Nilpotent Class Field Theory

Dissertation

zur Erlangung des akademischen Grades
doctor rerum naturalium (Dr. rer.nat.)
im Fach Mathematik

eingereicht an der

Mathematisch-Naturwissenschaftlichen Fakultät II
der Humboldt-Universität zu Berlin

von

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Eingereicht am: 03.06.1998
Tag der mündlichen Prüfung: 13.01.1999

ABSTRACT

Let G be a Galois group of some Galois extension F/K of a global algebraic field K. Consider the projective limit

$$C_{F/K} := \varprojlim_{L} C(K)/N_{L/K}C(L)$$

over finite subextensions L/K of F/K, where $C(L)$ is the idele class group of L. One has an Artin isomorphism $\varphi : C_{F/K} \to G^{\mathrm{ab}}$, where G^{ab} is the Galois group of the maximal abelian subextension of F/K. This map can be uniquely lifted to a map of graded Lie algebras, $\varphi : \mathcal{L}(C_{F/K}) \to L(G)$, from the free graded Lie algebra generated by the module $C_{F/K}$ to the graded Lie algebra of the lower central filtration of G. The obtained map is always surjective, thus the determination of its kernel gives the complete description of the lower central series filtration of the Galois group G in terms of the inner structure of the ground field K.

In the work [H. Koch, S. Kukkuk, J. Labute, *Nilpotent local class field theory*, Acta Arith. **83** (1998), no. 1, 45–64], the analogous kernel for a local algebraic field K and the Galois group of its maximal nilpotent extension K^{nil} was given. In this thesis, this kernel is studied for the maximal p-extension $K_S(p)$ over a global algebraic field K unramified at places outside of a specified set S. The essential difference from the work above is the following: the complete abstract structure of the Galois group $G_{K,S}(p)$ of the extension $K_S(p)/K$ is not known and in general this group has more then one relation in its minimal free representation.

The thesis contains sufficient conditions for the map φ in degree 2 to be isomorphism. Under these conditions, the kernel in degree 3 is given. The main result is the description of the kernel of φ on the whole Lie algebra under some assumptions. As an application, an example with complete structure of $L(G_{\mathbb{Q},S}(p))$ for some 3-element sets S is given.

ZUSAMMENFASSUNG

Sei K ein globaler, algebraischer Körper, F eine Galoische Erweiterung von K und G eine Galoische Gruppe dieser Erweiterung. Sei

$$C_{F/K} := \varprojlim_{L} C(K)/N_{L/K}C(L)$$

ein projektiver Limes über endlichen Teilerweiterungen L/K von F/K, wobei $C(L)$ die Idele Klassengruppe von L ist. Dann existiert der Artin Isomorphismus $\varphi : C_{F/K} \to G^{\mathrm{ab}}$, wobei G^{ab} eine Galoische Gruppe der maximalen abelschen Teilerweiterung von F/K ist.

Diese Abbildung kann eindeutig auf die graduierten Lieschen Algebren erweitert werden, $\varphi : \mathcal{L}(C_{F/K}) \to L(G)$, wobei $\mathcal{L}(C_{F/K})$ die von dem Modul $C_{F/K}$ erzeugte freie Liesche Algerba und $L(G)$ die graduierte Liesche Algebra der zentralen Filtrierung von G ist. Die induzierte Abbildung ist immer surjektiv. Also ergibt die Kenntnis ihres Kernes die vollständige Beschreibung der zentralen Filtrierung der Galoischen Gruppe G aufgrund der inneren Struktur des Grundkörpers K.

In der Arbeit [H. Koch, S. Kukkuk, J. Labute, *Nilpotent local class field theory*, Acta Arith. **83** (1998), no. 1, 45–64], wurde ein ähnlicher Kern für einen lokalen algebraischen Körper K und die Galoische Gruppe ihrer maximalen nilpotenten Erweiterung K^{nil} beschrieben. In der vorliegenden Dissertation wurde der Kern für die maximale p-Erweiterung $K_S(p)$ von eines globalen, algebraischen Körpers K, welcher unverzweigt außerhalb einer gegebenen Menge S von Primstellen ist, untersucht. Der wesentliche Unterschied zu der obengenannten Arbeit besteht darin, daßdie vollständige abstrakte Struktur der Galoischen Gruppe $G_{K,S}(p)$ der Erweiterung $K_S(p)/K$ nicht bekannt ist, und im allgemeinen diese Gruppe mehr als eine Relation in ihrer minimalen freien Repräsentation hat.

Diese Dissertation enthält hinreichende Bedingungen, bei denen die Abbildung φ im Grad 2 ein Isomorphismus ist. Der Kern in Grad 3 wird unter denselben Voraussetzungen bestimmt. Der Hautsatz enthält die Beschreibung des Kernes von φ auf der ganzen Lieschen Algebra unter zusätlichen Voraussetzungen. Als Beispiel wird die vollständige Struktur von $L(G_{\mathbb{Q},S}(p))$ für 3-elementige Mengen S angegeben.

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INTRODUCTION

The notion "Class Field Theory" refers to the description of the Galois group of the maximal abelian extension of a given field in connection with its algebraic extensions, given only in the terms of the inner structure of that field. In other words, for "Class Field Theory", one has, using only the inner structure of the given field, to construct a group, which is isomorphic to the Galois group of the maximal abelian extension of that field, and give a *canonical* isomorphism between these groups. Arithmetical properties of such an isomorphism are of special interest.

The most deeply investigated cases are the classical ones: global algebraic fields, which are finite extensions of the field of rational numbers $\mathbb{Q}$ and finite extensions of the fields $\mathbb{F}_p(t)$ of rational functions in one variable over finite fields, and local algebraic fields, which are the completions of global algebraic fields under some valuations. For these cases the theory is complete. Of course, there are other interesting types of fields, such as n-dimensional local fields (the subclass of local fields with imperfect residue field), investigated by K. Kato and A. Parshin.

For the classical case, the next step is considering Nilpotent Class Field Theory, that is, the description of the Galois group G of the maximal nilpotent extension K^{nil} of a given field K. Obviously, one can not define a *canonical* isomorphism from the group, defined only by the inner structure of the field K, to the Galois group G, as one can take the conjugated isomorphism (which corresponds to some automorphism of K^{nil} over K). Thus, one has either to use some additional information for "rigidity", like in [**KdS**], where some continuation of a Frobenius automorphism on the extension considered was fixed (for metabelian extensions in case of local algebraic fields), or to investigate not the Galois group itself, but some object associated to this group. The latter makes sense if that object retains enough information about the initial group. In this way, Langlands Theory treats the representations of Galois groups. In this thesis, the Lie algebras associated to the lower central series of Galois groups of nilpotent extensions will be investigated.

Since every pro-finite nilpotent group is the product of pro-p-groups (which are always nilpotent), it is enough to investigate the maximal p-extensions. The task of determining the Galois group of the maximal p-extension of a given field in connection with the arithmetical information is currently far from having a complete solution. In the local case, the abstract structure of these groups is well known: they are either free pro-p-groups or Demushkin groups (see [**De**]). But this description is not actually related with arithmetical properties. Some progress has been made in obtaining arithmetical information for the local case however. The ramification filtration was given for the second step in the central series filtration of the Galois group of the maximal p-extension of local algebraic fields of characteristic 0 in the works of E.-W. Zink [**Zi**] and G.-M. Cram [**Cr**], and for the Galois groups modulo p-th commutators of the maximal p-extensions of local fields of positive characteristic in the work of V. Abrashkin [**Ab**]. The situation in the global case is even more complicated.

The purpose of this work is to give the construction of the graded Lie algebra associated with the lower central series of the group $G = G_{K,S}(p)$, where $G_{K,S}(p)$ is the Galois group of the maximal p-extension of a global algebraic field which does not ramify at places outside some specified subset S of all places of that field. The main part of this work (section 5) can be considered as a generalization of the article of H. Koch, S. Kukkuk, J.P. Labute [**KKL**], where the analogous construction for the maximal p-extensions of local algebraic fields was given.

At the beginning, the degree 2 in this Lie algebra will be studied; namely, sufficient conditions for the epimorphism

$$\begin{aligned} G/G^{(2)} \wedge G/G^{(2)} &\to G^{(2)}/G^{(3)}, \\ x \mod G^{(2)} \wedge y \mod G^{(2)} &\mapsto (x,y) := xyx^{-1}y^{-1} \mod G^{(3)} \end{aligned}$$

to be an isomorphism will be given. The question of the injectivity of such homomorphism first appeared in the work [**Zi**] for the group G of the maximal p-extension of a local algebraic field, with the positive answer, due to the existence of a lifting of any projective representation

$$G \to \mathrm{PGL}_n(\mathbb{C})$$

to a representation

$$G \to \mathrm{GL}_n(\mathbb{C}).$$

In the global case, that argument works only if S coincides with the set of all places of the field.

In this thesis, two approaches for the degree 2 will be studied: first (section 2), the relation of this problem to the Schur multiplicator and multiplicator free groups will be given. Following the works of S.B. Watt and S.V. Ullom ([**Wt**],[**UW**]), with slightly modernized proofs, which can be used to avoid unnecessary restrictions, some sufficient conditions for the group $G_{K,S}(p)$ to be multiplicator free will be obtained. Also, the triviality of the Schur multiplicator for empty S will be connected with the surjectivity of the norm maps on global units.

The second approach (section 3), see [**AKL**], gives a weaker result, as it requires the triviality of the Shafarevich group, see [**Ko**, Chapter 11], but is more explicit and is closer to the last part of this thesis. The main idea of Theorem 3.1 belongs to H. Koch, I refined an unclear point on p-completion in the initial proof, which necessitated the additional assumption of the Leopoldt Conjecture.

If the group $G_{K,S}(p)$ is multiplicator free, the description of the degree 3 of the considered Lie algebra is obtained in section 4.

In the last part of this work, sections 5 and 6, the description of the whole algebra will be given in some special cases. For simplicity, it is supposed in that part that p is an *odd* prime.

Section 5 is based on the results of the articles of J.P. Labute [**La2**], [**La3**], and uses a technique developed in [**La1**], [**KKL**]. The idea is the following: for some groups G, represented as a factor group of a free group F by a relations subgroup R, the lower central series can be given by using only information obtained from R modulo third commutators of F. Moreover, the result of section 5 shows that the question of whether some chosen group is in this class depends only on R modulo third commutators and p-th powers of second commutators of F.

Of course, one has to consider only a subclass of all pro-p-groups, otherwise some uncontrolled relations, lying in the group of the third commutators $F^{(3)}$, can appear. It is very natural to use some "multiplicator free"-like condition, especially as it could be checked for many Galois groups. The main result, Theorem 5.8, requires some nondegenerance conditions of a technical nature. The really unpleasant restriction is the assumption that the torsion of the abelian factorgroup G^{ab} of G has to be of the form $\bigoplus_i \mathbb{Z}/p^{\kappa}\mathbb{Z}$ with the same κ. Hopefully, by refining of the technique, developed in [**La3**], [**KKL**] and here, it will be possible to avoid this assumption.

The main result and notations needed to formulate it are given in subsection 5.6.

The fundamental work of H. Koch, [**Ko**], makes the construction above useful, as the groups $G_{K,S}(p)$ modulo third commutators and p-th powers of the second commutators (in some situations, modulo third commutators) were investigated there. As an example, the complete structure of the lower central series of some groups $G_{\mathbb{Q},S}(p)$ with 2 relations will be obtained by means of Theorem 5.8.

At this point, I would like to thank my teacher, Prof. Helmut Koch, for giving me the theme of this thesis and directing me during the last three years to these results, Prof. Victor Abrashkin from Steklov Institute for the interesting discussions, Prof. Ernst-Wilhelm Zink and all members of the former Max Planck Institute working group "Algebraische Geometrie und Zahlentheorie", the warm atmosphere of which allowed me to more easily orient myself in the new environment, country and language, and Prof. S.V. Vostokov, my scientific advisor during my undergraduate studies in Saint-Petersburg State University. I would also like to thank Max Planck Institute for Mathematics in Bonn, which provided the doctoral stipend that allowed me to complete this work.

1. GENERAL

1.1. Filtrations

Let G be pro-p-group. A *filtration* of G is a sequence of descending closed subgroups G_i $(i \geq 1)$ such that the following conditions are fulfilled:

1) $G_1 = G$
2) $[G_i, G_j] \subseteq G_{i+j}$ for $i, j \in \mathbb{N}$,

where $[G_i, G_j]$ denotes the closed subgroup of G generated by the commutators

$$(g, h) := g^{-1}h^{-1}gh \text{ for } g \in G_i, h \in G_j.$$

The lower central series filtration $(G^{(i)})$, is defined by induction:

$$G^{(1)} := G, \qquad G^{(i+1)} := [G, G^{(i)}].$$

One proves by induction that $(G^{(i)})$ is a filtration of G using the following well known rules for commutators (see e.g. [**Ha**], 10.2), where x^y means $y^{-1}xy$.

$$\begin{aligned} (h, g) &= (g, h)^{-1}, \\ h^g &= h(h, g), \\ (f, gh) &= (f, h)(f, g)((f, g), h), \\ (fg, h) &= (f, h)((f, h), g)(g, h), \\ (f^g, (g, h))(g^h, (h, f))(h^f, (f, g)) &= 1, \end{aligned} \tag{1}$$

for $f, g, h \in G$.

One can associate to a filtered group G a graded Lie algebra $L(G)$ as follows. By definition, the groups G_i are normal subgroups of G. Let

$$L_m(G) := G_m/G_{m+1}$$

and

$$[\overline{g}, \overline{h}] := \overline{(g, h)}$$

for $g \in G_i,\ h \in G_j$. Using equations (1), it is easy to see that this definition does not depend on the choice of g and h in the classes

$\overline{g} \in L_i(G)$ and $\overline{h} \in L_j(G)$ and that it defines the structure of a graded Lie algebra on

$$L(G) := \bigoplus_{m=1}^{\infty} L_m(G).$$

Note that the Lie algebra associated to the lower central series of G is generated as a Lie algebra by the elements of $L_1(G) = G^{\mathrm{ab}} = G/[G,G]$.

1.2. Free Lie algebras

In this subsection the construction of a free Lie algebra associated to a module will be given.

Let k be a commutative, associative ring with unity and let A be a k-module. Let $\mathcal{T}(A)$ be the non-associative tensor algebra of A considered as a k-module, i.e.

$$\begin{aligned} \mathcal{T}(A) &:= \bigoplus_{m=1}^{\infty} \mathcal{T}_m(A), \\ \mathcal{T}_1(A) &:= A, \\ \mathcal{T}_m(A) &:= \bigoplus_{p+q=m} \mathcal{T}_p \otimes_k \mathcal{T}_q. \end{aligned}$$

One can define the Lie algebra $\mathcal{L}(A)$ as the factor algebra of $\mathcal{T}(A)$ by the ideal $\mathcal{I}(A)$ of $\mathcal{T}(A)$ generated by all elements of the form

$$a \otimes a, (a \otimes b) \otimes c + (b \otimes c) \otimes a + (c \otimes a) \otimes b,$$

with $a, b, c \in \mathcal{T}(A)$. Note that this ideal is homogeneous, thus

$$\begin{aligned} \mathcal{L}(A) &= \bigoplus_{m=1}^{\infty} \mathcal{L}_m(A), \\ \mathcal{L}_m(A) &:= (\mathcal{T}_m(A) + \mathcal{I}(A))/\mathcal{I}(A), \end{aligned}$$

and so $\mathcal{L}$ is a graded Lie algebra over k.

Note that if A is free k-module with basis $\{x_i\}_{i\in I}$, then $\mathcal{L}(A)$ coincides with the usual free Lie algebra over k with generators $\{x_i\}_{i\in I}$ (see [**Se2**] for the definition).

The algebra $\mathcal{L}(A)$ has the universal property; that is, for any graded Lie algebra $L = \bigoplus_{n=1}^{\infty} L_n$ over k and for any k-module homomorphism $f : A \to L_1$, there exists a unique degree preserving graded Lie algebra homomorphism $g : \mathcal{L}(A) \to L$, such that $g|_{\mathcal{L}_1(A)} = f$.

In the special situation of a free pro-p-group F (see [**Se1**], for the definition), where p denotes a prime number, the following theorem (see [**KKL**], Theorem 2.1) holds:

THEOREM 1.1. *Let $L(F)$ be the Lie algebra associated to the lower central series of F. The natural map $\psi : \mathcal{L}(F/F^{(2)}) \to L(F)$ is an isomorphism of graded Lie algebras over $\mathbb{Z}_p$.*

1.3. Artin reciprocity map

Let J_K be the idele group of the global algebraic field K (see [**CF**, Chapter VII] for definitions). Classical Class Field Theory states that there exists a canonical homomorphism, called Artin reciprocity map, from the idele class group of K

$$C_K = J_K/K^*$$

to the Galois group G_K^{ab} of the maximal abelian extension of K, which becomes an isomorphism under profinite completion (recall that C_K has topological structure).

