mathcal{H}, J)$, where $\mathcal{H} \subset TM$ is a smooth distribution of codimension 1 on M and $J : \mathcal{H} \to \mathcal{H}$ is a smooth bundle isomorphism on $\mathcal{H}$ with $J^2 = -id$, is called almost CR-structure on M^{2n-1}.

(2) An almost CR-structure $(\mathcal{H}, J)$ on M^{2n-1} is said to be integrable if and only if the Nijenhuis tensor N of J on $\mathcal{H}$ vanishes, i.e.

$$[JX,Y]+[X,JY] \in \Gamma(\mathcal{H}) \qquad \textit{and}$$
$$N(X,Y) := J([X,JY]+[JX,Y]) - [JX,JY] + [X,Y] = 0$$

for all $X, Y \in \Gamma(\mathcal{H})$. In this case we call $(M, \mathcal{H}, J)$ a CR-manifold.

(3) A C^∞-map f between two almost CR-manifolds $(M, \mathcal{H}, J)$ and $(\tilde{M}, \tilde{\mathcal{H}}, \tilde{J})$ is called a CR-map if and only if $df(\mathcal{H}) \subset \tilde{\mathcal{H}}$ and $df \circ J = \tilde{J} \circ df$.

Example. Let (N^3, h) be a 3-dimensional, oriented Riemannian manifold. The unit sphere bundle of N^3 is given by

$$S^2(TN) := SO(N) \times_{SO(3)} S^2 = SO(N) \times_{SO(3)} SO(3)/SO(2) \subset TN,$$

where $SO(N)$ denotes the $SO(3)$-principal fibre bundle of orthonormal frames on N. Let $\pi : S^2(TN) \to N$ denote the natural projection. The Levi-Civita connection of (N^3, h) decomposes the tangent bundle $TS^2(TN)$ into a horizontal and a vertical part:

$$TS^2(TN) = T^V S^2(TN) \oplus T^H S^2(TN).$$

On $S^2(TN)$, it exists a natural smooth distribution $\mathcal{H}^{S^2(TN)} \subset TS^2(TN)$ of codimension 1 given in a point $l \in S^2(TN)$ by

$$\mathcal{H}_l^{S^2(TN)} = \pi_*^{-1}((\mathbb{R}l)^\perp) = T^V S^2(TN) \oplus (T^H S^2(TN) \cap \pi_*^{-1}(\mathbb{R}l^\perp)).$$

Let J^{S^2} denote the standard $SO(3)$-invariant complex structure on S^2. It exists a natural complex structure on the distribution $\mathcal{H}^{S^2(TN)}$, which is pointwise defined by

$$J_l^{S^2(TN)} = \pi_*^{-1} \circ J_l \circ \pi_* + [s]^{-1} \circ J^{S^2} \circ [s],$$

where $[s]$ denotes the identification of a fibre with S^2 by an orthonormal frame s and $\pi_*^{-1} \circ J_l \circ \pi_*$ is the horizontal lift of the orthogonal complex structure J_l on $(\mathbb{R}l)^\perp \subset T_{\pi(l)}N$, which is the rotation around the angle $\frac{\pi}{2}$ in positive direction. This CR-structure $(\mathcal{H}^{S^2(TN)}, J^{S^2(TN)})$ on $S^2(TN)$ is always integrable. Furthermore, the unit sphere bundle together with its natural CR-structure is conformally invariant.

We continue with the definition of almost optical structures:

Definition 5.2.2. — *(comp. [Nur96]) Let M^{2n} be a 2n-dimensional C^∞-manifold.*

(1) A triple $\mathcal{O} = (\mathcal{K}, \mathcal{L}, J)$ consisting of subbundles $\mathcal{K}$ and $\mathcal{L}$ in TM such that

$$\mathcal{K} \subset \mathcal{L}, \qquad \dim \mathcal{K} = 1, \qquad \dim \mathcal{L} = 2n - 1$$

and an almost complex structure $J : \mathcal{L}/\mathcal{K} \to \mathcal{L}/\mathcal{K}$, $J^2 = -id$, on the quotient bundle $\mathcal{L}/\mathcal{K}$, is called an almost optical structure on M^{2n}.

(2) An almost optical structure $\mathcal{O} = (\mathcal{K}, \mathcal{L}, J)$ on M^{2n} is said to be integrable if and only if the following two conditions are satisfied:

(A)

$$[\Gamma(\mathcal{K}), \Gamma(\mathcal{L})] \subset \Gamma(\mathcal{L}) \quad \textit{and} \quad \Phi_t^{k*} J = J \qquad \textit{for all } k \in \Gamma(\mathcal{K}),$$

where Φ_t^k denotes the flow of the field k to the time t. This condition means that locally in every point $m \in M$ the almost optical structure $\mathcal{O}$ may be pushed down to an almost CR-structure $(\mathcal{H}^m, J^m)$ on a locally induced quotient manifold $U_m/{\sim^{\mathcal{K}}}$, where $U_m \subset M$ is a suitable neighborhood of $m \in M$ and two points $u_1, u_2 \in U_m$ are $\sim^{\mathcal{K}}$-related if and only if both belong to the same integral curve of the distribution $\mathcal{K}|_{U_m}$.

(B) The locally induced CR-structures $(\mathcal{H}^m, J^m)$ are integrable in every point $m \in M$.

(3) A C^∞-map $f : M \to \tilde{M}$ between two almost optical manifolds $(M, \mathcal{K}, \mathcal{L}, J)$ and $(\tilde{M}, \tilde{\mathcal{K}}, \tilde{\mathcal{L}}, \tilde{J})$ is called an optical map if and only if

$$df(\mathcal{K}) \subset \tilde{\mathcal{K}}, \quad df(\mathcal{L}) \subset \tilde{\mathcal{L}} \quad \textit{and} \quad df \circ J = \tilde{J} \circ df.$$

It is instantly clear from the definition that optical structures are closely related to CR-structures. We point this out by the following facts:

(1) An almost CR-manifold $(N^{2n-1}, \mathcal{H}, J)$ gives rise to a canonical almost optical structure on the manifold $M := \mathbb{R} \times N$. This almost optical structure on M is defined by

$$\mathcal{K}^M := T\mathbb{R}, \quad \mathcal{L}^M := T\mathbb{R} \oplus \mathcal{H} \quad \text{and} \quad J^M \cong J : \ \mathcal{L}^M/\mathcal{K}^M \cong \mathcal{H} \to \mathcal{L}^M/\mathcal{K}^M \cong \mathcal{H}.$$

(2) If an almost optical structure $(\mathcal{K}, \mathcal{L}, J)$ on M satisfies condition (A), the locally induced almost CR-structure on $U_m/{\sim^{\mathcal{K}}}$, $m \in M$, is given by

$$\mathcal{H}^m := \tilde{\pi}_*(\mathcal{L}) \quad \text{and} \quad J^m := \tilde{\pi}_* \circ J \circ \tilde{\pi}_*^{-1},$$

where $\tilde{\pi} : U_m \to U_m/{\sim^{\mathcal{K}}}$ is the natural projection. The open neighborhood $U_m \subset M$ of $m \in M$ may be chosen in such a way that $(U_m, \mathcal{K}|_{U_m}, \mathcal{L}|_{U_m}, J|_{U_m})$ is optically equivalent to $\mathbb{R} \times U_m/{\sim^{\mathcal{K}}}$ with the induced almost optical structure given as above.

(3) Let N^{2n-1} be a submanifold of codimension 1 in the almost optical manifold $(M^{2n}, \mathcal{K}, \mathcal{L}, J)$ such that

$$TN \oplus \mathcal{K}|_N = TM|_N.$$

It follows that $\mathcal{H}^N := TN \cap \mathcal{L}|_N$ is a distribution in TN with codimension 1. The distribution $\mathcal{H}^N$ is naturally identified with the restricted bundle $\mathcal{L}/\mathcal{K}|_N$ and J induces an almost complex structure J^N on $\mathcal{H}^N$. The pair $(\mathcal{H}^N, J^N)$ is a naturally induced almost CR-structure on N^{2n-1}. We say that $(N, \mathcal{H}^N, J^N)$ is an almost CR-submanifold of the almost optical manifold $(M, \mathcal{K}, \mathcal{L}, J)$. If $(\mathcal{K}, \mathcal{L}, J)$ satisfies condition (A), the almost CR-structure $(\mathcal{H}^N, J^N)$ in $m \in N \subset M$ is locally equivalent to the naturally induced almost CR-structure $(\mathcal{H}^m, J^m)$ on the quotient manifold $U_m/\sim^{\mathcal{K}}$ in the point $\{m\} \in U_m/\sim^{\mathcal{K}}$.

(4) An 1-dimensional distribution $\mathcal{K}$ on a manifold M is called regular if there is a smooth differentiable structure on the quotient set $M/\sim^{\mathcal{K}}$ such that $\tilde{\pi} : M \to M/\sim^{\mathcal{K}}$ is a C^∞-submersion. If $(M, \mathcal{K}, \mathcal{L}, J)$ is an optical manifold with regular distribution $\mathcal{K}$, the quotient manifold $M/\sim^{\mathcal{K}}$ admits globally a natural CR-structure.

After we have defined CR- and optical manifolds, we want to consider mappings between them and define for this an appropriate notion of holomorphicity.

Definition 5.2.3. — *Let (P, J^P) denote an almost complex manifold, $(N, \mathcal{H}^N, J^N)$ an almost CR-manifold and $(M, \mathcal{K}^M, \mathcal{L}^M, J^M)$ an almost optical manifold.*

(1) A C^∞-map $f : P \to N$ is called holomorphic if $df(TP) \subset \mathcal{H}^N$ and $df \circ J^P = J^N \circ df$.

(2) A C^∞-map $g : P \to M$ is called holomorphic if $dg(TP) \subset \mathcal{L}^M$, $dg(TP) \cap \mathcal{K}^M = 0$ and $\hat{\pi} \circ dg \circ J^P = J^M \circ \hat{\pi} \circ dg$, where $\hat{\pi} : \mathcal{L} \to \mathcal{L}/\mathcal{K}$ is the natural projection.

(3) A C^∞-map $h : N \to M$ is called holomorphic if $dh(\mathcal{H}^N) \subset \mathcal{L}^M$, $dh(\mathcal{H}^N) \cap \mathcal{K}^M = 0$ and $\hat{\pi} \circ dh \circ J^N = J^M \circ \hat{\pi} \circ df$.

With the same notations as in the definition, we have

Proposition 5.2.4. —

(1) If the mappings $f : P \to N$ and $h : N \to M$ are holomorphic, the map $g := h \circ f : P \to M$ is also holomorphic.

(2) Let $(\mathcal{K}^M, \mathcal{L}^M, J^M)$ on M satisfy condition (A) and let $k \in \Gamma(\mathcal{K}^M)$ be a vector field on M. If the flow Φ_t^k of k is defined on $(-\epsilon, \epsilon) \times U$, where $\epsilon > 0$ and $U \subset M$ is an open subset, and if $g : P \to M$ is holomorphic, then the map

$$g_t := \Phi_t^k \circ g|_{g^{-1}(U)} : g^{-1}(U) \subset P \to M$$

is holomorphic for every $t \in (-\epsilon, \epsilon)$.

The proof of this is obvious. The second part of the proposition says that the deformation of a holomorphic map by an arbitrary smooth flow along the distribution $\mathcal{K}$ is still holomorphic.

5.3 The twistor space of a Lorentzian manifold M_1^4

The twistor space of an oriented Lorentzian 4-manifold (M_1^4, g) can be defined as the fibre bundle of null directions in the tangent space TM_1^4 (comp. [Nur96]). We give here an equivalent definition of the Lorentzian twistor space as bundle of positive projective spinors on M_1^4. The twistor space admits natural almost optical structures. The integrability, the conformal invariance and the underlying CR-hypersurfaces of these optical structures are investigated. At the end of this part, we discuss explicitly the twistor space of the Minkowski space $\mathbb{R}^{1,3}$, the pseudosphere $S^{1,3}$ and the pseudohyperbolic space $H^{1,3}$.

