A System of Non-linear Partial Differential Equations Modeling Chemotaxis with Sensitivity Functions

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Introduction

In this work, we are going to study a mathematical model for chemotaxis. This phenomenon is the oriented motion of organisms sensitive to a concentration gradient of a chemical substance and appears in various biological processes. For instance, in the case of an infection in the human body, white blood cells move to the source of inflammation, that is, to regions where the concentration of bacteria is high. Also, several animals (e.g. insects and fish) are able to sense concentration gradients of given chemicals, so that information can be conveyed between members of the species by exuding and sensing so-called *pheromones*. Furthermore, chemotaxis is known to be responsible for aggregation processes in the life cycle of certain unicellular organisms. The chemotactic behaviour of these amoebae serves very well for studies of cell aggregation leading to generation of form.

Mathematical models have been developed during recent years in order to predict aggregation patterns of cells as well as to test different biological hypotheses trying to explain chemotaxis. These models consist of two coupled non-linear partial differential equations of reaction-diffusion type based on the Keller-Segel [22] model and have proved to be of great mathematical interest in their own right. As a prototype, we state the following system for $U = U(t,x)$ and $V = V(t,x)$:

$$\begin{aligned} U_t &= \Delta U - \nabla(\chi U \nabla V) \\ V_t &= \alpha\Delta V - \beta V + \delta U \end{aligned} \qquad (1)$$

on $(0,T) \times \Omega \subset I\!R \times I\!R^n$ with homogeneous Neumann boundary conditions

$$\nu \cdot \nabla U = \nu \cdot \nabla V = 0 \quad \text{on} \quad (0,T) \times \partial\Omega \qquad (2)$$

(ν is the unit outer normal at points of $\partial\Omega$.) and initial conditions $U(0,x) = U_0(x)$ and $V(0,x) = V_0(x)$ for all $x \in \Omega$. χ, α, β and δ are usually positive constants. (See Section 1.1 for details on the derivation of the model.)

The model describes the behaviour of the cellular slime molds *Dictyostelium discoideum* U moving towards regions of high concentration of the chemical substance cAMP (*cyclic adenosine monophosphate*) V, which they themselves produce. In times of shortage of food-supply, *Dictyostelium discoideum* cells appear that spontaneously secrete cAMP. Neighbouring cells now also start exuding the chemical in response to its increasing concentration. At the same time, the released cAMP is slowly destroyed by an enzyme. The amoebae form migrating multi-cellular slugs and move in streams towards the centre of highest concentration of cAMP. After a while, they come to a rest and erect themselves to a fruiting body, which aids the distribution of germinating spore cells.

In system (1), the U-flux moves in the direction of the concentration gradient of V. Therefore, chemotaxis can be viewed as a kind of negative drift as it appears for instance in equations modeling reaction-diffusion processes of electrically charged species in semiconductors, which can be written in a simplified form as

$$\begin{aligned}(u_1)_t &= \mu_1 \Delta u_1 - \nabla(u_1 \nabla \varphi) + R_1 \\ (u_2)_t &= \mu_2 \Delta u_2 + \nabla(u_2 \nabla \varphi) + R_2 \\ -\nabla(\varepsilon \nabla \varphi) &= u_2 - u_1,\end{aligned}$$

where u_1 and u_2 are the densities of electrons and holes, respectively, and φ is the electrostatic potential. The μ_i are positive diffusion coefficients and the R_i are reaction terms depending on the carrier densities.[†] The latter model has a strong resemblance with the whole system (1). The essential difference is that the production terms for each u_i in the Poisson equation has the same sign as the drift term in the respective continuity equation, whereas, in our equations, the chemotactic term and the production term in the V-equation are of opposite sign. This makes the chemotaxis-system more difficult to handle mathematically, and blow-up (explosion of solutions in finite time) cannot be excluded. In fact, the interaction of destabilizing chemotaxis and stabilizing diffusion determines the solution behaviour in the chemotaxis-setting and it is crucial which effect dominates.

In 1992, Jäger and Luckhaus [21] first studied the above system in a smooth domain $\Omega \subset \mathbb{R}^2$ after replacing the second equation by a stationary one, i.e.,

$$\begin{aligned}U_t &= \Delta U - \chi \nabla(U \nabla V) \\ 0 &= \alpha \Delta V - \beta V + \delta U\end{aligned} \tag{3}$$

with homogeneous Neumann conditions and $U(0) = U_0$. They showed global existence of solutions U and V for a class of initial values U_0 and, under the assumption of radial symmetry ($\Omega = B_R(0)$), existence of blow-up of solutions for large initial values U_0.

In 1994, Nagai [30] found conditions for blow-ups depending on the dimension n of the ball $\Omega = B_R(0)$. (Blow-up does not occur at all for $n = 1$, and for $n = 2$ only if $\|U_0\|_{L^1(\Omega)}$ is greater than a given threshold value or, equivalently, in the case of a fixed initial mass of U, when the chemotactic coefficient χ is sufficiently large.)

[†]For more details on the semiconductor model see for example van Roosbroeck [38] or Gajewski and Gröger [10].

Nagai and Diaz [7] studied system (3) for a smooth Ω without radial symmetry, but under homogeneous Dirichlet boundary conditions, where the only stationary solution is $(U, V) = (0, 0)$. In this case, they showed global existence of solutions and convergence to the trivial solution as time t tends to infinity.

However, one knows from a work by Schaaf [36], who investigated the bifurcation behaviour of inhomogeneous steady states of chemotactic systems in one dimension, that in the Neumann case, there are *non-trivial* stationary solutions.

There are some further works dealing with system (3) with (2) on a radially symmetric domain in two dimensions, where the second equation can be reduced to an ordinary differential equation: Mizutani and Nagai [28] proved convergence to trivial stationary solutions, Herrero and Velazquez showed existence of δ-distribution blow-ups in the disc centre by inverting the Δ- operator[†]. In reference [18], the latter discussed, for the first time, the system with an instationary V-equation, i.e., system (1). Also only in the case of a disc, Nagai and Senba [31] replaced V by $\phi(V)$ in the first equation of (3) with $\phi(V) = V^p$ $(p > 0)$ or $\phi(V) = \log V$.

In 1996, Gajewski and Zacharias [14] first found a Lyapunov function for the fully instationary system (1) with (2). For a more general domain Ω with possible singularities, they proved local existence and uniqueness of solutions. Moreover, they showed under quite natural conditions on the coefficients of the system convergence to trivial and non-trivial steady states, thus excluding blow-up in finite time. In the same year, Nagai, Senba and Yoshida [32] found a Lyapunov function, closely related to the one by Gajewski and Zacharias, and achieved estimates for the solutions U and V of system (1) in the Bochner space $L^\infty(0, \infty; L^\infty(\Omega))$, where they only considered a smooth domain. These authors obtained their results on the grounds of a theorem of existence by Yagi [40] for the case of a C^2-domain in $\mathbb{R}^2$.

We are going to study the fully instationary system (1) where the chemotactic coefficient χ and the δ in the production term for V are non-constant but of the form

$$\tilde{\chi}(V) := \chi S'(V) \quad \text{and} \quad \tilde{\delta}(V) := \delta S'(V). \tag{4}$$

The function S is called the *sensitivity function.* It specifies the ability of the amoebae U to sense the V-concentration and $S'(V)$ denotes its first derivative with respect to V.

In biological literature (see Murray [29], Schaaf [36]) , one finds the following forms of the sensitivity function:

$$S_1(V) = \frac{V}{1 + cV}, \quad S_2(V) = \frac{V^2}{1 + cV^2}, \quad S_3(V) = \log(V + c),$$

[†]See reference [19].

with constants $c \geq 1$. The above references suggest a chemotactic coefficient χ as in (4). In addition, the coefficient δ in the production term for V will in reality depend on the V-concentration. Rather than treating a more general production term, we confine ourselves to a case in which there exists a Lyapunov function for the system. Therefore, we assume the same V-dependence of χ and δ. (See Chapter 1 for a more detailed discussion.) We can now write our system as follows:

$$\begin{aligned} U_t &= \Delta U - \chi \nabla (U \nabla S(V)) \\ V_t &= \alpha \Delta V - \beta V + \delta U S'(V) \end{aligned} \tag{5}$$

in $(0,T) \times \Omega$, with Neumann boundary conditions

$$\nu \cdot \nabla U = \nu \cdot \nabla V = 0 \quad \text{on } (0,T) \times \partial\Omega, \tag{6}$$

and initial values $U(0,x) = U_0(x),\ V(0,x) = V_0(x)$. The system corresponds with general mathematical models for chemotaxis stated (but never studied) in literature.†

Although the sensitivity functions introduce additional non-linearities into the system, we will see that, at some stages, they facilitate proofs of higher regularity of the solutions.

We will treat a general bounded Lipschitz domain Ω, so that techniques for smooth domains cannot be applied. In the major part of this work, we will have to confine ourselves to a two-dimensional setting, but, at some stages, we will obtain results for higher dimensional domains.

Note that equations (1) are the special case of the *direct measurement*, where $S(V) = S_0(V) = V$ in (5). To state results for this (less realistic) case, we will need to require that $\Omega \subset I\!R^2$ has a piecewise C^2-boundary $\partial\Omega$ as in the work by Gajewski and Zacharias [14].

Formulating our results, we will always state the properties required of the sensitivity function S. The most general class of functions treated in this work will be the set

$$\mathcal{S} \ = \ \left\{ S \in C^1(I\!R, I\!R) : 0 \leq S(V),\ 0 \leq S'(V) \leq C' \ \text{ for all } V \geq 0 \right\}.$$

For some results, we will also require $S \in C^2(I\!R, I\!R)$ and

$$|S''(V)| \ \leq \ C'', \tag{7}$$

for all $V \geq 0$, with a positive C''.

The main achievement of this work lies in the existence theorem of global solutions of system (5) with (6) on a general two-dimensional Lipschitz domain for different natural

†See for example Childress and Percus [5] or Stevens [37].

classes of sensitivity functions. This result enables us to study the asymptotic solution behaviour and we are able to prove convergence of the trajectories of the solutions to trivial *and* non-trivial steady states under varying conditions on the data of the system. The non-trivial stationary states seem to be of principle interest to us as their inhomogeneous distribution of the species can be viewed as a starting position for the erection of a fruiting body.

Furthermore, the work contains, for the first time, results for the fully instationary chemotaxis-system (with or without sensitivity function) on higher dimensional domains. In the following, we want to give a brief summary of all our results.

The biological background of chemotaxis as well as a detailed derivation of the model used is described in Chapter 1. Furthermore, we will introduce some mathematical notations in this chapter. Finally, our three kinds of specific sensitivity functions, i.e., the identity sensitivity function S_0, bounded sensitivity functions (as S_1 and S_2) and the logarithmic sensitivity function S_3 will be shown to belong to the class $\mathcal{S}$ and to fulfill additionally property (7).

In Chapter 2, we will prove the existence of a pair of solution (U, V) on a set $(0,T) \times \Omega \subset I\!R \times I\!R^2$ for a general function $S \in \mathcal{S}$. Assuming additionally (7), we can also show uniqueness and further regularity of the solution.

Also for a general $S \in \mathcal{S}$ and in a possibly higher dimensional domain $\Omega \subset I\!R^n$, we will show in Chapter 3 that the function

$$F(U,V) = \int_\Omega \left\{ U \log U - \chi U S(V) + \frac{\chi}{2\delta}(\alpha |\nabla V|^2 + \beta V^2) \right\} dx$$

is a Lyapunov function for system (5), (6). All terms in the function F are bounded if the key condition

$$\int_\Omega e^{\chi S(V)} dx < \exp\left(\frac{\chi\alpha}{2\delta \|U_0\|_{L^1(\Omega)}} \|\nabla V\|^2_{L^2(\Omega)} + \frac{\chi\beta}{2\delta \|U_0\|_{L^1(\Omega)}} \|V\|^2_{L^2(\Omega)} + c \right)$$

is fulfilled with a positive constant c. From the boundedness of all terms in the Lyapunov function, we can derive important a-priori-estimates, and the above inequality will therefore be established in Section 3.2 for our three classes of specific sensitivity functions under different conditions on the data of the problem.

In Chapter 4, we will prove for the three classes of specific sensitivity functions in a two-dimensional domain Ω that the solutions are global. As we will see, one can show that $U \in L^\infty(0,\infty; L^p(\Omega))$ for all $1 \le p < \infty$ and $V \in L^\infty(0,\infty; L^\infty(\Omega))$ if S is bounded. For $S_0(V) = V$, we need a slightly more regular domain $\Omega \subset I\!R^2$ to obtain these estimates.

Furthermore, the result only holds for sufficiently small initial values of U.[†] For the logarithmic sensitivity function $S_3(V) = \log(V + c)$, with $c \geq 1$, we will prove that $U \in L^\infty(0,T;L^p(\Omega))$ for all $1 \leq p < \infty$ and $V \in L^\infty(0,T;L^\infty(\Omega))$ for any $T \geq 0$, which suffices to guarantee globality of the solution. In Section 4.4, we will replace the V-equation in system (5) by the stationary one. We will prove that if all terms in the Lyapunov function for this system are bounded, there exist time-independent estimates of $\|U\|_{L^\infty(0,T;L^2(\Omega))}$ and $\|V\|_{L^\infty(0,T;L^\infty(\Omega))}$ for all $T \geq 0$ and with any function $S \in \mathcal{S}$.

We believe that a realistic mathematical model for chemotaxis should be able to exclude blow-up of solutions in finite time. Thus, we do not agree with interpretations by other authors[††] that a δ-distribution explosion at a point can be viewed as an approximation of the erection of the fruiting body. Although chemotactic effects are also known to play a role in the fruiting body formation, the special form of our equations was chosen to model solely the preceding aggregation process. Besides, it becomes more obvious that one should not wish for explosion of solutions when considering examples for chemotaxis other than the amoebae *Dictyostelium discoideum*, where an equivalent phenomenon of generation of form is missing.

In Chapter 5, we will study the asymptotic behaviour of the solution and show its convergence to a possibly non-trivial steady state. In Section 5.2, examples will be given which demonstrate that, under different conditions on the coefficients of the system, this steady state can be trivial, i.e., spatially constant, as well as non-trivial.

In the appendix, we have collected results which complement statements in the main part. For example, we will need a higher dimensional existence theorem by Amann [1], which will be shown to be applicable to our system of equations in Appendix A. We can thus justify our results obtained for space dimensions n greater than 2 even though the theorem is only valid for a smooth domain $\Omega \subset \mathbb{R}^n$.

In Appendix B, we will prove a technical lemma and its corollary for functions in an Orlicz space $L^\Phi(\Omega)$. Both results will be needed in Chapter 4 in order to show globality of solutions for different sensitivity functions.

Finally, in Appendix C we will offer an additional motivation for the special form of our system of equations. Starting from a three-species-system, where states of different sensitivity of the amoebae are distinguished, we derive system (5).

[†]Compare with Gajewski and Zacharias [14], who showed time-dependent global estimates without sensitivity function.

[††]e.g. Nanjundiah [33], Herrero and Velazquez [19].

Chapter 1

Biological and Mathematical Background

In the following section we are going to describe the biological phenomenon of chemotaxis and give a detailed derivation of our mathematical model.

1.1 The Model of Chemotaxis

Chemotaxis is the oriented motion of organisms sensitive to a concentration gradient of a chemical substance. The phenomenon appears in various biological processes. For instance, in the case of an infection in the human body, *leukocytes* (white blood cells) move to the source of inflammation, that is, to regions where the concentration of bacteria is high. Also, several animals (e.g. insects and fish) are able to sense concentration gradients of given chemicals, so that information can be conveyed between members of the species by exuding and sensing so-called *pheromones*.

Furthermore, chemotaxis is known to be responsible for aggregation processes in the life cycle of certain unicellular organisms. The best-known example for this are the cellular slime molds *Dictyostelium discoideum*. From the biological point of view, the chemotactic behaviour of these amoebae seems to be ideal for studying interactions between cells during *morphogenesis* (the generation of form), which is an important component in the development of multi-cellular organisms.[†] We are going to give a brief description of the rather simple stages in the life cycle of *Dictyostelium discoideum*.[††]

[†] See Nanjundiah [33].

[††] For more details we refer to Martiel and Goldbeter [26], Nanjundiah [33], Savill and Hogeweg [35], Vasieva et al. [39].

Groups of free-living amoebae feed and divide until the food supply is exhausted. The starving population then enters into a social phase. Cells appear that spontaneously secrete pulses of the chemical cAMP (*cyclic adenosine monophosphate*). Neighbouring cells now exude cAMP in response to the increase in concentration of the substance to a threshold value. At the same time, the released cAMP is slowly destroyed by an enzyme (*phosphodiesterase*), which is also produced by the cells. The amoebae form migrating multi-cellular slugs and move in streams towards the forming mound of cells at the centre of highest concentration of cAMP (*streaming aggregation*). After a while, they come to a rest and erect themselves to a fruiting body of spore cells sitting atop a slender tapering stalk which consists of dead cells. This aids the distribution of the cells and, given the right conditions of moisture, temperature etc., the spores germinate, each one yielding an amoeba and the cycle then repeats itself.

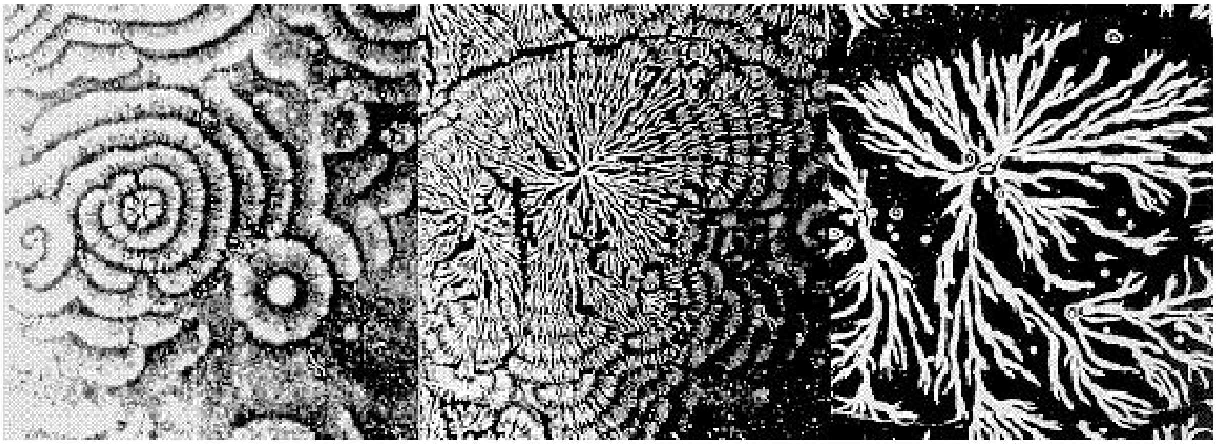

Figure 1.1: Chemotactic Aggregation of Slime Molds (Courtesy of P. C. Newell)

The stage we shall be concerned with is the aggregation phase, which is mediated by *positive chemotaxis* (movement *up* a concentration gradient), since the amoebae move towards regions of high concentration of cAMP. Taking random migration (diffusion) of both the amoebae and the chemical substance into account, one obtains the following general mathematical model, where $U = U(t,x)$ stands for the concentration of the amoebae and $V = V(t,x)$ for cAMP:

$$\begin{aligned} U_t &= \nabla(D_U \nabla U) - \nabla(U\chi(V)\nabla V) + f(U) \\ V_t &= \nabla(D_V \nabla V) + g(U,V) \end{aligned}$$

in $(0,T) \times \Omega$ with initial conditions $U(0,x) = U_0(x)$ and $V(0,x) = V_0(x)$.

We assume that there is no interaction of the system with its environment via the boundary of the domain Ω, and thus use natural no-flux boundary conditions

$$\nu \cdot \nabla U = \nu \cdot \nabla V = 0 \quad \text{on} \quad (0,T) \times \partial\Omega,$$

where ν is the unit outer normal at points of $\partial\Omega$. f and g are the production rates for the two species, D_U and D_V their diffusion coefficients and $\chi(V)$ is the chemotactic coefficient, which determines the intensity of the flux in response to the chemical gradient.

As we only model the aggregation phase of the cells, we can neglect production and mortality of the amoebae, so that we set $f(U) = 0$. For the generation term of the attracting chemical substance, the authors Keller and Segel [22] presented in 1970 the form $g(U,V) = \delta U - \beta V$, with positive constants δ and β. Here, δU stands for the spontaneous production of V by the amoebae U, which is taken to be proportional to the U-concentration, and $-\beta V$ describes the exponential decay in the activity of V. We assume constant diffusion coefficients and take without loss of generality (w.l.o.g.) $D_U = 1$ and write $\alpha := D_V << 1$, as it is suggested by Stevens [37] that the diffusion of V is considerably slower than that of U, which leads to stable aggregation centres. The model then reads as follows:

$$\begin{aligned} U_t &= \Delta U - \nabla(U\chi(V)\nabla V) \\ V_t &= \alpha\Delta V - \beta V + \delta U. \end{aligned} \tag{1.1}$$

In this system, the U-flux moves into the direction of the concentration gradient of V. Therefore, chemotaxis can be viewed as a kind of negative drift as it appears for instance in equations modeling reaction-diffusion processes of electrically charged species in semiconductors. The interaction of destabilizing chemotaxis and stabilizing diffusion determines the solution behaviour and it is crucial which effect dominates. In the U-equation, this is reflected in the opposite signs of the two terms. The negative sign in front of the chemotactic term makes the system more difficult to handle mathematically, and blow-up cannot be excluded.

Biologists suggest setting $\chi(V) = \chi S'(V)$ in (1.1), where $S : I\!R \to I\!R$ is a so-called sensitivity function and S' its first derivative. Since the first equation in (1.1) can now be written as

$$U_t = \Delta U - \chi\nabla(U\nabla S(V)),$$

one can see that, in this case, the amoebae U do not react directly to the concentration gradient of V but to the gradient of $S(V)$, which is biologically reasonable to assume.

As stated in the Introduction, some mathematicians have dealt with equations where a kind of sensitivity function is introduced into the U-equation. In *our* model, S will also appear in the second equation.

It is obvious that, in a realistic setting, δ will also depend on V. Since any response of the cells (be it chemotactic reaction or production) is mediated through receptors at the cell exterior,[†] one can expect that, in a first approximation, χ and δ depend on V in the same way, so that we set $\delta(V) := \delta S'(V)$.