For some subset S of primes of the field K (nonarchimedean or archimedean), let $G_{K,S}$ denote the group of the maximal Galois extension of K, which ramifies only at places from the set S. Let $G_{K,S}(p)$ be the pro-p-completion of $G_{K,S}$. Thus $G_{K,S}(p)$ is the Galois group of the maximal Galois p-extension $K_S(p)$ of K (see [**Ko**] for details).

Let

$$C_{K,S} := J_K/K^* U_{K,\bar{S}},$$

where

$$U_{K,\bar{S}} = \prod_{\mathfrak{p} \notin S} U_{\mathfrak{p},K}$$

with $U_{\mathfrak{p},K}$ be the local units (whole multiplicative group for archimedean places). Classical Class Field Theory gives an homomorphism

$$\varphi : C_{K,S} \to G_{K,S}^{\mathrm{ab}}, \tag{2}$$

where the group $G_{K,S}^{\mathrm{ab}}$ is the Galois group of the maximal abelian extension of K which ramifies only in places contained in S.

Let $C_{K,S}(p)$ denote the pro-p-completion of $C_{K,S}$. Obviously, one gets the isomorphism

$$\varphi : C_{K,S}(p) \to G_{K,S}^{\mathrm{ab}}(p),$$

induced by (2).

This isomorphism can be uniquely extended to an isomorphism of the graded Lie algebras:

$$\varphi : \mathcal{L}(C_{K,S}(p)) \to \mathcal{L}(G_{K,S}^{\mathrm{ab}}(p)).$$

One gets thus the graded Lie algebra homomorphism Φ to the Lie algebra associated with the lower central series of $G_{K,S}(p)$:

$$\Phi : \mathcal{L}(C_{K,S}(p)) \xrightarrow{\varphi} \mathcal{L}(G^{\mathrm{ab}}_{K,S}(p)) \to L(G_{K,S}(p)).$$

As $\mathcal{L}_1(G^{\mathrm{ab}}_{K,S}(p)) = G^{\mathrm{ab}}_{K,S}(p) = L_1(G_{K,S}(p))$, and $L(G_{K,S}(p))$ is generated by the elements from $L_1(G_{K,S}(p))$ as a Lie algebra, Φ is surjective.

The main purpose of this thesis is to obtain the description of the kernel of this homomorphism, thus getting the description of the lower central filtration of $G_{K,S}(p)$ by means of the inner structure of K.

1.4. The group $G_{K,S}(p)$ modulo 3rd commutators

Some of H. Koch's results on the structure of the group $G = G_{K,S}(p)$ modulo third commutators will be described here (see [**Ko**, Chapter 11]).

For any field L let $\delta(L) = 1$ if L contains the p-th roots of unity, and $\delta(L) = 0$ if not.

A place $\mathfrak{p}$ of K is ramified in the maximal p-extension $K^{\mathrm{sep}}(p)$ of K only if $N(\mathfrak{p}) \equiv 1 \pmod p$ (here $N(\mathfrak{p})$ is the norm of $\mathfrak{p}$ in $\mathbb{Q}$), if $\mathfrak{p}|p$, or if $\mathfrak{p}$ is real (in the case $p = 2$). Hence throughout this thesis it is often assumed that S contains only the places of these types.

Let

$$\kappa_{\mathfrak{p}} = \begin{cases} v_p(N(\mathfrak{p}-1)), & \text{if } N(\mathfrak{p}) \equiv 1 \pmod p \\ & \text{(here } v_p \text{ is the } p\text{-adic valuation, } v_p(p) = 1), \\ k, & \text{if } \mathfrak{p}|p \text{ and } K \text{ contains } p^k\text{-th, but not } p^{k+1}\text{-th} \\ & \text{roots of unity,} \\ 1, & \text{if } p = 2 \text{ and } \mathfrak{p} \text{ is real,} \end{cases}$$

and let $e_{\mathfrak{p}} = p^{\kappa_{\mathfrak{p}}}$.

Let

$$1 \to R \to F \to G \to 1$$

be a minimal representation of G by free pro-p-group F. One can construct it by means of a minimal system $\{\overline{\tau}_i\}_{i\in I}$, $\tau_i \in G$, of generators of $G/G^{(2)}G^p \cong C_{K,S}(p)/C_{K,S}(p)^p$, taking the generator system $\{s_i\}_{i\in I}$ of the free pro-p-group F such that the images of s_i are τ_i for $i \in I$.

For any $\mathfrak{p} \in S$, choose an extension $\mathfrak{P}$ in $K_S(p)$. Let $G_{\mathfrak{p}}$ be the Galois group of the maximal p-extension $K^{\mathrm{sep}}_{\mathfrak{p}}$ of $K_{\mathfrak{p}}$. The inclusions

$$K_S(p) \to (K_S(p))_{\mathfrak{P}} \to K^{\mathrm{sep}}_{\mathfrak{p}}$$

induce the homomorphism $\psi_{\mathfrak{p}}$ of $G_{\mathfrak{p}}$ into G.

Let

$$1 \to R_{\mathfrak{p}} \to F_{\mathfrak{p}} \to G_{\mathfrak{p}} \to 1 \tag{3}$$

be a minimal presentation of $G_{\mathfrak{p}}$ by the free pro-p-group $F_{\mathfrak{p}}$. Then one has a diagram

$$\begin{array}{ccccccccc} 1 & \to & R_{\mathfrak{p}} & \to & F_{\mathfrak{p}} & \to & G_{\mathfrak{p}} & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & R & \to & F & \to & G & \to & 1. \end{array} \tag{4}$$

Let $\chi_{\mathfrak{p}} : F_{\mathfrak{p}} \to F$ be the middle vertical map from this diagram.

The representations (3) of $G_{\mathfrak{p}}$ are well known (see [**Ko**] for references). For section 3 the following fact is essential: $G_{\mathfrak{p}}$ is free in the case $\mathfrak{p}|p$ and $\delta(K_{\mathfrak{p}}) = 0$, and $R_{\mathfrak{p}}$ is generated as a closed normal subgroup of $F_{\mathfrak{p}}$ by one relation $r_{\mathfrak{p}}$ otherwise. The relation $r_{\mathfrak{p}}$ has the form

$$r_{\mathfrak{p}} = t_{\mathfrak{p}}^{e_{\mathfrak{p}}} r'_{\mathfrak{p}} \text{ with } r'_{\mathfrak{p}} \in F_{\mathfrak{p}}^{(2)},$$

where $t_{\mathfrak{p}}$ is an element of $F_{\mathfrak{p}}$, which is mapped onto $\zeta_{\mathfrak{p}}$ by the homomorphism

$$F_{\mathfrak{p}} \to G_{\mathfrak{p}}/G_{\mathfrak{p}}^{(2)} \to \hat{K}_{\mathfrak{p}}^{*},$$

where $\zeta_{\mathfrak{p}}$ denotes a primitive root of unity of the order $e_{\mathfrak{p}}$ in $K_{\mathfrak{p}}$, $\hat{K}_{\mathfrak{p}}^{*}$ denotes the pro-p-completion of the multiplicative group $K_{\mathfrak{p}}^{*}$. Set $r_{\mathfrak{p}} := 1$ in the case $\mathfrak{p}|p$, $\delta(K_{\mathfrak{p}}) = 0$.

The complete form of the relations modulo third commutators is not given here in general, but will be given in section 6 for the special case, as it is essential for application.

Let

$$V_S = \{a \in K^* | (a) = \mathfrak{a}^p \text{ for some ideal } \mathfrak{a}, a \in K_{\mathfrak{p}}^p \text{ for } \mathfrak{p} \in S\},$$

where (a) is the principal ideal generated by a, and let B_S be the character group of V_S/K^p. According to [**Ko**, Chapter 11], if $B_S = \{0\}$, the relation module R in (4) is generated as a normal subgroup of F by the local relations $\chi_{\mathfrak{p}} r_{\mathfrak{p}}, \mathfrak{p} \in S$. If the p-th roots of unity are contained in K, the relation $\chi_{\mathfrak{p}_0} r_{\mathfrak{p}_0}$ for one arbitrary $\mathfrak{p}_0 \in S$ can be omitted.

1.5. The Leopoldt Conjecture

One of the equivalent formulations of the Leopoldt Conjecture is the following (see [**Wa**]): let $\hat{E}_K$ be the pro-p-completion of the group E_K of global units of the field K. One can consider the homomorphism ℓ of $\hat{E}_K$ to the pro-p-completion of the product of local units $U_{\mathfrak{p}}$ of the fields $K_{\mathfrak{p}}$ with $\mathfrak{p}|p$:

$$\prod_{\text{all } \mathfrak{p}:\ \mathfrak{p}|p} \hat{U}_{\mathfrak{p}} = \prod_{\text{all } \mathfrak{p}:\ \mathfrak{p}|p} U_{\mathfrak{p}}^1,$$

here $U_{\mathfrak{p}}^1$ is the group of principal local units ($u \in U_{\mathfrak{p}}^1$ if and only if $u \equiv 1 \pmod{\mathfrak{p}}$). Then the Leopoldt Conjecture says that the $\mathbb{Z}_p$-rank of the image of $\hat{E}_K$ under this homomorphism is equal to the $\mathbb{Z}$-rank of E_K, or equivalently, that the homomorphism ℓ is injective.

The conjecture is proved for abelian extensions of the rational number field $\mathbb{Q}$ and imaginary quadratic fields.

2. DEGREE 2: MULTIPLICATOR FREE GROUPS

In this and in the next section the question of the injectivity of $\Phi_2 = \Phi|_{\mathcal{L}_2(C_{K,S}(p))}$ will be investigated. In other words, if $G = G_{K,S}(p)$, $G^{\text{ab}} = G/G^{(2)}$, then the injectivity of the homomorphism

$$\begin{aligned} G^{\text{ab}} \wedge G^{\text{ab}} &\to G^{(2)}/G^{(3)} \\ x \mod G^{(2)} \wedge y \mod G^{(2)} &\mapsto (x,y) \mod G^{(3)} \end{aligned}$$

for $x, y \in G$ is considered.

2.1. Connection between triviality of Schur multiplicator and the injectivity of Φ_2

In this section G will be a pro-finite-group (resp. pro-p-group), represented by $G = F/R$, where the group F is free as a pro-finite-group (resp. pro-p-group). The following equations hold:

$$G/G^{(2)} = F/R\Big/(F/R, F/R) = F/RF^{(2)},$$
$$G^{(2)}/G^{(3)} = F^{(2)}/(R \cap F^{(2)})F^{(3)}.$$

Thus,

$$\begin{aligned} G/G^{(2)} \wedge G/G^{(2)} &= F/RF^{(2)} \wedge F/RF^{(2)} \\ &= F^{(2)}/(RF^{(2)}, F) = F^{(2)}/(R,F)F^{(3)}. \end{aligned}$$

This means, that $G/G^{(2)} \wedge G/G^{(2)}$ is isomorphic to $G^{(2)}/G^{(3)}$ exactly when

$$(R \cap F^{(2)})F^{(3)} = (R,F)F^{(3)}. \tag{5}$$

Let $M(G) = H_2(G, \mathbb{Z})$ be the *Schur multiplicator*. Following [**Fr**], [**Wt**], [**UW**] I use the word *"multiplicator"*, though its synonym *"multiplier"* is more popular. The group G is called *multiplicator free* if its Schur multiplicator is trivial.

It is well known that if $G = F/R$ is the representation of G as the factor of a free group by a relation subgroup, then $M(G) = \dfrac{R \cap F^{(2)}}{(R,F)}$.

Thus, if G is a multiplicator free pro-p-group, then

$$R \cap F^{(2)} = (R, F),$$

and (5) holds, i.e. the kernel of considered homomorphism Φ_2 is trivial.

REMARK. One important property of multiplicator free group G is the following: if for $G = F/R$ the group R modulo $F^{(2)}$ is generated by $r_1, \ldots, r_s \in RF^{(2)}/F^{(2)}$, then there exist liftings $R_1, \ldots, R_s \in R$, $R_i \bmod F^{(2)} = r_i$, such that these liftings generate R as normal subgroup of G.

The results of this section give some sufficient conditions for the group $G_{K,S}(p)$ to be multiplicator free. The proofs follows the works of S.B. Watt and S.V. Ullom ([**Wt**],[**UW**]), with some modification which allows to avoid some unnecessary restrictions.

2.2. Some facts

Let again G be pro-finite group.

If $G = \Omega/\Lambda$, where Ω, Λ are profinite groups, this gives immediately the uniquely defined map $M(\Omega/\Lambda) \to \dfrac{\Lambda \cap (\Omega, \Omega)}{(\Omega, \Lambda)}$ which can be shown to be surjective.

The profinite group Ω is multiplicator free if and only if for any normal open subgroup Λ the map $M(\Omega/\Lambda) \to \dfrac{\Lambda \cap (\Omega, \Omega)}{(\Omega, \Lambda)}$ is isomorphism (see [**Fr**, §3]).

Let K be global field of characteristic 0, S be some subset of its places, p some prime and $G_{K,S}(p)$ be the Galois group of the maximal p-extension of K, which is ramified only at places in S. The question to be investigated here is the condition for the the Galois group $G_{K,S}(p)$ of maximal ramified only in places from S p-extension of K, to be multiplicator free.

If L/K is Galois extension with Galois group G, and

$$\Omega = \mathrm{Gal}(K^{\mathrm{sep}}/K), \qquad \Lambda = \mathrm{Gal}(K^{\mathrm{sep}}/L),$$

put

$$\mathfrak{m}(L/K) = \frac{\Lambda \cap (\Omega, \Omega)}{(\Omega, \Lambda)}.$$

Theorem 3.3 of [**Fr**] says that the homomorphism $M(G) \to \mathfrak{m}(L/K)$ is the isomorphism.

If E is central extension of L/K (that is, $\mathrm{Gal}(E/L)$ is a central subgroup of $\mathrm{Gal}(E/K)$), and $\mathfrak{m}(L/K) \to \mathrm{Gal}(E/L \cdot (K^{\mathrm{ab}} \cap E))$ is isomorphism, we say that E *realizes* $\mathfrak{m}(L/K)$.

It follows from above that for some (not necessary finite) Galois extension $\mathcal{F}$ of K, $\mathrm{Gal}(\mathcal{F}/K)$ is multiplicator free if and only if for every finite Galois subextension L/K of $\mathcal{F}/K$ there exists finite central extension E of L/K, $E \subseteq \mathcal{F}$, such that E realizes $\mathfrak{m}(L/K)$ (see again [**Fr**]).

2.3. Topological Lemmas

For an abelian locally compact group A, let $X(A)$ (resp. $X_p(A)$) denote the torsion (resp. p-torsion) subgroup of its character group. For an abelian group homomorphism $f : A \to B$ the dual homomorphism will be denoted by $\hat{f} : X(B) \to X(A)$.

LEMMA 2.1. *If L/K is a finite Galois extension with Galois group Γ, then the group $L^* J_L^{\Gamma - 1}$ is closed in J_L.*

PROOF. Following [**CF**, Chapter II], let us define the content $c(\alpha)$ of an idele α to be $\prod_{\text{all } \mathfrak{p}} |\alpha_\mathfrak{p}|_\mathfrak{p}$, where the valuations $|\cdot|_\mathfrak{p}$ are multiplicative and normed so that $c(L^*) = 1$. Denote $\ker c$ with J_L^1. Then, by the same source, J_L^1/L^* is compact. Obviously, $J_L^{\Gamma-1} = {J_L^1}^{\Gamma-1}$, and the image of $L^* J_L^{\Gamma-1}$ in J_L^1/L^* is compact, hence closed. □

Let D_L be the connected component of unity in C_L.

LEMMA 2.2. *If L/K is a finite Galois extension with Galois group Γ, then* $\ker N_{L/K}(C_L) \cap D_L \subseteq C_L^{\Gamma-1}$.