We start with the description of the fibre type of the twistor bundle. Let

$$Q := \{\mathbb{R} \cdot v \in P^3(\mathbb{R}) : v \neq 0,\ \langle v, v\rangle_{1,3} = 0\}$$

denote the set of null directions in $\mathbb{R}^{1,3}$. The space Q is a submanifold of the projective space $P^3(\mathbb{R})$. There are several characterizations of the space Q. For the first, Q may be written as homogenous space, since the Lorentzian group $SO(1,3)$ acts in a natural way transitively on Q. Let H be the isotropy group of the natural $SO(1,3)$-action in $o := \mathbb{R} \cdot (e_1 + e_2) \in Q$. It holds

$$Q \cong SO(1,3)/H.$$

The Lie algebra $\mathfrak{h}$ of the isotropy group H is given by

$$\mathfrak{h} = Span\{E_{12}, E_{13} + E_{23}, E_{14} + E_{24}, E_{34}\},$$

where $\{E_{ij} : i < j\}$ denotes the standard basis in $\mathfrak{o}(1,3)$. It is $\mathfrak{o}(1,3) = \mathfrak{m} \oplus \mathfrak{h}$, where $\mathfrak{m} = Span\{E_{13} - E_{23}, E_{14} - E_{24}\}$ is the complement to $\mathfrak{h}$ in $\mathfrak{o}(1,3)$. This decomposition of the Lie algebra $\mathfrak{o}(1,3)$ is not reductive.

The space Q of null directions in $\mathbb{R}^{1,3}$ is also naturally identified with the set of positive projective spinors. For this let $\Delta^+_{1,3} \cong \mathbb{C}^2$ be the standard $Spin(1,3)$-module of positive half spinors. A spinor $0 \neq v \in \Delta^+_{1,3}$ induces the $\mathbb{R}$-linear map

$$\begin{array}{cccc} \hat{v}: & \mathbb{R}^{1,3} & \to & \Delta^+ \\ & x & \mapsto & x \cdot v \end{array}.$$

The kernel $ker(\hat{v})$ of this mapping is a null direction in $\mathbb{R}^{1,3}$ and the map

$$\begin{array}{cccc} \iota: & P(\Delta^+) & \to & Q \\ & [v] & \mapsto & ker(\hat{v}), \quad v \in [v] \end{array}$$

is bijective and $SO(1,3)$-equivariant. The natural complex structure $J^{P(\Delta^+)}$ on $P(\Delta^+) \cong P^1(\mathbb{C})$ is invariant by the natural $SO(1,3)$-action and is given on $\mathfrak{m} \cong T_oP$ by

$$J^{P(\Delta^+)}(E_{13} - E_{23}) = -(E_{14} - E_{24}) \quad \text{and} \quad J^{P(\Delta^+)}(E_{14} - E_{24}) = E_{13} - E_{23}.$$

Every unit timelike vector $T \in \mathbb{R}^{1,3}$, $\langle T, T\rangle_{1,3} = -1$, gives an identification of the space of null directions Q and the 2-sphere $S^2(T^\perp) \cong S^2$ in the orthogonal complement $T^\perp \cong \mathbb{R}^3$:

$$\begin{array}{cccc} \iota: & Q & \cong & S^2(T^\perp) \\ & \mathbb{R}(T+s) & \mapsto & s \end{array}.$$

The twistor space of a 4-dimensional oriented Lorentzian manifold (M_1^4, g) is defined to be the positive projective spinor bundle

$$\mathcal{Z}(M_1^4) := P(S^+) = SO(M_1^4) \times_{SO(1,3)} P(\Delta^+).$$

Let $\pi : \mathcal{Z}(M_1^4) \to M_1^4$ denote the natural projection. We may interpret the twistor bundle also as the bundle of null directions in TM_1^4

$$\mathcal{Z}(M_1^4) = SO(M_1^4) \times_{SO(1,3)} Q.$$

The tangent space $T\mathcal{Z}(M_1^4)$ decomposes with respect to the Levi-Civita connection of M_1^4 in a vertical and a horizontal part:

$$T\mathcal{Z}(M_1^4) = T^V\mathcal{Z}(M) \oplus T^H\mathcal{Z}(M).$$

On the twistor space $\mathcal{Z}(M_1^4)$, there are given two natural almost optical structures $\mathcal{O}^+ = (\mathcal{K}, \mathcal{L}, J^+)$ and $\mathcal{O}^- = (\mathcal{K}, \mathcal{L}, J^-)$. We define them pointwise as follows. Let $[\psi] \in \mathcal{Z}(M_1^4)$ be an arbitrary positive projective spinor. For a suitable orthonormal basis $s = (s_1, \ldots, s_4) \in SO(M)$ we may write $[\psi] = [s, \mathbb{R}(e_1 + e_2)]$. To $[\psi] \in \mathcal{Z}(M_1^4)$, it corresponds the orthogonal optical structure $\mathcal{O}^{[\psi]} = (K^{[\psi]}, L^{[\psi]}, J^{[\psi]})$ on the tangent space $T_{\pi([\psi])}M$, which is defined by

$$\begin{aligned} &K^{[\psi]} = \mathbb{R}(s_1 + s_2), \quad L^{[\psi]} = Span\{s_1 + s_2, s_3, s_4\}, \\ &J^{[\psi]}(s_3 + K^{[\psi]}) = s_4 + K^{[\psi]} \quad \text{and} \quad J^{[\psi]}(s_4 + K^{[\psi]}) = -s_3 + K^{[\psi]}. \end{aligned}$$

The almost optical structure $\mathcal{O}^+ = (\mathcal{K}, \mathcal{L}, J^+)$ is given in $T_{[\psi]}\mathcal{Z}(M)$ by

$$\mathcal{O}^+_{[\psi]} = \pi_*^{-1} \circ \mathcal{O}^{[\psi]} \circ \pi_* - [s]^{-1} \circ J^{P(\Delta^+)} \circ [s],$$

where $\pi_*^{-1} \circ \mathcal{O}^{[\psi]} \circ \pi_*$ denotes the horizontal lift of the optical structure $\mathcal{O}^{[\psi]}$ in $T_{\pi([\psi])}M$ to $T^H_{[\psi]}\mathcal{Z}(M)$. The almost optical structure $\mathcal{O}^-$ is given in $T_{[\psi]}\mathcal{Z}(M)$ by

$$\mathcal{O}^-_{[\psi]} = \pi_*^{-1} \circ \mathcal{O}^{[\psi]} \circ \pi_* + [s]^{-1} \circ J^{P(\Delta^+)} \circ [s].$$

Theorem 5.3.1. — *(comp. [Nur96] and [Lei98]) Let M_1^4 be an oriented Lorentzian 4-manifold and let $\mathcal{Z}(M_1^4)$ be its twistor space.*

(1) The almost optical structure $\mathcal{O}^+$ is integrable in $p \in \mathcal{Z}(M_1^4)$ if and only if p is a principal null direction in TM_1^4 (comp. [ON95]).

(2) The almost optical structure $\mathcal{O}^+$ on $\mathcal{Z}(M_1^4)$ is integrable if and only if M_1^4 is conformally flat.

(3) The almost optical structure $\mathcal{O}^-$ does not satisfy condition (A) and is never integrable.

We want to consider the conformal invariance of the twistor space $\mathcal{Z}(M_1^4)$ with its almost optical structures. For this, let $\tilde{g} = e^{2\rho}g$, where $\rho : M_1^4 \to \mathbb{R}$ is a smooth function, be a conformally equivalent metric to g on M_1^4. The twistor spaces $\mathcal{Z}(M_1^4, g)$ and $\mathcal{Z}(M_1^4, \tilde{g})$ are naturally identified, since the null directions in TM_1^4 are conformally invariant.

Theorem 5.3.2. — *Let (M_1^4, g) be an oriented Lorentzian 4-manifold and let $\tilde{g} = e^{2\rho}g$ be a conformally equivalent metric to g. The corresponding almost optical structures $\mathcal{O}^+$ and $\tilde{\mathcal{O}}^+$ on $\mathcal{Z}(M_1^4)$ are equal.*

Proof. The distribution $\mathcal{K}$ is the lightlike geodesic spray of the Lorentzian manifold M_1^4. The lightlike geodesic spray is conformally invariant. It follows that the optical flag $\mathcal{K} \subset \mathcal{L} \subset T\mathcal{Z}(M)$ is conformally invariant. It remains to show that the almost complex structures J^+ and $\tilde{J}^+$ on the screen space $\mathcal{L}/\mathcal{K}$ are equal. On the vertical part this is clear by definition. To prove it on the horizontal part, let

$$s = (s_1, s_2, s_3, s_4) : U \subset M \to SO(M, g)$$

be a local frame and denote by $\tilde{s} = e^{-\rho}s$ the conformally changed frame in $SO(M, \tilde{g})$. In the trivialization $U \times SO(1,3)/H$ of $\mathcal{Z}(M_1^4)$ induced by s, the horizontal lift s_i^H of s_i to $U \times SO(1,3)/H$ is given by

$$s_i^H = s_i + g(\nabla_{s_i}(s_1 + s_2), s_3)\frac{E_{13} - E_{23}}{2}H + g(\nabla_{s_i}(s_1 + s_2), s_4)\frac{E_{14} - E_{24}}{2}H.$$

For the conformal change of the Levi-Civita connection holds (comp. [Bes87])

$$\tilde{\nabla}_X Y = \nabla_X Y + d\rho(X)Y + d\rho(Y)X - g(X, Y)grad(\rho).$$

For the horizontal lift $\tilde{s}_i^H$ of $\tilde{s}_i$ with respect to the connection $\tilde{\nabla}$, we have

$$\begin{aligned} \tilde{s}_i^H = e^{-\rho} \cdot \big(s_i^H \quad &+ \quad [g((s_1 + s_2)(\rho) \cdot s_i, s_3) - g(s_3(\rho) \cdot s_i, s_1 + s_2)] \frac{E_{13} - E_{23}}{2} H \\ &+ \quad [g((s_1 + s_2)(\rho) \cdot s_i, s_4) - g(s_4(\rho) \cdot s_i, s_1 + s_2)] \frac{E_{14} - E_{24}}{2} H\big) \end{aligned}$$

and we see that

$$J^+(\tilde{s}_3^H + \mathcal{K}) = \tilde{s}_4^H + \mathcal{K} = \tilde{J}^+(\tilde{s}_3^H + \mathcal{K}), \qquad J^+(\tilde{s}_4^H + \mathcal{K}) = -\tilde{s}_3^H + \mathcal{K} = \tilde{J}^+(\tilde{s}_4^H + \mathcal{K}),$$

i.e. J^+ and $\tilde{J}^+$ are equal on the horizontal part of $\mathcal{L}/\mathcal{K}$. □

The proof also shows that the almost optical structure $\mathcal{O}^-$ is not conformally invariant.

We are now interested in the underlying CR-hypersurfaces of $(\mathcal{Z}(M_1^4), \mathcal{O}^+)$. Let $T \in \Gamma(TM)$, $g(T, T) = -1$, be a timelike unit vector field on M_1^4. The choice of such a field T is equivalent to a $SO(3)$-reduction $SO_T(M_1^4)$ of the frame bundle $SO(M_1^4)$ over M_1^4. The twistor bundle may then be written as

$$\mathcal{Z}(M) = SO_T(M) \times_{SO(3)} Q \cong SO_T(M) \times_{SO(3)} S^2.$$

The twistor fibre $\pi^{-1}(m)$ over a point $m \in M_1^4$ is identified via $T_m \in T_mM$ with the sphere $S^2(T_m^\perp) \subset T_m^\perp$, i.e. we can think of the twistor bundle as the bundle of 2-dimensional unit spheres, which are orthogonal to the timelike vector field $T \in \Gamma(TM)$.