It can be seen from the forms of sensitivity functions treated in this work that this is in accordance with the biological expectation of a fall-off in the V-production at high concentrations of V. (See Nanjundiah [33].)

As suggested by its name, S stands for the ability of the cells U to sense the V-concentration. In biological literature (see Murray [29], Schaaf [36]) , one finds the following forms of the sensitivity function:

$$\begin{aligned} S_1(V) &= \frac{V}{1+cV}, && \textit{(receptor kinetics)} \\ S_2(V) &= \frac{V^2}{1+cV^2}, && \textit{(cooperative binding)} \\ S_3(V) &= \log(V+c) && \textit{(logarithmic sensitivity)}, \end{aligned}$$

with constants $c \geq 1$. In all three cases, the chemotactic effect decreases with increasing concentration of V. For S_1 and S_2, there is even a saturation point which cannot be exceeded for V tending to infinity.

By introducing these sensitivity functions into system (1), we obtain the following pair of non-linear partial differential equations:

$$\begin{aligned} U_t &= \Delta U - \chi \nabla (U \nabla S(V)) \\ V_t &= \alpha \Delta V - \beta V + \delta U S'(V) \end{aligned} \qquad (1.2)$$

with positive constants χ, α, β and δ.

Note that the equations

$$\begin{aligned} U_t &= \Delta U - \chi \nabla (U \nabla V) \\ V_t &= \alpha \Delta V - \beta V + \delta U \end{aligned} \qquad (1.3)$$

are the special case of the *direct measurement*, where $S(V) = S_0(V) = V$.

Equations (1.2) are a special case of more general systems for chemotaxis stated (but never studied) in the following literature.

[†]See Newell et al. [34], Devreotes and Zigmond [6].

Childress and Percus [5] wrote down in 1981 the following general model although they then proceeded to study the form (1.3).

$$\begin{aligned} U_t &= \nabla[\mu(U,V)\nabla U - \chi(U,V)\nabla V] \\ V_t &= \alpha\Delta V - g(V) + f(U,V) \end{aligned} \tag{1.4}$$

with the conditions

$$\mu(U,V) \geq \mu_0, \quad \chi(U,V) \leq \chi_0 U, \quad f(U,V) \leq \delta_0 U, \quad g(V) \geq g_0 V. \tag{1.5}$$

$\alpha, \mu_0, \chi_0, \delta_0$ and g_0 are positive constants.

In 1992, Stevens [37] derived system (1.4) as limit dynamics of a stochastic model for chemotaxis, which is discrete in time, space and population size. In this system, $\mu(U,V) = \mu_0 > 0$, $g(V) = \beta(U,V)V$ and $f(U,V) = \delta(U,V)U$ with χ twice continuously differentiable and β and δ once continuously differentiable on $\mathbb{R}^2_+$, all three being bounded in $\mathbb{R}_+$ together with their derivatives.

Note that in our system, $\chi(U,V) = \chi U S'(V) \leq \chi C' U$, $g(V) = \beta V$ and $f(U,V) = \delta U S'(V) \leq \delta C' U$, so that conditions (1.5) hold, and since all sensitivity functions considered are infinitely often differentiable on $\mathbb{R}_+$, the regularity properties stated by Stevens are also fulfilled.

Martiel and Goldbeter [26] deal with a three-variable-system, where intra- and extracellular cAMP-concentrations are distinguished. Assuming some of the reactions to be very fast, the equation for intracellular cAMP can be eliminated. According to the authors, the two-variable-system obtained in this manner is a good approximation of the three-variable-system.

1.2 Mathematical Preliminaries

1.2.1 Notations

We are going to work in a bounded domain $\Omega \subset \mathbb{R}^2$ with a Lipschitz continuous boundary $\partial\Omega$. We denote by $C^k(\bar{\Omega})$, for k a non-negative integer, the usual spaces of k-times continuously differentiable functions on Ω and write $C(\bar{\Omega}) := C^0(\bar{\Omega})$ and $C^\infty(\bar{\Omega})$ for the space of infinitely often differentiable functions. By $L^p(\Omega)$ and $W^{1,p}(\Omega)$ we denote the Lebesgue spaces and Sobolev spaces of functions on Ω, where $H^1(\Omega) := W^{1,2}(\Omega)$. $L^p_+(\Omega)$ stands for the positive cone in the Banach space $L^p(\Omega)$, where $1 \le p \le \infty$. For $1 \le p \le \infty$, we write p' for its conjugate exponent satisfying $\frac{1}{p} + \frac{1}{p'} = 1$. (The value of $\frac{1}{\infty}$ is defined to be 0.)

For a general Banach space X, $\|\cdot\|_X$ denotes its norm and X^* its dual, and the dual pairing between $f \in X^*$ and $g \in X$ will be denoted by $\langle f, g\rangle$. Furthermore, we write $L^p(0,T;X)$ (for $T > 0$ and $1 \le p \le \infty$) for the Banach space of all (equivalence classes of) Bochner measurable functions $f : (0,T) \to X$ such that $\|f(\cdot)\|_X \in L^p(0,T)$. Correspondingly, we denote by $C([0,T];X)$ the Banach space of continuous functions on $[0,T]$ with values in X.

In the estimates of this work, we are going to write C for possibly different constants whose exact form has no importance.

We will study the following system of non-linear partial differential equations:

$$\begin{aligned} U_t &= \Delta U - \chi\nabla(U\nabla S(V)) \\ V_t &= \alpha\Delta V - \beta V + \delta U S'(V) \end{aligned} \tag{1.6}$$

in $(0,T) \times \Omega$ with positive constants χ, α, β and δ. The equations are complemented by initial values $U(0,x) = U_0(x)$, $V(0,x) = V_0(x)$ for all $x \in \Omega$ and the homogeneous Neumann boundary conditions

$$\nu \cdot \nabla U = \nu \cdot \nabla V = 0 \quad \text{on } (0,T) \times \partial\Omega, \tag{1.7}$$

where ν is the unit outer normal at points of $\partial\Omega$.

Definition 1.1 *A pair of functions* (U,V) *with*

$$\begin{aligned} U &\in L^2(0,T;H^1(\Omega))\cap C([0,T];L^2(\Omega)), \quad U_t \in L^2(0,T;(H^1(\Omega))^*) \\ V &\in C([0,T];H^1(\Omega)), \quad V_t \in L^2(0,T;L^2(\Omega)) \end{aligned}$$

is called a (weak) solution of (1.6), (1.7) if the following identities hold

$$\begin{aligned} \int_0^T \langle U_t, H\rangle ds &+ \int_0^T \int_\Omega (\nabla U - \chi U \nabla S(V))\nabla H \; dx \; ds = 0 \\ \int_0^T \int_\Omega V_t H dx ds &+ \int_0^T \int_\Omega (\alpha \nabla V \nabla H + \beta V H - \delta U S'(V) H) \; dx \; ds = 0 \end{aligned}$$

for all $H \in L^2(0,T;H^1(\Omega))$.

1.2.2 Properties of the Sensitivity Functions

The various results of this work will be applicable to different classes of sensitivity functions. The most general class of functions considered will be the set

$$\mathcal{S} \;=\; \Big\{ S \in C^1(I\!R, I\!R) : 0 \le S(V),\ 0 \le S'(V) \le C' \ \text{ for all } V \ge 0 \Big\}. \tag{1.8}$$

At some stages, we will have to require additionally that $S \in \mathcal{S}$ is twice continuously differentiable and that

$$|S''(V)| \;\le\; C'' \tag{1.9}$$

for all $V \ge 0$.

We will not be able to prove global existence of solutions to system (1.6) with (1.7) for all functions $S \in \mathcal{S}$ which also satisfy (1.9), but will give three different proofs for bounded sensitivity functions, the identical sensitivity function and the logarithmic sensitivity function, respectively. In order to show that the members of these three classes belong to $\mathcal{S}$ and fulfill (1.9), we are going to write down all the derivatives up to order 2 of these functions.

We have for the identity sensitivity function

$$S_0(V) = V \ge 0, \quad S_0'(V) \;=\; 1, \quad S_0''(V) \;=\; 0$$

for all $V \ge 0$. For the bounded sensitivity functions S_1 and S_2 we calculate

$$\begin{aligned} S_1(V) &= \frac{V}{1+cV} \ \ge 0, \quad 0 \le S_1'(V) = \frac{1}{(1+cV)^2} \ \le 1,\\ S_2(V) &= \frac{V^2}{1+cV^2} \ge 0, \ \ 0 \le S_2'(V) = \frac{2V}{(1+cV^2)^2} \le \frac{1}{\sqrt{c}}, \end{aligned}$$

and for their second derivatives

$$|S_1''(V)| = \frac{2c}{(1+cV)^3} \le 2c, \qquad |S_2''(V)| = \left| \frac{2(1-3cV^2)}{(1+cV^2)^3} \right| \ \le 2.$$

Finally, we have for the logarithmic sensitivity function

$$S_3(V) = \log(V+c) \ge 0, \quad 0 \le S_3'(V) = \frac{1}{(V+c)} \le \frac{1}{c}, \quad |S_3''(V)| = \frac{1}{(V+c)^2} \le \frac{1}{c^2}.$$

Chapter 2

Local Existence and Uniqueness of Solutions

Theorem 2.1 *Let the sensitivity function S belong to the set $\mathcal{S}$. For positive initial values $U_0 \in L^2(\Omega)$ and $V_0 \in L^\infty(\Omega) \cap H^1(\Omega)$, there exist a $T > 0$ and a corresponding (weak) solution (U, V) of system (1.6), (1.7) with initial values $U(0,x) = U_0(x)$, $V(0,x) = V_0(x)$ for all $x \in \Omega$ satisfying*

$$U \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)), \quad U_t \in L^2(0,T; (H^1(\Omega))^*),$$

$$V \in C([0,T]; H^1(\Omega)) \cap L^\infty(0,T; L^\infty(\Omega)), \quad V_t \in L^2(0,T; L^2(\Omega)).$$

U and V are both positive in the L^2-sense.

Furthermore, we can show that V belongs to $L^q(t,T; W^{1,p}(\Omega))$ for some $p > 2$ and $q > \dfrac{2p}{p-2}$ and that $U \in L^\infty(t,T,L^\infty(\Omega))$ for any $0 < t \leq T$. (The system has a smoothing effect on the initial values.)

Assuming additionally that $S \in C^2(\mathbb{R},\mathbb{R})$ with $|S''(V)| \leq C''$ for all $V \geq 0$, and if $U_0 \in L^\infty(\Omega)$ and $V_0 \in W^{1,p}(\Omega)$ for some $p > 2$, we obtain $U \in L^\infty(0,T; L^\infty(\Omega))$, and we can show uniqueness of the solution.

2.1 Proof of Existence and Regularity

For simplicity, we set w.l.o.g. $\chi = 1$ throughout the proof. (We can take $\tilde{S}(V) := \chi S(V)$ and $\tilde{\delta} := \dfrac{\delta}{\chi}$ without changing any of the prerequisites.)
With $W := \dfrac{U}{e^{S(V)}}$ the system (1.6) is transformed to

$$\begin{aligned} (We^{S(V)})_t &= \nabla(e^{S(V)}\nabla W) \\ V_t &= \alpha\Delta V - \beta V + \delta e^{S(V)} S'(V) W \end{aligned} \tag{2.1}$$

on $(0,T) \times \Omega$, with $V(0) = V_0 \in L^\infty_+(\Omega) \cap H^1(\Omega)$ and $W(0) = W_0 := \dfrac{U_0}{e^{S(V_0)}} \in L^2_+(\Omega)$ and

$$\nu \cdot \nabla W = \nu \cdot \nabla V = 0 \tag{2.2}$$

on $(0,T) \times \partial\Omega$.

For appropriate $T > 0$ and $K > 0$, we define $X := L^2(0,T; L^p_+(\Omega))$, for a $p > 2$, $V_K := \operatorname{sgn}(V)\min\{|V|, K\}$ and the mapping $A_K : X \to X$, by $f \longmapsto W = A_K f$, where W is the solution of the first equation of the regularized system

$$\begin{aligned} (We^{S(V_K)})_t &= \nabla(e^{S(V_K)}\nabla W) \\ V_t &= \alpha\Delta V - \beta V + \delta e^{S(V_K)} S'(V) f \end{aligned} \tag{2.3}$$

on $(0,T) \times \Omega$, with $V(0) = V_0, W(0) = W_0$ and the homogeneous Neumann boundary conditions (2.2).

In the case of bounded sensitivity functions, we evidently do not need the cut-off function for V. But as the proof is still correct, for simplicity, we will not present a second one differing only in this technical detail.

(i) For all $f \in X$, there exists a unique positive function $V \in L^2(0,T; H^1(\Omega))$ with $V_t \in L^2(0,T; L^2(\Omega))$ which solves the second equation in (2.3) with $\nu \cdot \nabla V = 0$ on $(0,T) \times \partial\Omega$. Furthermore, the mapping $t \mapsto \|\nabla V(t)\|^2_{L^2(\Omega)}$ is absolutely continuous on $[0,T]$ and $V \in C([0,T]; H^1(\Omega))$.

By standard results, since $V_0 \in H^1(\Omega)$,† there exists for every $Y \in L^2(0,T; H^1(\Omega))$ a unique solution $V \in L^2(0,T; H^1(\Omega))$ with $V_t \in L^2(0,T; L^2(\Omega))$ of the equation

$$V_t = \alpha\Delta V - \beta V + \delta e^{S(Y_K)} S'(Y) f$$

†See for example Ladyženskaja, Solonnikov and Ural'ceva [24], Chapter III, Theorem 6.1 for the proof with Dirichlet boundary conditions.

and by a simple fixed point argument, there exists a unique V with the same regularity which solves the second equation of (2.3). One also knows that $V \in C([0,T]; L^2(\Omega))$ from Gajewski, Gröger and Zacharias [13], Chapter IV, Theorem 1.17.

It follows now from the equation that $\Delta V \in L^2(0,T; L^2(\Omega))$. We can therefore apply Lemma 3.3 in Brézis [3], p. 73, and obtain the absolute continuity of $t \mapsto \|\nabla V(t)\|^2_{L^2(\Omega)}$ on $[0,T]$, so that $V \in C([0,T]; H^1(\Omega))$.

To obtain the positivity of V, we test the equation with $V^- := \max\{-V, 0\}$:

$$\begin{aligned}\int_\Omega V_t V^- dx &= -\alpha \int_\Omega \nabla V \nabla V^- dx - \beta \int_\Omega V V^- dx + \delta \int_\Omega e^{S(V_K)} S'(V) f V^- dx \\ &= \alpha \int_\Omega |\nabla V^-|^2 dx + \beta \int_\Omega (V^-)^2 dx + \delta \int_\Omega e^{S(V_K)} S'(V) f V^- dx \\ &\geq 0,\end{aligned}$$

as f is positive almost everywhere. Formally differentiating $\|V^-(t)\|^2_{L^2(\Omega)}$ with respect to t leads us to

$$\frac{1}{2}\frac{d}{dt}\|V^-(t)\|^2_{L^2(\Omega)} = \frac{1}{2}\frac{d}{dt}\int_\Omega (V^-(t))^2 dx = -\int_\Omega V_t(t) V^-(t) dx \leq 0$$

and we obtain after integration $\|V^-(t)\|_{L^2(\Omega)} = 0$ for all $t \in [0,T]$.[†] As a consequence, it follows that $V_K = \min\{V, K\}$.

(ii) For every $f \in X$ and the corresponding V found in step (i), the first equation in (2.3) has a unique solution W belonging to the space $L^2(0,T; H^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$ with $(We^{S(V_K)})_t \in L^2(0,T; (H^1(\Omega))^*)$ and satisfying $\nu \cdot \nabla W = 0$ on $(0,T) \times \partial\Omega$.

This is shown by a modification of the proof of Theorem 4.1 in Chapter III, Ladyženskaja et al. [24]. As will be pointed out later, we need for our proof that the space dimension $n = 2$, whereas the original theorem in [24], which deals with a linear equation, holds independently of n.

We apply the Galerkin method to the equation

$$(aW)_t = \nabla(a\nabla W), \quad a = e^{S(V_K)}.$$

Let $\{\psi_k\}, k \in I\!N$, be a complete orthonormal system in $L^2(\Omega)$. We are looking for approximations $w_N(t,x) = \sum_{k=1}^{N} c_N^k(t)\psi_k(x)$ for the solution W, where the coefficients

[†]Approximating V^- by a sequence of smooth functions, for which the time-derivative is well-defined, and passing to the limit by the Dominated Convergence Theorem, we can justify the formal argument.

$c_N^k(t) = \int_\Omega w_N(t,x)\psi_k(x)dx$ are determined through the equations

$$a\frac{d}{dt}c_N^k(t) + a'c_N^k(t) + \tilde{A}_{kl}(t)c_N^l(t) = 0 \tag{2.4}$$

where $c_N^k(0) = \int_\Omega W_0\psi_k dx$ or, with $A_{kl} = \frac{1}{a}\left(\tilde{A}_{kl} + \delta_{kl}a'\right)$,

$$\frac{d}{dt}c_N^k(t) + A_{kl}(t)c_N^l = 0.^\dagger$$

This system of linear ordinary differential equations is uniquely solvable for every $N \in I\!N$ (We have $c_N(t) = e^{-At}c_N(0)$.), and we need to show that the w_N are uniformly bounded in $L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$. From the energy identity

$$\langle (aw_N)_t, w_N\rangle + \int_\Omega a|\nabla w_N|^2 dx = 0$$

$$\iff \quad \frac{1}{2}\frac{d}{dt}\int_\Omega aw_N^2 dx + \int_\Omega a|\nabla w_N|^2 dx + \frac{1}{2}\int_\Omega a'w_N^2 dx = 0$$

we obtain by integration from 0 to t:

$$\frac{1}{2}\int_\Omega aw_N^2 dx\Big|_0^t + \int_0^t\int_\Omega a|\nabla w_N|^2 dxds + \frac{1}{2}\int_0^t\int_\Omega a'w_N^2 dxds = 0,$$

so that (Note that $a \geq 1$.)

$$\begin{aligned}
\frac{1}{2}\int_\Omega w_N^2(t)dx \; - \; \frac{1}{2}\int_\Omega e^{S(V_0)}W_0^2 dx \; + \; \int_0^t\int_\Omega |\nabla w_N|^2 dxds \; &\leq \; \frac{1}{2}\int_0^t\int_\Omega |a'|w_N^2 dxds \\
&\leq \; \frac{1}{2}\|a'\|_{L^2(0,t;L^2(\Omega))}\|w_N^2\|_{L^2(0,t;L^2(\Omega))} \\
&= \; \frac{1}{2}\|a'\|_{L^2(0,t;L^2(\Omega))}\|w_N\|^2_{L^4(0,t;L^4(\Omega))}.
\end{aligned} \tag{2.5}$$

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†Note that $1 \leq a \leq e^{S(K)}$, $a' = e^{S(V_K)}S'(V_K)(V_K)_t \in L^2(0,T;L^2(\Omega))$ and for the w_N we have:

$$\int_\Omega (aw_N)_t\psi_k dx + \mathcal{L}(w_N,\psi_k) = 0 \text{ for all } k = 1,...,N$$

with $\mathcal{L}(w_N,\psi_k) = \int_\Omega a\nabla w_N\nabla\psi_k dx$. Since

$$\frac{d}{dt}\int_\Omega a\sum_{l=1}^N c_N^l(t)\psi_l(x)\psi_k(x)\; dx = \frac{d}{dt}(ac_N^k(t)) = a\frac{d}{dt}c_N^k(t) + a'c_N^k(t)$$

and

$$\mathcal{L}(w_N,\psi_k) = \mathcal{L}(\sum_{l=1}^N c_N^l(t)\psi_l(x),\psi_k(x)) = \sum_{l=1}^N c_N^l(t)\mathcal{L}(\psi_l(x),\psi_k(x)),$$

we obtain (2.4) with $\tilde{A}_{kl} = \mathcal{L}(\psi_k,\psi_l)$.

We will write for simplicity

$$Z_{0,t} := L^\infty(0,t;L^2(\Omega)) \cap L^2(0,t;H^1(\Omega))$$

and define the norm $\|w\|^2_{Z_{0,t}} := \left(\|w\|^2_{L^\infty(0,t;L^2(\Omega))} + \|w\|^2_{L^2(0,t;H^1(\Omega))}\right)$ for all $w \in Z_{0,t}$.