PROOF. It is sufficient to show that for every archimedean place $\mathfrak{p}$ of K,

$$\ker N_{L/K}\left(\prod_{\mathfrak{P}|\mathfrak{p}} L_\mathfrak{P}^*\right) = \left(\prod_{\mathfrak{P}|\mathfrak{p}} L_\mathfrak{P}^*\right)^{\Gamma-1},$$

i.e., to show the triviality of

$$\hat{H}^{-1}\left(\Gamma, \left(\prod_{\mathfrak{P}|\mathfrak{p}} L_\mathfrak{P}^*\right)\right) = \hat{H}^{-1}(\Gamma_\mathfrak{P}, L_\mathfrak{P}) = \hat{H}^1(\Gamma_\mathfrak{P}, L_\mathfrak{P}) = 0,$$

here $\Gamma_\mathfrak{P}$ denotes the decomposition group of $\mathfrak{P}$, which is obviously cyclic as $\mathfrak{p}$ is archimedean. □

2.4. Working criteria

PROPOSITION 2.3. *If E is a finite central extension of a finite Galois extension L/K with $\Gamma = \mathrm{Gal}(L/K)$, then E realizes $\mathfrak{m}(L/K)$ if and only if*

$$X(C_L)^\Gamma \subseteq \left(\hat{N}_{L/K} X(C_K)\right) \cdot X\left(\frac{C_L}{N_{E/L}(C_E)}\right).$$

REMARK.

$$X(C_L)^\Gamma \supseteq \left(\hat{N}_{L/K} X(C_K)\right) \cdot X\left(\frac{C_L}{N_{E/L}(C_E)}\right)$$

always holds.

PROOF. Using the notations of [**Fr**], for the finite Galois extension L/K with Galois group Γ and its finite central extension E/K let

$$\Phi(E/L) = X(C_L/N_{E/L}(C_E)) \subseteq X(C_L)^\Gamma,$$

and

$$\Psi(L/K) = X(\hat{H}^{-1}(\Gamma, C(L))) = X\left(\frac{\ker N_{L/K}(C_L)}{C_L^{\Gamma-1}}\right).$$

Note also that the condition of E/K being a central extension of L/K can be written in the form $C_L^{\Gamma-1} \subseteq N_{E/L}C_E$, and one can consider the natural homomorphism from $\Phi(E/L)$ to $\Psi(L/K)$. Obviously

$$\hat{H}^{-1}(\Gamma, C(L)) = \hat{H}^1(\Gamma, \mathbb{Z})$$

is finite. Proposition 3.4 of [**Fr**] says that a central extension E of L/K realizes $\mathfrak{m}(L/K)$ exactly when that homomorphism is surjective.

By Pontryagin duality and finiteness of the groups the equivalent condition is the injectivity of the homomorphism

$$\frac{\ker N_{L/K}(C_L)}{C_L^{\Gamma-1}} \to \frac{C_L}{N_{E/L}(C_E)}. \tag{6}$$

By Lemma 2.2 the homomorphism

$$\frac{\ker N_{L/K}(C_L)}{C_L^{\Gamma-1}} \to \frac{\ker N_{L/K}(C_L) \cdot D_L}{C_L^{\Gamma-1} \cdot D_L}$$

is injective. Hence, the injectivity of (6) is equivalent to

$$\ker N_{L/K}(C_L) \cdot D_L \cap N_{E/L}(C_E) \subseteq C_L^{\Gamma-1} \cdot D_L.$$

Again, by Pontryagin duality, the last inclusion is equivalent to

$$X\left(\frac{C_L}{\ker N_{L/K}(C_L)}\right) \cdot X\left(\frac{C_L}{N_{E/L}(C_E)}\right) \supseteq X\left(\frac{C_L}{C_L^{\Gamma-1}}\right),$$

(as $N_{E/L}(C_E)$ is open in C_L and hence contains D_L), i.e., to

$$\left(\hat{N}_{L/K}X(C_K)\right)\cdot X\left(\frac{C_L}{N_{E/L}(C_E)}\right)\supseteq X(C_L)^\Gamma.$$

□

For some set S of places (finite or infinite) of K let $K_S(p)$ denote the maximal p-extension of K unramified outside places from S. Let $\bar{S}$ be the complement of S in the set of all places of K. For some finite extension L/K, let

$$U_{L,S}=\prod_{\substack{\mathfrak{P}|\mathfrak{p}\\ \mathfrak{p}\in S}}U_{\mathfrak{P},L}.$$

THEOREM 2.4. *If L/K is a finite Galois subextension of $K_S(p)$, then some central finite extension E of L/K realizes $\mathfrak{m}(L/K)$ if and only if*

$$X_p(C_L)^\Gamma\subseteq\left(\hat{N}_{L/K}X_p(C_K)\right)\cdot X_p\left(C_L\right)^\Gamma_S, \tag{7}$$

where $X(C_L)_S$ denotes the subgroup of characters unramified for places outside S.

PROOF. Note first that

$$X_\ell(C_L)^\Gamma=\hat{N}_{L/K}X_\ell(C_K)$$

for $p\neq\ell$ because

$$\frac{\ker N_{L/K}(C_L)}{C_L^{\Gamma-1}}=\hat{H}^{-1}(\Gamma,C_L)=\hat{H}^{-3}(\Gamma,\mathbb{Z})=M(\Gamma)$$

is a finite p-group. Thus one can omit the restriction on p-torsion characters in the equation (7).

By Class Field Theory, the condition that E is central over L/K is equivalent to $N_{E/L}C_E\supseteq C_L^{\Gamma-1}$, and the restricted ramification means that $N_{E/L}C_E$ contains the image W of $U_{\bar{S},L}$ in $C(L)$; by proposition 2.3 it is sufficient to show that

$$X\left(\frac{C_L}{W\cdot C_L^{\Gamma-1}}\right)\to X\left(\frac{\ker N_{L/K}(C_L)}{C_L^{\Gamma-1}}\right)$$

is surjective if and only if

$$X\left(\frac{C_L}{A}\right)\to X\left(\frac{\ker N_{L/K}(C_L)}{C_L^{\Gamma-1}}\right)$$

is surjective for some open subgroup A of C_L containing $W \cdot C_L^{\Gamma-1}$, which follows again from the fact that

$$\frac{\ker N_{L/K}(C_L)}{C_L^{\Gamma-1}}$$

is finite. □

Theorem 2.4 was proved by Watt, [**Wt**], in the case of finite S.

2.5. Sufficient conditions for the triviality of the Schur multiplicator

Let L/K be a finite Galois extension with $\Gamma = \mathrm{Gal}(L/K)$ as above. For a field F let E_F denote its group of global units.

As the norm map is surjective on local units for unramified extensions, the map

$$\hat{N}_{L/K} : X_p\left(U_{K,\bar{S}}\right) \to X_p\left(U_{L,\bar{S}}\right)^{\Gamma}$$

is an isomorphism. Combining restriction on $U_{L,\bar{S}}$ with $\hat{N}_{L/K}^{-1}$ one gets a well defined homomorphism

$$\alpha : X_p(C_L)^{\Gamma} \to X_p(U_{K,\bar{S}})$$

Obviously, $\ker \alpha = X_p(C_L)_S^{\Gamma}$.

The next observation is that the equation (7) holds exactly when

$$\alpha\left(X_p(C_L)^{\Gamma}\right) \subseteq \alpha\left(\hat{N}_{L/K} X_p(C_K)\right) = \mathrm{Im}\left(X_p\left(U_K/E_K\right) \to X_p(U_{K,\bar{S}})\right).$$

Note that the homomorphism $X_p\left(U_K/E_K\right) \to X_p(U_{K,\bar{S}})$ is surjective if every character from $X_p(E_K)$ can be lifted to a character from $X_p(U_{K,S})$ (the difficulty is that E_K has the discrete topology, the image of the diagonal embedding of E_K in $U_{K,S}$ does not necessarily has it).

This property evidently holds, if E_K is a finite group and its maximal p-factor maps injectively to the maximal pro-p-factor of $U_{K,\bar{S}}$, or S contains all places of the field K lying above p and the Leopoldt Conjecture holds for the field K.

THEOREM 2.5. *The Galois group of the maximal S-ramified p-extension of K is multiplicator free if one of the following conditions holds:*

a). *$p = 2$, S contains a finite prime, $K = \mathbb{Q}$ or K is an imaginary quadratic field, $K \neq \mathbb{Q}(i)$.*
b). *p is odd, $K = \mathbb{Q}$ or K is an imaginary quadratic field .*
c). *S contains all places of the field K which lay above p and the Leopoldt Conjecture holds for K.*

PROOF. The only case to consider is $K = \mathbb{Q}(\sqrt{-3})$, $p = 3$. If S does not contain the finite prime $\mathfrak{q}$ with norm congruent to 0 or 1 modulo 3, then the 3-extension of K cannot ramify, and thus is trivial. In the other case the map from (ζ_3) to the maximal pro-p-factor of $U_{\mathfrak{q},K}$ is injective. $\square$

Theorem 2.5.a,b is a result from [**Wt**] (with the exception $K = \mathbb{Q}(\sqrt{-3})$ if $p = 3$). Theorem 2.5.c was proved in [**UW**] for a totally imaginary field K and finite set S.

LEMMA 2.6. *If L/K is a finite p-extension and H_p is the Hilbert p-class field of L, then*

$$N_{H_p/K}H_p^* \cap E_K \subseteq N_{L/K}E_L.$$

PROOF. The result of [**Fo**] states that if L/K is a finite Galois extension and H is the Hilbert class field of L, then $N_{H/K}H^* \cap E_K \subseteq N_{L/K}E_L$. If ℓ is the degree of the Hilbert class field H over the Hilbert p-class field H_p, then

$$\left(N_{H_p/K}H_p^*\right)^\ell \cap E_K \subseteq N_{L/K}E_L.$$

If $a \in N_{H_p/K}H_p^* \cap E_K$, then $a^\ell \in N_{L/K}E_L$. If p^m is the degree of the extension L/K, then $a^{p^m} \in N_{L/K}E_L$. As $\gcd(\ell, p) = 1$, the considered element a lies in $N_{L/K}E_L$. $\square$

The next Theorem treats the case of empty S.

THEOREM 2.7. *The Galois group Γ of the maximal unramified p-extension of K is multiplicator free if and only if for every finite unramified p-extension L of K the homomorphism*

$$X_p\left(\frac{U_K}{E_K}\right) \to X_p\left(\frac{U_K}{N_{L/K}E_L}\right) \tag{8}$$

is the isomorphism. If the Leopoldt Conjecture holds for K, Γ is multiplicator free if and only if $N_{L/K}E_L = E_K$ for such L.

PROOF. Note that $U_L \cdot L^* J_L^{\Gamma-1}$ is an open (and hence closed) subgroup of J_L, because U_L is an open subgroup of J_L. As U_L contains the connected component of identity of J_L, the map

$$X_p\left(\frac{J_L}{L^* J_L^{\Gamma-1}}\right) \to X_p\left(\frac{U_L}{U_L \cap L^* J_L^{\Gamma-1}}\right) \tag{9}$$

is surjective. Note that

$$N_{L/K}(U_L \cap L^* J_L^{\Gamma-1}) \supseteq N_{L/K}(U_L \cap E_L J_L^{\Gamma-1}) = N_{L/K}E_L.$$

On the other side, if H_p is the Hilbert p-class field for L, then

$$N_{H_p/K}(U_{H_p} \cap H_p^* J_{H_p}^{\Gamma-1}) \subseteq U_K \cap N_{H_p/K} L^* \subseteq N_{L/K} E_L$$

by Lemma 2.6. Thus, the group Γ is multiplicator free if and only if the inclusions (8) hold.

Let $\overline{N_{L/K} E_L}$ and $\overline{E_K}$ denote the closure of the inclusions of $N_{L/K} E_L$ and E_K into $\prod_{\mathfrak{P}|\mathfrak{p}} U_{\mathfrak{P},L}$ respectively, $\overline{N_{L/K} E_L}^{(1)}$ and $\overline{E_K}^{(1)}$ be their intersections with the product of principal units $\prod_{\mathfrak{P}|\mathfrak{p}} U^1_{\mathfrak{P},L}$. If the group Γ is multiplicator free, then by the first part of the theorem,

$$\overline{N_{L/K} E_L}^{(1)} \supseteq \overline{E_K}^{(1)} \tag{10}$$

for every L. The Leopoldt Conjecture for K means that the $\mathbb{Z}_p$-rank of $\overline{E_K}^{(1)}$ coincides with the $\mathbb{Z}$-rank of E_K. This means that (10) holds exactly when the index of $N_{L/K} E_L$ in E_K is prime to p. As L/K is a p-extension, this index is always p-primary. We obtain thus, if Γ is multiplicator free, then $N_{L/K} E_L = E_K$ for every L. The reverse conclusion is obvious from the first part of the theorem. $\square$

Note that the group $G_{K,S}(p)$ is not always multiplicator free. For example, if $K = \mathbb{Q}(\sqrt{-21})$, $S = \emptyset$, $p = 2$, then this group is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

3. DEGREE 2: LOCAL RELATIONS

In this section the triviality of the kernel of the homomorphism $\Phi_2 : \mathcal{L}_2(C_{K,S}(p)) \to L_2(G_{K,S}(p))$ will be investigated by means of the technique developed by H. Koch in [**Ko**]. This technique gives, under some assumptions, the description of the group $G = G_{K,S}(p)$ modulo $G^{(3)}{G^{(2)}}^p$, and even modulo $G^{(3)}$ in some cases, which is essential for the next sections.

Though Theorem 3.1, which is the main result of [**AKL**], is weaker than Theorem 2.5 (as it requires $B_S = \{0\}$), it is probably closer to the main stream of this thesis and is more important for possible future generalizations.

See subsection 1.4 for notations.

THEOREM 3.1. *Let $B_S = \{0\}$ and one of the following conditions hold:*

a). *K is the field of rational numbers $\mathbb{Q}$ or an imaginary quadratic field. If $p = 2$ or if $p = 3$ and $K = \mathbb{Q}(\sqrt{-3})$, then $\mathfrak{p} \in S$ for one place $\mathfrak{p}$ of K with $\mathfrak{p}|p$.*

b). *S contains all places $\mathfrak{p}$ of K with $\mathfrak{p}|p$ and the Leopoldt Conjecture holds for K.*

Then the Artin map $\Phi_2 : \mathcal{L}_2(C_{K,S}(p)) \to L_2(G_{K,S}(p))$ is an isomorphism.

PROOF. Note that under the conditions of the theorem, if $\delta(K) = 1$, then there exists a place $\mathfrak{p}_0 \in S$ with $\mathfrak{p}_0|p$. Let us choose such a place and put $S^* := S$ if $\delta(K) = 0$, $S^* := S - \{\mathfrak{p}_0\}$ if $\delta(K) = 1$.

To prove the theorem, one has to show that

$$(R \cap F^{(2)})F^{(3)} = (R, F)F^{(3)}$$

(see subsection 2.1).

As R is generated by the elements $r_\mathfrak{p}$ for $\mathfrak{p} \in S^*$ as the normal subgroup of F, any class of $(R \cap F^{(2)})F^{(3)}/F^{(3)}$ is of the form

$$\prod_{\mathfrak{p} \in S^*} (\chi_\mathfrak{p} r_\mathfrak{p})^{\alpha_\mathfrak{p}} r F^{(3)}, \alpha_\mathfrak{p} \in \mathbb{Z}_p,$$

where $r \in (R, F)$ and

$$\prod_{\mathfrak{p}\in S^*} (\chi_\mathfrak{p} r_\mathfrak{p})^{\alpha_\mathfrak{p}} \in F^{(2)},$$

or equivalently,

$$\prod_{\mathfrak{p}\in S^*} (\chi_\mathfrak{p} t_\mathfrak{p})^{e_\mathfrak{p}\alpha_\mathfrak{p}} \in F^{(2)}. \tag{11}$$

To prove the theorem it is sufficient to show that if (11) holds, then $\alpha_\mathfrak{p} = 0$ for $\mathfrak{p} \in S^*$.

Now the problem is reduced to a question of Classical Class Field Theory. One has to study $G/G^{(2)} \cong C_{K,S}(p)$ in connection with $F/F^{(2)}$.

First a generator system of $G/G^{(2)}$ according to [**Ko**, Chapter 11, p. 4] will be constructed.

Let $\zeta_\mathfrak{p}$ be a primitive root of unity of the order $e_\mathfrak{p}$ in $K_\mathfrak{p}$. Denote the element corresponding to $\zeta_\mathfrak{p}$ in $C_{K,S}(p)$ again with $\zeta_\mathfrak{p}$ and the corresponding element in $G/G^{(2)}$ with $\tau'_\mathfrak{p}$.

If $N(\mathfrak{p}) \equiv 1 \pmod p$ or $\mathfrak{p}$ is real, the inertia group of $G_\mathfrak{p}$ is cyclic.