Let us consider an oriented spacelike hypersurface P^3 in M_1^4, i.e. the restriction $g|_P$ is positive definite. There is a unique timelike unit normal field T on P such that (T, s_2, s_3, s_4) is positive oriented on M_1^4 if $(s_2, s_3, s_4) \in SO(P)$. The field T induces a $SO(3)$-reduction of the restricted frame bundle $SO(M)|_P$ and we have a natural identification

$$\Psi : \mathcal{Z}(M)|_P \cong SO(P) \times_{SO(3)} S^2$$

of the restricted twistor bundle and the unit sphere bundle over P^3. The restriction $\mathcal{Z}(M_1^4)|_P$ is a submanifold of codimension 1 in $\mathcal{Z}(M_1^4)$ and obviously, it holds

$$T(\mathcal{Z}(M)|_P) \cap \mathcal{K} = \{0\}.$$

It follows that the almost optical structure $\mathcal{O}^+$ induces an almost CR-structure $(\mathcal{H}, J^+)$ on $\mathcal{Z}(M_1^4)|_P$ (comp. 5.2).

Theorem 5.3.3. — *Let M_1^4 be an oriented Lorentzian 4-manifold and let P^3 be an oriented spacelike hypersurface of M_1^4.*

(1) The almost CR-structure $(\mathcal{H}, J^+)$ on the restricted twistor bundle $\mathcal{Z}(M_1^4)|_P$ is always integrable.

(2) The CR-structure $(\mathcal{H}, J^+)$ on $\mathcal{Z}(M_1^4)|_P$ and the natural CR-structure $(\mathcal{H}^{S^2(TP)}, J^{S^2(TP)})$ on the unit sphere bundle $S^2(TP)$ are equivalent under the identification Ψ if and only if P^3 is a totally umbilic hypersurface in M_1^4.

Proof. First, we compare the CR-structure $(\mathcal{H}^{S^2(TP)}, J^{S^2(TP)})$ and the almost CR-structure $(\mathcal{H}, J^+)$ on $\mathcal{Z}(M)|_P \cong S^2(TP)$. To every point $n \in P \subset M$, it exists an open neighborhood $U \subset M$ and a local frame

$$s = (s_1, s_2, s_3, s_4) : U \subset M \to SO(M)$$

such that $T = s_1$ on $U \cap P$ and $(s_2, s_3, s_4) : U \cap P \to SO(P)$ is an orthonormal frame. An arbitrary point $p \in \mathcal{Z}(M)|_{P\cap U}$ may be written as $[s \cdot A, \mathbb{R}(e_1 + e_2)]$ for some $A \in SO(3)$. We have

$$\Psi(p) = \Psi\left(\mathbb{R}(s_1 + (s \cdot A)_2)\right) = (s \cdot A)_2 \in S^2(TP)$$

and it holds

$$\pi_*(\mathcal{H}_p) = \left(\mathbb{R}T_{\pi(p)} \oplus \mathbb{R}(s_{\pi(p)} \cdot A)_2\right)^\perp = Span\{(s_{\pi(p)} \cdot A)_3, (s_{\pi(p)} \cdot A)_4\} = \pi_*\left(\mathcal{H}^{S^2(TP)}_{\Psi(p)}\right).$$

It follows $\Psi_*(\mathcal{H}_p) = \mathcal{H}^{S^2(TN)}_{\Psi(p)}$ for every $p \in \mathcal{Z}(M)|_P$. A local frame s as above induces trivializations of $\mathcal{Z}(M)|_P$ and $S^2(TP)$:

$$\begin{array}{ccc} \mathcal{Z}(M)|_{P\cap U} & \cong & S^2(TN)|_{P\cap U} \\ \updownarrow & & \updownarrow \\ (P\cap U) \times SO(1,3)/H & \cong & (P \cap U) \times SO(3)/SO(2). \end{array}$$

The horizontal lifts of s_i, $i=2,3,4$, with respect to the Levi-Civita connection on P are given in the trivialization of $S^2(TP)|_{U\cap P}$ by

$$t_i^H = s_i - g(\nabla_{s_i}s_2, s_3)E_{23}SO(2) - g(\nabla_{s_i}s_2, s_4)E_{24}SO(2).$$

Let s_i^H denote the horizontal lift of s_i, $i=2,3,4$, with respect to the Levi-Civita connection on M_1^4. It holds

$$t_3^H = \Psi_*(s_3^H) + g(\nabla_{s_3}s_1, s_3)E_{23}SO(2) + g(\nabla_{s_3}s_1, s_4)E_{24}SO(2),$$
$$t_4^H = \Psi_*(s_4^H) + g(\nabla_{s_3}s_1, s_4)E_{23}SO(2) + g(\nabla_{s_4}s_1, s_4)E_{24}SO(2)$$

and the almost complex structure J^+ on $\mathcal{H}$ is given on $\mathcal{H}^{S^2(TP)}$ in $TS^2(TP)$ via the identification Ψ by

$$J^+(t_3^H) = t_4^H - 2g(\nabla_{s_3}s_1, s_4)E_{23}SO(2) + (g(\nabla_{s_3}s_1, s_3) - g(\nabla_{s_4}s_1, s_4))E_{24}SO(2),$$
$$J^+(t_4^H) = -t_3^H + 2g(\nabla_{s_3}s_1, s_4)E_{24}SO(2) + (g(\nabla_{s_3}s_1, s_3) - g(\nabla_{s_4}s_1, s_4))E_{23}SO(2),$$
$$J^+(E_{23}SO(2)) = E_{24}SO(2), \quad \text{and} \quad J^+(E_{24}SO(2)) = -E_{23}SO(2).$$

The calculation of the corresponding Nijenhuis tensor shows that the almost CR-structure $(\mathcal{H}, J^+)$ on $\mathcal{Z}(M)|_P \cong S^2(TP)$ is integrable.

Moreover, the above formulas show that J^+ and $J^{S^2(TP)}$ on $\mathcal{H}^{S^2(TP)}$ are identical if and only if

$$g(\nabla_{s_3}s_1, s_3) = g(\nabla_{s_4}s_1, s_4) \quad \text{and} \quad g(\nabla_{s_3}s_1, s_4) = 0$$

for every orthonormal basis (s_2, s_3, s_4) on P. But this condition just means that the second fundamental form II of P^3 in M_1^4 looks like $II = g \otimes H$, where H is the mean curvature, i.e. the hypersurface P^3 in M_1^4 is totally umbilic. □

The almost CR-structure on the restriction $\mathcal{Z}(M)|_P$ of the twistor bundle to a spacelike oriented hypersurface P that is induced by the almost optical structure $\mathcal{O}^-$ on $\mathcal{Z}(M_1^4)$ is never integrable.

Remark. The twistor bundle $\mathcal{Z}(M_1^4)$ on M_1^4 can also be identified with the bundle $P(S^-)$ of negative projective spinors. The naturally induced almost optical structures on $P(S^-)$ are anti-holomorphic to $\mathcal{O}^+$ and $\mathcal{O}^-$ on $\mathcal{Z}(M_1^4) = P(S^+)$ (comp. [Lei98]) and the results on $P(S^+)$ in this section carry over to the bundle $P(S^-)$.

Example A. The twistor space $\mathcal{Z}(\mathbb{R}^{1,3})$ of the Minkowski space $\mathbb{R}^{1,3}$ with optical structure $\mathcal{O}^+$
Let $\mathbb{R}^{1,3}$ be the flat Minkowski space. The twistor space $\mathcal{Z}(\mathbb{R}^{1,3})$ of $\mathbb{R}^{1,3}$ is given as

$$\begin{array}{ccccccc} \mathbb{R}^{1,3} \times P(\Delta^+) & \cong & \mathbb{R}^{1,3} \times Q & \cong & \mathbb{R}^{1,3} \times SO(1,3)/H & \cong & \mathbb{R}^{1,3} \times S^2 \\ (x,[v]) & \leftrightarrow & (x, ker(\hat{v})) = (x, \mathbb{R}(e_1+s)) & \leftrightarrow & (x, AH) & \leftrightarrow & (x,s) \\ & & = (x, \mathbb{R}[A\cdot(e_1+e_2)]) & & & & \end{array}$$

Let e_i denote the standard basis in $T_x\mathbb{R}^{1,3} \cong \mathbb{R}^{1,3}$ and let $e_i^A = (e_1, \dots, e_4)\cdot A$ be the orthonormal basis transformed by $A \in SO(1,3)$. The optical structure $\mathcal{O}^+$ on $\mathcal{Z}(\mathbb{R}^{1,3})$ is given in $[\psi] =$

$(x, AH) \in \mathcal{Z}(\mathbb{R}^{1,3})$ by

$$\mathcal{K}_{[\psi]} = \mathbb{R}(e_1^A + e_2^A), \quad \mathcal{L}_{[\psi]} = Span\{e_1^A + e_2^A, e_3^A, e_4^A\}, \quad J^+|_{T^V\mathcal{Z}(\mathbb{R}^{1,3})} \cong -J^{P(\Delta^+)},$$
$$J^+(e_3^A + \mathcal{K}_{[\psi]}) = e_4^A + \mathcal{K}_{[\psi]} \quad \text{and} \quad J^+(e_4^A + \mathcal{K}_{[\psi]}) = -e_3^A + \mathcal{K}_{[\psi]}.$$

The twistor space $(\mathcal{Z}(\mathbb{R}^{1,3}), \mathcal{O}^+)$ is optically diffeomorphic to $\mathbb{R} \times (\mathbb{R}^3 \times S^2)$ with the optical structure

$$(T\mathbb{R}, T\mathbb{R} \oplus H^{S^2(T\mathbb{R}^3)}, J^{S^2(T\mathbb{R}^3)}),$$

where $(\mathbb{R}^3 \times S^2, H^{S^2(T\mathbb{R}^3)}, J^{S^2(T\mathbb{R}^3)})$ is the unit sphere bundle over the Euclidean space $\mathbb{R}^3$ with the natural induced CR-structure (comp. 5.2).

Example B. The twistor space $\mathcal{Z}(S^{1,3})$ of the pseudosphere $S^{1,3}$ with optical structure $\mathcal{O}^+$
The hypersurface

$$S^{1,3} := \{x \in \mathbb{R}^{1,4} : \ \langle x, x \rangle_{1,4} = 1\}$$

in $\mathbb{R}^{1,4}$ with the induced metric is an oriented Lorentzian 4-space of constant sectional curvature 1 and is called the 4-dimensional pseudosphere (comp. [ON83]). The pseudosphere $S^{1,3} \cong SO(1,4)/SO(1,3)$ is a symmetric space, where we choose the embedding

$$\iota : \quad SO(1,3) \hookrightarrow SO(1,4)$$
$$(A_{ij}) \mapsto \begin{pmatrix} A_{11} & 0 & A_{1i} \\ 0 & 1 & 0 \\ A_{i1} & 0 & A_{ij} \end{pmatrix}$$

for the isotropy group $SO(1,3)$. The twistor space $\mathcal{Z}(S^{1,3})$ is given as homogenous space by $SO(1,4)/H$ and $\mathcal{Z}(S^{1,3})$ is diffeomorphic to

$$\mathbb{R} \times S^2(TS^3) \cong \mathbb{R} \times SO(4)/SO(2) \cong \mathbb{R} \times S^3 \times S^2,$$

where $S^2(TS^3) \cong SO(4)/SO(2)$ is the unit sphere bundle over the sphere $S^3 \cong SO(4)/SO(3)$.