By Gagliardo-Nirenberg's (Here we need $n = 2$.), Hölder's and Young's Inequality, we estimate

$$\begin{aligned}
\|w_N\|^4_{L^4(0,t;L^4(\Omega))} &= \int_0^t \|w_N(s)\|^4_{L^4(\Omega)} ds \;\le\; \tilde{C}^2 \int_0^t \|w_N(s)\|^2_{L^2(\Omega)} \|w_N(s)\|^2_{H^1(\Omega)} ds \\
&\le \tilde{C}^2 \|w_N\|^2_{L^\infty(0,t;L^2(\Omega))} \|w_N\|^2_{L^2(0,t;H^1(\Omega))} \\
&\le \tilde{C}^2 \left(\|w_N\|^4_{L^\infty(0,t;L^2(\Omega))} + \|w_N\|^4_{L^2(0,t;H^1(\Omega))}\right) \;\le\; \tilde{C}^2 \|w_N\|^4_{Z_{0,t}},
\end{aligned}$$

so that, taking the supremum over the interval $[0,t]$, we get in (2.5)

$$\|w_N\|^2_{L^\infty(0,t;L^2(\Omega))} + 2\,\|\nabla w_N\|^2_{L^2(0,t;L^2(\Omega))} \le \|e^{S(V_0)} W_0\|^2_{L^2(\Omega)} + \tilde{C}\|a'\|_{L^2(0,t;L^2(\Omega))} \|w_N\|^2_{Z_{0,t}}.$$

We estimate the left hand side from below:

$$\begin{aligned}
\|w_N\|^2_{L^\infty(0,t;L^2(\Omega))} &+ 2\,\|\nabla w_N\|^2_{L^2(0,t;L^2(\Omega))} \\
&\ge \frac{1}{2}\,\|w_N\|^2_{L^\infty(0,t;L^2(\Omega))} + \frac{1}{2t}\|w_N\|^2_{L^2(0,t;L^2(\Omega))} + \|\nabla w_N\|^2_{L^2(0,t;L^2(\Omega))} \\
&\ge \frac{1}{2}\min\left(1, \frac{1}{T}\right)\left\{\|w_N\|^2_{L^\infty(0,t;L^2(\Omega))} + \|w_N\|^2_{L^2(0,t;H^1(\Omega))}\right\} \\
&= \frac{1}{2}\,C(T)\|w_N\|^2_{Z_{0,t}}
\end{aligned}$$

with $C(T) = \min\left(1, \frac{1}{T}\right)$. We thus obtain

$$\|w_N\|^2_{Z_{0,t}} \le \frac{2}{C(T)}\|e^{S(V_0)} W_0\|^2_{L^2(\Omega)} + \frac{2\tilde{C}}{C(T)}\|a'\|_{L^2(0,t;L^2(\Omega))}\|w_N\|^2_{Z_{0,t}}.$$

Let $t = t_1$ be so small that $\|a'\|_{L^2(0,t_1;L^2(\Omega))} \le \dfrac{C(T)}{4\tilde{C}}$. We then have

$$\|w_N\|^2_{Z_{0,t_1}} \le \frac{4}{C(T)}\|e^{S(V_0)} W_0\|^2_{L^2(\Omega)} \;=\; \frac{4}{C(T)}\|U_0\|^2_{L^2(\Omega)}.$$

We can now partition $[0,T]$ into finitely many intervals $[t_{k-1}, t_k]$ for $k = 1, ..., s$, where $t_0 = 0$ and $t_s = T$, such that

$$\|a'\|_{L^2(t_{k-1},t_k;L^2(\Omega))} \le \frac{C(T)}{4\tilde{C}}$$

and consequently

$$\|w_N\|^2_{Z_{t_{k-1},t_k}} \;\le\; \frac{4}{C(T)}\|e^{S(V_K(t_{k-1}))}w_N(t_{k-1})\|^2_{L^2(\Omega)} \;\le\; \frac{4e^{S(K)}}{C(T)}\|w_N(t_{k-1})\|^2_{L^2(\Omega)}$$

for $k = 1, ..., s$ but

$$\|a'\|_{L^2(t_{k-1},t_k;L^2(\Omega))} \ge \frac{C(T)}{8\tilde{C}} \quad \text{for} \quad k = 1, ..., s-1.$$

One has the following bound for the number s of intervals:

$$(s-1)\left(\frac{C(T)}{8\tilde{C}}\right)^2 \le \sum_{k=1}^{s-1}\|a'\|^2_{L^2(t_{k-1},t_k;L^2(\Omega))} \le \sum_{k=1}^{s}\|a'\|^2_{L^2(t_{k-1},t_k;L^2(\Omega))} = \|a'\|^2_{L^2(0,T;L^2(\Omega))},$$

i.e.,

$$s \le 1 + \left(\frac{8\tilde{C}}{C(T)}\right)^2 \|a'\|^2_{L^2(0,T;L^2(\Omega))}.$$

An iteration argument, using

$$\|w_N(t_{k-1})\|^2_{L^2(\Omega)} \;\le\; \|w_N\|^2_{L^\infty(t_{k-2},t_{k-1};L^2(\Omega))} \;\le\; \|w_N\|^2_{Z_{t_{k-2},t_{k-1}}},$$

yields the uniform estimate

$$\|w_N\|^2_{Z_{0,T}} \;\le\; C.$$

With the arguments of Ladyženskaja et al. [24], Chapter III, Theorem 4.1, we now obtain a solution W of the first equation in (2.3) with homogeneous Neumann boundary conditions belonging to $L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$. It follows from the equation itself that $(We^{S(V_K)})_t \in L^2(0,T;(H^1(\Omega))^*)$.

(iii) $W \ge 0$ *almost everywhere in* $[0,T]\times\Omega$.

Testing the first equation in (2.3) with $W^- = \max\{-W, 0\}$ gives

$$\langle (We^{S(V_K)})_t, W^-\rangle \;=\; \int_\Omega e^{S(V_K)}|\nabla W^-|^2 dx$$

so that formally (See the footnote in part *(i)* of this proof.)

$$\begin{aligned}
\frac{d}{dt}\int_\Omega (W^-)^2 e^{S(V_K)} dx \;&=\; \frac{d}{dt}\int_\Omega (W^- e^{S(V_K)})^2 e^{-S(V_K)} dx \\
&=\; -\, 2\langle (We^{S(V_K)})_t, W^-\rangle \\
&\quad -\int_\Omega (W^- e^{S(V_K)})^2 e^{-S(V_K)} S'(V_K)(V_K)_t dx \\
&=\; -2\int_\Omega e^{S(V_K)}|\nabla W^-|^2 dx - \int_\Omega (W^-)^2 e^{S(V_K)} S'(V_K)(V_K)_t dx.
\end{aligned}$$

Integration over $(0,t)$, using $W^-(0) = 0$ and Gagliardo-Nirenberg's and Young's Inequality, yields

$$\begin{aligned}\int_\Omega W^-(t)^2 e^{S(V_K)}dx &+ 2\int_0^t\int_\Omega e^{S(V_K)}|\nabla W^-(s)|^2 dxds \\ &\leq \int_0^t\int_\Omega (W^-(s))^2 e^{S(V_K(s))}S'(V_K(s))|(V_K)_t|dxds \\ &\leq e^{S(K)}C'\int_0^t \|(V_K)_t(s)\|_{L^2(\Omega)}\|W^-(s)\|_{L^4(\Omega)}^2 ds \\ &\leq C\int_0^t \|V_t(s)\|_{L^2(\Omega)}(\|\nabla W^-(s)\|_{L^2}+\|W^-(s)\|_{L^2})\|W^-(s)\|_{L^2(\Omega)}ds \\ &\leq C\int_0^t C(\varepsilon)\left(\|V_t(s)\|_{L^2(\Omega)}^2+1\right)\|W^-(s)\|_{L^2(\Omega)}^2 ds \\ &\quad + \varepsilon\|\nabla W^-\|_{L^2(0,t;L^2(\Omega))}^2.\end{aligned}$$

If we choose $\varepsilon = 2$, having in mind that $e^{S(V_K)} \geq 1$, we obtain

$$\|W^-(t)\|_{L^2(\Omega)}^2 \leq C\int_0^t \left(\|V_t\|_{L^2(\Omega)}^2+1\right)\|W^-(s)\|_{L^2(\Omega)}^2 ds$$

for all $t \in (0,T)$, and from Gronwall's Lemma $\|W^-(t)\|_{L^2(\Omega)}^2 = 0$ follows, so that $W(t)$ belongs to $L^2_+(\Omega)$. Note that we have now also shown that A_K is well-defined.

(iv) For any $R > 0$ and $T > 0$ small enough, the mapping A_K sends the ball $B := \{f \in X : \|f\|_{L^2(0,T;L^p(\Omega))} \leq R\}$ into itself.

Testing the second equation with V_t gives

$$\begin{aligned}\|V_t\|_{L^2(0,t;L^2(\Omega))}^2 &+ \frac{\alpha}{2}\left(\|\nabla V(t)\|_{L^2(\Omega)}^2 - \|\nabla V_0\|_{L^2(\Omega)}\right) + \frac{\beta}{2}\left(\|V(t)\|_{L^2(\Omega)}^2 - \|V_0\|_{L^2(\Omega)}^2\right) \\ &= \delta\int_0^t\int_\Omega e^{S(V_K)}S'(V)fV_t dxds \\ &\leq \delta e^{S(K)}C'\|f\|_{L^2(0,t;L^2(\Omega))}\|V_t\|_{L^2(0,t;L^2(\Omega))} \leq \frac{1}{2}(CR)^2 + \frac{1}{2}\|V_t\|_{L^2(0,t;L^2(\Omega))}^2,\end{aligned}$$

i.e.,

$$\|V_t\|_{L^2(0,T;L^2(\Omega))}^2 + \sup_{t\in(0,T)}\alpha\|\nabla V(t)\|_{L^2(\Omega)}^2 \leq C(\|V_0\|_{H^1(\Omega)}) + CR^2. \tag{2.6}$$

On the other hand, testing the first equation with W shows

$$\begin{aligned}\frac{d}{dt}\int_\Omega W^2 e^{S(V_K)}dx &+ 2\int_\Omega |\nabla W|^2 e^{S(V_K)}dx = -\int_\Omega W^2 e^{S(V_K)}S'(V_K)(V_K)_t dx \\ &\leq e^{S(K)}C'\|V_t\|_{L^2(\Omega)}\|W\|_{L^4(\Omega)}^2 \\ &\leq C(\|V_t\|_{L^2(\Omega)}^2+1)\|W\|_{L^2(\Omega)}^2 + \|\nabla W\|_{L^2(\Omega)}^2.\end{aligned}$$

Integration over $(0,t)$ leads to

$$\begin{aligned}\int_\Omega W^2(t)dx &- \int_\Omega e^{S(V_0)}W_0^2dx + \int_0^t \|\nabla W\|^2_{L^2(\Omega)}ds \le C\int_0^t (\|V_t\|^2_{L^2(\Omega)}+1)\|W\|^2_{L^2(\Omega)}ds \\ &\le C\int_0^t (\|V_t\|^2_{L^2(\Omega)}+1)\left[\|W(s)\|^2_{L^2(\Omega)} + \int_0^s \|\nabla W(\tau)\|^2_{L^2(\Omega)}d\tau\right]ds\end{aligned}$$

and by Gronwall's Lemma and estimate (2.6) for $\|V_t\|_{L^2(0,T;L^2(\Omega))}$,

$$\begin{aligned}\|W(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla W(s)\|^2_{L^2(\Omega)}ds &\le \|e^{\frac{S(V_0)}{2}}W_0\|^2_{L^2(\Omega)}\exp\left(C\int_0^t (\|V_t\|^2_{L^2(\Omega)}+1)ds\right) \\ &\le e^{S(K)}\|W_0\|^2_{L^2(\Omega)}e^{(C(\|V_0\|_{H^1})+CR^2+t)} \\ &\le C_1e^{CR^2}e^t. \qquad (2.7)\end{aligned}$$

Using Gagliardo-Nirenberg's (Remember that $p>2$.), Hölder's Inequality and (2.7), we can further estimate

$$\begin{aligned}\|W\|^2_{L^2(0,T;L^p(\Omega))} &= \int_0^T \|W\|^2_{L^p(\Omega)}ds \le C\int_0^T \|W(s)\|^{\frac{2(p-2)}{p}}_{H^1(\Omega)}\|W(s)\|^{\frac{4}{p}}_{L^2(\Omega)}ds \\ &\le C\left(\int_0^T \|W(s)\|^2_{H^1(\Omega)}ds\right)^{\frac{p-2}{p}}\left(\int_0^T \|W(s)\|^2_{L^2(\Omega)}ds\right)^{\frac{2}{p}} \\ &\le C\left(\int_0^T \|\nabla W(s)\|^2_{L^2(\Omega)}ds\right)^{\frac{p-2}{p}}\left(\int_0^T \|W(s)\|^2_{L^2(\Omega)}ds\right)^{\frac{2}{p}} \\ &\quad + C\int_0^T \|W(s)\|^2_{L^2(\Omega)}ds \\ &\overset{(2.7)}{\le} CC_1^{\frac{p-2}{p}}e^{\frac{p-2}{p}(CR^2+T)}C_1^{\frac{2}{p}}e^{\frac{2}{p}CR^2}(e^T-1)^{\frac{2}{p}} + CC_1e^{CR^2}(e^T-1) \\ &= CC_1e^{CR^2}e^{\frac{p-2}{p}T}(e^T-1)^{\frac{2}{p}} + CC_1e^{CR^2}(e^T-1),\end{aligned}$$

so that $\|W\|_{L^2(0,T;L^p(\Omega))} \le R$ if

$$e^{\frac{p-2}{p}T}(e^T-1)^{\frac{2}{p}} \le \frac{R^2}{2CC_1}e^{-CR^2} \quad \text{and} \quad (e^T-1) \le \frac{R^2}{2CC_1}e^{-CR^2}.$$

(v) There exist a $T>0$ and positive functions $W \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ and $V \in C([0,T];H^1(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega))$ with $(We^{S(V_K)})_t \in L^2(0,T;(H^1(\Omega))^*)$ and $V_t \in L^2(0,T;L^2(\Omega))$, solving the following system

$$\begin{aligned}(We^{S(V_K)})_t &= \nabla(e^{S(V_K)}\nabla W) \\ V_t &= \alpha\Delta V - \beta V + \delta e^{S(V_K)}S'(V)W \qquad (2.8)\end{aligned}$$

on $(0,T)\times\Omega$ with initial conditions $W(0)=W_0 \in L^2(\Omega)$, $V(0)=V_0 \in L^\infty(\Omega)\cap H^1(\Omega)$ and homogeneous Neumann boundary conditions.

We want to apply Schauder's fixed point theorem to the mapping A_K. In order to prove that $A_K : X \longrightarrow X$ is continuous and compact, we will, in a first step, show that the mapping $f \mapsto V$ is Lipschitz continuous from the space $L^2(0,T;L^p(\Omega))$ to $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$.

For f_1 and f_2 in $L^2(0,T;L^p(\Omega))$ we obtain from part *(i)* of this proof corresponding V_1 and V_2. Testing the equation for $(V_1 - V_2)$ with the difference $(V_1 - V_2)$ itself yields

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\int_\Omega (V_1 - V_2)^2 dx &= \int_\Omega (V_1 - V_2)_t (V_1 - V_2) dx \\ &= -\alpha \int_\Omega |\nabla (V_1 - V_2)|^2 dx - \beta \int_\Omega (V_1 - V_2)^2 dx \\ &\quad + \delta \int_\Omega \left(e^{S(V_{1K})} S'(V_1) f_1 - e^{S(V_{2K})} S'(V_2) f_2 \right) (V_1 - V_2)\, dx. \quad (2.9) \end{aligned}$$

And since the function $V \mapsto e^{S(V_K)} S'(V)$ is Lipschitz continuous, we can estimate

$$\begin{aligned} |e^{S(V_{1K})} S'(V_1) f_1 - e^{S(V_{2K})} S'(V_2) f_2| &\le |(e^{S(V_{1K})} S'(V_1) - e^{S(V_{2K})} S'(V_2)) f_1| \\ &\quad + |e^{S(V_{2K})} S'(V_2)(f_1 - f_2)| \\ &\le e^{S(K)} \left[(C')^2 + C'' \right] |f_1|\, |V_1 - V_2| + e^K C' |f_1 - f_2|, \end{aligned}$$

so that we obtain after integrating equality (2.9) over $[0,t]$

$$\begin{aligned} \|(V_1 - V_2)(t)\|^2_{L^2(\Omega)} + \|V_1 - V_2\|^2_{L^2(0,t;H^1(\Omega))} &\le C \int_0^t \int_\Omega |f_1|\, (V_1 - V_2)^2 dx ds \\ &\quad + C \int_0^t \int_\Omega |f_1 - f_2|\, |V_1 - V_2| dx ds. \end{aligned}$$

To the first term on the right hand side of this estimate, we apply Hölder's and the Gagliardo-Nirenberg's Inequality and continue by using Young's Inequality:

$$\begin{aligned} \|(V_1 - V_2)(t)\|^2_{L^2(\Omega)} + \|V_1 - V_2\|^2_{L^2(0,t;H^1(\Omega))} &\le C \int_0^t \|f_1(s)\|_{L^2(\Omega)} \|(V_1 - V_2)(s)\|^2_{L^4(\Omega)} ds \\ &\quad + C \int_0^t \int_\Omega |(f_1 - f_2)(s)|\, |(V_1 - V_2)(s)| dx ds \\ &\le C \int_0^t \|f_1(s)\|_{L^2(\Omega)} \|(V_1 - V_2)(s)\|_{L^2(\Omega)} \|(V_1 - V_2)(s)\|_{H^1(\Omega)} ds \\ &\quad + \int_0^t \int_\Omega (f_1 - f_2)^2(s) dx ds + C \int_0^t \int_\Omega (V_1 - V_2)^2(s) dx ds \\ &\le C \int_0^t \left(1 + \|f_1(s)\|^2_{L^2(\Omega)} \right) \|(V_1 - V_2)(s)\|^2_{L^2(\Omega)} ds + \frac{1}{2} \|V_1 - V_2\|^2_{L^2(0,t;H^1(\Omega))} \\ &\quad + C \|f_1 - f_2\|^2_{L^2(0,T;L^p(\Omega))}. \end{aligned}$$

Finally, we obtain

$$\begin{aligned}
\|(V_1 - V_2)(t)\|^2_{L^2(\Omega)} &+ \frac{1}{2}\|V_1 - V_2\|^2_{L^2(0,t;H^1(\Omega))} \\
&\le C\int_0^t \left(1 + \|f_1(s)\|^2_{L^2(\Omega)}\right) \|(V_1 - V_2)(s)\|^2_{L^2(\Omega)} ds + C\|f_1 - f_2\|^2_{L^2(0,T;L^p(\Omega))} \\
&\le C\int_0^t \left(1 + \|f_1(s)\|^2_{L^2(\Omega)}\right) \left[\|(V_1 - V_2)(s)\|^2_{L^2(\Omega)} + \frac{1}{2}\|V_1 - V_2\|^2_{L^2(0,s;H^1(\Omega))}\right] ds \\
&\quad + C\|f_1 - f_2\|^2_{L^2(0,T;L^p(\Omega))}.
\end{aligned}$$

Here we are in a Gronwall constellation, so that Lipschitz continuity of the mapping $f \in L^2(0,T;L^p(\Omega)) \mapsto V \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ follows after taking the supremum over $[0,T]$ on both sides:

$$\begin{aligned}
\|V_1 - V_2\|^2_{L^\infty(0,T;L^2(\Omega))} &+ \frac{1}{2}\|V_1 - V_2\|^2_{L^2(0,T;H^1(\Omega))} \\
&\le C\|f_1 - f_2\|^2_{L^2(0,T;L^p(\Omega))} \exp\left(C\int_0^T \left(1 + \|f_1\|^2_{L^2(\Omega)}\right) ds\right) \\
&\le C\|f_1 - f_2\|^2_{L^2(0,T;L^p(\Omega))} \exp\left[\left(T + \|f_1\|^2_{L^2(0,T;L^p(\Omega))}\right) C\right] \\
&\le C(T)\|f_1 - f_2\|^2_{L^2(0,T;L^p(\Omega))}. \qquad (2.10)
\end{aligned}$$

Now we can prove the claim of this part of the proof. Suppose there is a sequence of functions $f_n \in L^2(0,T;L^p(\Omega))$ converging to an $f \in L^2(0,T;L^p(\Omega))$ as $n \to \infty$. We need to show that the $W_n := A_K(f_n)$ converge to $W := A_K(f)$ in the same space. From estimate (2.10) it follows that the corresponding $V_n \longrightarrow V$ in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$, where $f \mapsto V$.

From part *(ii)* of this proof we know that the sequence W_n is uniformly bounded in $L^2(0,T;H^1(\Omega))$. We want to use the fact that the embedding

$$\{W : W \in L^2(0,T;H^1(\Omega)), W_t \in L^2(0,T;(W^{1,p}(\Omega))^*)\} \hookrightarrow L^2(0,T;L^p(\Omega)) \qquad (2.11)$$

is compact† and we now first need to show that there exists a positive constant $C(T)$ so that $\|W_t\|_{L^2(0,T;(W^{1,p}(\Omega))^*)} \le C(T)$ for the solution W of the first equation in (2.3). Then we have a uniform estimate for the sequence W_n in that space, too.

Testing the first equation with $He^{-S(V_K)}$ for a $H \in L^2(0,T;W^{1,p}(\Omega))$, we obtain

$$\begin{aligned}
\langle W_t, H\rangle &= \langle e^{S(V_K)}W_t, e^{-S(V_K)}H\rangle \\
&= \langle (e^{S(V_K)}W)_t, e^{-S(V_K)}H\rangle - \int_\Omega (V_K)_t S'(V_K) W H dx \\
&= -\int_\Omega e^{S(V_K)}\nabla W \nabla(e^{-S(V_K)}H) dx - \int_\Omega (V_K)_t S'(V_K) W H dx \\
&= -\int_\Omega \nabla W(\nabla H - HS'(V_K)\nabla V_K) dx - \int_\Omega (V_K)_t S'(V_K) W H dx
\end{aligned}$$

†See Lions [25], Chapter I, Theorem 5.1.

It follows that

$$\begin{aligned}
\int_0^T |\langle W_t, H\rangle| ds &\le \int_0^T \Big\{\|\nabla W\|_{L^2(\Omega)}(\|\nabla H\|_{L^2(\Omega)} + C'\|H\|_{L^\infty(\Omega)}\|\nabla V\|_{L^2(\Omega)}) \\
&\quad + C'\|V_t\|_{L^2(\Omega)}\|W\|_{L^2(\Omega)}\|H\|_{L^\infty(\Omega)}\Big\} ds \\
&\le \|\nabla W\|_{L^2(0,T;L^2(\Omega))}(\|\nabla H\|_{L^2(0,T;L^2(\Omega))} \\
&\quad + C'\|H\|_{L^2(0,T;L^\infty(\Omega))}\|\nabla V\|_{L^\infty(0,T;L^2(\Omega))}) \\
&\quad + C'\|V_t\|_{L^2(0,T;L^2(\Omega))}\|W\|_{L^\infty(0,T;L^2(\Omega))}\|H\|_{L^2(0,T;L^\infty(\Omega))}.
\end{aligned}$$

Using the embeddings $W^{1,p}(\Omega) \hookrightarrow H^1(\Omega)$ and $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ for $p > 2$ and the estimates obtained for W and V in step *(iv)*, we finally get

$$\int_0^T |\langle W_t, H\rangle| ds \le Ce^T \|H\|_{L^2(0,T;W^{1,p}(\Omega))},$$

which demonstrates the alleged boundedness of W_t in $L^2(0,T;(W^{1,p}(\Omega))^*)$.