For $\mathfrak{p}|p$ let $\zeta_\mathfrak{p}, \alpha_{1\mathfrak{p}}, \dots, \alpha_{n_\mathfrak{p}\mathfrak{p}}$ with $n_\mathfrak{p} := [K_\mathfrak{p} : \mathbb{Q}_\mathfrak{p}]$ be a generator system of the principal unit group of $K_\mathfrak{p}$. Denote the corresponding elements in $C_K(p)$ again by $\zeta_\mathfrak{p}, \alpha_{1\mathfrak{p}}, \dots, \alpha_{n_\mathfrak{p}\mathfrak{p}}$ and the corresponding elements in $G/G^{(2)}$ with $\tau'_\mathfrak{p}, \tau'_{1\mathfrak{p}}, \dots, \tau'_{n_\mathfrak{p}\mathfrak{p}}$.

Furthermore, let $\mathrm{Cl}_K(p)$ be the p-component of the ideal class group of K and let $\mathfrak{a}_1, \dots, \mathfrak{a}_h$ be ideles in J_K such that the corresponding ideals $\mathfrak{a}'_1, \dots, \mathfrak{a}'_h$ form a basis of $\mathrm{Cl}_K(p)$, that is, for some p-primary q_i; $\mathrm{Cl}_K(p) \cong \bigoplus_{i=1}^{h} \mathbb{Z}/q_i\mathbb{Z} \cdot \mathfrak{a}''_i$, where $\mathfrak{a}''_i$ are the classes of $\mathfrak{a}'_i$ modulo principal ideals. Let $\omega_1, \dots, \omega_h$ be the corresponding elements in $G/G^{(2)}$. Then by Class Field Theory

$$\{\omega_1, \dots, \omega_h\} \cup \{\tau'_\mathfrak{p} | \mathfrak{p} \in S\} \cup \{\tau'_{1\mathfrak{p}}, \dots, \tau'_{n_\mathfrak{p}\mathfrak{p}} | \mathfrak{p} \in S, \mathfrak{p}|p\}$$

is a generator system of $G/G^{(2)}$ (not necessarily minimal).

One can represent $G/G^{(2)}$ as the quotient of the free abelian pro-p-group A with the generator system

$$\{w_1, \dots, w_h\} \cup \{t'_\mathfrak{p} | \mathfrak{p} \in S\} \cup \{t'_{1\mathfrak{p}}, \dots, t'_{n_\mathfrak{p}\mathfrak{p}} | \mathfrak{p} \in S, \mathfrak{p}|p\}.$$

Define the map $\alpha : A \to G/G^{(2)}$ by

$$\begin{aligned} \alpha(w_i) &= \omega_i, \quad i = 1, \dots, h, \\ \alpha(t'_\mathfrak{p}) &= \tau'_\mathfrak{p}, \quad \mathfrak{p} \in S, \\ \alpha(t'_{i\mathfrak{p}}) &= \tau'_{i\mathfrak{p}}, \quad i = 1, \dots, n_\mathfrak{p}, \quad \mathfrak{p}|p, \quad \mathfrak{p} \in S. \end{aligned}$$

Then the kernel of α is generated by the local relations

$$t'^{e_{\mathfrak{p}}}_{\mathfrak{p}}, \mathfrak{p} \in S, \tag{12}$$

and the relations which correspond to the finiteness of $\mathrm{Cl}_K(p)$ and the group E_K of global units of K:

Let $\varepsilon \in E_K$. Then we have the following equation in $J_K(p)$:

$$(\varepsilon) = \prod_{\mathfrak{p}\in S} \zeta_{\mathfrak{p}}^{a_{\mathfrak{p}\varepsilon}} \prod_{\substack{\mathfrak{p}|p\\ \mathfrak{p}\in S}} \prod_{i=1}^{n_{\mathfrak{p}}} \alpha_{i\mathfrak{p}}^{b_{i\mathfrak{p}\varepsilon}}, \;\; a_{\mathfrak{p}\varepsilon} \in \mathbb{Z}, \;\; b_{i\mathfrak{p}\varepsilon} \in \mathbb{Z}_p.$$

Let q_j be the order of $\mathfrak{a}''_j$ and $\beta_j \in K^*$ such that $\mathfrak{a}'^{q_j}_j = (\beta_j)$ for $j = 1, \dots, h$. Then $\mathfrak{a}_j^{q_j}(\beta_j^{-1})$ is a unit idele whose image in $J_K(p)$ has the form

$$\prod_{\mathfrak{p}\in S} \zeta_{\mathfrak{p}}^{c_{\mathfrak{p}j}} \prod_{\substack{\mathfrak{p}|p\\ \mathfrak{p}\in S}} \prod_{i=1}^{n_{\mathfrak{p}}} \alpha_{i\mathfrak{p}}^{d_{i\mathfrak{p}j}}, \;\; c_{\mathfrak{p}j} \in \mathbb{Z}, \;\; d_{i\mathfrak{p}j} \in \mathbb{Z}_p.$$

The corresponding relations in A are

$$\prod_{\mathfrak{p}\in S} t'^{\,a_{\mathfrak{p}\varepsilon}}_{\mathfrak{p}} \prod_{\substack{\mathfrak{p}|p\\ \mathfrak{p}\in S}} \prod_{i=1}^{n_{\mathfrak{p}}} t'^{\,bi\mathfrak{p}\varepsilon}_{i\mathfrak{p}} \tag{13}$$

for $\varepsilon \in E_K$ and

$$w_j^{-q_j} \prod_{\mathfrak{p}\in S} t'^{\,c_{\mathfrak{p}j}}_{\mathfrak{p}} \prod_{\substack{\mathfrak{p}|p\\ \mathfrak{p}\in S}} \prod_{i=1}^{n_{\mathfrak{p}}} t'^{\,b_{i\mathfrak{p}\varepsilon}}_{i\mathfrak{p}} \tag{14}$$

for $j = 1, \dots, n$. The relations (12), (13), (14) generate $\ker\alpha$ by construction.

$F/F^{(2)}$ has to be generated by a minimal system of generators. One can do this by choosing a minimal subsystem $\{s_i\}_{i\in I}$ among the generators of A. This defines a homomorphism β of A onto $F/F^{(2)}$ with the kernel generated by the relations (13) and (14) (see the procedure in [**Ko**, Chapter 11, p. 4]).

The condition (11) is equivalent to

$$\prod_{\mathfrak{p}\in S^*} t'^{\,e_{\mathfrak{p}}\alpha_{\mathfrak{p}}}_{\mathfrak{p}} \in \ker\beta. \tag{15}$$

Since the generators $\omega_1, \dots, \omega_h$ do not appear in (15), the relation has to be of the form

$$\prod_{j=1}^{m} \left(\prod_{\mathfrak{p}\in S} t'^{\,a_{\mathfrak{p}\varepsilon_j}}_{\mathfrak{p}} \prod_{\substack{\mathfrak{p}|p\\ \mathfrak{p}\in S}} \prod_{i=1}^{n_{\mathfrak{p}}} t'^{\,b_{i\mathfrak{p}\varepsilon_j}}_{i\mathfrak{p}} \right)^{c_j}, \tag{16}$$

where $\varepsilon_1, \ldots, \varepsilon_m$ generate E_K as a multiplicative $\mathbb{Z}$ module, $c_1, \ldots, c_m \in \mathbb{Z}_p$.

The relation (16) is actually of the form

$$\prod_{\mathfrak{p} \in S} t'^{\,a_{\mathfrak{p}\varepsilon}}_{\mathfrak{p}} \prod_{\substack{\mathfrak{p}|p \\ \mathfrak{p} \in S}} \prod_{i=1}^{n_{\mathfrak{p}}} t'^{\,b_{i\mathfrak{p}\varepsilon}}_{i\mathfrak{p}},$$

where ε is some root of unity of p-th power order. In case a) of this theorem, it is evident, as the unit group E_K is finite. Consider case b). Denote the images of $\varepsilon, \varepsilon_1, \ldots, \varepsilon_m$ in $\hat{E}_K$ with $\hat{\varepsilon}, \hat{\varepsilon}_1, \ldots, \hat{\varepsilon}_m$ (see subsection 1.5).

Taking $\mathfrak{p} \in S$, $\mathfrak{p}|p$, one gets $\sum_{j=1}^{m} c_j b_{i\mathfrak{p}\varepsilon_j} = 0$ for $i = 1, \ldots, n_{\mathfrak{p}}$. This implies that the image of $\hat{\varepsilon}_1^{c_1} \cdot \cdots \cdot \hat{\varepsilon}_m^{c_m}$ in $U_{\mathfrak{p}}$ is a root of unity of p-power order, which means that for big enough natural n, the image of $(\hat{\varepsilon}_1^{c_1} \cdot \cdots \cdot \hat{\varepsilon}_m^{c_m})^{p^n}$ in $U_{\mathfrak{p}}$ is unity. Varying $\mathfrak{p}$, it follows that for big enough natural n

$$\ell\,(\hat{\varepsilon}_1^{c_1} \cdot \cdots \cdot \hat{\varepsilon}_m^{c_m})^{p^n} = 1$$

(see subsection 1.5). The Leopoldt Conjecture states the injectivity of ℓ, thus one gets that $\hat{\varepsilon}_1^{c_1} \cdot \cdots \cdot \hat{\varepsilon}_m^{c_m} = \hat{\varepsilon}$ for some root of unity ε. Since only powers of p contribute to pro-p-completion, it may be assumed that ε is a root of unity of p-th power order.

In the case $\delta(K) = 0$, this implies $a_{\mathfrak{p}\varepsilon} = 0$ and therefore $\alpha_{\mathfrak{p}} = 0$. In the case $\delta(K) = 1$ let the product in (15) run through $S^* = S - \{\mathfrak{p}_0\}$. This means $a_{\mathfrak{p}_0\varepsilon} = 0$ and $\varepsilon = 1$, hence $\alpha_{\mathfrak{p}} = 0$. This proves Theorem 3.1 □

Remark. From the proof of the Theorem 3.1 (or from the Theorem 2.5 and Remark from subsection 2.1), it can be seen that under the condition of the Theorem 3.1, the group G can be represented in the form $G = F/R$, F is a free pro-p-group generated by $\{s_i\}_{i \in I}$, and R is generated by the relations $r_j = s_j^{p^{\kappa_j}} t_j$, $j \in J \subseteq I$, $\kappa_j \in \mathbb{N}$, $t_j \in F^{(2)}$.

4. DEGREE 3

For the simplification of notations only, let S be finite in this section. Suppose that the group $G = G_{K,S}(p)$ has a representation

$$1 \to R \to F \to G \to 1, \tag{17}$$

where the generators of F are the elements

$$s_1, \dots, s_n, s_{n+1}, \dots, s_N, \tag{18}$$

and the relations have the form

$$r_i = s^{p^{\kappa_i}} t_i, \quad t_i \in F^{(2)}, \quad \kappa_i \in \mathbb{N}, \quad i = 1, \dots, n. \tag{19}$$

The multiplicator free groups and the groups satisfying the conditions of Theorem 3.1 can be represented this way.

One can describe $G^{(3)}/G^{(4)} \cong \dfrac{F^{(3)}}{(R \cap F^{(3)})F^{(4)}}$ as follows: every element ρ of $R \cap F^{(3)}$ modulo $F^{(4)}$ can be written in the form

$$\rho = r_1^{a_1} \dots r_n^{a_n} (x_1, r_1) \dots (x_n, r_n)(y_1, (z_1, r_1)) \dots (y_n, (z_n, r_n))F^{(4)},$$

where $a_i \in \mathbb{Z}$, $x_i, y_i, z_i \in F$ for $1 \leq i \leq n$.

As $\rho \in F^{(2)}$, the form of the relations r_i gives that $a_i = 0$ for $1 \leq i \leq n$. Note that $(x_1, r_1) \dots (x_n, r_n) \in F^{(3)}$ is equivalent to

$$(x_1, s_1)^{p^{e_1}} \cdot \dots \cdot (x_n, s_n)^{p^{e_n}} \in F^{(3)}.$$

Using the isomorphism $F^{(2)}/F^{(3)} \cong F/F^{(2)} \wedge F/F^{(2)}$, one can write this in the form

$$\sum_{i=1}^{n} p^{e_i} \bar{x}_i \wedge \bar{s}_i = 0,$$

where $\bar{x}_i, \bar{s}_i$ are the images of x_i, s_i in $F/F^{(2)}$ respectively.

Every solution $(\bar{x}_1, \dots, \bar{x}_n)$ of this linear equation is obviously the linear combination of

$$(0, \dots, 0, \underbrace{\bar{s}_i}_{i\text{-th position}}, 0, \dots, 0)$$

for $1 \leq i \leq n$ and

$$(0, \dots, 0, \underbrace{\bar{s}_j}_{i\text{-th position}}, 0, \dots, 0, \underbrace{p^{e_i - e_j} \bar{s}_i}_{j\text{-th position}}, 0, \dots, 0)$$

for $1 \le i \le n$, $1 \le j \le n$, $i \ne j$, $e_i \ge e_j$.

Thus, $(R \cap F^{(3)})F^{(4)}/F^{(4)}$ is generated as a $\mathbb{Z}_p$ module by

$$(x,(y,r_i)), \quad x, y \in F, \quad 1 \le i \le n, \tag{20}$$

and

$$\begin{gathered}(s_i,t_i), \quad 1 \le i \le n,\\ (s_j,t_i)\cdot(s_i^{p^{e_i-e_j}},t_j), \quad 1 \le i,j \le n, \quad i \ne j, \quad e_i \ge e_j.\end{gathered} \tag{21}$$

Note that the relations (20) are trivial in $\mathcal{L}_3(G^{\mathrm{ab}})$, thus one gets

Proposition 4.1. *If a pro-p-group G has a representation* (17) *with generators* (18) *and relations* (19), *then the kernel of the map*

$$\mathcal{L}_3(G^{\mathrm{ab}}) \to L_3(G)$$

is generated by the images of the elements (21) *in* $G^{(3)}/G^{(4)}$.

Remark. Under assumptions of Theorem 3.1, one can apply the results of [**Ko**]. Namely, one can use the procedure described there to obtain the representation (17). The generator images in G^{ab} are determined by the Artin map, and a minimal system of relations can be given. The form of these relations is determined by the procedure only modulo $F^{(3)}F^{(2)^p}$ (modulo $F^{(3)}$ in some cases). One can then easily change generators so that the representation in new generators will be of the form (18), (19). Taking the inverse image in $C_{K,S}(p)$ of the elements (21) written in these new generators, one obtain the kernel of the map

$$\Phi_3 : \mathcal{L}_3(C_{K,S}(p)) \to L_3(G_{K,S}(p)).$$

5. THE KERNEL OF Φ FOR SOME GROUPS $G_{K,S}(P)$

In this section, the kernel of the homomorphism $\mathcal{L}(G) \to L(G)$ for a specific class of pro-p-groups $G = F/R$ will be computed in terms of generators and relations. The result shows that only the initial form modulo $F^{(3)}$ of relations from R is essential. Moreover, the membership of the group G in the above class depends only on the initial forms of relations modulo $F^{(2)^p}F^{(3)}$. Thus, the information obtained by the method of [**Ko**], is sufficient to answer this question for the group $G_{K,S}(p)$. In the next section, examples of such groups $G_{K,S}(p)$ will be given.

The structure of this section follows [**KKL**]. Therefore, some proofs are similar to ones there. In particular, Propositions 5.2 – 5.4 and Lemma 5.13 are taken from that article (for Lemma 5.13, I give a more detailed proof than in [**KKL**]).

In this section it is supposed that $p > 2$.

5.1. p-filtrations

The special filtrations (G_i) of a pro-p group G with the property

$$G_i^p \subseteq G_{i+1}$$

are called p-filtrations.

If (G_i) is a p-filtration of G, then $L(G)$ is an $\mathbb{F}_p$-Lie algebra with a homogeneous operator π of degree 1 defined by

$$\pi(gG_{i+1}) = g^p G_{i+2}, \;\; i = 1, 2, \dots .$$

Using induction over s, one can prove that

$$(gh)^s \equiv g^s h^s (g,h)^{\frac{s(s-1)}{2}} (\mathrm{mod}\;\; G_{i+j+1}) \;\text{ for }\; g \in G_i, h \in G_j.$$

This shows that π is linear for $p > 2$. Using the equations (1), one can prove by induction over s that

$$(g^s, h) \equiv (g,h)^s ((g,h),g)^{\frac{s(s-1)}{2}} (\mathrm{mod} G_{2i+j+1}) \;\text{ for }\; g \in G_i, h \in G_j.$$

This shows that

$$\pi[a,b] = [\pi a, b]$$

if $a \in L_i(G)$, $b \in L_j(G)$. Summarizing, $L(G)$ is a graded $\mathbb{F}_p[\pi]$-Lie algebra if $p > 2$ and (G_m) $(m \geq 1)$ is a p-filtration.