The optical structure $\mathcal{O}^+ = (\mathcal{K}, \mathcal{L}, J^+)$ on $\mathcal{Z}(S^{1,3})$ is $SO(1,4)$-equivariant and is given as follows. The Lie algebra $\mathfrak{o}(1,4) = Span\{E_{ij} : 1 \leq i < j \leq 5\}$ decomposes to

$$\mathfrak{o}(1,4) = \mathfrak{m} \oplus \mathfrak{h} \oplus \mathfrak{b},$$

where $\mathfrak{o}(1,3) = \mathfrak{m} \oplus \mathfrak{h}$ and $\mathfrak{b} = Span\{E_{12}, E_{23}, E_{24}, E_{25}\}$. The subspace $\mathfrak{b}$ is $Ad(H)$-invariant and there exists an $Ad(H)$-equivariant optical structure $(\mathfrak{k}, \mathfrak{l}, \underline{J})$ on $\mathfrak{b}$ defined by

$$\mathfrak{k} := \mathbb{R}(-E_{12} + E_{23}), \quad \mathfrak{l} := Span\{E_{24}, E_{25}\} \oplus \mathfrak{k}, \quad \text{and}$$
$$\underline{J}(E_{24} + \mathfrak{k}) = E_{25} + \mathfrak{k}, \quad \underline{J}(E_{25} + \mathfrak{k}) = -E_{24} + \mathfrak{k}.$$

The twistor bundle $\mathcal{Z}(S^{1,3})$ splits into a horizontal and a vertical part, where the horizontal bundle is given by

$$T^H SO(1,4)/H = SO(1,4) \times_{Ad(H)} \mathfrak{b}.$$

The distributions $\mathcal{K}$ and $\mathcal{L}$ of the optical structure $\mathcal{O}^+$ are defined by

$$\mathcal{K} = SO(1,4) \times_{Ad(H)} \mathfrak{k} \quad \text{and} \quad \mathcal{L} = (T^V SO(1,4)/H) \oplus (SO(1,4) \times_{Ad(H)} \mathfrak{l}).$$

The complex structure $J^+ : \mathcal{L}/\mathcal{K} \to \mathcal{L}/\mathcal{K}$ is given on the vertical part by $J^+|_{T^V SO(1,4)/H} \cong -J^{P(\Delta^+)}$ and on the horizontal part by

$$J^+([A, l + \mathfrak{k}]) = [A, \underline{J}(l + \mathfrak{k})], \quad A \in SO(1,4),\ l \in \mathfrak{l}.$$

The distribution $\mathcal{K}$ is regular on $\mathcal{Z}(S^{1,3})$ and the underlying CR-manifold of $(\mathcal{Z}(S^{1,3}), \mathcal{O}^+)$ is equivalent to the unit sphere bundle of S^3 with natural CR-structure. We describe this CR-structure on $SO(4)/SO(2) \cong S^2(TS^3)$. Let $\{E_{ij}\}$ be the standard basis in $\mathfrak{o}(4)$,

$$\begin{array}{cccc} \iota : & SO(2) & \hookrightarrow & SO(4) \\ & \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} & \mapsto & \exp tE_{34} \end{array}$$

the embedding of $SO(2)$ and let $\mathfrak{a} := Span\{E_{13}, E_{14}, E_{23}, E_{24}\}$ be a subspace of $\mathfrak{o}(4)$. The subspace $\mathfrak{a}$ is $Ad(SO(2))$-invariant and the linear map $\underline{J} : \mathfrak{a} \to \mathfrak{a}$ given by

$$\underline{J}(E_{13}) = E_{14}, \quad \underline{J}(E_{23}) = E_{24}, \quad \underline{J}(E_{14}) = -E_{13} \quad \text{and} \quad \underline{J}(E_{24}) = -E_{23}$$

is an $Ad(SO(2))$-equivariant complex structure on $\mathfrak{a}$. The canonical CR-structure on $SO(4)/SO(2)$ is given by

$$\mathcal{H}^{S^2(TS^3)} = SO(4) \times_{Ad(SO(2))} \mathfrak{a}, \qquad \begin{array}{cccc} J^{S^2(TS^3)} : & \mathcal{H}^{S^2(TS^3)} & \to & \mathcal{H}^{S^2(TS^3)} \\ & [A, a] & \mapsto & [A, \underline{J}a], \quad a \in \mathfrak{a} \end{array}.$$

Example C. The twistor space $\mathcal{Z}(H^{1,3})$ with optical structure $\mathcal{O}^+$

The hypersurface

$$H^{1,3} := \{x \in \mathbb{R}^{2,3} : \langle x, x \rangle_{2,3} = -1\}$$

in $\mathbb{R}^{2,3}$ with the induced metric is an oriented Lorentzian 4-manifold of constant sectional curvature -1 and is called the 4-dimensional pseudohyperbolic space. The pseudohyperbolic space $H^{1,3} \cong SO(2,3)/SO(1,3)$ is a symmetric space, where we choose the embedding

$$\begin{array}{cccc} \iota : & SO(1,3) & \hookrightarrow & SO(2,3). \\ & A & \mapsto & \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \end{array}$$

The twistor space $\mathcal{Z}(H^{1,3})$ is given as homogenous space by $SO(2,3)/H$ and is diffeomorphic to

$$S^1 \times S^2(TH^3) \cong S^1 \times SO^+(1,3)/SO(2) \cong S^1 \times \mathbb{R}^3 \times S^2,$$

where $S^2(TH^3) \cong SO^+(1,3)/SO(2)$ is the unit sphere bundle over the hyperbolic space $H^3 \cong SO^+(1,3)/SO(3)$.

The optical structure $\mathcal{O}^+ = (\mathcal{K}, \mathcal{L}, J^+)$ on $\mathcal{Z}(H^{1,3})$ is $SO(2,3)$-equivariant. We describe $\mathcal{O}^+$ as follows. The Lie algebra of $SO(2,3)$ splits into $\mathfrak{o}(2,3) = \mathfrak{m} \oplus \mathfrak{h} \oplus \mathfrak{b}$, where the subspace $\mathfrak{b} = Span\{E_{12}, E_{13}, E_{14}, E_{15}\}$ is $Ad(H)$-invariant. The subspaces

$$\mathfrak{k} := \mathbb{R}(E_{12} + E_{13}), \qquad \mathfrak{l} := Span\{E_{12} + E_{13,} E_{14}, E_{15}\}$$

of $\mathfrak{b}$ and the complex structure $\underline{J} : \mathfrak{l}/\mathfrak{k} \to \mathfrak{l}/\mathfrak{k}$ given by

$$\underline{J}(E_{14} + \mathfrak{k}) = E_{15} + \mathfrak{k} \quad \text{and} \quad \underline{J}(E_{15} + \mathfrak{k}) = -E_{14} + \mathfrak{k}$$

form together an $Ad(H)$-equivariant optical structure $(\mathfrak{k}, \mathfrak{l}, \underline{J})$ on $\mathfrak{b}$. The optical structure $\mathcal{O}^+$ on $\mathcal{Z}(H^{1,3}) = SO(2,3)/H$ is then given by

$$\mathcal{K} := SO(2,3) \times_{Ad(H)} \mathfrak{k}, \quad \mathcal{L} := (T^V SO(2,3)/H) \oplus (SO(2,3) \times_{Ad(H)} \mathfrak{l}),$$
$$J^+|_{T^V SO(2,3)/H} \cong -J^{P(\Delta^+)} \quad \text{and} \quad J^+([A, l + \mathfrak{k}]) = [A, \underline{J}(l + \mathfrak{k})],$$

where $[A, l + \mathfrak{k}] \in \mathcal{L} \cap T^H SO(2,3)/H$.

The locally induced CR-structures of $\mathcal{O}^+$ are equivalent to the natural CR-structure on the unit sphere bundle of H^3. We describe this CR-structure on $SO^+(1,3)/SO(2) \cong S^2(TH^3)$. Let

$$\begin{array}{rccc} \iota: & SO(2) & \hookrightarrow & SO^+(1,3) \\ & A & \mapsto & \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix} \end{array}$$

be the embedding of $SO(2)$ in $SO^+(1,3)$. The pair $(\mathfrak{a}, \underline{J})$ defined by

$$\mathfrak{a} := Span\{E_{13}, E_{14}, E_{23}, E_{24}\} \subset \mathfrak{o}(1,3),$$
$$\underline{J}(E_{13}) = E_{14}, \quad \underline{J}(E_{14}) = -E_{13}, \quad \underline{J}(E_{23}) = E_{24} \quad \text{and} \quad \underline{J}(E_{24}) = -E_{23}$$

is an $Ad(SO(2))$-equivariant CR-structure on $Span\{E_{13}, E_{14}, E_{23}, E_{24}, E_{34}\} \subset \mathfrak{o}(1,3)$. The CR-structure $(\mathcal{H}^{S^2(TH^3)}, J^{S^2(TH^3)})$ is then given by

$$\mathcal{H}^{S^2(TH^3)} = SO^+(1,3) \times_{Ad(SO(2))} \mathfrak{a}, \qquad \begin{array}{rccc} J^{S^2(TH^3)}: & \mathcal{H}^{S^2(TH^3)} & \to & \mathcal{H}^{S^2(TH^3)} \\ & [A,a] & \mapsto & [A, \underline{J}a], \quad a \in \mathfrak{a} \end{array}.$$

5.4 Optical-geometric interpretation of the twistor equation

We proved in [Lei98] an optical-geometric interpretation of the twistor equation on a Lorentzian spin 4-manifold. We briefly recall this result for the twistor equation here. Let $(S^+)^*$ denote the dual spinor bundle to S^+. The bundle $(S^+)^* \setminus 0$ without the zero section is in a natural way a $\mathbb{C}^*$-principal fibre bundle over the twistor space $\mathcal{Z}(M_1^4)$ with respect to the natural projection

$$\begin{array}{rccc} \pi: & (S^+)^* \setminus 0 & \to & \mathcal{Z}(M_1^4). \\ & \xi & \mapsto & ker_{\mathbb{C}}(\xi) \end{array}$$

To the principal fibre bundle $(S^+)^* \setminus 0$, we can associate the canonical line bundle $\mathcal{B}$ over the twistor space $\mathcal{Z}(M^4_1)$, which is defined by

$$\mathcal{B} := \big((S^+)^* \setminus 0\big) \times_{\mathbb{C}^*} \mathbb{C}.$$

The line bundle $\mathcal{B}$ admits a natural almost optical structure $\mathcal{O}^+_{\mathcal{B}}$ such that the projection to $(\mathcal{Z}(M^4_1), \mathcal{O}^+)$ is a holomorphic map. We observe that there exists a bijective correspondence between spinor fields $\Gamma(S^+)$ and linear sections $\Gamma_{Lin}(\mathcal{Z}(M^4_1); \mathcal{B})$ in the following way. Let $\psi \in \Gamma(S^+)$ be a spinor field. The function

$$\begin{array}{rccc} \hat{\psi}: & (S^+)^* \setminus 0 & \to & \mathbb{C} \\ & \xi & \mapsto & \xi(\psi) \end{array}$$

on $(S^+)^* \setminus 0$ is linear and, in particular, $\hat{\psi}$ is $\mathbb{C}^*$-equivariant. That means $\hat{\psi}$ corresponds to a linear section in the line bundle $\mathcal{B}$. A twistor spinor $\varphi \in \Gamma(S^+)$ interpreted as linear section in the bundle $\mathcal{B}$ can be characterized in the following way.

Theorem 5.4.1. *— (see [Lei98]) Let $\varphi \in \Gamma(S)$ be a spinor field on a Lorentzian spin 4-manifold M^4_1. Then φ is a twistor spinor ($\varphi \in ker(P)$) if and only if the corresponding linear section $\hat{\varphi} \in \Gamma_{Lin}(\mathcal{Z}(M^4_1); \mathcal{B})$ is holomorphic with respect to the almost optical structures $\mathcal{O}^+$ and $\mathcal{O}^+_{\mathcal{B}}$.*

Notice that different then in the Riemannian case the existence of a twistor spinor is not an integrability condition for the almost optical structure $\mathcal{O}^+$ on the twistor space $\mathcal{Z}(M^4_1)$.