It follows by reflexivity of these spaces that there exists a $\tilde{W}$ such that a subsequence

$$W_{n_k} \rightharpoonup \tilde{W} \quad \text{in } L^2(0,T;H^1(\Omega))$$

and

$$(W_{n_k})_t \rightharpoonup \tilde{W}_t \quad \text{in } L^2(0,T;(W^{1,p}(\Omega))^*) \quad \text{as } n_k \to \infty.$$

Testing the equations for the W_{n_k} with a function $H \in C^\infty([0,T]\times\bar{\Omega})$ that vanishes at the endpoints of the interval $[0,T]$ gives

$$\int_0^T \Big\langle \Big(W_{n_k} e^{S(V_{n_k}K)}\Big)_t, H\Big\rangle dt \;=\; -\int_0^T \int_\Omega e^{S(V_{n_k}K)} \nabla W_{n_k} \nabla H dxdt$$

We can pass to the limit $n_k \to \infty$ on both sides of this equality as follows

$$\begin{aligned}
\int_0^T \Big\langle \Big(W_{n_k} e^{S(V_{n_k}K)}\Big)_t, H\Big\rangle dt &= -\int_0^T \int_\Omega W_{n_k} e^{S(V_{n_k}K)} H_t dxdt \\
&\longrightarrow -\int_0^T \int_\Omega \tilde{W} e^{S(V_K)} H_t dxdt \\
&= \int_0^T \Big\langle \Big(\tilde{W} e^{S(V_K)}\Big)_t, H\Big\rangle dt
\end{aligned}$$

and

$$-\int_0^T \int_\Omega e^{S(V_{n_k}K)} \nabla W_{n_k} \nabla H dxdt \;\longrightarrow\; -\int_0^T \int_\Omega e^{S(V_K)} \nabla \tilde{W} \nabla H dxdt,$$

so that we obtain

$$\int_0^T \Big\langle \Big(\tilde{W} e^{S(V_K)}\Big)_t, H\Big\rangle dt \;=\; -\int_0^T \int_\Omega e^{S(V_K)} \nabla \tilde{W} \nabla H dxdt, \tag{2.12}$$

for all $H \in L^2(0,T;W^{1,p}(\Omega))$ by a density argument. From part *(ii)* of the proof, we also know that $\left(W_n e^{S(V_{n_K})}\right)_t$ is uniformly bounded in $L^2(0,T;(H^1(\Omega))^*)$ and we can assume w.l.o.g. that $\left(W_{n_k} e^{S(V_{n_k K})}\right)_t$ converges weakly to a Y in that space. Via another density argument, it then follows that $\left(\tilde{W} e^{S(V_K)}\right)_t = Y \in L^2(0,T;(H^1(\Omega))^*)$ and (2.12) holds for all $H \in L^2(0,T;H^1(\Omega))$.

From the uniqueness of the solution W of the first equation in (1.6) with (1.7), it follows that $\tilde{W} = W$ and we can deduce convergence of the whole sequence $W_n \rightharpoonup W$ in $L^2(0,T;H^1(\Omega))$ and $(W_n)_t \rightharpoonup \tilde{W}_t$ in $L^2(0,T;(W^{1,p}(\Omega))^*)$ as $n \to +\infty$. By compactness of the embedding (2.11), we conclude that $W_n \longrightarrow W$ in $L^2(0,T;L^p(\Omega))$ and have thus proven the claimed continuity of the mapping A_K.

Also by the compactness of the embedding (2.11), the image of the ball $B := \{f \in X : \|f\|_{L^2(0,T;L^p(\Omega))} \leq R\}, A_K(B)$, is compact, so that with Schauder's Theorem the existence of a fixed point $W \in B$ follows, which solves with the corresponding V system (2.8) with homogeneous Neumann boundary conditions.

The claimed regularities $W \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ with $(We^{S(V_K)})_t$ belonging to $L^2(0,T;(H^1(\Omega))^*)$ and $V \in C([0,T];H^1(\Omega))$ with $V_t \in L^2(0,T;L^2(\Omega))$ follow from steps *(i)* - *(iii)*.

Since $e^{S(V_K)} S'(V) \leq e^{S(K)} C'$, we have for the right hand side of the V-equation in (2.8): $e^{S(V_K)} S'(V) W \in L^2(0,T;L^p(\Omega))$ for a $p > 2$, too, and it follows by standard results that $V \in L^\infty(0,T;L^\infty(\Omega))$.†

(vi) For a sufficiently small $T > 0$, we obtain $\|V\|_{L^\infty(0,T;L^\infty(\Omega))} \leq K$, so that $V_K = V$ and W and V also solve (2.1) with (2.2).

The function $U = We^{S(V)}$ belongs to $L^2(0,T;H^1(\Omega))$ with $U_t \in L^2(0,T;(H^1(\Omega))^*)$ and it follows that $U \in C([0,T];L^2(\Omega))$. Furthermore, U is positive in the L^2-sense and solves system (1.6) with (1.7) together with V.

We have chosen $K > \|V_0\|_{L^\infty(\Omega)}$. Since the right hand side of the equation

$$V_t - \alpha \Delta V + \beta V = \delta S'(V) e^{\chi S(V_K)} W$$

is bounded by the $L^\infty(0,T;L^2(\Omega))$-function $\delta C' e^{\chi S(K)} W$, we obtain by a comparison theorem that $\|V\|_{L^\infty(0,T;L^\infty(\Omega))} \leq K$, too, provided $T > 0$ is chosen small enough.

†In the book by Ladyženskaja et al. [24], Chapter III, Theorem 7.1, we find the proof of this result for Dirichlet boundary conditions, which can be modified for equations with homogeneous Neumann boundary conditions.

Hence, $V(t) \leq K$ for all $t \in [0,T]$, so that $V_K = V$ on this interval and the systems (2.8) and (2.1) are equivalent.

Since $U = We^{S(V)}$ and we have shown that the positive function $e^{S(V)}$ belongs to the spaces $L^2(0,T;H^1(\Omega))$ and $L^\infty(0,T;L^\infty(\Omega))$, the claimed regularity and positivity of U follow from the properties of W.† In part *(iii)*, it was shown that $U_t = (We^{S(V)})_t$ belongs to the space $L^2(0,T;(H^1(\Omega))^*)$.

(vii) If $U_0 \in L^\infty(\Omega)$ and $V_0 \in W^{1,p}(\Omega)$ for a $p > 2$, then $|\nabla V| \in L^q(0,T;L^p(\Omega))$ with $q > \dfrac{2p}{p-2}$ and it follows that $U \in L^\infty(0,T;L^\infty(\Omega))$.

If we regard $\nabla S(V)$ as a simple coefficient in the first equation of (1.6), it follows from Theorem 7.1, Chapter III in Ladyženskaja et al. [24] that $U \in L^\infty(0,T;L^\infty(\Omega))$ if $|\nabla S(V)| \in L^q(0,T;L^p(\Omega))$ for a $p > 2$ and a $q > \dfrac{2p}{p-2}$, or equivalently,

$$|\nabla V| \in L^q(0,T;L^p(\Omega)) \quad \text{for} \quad p > 2 \quad \text{and} \quad q > \frac{2p}{p-2}. \tag{2.13}$$

Hence, let us prove (2.13). Extending V_0 to $[0,T]$ by $V_0(t,x) := V_0(x)$ for all $t \in [0,T]$, we define $\tilde{V} := V - V_0$ and thus obtain the following equation

$$\tilde{V}_t - \alpha\Delta\tilde{V} + \beta\tilde{V} = \alpha\Delta V_0 - \beta V_0 + \delta S'(\tilde{V} + V_0)U =: f$$

with $\tilde{V}_0 = 0$. The right hand side f of the equation is in $L^\infty(0,T;(W^{1,p'}(\Omega))^*)$. From a parabolic regularity result by Gröger [16], it follows that the mapping $f \mapsto \tilde{V}$ is continuous between the spaces $L^p(0,T;(W^{1,p'}(\Omega))^*)$ and $L^p(0,T;W^{1,p}(\Omega))$. According to a result by Dore [8], it is then also continuous from $L^q(0,T;(W^{1,p'}(\Omega))^*)$ to $L^q(0,T;W^{1,p}(\Omega))$ for all $q > 1$ and in particular for a $q > \dfrac{2p}{p-2}$. We conclude that $V \in L^q(0,T;W^{1,p}(\Omega))$, so that (2.13) holds.

(viii) Even without the further regularity of the initial values, we can show that $V \in L^q(t,T;W^{1,p}(\Omega))$ for some $p > 2$ and a $q > \dfrac{2p}{p-2}$ and $U \in L^\infty(t,T;L^\infty(\Omega))$ for all $0 < t \leq T$.

On one hand, we have $U \in C([0,T];L^2(\Omega))$ and $V \in C([0,T];H^1(\Omega))$. On the other hand, one concludes from $U_t \in L^2(0,T;(H^1(\Omega))^*)$ with an argument by Gajewski and Gröger (Theorem 2 in [9]) that tV_t belongs to $C([0,T];L^2(\Omega))$.

†$U \in C([0,T];L^2(\Omega))$ follows by standard arguments from the fact that $U \in L^2(0,T;H^1(\Omega))$ and $U_t \in L^2(0,T;(H^1(\Omega))^*)$, see e.g. Gajewski, Gröger and Zacharias [13], Chapter IV, Theorem 1.17.

In particular, we have $-\beta V(t) + \delta U(t) S'(V(t)) - V_t(t) \in L^2(\Omega)$, for $0 < t \leq T$, and as $V(t)$ is the (weak) solution of the problem

$$-\alpha \Delta V(t) = -\beta V(t) + \delta U(t) S'(V(t)) - V_t(t) \quad \text{on} \quad \Omega, \qquad \nu \cdot \nabla V = 0 \quad \text{on} \quad \partial\Omega,$$

an elliptic regularity result by Gröger [15] yields that $V(t) \in W^{1,p}(\Omega)$ for some $p > 2$ since $L^2(\Omega) \subset (W^{1,p}(\Omega))^*$.

As in part *(vii)*, we can now deduce that V belongs to $L^q(t, T; W^{1,p}(\Omega))$ with $q > \dfrac{2p}{p-2}$ for any $0 < t \leq T$ and it follows analogously that $U \in L^\infty(t, T; L^\infty(\Omega))$. □

2.2 Proof of Uniqueness

Suppose there were two solutions (U_1, V_1) and (U_2, V_2) of system (1.6) with the same initial values $(U_i(0), V_i(0)) = (U_0, V_0)$ for $i = 1, 2$.

Integrating the identity

$$\begin{aligned}
&\frac{d}{dt} \left\{U_1(\log U_1 - 1) + U_2(\log U_2 - 1) - (U_1 + U_2)\left(\log\left(\frac{U_1 + U_2}{2}\right) - 1\right)\right\} \\
&= U_{1t} \log U_1 + U_{2t} \log U_2 - (U_1 + U_2)_t \log\left(\frac{U_1 + U_2}{2}\right) \\
&= U_{1t} \log \frac{2U_1}{U_1 + U_2} + U_{2t} \log \frac{2U_2}{U_1 + U_2}
\end{aligned}$$

over $(0, t)$ and using the equations for the U_i, we obtain

$$\begin{aligned}
&\int_\Omega \left\{U_1(\log U_1 - 1) + U_2(\log U_2 - 1) - (U_1 + U_2)\left(\log\left(\frac{U_1 + U_2}{2}\right) - 1\right)\right\}(t)dx \\
&= \int_0^t \int_\Omega \left[U_{1t} \log \frac{2U_1}{U_1 + U_2} + U_{2t} \log \frac{2U_2}{U_1 + U_2}\right] dxds \\
&= -\int_0^t \int_\Omega \Big\{(\nabla U_1 - \chi U_1 \nabla S(V_1))\nabla \log \frac{2U_1}{U_1 + U_2} \\
&\qquad + (\nabla U_2 - \chi U_2 \nabla S(V_2))\nabla \log \frac{2U_2}{U_1 + U_2}\Big\} dxds \\
&= -\int_0^t \int_\Omega \frac{1}{U_1 + U_2} \Big[(\nabla U_1 - \chi U_1 \nabla S(V_1))\left(\frac{U_2}{U_1}\nabla U_1 - \nabla U_2\right) \\
&\qquad + (\nabla U_2 - \chi U_2 \nabla S(V_2))\left(\frac{U_1}{U_2}\nabla U_2 - \nabla U_1\right)\Big] dxds \\
&= -\int_0^t \int_\Omega \frac{1}{U_1 + U_2} \Big[(\nabla U_1 - \chi U_1 \nabla S(V_1))U_2 \nabla \log \frac{U_1}{U_2} \\
&\qquad - (\nabla U_2 - \chi U_2 \nabla S(V_2))U_1 \nabla \log \frac{U_1}{U_2}\Big] dxds \\
&= -\int_0^t \int_\Omega \frac{U_1 U_2}{U_1 + U_2} \left[\nabla\left(\log \frac{U_1}{U_2} - \chi(S(V_1) - S(V_2))\right)\nabla \log \frac{U_1}{U_2}\right] dxds \\
&= -\int_0^t \int_\Omega \frac{U_1 U_2}{U_1 + U_2} \left[\left|\nabla\left(\log \frac{U_1}{U_2}\right)\right|^2 - \chi \nabla(S(V_1) - S(V_2))\nabla \log \frac{U_1}{U_2}\right] dxds \\
&\leq -\frac{1}{2}\int_0^t \int_\Omega \frac{U_1 U_2}{U_1 + U_2} \left[\left|\nabla\left(\log \frac{U_1}{U_2}\right)\right|^2 - \chi^2 |\nabla(S(V_1) - S(V_2))|^2\right] dxds \\
&\leq \frac{\chi^2}{2}\int_0^t \int_\Omega \frac{U_1 U_2}{U_1 + U_2} |\nabla(S(V_1) - S(V_2))|^2 dxds. \qquad (2.14)
\end{aligned}$$

Note that we used $\log \frac{2U_i}{U_1 + U_2}$ as a test function in the above estimate. This comes down to testing with a function like $\log U_i$. Strictly speaking, $\log U_i$ does not belong to the test-space $L^2(0,T;H^1(\Omega))$. Nevertheless, we can justify this test function by using, in a first step, the admissible function $\log(U_i+\varepsilon)$ with a positive $\varepsilon \le 1$ and then passing to the limit $\varepsilon \to 0$ by the Dominated Convergence Theorem.

To estimate the left hand side of inequality (2.14), we need the following result about the function $f(x) = x(\log x - 1)$:

$$f(x) - 2f\left(\frac{x+y}{2}\right) + f(y) \ge \frac{1}{4}(\sqrt{x}-\sqrt{y})^2. \tag{2.15}$$

This is shown by setting $z := \dfrac{x}{y}$ and calculating

$$\begin{aligned}
\left(\sqrt{\frac{x}{y}}-1\right) &= (\sqrt{z}-1)^2 = \left(\frac{z-1}{\sqrt{z}+1}\right)^2 \le \frac{(z-1)^2}{z+1} \le \frac{(z-1)^2}{z} \\
&= \frac{1}{z}\int_1^z \left(\int_1^z dr\right) ds \le 2\int_1^z \left(\int_1^z \frac{dr}{r+s}\right) ds \\
&= 4\{z\log z - (z+1)\log(1+z) + (1+z)\log 2\} \\
&= 4\left\{\frac{x}{y}(\log x - \log y) - \left(\frac{x+y}{y}\right)[\log(x+y) - \log y] + \frac{x+y}{y}\log 2\right\} \\
&= 4\left\{\frac{x}{y}\log x - \left(\frac{x+y}{y}\right)\log\left(\frac{x+y}{2}\right) + \log y\right\},
\end{aligned}$$

so that multiplying this last result by y gives (2.15).

To the right hand side of inequality (2.14) we can apply the arithmetic-geometric mean inequality for U_1U_2:

$$\sqrt{U_1U_2} \le \frac{1}{2}(U_1+U_2) \implies U_1U_2 \le \frac{1}{4}(U_1+U_2)^2 \tag{2.16}$$

and with (2.16) and (2.15) we go on calculating:

$$\begin{aligned}
\frac{1}{4}\|(\sqrt{U_1} &- \sqrt{U_2})(t)\|^2_{L^2(\Omega)} \\
&\le \frac{\chi^2}{8}\|U_1+U_2\|_{L^\infty(0,T;L^\infty(\Omega))}\int_0^t \|\nabla(S(V_1)-S(V_2))\|^2_{L^2(\Omega)}ds \\
&\le C\int_0^t \|S'(V_1)\nabla(V_1-V_2) + \nabla V_2(S'(V_1)-S'(V_2))\|^2_{L^2(\Omega)}ds \\
&\le C(C')^2\int_0^t \|\nabla(V_1-V_2)\|^2_{L^2(\Omega)}ds + C(C'')^2\int_0^t\int_\Omega |\nabla V_2|^2(V_1-V_2)^2dxds.
\end{aligned}$$

Using Hölder's Inequality and the embedding $H^1(\Omega) \hookrightarrow L^{\frac{2p}{p-2}}(\Omega)$, we obtain for the U_i:

$$
\begin{aligned}
\frac{1}{4}\|(\sqrt{U_1} \;&-\; \sqrt{U_2})(t)\|^2_{L^2(\Omega)} \\
&\le C\left(\int_0^t \|\nabla(V_1-V_2)\|^2_{L^2(\Omega)}ds + \int_0^t \|\nabla V_2\|^2_{L^p(\Omega)}\|V_1-V_2\|^2_{L^{\frac{2p}{p-2}}(\Omega)}ds\right) \\
&\le C\left(\int_0^t \|\nabla(V_1-V_2)\|^2_{L^2(\Omega)}ds + \int_0^t \|\nabla V_2\|^2_{L^p(\Omega)}\|V_1-V_2\|^2_{H^1(\Omega)}ds\right) \\
&\le C\int_0^t (1+\|\nabla V_2\|^2_{L^p(\Omega)})(\|V_1-V_2\|^2_{L^2(\Omega)} + \|\nabla(V_1-V_2)\|^2_{L^2(\Omega)})ds. \qquad (2.17)
\end{aligned}
$$

On the other hand, testing the equation for $V_1 - V_2$ with $(V_1-V_2)_t$ (and using the absolute continuity of the mapping $t \mapsto \|\nabla V(t)\|^2_{L^2(\Omega)}$) gives

$$
\begin{aligned}
\int_0^t \|(V_1 \;-\; V_2)_t\|^2_{L^2(\Omega)}ds &+ \frac{\alpha}{2}\|\nabla(V_1-V_2)(t)\|^2_{L^2(\Omega)} + \frac{\beta}{2}\|(V_1-V_2)(t)\|^2_{L^2(\Omega)} \\
&= \delta\int_0^t\int_\Omega (U_1S'(V_1) - U_2S'(V_2))(V_1-V_2)_t dxds \\
&\le \int_0^t \left(\delta^2\|U_1S'(V_1)-U_2S'(V_2)\|^2_{L^2(\Omega)} + \|(V_1-V_2)_t\|^2_{L^2(\Omega)}\right) ds \qquad (2.18)
\end{aligned}
$$

by Young's Inequality. Since

$$
\begin{aligned}
\int_0^t \|U_1S'(V_1) - U_2S'(V_2)\|^2_{L^2(\Omega)}ds &= \int_0^t \|U_1(S'(V_1)-S'(V_2)) + S'(V_2)(U_1-U_2)\|^2_{L^2(\Omega)}ds \\
&\le \int_0^t\int_\Omega U_1^2(C'')^2(V_1-V_2)^2dxds + \int_0^t\int_\Omega (C')^2(U_1-U_2)^2dxds \\
&\le \|U_1\|^2_{L^\infty(0,T;L^\infty(\Omega))}(C'')^2\int_0^t \|V_1-V_2\|^2_{L^2(\Omega)}ds + (C')^2\int_0^t\int_\Omega (U_1-U_2)^2dxds \\
&\le C\int_0^t \|V_1-V_2\|^2_{L^2(\Omega)}ds + C\int_0^t\int_\Omega (\sqrt{U_1}+\sqrt{U_2})^2(\sqrt{U_1}-\sqrt{U_2})^2dxds \\
&\le C\int_0^t \|V_1-V_2\|^2_{L^2(\Omega)}ds \\
&\quad + C\|U_1+U_2\|_{L^\infty(0,T;L^\infty(\Omega))}\int_0^t \|\sqrt{U_1}-\sqrt{U_2}\|^2_{L^2(\Omega)}ds,
\end{aligned}
$$

inequality (2.18) becomes

$$
\begin{aligned}
\|(V_1-V_2)(t)\|^2_{L^2(\Omega)} \;&+\; \|\nabla(V_1-V_2)(t)\|^2_{L^2(\Omega)} \\
&\le C\int_0^t \left(\|\sqrt{U_1}-\sqrt{U_2}\|^2_{L^2(\Omega)} + \|V_1-V_2\|^2_{L^2(\Omega)}\right) ds. \qquad (2.19)
\end{aligned}
$$

Adding now inequality (2.17) to (2.19) we finally obtain

$$
\begin{aligned}
\|(\sqrt{U_1} \;-\; \sqrt{U_2})(t)\|^2_{L^2(\Omega)} &+ \|(V_1-V_2)(t)\|^2_{L^2(\Omega)} + \|\nabla(V_1-V_2)(t)\|^2_{L^2(\Omega)} \\
\le C\int_0^t (2+\|\nabla V_2\|^2_{L^p(\Omega)}) &\Big(\|(\sqrt{U_1}-\sqrt{U_2})(s)\|^2_{L^2(\Omega)} \\
&+\|(V_1-V_2)(s)\|^2_{L^2(\Omega)} + \|\nabla(V_1-V_2)(s)\|^2_{L^2(\Omega)}\Big) ds.
\end{aligned}
$$

As

$$\int_0^t (2 + \|\nabla V_2\|^2_{L^p(\Omega)})ds \;\leq\; 2t + t^{\frac{q-2}{q}} \|\nabla V_2\|^2_{L^q(0,T;L^p(\Omega))} \;\leq\; C(t),$$

we can conclude by Gronwall's Lemma that

$$\|(\sqrt{U_1} - \sqrt{U_2})(t)\|^2_{L^2(\Omega)} + \|(V_1 - V_2)(t)\|^2_{L^2(\Omega)} + \|\nabla(V_1 - V_2)(t)\|^2_{L^2(\Omega)} = 0$$

so that $U_1 = U_2$ and $V_1 = V_2$ almost everywhere, and hence uniqueness follows. □

Chapter 3

A Lyapunov Function for the System

As we will obtain results on $\Omega \subset \mathbb{R}^n$ with $n \in \mathbb{N}$ possibly bigger than 2, we refer to a theorem of existence by Amann [1] for smooth domains. We show in Appendix A that this result can be applied to our system of equations so that studying solutions in higher dimensions has a justification although we consider general Lipschitz domains.