5.2. κ and(κ, p)-filtrations

Let F be a free pro-p-group with generators $s_1, \dots, s_N$. For our purposes we need special filtrations called κ-filtrations, and corresponding p-filtrations, called (κ, p)-filtrations, introduced in [**Lz**, II.3.2], in greater generality.

For the definitions of these filtrations, consider the completed group algebra $A := \mathbb{Z}_p[[F]]$, which is isomorphic to the ring $\mathbb{Z}_p[[X_1, \dots, X_N]]$ of associative noncommutative formal power series in the variables $X_1, \dots, X_N$ with coefficients in $\mathbb{Z}_p$. The isomorphism α is defined by $\alpha(s_i) = 1 + X_i$ (see [**Ko**] or [**Se2**]). One can identify A and $\mathbb{Z}_p[[X_1, \dots, X_N]]$ by means of α. The restriction of α to F yields the Magnus representation of F.

For some natural number κ, let v be the valuation of A in the sense of Lazard ([**Lz**], I.2.2), so that

$$v(\sum_{i_1,\dots,i_k} a_{i_1,\dots,i_k} X_{i_1} \cdots X_{i_k}) = \inf_{i_1,\dots,i_k} \{b_{i_1,\dots,i_k}\}$$

with

$$b_{i_1,\dots,i_k} = \nu_p(a_{i_1,\dots,i_k}) + k \cdot \kappa,$$

where ν_p denotes the p-adic valuation of $\mathbb{Z}_p$. Then v defines a filtration (A^i) of A with

$$A^i := \{u \in A \mid v(u) \geq i\}.$$

One can define the (κ, p)-filtration of F by

$$\hat{F}^{(i)} := \{x \in F \mid v(x-1) \geq i\}.$$

The associated Lie algebra $\hat{L} = \sum_{m=1}^{\infty} \hat{L}_m$ is an $\mathbb{F}_p[\pi]$-Lie algebra.

In the same way, one can define the κ-filtration $(\tilde{F}^{(n)})$ by means of the valuation w of A that is given by

$$w(\sum_{i_1,\dots,i_k} a_{i_1,\dots,i_k} X_{i_1} \cdots X_{i_k}) = \inf_{\substack{i_1,\dots,i_k \\ a_{i_1,\dots,i_k} \neq 0}} \{c_{i_1,\dots,i_k}\}$$

with

$$c_{i_1,\dots,i_k} = k \cdot \kappa.$$

Define a filtration (B^i) of A as follows:

$$B^i := \{u \in A \mid w(u) \geq i\}.$$

Then

$$\tilde{F}^{(i)} := \{x \in F \mid w(x-1) \geq i\}.$$

The associated graded Lie algebra will be denoted with $\tilde{L} = \sum_{m=1}^{\infty} \tilde{L}_m$. The Lie algebra $\tilde{L}$ is a free Lie algebra over $\mathbb{Z}_p$ on the images of $s_1, \dots, s_N$ in $L_\kappa = \tilde{F}^{(\kappa)}/\tilde{F}^{(\kappa+1)}$. Note that the κ-filtration $(\tilde{F}^{(m)})$ actually only differs from the filtration $(F^{(m)})$ by scaling: if $(\ell-1)\kappa + 1 \leq m \leq \ell\kappa$, then $\tilde{F}^{(m)} = F^{(\ell)}$.

Let $\overline{L}$ be the sub-Lie-algebra of $\hat{L}$ generated by $\hat{\sigma}_i := s_i \hat{F}^{(\kappa+1)}$ ($i = 1, \dots, N$), and let

$$\overline{L}_m := \hat{L}_m \cap \overline{L}, \quad m = 1, 2, \dots .$$

Then $\overline{L}_m = \{0\}$ if $m \not\equiv 0 \pmod{\kappa}$.

The following structure theorem for $\hat{L}$ holds:

THEOREM 5.1. *$\overline{L}$ is the free $\mathbb{F}_p$-Lie algebra with generators $\hat{\sigma}_1, \dots, \hat{\sigma}_N$, and $\hat{L}$ is the free $\mathbb{F}_p[\pi]$-Lie algebra with generators $\hat{\sigma}_1, \dots, \hat{\sigma}_N$.*

This result is well known. It is proved in [**Lz**, II.3.2], and goes back to A. Skopin ([**Sk**]).

5.3. Relation between κ- and (κ, p)-filtrations

PROPOSITION 5.2.

$$\begin{aligned} p^h \tilde{L}_m &= (\tilde{F}^{(m)} \cap \hat{F}^{(m+h)})\tilde{F}^{(m+1)}/\tilde{F}^{(m+1)} \\ &= (\tilde{F}^{(m)} \cap \hat{F}^{(m+h)}\tilde{F}^{(m+1)})/\tilde{F}^{(m+1)}. \end{aligned}$$

PROOF. An element in $p^h \tilde{L}_m$ has the form $x^{p^h}\tilde{F}^{m+1}$ with $x \in \tilde{F}^{(m)}$. Therefore, $x^{p^h} \in \hat{F}^{(m+h)}$. Let y be an element of $\tilde{F}^{(m)} \cap \hat{F}^{(m+h)}$. One has to show that $y\tilde{F}^{(m+1)}$ is in $p^h \tilde{L}_m$.

Assume that $m = \kappa l$ with $l \in \mathbb{N}$. Then

$$y \equiv 1 + y_m \pmod{B^{m+1}},$$

where y_m is a homogeneous polynomial of degree l in A. Furthermore, $y \in \hat{F}^{(m+h)}$ if and only if $y_m \in A^{m+h}$. This is possible only if each coefficient of the polynomial y_m is divisible by p^h. Hence, y has the form

$$y \equiv 1 + p^h z_m \pmod{B^{m+1}}$$

with $z_m \in B^m$. By theorem of Witt ([**Wi**]), z_m is a Lie polynomial in A. Hence, there is a $z \in \tilde{F}^{(m)}$ such that $z \equiv 1 + z_m \pmod{B^{m+1}}$, and this implies $z^{p^h}\tilde{F}^{(m+1)} = y\tilde{F}^{(m+1)} \in p^h \tilde{L}_m$. $\square$

By Theorem 5.1, the group $\hat{L}_m$ has the form

$$\hat{L}_m = \bigoplus_{l=0}^{m-1} \pi^l \overline{L}_{m-l}.$$

Define a filtration $(\hat{L}_m^{(h)})_{1\le h\le m}$ of $\hat{L}_m$ by

$$\hat{L}_m^{(h)} := \bigoplus_{l=0}^{m-h} \pi^l \overline{L}_{m-l}.$$

PROPOSITION 5.3.

$$\hat{L}_m^{(h)} = (\hat{F}^{(m)} \cap \tilde{F}^{(h)})\hat{F}^{(m+1)}/\hat{F}^{(m+1)} = (\hat{F}^{(m)} \cap \tilde{F}^{(h)}\hat{F}^{(m+1)})/\hat{F}^{(m+1)}.$$

The proof of this proposition is a variation of the proof of Theorem 5.1.

Now define the following maps $\omega_{h,m}$ from $p^h\tilde{L}_m$ onto $\pi^h\overline{L}_m$, which allow us to compare $\tilde{L}$ with $\hat{L}$:

$$\begin{aligned}\omega_{h,m} : p^h\tilde{L}_m &= (\tilde{F}^{(m)} \cap \hat{F}^{(m+h)})\tilde{F}^{(m+1)}/\tilde{F}^{(m+1)} \\ &\longrightarrow (\tilde{F}^{(m)} \cap \hat{F}^{(m+h)})\tilde{F}^{(m+1)}\hat{F}^{(m+h+1)}/\tilde{F}^{(m+1)}\hat{F}^{(m+h+1)} \\ &\xrightarrow{\cong} (\tilde{F}^{(m)}\hat{F}^{(m+h+1)} \cap \hat{F}^{(m+h)})/(\tilde{F}^{(m+1)}\hat{F}^{(m+h+1)} \cap \hat{F}^{(m+h)}) \\ &\xrightarrow{\cong} \hat{L}_{m+h}^{(m)}/\hat{L}_{m+h}^{(m+1)} \xrightarrow{\cong} \pi^h\overline{L}_m,\end{aligned}$$

where the arrows denote the corresponding natural maps.

PROPOSITION 5.4. $\ker\omega_{h,m} = p^{h+1}\tilde{L}_m$.

PROOF. By definition

$$\ker\omega_{h,m} = (\tilde{F}^{(m)} \cap \hat{F}^{(m+h+1)}\tilde{F}^{(m+1)})/\tilde{F}^{(m+1)} = p^{h+1}\tilde{L}_m.$$

□

5.4. Annulators in free Lie algebras

We need the following result for abstract free Lie algebras.

Let L be a free graded Lie algebra over a commutative ring with unit k, and U be its universal enveloping algebra.

Let $e_1, \ldots, e_n$ be the standard basis of U^n, so that

$$e_i = (0, \ldots, 0, \underset{i\text{-th. position}}{1}, 0\ldots, 0).$$

For a vector $x = (x_1, \ldots, x_n) \in L^n$, let

$$e_i^{(x)}(u) := (\mathrm{ad}(u)x_i)ue_i \quad (22)$$

$$e_{ij}^{(x)}(u,v) := (\mathrm{ad}(v)x_j)ue_i + (\mathrm{ad}(u)x_i)ve_j. \quad (23)$$

Denote by $E(x)$ the U-submodule of U^n generated by $e_i^{(x)}(u)$ and $e_{ij}^{(x)}(u,v)$ for $1 \le i,j \le n$, $u,v \in U$.

Obviously, if $(u_1, \ldots, u_n) \in E(x)$, then $\mathrm{ad}(u_1)x_1 + \cdots + \mathrm{ad}(u_n)x_n = 0$.

Note that if I is some ideal of L, then $I/[I, I]$ has the module structure over the universal enveloping algebra of L/I induced by the adjoint representation of U in L.

THEOREM 5.5. *Let $x = (x_1, \dots, x_n)$, where $x_1, \dots, x_n \in L$ and the following four conditions hold:*

(1). *$x_1, \dots, x_n$ are homogeneous.*
(2). *The ideal $I \subseteq L$ generated by $x_1, \dots, x_n$ is a free Lie algebra over k.*
(3). *L/I is a free k-module.*
(4). *$I/[I, I]$ is a free W-module with generators $\bar{x}_1, \dots, \bar{x}_n$, where W is the universal enveloping algebra of L/I and $\bar{x}_i$ are the images of x_i in $I/[I, I]$.*

Then

$$\mathrm{ad}(u_1)x_1 + \cdots + \mathrm{ad}(u_n)x_n = 0$$

if and only if

$$(u_1, \dots, u_n) \in E(x).$$

For a proof, see Theorem 2 of [**La2**]

COROLLARY 5.6. *If $x_1, \dots, x_n$ is a part of a free generating system of L, then* $\mathrm{ad}(u_1)x_1 + \cdots + \mathrm{ad}(u_n)x_n = 0$ *if and only if $(u_1, \dots, u_n) \in E(x)$.*

The proof of Corollary 5.6 is obvious. Note also, that it is just Corollary 1 from Theorem 2, [**La2**].

Note that later Theorem 5.5 and Corollary 5.6 will be applied only for the Lie algebra over $\mathbb{F}_p$ with $p > 2$, and in this case the elements (22) could be expressed via the elements (23), and so the module $E(x)$ is generated by the elements (23) only. By [**Reu**, Theorem 2.5], the condition (2) of Theorem 5.5 holds automatically, if k is a field, as well as the condition (3).

5.5. Triviality of the kernel for (κ, p)-filtration

Let

$$1 \to R \to F \to G \to 1$$

be the representation of the pro-p-group G as a factor group of the free pro-p-group F. Let $r_1, \dots, r_n$ be the generators of R. Let $\hat{\rho}_1, \dots, \hat{\rho}_n$ be the their initial forms relative to the (κ, p)-filtration $(\hat{F}^{(m)})$; that is, if $r_i \in \hat{F}^{(m_i)}$, $r_i \notin \hat{F}^{(m_i+1)}$, then

$$\hat{\rho}_i = r_i \hat{F}^{(m_i+1)}.$$

Let $\mathfrak{r}$ be the $\mathbb{F}_p[\pi]$ ideal of $\hat{L}$ generated by the elements $\hat{\rho}_i$ $(1 \le i \le n)$ and $\hat{V}$ be the universal enveloping algebra of $\hat{L}/\mathfrak{r}$. Obviously, $\mathfrak{r}/[\mathfrak{r},\mathfrak{r}]$ has a structure of a $\hat{V}$-module via adjoint representation. The following theorem is then a special case of [**La3**, Theorem 3]:

THEOREM 5.7. *If the conditions*

1). $\hat{L}/\mathfrak{r}$ *is a free* $\mathbb{F}_p[\pi]$ *module,*
2). $\mathfrak{r}/[\mathfrak{r},\mathfrak{r}]$ *is a free* $\hat{V}$*-module with generators* $\hat{\rho}_1, \dots, \hat{\rho}_n$*,*

hold, then

$$\mathfrak{r} = \mathrm{gr}(R),$$

where $\mathrm{gr}(R) = \bigoplus_{i=1}^{\infty} (R \cap \hat{F}^{(i)})\hat{F}^{(i)}/\hat{F}^{(i+1)}$.

5.6. Formulation of the main result

Now the main result of this thesis can be formulated. Some notations necessary for its formulation are duplicated in this subsection. Recall that $p > 2$.

Let G be a finitely generated pro-p-group represented as the factor of free pro-p-group F with generators $s_1, \dots, s_N$, by the relation subgroup R, generated by $r_1, \dots, r_n$, $n \le N$, of the form

$$r_i = s_i^{p^\kappa} t_i, \quad t_i \in F^{(2)}$$

for some $\kappa \in \mathbb{N}$. Recall that κ is the same for all relations.

Let $L(F)$ (resp. $L(G)$) be the $\mathbb{Z}_p$ Lie algebra associated to the lower central series of F (resp. G) and U be the universal enveloping algebra of $L(F)$. Let $\sigma_j = s_j F^{(2)}$, $\tau_i = t_i F^{(3)}$.

Let $\hat{L}$ be the $\mathbb{F}_p[\pi]$ Lie algebra associated to the (κ, p)-filtration $\hat{F}^{(m)}$ of F (see subsection 5.2). Let $\hat{\sigma}_j = s_j \hat{F}^{(\kappa+1)}$, $\hat{\tau}_i = t_i \hat{F}^{(2\kappa+1)}$, $\hat{\rho}_i = r_i \hat{F}^{(2\kappa+1)}$. Let $\mathfrak{r}$ be the ideal generated by $\hat{\rho}_1, \dots, \hat{\rho}_n$ and $\hat{V}$ be the universal enveloping algebra of $\hat{L}/\mathfrak{r}$.

Let $\overline{L}$ be the $\mathbb{F}_p$ Lie subalgebra generated by $\hat{\sigma}_1, \dots, \hat{\sigma}_N$ in $\hat{L}$. Let $\overline{I}$ be the ideal generated by $\hat{\tau}_1 \dots \hat{\tau}_n$ and $\overline{W}$ be the universal enveloping algebra of $\overline{L}/\overline{I}$.

THEOREM 5.8. *If*
(1). $\overline{I}/[\overline{I},\overline{I}]$ *is a free* $\overline{W}$*-module,*
(2). $\mathfrak{r}/[\mathfrak{r},\mathfrak{r}]$ *is a free* $\hat{V}$*-module,*
then the kernel of the surjective homomorphism

$$L(F) \to L(G),$$

induced by the surjection $F \to G$*, is generated as an ideal by the elements*

$$p^{\kappa}\sigma_i, \;\; [\mathrm{ad}(\lambda)\sigma_i, \mathrm{ad}(\mu)\tau_j] + [\mathrm{ad}(\mu)\sigma_j, \mathrm{ad}(\lambda)\tau_i]$$

for $\lambda, \mu \in U$, $1 \leq i, j \leq n$.

REMARKS. It is sufficient to consider only homogeneous elements $\lambda, \mu \in U$. The kernel is generated as a $\mathbb{Z}_p$-module by the elements

$$p^{\kappa}\mathrm{ad}(\lambda)\sigma_i, \;\; [\mathrm{ad}(\lambda)\sigma_i, \mathrm{ad}(\mu)\tau_j] + [\mathrm{ad}(\mu)\sigma_j, \mathrm{ad}(\lambda)\tau_i]$$

for $\lambda, \mu \in U$, $1 \leq i, j \leq n$.