5.5 Spacelike immersed surfaces

We study the second fundamental form of a spacelike immersed surface in a Lorentzian 4-space. Let (M^4_1, g) be an oriented Lorentzian 4-space and let $f : N^2 \to M^4_1$ be an isometric immersion of an oriented Riemannian 2-manifold (N^2, h). A local Darboux frame on N^2 is a map

$$s = (s_1, \dots, s_4) : U \subset N \to SO(M)|_N$$

such that (s_3, s_4) is locally a positive oriented orthonormal frame in TN. Let $TN^\perp$ denote the normal bundle of the immersion f in TM^4_1. The metric g on TM^4_1 induces a metric $h^\perp$ of signature $(1,1)$ on $TN^\perp$. The normal bundle splits into a positive and a negative line bundle of lightlike normal vectors, $TN^\perp = TN^\perp_+ \oplus TN^\perp_-$, where these line bundles over N are locally defined by

$$TN^\perp_+ = \mathbb{R}(s_1 + s_2) \quad \text{and} \quad TN^\perp_- = \mathbb{R}(s_1 - s_2)$$

with respect to a Darboux frame s. The second fundamental form of the immersion f is defined to be the normal part of the Levi-Civita connection ∇ on M^4_1

$$I\!I(X,Y) = \mathcal{N}\nabla_X Y, \qquad X, Y \in \Gamma(TN).$$

Let $h^\alpha_{ij} := \epsilon_\alpha g(\nabla_{s_i} s_j, s_\alpha)$, $\alpha \in \{1,2\}$, $i,j \in \{3,4\}$, denote the components of $I\!I$ with respect to a Darboux frame s. The mean curvature vector H of the isometric immersion f is given by

$$\begin{aligned} H &= \tfrac{1}{2} tr I\!I = \tfrac{1}{2} g^{ij} h^\alpha_{ij} s_\alpha = H_+ + H_- \\ &:= \tfrac{h^1_{33}+h^1_{44}+h^2_{33}+h^2_{44}}{4}(s_1 + s_2) + \tfrac{-h^1_{33}-h^1_{44}+h^2_{33}+h^2_{44}}{4}(s_2 - s_1), \end{aligned}$$

where $H_- \in TN^\perp_-$ and $H_+ \in TN^\perp_+$. Let $t := \frac{-is_3^*+s_4^*}{\sqrt{2}} \in \mathbb{C} \otimes T^*M$. It is

$$\begin{aligned} II &= h \otimes H + Re\left[\left(\tfrac{1}{2}(-h^1_{33} + h^1_{44} - h^2_{33} + h^2_{44}) + i(h^1_{34} + h^2_{34})\right) \cdot t \circ t\right] \otimes (s_1 + s_2) \\ &\quad + Re\left[\left(\tfrac{1}{2}(h^1_{33} - h^1_{44} - h^2_{33} + h^2_{44}) + i(-h^1_{34} + h^2_{34})\right) \cdot t \circ t\right] \otimes (s_2 - s_1) \\ &:= h \otimes H_+ + h \otimes H_- + L_+ + L_-, \end{aligned}$$

where $L_- \in Sym^2(TN) \otimes TN^\perp_-$ and $L_+ \in Sym^2(TN) \otimes TN^\perp_+$ are symmetric 2-forms with values in the normal null line bundles.

Definition 5.5.1. — *Let $f : N^2 \to M^4_1$ be a spacelike immersion of an oriented surface, i.e. the induced metric $h := f^*g$ is positive definite on N^2. We call the spacelike immersion f*

null-stationary $\Leftrightarrow H_- = 0$
null-umbilic $\Leftrightarrow L_- = 0$
isotropic $\Leftrightarrow H_- = L_- = 0$
stationary $\Leftrightarrow H = 0$
totally umbilic $\Leftrightarrow L_+ = L_- = 0$.

Remark.

(1) If we change the orientation on N^2, the null line bundles $TN^\perp_+$ and $TN^\perp_-$ switch. A null-stationary (null-umbilic) surface satisfies then $H_+ = 0$ ($L_+ = 0$).

(2) Let us consider a spacelike immersion $f : N^2 \to M^4_1$ and let $\tilde{g} := e^{2\rho}g$ be a conformally equivalent metric to g on M^4_1. Denote by $\tilde{II}$ the second fundamental form of the isometric immersion $f : (N, f^*\tilde{g}) \to (M, \tilde{g})$. The comparison of the covariant derivatives ∇ and $\tilde{\nabla}$ yields

$$\tilde{II} = II - g \otimes \mathcal{N} grad\rho = f^*\tilde{g} \otimes [e^{-2\rho} \cdot (H - \mathcal{N} grad\rho)] + L_+ + L_-.$$

This shows that the vanishing of the components L_+ and L_- is invariant under conformal change of the metric g on M^4_1. In particular, the property of a spacelike immersion to be totally umbilic is conformally invariant, whereas the stationary condition is not conformally invariant.

5.6 Holomorphic Gauss lifts of spacelike immersed surfaces

We define now the Gauss lift of a conformally spacelike immersed surface to the twistor space and relate geometric properties of the conformal immersion to the holomorphicity of its Gauss lift.

Let (N^2, J^N) be a Riemannian surface and let $f : N^2 \to M^4_1$ be a conformal spacelike immersion into an oriented Lorentzian 4-manifold, i.e. the complex structure J^N is orthogonal with respect to the induced metric f^*g on N. We have the positive line bundle $TN^\perp_+$ on N, where the space $T_nN^\perp_+$ is for every $n \in N$ a null direction in TM^4_1, i.e. an element of the twistor space $\mathcal{Z}(M^4_1)$. We define the Gauss lift γ_f of f into the twistor space $\mathcal{Z}(M^4_1)$ by

$$\begin{array}{rcl} \gamma_f : N & \to & \mathcal{Z}(M) \\ n & \mapsto & T_nN^\perp_+ \end{array}, \qquad \begin{array}{ccc} & & \mathcal{Z}(M) \\ & \overset{\gamma_f}{\nearrow} & \downarrow \\ f : N & \to & M \end{array}.$$

Definition 5.6.1. — *Let $f : N^2 \to M^4_1$ be a conformal spacelike immersion of a Riemannian surface into an oriented Lorentzian 4-space. The immersion f is called*

$$\begin{aligned}&\mathcal{O}^+\text{-holomorphic if } \gamma_f : (N, J^N) \to (\mathcal{Z}(M), \mathcal{O}^+) \text{ is holomorphic} \quad \text{or}\\ &\mathcal{O}^-\text{-holomorphic if } \gamma_f : (N, J^N) \to (\mathcal{Z}(M), \mathcal{O}^-) \text{ is holomorphic.}\end{aligned}$$

The conformal immersion $f : N^2 \to M^4_1$ is called horizontal if the lift γ_f to $\mathcal{Z}(M^4_1)$ is horizontal, i.e. $d\gamma_f(T_nN) \subset T^H_{f(n)}\mathcal{Z}(M)$ for all $n \in N$.

That a conformal spacelike immersion $f : N^2 \to M^4_1$ is horizontal means that the parallel displacement of a positive normal null vector along an arbitrary curve on N^2 in M^4_1 remains a positive normal null vector.

Let us consider the differential of the lift γ_f. The horizontal part $d\gamma^H_f$ is simply the horizontal lift of df. For the vertical part $d\gamma^V_f$ holds

$$d\gamma^V_f = \frac{-\omega_{13} - \omega_{23}}{2}(E_{13} - E_{23})H + \frac{-\omega_{14} - \omega_{24}}{2}(E_{14} - E_{24})H,$$

where $\omega_{ij} := g(\nabla^M s_i, s_j)$ are the connections forms of the Levi-Civita connection on M^4_1 with respect to a local Darboux frame s. From this formula we obtain by a direct calculation

Proposition 5.6.2. — *Let $f : N^2 \to M^4_1$ be a conformal immersion. The following relations hold:*

(1) f is $\mathcal{O}^-$-holomorphic $\Leftrightarrow$ $H_- = 0$ $\Leftrightarrow$ f is null-stationary

(2) f is $\mathcal{O}^+$-holomorphic $\Leftrightarrow$ $L_- = 0$ $\Leftrightarrow$ f is null-umbilic

(3) f is horizontal $\Leftrightarrow$ $H_- = L_- = 0 \Leftrightarrow$ f is isotropic

Remark. Surfaces in a Riemannian 4-manifold, that have a horizontal Gauss lift to the Riemannian twistor space, are called superminimal surfaces, since they satisfy a stronger curvature condition then minimality. In this sense we could say that isotropic surfaces in M^4_1 are super-null-stationary.

5.7 Twistoriel construction of spacelike surfaces

The Proposition 5.6.2 of the previous part relates geometric properties of a conformally and spacelike immersed surface to the holomorphicity of its Gauss lift. In the following the reconstruction of null-stationary and null-umbilic surfaces in Lorentzian 4-spaces from holomorphic curves in the Lorentzian twistor space is established. We give also a local description of null-umbilic surfaces in conformally flat Lorentzian 4-spaces. Moreover, we describe isotropic surfaces in the Lorentzian space forms $\mathbb{R}^{1,3}$, $S^{1,3}$ and $H^{1,3}$.

Theorem 5.7.1. — *Let (N^2, J^N) be a Riemannian surface and let M^4_1 be an oriented Lorentzian 4-manifold.*

(1) *If* $\gamma : (N^2, J^N) \to (\mathcal{Z}(M), \mathcal{O}^-)$ *is a holomorphic map into the twistor space over* M_1^4 *such that* $d\gamma \neq 0$ *on* N *and* γ *is non-vertical, that means* $d\gamma(T_nN) \not\subset T^V_{\gamma(n)}\mathcal{Z}(M)$ *for all* $n \in N$*, then the projection*

$$f := \pi \circ \gamma : N^2 \to M_1^4$$

is a conformal spacelike immersion with $H_- = 0$*. In particular, there is a bijective correspondence between* $\mathcal{O}^-$*-holomorphic, non-vertical curves in* $\mathcal{Z}(M_1^4)$ *and conformally immersed null-stationary surfaces in* M_1^4*.*

(2) *There is a bijective correspondence between* $\mathcal{O}^+$*-holomorphic, non-vertical curves in* $\mathcal{Z}(M_1^4)$ *and conformally immersed surfaces in* M_1^4 *with* $L_- = 0$*.*

Proof. We prove only the first statement, since the proof of the second statement works in the same way. Let $\gamma : (N, J^N) \to (\mathcal{Z}(M), \mathcal{O}^-)$ be a non-vertical, holomorphic curve with $d\gamma \neq 0$ on N. Obviously, the projected map $f := \pi \circ \gamma$ is an immersion. It is $d\gamma(TN) \subset \mathcal{L}$ and it follows

$$\pi_*(d\gamma(T_nN)) \subset L^{\gamma(n)} \qquad \text{for all } n \in N.$$

Moreover, it is $\hat{\pi} \circ d\gamma \circ J^N = J^- \circ \hat{\pi} \circ d\gamma$, which means that

$$pr_{\gamma(n)} \circ df \circ J^N_n = J^{\gamma(n)} \circ pr_{\gamma(n)} \circ df : T_nN \to L^{\gamma(n)}/K^{\gamma(n)} \qquad \text{for all } n \in N,$$

where $pr_{\gamma(n)} : L^{\gamma(n)} \to L^{\gamma(n)}/K^{\gamma(n)}$ is the natural projection. $J^{\gamma(n)}$ is orthogonal with respect to $g_{f(n)}$ and therefore the immersion $f : N^2 \to M_1^4$ is conformal. The Gauss lift γ_f of the immersion $f = \pi \circ \gamma$ is equal to the original map γ and we can apply Proposition 5.6.2 to obtain that f is a null-stationary immersion. □

Parts of this theorem can also be found in [Bob98]. Similar, it can be proved that there is a bijective correspondence between conformally immersed totally umbilic surfaces in a Riemannian 3-manifold P^3 and non-vertical, holomorphic curves in the unit sphere bundle $(S^2(TP), \mathcal{H}^{S^2(TP)}, J^{S^2(TP)})$.