Theorem 3.1 *If $S \in C^1(\mathbb{R}, \mathbb{R})$, then*

$$F(U(t), V(t)) = \int_\Omega \left\{ U(t) \log U(t) - \chi U(t) S(V(t)) + \frac{\chi}{2\delta}(\alpha |\nabla V(t)|^2 + \beta (V(t))^2) \right\} dx \tag{3.1}$$

is a Lyapunov function for the system (1.6), (1.7).

Proof: We formally differentiate F with respect to t:

$$\begin{aligned} \frac{d}{dt} F(U(t), V(t)) &= \int_\Omega U_t(t)[\log U(t) - \chi S(V(t))] dx \; + \; \int_\Omega U_t(t) dx \\ &\quad + \chi \int_\Omega \left[\frac{\alpha}{2\delta} \frac{d}{dt} |\nabla V(t)|^2 + \frac{\beta}{\delta} V(t) V_t(t) - U(t) S'(V(t)) V_t(t) \right] dx \end{aligned}$$

Testing the first equation in (1.6) with $[\log U - \chi S(V)]$ and 1, respectively, gives

$$\begin{aligned} \int_\Omega U_t [\log U - \chi S(V)] dx &= -\int_\Omega (\nabla U - U \chi S(V)) \nabla (\log U - \chi S(V)) dx \\ &= -\int_\Omega U(t) |\nabla [\log U(t) - \chi S(V(t))]|^2 dx \end{aligned}$$

and

$$\int_\Omega U_t(t) dx = 0.$$

(Remember that we justified testing with $\log U$ in the proof of uniqueness in Chapter 2.)

By the absolute continuity of the mapping $t \mapsto \|\nabla V(t)\|^2_{L^2(\Omega)}$, using the $V-$equation, we can write

$$\int_\Omega \frac{\alpha}{2\delta}\frac{d}{dt}|\nabla V(t)|^2 dx = \int_\Omega \left\{ -\frac{\beta}{\delta} V(t)V_t(t) + U(t)S'(V(t))V_t(t) - \frac{1}{\delta}V_t^2 \right\} dx,$$

so that we obtain

$$\frac{d}{dt}F(U(t),V(t)) = -\int_\Omega U(t)|\nabla[\log U(t) - \chi S(V(t))]|^2 dx \;-\; \frac{\chi}{\delta}\int_\Omega (V_t(t))^2 dx \;\le\; 0,$$

i.e., F decreases along solution trajectories. □

Remark: The function F is very similar to the Lyapunov function for the semiconductor equations found by Gajewski and Gröger [12]

$$\Psi(u_1,u_2,\varphi) = \int_\Omega \frac{1}{2}(\varepsilon|\nabla\varphi|^2 + \kappa\varphi^2)dx \;+\; \sum_{i=1}^{2}\int_\Omega u_i^*[u_i(\log u_i - 1) + 1]dx.$$

But whereas Ψ is always bounded from below ($x(\log x - 1) \ge C$), our Lyapunov function F could tend to $-\infty$.

The following proposition shows that a solution to system (1.6), (1.7) ceases to exist in finite time if the Lyapunov function F becomes $-\infty$ for a finite t_0. This is because we can find a $p > 1$ (but close to 1) such that the L^p-norm of the function U will explode at this point of the time scale. The proof requires *sublinearity* of the sensitivity function, that is, there must be positive constants c_1 and c_2 such that

$$S(V) \;\le\; c_1 V + c_2 \tag{3.2}$$

for all $V \ge 0$. Note that all sensitivity functions $S \in \mathcal{S}$ are sublinear.

Proposition 3.2 *Let us consider a Lipschitz domain $\Omega \subset \mathbb{R}^n$ and let S satisfy condition (3.2). If there exists a solution (U,V) of system (1.6), (1.7) such that the Lyapunov function $F(U(t),V(t)) \longrightarrow -\infty$ for $t \to t_0$, for a $t_0 \in (0,\infty]$, then*

$$\|U(t)\sqrt{\log U(t)}\|_{L^1(\Omega)} \longrightarrow +\infty \quad \text{as } t \to t_0 \quad \text{if } n = 2$$

so that

$$\|U(t)\|_{L^p(\Omega)} \longrightarrow +\infty \quad \text{for every } p > 1$$

and

$$\|U(t)\|_{L^p(\Omega)} \longrightarrow +\infty \quad \text{as } t \to t_0 \quad \text{for every } \; p > \frac{2n}{n+2} \quad \text{if } n > 2.$$

Proof: On one hand, $x \log x$ attains its minimum at $x = \frac{1}{e}$, so that

$$\int_\Omega U(t) \log U(t) dx \geq -\frac{|\Omega|}{e},$$

and

$$\begin{aligned} F(U(t), V(t)) &= \int_\Omega \left\{ U(t) \log U(t) - \chi U(t) S(V(t)) + \frac{\chi}{2\delta}(\alpha |\nabla V(t)|^2 + \beta (V(t))^2) \right\} dx \\ &\geq -\frac{|\Omega|}{e} - \chi \int_\Omega U(t) S(V(t)) dx - C \|V(t)\|^2_{H^1(\Omega)}, \end{aligned} \tag{3.3}$$

On the other hand, if $n = 2$, we can use the Trudinger Inequality (See for instance Kufner et al. [23], Remark 5.7.9 (v).), by which

$$\|V\|_{L^{\tilde{\Phi}}(\Omega)} \leq C \|V\|_{H^1(\Omega)},$$

where $\tilde{\Phi}$ is the Young function $\tilde{\Phi}(t) = e^{t^2} - 1$. Applying the extended Hölder Inequality for Young functions (B.4), sublinearity of S and Young's Inequality to the second term on the right hand side of (3.3) gives

$$\begin{aligned} F(U(t), V(t)) &\geq -\frac{|\Omega|}{e} - \chi \|U(t)\|_{L^{\tilde{\Psi}}(\Omega)} \|S(V(t))\|_{L^{\tilde{\Phi}}(\Omega)} + \bar{C} \|V(t)\|^2_{L^{\tilde{\Phi}}(\Omega)} \\ &\overset{(3.2)}{\geq} -\frac{|\Omega|}{e} - C \|U(t)\|_{L^{\tilde{\Psi}}(\Omega)} \left(\|V(t))\|_{L^{\tilde{\Phi}}(\Omega)} + 1 \right) + \bar{C} \|V(t)\|^2_{L^{\tilde{\Phi}}(\Omega)} \\ &\geq -\frac{|\Omega|}{e} - C \|U(t)\|^2_{L^{\tilde{\Psi}}(\Omega)} - C. \end{aligned}$$

$\tilde{\Psi}(t)$ is the Young function that is complementary to $\tilde{\Phi}(t)$. One can show that it is asymptotically equal to $t\sqrt{\log t}$, so that we obtain for $n = 2$

$$\|U(t) \sqrt{\log U(t)}\|^2_{L^1(\Omega)} \geq -C \left[1 + F(U(t), V(t))\right] \longrightarrow +\infty \quad \text{as } t \to t_0.$$

Since

$$\|U \sqrt{\log U}\|_{L^1(\Omega)} \leq C \|U \log U\|_{L^1(\Omega)} \leq C \|U\|_{L^p(\Omega)}$$

by the continuous embedding $L^p(\Omega) \hookrightarrow L^\Phi(\Omega)$ for every $p > 1$ with the Young function $\Phi(t) = (1 + t) \log(1 + t) - t$, we have proven blow-up of every L^p-norm of U with $p > 1$.[†]

If $n > 2$, we use the Sobolev Embedding Theorem $H^1(\Omega) \hookrightarrow L^{p'}(\Omega)$ for all $p' \leq \frac{2n}{n-2}$ (It then follows that the conjugate exponent $p \geq \frac{2n}{n+2}$.) and the usual Hölder Inequality

[†]See Section B.1 in Appendix B for more information on Orlicz spaces.

to obtain from (3.3)

$$
\begin{aligned}
F(U(t),V(t)) &\geq -\frac{|\Omega|}{e} - \chi\|U(t)\|_{L^p(\Omega)}\|S(V(t))\|_{L^{p'}(\Omega)} + \bar{C}\|V(t)\|^2_{L^{p'}(\Omega)} \\
&\overset{(3.2)}{\geq} -\frac{|\Omega|}{e} - C\|U(t)\|_{L^p(\Omega)}\left(\|V(t))\|_{L^{p'}(\Omega)} + 1\right) + \bar{C}\|V(t)\|^2_{L^{p'}(\Omega)} \\
&\geq -\frac{|\Omega|}{e} - C\|U(t)\|^2_{L^p(\Omega)} - C.
\end{aligned}
$$

Hence, for $n > 2$:

$$
\|U(t)\|^2_{L^p(\Omega)} \geq -C\,[1 + F(U(t),V(t))] \longrightarrow +\infty
$$

as $t \to t_0$ for all $p \geq \dfrac{2n}{n+2}$ as claimed. □

As we will want to study at a later stage the long time behaviour of a solution of system (1.6), (1.7) we are interested in finding conditions ensuring the boundedness of F. This is the principal aim of the next section.

3.1 A-priori-Estimates

In the following, let S belong to the class $\mathcal{S}$ introduced in Section 1.2.2. (Note that all results obtained up to now in this chapter are valid for the functions in $\mathcal{S}$.)

Lemma 3.3 *We have for the solutions of (1.6), (1.7):*

$$\|U(t)\|_{L^1(\Omega)} = \|U_0\|_{L^1(\Omega)}$$

and there exists a constant $C > 0$, depending only on $\|U_0\|_{L^1(\Omega)}$ and $\|V_0\|_{L^1(\Omega)}$ such that

$$\|V(t)\|_{L^1(\Omega)} \le C.$$

Proof: Since U and V are positive, we obtain the assertions by testing both equations with $H \equiv 1$, respectively:

$$\frac{d}{dt}\int_\Omega U(t)dx = 0 \implies \|U(t)\|_{L^1(\Omega)} = \|U_0\|_{L^1(\Omega)} \tag{3.4}$$

and from

$$\frac{d}{dt}\int_\Omega V(t)dx = -\beta\int_\Omega V dx \ + \ \delta\int_\Omega U S'(V) dx$$

we get

$$\begin{aligned}
\|V(t)\|_{L^1(\Omega)} &= \|V_0\|_{L^1(\Omega)} e^{-\beta t} + e^{-\beta t}\int_0^t e^{\beta s}\delta\int_\Omega U(s)S'(V(s))dxds \\
&\overset{(3.4)}{\le} \|V_0\|_{L^1(\Omega)} + C'\|U_0\|_{L^1(\Omega)}\frac{\delta}{\beta}(1-e^{-\beta t}) \\
&\le \|V_0\|_{L^1(\Omega)} + \frac{C'\delta}{\beta}\|U_0\|_{L^1(\Omega)} \ \le \ C
\end{aligned}$$

by the formula of variation of the constant. □

In the next lemma we give a technical condition with which we can ensure boundedness of the Lyapunov function F and all its terms.

Lemma 3.4 *If there exists a positive constant c such that the inequality*

$$\int_\Omega e^{aS(V)}dx \ \le \ \exp\left(\frac{\chi\alpha}{2\delta\|U_0\|_{L^1(\Omega)}} \ \|\nabla V\|^2_{L^2(\Omega)} + \frac{\chi\beta}{2\delta\|U_0\|_{L^1(\Omega)}}\|V\|^2_{L^2(\Omega)} + c\right) \tag{3.5}$$

holds with $a = \chi$, then the Lyapunov function F is bounded from below and there exists a $C > 0$, independent of t, such that

$$|F(U(t), V(t))| \le C \ \ \textit{for all} \ \ t \ge 0. \tag{3.6}$$

If we can choose $a > \chi$ in (3.5), then we have additionally

$$(\ 0 \le \ ) \int_\Omega U(t)S(V(t))dx \ \le \ C \text{ for all } t \ge 0 \tag{3.7}$$

and the boundedness of all terms in F follows.

Proof: Suppose inequality (3.5) holds for an a, which will be determined later on. For fixed $t \in (0,T)$, we set $\psi(x) := \frac{M}{\mu} e^{aS(V(x,t))}$, where $M := \|U(t)\|_{L^1(\Omega)} = \|U_0\|_{L^1(\Omega)}$ and $\mu := \int_\Omega e^{aS(V(x,t))}dx$. As $-\log(x)$ is a convex function, we know from Jensen's inequality that

$$0 = -\log \int_\Omega \frac{\psi}{U}\frac{U}{M}dx \le \int_\Omega \left(-\log\frac{\psi}{U}\right)\frac{U}{M}dx = \frac{1}{M}\int_\Omega U \log\frac{U}{\psi}dx,$$

so that

$$\begin{aligned} 0 &\le \int_\Omega U(\log U - \log\psi)dx \\ &= \int_\Omega U(\log U - \log M + \log\mu - aS(V))dx \\ &= \int_\Omega U\log U dx + M(\log\mu - \log M) - a\int_\Omega US(V)dx \\ &\overset{(3.5)}{\le} \int_\Omega U\log U dx - M\log M - a\int_\Omega US(V)dx \\ &\quad + M\log c + \frac{\chi\alpha}{2\delta}\|\nabla V\|^2_{L^2(\Omega)} + \frac{\chi\beta}{2\delta}\|V\|^2_{L^2(\Omega)} + Mc. \end{aligned}$$

It follows that

$$\begin{aligned} (a-\chi)\int_\Omega U(t)S(V(t))dx &\le F(U(t),V(t)) + M(\log\frac{c}{M} + c) \\ &\le F(U_0,V_0) + M(\log\frac{c}{M} + c) \le \tilde{C} \end{aligned}$$

with a positive constant $\tilde{C}$ and we obtain

$$-M(\log\frac{c}{M} + c) \le F(U(t),V(t)) \le \tilde{C}$$

by choosing $a = \chi$. Furthermore, it is clear, that if $a > \chi$, estimate (3.7) also holds:

$$0 \ \le \ \int_\Omega U(t)S(V(t))dx \ \le \ \frac{\tilde{C}}{a-\chi}$$

and it follows that

$$\int_\Omega \left\{U(t)\log U(t) + \frac{\chi}{2\delta}(\alpha|\nabla V(t)|^2 + \beta(V(t))^2)\right\}dx \ \le \ C + \chi\int_\Omega U(t)S(V(t))dx \ \le \ C,$$

so that all terms in F are bounded. (Remember that $U\log U \ge -\frac{1}{e}$.) □

Corollary 3.5 Since

$$\frac{\chi}{\delta}\int_\Omega V_t^2 dx \;+\; \int_\Omega U|\nabla(\log U - \chi S(V))|^2 dx \;=\; -\frac{d}{dt}F(U,V),$$

it follows from Lemma 3.4, that

$$\frac{\chi}{\delta}\int_0^t \|V_t(s)\|^2_{L^2(\Omega)} ds \;+\; \int_0^t\int_\Omega U|\nabla(\log U - \chi S(V))|^2 dxds \;\le\; F(U_0,V_0) - F(U,V) \;\le\; C$$

whenever the Lyapunov function is bounded from below, where the constant C does not depend on t.

If additionally (3.7) holds, i.e., if all terms in the Lyapunov function are bounded, then $\|V(t)\|_{H^1(\Omega)}$ and $\|U(t)\log U(t)\|_{L^1(\Omega)}$ are bounded independently of t.

Remark: In general, we cannot exclude the possibility that particular terms in F become unbounded even if the whole Lyapunov function is bounded. If we can bound the term $US(V)$, however, boundedness of all terms in F follows.

3.2 Specific Sensitivity Functions

All three specific sensitivity functions we consider belong to the class $\mathcal{S}$ as was shown in Section 1.2.2, so that (3.1) is a Lyapunov function for each S. In the following, we will show under different conditions that inequality (3.5) is valid in the three cases, that is, that F stays bounded for all times.

Proposition 3.6 *If S is bounded, then the Lyapunov function $F(U(t), V(t))$ and all its terms are bounded independently of $t > 0$.*

Proof: Here, inequality (3.5) holds trivially for every $a > \chi$:

$$\int_\Omega e^{aS(V)}dx \le e^{aC}|\Omega| \le C(|\Omega|) \le \exp\left(\frac{\chi\alpha}{2\delta M}\,\|\nabla V\|^2_{L^2(\Omega)} + \frac{\chi\beta}{2\delta M}\|V\|^2_{L^2(\Omega)} + c\right)$$

and the assertion follows from Lemma 3.4. □

Proposition 3.7 *For $S(V) = \log(V + c)$, all terms in the Lyapunov function F are bounded for all $\chi < \infty$ if $n = 2$. If $n > 2$, then (3.6) holds for $\chi \le \frac{2n}{n-2}$. If $\chi < \frac{2n}{n-2}$, then (3.7) is also true.*

Proof: For $1 \le a \le \dfrac{2n}{n-2}$ (If $n = 2, a < \infty$.), we have with an arbitrary $\varepsilon > 0$

$$\begin{aligned}
\int_\Omega e^{aS(V)}dx &= \int_\Omega (V+c)^a dx \;=\; C\left(\|V\|^a_{L^a(\Omega)} + 1\right) \;\le\; C\left(\|V\|^{a\theta}_{H^1(\Omega)}\|V\|^{a(1-\theta)}_{L^1(\Omega)} + 1\right) \\
&\le C\left[\left(\|\nabla V\|^{a\theta}_{L^2(\Omega)} + \|V\|^{a\theta}_{L^1(\Omega)}\right)\|V\|^{a(1-\theta)}_{L^1(\Omega)} + 1\right] \\
&= C\left(\|\nabla V\|^{a\theta}_{L^2(\Omega)}\|V\|^{a(1-\theta)}_{L^1(\Omega)} + \|V\|^a_{L^1(\Omega)} + 1\right) \\
&\le C(\|V_0\|_{L^1(\Omega)})\left(\|\nabla V\|^{a\theta}_{L^2(\Omega)} + 1\right) \\
&\le \varepsilon\|\nabla V\|^{2a\theta}_{L^2(\Omega)} + C,
\end{aligned}$$

by the Gagliardo-Nirenberg Inequality with $\theta = \dfrac{(1-\frac{1}{a})n}{1+\frac{n}{2}}$† and Young's Inequality. For the last exponent we calculate

$$2a\theta \;=\; \frac{(a-1)2n}{1+\frac{n}{2}} \;<\; 4(a-1).$$

Since we can choose $\varepsilon > 0$ so small that

$$\varepsilon x^{4(a-1)} \le \exp\left(\frac{\chi\alpha}{2\delta\|U_0\|_{L^1(\Omega)}}x^2\right)$$

†Note that we need here the condition for a.

for all $x \in [0, +\infty)$, we get

$$\begin{aligned}\int_\Omega e^{aS(V)} dx &\le \exp\left(\frac{\chi\alpha}{2\delta\|U_0\|_{L^1(\Omega)}}\|\nabla V\|^2_{L^2(\Omega)}\right) + C \\ &\le \exp\left(\frac{\chi\alpha}{2\delta\|U_0\|_{L^1(\Omega)}}\|\nabla V\|^2_{L^2(\Omega)} + C\right),\end{aligned}$$

i.e., (3.5) holds and the assertions follow. □

In order to deal with $S(V) = V$, we need a Trudinger-Moser-type inequality, which only holds in two dimensions and for a more regular Ω. Hence, we consider here the case of a bounded, finitely connected domain $\Omega \subset \mathbb{R}^2$ with boundary $\partial\Omega$. We assume that $\partial\Omega$ is piecewise C^2 with a finite number of vertices with non-vanishing interior angles. Let θ be the minimum interior angle at the vertices of Ω.

We quote the following lemma from Gajewski and Zacharias [14], where a result by Chang and Yang [4] was generalized.

Lemma 3.8 *There exists a constant $c(\Omega)$ such that we have*

$$\int_\Omega e^{2\theta v^2} dx \le c(\Omega)$$

for all $v \in H^1(\Omega)$ with spatial mean $\bar{v} = \frac{1}{|\Omega|}\int_\Omega v\ dx = 0$ and $\int_\Omega |\nabla v|^2 dx \le 1$.

Proposition 3.9 *Let $\Omega \subset \mathbb{R}^2$. Its boundary $\partial\Omega$ be piecewise C^2 with a finite number of vertices with non-vanishing interior angles. Let θ be the minimum interior angle at the vertices of Ω.*

In the case of the identity sensitivity function $S(V) = V$, the estimates (3.6) and (3.7) hold if $M = \|U_0\|_{L^1(\Omega)} < \frac{4\theta\alpha}{\delta\chi}$. (If equality holds, estimate (3.6) remains true.)

Proof: Since

$$\begin{aligned}aV &\le a\,|V - \bar{V}| + a\bar{V} \\ &\le 2\theta\left(\int_\Omega |\nabla V|^2 dx\right)^{-1}|V - \bar{V}|^2 + \frac{a^2}{8\theta}\left(\int_\Omega |\nabla V|^2 dx\right) + \frac{a}{|\Omega|}\|V\|_{L^1(\Omega)},\end{aligned}$$

we obtain by Lemma 3.8 the following Trudinger-Moser-Inequality

$$\int_\Omega e^{aV} dx \le c(\Omega)\exp\left(\frac{a^2}{8\theta}\|\nabla V\|^2_{L^2(\Omega)} + C(a, \Omega, \|V\|_{L^1(\Omega)})\right)$$

for all positive $V \in H^1(\Omega)$.

Now, it is possible to choose an $a > \chi$ with $\frac{a^2}{8\theta} \leq \frac{\chi\alpha}{2\delta M}$ if and only if $\frac{\chi^2}{8\theta} < \frac{\chi\alpha}{2\delta M}$, which is equivalent to $M \leq \frac{4\theta\alpha}{\delta\chi}$. It is obvious that a can be chosen to be strictly greater than χ if and only if $M < \frac{4\theta\alpha}{\delta\chi}$. Thus, the assertion of the proposition holds. □

Chapter 4

Global Existence of Solutions

In this chapter, we will show that the solution (U, V) of system (1.6), (1.7) in a two-dimensional domain $\Omega \subset I\!R^2$ is global if all terms appearing in the Lyapunov function are bounded.

Unfortunately, we are not able to show this result for a general $S \in \mathcal{S}$. Nor can the arguments be carried over to the three-dimensional case due to dimension-dependent estimates (essentially Galiardo-Nirenberg's Inequality).

As we have to argue for all three classes of specific sensitivity functions in a different way, the result will be shown in the following sections for each class separately.