The author is thankful to J. P. Labute for pointing out that analogous results were obtained in [**La4**] for link group. Though these groups are different from the groups considered here, the author hopes that for some "mixed" case, with some relations like in this thesis and some like in [**La4**], the analogous could be obtained. This will allow us to weaken or avoid the "multiplicator free" assumption.

The rest of this section gives the proof of Theorem 5.8.

5.7. The ideals $\mathcal{N}(R)$, $\mathcal{N}'(R)$

The projection $F \to G$ induces a surjective homomorphism

$$\theta : L(F) \to L(G).$$

Let

$$\begin{aligned} R^{(m)} &:= R \cap F^{(m)}, \\ \mathcal{N}_m(R) &:= R^{(m)}F^{(m+1)}/F^{(m+1)}, \\ \mathcal{N}(R) &:= \sum_{m=1}^{\infty} \mathcal{N}_m(R). \end{aligned}$$

The next proposition gives the description of θ:

PROPOSITION 5.9. $\mathcal{N}_m(R)$ *is the kernel of* $\theta_m : L_m(F) \to L_m(F/R)$.

PROOF. One has

$$\begin{aligned} L_m(F/R) &= (F/R)^{(m)}/(F/R)^{(m+1)} \\ &\cong F^{(m)}R/F^{(m+1)}R \cong F^{(m)}/F^{(m+1)}(F^{(m)} \cap R). \end{aligned}$$

Hence,

$$\ker\theta_m = F^{(m+1)}(F^{(m)} \cap R)/F^{(m+1)}.$$

□

Let U be the universal enveloping algebra of $L(F)$. Since the $\mathbb{Z}_p$-Lie algebra $L(F)$ is the free algebra generated by

$$\{\sigma_i := s_i F^{(2)} \mid i = 1, \ldots, N\},$$

one can identify U with the ring of polynomials in the non-commutative indeterminants $\sigma_1, \ldots, \sigma_N$ with coefficients in $\mathbb{Z}_p$. The ring U operates on $L(F)$ by adjoint representation.

Let

$$\sigma_i := s_i F^{(2)} \in L_1(F), \qquad 1 \le i \le N,$$
$$\tau_i := t_i F^{(3)} \in L_2(F), \qquad 1 \le i \le n.$$

Let $\mathcal{N}'(R)$ be the ideal of $L(F)$ generated by the elements

$$p^\kappa \sigma_i, \ \ [\mathrm{ad}(\lambda)\sigma_i, \mathrm{ad}(\mu)\tau_j] + [\mathrm{ad}(\mu)\sigma_j, \mathrm{ad}(\lambda)\tau_i], \tag{24}$$

where $\lambda, \mu \in U$, $1 \le i, j \le n$. By linearity, one can assume of course that λ, μ are homogeneous.

The goal of the next four subsections is the proof of the following theorem:

THEOREM 5.10. *If the conditions of Theorem 5.7 hold, then*

$$\mathcal{N}(R) = \mathcal{N}'(R).$$

5.8. Inclusion $\mathcal{N}'(R) \subseteq \mathcal{N}(R)$

Obviously,

$$r_i F^{(2)} = s_i^{p^\kappa} F^{(2)} = p^\kappa \sigma_i$$

for $1 \le i \le n$.

To show that the elements of the form

$$[\mathrm{ad}(\lambda)\sigma_i, \mathrm{ad}(\mu)\tau_j] + [\mathrm{ad}(\mu)\sigma_j, \mathrm{ad}(\lambda)\tau_i]$$

lie in $\mathcal{N}(R)$, assume that

$$\lambda = \sigma_{i_1} \cdots \sigma_{i_l} \ \text{ and } \ \mu = \sigma_{j_1} \cdots \sigma_{j_k}.$$

Then

$$\begin{aligned}
[\mathrm{ad}(\lambda)\sigma_i, &\mathrm{ad}(\mu)(\tau_j)] + [\mathrm{ad}(\mu)\sigma_j, \mathrm{ad}(\lambda)\tau_i] \\
&= ((s_{i_1}, \ldots, (s_{i_l}, s_i) \ldots), (s_{j_1}, \ldots, (s_{j_k}, t_j) \ldots)) \\
&\cdot ((s_{j_1}, \ldots, (s_{j_k}, s_j) \ldots), (s_{i_1}, \ldots, (s_{i_l}, t_i) \ldots)) F^{(l+k+4)}.
\end{aligned}$$

Since $r_i = s_i^{p^\kappa} t_i$, we have

$$\begin{aligned} (s_{i_l}, t) &\in (s_{i_l}, s_1^{-p^\kappa} r) F^{(4)} R \\ &= (s_{i_l}, s_1^{-p^\kappa}) F^{(4)} R \end{aligned}$$

and

$$\begin{aligned} (s_{i_l}, s_i^{-p^\kappa}) F^{(4)} &= (s_{i_l}, s_i)^{-p^\kappa} ((s_{i_l}, s_i), s_i)^{-p^\kappa(-p^\kappa+1)/2} F^{(4)} \\ &= (s_{i_l}, s_i)^{-p^\kappa} ((s_{i_l}, s_i), r)^{(p^\kappa-1)/2}) F^{(4)}. \end{aligned}$$

Proceeding analogously by induction, one gets:

$$\begin{aligned} &((s_{i_1}, \dots, (s_{i_l}, s_i) \dots), (s_{j_1}, \dots, (s_{j_k}, s_j^{-p^\kappa}) \dots)) \\ &\cdot ((s_{j_1}, \dots, (s_{j_k}, s_j) \dots), (s_{i_1}, \dots, (s_{i_l}, s_i^{-p^\kappa}) \dots)) RF^{(l+k+4)} \\ =&((s_{i_1}, \dots, (s_{i_l}, s_i) \dots), (s_{j_1}, \dots, (s_{j_k}, s_j) \dots))^{-p^\kappa} \\ &\cdot ((s_{j_1}, \dots, (s_{j_k}, s_j) \dots), (s_{i_1}, \dots, (s_{i_l}, s_i) \dots))^{-p^\kappa} RF^{(l+k+4)} \\ =&RF^{(l+k+4)}, \end{aligned}$$

which implies

$$[\mathrm{ad}(\lambda)\sigma_i, \mathrm{ad}(\mu)\tau_j] + [\mathrm{ad}(\mu)\sigma_j, \mathrm{ad}(\lambda)\tau_i] \in \mathcal{N}_{l+k+3}(R).$$

5.9. Reverse inclusion: notations

To show that $\mathcal{N}(R) \subseteq \mathcal{N}'(R)$ the technique of [**La1**] and [**KKL**] consisting of the comparison of the κ- and (κ, p)-filtrations of F, is used.

Let

$$\begin{aligned} \tilde{\sigma}_i &:= s_i \tilde{F}^{(\kappa+1)} \in \tilde{L}_\kappa,\ i = 1, \dots, N, \\ \tilde{\tau}_i &:= t_i \tilde{F}^{(2\kappa+1)} \in \tilde{L}_{2\kappa},\ i = 1, \dots, n, \\ \tilde{\mathcal{N}}_m(R) &:= (R \cap \tilde{F}^{(m)}) \tilde{F}^{(m+1)} / \tilde{F}^{(m+1)}, \\ \tilde{\mathcal{N}}(R) &:= \bigoplus_{m=1}^{\infty} \tilde{\mathcal{N}}_m(R). \end{aligned}$$

Let $\tilde{\mathcal{N}}'(R)$ be the ideal of $\tilde{L}$ generated by $p^\kappa \tilde{\sigma}_i$ and $[\mathrm{ad}(\lambda)\tilde{\sigma}_i, \mathrm{ad}(\mu)\tilde{\tau}_j] + [\mathrm{ad}(\mu)\tilde{\sigma}_j, \mathrm{ad}(\lambda)\tilde{\tau}_i]$ for $1 \le i, j \le n$, $\lambda, \mu \in \tilde{U}$, where $\tilde{U}$ denotes the universal enveloping algebra of $\tilde{L}$.

$$\begin{aligned}\hat{\sigma}_i &:= s_i\hat{F}^{(\kappa+1)} \in \hat{L}_\kappa,\ i=1,\dots,N,\\ \hat{\tau}_i &:= t_i\hat{F}^{(2\kappa+1)} \in \hat{L}_{2\kappa},\ i=1,\dots,n,\\ \hat{\mathcal{N}}_m(R) &:= (R\cap\hat{F}^{(m)})\hat{F}^{(m+1)}/\hat{F}^{(m+1)},\\ \hat{\mathcal{N}}(R) &:= \bigoplus_{m=1}^{\infty}\hat{\mathcal{N}}_m(R).\end{aligned}$$

Let also

$$\begin{aligned}\hat{\sigma} &:= (\hat{\sigma}_1,\dots\hat{\sigma}_n),\\ \hat{\tau} &:= (\hat{\tau}_1,\dots\hat{\tau}_n)\end{aligned}$$

be n-dimensional vectors.

The homogeneous component $\hat{\mathcal{N}}_{2\kappa}(R)$ contains the elements

$$r_i\hat{F}^{(2\kappa+1)} = \pi^\kappa\hat{\sigma}_i + \hat{\tau}_i, \quad 1\le i\le n,$$

and, by assumption on the structure of $G = F/R$, the Theorem 5.7 holds for the group G. Thus, $\hat{\mathcal{N}}(R)$ is generated as an ideal of $\hat{L}$ by the elements $\pi^\kappa\hat{\sigma}_i + \hat{\tau}_i$, which is the initial point of the proof of Theorem 5.10.

Note that since the grading of $\tilde{L}$ is only the rescaling of the grading of L, to prove Theorem 5.10, it is enough to show that $\tilde{\mathcal{N}}'(R) \supseteq \tilde{\mathcal{N}}(R)$.

Let $\overline{U}$ be the universal enveloping algebra of $\overline{L}$. Then $\overline{U}$ can be identified with the $\mathbb{F}_p$-subalgebra of the universal enveloping algebra $\hat{U}$ of $\hat{L}$ generated by $\hat{\sigma}_1,\cdots,\hat{\sigma}_N$. Any non-zero homogeneous element λ of $\hat{U}$ can be uniquely written in the form

$$\lambda = \lambda^{(0)} + \pi\lambda^{(1)} + \cdots + \pi^\ell\lambda^{(\ell)}$$

with $\lambda^{(0)},\lambda^{(1)},\cdots,\lambda^{(\ell)}\in\overline{U}$ and $\lambda_{(\ell)}\neq 0$. Thus, for any n-dimensional vector of homogeneous elements $(\lambda_1,\dots,\lambda_n)$, where all nonzero elements have equal degrees, there exist unique degree ℓ and coefficients $\lambda_i^{(j)}\in\overline{U}$, $1\le i\le n$, $0\le j\le\ell$, such that

$$\lambda_i = \lambda_i^{(0)} + \pi\lambda_i^{(1)} + \cdots + \pi^\ell\lambda_i^{(\ell)}$$

and $\lambda_i^{(\ell)}\neq 0$ for some i.

Since $\deg(\lambda_i^{(\ell)}) \equiv 0 \pmod{\kappa}$ and $\deg(\lambda_i^{(\ell-j)}) = \deg(\lambda_i^{(\ell)}) + j$, one has $\lambda_i^{(j)} = 0$ if $j\not\equiv\ell \pmod{\kappa}$.

Let $\overline{\mathcal{I}}$ be the ideal of $\overline{L}$ generated by the elements $\hat{\sigma}_1,\dots,\hat{\sigma}_n$, and let $\overline{\mathcal{N}}$ be the ideal of $\overline{L}$ generated by the elements of the form

$$[\mathrm{ad}(\lambda)\sigma_i, \mathrm{ad}(\mu)\tau_j] + [\mathrm{ad}(\mu)\sigma_j, \mathrm{ad}(\lambda)\tau_i]$$

with $1\le i,j\le n$, $\lambda,\mu\in\overline{U}$.

5.10. Reverse inclusion: key lemma

The following lemma plays the key role and determines the form of the elements (24) which generate the kernel under investigation.

LEMMA 5.11.

$$\hat{\mathcal{N}}_m(R) \cap \hat{L}_m^{(m-j)} \subseteq \begin{cases} \pi^j \overline{\mathcal{N}}_{m-j} + \hat{L}_m^{(m-j+1)} & \textit{if } j < \kappa \\ \pi^j \,\overline{\mathcal{I}}_{m-j} + \hat{L}_m^{(m-j+1)} & \textit{if } j \geq \kappa. \end{cases}$$

PROOF. Any element ρ of $\hat{\mathcal{N}}_m(R)$ has the form

$$\sum_{i=1}^{n} \mathrm{ad}(\lambda_i)(\pi^\kappa \hat{\sigma}_i + \hat{\tau}_i)$$

with λ_i as above. If $\ell = d\kappa + e$ with $0 \leq e < \kappa$, then

$$\begin{aligned} \rho = \pi^e \sum_{i=1}^{n} \mathrm{ad}(\lambda_i^{(e)})\hat{\tau}_i \\ + \sum_{j=1}^{d} \pi^{e+j\kappa} \sum_{i=1}^{n} \Big(\mathrm{ad}(\lambda_i^{(e+(j-1)\kappa)})\hat{\sigma}_i + \mathrm{ad}(\lambda_i^{(e+j\kappa)})\hat{\tau}_i \Big) \\ + \pi^{\ell+\kappa} \sum_{i=1}^{n} \mathrm{ad}(\lambda_i^{(\ell)})\hat{\sigma}_i. \end{aligned}$$

If $\sum_{i=1}^{n} \mathrm{ad}(\lambda_i^{(\ell)})\hat{\sigma}_i \neq 0$, then

$$\rho \in \pi^{\ell+\kappa} \overline{\mathcal{I}}_{m-(\ell+\kappa)} + \hat{L}_m^{(m-(\ell+\kappa)+1)},$$

which yields the required result.

Now suppose that $\sum_{i=1}^{n} \mathrm{ad}(\lambda_i^{(\ell)})\hat{\sigma}_i = 0$.

Then, by Corollary 5.6 (applied to the Lie algebra $\overline{L}$ over $\mathbb{F}_p$ and the elements $\hat{\sigma}_1, \ldots, \hat{\sigma}_n$),

$$\lambda^{(\ell)} := (\lambda_1^{(\ell)}, \ldots, \lambda_n^{(\ell)}) = \sum_{i=1}^{n} \lambda_i^{(\ell)} e_i \in E(\hat{\sigma});$$

i.e.,

$$\lambda^{(\ell)} = \sum_{\substack{1 \leq i,j \leq n \\ u,v \in \overline{U}}} c_{ijuv} \left((\mathrm{ad}(v)\hat{\sigma}_j) u e_i + (\mathrm{ad}(u)\hat{\sigma}_i) v e_j \right)$$

for some $c_{ijuv} \in \overline{U}$.