One way to obtain holomorphic curves in an almost optical manifold is to construct holomorphic curves in a CR-hypersurface. For example, if $P^3 \subset M_1^4$ is an oriented spacelike hypersurface in a Lorentzian 4-manifold then the unit sphere bundle $S^2(TP)$ with CR-structure $(\mathcal{H}, J^+)$ is a CR-hypersurface in $(\mathcal{Z}(M), \mathcal{O}^+)$ (Theorem 5.3.3). A non-vertical, holomorphic curve in $(S^2(TP), \mathcal{H}, J^+)$ is then a non-vertical, holomorphic curve in $(\mathcal{Z}(M), \mathcal{O}^+)$ and projects to a null-umbilic immersed surface in M_1^4.

The existence of a holomorphic curve in an integrable optical manifold gives rise to a whole family of holomorphic curves. This is the idea to

Corollary 5.7.2. — *Let* M_1^4 *be an oriented conformally flat Lorentzian 4-manifold and let* $k \in \Gamma(\mathcal{K})$ *be a vector field in the distribution* $\mathcal{K}$ *on the twistor space* $\mathcal{Z}(M_1^4)$*. If* $f : N^2 \to M_1^4$ *is a conformally immersed Riemannian surface with* $L_- = 0$ *then the map*

$$f_t := \pi \circ \Phi^k_t \circ \gamma_f : N^2 \to M_1^4,$$

where Φ_t^k denotes the flow of the field k, is at least locally for small $t \in \mathbb{R}$ a conformally immersed surface in M_1^4 with $L_- = 0$.

Proof. If $f : N^2 \to M_1^4$ is a conformally immersed surface with $L_- = 0$ then the Gauss lift

$$\gamma_f : (N, J^N) \to (\mathcal{Z}(M), \mathcal{O}^+)$$

is holomorphic. Since M_1^4 is conformally flat, $\mathcal{O}^+$ on $\mathcal{Z}(M_1^4)$ is integrable and therefore

$$\Phi_t^k \circ \gamma_f : (N, J^N) \to (\mathcal{Z}(M), \mathcal{O}^+)$$

is locally and for small $t \in \mathbb{R}$ a non-vertical, holomorphic map (Proposition 5.2.4), which projects to a null-umbilic immersed surface in M_1^4. □

Corollary 5.7.2 may be interpreted as follows. Let N^2 be an oriented spacelike surface in M_1^4. The normal bundle $TN^\perp$ over N^2 in M_1^4 decomposes to the bundle of positive and negative normal null directions on the surface N^2 in M_1^4. Every positive normal null vector on N^2 defines a geodesic that intersects the surface N^2. We call such a geodesic through N^2 positive normal null. The distribution $\mathcal{K}$ on $\mathcal{Z}(M_1^4)$ is the lightlike geodesic spray of M_1^4. Hence, Corollary 5.7.2 says that a smooth bijective deformation of a null-umbilic surface N^2 along its positive normal null geodesics is also null-umbilic.

In case that the integral curves to $k \in \Gamma(\mathcal{K})$ on $\mathcal{Z}(M_1^4)$ are complete, the map $f_t = \pi \circ \Phi_t^k \circ \gamma_f$ in Corollary 5.7.2 is defined for every $t \in \mathbb{R}$. But it may happen that the holomorphic curve $\Phi_t^k \circ \gamma_f$ in $\mathcal{Z}(M_1^4)$ is not any more a non-vertical curve for some $t \in \mathbb{R}$, i.e. the deformation f_t is not in general an immersion for every $t \in \mathbb{R}$.

Corollary 5.7.3. *— Let M_1^4 be conformally flat. Every null-umbilic surface N^2 in M_1^4 is locally a deformation of a totally umbilic surface $\tilde{N}^2$ in M_1^4 along the positive normal null geodesics on $\tilde{N}^2$.*

Proof. Since M_1^4 is conformally flat, there exists a hypersphere segment $P_m^3 \subset M_1^4$ in every point $m \in M_1^4$. So let $n \in N^2$ be arbitrary and let P_n^3 be a hypersphere segment in n. Then an open neighborhood $U \subset N^2$ of $n \in N^2$ exists such that every positive normal null geodesic through U intersects the hypersphere P_n^3. Moreover, we can find a vector field k, which is tangential to the positive normal null geodesics through U, and a neighborhood $\tilde{U} \subset U$ of n such that $\tilde{N}^2 := \Phi_{t_1}^k(\tilde{U}) \subset P_n^3$ for some $t_1 > 0$ and such that the map $\Phi_t^k : \tilde{U} \to \Phi_t^k(\tilde{U})$ is a diffeomorphism for every $t \in [0, t_1]$. Since $\tilde{N}^2$ is null-umbilic, it follows that $\tilde{N}^2$ is totally umbilic in P_n^3. This implies that $\tilde{N}^2$ is even totally umbilic in M_1^4 and the open set $\tilde{U} \subset N^2$ is a deformation of $\tilde{N}^2$.□

There is no analogous version of Corollary 5.7.2 for null-stationary surfaces, sine the almost optical structure $\mathcal{O}^-$ on $\mathcal{Z}(M_1^4)$ does not satisfy condition (A). However, we will prove at the end of this section an analogous version of Corollary 5.7.2 and Corollary 5.7.3 for isotropic surfaces in case that M_1^4 has constant sectional curvature!

Example A. Null-umbilic surfaces and isotropic surfaces in the Minkowski space $\mathbb{R}^{1,3}$

Let us consider the flat Minkowski space $\mathbb{R}^{1,3}$. The closed totally umbilic spacelike hypersurfaces in $\mathbb{R}^{1,3}$ are isometric embeddings of the Euclidean or hyperbolic 3-space. We choose the totally geodesic embedding

$$\begin{array}{rcl} P^3 = \mathbb{R}^3 & \hookrightarrow & \mathbb{R}^{1,3}. \\ (y_1, y_2, y_3) & \mapsto & (0, y_1, y_2, y_3) \end{array}$$

The closed totally umbilic surfaces in the Euclidean 3-space $\mathbb{R}^3$ are isometric embeddings of the 2-sphere or the Euclidean plane.

The embedding

$$\begin{array}{rrcl} j: & S^2 & \hookrightarrow & \mathbb{R}^{1,3} \\ & (y_1, y_2, y_3) & \mapsto & (0, y_1, y_2, y_3), \quad y_1^2 + y_2^2 + y_3^2 = 1 \end{array}$$

is totally umbilic. The null vector field

$$\frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial x_3} + y_3 \frac{\partial}{\partial x_4} \in \Gamma(T\mathbb{R}^{1,3}|_{j(S^2)})$$

is positive normal to the surface $j(S^2) \subset \mathbb{R}^{1,3}$. The positive normal null geodesics intersecting the sphere $j(S^2)$ are given by

$$\gamma_{(a,b,c)}(t) = (t, ta, tb, tc), \quad t \in \mathbb{R},\ a^2 + b^2 + c^2 = 1.$$

Every deformation of the surface $j(S^2) \subset \mathbb{R}^{1,3}$ along the positive normal null geodesics has the form

$$\begin{array}{rrcl} j_\lambda: & S^2 & \hookrightarrow & \mathbb{R}^{1,3}, \\ & y = (y_1, y_2, y_3) & \mapsto & (\lambda, \lambda y_1, \lambda y_2, \lambda y_3) \end{array}$$

where $\lambda(y)$ is a smooth function on S^2. If $\lambda \neq 0$ is a positive or negative function then the map j_λ is a conformal embedding. By Corollary 5.7.2, it follows that those conformal embeddings of S^2 are null-umbilic surfaces in $\mathbb{R}^{1,3}$. Since the Gauss lift of the embedding j is not horizontal, the embedding j_λ has no horizontal Gauss lift for any function $\lambda \neq 0$ and the surface $j_\lambda(S^2)$ is nowhere isotropic in $\mathbb{R}^{1,3}$. In case that $\lambda(y) = 0$ for some $y \in S^2$, the map j_λ is not an immersion.

The isometric embedding

$$\begin{array}{rrcl} i: & (\mathbb{R}^2, \langle\cdot,\cdot\rangle) & \hookrightarrow & (\mathbb{R}^{1,3}, \langle\cdot,\cdot\rangle_{1,3}) \\ & (z_1, z_2) & \mapsto & (0, 0, z_1, z_2) \end{array}$$

of the Euclidean plan is totally geodesic. The positive normal null geodesics intersecting the plane $i(\mathbb{R}^2)$ are given by

$$\gamma_{(a,b)}(t) = (t, t, a, b), \quad t \in \mathbb{R}.$$

Every surface that is a deformation of $i(\mathbb{R}^2) \subset \mathbb{R}^{1,3}$ along the positive normal null geodesics on $i(\mathbb{R}^2)$ has the form

$$\begin{array}{rrcl} i_\lambda: & \mathbb{R}^2 & \rightarrow & \mathbb{R}^{1,3}, \\ & (z_1, z_2) & \mapsto & (\lambda(z_1, z_2), \lambda(z_1, z_2), z_1, z_2) \end{array}$$

where $\lambda(z_1, z_2)$ is a smooth function on $\mathbb{R}^2$. The embedding i_λ is isometric and $i_\lambda(\mathbb{R}^2)$ is a null-umbilic surface in $\mathbb{R}^{1,3}$ for every function λ on $\mathbb{R}^2$. By calculating the second fundamental form of i_λ into $\mathbb{R}^{1,3}$ we obtain

$$II = \langle \cdot, \cdot \rangle_2 \otimes \Big(\Delta^{\mathbb{R}^2} \lambda \Big) \cdot \frac{e_1 + e_2}{2} \; + \; L_+.$$

That means an embedding of the form i_λ is even isotropic.

Let $f : N^2 \to \mathbb{R}^{1,3}$ be an arbitrary conformal null-umbilic immersion of a connected Riemannian surface N^2. It follows from Corollary 5.7.3 that the immersion f has locally the form j_λ or i_λ up to an isometry of $\mathbb{R}^{1,3}$ (comp. [Elg96]). Furthermore, the subset of N^2, on which f is isotropic, is open and closed, i.e. the immersion f is globally isotropic or only null-umbilic. In case that the immersion f is only null-umbilic, it can be locally deformated along the normal null geodesics to a unique point in $\mathbb{R}^{1,3}$.