Theorem 4.1 *We consider the system*

$$\begin{aligned} U_t &= \Delta U - \chi \nabla (U \nabla S(V)) \\ V_t &= \alpha \Delta V - \beta V + \delta U S'(V) \end{aligned} \tag{4.1}$$

in $(0, T) \times \Omega \subset I\!R \times I\!R^2$, *with homogeneous Neumann boundary conditions*

$$\nu \cdot \nabla U = \nu \cdot \nabla V = 0 \tag{4.2}$$

on $(0, T) \times \partial\Omega$ *and initial conditions* $U_0 \in L^\infty(\Omega)$, $V_0 \in W^{1,p}(\Omega)$ *for some* $p > 2$, *where the sensitivity function* S *is either bounded or* $S(V) = V$ *or* $S(V) = \log(V + c)$, $c \geq 1$. *If the Lyapunov function* $F(U, V)$ *of this system and all its terms are bounded, then the solution* (U, V) *of (4.1) is global in time.*

In case the sensitivity function S *is bounded or if* $S(V) = V$, *we obtain for every* $1 \leq p < \infty$ *a constant* C *such that*

$$\|U\|_{L^\infty(0,\infty;L^p(\Omega))} + \|V\|_{L^\infty(0,\infty;L^\infty(\Omega))} \leq C.$$

If $S(V) = \log(V + c)$*, then there exist constants* $C > 0$ *and* $r > 2$*, independent of* T*, such that*

$$\|U\|_{L^\infty(0,T;L^2(\Omega))} + \|V\|_{L^\infty(0,T;L^\infty(\Omega))} \leq Ce^{CT^r} \tag{4.3}$$

and

$$\|U\|_{L^\infty(0,T;L^p(\Omega))} \leq C\exp(pCe^{CT^r}) \tag{4.4}$$

for all $T > 0$ *and every* $2 < p < \infty$*.*

In case $\chi \leq 1$*, the exponential time-dependence for* V *in (4.3) can be improved to a polynomial dependence.*

Remark: As we have seen in Section 3.2, in two space dimensions, we need an additional condition to ensure boundedness of all terms in the Lyapunov function only in the case of the identical sensitivity function. But since globality of solutions is impossible if the Lyapunov function is unbounded on a finite interval (See Proposition 3.2.) and the a-priori-estimates obtained from the boundedness of all terms in F are essential to all three proofs, we have formulated Theorem 4.1 with this general condition.

4.1 Bounded Sensitivity Functions

Proposition 4.2 *Let Ω be a Lipschitz domain in $I\!R^2$. If $S(V)$ is bounded, then there exists a positive constant K, independent of $T > 0$, such that for every $1 \le p < \infty$ we have for the solution U of the first equation in (4.1) with homogeneous Neumann boundary conditions*

$$\|U\|_{L^\infty(0,T;L^p(\Omega))} \le K^p.$$

Proof: Let $e^{\chi S(V)} \le C_e$. (Note that $C_e \ge 1$.)

As we want to demonstrate the exact dependence on p of the bound for U in the space $L^\infty(0,T;L^p(\Omega))$, we are going to distinguish scrupulously between the different constants in the estimates of this proof.

W.l.o.g., we take $\chi = 1$ throughout the proof. (We can define $\tilde{S}(V) := \chi S(V)$ and $\tilde{\delta} := \frac{\delta}{\chi}$.)

We will show by induction for all $p = 2^k$ with $k \in I\!N \cup \{0\}$ that

$$\|U(t)\|_{L^p(\Omega)} \le K^p \text{ for all } t \in [0,T]. \tag{4.5}$$

For $k = 0$, i.e., $p = 1$, this is true if

$$\|U(t)\|_{L^1(\Omega)} = \|U_0\|_{L^1(\Omega)} \le K.$$

Now let $p = 2^k \ge 2$ and suppose that (4.5) holds for $\hat{p} = 2^{k-1} = \frac{p}{2}$.

Testing the first equation of (4.1), (4.2) with $p\left(\frac{U}{e^{S(V)}}\right)^{p-1}$ gives

$$p\int_\Omega U_t \left(\frac{U}{e^{S(V)}}\right)^{p-1} dx + p\int_\Omega e^{S(V)} \nabla\left(\frac{U}{e^{S(V)}}\right) \nabla\left(\frac{U}{e^{S(V)}}\right)^{p-1} dx = 0.^\dagger \tag{4.6}$$

We calculate for the two terms separately:

$$p\int_\Omega U_t \left(\frac{U}{e^{S(V)}}\right)^{p-1} dx = \frac{d}{dt}\int_\Omega \left(\frac{U}{e^{S(V)}}\right)^p e^{S(V)} dx + (p-1)\int_\Omega \left(\frac{U}{e^{S(V)}}\right)^p e^{S(V)} S'(V) V_t dx$$

and

$$\begin{aligned} p\int_\Omega e^{S(V)} \nabla\left(\frac{U}{e^{S(V)}}\right) \nabla\left(\frac{U}{e^{S(V)}}\right)^{p-1} dx &= p\int_\Omega e^{S(V)} \left|\nabla\left(\frac{U}{e^{S(V)}}\right)\right|^2 (p-1)\left(\frac{U}{e^{S(V)}}\right)^{p-2} dx \\ &= \frac{4(p-1)}{p}\int_\Omega e^{S(V)} |\nabla W|^2 dx \end{aligned}$$

---

†Note that $e^{S(V)}\nabla\left(\frac{U}{e^{S(V)}}\right) = e^{S(V)}\left(\frac{\nabla U}{e^{S(V)}} - \frac{U\nabla S(V)}{e^{S(V)}}\right) = \nabla U - U\nabla S(V)$.

with $W := \left(\frac{U}{e^{S(V)}}\right)^{\frac{p}{2}}$ † .

Inserting this into (4.6) gives

$$\begin{aligned}
\frac{d}{dt}\int_\Omega W^2 e^{S(V)}dx \;+\; \frac{4(p-1)}{p}\int_\Omega e^{S(V)}|\nabla W|^2 dx \;&=\; -(p-1)\int_\Omega W^2 S'(V)e^{S(V)}V_t dx \\
&\le\; (p-1)C'C_e\int_\Omega |W^2V_t|dx \;\le\; pC'C_e\|W\|^2_{L^4(\Omega)}\|V_t\|_{L^2(\Omega)} \\
&\le\; pC_1C_e\|W\|_{H^1(\Omega)}\|W\|_{L^2(\Omega)}\|V_t\|_{L^2(\Omega)} \\
&\le\; pC_1C_e(\|W\|_{L^2(\Omega)}+\|\nabla W\|_{L^2(\Omega)})\|W\|_{L^2(\Omega)}\|V_t\|_{L^2(\Omega)} \\
&\le\; pC_1C_e\|W\|^2_{L^2(\Omega)}\|V_t\|_{L^2(\Omega)} + \frac{3p-4}{p}\|\nabla W\|^2_{L^2(\Omega)} \\
&\quad +\frac{p^2p}{3p-4}(C_1C_e)^2\|W\|^2_{L^2(\Omega)}\|V_t\|^2_{L^2(\Omega)} \\
&\le\; pC_1C_e\|W\|^2_{L^2(\Omega)}\|V_t\|_{L^2(\Omega)} + \frac{3p-4}{p}\|\nabla W\|^2_{L^2(\Omega)} \\
&\quad +(C_1C_ep)^2\|W\|^2_{L^2(\Omega)}\|V_t\|^2_{L^2(\Omega)},
\end{aligned}$$

applying Hölder's, Gagliardo-Nirenberg's and Young's Inequality. Since $1 \le e^{S(V)}$, we obtain

$$\frac{d}{dt}\int_\Omega W^2e^{S(V)}dx + \|\nabla W\|^2_{L^2(\Omega)} \le \|W\|^2_{L^2(\Omega)}(C_1C_ep\|V_t\|_{L^2(\Omega)} + (C_1C_ep)^2\|V_t\|^2_{L^2(\Omega)}). \quad (4.7)$$

By Gagliardo-Nirenberg's, Poincaré's and Young's Inequality, we have, using the inductive assumption in the last step:

$$\begin{aligned}
\|W\|^2_{L^2(\Omega)} \;&\le\; C_2\left(\|\nabla W\|^2_{L^2(\Omega)}+\|W\|^2_{L^1(\Omega)}\right)^{\frac{1}{2}}\|W\|_{L^1(\Omega)} \\
&\le\; C_2\left(\|\nabla W\|_{L^2(\Omega)}\|W\|_{L^1(\Omega)}+\|W\|^2_{L^1(\Omega)}\right) \\
&\le\; \|\nabla W\|^2_{L^2(\Omega)} + C_3\|W\|^2_{L^1(\Omega)} \\
&\le\; \|\nabla W\|^2_{L^2(\Omega)} + C_3\|U\|^p_{L^{\frac{p}{2}}(\Omega)} \\
&\le\; \|\nabla W\|^2_{L^2(\Omega)} + C_3K^{\frac{p^2}{2}} \qquad (4.8)
\end{aligned}$$

On the other hand, since $\left(1-cp\|V_t\|_{L^2(\Omega)}\right)^2 \ge 0$,

$$C_1C_ep\|V_t\|_{L^2(\Omega)} \le \frac{1}{2} + \frac{(C_1C_ep)^2}{2}\|V_t\|^2_{L^2(\Omega)}. \quad (4.9)$$

† We have $\nabla W = \frac{p}{2}\left(\frac{U}{e^{S(V)}}\right)^{\frac{p-2}{2}}\nabla\left(\frac{U}{e^{S(V)}}\right) \Longrightarrow |\nabla W|^2 = \frac{p^2}{4}\left(\frac{U}{e^{S(V)}}\right)^{p-2}\left|\nabla\left(\frac{U}{e^{S(V)}}\right)\right|^2.$

Estimating the left hand side of (4.7) by (4.8) and using (4.9) on the right hand side, one obtains

$$\begin{aligned}\frac{d}{dt}\int_\Omega W^2 e^{S(V)}dx &\le \|W\|^2_{L^2(\Omega)}\left(\frac{3}{2}(C_1C_ep)^2\|V_t\|^2_{L^2(\Omega)}-\frac{1}{2}\right)+C_3K^{\frac{p^2}{2}}\\ &= \frac{3}{2}(C_1C_ep)^2\|V_t\|^2_{L^2(\Omega)}\int_\Omega W^2dx-\frac{1}{2}\int_\Omega W^2dx+C_3K^{\frac{p^2}{2}}\\ &\le \left(\frac{3}{2}(C_1C_ep)^2\|V_t\|^2_{L^2(\Omega)}-\frac{1}{2C_e}\right)\int_\Omega W^2e^{S(V)}dx+C_3K^{\frac{p^2}{2}}\end{aligned}$$

since $1\le e^{S(V)}\le C_e$. Putting $\varphi(s):=\left(\frac{3}{2}(C_1C_ep)^2\|V_t\|^2_{L^2(\Omega)}-\frac{1}{2C_e}\right)$, we can apply a variant of Gronwall's Lemma: With $g(t):=\int_\Omega W^2(t,x)e^{S(V(t,x))}dx$, the last inequality can be written as

$$g'(t)\ \le\ \varphi(t)g(t)+C_3K^{\frac{p^2}{2}}. \tag{4.10}$$

Defining now $h(t):=g(t)\ e^{-\int_0^t\varphi(s)ds}$, we calculate

$$h'(t)\ =\ g'(t)\ e^{-\int_0^t\varphi(s)ds}\ -\ g(t)\ \varphi(t)\ e^{-\int_0^t\varphi(s)ds}\ \overset{(4.10)}{\le}\ C_3K^{\frac{p^2}{2}}\ e^{-\int_0^t\varphi(s)ds}.$$

Integrating this last inequality, we obtain

$$h(t)\ \le\ h(0)+C_3K^{\frac{p^2}{2}}\int_0^t e^{-\int_0^s\varphi(\tau)d\tau}ds\ =\ g(0)+C_3K^{\frac{p^2}{2}}\int_0^t e^{-\int_0^s\varphi(\tau)d\tau}ds,$$

so that

$$g(t)\ =\ e^{\int_0^t\varphi(s)ds}h(t)\ \le\ e^{\int_0^t\varphi(s)ds}g(0)\ +\ C_3K^{\frac{p^2}{2}}e^{\int_0^t\varphi(s)ds}\int_0^t e^{-\int_0^s\varphi(\tau)d\tau}ds,$$

i.e.,

$$\int_\Omega W^2(t)e^{S(V(t))}dx\ \le\ e^{\int_0^t\varphi(s)ds}\int_\Omega W_0^2e^{S(V_0)}dx\ +\ e^{\int_0^t\varphi(s)ds}C_3K^{\frac{p^2}{2}}\int_0^t e^{-\int_0^\tau\varphi(s)ds}d\tau.$$

From the boundedness of the Lyapunov function F we know (see Conclusion 3.5) that V_t is bounded in $L^2(0,T;L^2(\Omega))$ independently of $T>0$, so that

$$\int_0^t\varphi(s)ds\le C_4(pC_e)^2-\frac{t}{2C_e}\le C_4(pC_e)^2.$$

On the other hand,

$$\int_0^t e^{-\int_0^\tau\varphi(s)ds}d\tau\le\int_0^t e^{\frac{\tau}{2C_e}}d\tau=2C_e\left(e^{\frac{t}{2C_e}}-1\right).$$

Since

$$\int_\Omega W_0^2e^{S(V_0)}dx\ =\ \int_\Omega\frac{U_0^p}{\left(e^{S(V_0)}\right)^{(p-1)}}dx\ \le\ \int_\Omega U_0^pdx\ \le\ |\Omega|\ \|U_0\|^p_{L^\infty(\Omega)},$$

we finally obtain

$$\begin{aligned}\int_\Omega W^2(t)e^{S(V(t))}dx &\leq e^{C_4(pC_e)^2}|\Omega|\;\|U_0\|^p_{L^\infty(\Omega)} + e^{C_4(pC_e)^2}C_3K^{\frac{p^2}{2}}2C_e\left(1-e^{-\frac{t}{2C_e}}\right)\\ &\leq C_5e^{C_4(pC_e)^2} + C_6e^{C_4(pC_e)^2}C_eK^{\frac{p^2}{2}}.\end{aligned}$$

It follows that if $K \geq \|U_0\|_{L^1(\Omega)}$ is chosen sufficiently big, that is, if

$$C_5e^{C_4C_e^2}C_e \leq \frac{1}{2}K \Longrightarrow C_5e^{C_4(pC_e)^2}C_e^{(p-1)} \leq C_5^{p^2}e^{C_4(pC_e)^2}C_e^{p^2} \leq \frac{1}{2^{p^2}}K^{p^2} \leq \frac{1}{2}K^{p^2},$$

so that

$$C_5e^{C_4(pC_e)^2} \leq \frac{1}{2}C_e^{-(p-1)}K^{p^2}$$

and if

$$C_6e^{C_4C_e^2}C_e \leq \frac{1}{2}K^{\frac{1}{2}} \Longrightarrow C_6e^{C_4(pC_e)^2}C_e^p \leq C_6^{p^2}e^{C_4(pC_e)^2}C_e^{p^2} \leq \frac{1}{2^{p^2}}K^{\frac{p^2}{2}} \leq \frac{1}{2}K^{\frac{p^2}{2}},$$

so that

$$C_6e^{C_4(pC_e)^2}C_eK^{\frac{p^2}{2}} \leq \frac{1}{2}C_e^{-(p-1)}K^{p^2},$$

we obtain

$$\begin{aligned}\|U(t)\|^p_{L^p(\Omega)} &\leq C_e^{p-1}\int_\Omega W^2e^{S(V)}dx \leq C_e^{p-1}\left(C_5e^{C_4(pC_e)^2} + C_6e^{C_4(pC_e)^2}C_eK^{\frac{p^2}{2}}\right)\\ &\leq \frac{1}{2}K^{p^2} + \frac{1}{2}K^{p^2} = K^{p^2}.\end{aligned}$$

Extracting the p-th root gives (4.5). □

Conclusion 4.3 Since $S'(V) \leq C'$, the right hand side $\delta US'(V)$ of the equation for V is in $L^\infty(0,T;L^p(\Omega))$ for a $p > 2$ and it follows by standard arguments that V is bounded in $L^\infty(0,T;L^\infty(\Omega))$ independently of $T > 0$. (See the remark at the end of part *(vi)* of the proof of Theorem 2.1.)

From the a-priori-estimates obtained in Section 3.1 (Corollary 3.5), we know that the estimate for V in $L^\infty(0,T;H^1(\Omega))$ neither depends on $T > 0$, so that we have proven the same regularity for the solutions $U(t)$ and $V(t)$, for almost all $t \in (0,\infty)$, as we had for the initial values. Therefore, we can, step by step, extend the interval of existence to $(0,\infty)$, and thus obtain global solutions. Furthermore, the norm estimates are valid on the whole positive real line, and Theorem 4.1 is proven for bounded sensitivity functions.

Remark: By the smoothing effect of the system proved in step *(viii)* of the proof of Theorem 2.1, we have automatically global uniqueness of the extended solution.

4.2 The Identity Sensitivity Function

We are going to prove Theorem 4.1 for the identity sensitivity function $S(V) = V$.

As in the case of bounded sensitivity functions in Section 4.1, we will use a Moser-type argument in order to bound increasing L^p-norms of $U(t)$ independently of the time t, which has not been accomplished up to now for a general Lipschitz domain.[†] Here we will need, however, an additional technical lemma proven in Appendix B.

Proposition 4.4 *Let $\Omega \subset I\!R^2$ be a Lipschitz domain. If all terms in the Lyapunov function F are bounded, then there exists for every $1 \leq p < \infty$ a positive constant $C = C(p)$, but independent of T, such that*

$$\|U\|_{L^\infty(0,T;L^p(\Omega))} \leq C \text{ for all } T \geq 0. \tag{4.11}$$

Proof: We will show inequality (4.11) for all $p = 2^k$ with $k = 0, 1, 2, \ldots$ by induction.

For $k = 0$, the assertion holds. (See Lemma 3.3.)

Now let $p \geq 2$ ($k \geq 1$) and suppose, (4.11) holds for $2^{k-1} = \frac{p}{2}$. Unlike the situation in Section 4.1, we do not have $e^{\chi V} \leq C$, so that we have to modify the proof of Proposition 4.2. Here, we multiply the first equation by pU^{p-1} and integration over Ω leads to

$$\frac{d}{dt}\int_\Omega U^p dx + (p-1)p\int_\Omega |\nabla U|^2 U^{p-2} dx = (p-1)p\,\chi\int_\Omega U^{p-1}\nabla U \nabla V dx,$$

or, with $W := U^{\frac{p}{2}}$ (Note that we have $\nabla W = \frac{p}{2}U^{\frac{p}{2}-1}\nabla U \implies |\nabla W|^2 = \frac{p^2}{4}U^{p-2}|\nabla U|^2$.),

$$\begin{aligned}\frac{d}{dt}\|W(t)\|^2_{L^2(\Omega)} \quad &+ \quad \frac{4(p-1)}{p}\|\nabla W\|^2_{L^2(\Omega)} = (p-1)\chi\int_\Omega \nabla V\nabla(U^p)dx \\ &= \quad \frac{(p-1)\chi}{\alpha}\left\{-\beta\int_\Omega VU^p dx + \delta\int_\Omega U^{p+1}dx - \int_\Omega V_t U^p dx\right\}, \end{aligned}\tag{4.12}$$

by using the second equation. Setting $\tilde{p} := \dfrac{2(p+1)}{p}$, the equality can be written for W only as follows:

$$\begin{aligned}\frac{d}{dt}\|W(t)\|^2_{L^2(\Omega)} \quad &+ \quad \frac{4(p-1)}{p}\|\nabla W\|^2_{L^2(\Omega)} \\ &= \quad \frac{(p-1)\chi}{\alpha}\left\{-\beta\int_\Omega VW^2 dx + \delta\int_\Omega W^{\tilde{p}}dx - \int_\Omega V_t W^2 dx\right\} \\ &\leq \quad \frac{(p-1)\chi}{\alpha}\left\{\delta\|W\|^{\tilde{p}}_{L^{\tilde{p}}(\Omega)} - \int_\Omega V_t W^2 dx\right\}. \end{aligned}\tag{4.13}$$

We will treat the last two terms separately.

[†]See Gajewski and Zacharias [14] for time-dependent estimates or Nagai et al. [32] for the case of a smooth domain.

If $p = 2$ ($k = 1$), then $\tilde{p} = 3$ and we apply Lemma B.1 from Appendix B with $r = 1$ and $q = \tilde{p} = 3$ to obtain

$$\begin{aligned}
\frac{\chi\delta}{\alpha}\|W\|^3_{L^3(\Omega)} &= \frac{\chi\delta}{\alpha}\|U\|^3_{L^3(\Omega)} \\
&\leq \varepsilon\|\nabla U\|^2_{L^2(\Omega)}\|U\log U\|_{L^1(\Omega)} + \varepsilon\|U\|^2_{L^1(\Omega)}\|U\log U\|_{L^1(\Omega)} + k(\varepsilon)\|U\|_{L^1(\Omega)} \\
&\leq \varepsilon_1\|\nabla U\|^2_{L^2(\Omega)} + C(\varepsilon_1) \;=\; \varepsilon_1\|\nabla W\|^2_{L^2(\Omega)} + C(\varepsilon_1),
\end{aligned}$$

where we used that $\|U\log U\|_{L^1(\Omega)} \leq C$ by the boundedness of all terms in the Lyapunov function. (See Corollary 3.5.)