If $\ell < \kappa$, one has

$$\rho = \pi^{\ell} \sum_{\substack{1\le i,j\le n\\ u,v\in\overline{\overline{U}}}} c_{ijuv}\left([\mathrm{ad}(v)\hat{\sigma}_j, \mathrm{ad}(u)\hat{\tau}_i] + [\mathrm{ad}(u)\hat{\sigma}_i, \mathrm{ad}(v)\hat{\tau}_j]\right) \in \pi^{\ell}\overline{\mathcal{N}}_{m-\ell},$$

as required. If $\ell \geq \kappa$ and $\sum_{i=1}^{n}\left(\mathrm{ad}(\lambda_i^{(\ell-\kappa)})\hat{\sigma}_i + \mathrm{ad}(\lambda_i^{(\ell)})\hat{\tau}_i\right) \neq 0$, then

$$\rho \in \pi^{\ell}\overline{\mathcal{I}}_{m-\ell} + \hat{L}_m^{(m-\ell+1)},$$

as required. If $\sum_{i=1}^{n}\left(\mathrm{ad}(\lambda_i^{(\ell-\kappa)})\hat{\sigma}_i + \mathrm{ad}(\lambda_i^{(\ell)})\hat{\tau}_i\right) = 0$, then

$$\begin{aligned}
\sum_{i=1}^{n}\mathrm{ad}(\lambda_i^{(\ell-\kappa)})\hat{\sigma}_i &= -\sum_{i=1}^{n}\mathrm{ad}(\lambda_i^{(\ell)})\hat{\tau} \\
&= -\sum_{\substack{1\le i,j\le n\\ u,v\in\overline{\overline{U}}}} c_{ijuv}\left(\mathrm{ad}((\mathrm{ad}(v)\hat{\sigma}_j)u)\hat{\tau}_i + \mathrm{ad}((\mathrm{ad}(u)\hat{\sigma}_i)v)\hat{\tau}_j\right) \\
&= -\sum_{\substack{1\le i,j\le n\\ u,v\in\overline{\overline{U}}}} c_{ijuv}\left([\mathrm{ad}(v)\hat{\sigma}_j, \mathrm{ad}(u)\hat{\tau}_i] + [\mathrm{ad}(u)\hat{\sigma}_i, \mathrm{ad}(v)\hat{\tau}_j]\right) \\
&= \sum_{\substack{1\le i,j\le n\\ u,v\in\overline{\overline{U}}}} c_{ijvu}\left([\mathrm{ad}(v)\hat{\tau}_j, \mathrm{ad}(u)\hat{\sigma}_i)] + [\mathrm{ad}(u)\hat{\tau}_i, \mathrm{ad}(v)\hat{\sigma}_j)]\right) \\
&= \sum_{\substack{1\le i,j\le n\\ u,v\in\overline{\overline{U}}}} c_{ijuv}\left(\mathrm{ad}((\mathrm{ad}(v)\hat{\tau}_j)u)\hat{\sigma}_i + \mathrm{ad}((\mathrm{ad}(u)\hat{\tau}_i)v)\hat{\sigma}_j\right).
\end{aligned}$$

Hence, there exists $y = (y_1, \ldots, y_n) \in E(\hat{\tau})$ such that

$$\sum_{i=1}^{n}\mathrm{ad}(\lambda_i^{(\ell-\kappa)})\hat{\sigma}_i = \sum_{i=1}^{n}\mathrm{ad}(y_i)\hat{\sigma}_i.$$

By Corollary 5.6, $\lambda^{(\ell-\kappa)} - y \in E(\hat{\sigma})$, thus $\lambda^{(\ell-\kappa)} \in E(\hat{\sigma}) + E(\hat{\tau})$.

If $\sum_{i=1}^{n}\left(\mathrm{ad}(\lambda_i^{(\ell-(j+1)\kappa}))\hat{\sigma}_i + \mathrm{ad}(\lambda_i^{(\ell-j\kappa)})\hat{\tau}_i\right) = 0$ for $1 \leq j \leq d$, then, repeating the above argument, one gets

$$\lambda^{(e)} \in E(\hat{\sigma}) + E(\hat{\tau}),$$

which yields $\rho \in \pi^{e}\overline{\mathcal{N}}_{m-e}$. Otherwise, there exists an index j such that

$$\sum_{i=1}^{n}\left(\mathrm{ad}(\lambda_i^{(\ell-(j-1)\kappa)})\hat{\sigma}_i + \mathrm{ad}(\lambda_i^{\ell-j\kappa})\hat{\tau}_i\right) \neq 0$$

and

$$\rho \in \pi^{\ell-j\kappa}\overline{\mathcal{I}}_{m-(\ell-j\kappa)} + \hat{L}_m^{(m-(\ell-j\kappa)+1)}.$$

□

COROLLARY 5.12.

$$\hat{\mathcal{N}}(R) \cap \overline{L} = \overline{\mathcal{N}}.$$

5.11. Reverse inclusion: rest of the proof

Consider now the homomorphism $\omega_{0,m}$ of $\tilde{L}_m$ onto $\overline{L}_m$. By Proposition 5.4, its kernel is $p\tilde{L}_m$. Furthermore, $\omega_{0,m}$ maps $\tilde{\mathcal{N}}_m(R)$ onto $\overline{L}_m \cap \hat{\mathcal{N}}_m(R) = \overline{\mathcal{N}}_m$. Hence,

$$\tilde{\mathcal{N}}(R) \subseteq \tilde{\mathcal{N}}'(R) + p\tilde{L}.$$

Using the homomorphisms $\omega_{h,m}$, it will be proved by induction that

$$\tilde{\mathcal{N}}(R) \subseteq \tilde{\mathcal{N}}'(R) + p^{1+h}\tilde{L},\ h = 0, 1, \ldots$$

LEMMA 5.13. $\tilde{\mathcal{N}}_m(R) \cap p^h\tilde{L}_m = (R \cap \tilde{F}^{(m)} \cap \hat{F}^{(m+h)})\tilde{F}^{(m+1)}/\tilde{F}^{(m+1)}$

PROOF. Let $\eta \in \tilde{\mathcal{N}}_m(R) \cap p^h\tilde{L}_m$. Then $\eta = yF^{(m+1)}$ with $y = u^{p^h}v \in R \cap \tilde{F}^{(m)}$, $u \in \tilde{F}^{(m)}$, $v \in \tilde{F}^{(m+1)}$.

Let ℓ be the largest natural number such that there exists $s \in R \cap \tilde{F}^{(m+1)}$ with $vs \in \hat{F}^{(m+\ell)} \cap \tilde{F}^{(m+1)}$. Note that $\ell \geq 1$, as $v \in \tilde{F}^{(m+1)} \cap \hat{F}^{(m+1)}$. If $\ell \geq h$, then $ys \in R \cap \tilde{F}^{(m)} \cap \hat{F}^{(m+h)}$, as required.

Assume that $\ell < h$. Let

$$M_\ell := \{s \in R \cap \tilde{F}^{(m+1)} | vs \in \hat{F}^{(m+\ell)} \cap \tilde{F}^{(m+1)}\},$$

and, for $s \in M_\ell$, let δ_s be the image of vs (and obviously also of ys) in $(\hat{F}^{(m+\ell)} \cap \tilde{F}^{(m+1)})\hat{F}^{(m+\ell+1)}/\hat{F}^{(m+\ell+1)}$. Then

$$\delta_s \in \hat{\mathcal{N}}_{m+\ell}(R) \cap \hat{L}^{(\ell-1)}_{m+\ell}. \tag{25}$$

Let c_s be the minimal integer such that

$$\delta_s \in \hat{\mathcal{N}}_{m+\ell}(R) \cap \hat{L}^{(c_s)}_{m+\ell}.$$

From (25) one gets $c_s \leq \ell - 1$.

Choose $s \in M_\ell$ such that $c := c_s$ is minimal. By Lemma 5.11,

$$\delta_s \in \begin{cases} \pi^c\overline{\mathcal{N}}_{m+\ell-c} + \hat{L}^{(c-1)}_{m+\ell} & \text{if } c < \kappa, \\ \pi^c\overline{\mathcal{I}}_{m+\ell-c} + \hat{L}^{(c-1)}_{m+\ell} & \text{if } c \geq \kappa. \end{cases}$$

As $\ell - c \geq 1$, using Corollary 5.12 in case $c < \kappa$ and obvious arguments in case $c \geq \kappa$, one gets that there exists an element $z \in R \cap \hat{F}^{(m+\ell)} \cap \tilde{F}^{(m+1)}$, such that for $\gamma = z\hat{F}^{(m+\ell+1)}$

$$\delta_s - \gamma \in \hat{L}^{(c-1)}_{m+\ell}.$$

As $\gamma, \delta_s \in \hat{\mathcal{N}}_{m+\ell}(R)$, this contradicts the minimality of c. That means $\delta_s = 0$, or equivalently, $vs \in \hat{F}^{(m+\ell+1)} \cap \tilde{F}^{(m+1)}$, contradicting the maximality of ℓ. $\square$

Now,

$$\begin{aligned} &\omega_{h,m}(\tilde{\mathcal{N}}_m(R) \cap p^h \tilde{L}_m) \\ =&\omega_{h,m}((R \cap \tilde{F}^{(m)} \cap \hat{F}^{(m+h)})\tilde{F}^{(m+1)}/\tilde{F}^{(m+1)}) \\ =&((R \cap \tilde{F}^{(m)} \cap \hat{F}^{(m+h)})\hat{F}^{(m+h+1)}/\hat{F}^{(m+h+1)})\hat{L}_{m+h}^{(m+1)}/\hat{L}_{m+h}^{(m+1)} \\ =&((\hat{\mathcal{N}}_{m+h}(R) \cap \hat{L}_{m+h}^{(m)}) + \hat{L}_{m+h}^{(m+1)})/\hat{L}_{m+h}^{(m+1)}. \end{aligned} \tag{26}$$

Assume that

$$\tilde{\mathcal{N}}(R) \subseteq \tilde{\mathcal{N}}'(R) + p^h \tilde{L}$$

for a certain h. One has to show that

$$\tilde{\mathcal{N}}(R) \subseteq \tilde{\mathcal{N}}'(R) + p^{h+1} \tilde{L}.$$

Let $\xi \in \tilde{\mathcal{N}}_m(R)$. Then there exists $\xi' \in \tilde{\mathcal{N}}'_m$ such that $\xi'' = \xi - \xi' \in p^h \tilde{L}_m$. It follows that $\xi'' \in \tilde{\mathcal{N}}_m(R) \cap p^h \tilde{L}_m$. By (26) and Lemma 5.11,

$$\omega_{h,m}(\xi'') \in \begin{cases} \pi^h \overline{\mathcal{N}}_m & \text{if } h < \kappa, \\ \pi^h \overline{\mathcal{I}}_m & \text{if } h \geq \kappa. \end{cases}$$

Hence, there exists $\delta \in \tilde{\mathcal{N}}'_m(R)$ such that $\omega_{h,m}(\xi'') = \omega_{h,m}(\delta)$, which implies that $\xi'' - \delta \in p^{h+1}\tilde{L}_m$. Consequently, $\xi - (\delta + \xi') \in p^{h+1}\tilde{L}_m$, which completes the inductive step. It follows that $\tilde{\mathcal{N}}(R) \subseteq \tilde{\mathcal{N}}'(R)$; hence $\tilde{\mathcal{N}}(R) = \tilde{\mathcal{N}}'(R)$.

Since the grading of $\tilde{L}$ is only a rescaling of the grading of L (see subsection 5.2), it follows that $\mathcal{N}(R) = \mathcal{N}'(R)$, which is just Theorem 5.10.

5.12. Conditions of Theorem 5.7

In this subsection, the meaning of the conditions of Theorem 5.7 in the context of Theorem 5.8 will be explained. In this context,

$$\hat{\rho}_i = \pi^\kappa \hat{\sigma}_i + \hat{\tau}_i$$

with $\hat{\tau}_i \in \overline{L}_{2\kappa}$.

Let $\overline{I}$ be the ideal of $\overline{L}$ generated by $\hat{\tau}_1, \ldots, \hat{\tau}_n$, and let $\overline{W}$ be the universal enveloping algebra of $\overline{L}/\overline{I}$ over $\mathbb{F}_p$over $\mathbb{F}_p$.

LEMMA 5.14. *If $\overline{I}/[\overline{I}, \overline{I}]$ is a free $\overline{W}$-module with basis $\hat{\tau}_1, \ldots, \hat{\tau}_n$, then $\hat{L}/\mathfrak{r}$ is a free $\mathbb{F}_p[\pi]$-module.*

PROOF. First we show that $\hat{L}/\mathfrak{r}$ has no π-torsion. Suppose it is not true. Then there exists $x \in \hat{L}$ such that $\pi x \in \mathfrak{r}$, but $x \notin \mathfrak{r}$. Let

$$\pi x = \sum_{i=1}^{n} \mathrm{ad}(\lambda_i)\hat{\rho}_i = \sum_{i=1}^{n} \mathrm{ad}(\lambda_i)(\pi^\kappa \hat{\sigma}_i + \hat{\tau}_i), \tag{27}$$

with $\lambda_i \in \hat{U}$. As in subsection 5.9, one has

$$\lambda_i = \lambda_i^{(0)} + \pi\lambda_i^{(1)} + \cdots + \pi^\ell \lambda_i^{(\ell)}$$

for some $\lambda_i^{(j)} \in \overline{U}$, $1 \le i \le n$, $1 \le j \le \ell$, $\ell \in \mathbb{Z}$. Thus, one gets from (27) the equation

$$\sum_{i=1}^{n} \mathrm{ad}(\lambda_i^{(0)})\hat{\tau}_i = 0.$$

According to the remark in subsection 5.4, under the condition of this lemma, Theorem 5.5 holds for the algebra $\overline{L}$ over $\mathbb{F}_p$ and the elements $\hat{\tau}_1, \ldots, \hat{\tau}_n$. Thus,

$$\lambda^{(0)} := (\lambda_1^{(0)}, \ldots, \lambda_n^{(0)}) \in E(\hat{\tau}),$$

which means one can write

$$\lambda^{(0)} = \sum_{\substack{1 \le i,j \le n \\ u,v \in \overline{U}}} c_{ijuv}\left((\mathrm{ad}(v)\hat{\tau}_j)ue_i + (\mathrm{ad}(u)\hat{\tau}_i)ve_j\right)$$

for some $c_{ijuv} \in \overline{U}$. Let

$$\begin{aligned}\mu =&(\mu_1, \ldots, \mu_n)\\ :=&\sum_{\substack{1 \le i,j \le n \\ u,v \in \overline{U}}} c_{ijuv}\Big((\mathrm{ad}(v)(\pi^\kappa\hat{\sigma}_j + \hat{\tau}_j))ue_i + (\mathrm{ad}(u)(\pi^\kappa\hat{\sigma}_i + \hat{\tau}_i))ve_j\Big).\end{aligned}$$

Obviously, $\mu \in E(\hat{\rho})$, where $\hat{\rho} = (\hat{\rho}_1, \ldots, \hat{\rho}_n)$. Thus, $\sum_{i=1}^{n} \mathrm{ad}(\mu_i)\hat{\rho}_i = 0$, and $\pi x = \sum_{i=1}^{n} \mathrm{ad}(\lambda_i - \mu_i)\hat{\rho}_i$. Note that by construction $\lambda_i - \mu_i \in \pi\hat{L}$, $i = 1, \ldots, n$. Hence, if $\pi\nu_i = \lambda_i - \mu_i$, then $x = \sum_{i=1}^{n} \mathrm{ad}(\nu_i)\hat{\rho}_i$, contradicting $x \notin \mathfrak{r}$.

Consider now $\hat{L}/(\mathfrak{r} + \pi\hat{L}) \cong \overline{L}/\mathfrak{a}$, where $\mathfrak{a}$ is the ideal of $\overline{L}$ generated by $\hat{\tau}_1, \ldots, \hat{\tau}_n$ (as $\hat{\rho}_i \equiv \hat{\tau}_i \mod \pi\hat{L}$). Obviously, $\mathfrak{a}$ is homogeneous.

Let $v_j \in \hat{L}$, $j \in J$ be the homogeneous $\mathbb{F}_p$-basis of $\overline{L}/\mathfrak{a}$. Note that $\{v_j\}_{j \in J}$ are $\mathbb{F}_p[\pi]$-independent in $\hat{L}/\mathfrak{r}$: if

$$\sum_{j \in J} q_j(\pi)v_j \in \mathfrak{r}$$

for some polynomials $q_j \in \mathbb{F}_p[\pi]$, not all equal to 0, then, dividing by some power of π and using the fact that $\hat{L}/\mathfrak{r}$ has no π-torsion, one can assume that $q_j(\pi) \equiv b_j \pmod{\pi\mathbb{F}_p[\pi]}$, $b_j \in \mathbb{F}_p$, not all of which are equal to 0. Thus, one gets that $\{v_j\}_{J\in J}$ are $\mathbb{F}_p$-linear dependent in $\overline{L}/\mathfrak{a}$.

One has to show now that $\{v_j\}_{j\in J}$ generates $\hat{L}/\mathfrak{r}$. Note that since the $\hat{\rho}_i$ are homogeneous, the ideal $\mathfrak{r}$ is homogeneous. It is sufficient to show that a homogeneous element $x \in \hat{L}_m$ is a linear combination over $\overline{U}$ of $\{v_j\}_{j\in J}$ modulo $\mathfrak{r}$.

As $\{v_j\}_{j\in J}$ is an $\mathbb{F}_p$ basis of $\hat{L}/(\mathfrak{r} + \pi\hat{L})$, there exist $c_j \in \hat{U}$ such that

$$x - \sum_{j\in J} c_j v_j \in \mathfrak{r} + \pi\hat{L}.$$

As x, $\mathfrak{r}$, v_j are homogeneous, one can take the suitable homogeneous components b_j of c_j, such that

$$x - \sum_{j\in J} b_j v_j \in \mathfrak{r} + \pi\hat{L}_{m-1}.$$

Proceeding by induction on m, one finishes the proof. □

Summarizing the results of this Lemma, Proposition 5.9 and Theorem 5.10, one gets the proof of Theorem 5.8.

For the example in the next section we need some observations about conditions (1), (2) of Theorem 5.8. Sufficient criterion for condition (2) will be given. The argument follows the proof of [**La2**, Theorem 1].