The fact that every complete lightlike geodesic in $\mathbb{R}^{1,3}$ intersects the hyperplane $P^3 \subset \mathbb{R}^{1,3}$ in a single point gives rise to a uniquely defined smooth map

$$\begin{array}{rccc} f' : & N^2 & \to & P^3 \cong \mathbb{R}^3, \\ & n & \mapsto & r_n \end{array}$$

where r_n is the intersection point of the positive normal null geodesic through $f(n)$ with the hyperplane $P^3 \subset \mathbb{R}^{1,3}$. If f is isotropic then the map f' is an immersion. In case that f is only null-umbilic, it may happen that f' has singularities. Since the inverse image of a singularity of f' is closed, it follows that f' has to be already constant on N^2. Because two null geodesics in $\mathbb{R}^{1,3}$ have at most one intersection point, we can find another totally geodesic hypersurface $P' \cong \mathbb{R}^3 \subset \mathbb{R}^{1,3}$ such that the unique deformation of f into $P' \cong \mathbb{R}^3$ is an immersion. Hence, we obtain in any case a smooth deformation $f' : N^2 \to \mathbb{R}^3$ of the null-umbilic immersion f, which is by Corollary 5.7.2 conformal and totally umbilic. Moreover, the immersion f is isotropic if and only if the induced immersion f' into $\mathbb{R}^3$ is totally geodesic. In this case the induced metric $f'^*(\langle \cdot, \cdot \rangle_{1,3})$ on N^2 is flat, whereas if f' is not totally geodesic the metric $f'^*(\langle \cdot, \cdot \rangle_{1,3})$ on N^2 has positive constant sectional curvature. We can conclude that in the class of compact Riemannian surfaces only the sphere S^2 admits a conformal null-umbilic immersion into $\mathbb{R}^{1,3}$. Such an immersion has the form $j_\lambda : S^2 \to \mathbb{R}^{1,3}$ up to an isometry of $\mathbb{R}^{1,3}$. Moreover, since the only complete flat Riemannian 2-manifold that admits a totally geodesic immersion into $\mathbb{R}^3$ is the plan $\mathbb{R}^2$, it follows that there exists no isotropic conformal immersion of a compact Riemannian surface into $\mathbb{R}^{1,3}$.

The property that a surface is null-umbilic is independent of the conformal class of the ambient Lorentzian 4-space. Hence, the immersions of the form i_λ and j_λ describe locally every null-umbilic surface in a conformally flat Lorentzian 4-space.

Example B. Null-umbilic surfaces and isotropic surfaces in the pseudosphere $S^{1,3}$

Let $S^{1,3}$ be the pseudoshere. The image of the isometric embedding

$$\begin{array}{rccc} \iota : & S^3 \subset \mathbb{R}^4 & \hookrightarrow & S^{1,3} \subset \mathbb{R}^{1,4} \\ & (y_1, y_2, y_3, y_4) & \mapsto & (0, y_1, y_2, y_3, y_4) \end{array}$$

is a totally geodesic spacelike hypersurface in $S^{1,3}$. Every closed totally umbilic surface in S^3 is up to an isometry of S^3 the image of a conformal embedding of the form

$$\begin{array}{rccl} j^c: & S^2 & \hookrightarrow & S^3 \subset \mathbb{R}^4. \\ & (y_1, y_2, y_3) & \mapsto & (c, \sqrt{1-c^2}\cdot y_1, \sqrt{1-c^2}\cdot y_2, \sqrt{1-c^2}\cdot y_3), \quad c \in \mathbb{R},\ |c| < 1 \end{array}$$

Only the surface $i(S^2) := j^0(S^2) \subset S^3$ is totally geodesic.

Every deformation of the embedding $j^c : S^2 \hookrightarrow S^{1,3}$ along its positive normal null geodesics has the form

$$\begin{array}{rccl} j^c_\lambda: & S^2 & \hookrightarrow & S^{1,3}, \\ & y = (y_1, y_2, y_3) & \mapsto & (\lambda, \lambda\sqrt{1-c^2} + c, (\sqrt{1-c^2} - \lambda c)y) \end{array}$$

where $\lambda : S^2 \to \mathbb{R}$ is an arbitrary smooth function. The map $j^c_\lambda : S^2 \to S^{1,3}$ is a conformal embedding if $c = 0$ or $\lambda \neq \frac{\sqrt{1-c^2}}{c}$ on S^2. Those embedded surfaces are null-umbilic in $S^{1,3}$. The embedding

$$\begin{array}{rccl} i_\lambda := j^0_\lambda: & S^2 & \hookrightarrow & S^{1,3} \subset \mathbb{R}^{1,4} \\ & y & \mapsto & (\lambda, \lambda, y), \quad \|y\| = 1 \end{array}$$

is isometric for every smooth function λ on S^2. Let

$$\begin{array}{rccc} sp: & S^{1,3}\backslash\{y \in S^{1,3} :\ y_3 = 1\} & \xrightarrow{\cong} & \mathbb{R}^{1,3}\backslash H^3_0 \\ & y = (y_1, y_2, y_3, y_4, y_5) & \mapsto & \frac{1}{1-y_3}(y_1, y_2, y_4, y_5) \\ & \frac{2}{1+\|x\|^2}(x_1, x_2, \frac{\|x\|^2-1}{2}, x_3, x_4) & \leftarrow & x = (x_1, x_2, x_3, x_4) \end{array}$$

denote the stereographic projection in $(0,0,1,0,0) \in S^{1,3}$. We set $\rho = \ln\frac{2}{1+\|x\|^2}$ and $\tilde{\lambda} := e^{-\rho}\cdot\lambda\circ sp$. It holds

$$\begin{array}{rccl} sp^{-1}\circ i_{\tilde{\lambda}}: & \mathbb{R}^2 & \hookrightarrow & S^{1,3}, \\ & (z_1, z_2) & \mapsto & \frac{2}{1+z_1^2+z_2^2}(\tilde{\lambda}, \tilde{\lambda}, \frac{z_1^2+z_2^2-1}{2}, z_1, z_2) = (\lambda, \lambda, y_3, y_4, y_5), \end{array}$$

where $y_3^2 + y_4^2 + y_5^2 = 1$, i.e. the stereographic projection sp maps the surface $i_\lambda(S^2)$ in $S^{1,3}$ to the surface $i_{\tilde{\lambda}}(\mathbb{R}^2)$ in $\mathbb{R}^{1,3}$. We can use this to calculate the second fundamental form $\tilde{I\!I}$ of the isometric embedding $i_\lambda : S^2 \hookrightarrow S^{1,3}$. It holds

$$\begin{aligned} \tilde{I\!I} &= g^{S^2} \otimes [e^{-2\rho}(H - \mathcal{N}grad(\rho))] \ +\ L_+ \\ &= g^{S^2} \otimes [\Delta^{S^2}\lambda + 2\lambda]\cdot\frac{1}{2}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right) \ +\ L_+. \end{aligned}$$

This proves that $i_\lambda : S^2 \hookrightarrow S^{1,3}$ is an isotropic embedding into $S^{1,3}$. The embeddings of the form j^c_λ, $c \neq 0$, are never isotropic. If $\hat{\lambda}$ is a spherical function to the eigenvalue -2 of the Laplace operator Δ^{S^2} on S^2, the surface $i_{\hat{\lambda}}(S^2) \subset S^{1,3}$ is stationary and null-umbilic.

Let $f : N^2 \to S^{1,3}$ be an arbitrary conformal null-umbilic immersion. Again, the immersion f can be deformated along its normal null geodesics to a conformal totally umbilic immersion

$f' : N^2 \to S^3$. The immersion f is isotropic if and only if the corresponding immersion f' is totally geodesic. The induced metric $f'^*(g^{S^{1,3}})$ on N^2 has positive constant sectional curvature. We can conclude that in the class of compact Riemannian surfaces only the sphere S^2 admits a conformal null-umbilic immersion into $S^{1,3}$. The sphere S^2 admits even isotropic embeddings into $S^{1,3}$.

Example C. Null-umbilic surfaces and isotropic surfaces in the pseudohyperbolic space $H^{1,3}$
Let us consider the pseudohyperbolic 4-space

$$H^{1,3} := \{x \in \mathbb{R}^{2,3} : \ \langle x, x\rangle_{2,3} = -1\} \subset \mathbb{R}^{2,3}.$$

The isometric embedding

$$\begin{array}{llll} \iota : & H^3 \subset \mathbb{R}^{1,3} & \hookrightarrow & H^{1,3} \subset \mathbb{R}^{2,3} \\ & (y_1, y_2, y_3, y_4) & \mapsto & (y_1, 0, y_2, y_3, y_4), \quad y_1 > 0, \ \|y\| = -1 \end{array}$$

is a spacelike totally geodesic hypersurface in $H^{1,3}$. Up to an isometry of H^3, every closed totally umbilic surface in H^3 has one of the following forms:

$$\begin{array}{llll} j : & S^2 & \hookrightarrow & H^3, \\ & (z_1, z_2, z_3) & \mapsto & (c, \sqrt{c^2-1}\cdot z_1, \sqrt{c^2-1}\cdot z_2, \sqrt{c^2-1}\cdot z_3), \quad c > 1 \\ i : & H^2 & \hookrightarrow & H^3. \\ & (z_1, z_2, z_3) & \mapsto & (z_1, 0, z_2, z_3) \end{array}$$

The deformations of the surfaces $j(S^2) \subset H^{1,3}$ and $i(H^2) \subset H^{1,3}$ along its positive normal null geodesics look as follows:

$$\begin{array}{llll} j_\lambda : & S^2 & \hookrightarrow & H^{1,3}, \\ & z = (z_1, z_2, z_3) & \mapsto & (\lambda, \lambda\sqrt{c^2-1} + c, (\sqrt{c^2-1} + \lambda c)\cdot z), \quad c > 1 \\ i_\lambda : & H^2 & \hookrightarrow & H^{1,3}, \\ & (z_1, z_2, z_3) & \mapsto & (z_1, \lambda, \lambda, z_2, z_3) \end{array}$$

where $\lambda(z)$ is a smooth function on S^2 resp. H^2. The map j_λ is a conformal null-umbilic embedding for every function $\lambda \neq -\frac{\sqrt{c^2-1}}{c}$ on S^2 and the map i_λ is an isometric null-umbilic embedding for every function λ on H^2. For the second fundamental form of the isometric embedding i_λ, we calculate

$$\tilde{II} = g^{H^2} \otimes \left[\Delta^{H^2}\lambda - 2\lambda\right] \cdot \frac{1}{2}\left(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}\right) \ + \ L_+,$$

i.e. the embeddings of the form i_λ are isotropic, whereas the embeddings j_λ are only null-umbilic. The only compact Riemannian surface that admits a conformal null-umbilic immersion into $H^{1,3}$ is the sphere S^2 and there is no isotropic immersion of a compact Riemannian surface into $H^{1,3}$.

The discussion of the isotropic surfaces in the space forms $\mathbb{R}^{1,3}$, $S^{1,3}$ and $H^{1,3}$ proves the following analogous result to Corollary 5.7.2 and 5.7.3:

Theorem 5.7.4. — *Let $M_1^4(k)$ be an oriented Lorentzian 4-manifold with constant sectional curvature k.*

(1) Every smooth deformation of an isotropic surface N^2 in $M_1^4(k)$ along the positive normal null geodesics of N^2 remains isotropic.

(2) Every isotropic surface in $M_1^4(k)$ is locally a smooth deformation of a totally geodesic surface N^2 in $M_1^4(k)$ along the positive normal null geodesics on N^2.

Remember that the property of a surface to be null-stationary is not a conformal invariant of the ambient Lorentzian manifold. In particular, a stereographic projection does not map every null-stationary surface again into such one. It is remarkable that Theorem 5.7.4 holds, although the almost optical structure $\mathcal{O}^-$ on $\mathcal{Z}(M_1^4(k))$ is not integrable. Theorem 5.7.4 is not true for arbitrary conformally flat Lorentzian manifolds.

References

[AFOHS98] B.S. Acharya, J.M. Figueroa-O'Farrill, C.M. Hull, and B. Spence. Branes at conical singularities and holography. *Adv. Theor. Math. Phys.*, 2:1249–1286, 1998.

[AHS78] M.F. Atiyah, N. Hitchin, and I.M. Singer. Self-duality in four-dimensional Riemannian geometry. *Proc. R. Soc. London*, A 362:425–461, 1978.

[Ale72] D. Alekseevsky. Groups of conformal transformations of Riemannian spaces. *Mat. Sbornik 89(131) 1972 (in Russian), English translation Math. USSR Sbornik*, 18:285–301, 1972.

[Bär93] Ch. Bär. Real Killing spinors and holonomy. *Comm. Math. Phys.*, 154:509–521, 1993.