If $p \geq 4$ ($k \geq 2$), then we can use the Gagliardo-Nirenberg Inequality and with the inductive assumption

$$\begin{aligned}
\frac{\chi\delta}{\alpha}\|W\|^{\tilde{p}}_{L^{\tilde{p}}(\Omega)} &\leq C\|W\|^{\tilde{p}-1}_{H^1(\Omega)}\|W\|_{L^1(\Omega)} \leq C\left(\|\nabla W\|^{\tilde{p}-1}_{L^2(\Omega)} + \|W\|^{\tilde{p}-1}_{L^1(\Omega)}\right)\|W\|_{L^1(\Omega)} \\
&\leq C\left(\|\nabla W\|^{\tilde{p}-1}_{L^2(\Omega)} + \|U\|^{\frac{(\tilde{p}-1)p}{2}}_{L^{\frac{p}{2}}(\Omega)}\right)\|U\|^{\frac{p}{2}}_{L^{\frac{p}{2}}(\Omega)} \leq C_1\|\nabla W\|^{\tilde{p}-1}_{L^2(\Omega)} + C_2.
\end{aligned}$$

Since $p \geq 4$ we have

$$2 < \tilde{p} = \frac{2(p+1)}{p} = 2 + \frac{2}{p} \leq \frac{5}{2} < 3,$$

and there exists a $1 \leq q < 2$ such that $q(\tilde{p}-1) = 2$. (For the conjugate exponent, we then have $2 < q' = \dfrac{2}{3-\tilde{p}} \leq 4$.) Applying Young's Inequality, we obtain for any $0 < \varepsilon_1 < 1$:

$$\begin{aligned}
\frac{\chi\delta}{\alpha}\|W\|^{\tilde{p}}_{L^{\tilde{p}}(\Omega)} &\leq C_1\|\nabla W\|^{\tilde{p}-1}_{L^2(\Omega)} + C_2 \leq \frac{1}{q}\varepsilon_1^q\,\|\nabla W\|^2_{L^2(\Omega)} + \frac{1}{q'}\frac{1}{\varepsilon_1^{q'}}C_1^{q'} + C_2 \\
&\leq \varepsilon_1\|\nabla W\|^2_{L^2(\Omega)} + \varepsilon_1^{-4}\bar{C}_1 + C_2 \;=\; \varepsilon_1\|\nabla W\|^2_{L^2(\Omega)} + C(\varepsilon_1).
\end{aligned}$$

The second term is treated as in Section 4.1: we apply Hölder's, Gagliardo-Nirenberg's and Young's Inequality:

$$\begin{aligned}
-\frac{\beta\chi}{\alpha}\int_\Omega V_t W^2 dx &\leq \frac{\beta\chi}{\alpha}\int_\Omega |V_t W^2| dx \;\leq\; C\;\|V_t\|_{L^2(\Omega)}\|W\|^2_{L^4(\Omega)} \\
&\leq C\;\|V_t\|_{L^2(\Omega)}(\|\nabla W\|_{L^2(\Omega)} + \|W\|_{L^2(\Omega)})\|W\|_{L^2(\Omega)} \\
&\leq \varepsilon_2\|\nabla W\|^2_{L^2(\Omega)} + C(\varepsilon_2)(\|V_t\|^2_{L^2(\Omega)} + \|V_t\|_{L^2(\Omega)})\|W\|^2_{L^2(\Omega)} \\
&\leq \varepsilon_2\|\nabla W\|^2_{L^2(\Omega)} + \left(C(\varepsilon_2)\|V_t\|^2_{L^2(\Omega)} + \frac{1}{2}\right)\|W\|^2_{L^2(\Omega)}.
\end{aligned}$$

Inserting these estimates into (4.13) gives

$$\frac{d}{dt}\|W(t)\|^2_{L^2(\Omega)} + (p-1)[\frac{4}{p} - \varepsilon_1 - \varepsilon_2]\|\nabla W\|^2_{L^2(\Omega)} \leq \left(C(\varepsilon_2)\|V_t\|^2_{L^2(\Omega)} + \frac{1}{2}\right)\|W\|^2_{L^2(\Omega)} + C(\varepsilon_1).$$

We choose ε_1 and ε_2 so small that $(p-1)[\frac{4}{p}-\varepsilon_1-\varepsilon_2]\geq 1$ (Note that the $\frac{1}{\varepsilon_i}$ behave like p.) and with

$$\begin{aligned}\|W\|^2_{L^2(\Omega)} &\leq C(\|\nabla W\|^{\frac{1}{2}}_{L^2(\Omega)}\|W\|^{\frac{1}{2}}_{L^1(\Omega)}+\|W\|_{L^1(\Omega)})^2\\ &\leq \|\nabla W\|^2_{L^2(\Omega)}+C\|W\|_{L^1(\Omega)}\leq \|\nabla W\|^2_{L^2(\Omega)}+C\end{aligned}$$

we obtain

$$\frac{d}{dt}\|W(t)\|^2_{L^2(\Omega)}\leq \left(C\|V_t\|^2_{L^2(\Omega)}-\frac{1}{2}\right)\|W(t)\|_{L^2(\Omega)}+C$$

Setting here $\varphi(t):=C\left(\|V_t(t)\|^2_{L^2(\Omega)}-\frac{1}{2}\right)$, we obtain by the Gronwall-type inequality used in the proof of Proposition 4.2

$$\|W(t)\|^2_{L^2(\Omega)}\leq e^{\int_0^t\varphi(s)ds}\|W_0\|^2_{L^2(\Omega)} \;+\; e^{\int_0^t\varphi(s)ds}C\int_0^t e^{-\int_0^s\varphi(\tau)d\tau}ds$$

and with the information $V_t\in L^2(0,T;L^2(\Omega))$,

$$\|U(t)\|^p_{L^p(\Omega)}=\|W(t)\|^2_{L^2(\Omega)}\;\leq\; C\;\|U_0\|^p_{L^p(\Omega)}\;+\;C(1-e^{-\frac{t}{2}})\;\leq\; C(\|U_0\|^p_{L^\infty(\Omega)}+1)\;\leq\; C$$

so that (4.11) follows. □

Conclusion 4.5 As in Conclusion 4.3, we can conclude from Proposition 4.4 that V belongs to $L^\infty(0,T;L^\infty(\Omega))$, bounded independently of T, since $\delta U\in L^\infty(0,T;L^p(\Omega))$ for a $p>1$. Globality of the solution and therewith

$$\|U\|_{L^\infty(0,\infty;L^p(\Omega))}\;+\;\|V\|_{L^\infty(0,\infty;L^\infty(\Omega))}\;\leq\; C$$

for all $1\leq p<\infty$ follows analogously. Moreover, the extension of the solution is unique. (See the remark after Conclusion 4.3.)

4.3 The Logarithmic Sensitivity Function

Since the arguments applied in Sections 4.1 and 4.2 cannot be used for the logarithmic sensitivity function, we will not be able to obtain time-independent estimates for all global solutions (U, V) of system (4.1), (4.2).

The first step to prove Theorem 4.1 for the logarithmic sensitivity function will consist in finding (time-dependent) bounds for the $L^2(0,T;L^2(\Omega))$-norm of $US'(V)$.

In order to do so, we will first prove three results for more general sensitivity functions fulfilling certain convexity and concavity conditions, respectively. These results will be applied to the logarithmic function in Proposition 4.9.

Lemma 4.6 *Let $S \in \mathcal{S}$ and suppose that $\chi < 1$.*

If all the terms in the Lyapunov function $F(U,V)$ for system (4.1) are bounded and if the sensitivity function is twice continuously differentiable and fulfills the condition $S''(V) + (S'(V))^2 \leq 0$ for all $V \geq 0$, then there exists a positive constant C, independent of $T > 0$, such that

$$\|\nabla\sqrt{U}\|_{L^2(0,T;L^2(\Omega))} \leq C(1+T)^{\frac{1}{2}}. \tag{4.14}$$

Proof: Obviously, the integral

$$\int_0^T \int_\Omega U|\nabla(\log U - \chi S(V))|^2 dxds. \tag{4.15}$$

is positive for all $T > 0$.

Using that $\nabla(\log U) = \dfrac{\nabla U}{U}$ and $\nabla\sqrt{U} = \dfrac{\nabla U}{2\sqrt{U}}$, we can rewrite the integral in the following way:

$$\begin{aligned}\int_0^T \int_\Omega U|\nabla(\log U - \chi S(V))|^2 dxds &= 4\int_0^T \int_\Omega |\nabla\sqrt{U}|^2 dxds - 2\chi\int_0^T \int_\Omega \nabla U \nabla S(V) dxds \\ &+ \chi^2 \int_0^T \int_\Omega U|\nabla S(V)|^2 dxds \\ &= \chi^2 \int_0^T \int_\Omega [U|\nabla V|^2 (S'(V))^2 - \nabla U \nabla S(V)] dxds \\ &- \chi(2-\chi)\int_0^T \int_\Omega \nabla U \nabla S(V) dxds + 4(2-\chi)\int_0^T \|\nabla\sqrt{U}\|^2_{L^2(\Omega)} ds \\ &- 4(1-\chi)\int_0^T \|\nabla\sqrt{U}\|^2_{L^2(\Omega)} ds \end{aligned} \tag{4.16}$$

We know that $\int_\Omega U_t \, dx = 0$ and by the first equation of (4.1) tested with $\log U$, we obtain

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega U\log U dx &= \int_\Omega U_t \log U dx + \int_\Omega U_t\ dx = \int_\Omega U_t \log U dx \\
&= \chi\int_\Omega U\nabla S(V)\nabla \log U dx - \int_\Omega \nabla U \nabla \log U dx \\
&= \chi\int_\Omega \nabla U\nabla S(V) dx - 4\int_\Omega |\nabla\sqrt{U}|^2 dx,
\end{aligned}
$$

so that

$$
\begin{aligned}
-\chi(2-\chi)\int_0^T\int_\Omega \nabla U\nabla S(V)dxds \ &+\ 4(2-\chi)\int_0^T \|\nabla\sqrt{U}\|^2_{L^2(\Omega)}ds \\
&= -(2-\chi)\left(\int_\Omega U(t)\log U(t)dx - \int_\Omega U_0\log U_0 dx\right). \quad (4.17)
\end{aligned}
$$

On the other hand, testing the second equation in (4.1) with $US'(V)$ gives

$$
\begin{aligned}
\frac{1}{\alpha}\int_\Omega [V_t+\beta V-\delta US'(V)]US'(V)dx &= -\int_\Omega \nabla(US'(V))\nabla V dx \\
&= -\int_\Omega US''(V)|\nabla V|^2 dx - \int_\Omega \nabla U\nabla S(V)dx,
\end{aligned}
$$

and hence

$$
\begin{aligned}
-\chi^2\int_0^T\int_\Omega \nabla U\nabla S(V)dxds &= \frac{\chi^2}{\alpha}\int_0^T\int_\Omega \left\{V_tUS'(V)+\beta VUS'(V)-\delta(US'(V))^2\right\}dxds \\
&+\chi^2\int_0^T\int_\Omega U|\nabla V|^2 S''(V)dxds. \quad (4.18)
\end{aligned}
$$

Inserting (4.17) and (4.18), identity (4.16) becomes

$$
\begin{aligned}
\int_0^T\int_\Omega U|\nabla(\log U - \chi S(V))|^2 dxds &= \chi^2\int_0^T\int_\Omega U|\nabla V|^2[(S'(V))^2+S''(V)]dxds \\
&+\frac{\chi^2}{\alpha}\int_0^T\int_\Omega \left\{V_tUS'(V)+\beta VUS'(V)-\delta(US'(V))^2\right\}dxds \\
&-(2-\chi)\left(\int_\Omega U(t)\log U(t)dx - \int_\Omega U_0\log U_0 dx\right) \\
&-4(1-\chi)\int_0^T \|\nabla\sqrt{U}\|^2_{L^2(\Omega)}ds. \quad (4.19)
\end{aligned}
$$

Using now the positivity of the left hand side and the condition $S''(V)+(S'(V))^2\le 0$, we obtain the estimate

$$
\begin{aligned}
4(1-\chi)\int_0^T \|\nabla\sqrt{U}\|^2_{L^2(\Omega)}ds &\le \frac{2\chi^4}{\delta\alpha^2}\|V_t\|^2_{L^2(0,T;L^2(\Omega))}+\frac{2\chi^4\beta^2}{\delta\alpha^2}\|V\|^2_{L^2(0,T;L^2(\Omega))} \\
&+(2-\chi)\ (\|U(T)\log U(T)\|_{L^1(\Omega)}+\|U_0\log U_0\|_{L^1(\Omega)}) \\
&\le C\Big(\|V_t\|^2_{L^2(0,T;L^2(\Omega))}+T\|V\|^2_{L^\infty(0,T;L^2(\Omega))} \\
&+\|U\log U\|_{L^\infty(0,T;L^1(\Omega))}\Big) \\
&\le C(1+T), \quad (4.20)
\end{aligned}
$$

where we used in the last step the a-priori-estimates from Corollary 3.5. Since $\chi < 1$, estimate (4.14) is proven. □

Lemma 4.7 *Let $S \in \mathcal{S}$ and suppose that $\chi > 1$.*

If all the terms in the Lyapunov function $F(U,V)$ are bounded and if the sensitivity function $S \in C^2(\mathbb{R},\mathbb{R})$ and fulfills the condition $S''(V) + (S'(V))^2 \geq 0$ for all $V \geq 0$, then there exists a constant C such that for all $T > 0$

$$\|\nabla\sqrt{U}\|_{L^2(0,T;L^2(\Omega))} \;\leq\; C(1+T)^{\frac{1}{2}}. \tag{4.21}$$

Proof: From the boundedness of the Lyapunov function (see Corollary 3.5) we know that the integral, whose positivity was used in the proof of Lemma 4.6, is also bounded from above:

$$\int_0^T \int_\Omega U|\nabla(\log U - \chi S(V))|^2 dxds \;\leq\; C$$

for all $T > 0$.

Using now identity (4.19), the assumed property $S''(V) + (S'(V))^2 \geq 0$ of the sensitivity function and the a-priori-estimates (See again Corollary 3.5.), we obtain

$$\begin{aligned} C \;\geq\; & -|\chi - 2|\;(\|U(t)\log U(t)\|_{L^1(\Omega)} + \|U_0 \log U_0\|_{L^1(\Omega)}) \\ & - C(\chi,\alpha,\beta,\delta)\|V_t\|^2_{L^2(0,T;L^2(\Omega))} - 2\delta\|US'(V)\|^2_{L^2(0,T;L^2(\Omega))} \\ & + 4(\chi-1)\int_0^T \|\nabla\sqrt{U}\|^2_{L^2(\Omega)}ds \\ \geq\; & - C - 2\delta C'\|U\|^2_{L^2(0,T;L^2(\Omega))} \\ & + 4(\chi-1)\int_0^T \|\nabla\sqrt{U}\|^2_{L^2(\Omega)}ds. \end{aligned} \tag{4.22}$$

By Corollary B.2 from Appendix B we infer that

$$2\delta C'\|U\|^2_{L^2(0,T;L^2(\Omega))} \;\leq\; (\chi-1)\int_0^T \|\nabla\sqrt{U}\|^2_{L^2(0,T;L^2(\Omega))}ds \;+\; C(\delta,\chi,C')\;T,$$

so that we have shown

$$\|\nabla\sqrt{U}\|^2_{L^2(0,T;L^2(\Omega))} \;\leq\; C(1+T),$$

which gives the claim. (Note that we used $\chi > 1$ in the last step.) □

Remarks:

1. The conditions for the sensitivity functions in Lemma 4.6 and Lemma 4.7 are equivalent to requiring that $e^{S(V)}$ be concave or convex, respectively, since

$$\frac{d^2}{dV^2}\left(e^{S(V)}\right) \;=\; e^{S(V)}(S''(V) + (S'(V))^2).$$

2. For our treatment of the logarithmic sensitivity function $S(V) = \log(V+c)$ we still need a result for the case $\chi = 1$. As in Lemma 4.6, we could use the positivity of the integral $\int_0^t \int_\Omega U|\nabla(\log U - \chi S(V))|^2 dxds$ and equality (4.19).

However, we will now prove a more general result, which is applicable to sensitivity functions for which $e^{\chi S(V)}$ is concave.

Lemma 4.8 *Let $S \in \mathcal{S}$ be twice continuously differentiable. If all the terms in the Lyapunov function for system (4.1) are bounded and if the sensitivity function fulfills the condition $S''(V) + \chi(S'(V))^2 \leq 0$, then there exists a positive constant C, independent of T, such that*

$$\|US'(V)\|_{L^2(0,T;L^2(\Omega))} \leq C(1+T)^{\frac{1}{2}}$$

for all $T > 0$.

Proof: Testing the first equation with $S(V)$ gives

$$\begin{aligned}
\int_\Omega U_t S(V)\,dx &= -\int_\Omega \nabla U \nabla S(V)\,dx + \chi \int_\Omega U|\nabla S(V)|^2 dx \\
&= -\int_\Omega S'(V)\nabla U \nabla V\,dx + \chi \int_\Omega U(S'(V))^2|\nabla V|^2 dx \\
&= -\int_\Omega \nabla(US'(V))\nabla V dx + \int_\Omega U|\nabla V|^2[S''(V) + \chi(S'(V))^2]dx \\
&= \frac{1}{\alpha}\int_\Omega US'(V)V_t dx + \frac{\beta}{\alpha}\int_\Omega US'(V)V dx - \frac{\delta}{\alpha}\int_\Omega U^2(S'(V))^2 dx \\
&\quad + \int_\Omega U|\nabla V|^2[S''(V) + \chi(S'(V))^2]dx,
\end{aligned}$$

where we used the second equation in the last step. Since

$$\int_\Omega U_t S(V)\,dx = \frac{d}{dt}\int_\Omega US(V)dx - \int_\Omega US'(V)V_t\,dx,$$

we now obtain by Hölder's and Young's Inequality

$$\begin{aligned}
\frac{d}{dt}\int_\Omega US(V)\,dx &= (1+\frac{1}{\alpha})\int_\Omega US'(V)V_t dx + \frac{\beta}{\alpha}\int_\Omega US'(V)V dx \\
&\quad - \frac{\delta}{\alpha}\int_\Omega U^2(S'(V))^2 dx + \int_\Omega U|\nabla V|^2[S''(V) + \chi(S'(V))^2]dx \\
&\leq C(\alpha,\delta)\,\|V_t\|^2_{L^2(\Omega)} + C(\alpha,\beta,\delta)\,\|V\|^2_{L^2(\Omega)} - \frac{\delta}{2\alpha}\,\|US'(V)\|^2_{L^2(\Omega)} \\
&\quad + \int_\Omega U|\nabla V|^2[S''(V) + \chi(S'(V))^2]dx.
\end{aligned}$$

Using now the hypothesis on S, integration from 0 to T yields:

$$\begin{aligned}\int_\Omega U(t)S(V(t))dx &\leq \int_\Omega U_0 S(V_0)dx + C\left(\|V_t\|^2_{L^2(0,T;L^2(\Omega))} + \|V\|^2_{L^2(0,T;L^2(\Omega))}\right) \\ &\quad -\frac{\delta}{2\alpha}\int_0^T \int_\Omega U^2(s)(S'(V(s)))^2 dx.\end{aligned}$$

And with the estimates obtained with the Lyapunov function (See Corollary 3.5.), we find

$$\begin{aligned}\frac{\delta}{2\alpha}\|US'(V)\|^2_{L^2(0,T;L^2(\Omega))} &\leq \int_\Omega U_0 S(V_0)dx + C\left(\|V_t\|^2_{L^2(0,T;L^2(\Omega))} + \|V\|^2_{L^2(0,T;L^2(\Omega))}\right) \\ &\leq C_0 + C\left(\|V_t\|^2_{L^2(0,T;L^2(\Omega))} + T\|V\|^2_{L^\infty(0,T;L^2(\Omega))}\right) \\ &\leq C(1+T),\end{aligned}$$

with C independent of T. □

Proposition 4.9 *Let $S(V) = \log(V + c)$. We can bound the solution U of the first equation in (4.1) in the space $L^2(0,T;L^2(\Omega))$ as follows*

$$\|U\|_{L^2(0,T;L^2(\Omega))} \leq C(1+T)^{\frac{1}{2}},$$

if $\chi \neq 1$.

For $\chi = 1$, we still have

$$\|US'(V)\|_{L^2(0,T;L^2(\Omega))} \leq C(1+T)^{\frac{1}{2}}.$$

The constant C in both estimates does not depend on the time $T > 0$.

Proof: We know that all the terms in the Lyapunov function are bounded for all χ in the case of the logarithmic sensitivity function. (See Proposition 3.7.)