Let $\varepsilon : \hat{U} \to \mathbb{F}_p[\pi]$ be the augmentation map, defined by $\varepsilon(\hat{\sigma}_j) = 0$, and let $\hat{I} \subseteq \hat{U}$ be the kernel of ε. Let $\mathfrak{r}$ (resp. $\mathfrak{R}$) be the ideal of $\hat{L}$ (resp. $\hat{U}$), generated by the elements $\hat{\rho}_1, \dots, \hat{\rho}_n$. Since $\mathfrak{R}$ is generated by $\mathfrak{r}$, the inclusion $\mathfrak{r} \subset \hat{I}$ induces the $\mathbb{F}_p[\pi]$-homomorphism

$$f : \mathfrak{r}/[\mathfrak{r},\mathfrak{r}] \to \hat{I}/\mathfrak{R}\hat{I},$$

which is $\hat{U}$-linear ($\hat{U}$ acts via the adjoint representation on $\mathfrak{r}$), since for $x \in \hat{L}$, $a \in \mathfrak{r}$

$$[x,a] = xa - ax \equiv xa \mod \mathfrak{R}\hat{I}.$$

Let $\hat{V}$ be the universal enveloping algebra of $\hat{L}/\mathfrak{r}$. As $\hat{V} \cong \hat{U}/\mathfrak{R}$, f is a homomorphism of $\hat{V}$-modules.

As $\hat{U}$ is the algebra of associative noncommutative polynomials from $\hat{\sigma}_1, \dots, \hat{\sigma}_N$ over $\mathbb{F}_p[\pi]$, the $\hat{U}$-module $\hat{I}$ is the direct sum $\bigoplus_{j=1}^{N} \hat{U}\hat{\sigma}_j$.

If $\bar{\sigma}_j$ are the images of $\hat{\sigma}_j$ in $\hat{I}/\mathfrak{R}\hat{I}$, $j = 1, \ldots, N$, then $\hat{I}/\mathfrak{R}\hat{I}$ is a free $\hat{V}$-module with basis $\{\bar{\sigma}_j\}_{i=1,\ldots,N}$. Obviously, the image of f in $\hat{I}/\mathfrak{R}\hat{I}$ is generated by the images of $\hat{\rho}_1, \ldots, \hat{\rho}_n$ in $\hat{I}/\mathfrak{R}\hat{I}$. As $\hat{\rho}_i \in \hat{I}$, one can uniquely write

$$\hat{\rho}_i = \sum_{j=1}^{N} c_{ij}\hat{\sigma}_j, \quad c_{ij} \in \hat{U}, \quad 1 \leq i \leq n.$$

As $\hat{I}/\mathfrak{R}\hat{I} = \bigoplus_{j=1}^{N} \hat{V}\bar{\sigma}_j$, $f(\hat{\rho}_i) = (\bar{c}_{i1}, \ldots, \bar{c}_{iN})$, where $\bar{c}_{ij}$ are the images of c_{ij} in $\hat{V}$. If $f(\mathfrak{r}/[\mathfrak{r},\mathfrak{r}])$ is free with the basis $f(\hat{\rho}_1), \ldots, f(\hat{\rho}_n)$, then obviously $\mathfrak{r}/[\mathfrak{r},\mathfrak{r}]$ is free with the basis $\hat{\rho}_1, \ldots, \hat{\rho}_n$.

Note that if $\hat{L}/\mathfrak{r}$ is a free $\mathbb{F}_p[\pi]$-module, the Birkhoff-Witt theorem (see [**Bou**, Chapter 1, §2.8, Cor. 7]) shows that $\hat{V}$ has no zero divisors.

Let $C \in \mathrm{Mat}_{nN}(\hat{U})$ be the $n \times N$ matrix with coefficients c_{ij} from above, and let $\overline{C}$ be its image in $\mathrm{Mat}_{nN}(\hat{V})$. If there exist invertible matrices $A \in \mathrm{Mat}_{nn}(\hat{V})$, $B \in \mathrm{Mat}_{NN}(\hat{V})$ such that $A\bar{C}B$ has the form

$$\begin{pmatrix} d_1 & 0 & \cdots & 0 & 0 & * & \cdots & * \\ 0 & d_2 & & 0 & 0 & * & & * \\ \vdots & & \ddots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & & d_{n-1} & 0 & * & & * \\ 0 & 0 & \cdots & 0 & d_n & * & \cdots & * \end{pmatrix}$$

with $d_i \in \hat{V}$, $d_i \neq 0$, $1 \leq i \leq n$, the fact that $\hat{V}$ has no zero divisors gives the condition (2) of Theorem 5.7. This fact will be used in the next section. Note also, that the same method could be applied for checking condition (1) of Theorem 5.8 and the condition of Lemma 6.1.

Remark. The problem whenever a module over an universal enveloping algebra of a graded Lie algebra is free looks difficult to check. Beside the methodic above and one from Lemma 6.1, Theorem 1 from [**La4**] could be pointed. It gives a criterion based on dimensions of the grade components of a Lie algebra in a similar situation (see also [**An**], [**HL**]).

6. EXAMPLE

Here, Example 11.11 of [**Ko**] will be used.

Let $K = \mathbb{Q}$, $p > 2$. The places, that can ramify in p-extensions are thus the ideals generated by p or natural prime numbers which are congruent to 1 modulo p. Let S be some set of such places. We will denote by the same letter the natural prime, the ideal, generated by it, and the corresponding place.

For $q \in S$, let

$$\alpha_q = \begin{cases} \text{some primitive root of 1 modulo } q, & q \equiv 1 \mod p, \\ 1+p, & q = p. \end{cases}$$

For $q_1, q_2 \in S$, choose $c_{q_1 q_2} \in \mathbb{Z}_p$ such that

$$\frac{1}{q_1} \equiv \alpha_{q_2}^{c_{q_2 q_1}}, \quad \text{if } q_2 \neq p, \tag{28}$$

$$c_{q_2 q_1} = 1 + p, \quad \text{if } q_2 = p. \tag{29}$$

Then [**Ko**] gives that $G = G_{\mathbb{Q},S}(p) = F/R$ with free pro-p-group F generated by $\{t_q\}_{q \in S}$ and the relations

$$t_q^{q-1} \prod_{q' \in S} (t_q, t_{q'})^{c_{q'q}} r'_q, \quad r'_q \in F^{(3)} \tag{30}$$

for $q \in S \setminus \{p\}$.

Let now S be the set consisting of three elements p, q_1, q_2 such that

$$v_p(q_1 - 1) = v_p(q_2 - 1) = 1.$$

As $v_p(q_i - 1) = 1$, the equation (29) shows that $c_{pq_i} \not\equiv 0 \mod p$, $i = 1, 2$.

To simplify the relations, one can change generators. Let

$$\begin{aligned} \alpha &= \frac{q_1-1}{p} c_{pq_1}, \\ \beta &= \frac{q_2-1}{p} c_{pq_2}, \\ s_1 &= t_{q_1}^{\frac{p}{q_1-1}}, \\ s_2 &= t_{q_2}^{\frac{p}{q_2-1}}, \\ s_3 &= t_p + \frac{c_{q_2q_1}}{c_{pq_1}} t_{q_2} + \frac{c_{q_1q_2}}{c_{pq_2}} t_{q_1}. \end{aligned}$$

Obviously, $\alpha, \beta \not\equiv 0 \mod p$. The substitution to (30) shows that the group $G = G_{\mathbb{Q},S}(p)$ is isomorphic to the factor group of the free pro-p-group F with three generators s_1, s_2, s_3, and two relations

$$s_1^p (s_1, s_3)^\alpha y_1, \quad s_2^p (s_2, s_3)^\beta y_2, \tag{31}$$

with $y_1, y_2 \in F^{(3)}$.

LEMMA 6.1. *If $\overline{L}$ is the free Lie algebra over $\mathbb{F}_p$ with generators $\hat\sigma_1, \hat\sigma_2, \hat\sigma_3$, $\overline{I}$ the ideal generated by the elements $[\hat\sigma_1, \hat\sigma_3]$ and $[\hat\sigma_2, \hat\sigma_3]$, and $\overline{W}$ the enveloping algebra of $\overline{L}/\overline{I}$, then $\overline{I}/[\overline{I}, \overline{I}]$ is a free $\overline{W}$-module.*

PROOF. Note that $\overline{W}$ is a "partially commutative" algebra, of polynomials from $\hat\sigma_1, \hat\sigma_2, \hat\sigma_3$ over $\mathbb{F}_p$, where $\hat\sigma_3$ commutes with $\hat\sigma_1, \hat\sigma_2$. Thus, each of its elements can be uniquely written as an $\mathbb{F}_p$-linear combination of elements

$$\hat\sigma_3^\ell u_1 \dots u_n, \tag{32}$$

where $\ell \geq 0$, $n \geq 0$, $u_i \in \{\hat\sigma_1, \hat\sigma_2\}$ for $1 \leq i \leq n$.

[**Reu**, Theorem 2.5] shows that $\overline{I}$ is a free Lie algebra. The generators of this algebra will be computed by applying the following elimination theorem [**Bou**, Chapter 2, §2.9, Prop. 10] twice:

Let $L(X)$ be a free Lie algebra with the set of generators X and $U \subset X$. Let T be the set of sequences $(u_1, \dots, u_n, x)$, $n \geq 0$, $u_1, \dots, u_n \in U$, $x \in X \setminus U$. Let $\mathfrak{a}$ be the ideal of $L(X)$ generated by the elements from $X \setminus S$. Then

$$L(X) = L(S) \oplus \mathfrak{a},$$

and one has an isomorphism

$$\begin{aligned} L(T) &\cong \mathfrak{a} \\ (u_1, \dots, u_n, x) &\mapsto \operatorname{ad}(u_1 \dots u_n)(x). \end{aligned}$$

First, consider the ideal $\overline{J}$ of $\overline{L}$ generated by the element $\hat{\sigma}_3$. It has the generator set $w_{u_1\dots u_n} := \mathrm{ad}(u_1 \cdot \dots \cdot u_n)\hat{\sigma}_3$, where $n \geq 0$, $u_i \in \{\hat{\sigma}_1, \hat{\sigma}_2\}$. Using Jacobi relations, one gets

$$\overline{J} = \overline{I} \oplus \mathbb{F}_p \hat{\sigma}_3.$$

Thus, $\overline{I}$ is the ideal of $\overline{J}$ generated by the elements $w_{u_1\dots u_n}$ for $n > 0$, and the generator set of $\overline{I}$ is the set of

$$\mathrm{ad}(\hat{\sigma}_3^\ell) w_{u_1\dots u_n} = \mathrm{ad}(\hat{\sigma}_3^\ell u_1 \dots u_n)(\hat{\sigma}_3)$$

for $\ell \geq 0$, $n > 0$, $u_i \in \{\hat{\sigma}_1, \hat{\sigma}_2\}$, or equivalently,

$$\{\mathrm{ad}(\hat{\sigma}_3^\ell u_1 \dots u_n)([\hat{\sigma}_1, \hat{\sigma}_3]) | \ell \geq 0, n \geq 0, u_i \in \{\hat{\sigma}_1, \hat{\sigma}_2\}\}$$
$$\cup \{\mathrm{ad}(\hat{\sigma}_3^\ell u_1 \dots u_n)([\hat{\sigma}_2, \hat{\sigma}_3]) | \ell \geq 0, n \geq 0, u_i \in \{\hat{\sigma}_1, \hat{\sigma}_2\}\}$$

The structure of $\overline{W}$, see (32), shows that $\overline{I}$ is a free $\overline{W}$-module with basis elements $[\hat{\sigma}_1, \hat{\sigma}_3]$, $[\hat{\sigma}_2, \hat{\sigma}_3]$. □

Let $\overline{\alpha}, \overline{\beta}$ be the residues of α, β in $\mathbb{F}_p$. The elements $\hat{\rho}_1, \hat{\rho}_2 \in \hat{L}$, which correspond to the relations (31), have the form

$$\hat{\rho}_1 = \pi\hat{\sigma}_1 + \overline{\alpha}[\hat{\sigma}_1, \hat{\sigma}_3] = (\pi - \overline{\alpha}\hat{\sigma}_3)\hat{\sigma}_1 + \overline{\alpha}\hat{\sigma}_1\hat{\sigma}_3,$$
$$\hat{\rho}_2 = \pi\hat{\sigma}_2 + \overline{\beta}[\hat{\sigma}_2, \hat{\sigma}_3] = (\pi - \overline{\beta}\hat{\sigma}_3)\hat{\sigma}_2 + \overline{\beta}\hat{\sigma}_2\hat{\sigma}_3.$$

Applying the decomposition from subsection 5.12, the matrix C has the form

$$\begin{pmatrix} \pi - \overline{\alpha}\hat{\sigma}_3 & 0 & \overline{\alpha}\hat{\sigma}_1 \\ 0 & \pi - \overline{\beta}\hat{\sigma}_3 & \overline{\beta}\hat{\sigma}_2 \end{pmatrix}.$$

As obviously $\pi - \overline{\alpha}\hat{\sigma}_3$ and $\pi - \overline{\beta}\hat{\sigma}_3$ do not lie in the ideal generated by $\hat{\rho}_1, \hat{\rho}_2$, subsection 5.12 states that $\mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$ is a free $\hat{V}$-module with basis $\hat{\rho}_1, \hat{\rho}_2$. This and Lemma 6.1 allow us to apply the main theorem.

Let $w_q \in C_{K,S}(p)$, $q \in S$ be such that

$$\varphi(w_q) = t_q RF^{(2)}.$$

The way in which the generators and relation of G were chosen in [**Ko**] gives that there exist $\zeta_1 \in \mathbb{Q}_{q_1}$, $\zeta_2 \in \mathbb{Q}_{q_2}$, $u \in \mathbb{Q}_p$ such that ζ_i is the primitive p-th root of 1 in $\mathbb{Q}_{q_i}$ $(i = 1, 2)$, u is the generator of the multiplicative group of local principal units in $\mathbb{Q}_p$, and w_{q_1}, w_{q_2}, w_p are the images of ζ_1, ζ_2, u respectively under the map

$$\mathbb{Q}_q^* \to J(\mathbb{Q}) \to C_{\mathbb{Q},S}(p),$$

$q = q_1, q_2, p$ respectively.

Now the main theorem gives the following (here $C_{\mathbb{Q},S}(p)$ is written additively):

PROPOSITION 6.2. *The kernel of the map*

$$\Phi : \mathcal{L}(C_{\mathbb{Q},S}(p)) \to L(G_{\mathbb{Q},S}(p))$$

is generated by the elements

$$pw_{q_1}, pw_{q_2},$$
$$\begin{aligned}&\big[\mathrm{ad}(\lambda)w_{q_1}, \mathrm{ad}(\mu)\big(c_{pq_1}[w_{q_1}, w_p] + c_{q_2q_1}[w_{q_1}, w_{q_2}]\big)\big]\\ &\qquad + \big[\mathrm{ad}(\mu)w_{q_1}, \mathrm{ad}(\lambda)\big(c_{pq_1}[w_{q_1}, w_p] + c_{q_2q_1}[w_{q_1}, w_{q_2}]\big)\big],\\ &\big[\mathrm{ad}(\lambda)w_{q_1}, \mathrm{ad}(\mu)\big(c_{pq_2}[w_{q_2}, w_p] + c_{q_1q_2}[w_{q_2}, w_{q_1}]\big)\big]\\ &\qquad + \big[\mathrm{ad}(\mu)w_{q_2}, \mathrm{ad}(\lambda)\big(c_{pq_1}[w_{q_1}, w_p] + c_{q_2q_1}[w_{q_1}, w_{q_2}]\big)\big],\\ &\big[\mathrm{ad}(\lambda)w_{q_2}, \mathrm{ad}(\mu)\big(c_{pq_2}[w_{q_2}, w_p] + c_{q_1q_2}[w_{q_2}, w_{q_1}]\big)\big]\\ &\qquad + \big[\mathrm{ad}(\mu)w_{q_2}, \mathrm{ad}(\lambda)\big(c_{pq_2}[w_{q_2}, w_p] + c_{q_1q_2}[w_{q_2}, w_{q_1}]\big)\big]\end{aligned}$$

with homogeneous λ, μ *from the enveloping algebra of* $\mathcal{L}(C_{\mathbb{Q},S}(p))$.

REMARK. If the set $S = \{p, q_1, q_2\}$ and $v_p(q_1 - 1) = v_p(q_2 - 1) > 1$, then $c_{pq_1} \equiv c_{pq_2} \equiv 0 \pmod p$. Thus Theorem 5.8 cannot be applied. This situation is probably connected with the groups considered in [**La1**].

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