[Bau81] H. Baum. *Spin-Strukturen und Dirac-Operatoren über pseudo-Riemannschen Mannigfaltigkeiten*, volume 41 of *Teubner-Texte zur Mathematik*. Teubner-Verlag, Leipzig, 1981.

[Bau89] H. Baum. Complete Riemannian manifolds with imaginary Killing spinors. *Ann. Glob. Anal. Geom.*, 7:205–226, 1989.

[Bau99] H. Baum. Lorentzian twistor spinors and CR-geometry. *J. Diff. Geom. and its Appl.*, 11(1):69–96, 1999.

[Bau00a] H. Baum. Twistor and Killing spinors on Lorentzian manifolds, in: Global analysis and harmonic analysis, eds. J.P. Bourguignon, T. Branson and O. Hijazi, Seminaires & Congres 4, SMF 2000, p. 35-52. *Seminaires et Congres, French Math. Soc.*, 4:35–52, 2000.

[Bau00b] H. Baum. Twistor spinors on Lorentzian symmetric spaces. *Journ. Geom. Phys.*, 34:270–286, 2000.

[BEE96] J. Beem, P. Ehrlich, and K. Easley. *Global Lorentzian Geometry*. Marcel Dekker. Inc., New York and Basel, 1996.

[Bes87] A.L. Besse. *Einstein Manifolds*, volume 10 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer, 1987.

[BFGK91] H. Baum, Th. Friedrich, R. Grunewald, and I. Kath. *Twistors and Killing Spinors on Riemannian Manifolds*, volume 124 of *Teubner-Texte zur Mathematik*. Teubner-Verlag, Stuttgart/Leipzig, 1991.

[BK99] H. Baum and I. Kath. Parallel spinors and holonomy groups on pseudo-Riemannian spin manifolds. *Ann. Glob. Anal. Geom.*, 17:1–17, 1999.

[Bob98] M. Bobienski. Twistorowa konstrukcja odwzorowan harmonicznych i jej odpowiedniki w sygnaturach pseudoeuklidesowych. Master thesis (Pohlish), University of Warsaw, 1998.

[Boh98] Ch. Bohle. Killing and twistor spinors on Lorentzian manifolds. Diplomarbeit, Freie Universität Berlin, 1998.

[Boh00] Ch. Bohle. Killing spinors on Lorentzian manifolds. *Journ. Geom. Phys.*, 2000.

[Bry82] R.L. Bryant. Conformal and minimal immersions of compact surfaces into the 4-sphere. *Journ. Diff. Geom.*, 17:455–473, 1982.

[Bry00] R.L. Bryant. Pseudo–Riemannian metrics with parallel spinor fields and vanishing Ricci tensor, in: Global Analysis and Harmonic Analysis, eds. J.P. Bourguignon, T. Branson, O. Hijazi. *Seminaires et Congres, French Math. Soc.*, 4:53–94, 2000.

[CSS97] A. Cap, J. Slovak, and V. Soucek. Invariant operators on manifolds with almost hermitian structures, I & II. No. 1, p. 33-69 & N. 2, p. 203-220. *Acta Math. Univ. Comenianae*, 66, 1997.

[DNP86] M.J. Duff, B. Nilsson, and C.N. Pope. Kaluza-Klein supergravity. *Phys. Rep.*, 130:1–142, 1986.

[EK62] J. Ehlers and W. Kundt. *Gravitation, an introduction to current research.* Wiley, New York, 1962.

[Elg96] R. Elghanmi. Spacelike surfaces in Lorentzian manifolds. *Diff. Geom. and Appl.*, 6:199–218, 1996.

[ES85] J. Eells and S. Samalson. Twistorial construction of harmonic maps of surfaces into four-spaces. *Ann. Scuola Norm. Sup. Pisa*, 12:589–640, 1985.

[Fef76] C. Fefferman. Monge–Ampère equations, the Bergman kernel, and geometry of pseudoconvex domains. *Ann. Math.*, 103:395–416, 1976.

[Fri75] F.G. Friedlander. *The Wave Equation on a curved Space Time.* Camb. Univ. Press, 1975.

[Fri77] H. Friedrich. Twistor connection and normal conformal cartan connection. *General Relativity and Gravitation*, 8:303–312, 1977.

[Fri84] Th. Friedrich. On surfaces in four-spaces. *Ann. Glob. Analysis and Geometry Vol. 2, No. 3*, pages 257–287, 1984.

[Fri89] Th. Friedrich. On the conformal relation between twistors and Killing spinors. *Suppl. Rend. Circ. Math. Palermo, Ser.II, No.22*, pages 59–75, 1989.

[Fri97] Th. Friedrich. On superminimal surfaces. *Archivum Mathematicum Vol. 33*, pages 41–56, 1997.

[Gra87] C.R. Graham. On Sparling's characterization of Fefferman metrics. *Amer. J. Math.*, 109:853–874, 1987.

[Gün88] P. Günther. *Huygens' Principle and Hyperbolic Equations*, volume 5 of *Persp. in Math.* Acad. Press Inc., Boston, 1988.

[Hab90] K. Habermann. The twistor equation on Riemannian manifolds. *J. Geom. Phys.*, 7:469–488, 1990.

[Hal90] G.S. Hall. Conformal symmetries and fixed points in space-time. *J. Math. Phys.*, 31(5):1198–1207, 1990.

[HCB97] G.S. Hall, M.S. Capocci, and R. Beig. Zeros of conformal vector fields. *Class. Quant. Grav.*, 14:L49–L52, 1997.

[Hij86] O. Hijazi. A conformal lower bound for the smallest eigenvalue of the Dirac operator and Killing spinors. *Comm. Math. Phys.*, 104:151–162, 1986.

[HPSW72] L.P. Hughston, R. Penrose, P. Sommers, and M. Walker. On an auartic first integral for the charged particle orbits in the charged kerr solution. *Comm. Math. Phys.*, 27:303–308, 1972.

[HS91] G.S. Hall and J.D. Steele. Conformal vector fields in general relativity. *J. Math. Phys.*, 37(2):1847–1853, 1991.

[Jac90] H. Jacobowitz. *An introduction to CR structures*, volume 32 of *Mathematical Surveys and Monographs*. Amer. Math. Soc., 1990.

[JR90] G.R. Jensen and M. Rigoli. Neutral surfaces in neutral 4-spaces. *Le Matematiche*, XLV(II):407–443, 1990.

[Kat99] I. Kath. Killing Spinors on Pseudo-Riemannian Manifolds. Habilitationsschrift Humboldt-Universität Berlin, 1999.

[Kob72] S. Kobayashi. *Transformation Groups in Differential Geometry.* Springer-Verlag Berlin, Heidelberg, 1972.

[KR94] W. Kühnel and H-B. Rademacher. Twistor spinors with zeros. *Int. J. Math.*, 5:887–895, 1994.

[KR95] W. Kühnel and H-B. Rademacher. Essential conformal fields in pseudo-Riemannian geometry. *J. Math. Pures Appl.*, 74:453–481, 1995.

[KR96] W. Kühnel and H-B. Rademacher. Twistor spinors and gravitational instantons. *Lett. Math. Phys.*, 38:411–419, 1996.

[KR97a] W. Kühnel and H-B. Rademacher. Conformal completion of $U(n)$-invariant Ricci-flat Kählerian metrics at infinity. *Zeitschr. Anal. Anwend.*, 16:113–117, 1997.

[KR97b] W. Kühnel and H-B. Rademacher. Essential conformal fields in pseudo-Riemannian geometry II. *J. Math. Sci. Univ. Tokyo*, 4:649–662, 1997.

[KR97c] W. Kühnel and H-B. Rademacher. Twistor spinors on conformally flat manifolds. *Illinois J. Math.*, 41:495–503, 1997.

[Lei98] F. Leitner. The twistor space of a Lorentzian manifold. Sfb 288 preprint No. 314, 1998.

[Lew91] J. Lewandowski. Twistor equation in a curved spacetime. *Class. Quant. Grav.*, 8:11–17, 1991.

[LF71] J. Lelong-Ferrand. Transformations conformes et quasi-conformes des varietes riemanniennes. *Acad. Roy. Belg. Sci. Mem. Coll.*, 39, 1971.

[Lic88] A. Lichnerowicz. Killing spinors, twistor spinors and Hijazi inequality. *J. Geom. Phys.*, 5:2–18, 1988.

[LM89] H.B. Lawson and M-L. Michelsohn. *Spin Geometry.* Princton Univ. Press, 1989.

[NP00] P. Nuowski and J. Plebanski. Non-vacuum twisting type N metrics. SFB 288 preprint Nr. 483, 2000.

[Nur96] Pawel Nurowski. Optical geometries and related structures. *J. Geom. Phys.*, 18:335–348, 1996.

[NW84] P. Nieuwenhuizen and N.P. Warner. Integrability conditions for Killing spinors. *Comm. Math. Phys.*, 93:277–284, 1984.

[Oba70] M. Obata. Conformal transformations of Riemannian manifolds. *J. Diff. Geom.*, 4:311–333, 1970.

[Och70] T. Ochiai. Geometry associated with semisimple flat homogenous spaces. *Trans. Amer. Math. Soc.*, 152:159–193, 1970.

[ON83] Barrett O' Neill. *Semi-Riemannian Geometry.* Academic Press, 1983.

[ON95] Barett O' Neill. *The geometry of Kerr black holes.* A K Peters Wellesley, Mass., 1995.

[Pen67] R. Penrose. Twistor algebra. *J. Math. Phys.*, 8:345–366, 1967.

[Pen72] R. Penrose. *Techniques of Differential Toplogy in Relativity*, volume 7 of *Regional Conference Series in Applied Math.* SIAM, Philadelphia, 1972.

[PM72] R. Penrose and M.A. MacCallum. *Phys. Rep.*, 6C:241, 1972.

[PR86] R. Penrose and W. Rindler. *Spinors and Space-time II.* Combr. Univ. Press, 1986.

[Sch74] R. Schimming. Riemannsche Räume mit ebenfrontiger und ebener Symmetrie. *Mathematische Nachrichten*, 59:129–162, 1974.

[Sch97] M. Schottenloher. *A mathematical introduction to conformal field theory.* Springer-Verlag, Berlin-Heidelberg, 1997.

[Sha96] R.W. Sharp. *Differential Geometry*, volume 166 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1996.

[Spa85] G.A. Sparling. Twistor theory and the characterization of Fefferman's conformal structures. Preprint Univ. Pittsburg, 1985.

[Tan75] N. Tanaka. *A differential geometric study on strongly pseudoconvex manifolds.* Kinokuniya Company Ltd., Tokyo, 1975.

[Tra85] A Trautman. Optical structures in relativistic theories. *Soc. Math. France, Asterisque, hors serie*, pages 401–420, 1985.

[Wan89] McKenzy Y. Wang. Parallel spinors and parallel forms. *Ann. Glob. Anal. and Geom.*, 7:59–68, 1989.

[Web78] S. Webster. Pseudohermitian structures on a real hypersurface. *J. Diff. Geom.*, 13:25–41, 1978.

[Yos76] Y. Yoshimatsu. On a theorem of Alekseevskii concerning conformal transformations. *J. Math. Soc. Japan*, 28:278–289, 1976.

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Erklärung

Hiermit versichere ich, daß ich die vorliegende Dissertation selbständig und nur unter Verwendung der angegebenen Hilfsmittel angefertigt habe.

Ich erkläre, daß ich die Arbeit erstmalig und nur an der Humboldt-Universität zu Berlin eingereicht habe und mich nicht anderenorts um einen Doktorgrad beworben habe. In dem angestrebten Promotionsfach besitze ich keinen Doktorgrad.

Der Inhalt der dem Verfahren zugrunde liegenden Promotionsordnung ist mir bekannt.

Berlin, den 16. Juli 2001