Also, $e^{S(V)}$ is obviously both concave and convex for $S(V) = \log(V + c)$ (See the first remark after Lemma 4.7.) and we have

$$S''(V) + (S'(V))^2 = (\log(V+c))'' + ((\log(V+c))')^2 = -\frac{1}{(V+c)^2} + \frac{1}{(V+c)^2} = 0.$$

Hence, we can apply Lemma 4.7 if $\chi > 1$ and Lemma 4.6 if $\chi < 1$ in order to obtain

$$\|\nabla\sqrt{U}\|^2_{L^2(0,T;L^2(\Omega))} \leq C(1+T).$$

Now, we can use Corollary B.2 with $\kappa = 1$ to conclude that

$$\int_0^T \|U\|^2_{L^2(\Omega)}ds \;\leq\; \int_0^T \|\nabla\sqrt{U}\|^2_{L^2(\Omega)}ds + CT \;\leq\; C(1+T).$$

For $\chi = 1$, we apply Lemma 4.8, and the claim follows. □

Proposition 4.10 *Let S be the logarithmic sensitivity function. There exists a constant C, which is independent of the time $T > 0$, and an exponent $r > 2$ such that we have the following estimate for the solution (U, V) of system (4.1), (4.2):*

$$\|U\|_{L^\infty(0,T;L^2(\Omega))} \ + \ \|V\|_{L^\infty(0,T;L^\infty(\Omega))} \ \leq \ Ce^{CT^r}.$$

Proof:

(i) Using U itself as test function for the first equation in (4.1), we obtain

$$\begin{aligned} \frac{1}{2}\left(\|U(t)\|^2_{L^2(\Omega)} - \|U_0\|^2_{L^2(\Omega)}\right) &= \int_0^t \int_\Omega \frac{d}{ds}\left(\frac{1}{2}U^2(s)\right) dxds \ = \ \int_0^t \langle U_t(s), U(s)\rangle ds \\ &= \int_0^t \int_\Omega [\chi U \nabla S(V) \nabla U - |\nabla U|^2] dxds. \qquad (4.23) \end{aligned}$$

We apply Hölder's Inequality to the first term on the right hand side, where the coefficients fulfill

$$\frac{1}{r} + \frac{1}{p} = \frac{1}{2} \ .^\dagger$$

$$\chi \int_0^t \int_\Omega U\nabla S(V)\nabla U dxds \ \leq \ \chi C' \int_0^t \|U(s)\|_{L^r(\Omega)} \|\nabla V(s)\|_{L^p(\Omega)} \|\nabla U(s)\|_{L^2(\Omega)} \ ds \qquad (4.24)$$

By Gagliardo-Nirenberg's Inequality, we have

$$\begin{aligned} \|U\|_{L^r(\Omega)} &\leq C\|U\|^{\frac{2}{r}}_{L^2(\Omega)} \|U\|^{1-\frac{2}{r}}_{H^1(\Omega)} \\ &\leq C\|U\|^{\frac{2}{r}}_{L^2(\Omega)} \left(\|U\|^{1-\frac{2}{r}}_{L^2(\Omega)} + \|\nabla U\|^{1-\frac{2}{r}}_{L^2(\Omega)}\right) \\ &= C\left(\|U\|_{L^2(\Omega)} + \|U\|^{\frac{2}{r}}_{L^2(\Omega)} \|\nabla U\|^{1-\frac{2}{r}}_{L^2(\Omega)}\right), \end{aligned}$$

so that estimate (4.24) becomes

$$\begin{aligned} \chi \int_0^t \int_\Omega U\nabla S(V)\nabla U dxds &\leq C\int_0^t \Big(\|U(s)\|_{L^2(\Omega)} \|\nabla V(s)\|_{L^p(\Omega)} \|\nabla U(s)\|_{L^2(\Omega)} \\ &\quad + \|U(s)\|^{\frac{2}{r}}_{L^2(\Omega)} \|\nabla V(s)\|_{L^p(\Omega)} \|\nabla U(s)\|^{2-\frac{2}{r}}_{L^2(\Omega)} \Big) ds \\ &\leq C(\varepsilon_1, \varepsilon_2) \int_0^t \left(\|\nabla V(s)\|^2_{L^p} + \|\nabla V(s)\|^r_{L^p}\right) \|U(s)\|^2_{L^2} \ ds \\ &\quad + (\varepsilon_1 + \varepsilon_2) \int_0^t \|\nabla U(s)\|^2_{L^2(\Omega)} ds, \qquad (4.25) \end{aligned}$$

where we used twice Young's Inequality in the last step. Choosing $\varepsilon_1 + \varepsilon_2 \leq 1$, we obtain from (4.23)

† Note that this condition is equivalent to

$$r \ = \ \frac{2p}{p-2} \ = \ \frac{2p'}{2-p'} \ ,$$

so that $W^{1,p'}(\Omega) \hookrightarrow L^r(\Omega)$ and $L^{r'}(\Omega)$ is the greatest Lebesgue space, that is, the space with the smallest exponent, still contained in the dual of $W^{1,p'}(\Omega)$.

$$
\begin{aligned}
\frac{1}{2}\left(\|U(t)\|^2_{L^2(\Omega)} \;-\; \|U_0\|^2_{L^2(\Omega)}\right) \;&\le\; \int_0^t (\varepsilon_1+\varepsilon_2-1)\,\|\nabla U(s)\|^2_{L^2(\Omega)}ds \\
&\quad + C\int_0^t \left(\|\nabla V(s)\|^2_{L^p(\Omega)} + \|\nabla V(s)\|^r_{L^p(\Omega)}\right)\|U(s)\|^2_{L^2(\Omega)}ds \\
&\le\; C\int_0^t \left(\|\nabla V(s)\|^2_{L^p(\Omega)} + \|\nabla V(s)\|^r_{L^p(\Omega)}\right)\|U(s)\|^2_{L^2(\Omega)}ds.
\end{aligned}
$$

It follows now by Gronwall's Inequality and after taking the supremum over $t \in [0,T]$ on both sides of the inequality that

$$
\|U\|^2_{L^\infty(0,T;L^2(\Omega))} \;\le\; \|U_0\|^2_{L^2(\Omega)} \exp\left(C\int_0^T \left[\|\nabla V(s)\|^2_{L^p(\Omega)} + \|\nabla V(s)\|^r_{L^p(\Omega)}\right]ds\right). \qquad (4.26)
$$

Thus, we need to control the right hand side of (4.26) to obtain a bound for U in the space $L^\infty(0,T;L^2(\Omega))$. In order to do so, we will use the following parabolic regularity result by Gröger [16]. The function $\tilde{V} := V - V_0$ solves the problem

$$
\tilde{V}_t - \alpha\Delta\tilde{V} + \beta\tilde{V} = \alpha\Delta V_0 - \beta V_0 + \delta U S'(\tilde{V}+V_0) \quad \text{in} \;\; (0,T)\times\Omega \qquad (4.27)
$$

with vanishing initial value $\tilde{V_0} = 0$. Using the information that $V_0 \in W^{1,p}(\Omega)$ for a $p > 2$, we obtain from reference [16]

$$
\begin{aligned}
\|\nabla\tilde{V}\|_{L^p(0,T;L^p(\Omega))} \;&\le\; C\|\alpha\Delta V_0 - \beta V_0 + \delta U S'(\tilde{V}+V_0)\|_{L^p(0,T;(W^{1,p'}(\Omega))^*)} \\
&\le\; C\left(\|V_0\|_{L^p(0,T;W^{1,p}(\Omega))} + \|US'(V)\|_{L^p(0,T;(W^{1,p'}(\Omega))^*)}\right) \\
&\le\; C\left(T^{\frac{1}{p}}\|V_0\|_{L^\infty(0,T;W^{1,p}(\Omega))} + \|US'(V)\|_{L^p(0,T;(W^{1,p'}(\Omega))^*)}\right) \\
&\le\; C\left(1 + T + \|US'(V)\|_{L^p(0,T;(W^{1,p'}(\Omega))^*)}\right).
\end{aligned}
$$

In the terminology of Dore [8], this means that there is L^p-regularity on the interval $[0,T]$ for problem (4.27). He shows in [8], Theorem 4.2, that L^p-regularity for one p implies $L^{\tilde{p}}$-regularity for any $1 < \tilde{p} < \infty$, so that in particular

$$
\begin{aligned}
\int_0^T \|\nabla V(s)\|^2_{L^p(\Omega)}ds \;&\le\; \|\nabla\tilde{V}\|^2_{L^2(0,T;L^p(\Omega))} + CT \\
&\le\; C\left(1 + T + \|US'(V)\|_{L^2(0,T;(W^{1,p'}(\Omega))^*)}\right)^2
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^T \|\nabla V(s)\|^r_{L^p(\Omega)}ds \;&\le\; \|\nabla\tilde{V}\|^r_{L^r(0,T;L^p(\Omega))} + CT \\
&\le\; C\left(1 + T + \|US'(V)\|_{L^r(0,T;(W^{1,p'}(\Omega))^*)}\right)^r
\end{aligned}
$$

From the embedding of $L^{r'}(\Omega)$ into $(W^{1,p'}(\Omega))^{*\dagger}$, we thus obtain

$$\begin{aligned}\int_0^T \left[\|\nabla V(s)\|^2_{L^p(\Omega)} + \|\nabla V(s)\|^r_{L^p(\Omega)}\right] ds \;&\le\; C\left(1+T+\|US'(V)\|_{L^2(0,T;L^{r'}(\Omega))}\right)^2 \\ &\quad + C\left(1+T+\|US'(V)\|_{L^r(0,T;L^{r'}(\Omega))}\right)^r \\ &\le\; C\left(1+T+\|US'(V)\|_{L^2(0,T;L^2(\Omega))}\right)^2 \\ &\quad + C\left(1+T+\|US'(V)\|_{L^r(0,T;L^{r'}(\Omega))}\right)^r\end{aligned}$$

By interpolation, we find with $\dfrac{1}{r'} = 1-\theta+\dfrac{\theta}{2} = 1-\dfrac{\theta}{2} \;\Leftrightarrow\; \theta = 2\left(1-\dfrac{1}{r'}\right) = \dfrac{2}{r}$

$$\begin{aligned}\|US'(V)\|^r_{L^r(0,T;L^{r'}(\Omega))} &= \int_0^T \|U(s)S'(V(s))\|^r_{L^{r'}(\Omega)} ds \\ &\le C\int_0^T \|U(s)S'(V(s))\|^{r(1-\theta)}_{L^1(\Omega)}\|U(s)S'(V(s))\|^{r\theta}_{L^2(\Omega)} ds \\ &\le C(C')^{r(1-\theta)}\|U\|^{r(1-\theta)}_{L^\infty(0,T;L^1(\Omega))}\int_0^T \|U(s)S'(V(s))\|^{r\theta}_{L^2(\Omega)} ds \\ &= C\|U\|^{r-2}_{L^\infty(0,T;L^1(\Omega))}\|US'(V)\|^2_{L^2(0,T;L^2(\Omega))} \\ &\le C\|US'(V)\|^2_{L^2(0,T;L^2(\Omega))}.\end{aligned}$$

Using the last two estimates, it follows from Proposition 4.9 that

$$\begin{aligned}\int_0^T \left[\|\nabla V(s)\|^2_{L^p(\Omega)} + \|\nabla V(s)\|^r_{L^p(\Omega)}\right] ds \;&\le\; C(1+T^r+\|US'(V)\|^2_{L^2(0,T;L^2(\Omega))}) \\ &\overset{Prop.4.9}{\le}\; C(1+T^r+(1+T)) \\ &\le\; C(1+T^r),\end{aligned}$$

so that inequality (4.26) becomes

$$\|U\|_{L^\infty(0,T;L^2(\Omega))} \;\le\; Ce^{CT^r}.$$

(ii) We have shown in part *(i)* of this proof that the right hand side of the equation for V in (4.1) is bounded in the space $L^\infty(0,T;L^2(\Omega))$. We can therefore conclude that

$$\|V\|_{L^\infty(0,T;L^\infty(\Omega))} \;\le C\;\|U\|_{L^\infty(0,T;L^2(\Omega))} \;\le\; Ce^{CT^r},$$

which completes the proof. □

†See the footnote at the beginning of this proof.

Proposition 4.11 *The solution (U, V) of system (4.1), (4.2) with the logarithmic sensitivity function $S(V) = \log(V + c)$ is global.*

Proof: As in step *(viii)* of the proof of Theorem 2.1, we can show that the pair $(U(T), V(T)) \in L^\infty(\Omega) \times W^{1,p}(\Omega)$ for a $p > 2$, so that we can extend the solution by the same theorem up to a $T_1 > T$ and this extension is unique.

This procedure can be repeated and from existence on $[0, T_{n-1}]$ one infers existence on $[0, T_n]$ for a sequence $T < T_1 < ... < T_{n-1} < T_n$ for all $n \in \mathbb{N}$.

Suppose that $T_n \longrightarrow T_{max} < +\infty$ as $n \to +\infty$. From the estimates obtained in Proposition 4.10 we know that

$$\|U\|_{L^\infty(0,T_{max};L^2(\Omega))} \; + \; \|V\|_{L^\infty(0,T_{max};L^\infty(\Omega))} \; \leq \; Ce^{C(T_{max})^r},$$

so that T_{max} can be taken as a new starting point for an extension of the interval of existence (We have excluded explosion in $T_{max} < +\infty$.) and it follows that the solution (U, V) exists for all times. □

In case $\chi \leq 1$, we can improve the T-dependence of the estimates with the following proposition.

Proposition 4.12 *Suppose $\chi \leq 1$. Defining $W := \frac{U}{e^{\chi S(V)}} = \frac{U}{(V+c)^\chi}$, we can rewrite system (4.1) in the following way*

$$\begin{aligned} (W(V+c)^\chi)_t &= \nabla((V+c)^\chi \nabla W) \\ V_t &= \alpha \Delta V - \beta V + \delta \frac{W}{(V+c)^{(1-\chi)}}. \end{aligned} \tag{4.28}$$

Now, there exist positive constants C and γ such that we have for the solution (W, V) of the system (4.28)

$$\|W\|_{L^\infty(0,T;L^2(\Omega))} + \|V\|_{L^\infty(0,T;L^\infty(\Omega))} \leq C(1+T)^\gamma,$$

for every $T > 0$.

Proof: The equivalence between systems (4.1) and (4.28) with $W = \frac{U}{e^{\chi S(V)}} = \frac{U}{(V+c)^\chi}$ is obvious.

(i) We have for the solution W of the first equation in (4.28) with homogeneous Neumann boundary conditions

$$\|W\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\nabla W\|^2_{L^2(0,T;L^2(\Omega))} \;\leq\; C(1+T)^\gamma,$$

where C and γ are independent of the time T.

We set $\psi := (V+c)^\chi$. We showed in Proposition 4.10 that V belongs to the space $L^\infty(0,T;L^\infty(\Omega))$ so that $\psi \in L^\infty(0,T;L^\infty(\Omega))$ follows. Moreover, because of $\chi \leq 1$,

$$\begin{aligned}\|\psi_t\|_{L^2(0,T;L^2(\Omega))} &= \chi\|(V+c)^{(\chi-1)}V_t\|_{L^2(0,T;L^2(\Omega))} = \chi\left\|\frac{V_t}{(V+c)^{(1-\chi)}}\right\|_{L^2(0,T;L^2(\Omega))} \\ &\leq \|V_t\|_{L^2(0,T;L^2(\Omega))} \leq C, \end{aligned} \tag{4.29}$$

by the a-priori-estimate obtained for V_t from the boundedness of the Lyapunov function (Corollary 3.5), where C does not depend on T.

Testing the first equation in (4.28) with W gives:

$$\langle(\psi W)_t, W\rangle \;+\; \int_\Omega \psi|\nabla W|^2 dx = 0,$$

which is because of

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \psi W^2 dx \;=\; \frac{1}{2}\int_\Omega \psi_t W^2 dx \;+\; \langle \psi W_t, W\rangle \;=\; \langle(\psi W)_t, W\rangle \;-\; \frac{1}{2}\int_\Omega \psi_t W^2 dx$$

equivalent to

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \psi W^2 dx + \int_\Omega \psi|\nabla W|^2 dx = -\frac{1}{2}\int_\Omega \psi_t W^2 dx.$$

and by integration from 0 to t we obtain:

$$\frac{1}{2}\int_\Omega \psi W^2 dx\Big|_0^t + \int_0^t\int_\Omega \psi|\nabla W|^2 dxds = -\frac{1}{2}\int_0^t\int_\Omega \psi_t W^2 dxds,$$

so that (Note that $\psi \geq 1$.)

$$\begin{aligned}\int_\Omega W^2(t)dx \;+\; 2\int_0^t\int_\Omega |\nabla W|^2 dxds &\leq 2\int_\Omega (V_0+c)^\chi W^2(0)dx \;+\; \int_0^t\int_\Omega \psi_t W^2 dxds \\ &\leq 2(\|V_0\|_{L^\infty(\Omega)}+c)^\chi \int_\Omega W^2(0)dx \;+\; \|\psi_t\|_{L^2(0,t;L^2(\Omega))}\|W^2\|_{L^2(0,t;L^2(\Omega))} \\ &= C_0\int_\Omega W^2(0)dx \;+\; \|\psi_t\|_{L^2(0,t;L^2(\Omega))}\|W\|^2_{L^4(0,t;L^4(\Omega))}. \end{aligned} \tag{4.30}$$

We will write for simplicity

$$Z_{0,t} := L^\infty(0,t;L^2(\Omega)) \cap L^2(0,t;H^1(\Omega))$$

and define the norm $\|w\|^2_{Z_{0,t}} := \Big(\|w\|^2_{L^\infty(0,t;L^2(\Omega))} + \|\nabla w\|^2_{L^2(0,t;L^2(\Omega))}\Big)$ for all $w \in Z_{0,t}$.

We know that

$$\|W\|^2_{L^2(0,T;L^2(\Omega))} = \left\|\frac{U}{e^{\chi S(V)}}\right\|^2_{L^2(0,T;L^2(\Omega))} \leq \|U\|^2_{L^2(0,T;L^2(\Omega))} \leq C(1+T),$$

for $\chi \neq 1$, and if $\chi = 1$,

$$\begin{aligned}\|W\|^2_{L^2(0,T;L^2(\Omega))} &= \left\|\frac{U}{e^{S(V)}}\right\|^2_{L^2(0,T;L^2(\Omega))} = \left\|\frac{U}{(V+c)}\right\|^2_{L^2(0,T;L^2(\Omega))} \\ &= \|US'(V)\|^2_{L^2(0,T;L^2(\Omega))} \leq C(1+T),\end{aligned}$$

by Proposition 4.9. Hence, we can apply the Gagliardo-Nirenberg Inequality to obtain

$$\begin{aligned}\|W\|^4_{L^4(0,t;L^4(\Omega))} &= \int_0^t \|W(s)\|^4_{L^4(\Omega)}ds \leq C_1\int_0^t \|W(s)\|^2_{L^2(\Omega)}\|W(s)\|^2_{H^1(\Omega)}ds \\ &= C_1\int_0^t \|W(s)\|^2_{L^2(\Omega)}\left(\|W(s)\|^2_{L^2(\Omega)} + \|\nabla W(s)\|^2_{L^2(\Omega)}\right)ds \\ &\leq C_1\|W\|^2_{L^\infty(0,t;L^2(\Omega))}\left(\|W\|^2_{L^2(0,t;L^2(\Omega))} + \|\nabla W\|^2_{L^2(0,t;L^2(\Omega))}\right) \\ &\leq C_1\|W\|^2_{L^2(0,T;L^2(\Omega))}\|W\|^2_{L^\infty(0,t;L^2(\Omega))} \\ &\quad + \frac{C_1}{2}\left(\|W\|^4_{L^\infty(0,t;L^2(\Omega))} + \|\nabla W\|^4_{L^2(0,t;L^2(\Omega))}\right) \\ &\leq C_2(1+T)\|W\|^2_{L^\infty(0,t;L^2(\Omega))} + \frac{C_1}{2}\|W\|^4_{Z_{0,t}} \\ &\leq C_2^2(1+T)^2 + \left(\frac{C_1}{2}+1\right)\|W\|^4_{Z_{0,t}}, \end{aligned} \tag{4.31}$$

so that taking the supremum over $(0,t)$ in (4.30) yields

$$\begin{aligned}\|W\|^2_{Z_{0,t}} &\leq C_0\|W(0)\|^2_{L^2(\Omega)} + \|\psi_t\|_{L^2(0,t;L^2(\Omega))}\|W\|^2_{L^4(0,t;L^4(\Omega))} \\ &\leq C_0\|W(0)\|^2_{L^2(\Omega)} + \left(\frac{C_1}{2}+1\right)^{\frac{1}{2}}\|\psi_t\|_{L^2(0,t;L^2(\Omega))}\|W\|^2_{Z_{0,t}} \\ &\quad + C_2(1+T)\|\psi_t\|_{L^2(0,t;L^2(\Omega))} \\ &\leq C_0\|W(0)\|^2_{L^2(\Omega)} + \tilde{C}\|\psi_t\|_{L^2(0,t;L^2(\Omega))}\|W\|^2_{Z_{0,t}} + C_2(1+T)\|\psi_t\|_{L^2(0,T;L^2(\Omega))} \\ &\leq C_0\|W(0)\|^2_{L^2(\Omega)} + \tilde{C}\|\psi_t\|_{L^2(0,t;L^2(\Omega))}\|W\|^2_{Z_{0,t}} + \hat{C}(1+T)\end{aligned}$$

by (4.29). Now, let $t = t_1$ be so small that $\|\psi_t\|_{L^2(0,t_1;L^2(\Omega))} \leq \frac{1}{2\tilde{C}}$. We then have

$$\|W\|^2_{Z_{0,t_1}} \leq 2(C_0\|W(0)\|^2_{L^2(\Omega)} + \hat{C}(1+T)). \tag{4.32}$$

Step by step, we can partition $[0,T]$ into finitely many intervals $[t_{k-1},t_k]$ for $k = 1,...,s$, where $t_0 = 0$ and $t_s = T$ such that

$$\|\psi_t\|_{L^2(t_{k-1},t_k;L^2(\Omega))} \leq \frac{1}{2\tilde{C}}$$

and consequently

$$\|W\|^2_{Z_{t_{k-1},t_k}} \le 2\left[2\int_\Omega (V(t_{k-1})+c)^\chi W^2(t_{k-1})dx + \hat{C}(1+T)\right] \tag{4.33}$$

for $k=1,...,s$. On the other hand, we can choose the partition of $[0,T]$ such that

$$\|\psi_t\|_{L^2(t_{k-1},t_k;L^2(\Omega))} \ge \frac{1}{4\tilde{C}} \quad \text{for} \quad k=1,...,s-1,$$

so that one has the following bound for the number s of intervals:

$$(s-1)\left(\frac{1}{4\tilde{C}}\right)^2 \le \sum_{k=1}^{s-1}\|\psi_t\|^2_{L^2(t_{k-1},t_k;L^2(\Omega))} \le \sum_{k=1}^{s}\|\psi_t\|^2_{L^2(t_{k-1},t_k;L^2(\Omega))} = \|\psi_t\|^2_{L^2(0,T;L^2(\Omega))},$$

i.e.,

$$s \le 1+(4\tilde{C})^2\|\psi_t\|^2_{L^2(0,T;L^2(\Omega))} \ \le \ C.$$

We will now show by induction that

$$\|W\|^2_{L^\infty(t_{k-1},t_k;L^2(\Omega))} \ \le \ \|W\|^2_{Z_{t_{k-1},t_k}} \ \le \ C(1+T)^{\left(\frac{3}{2}\right)^{(k-1)}}. \tag{4.34}$$

For $k=1$, the estimate is true since

$$\|W\|^2_{L^\infty(0,t_1;L^2(\Omega))} \le \|W\|^2_{Z_{0,t_1}} \le C(1+T)$$

and because of equation (4.32).

Let us assume that

$$\|W\|^2_{L^\infty(t_{k-2},t_{k-1};L^2(\Omega))} \ \le \ \|W\|^2_{Z_{t_{k-2},t_{k-1}}} \ \le \ C(1+T)^{\left(\frac{3}{2}\right)^{(k-2)}}.$$

As by standard results on parabolic equations,

$$\|V(t_{k-1})\|_{L^\infty(\Omega)} \ \le \ C(k)\|W\|_{L^\infty(t_{k-2},t_{k-1};L^2(\Omega))} \ \le \ C\|W\|_{L^\infty(t_{k-2},t_{k-1};L^2(\Omega))},$$

we calculate with inequality (4.33):

$$\begin{aligned}
\|W\|^2_{L^\infty(t_{k-1},t_k;L^2(\Omega))} \ &\le \ \|W\|^2_{Z_{t_{k-1},t_k}} \\
&\le \ 2\left[2(\|V(t_{k-1})\|_{L^\infty(\Omega)}+c)^\chi\|W(t_{k-1})\|^2_{L^2(\Omega)} + \hat{C}(1+T)\right] \\
&\le \ 4\left(C(1+T)^{\frac{\chi}{2}\left(\frac{3}{2}\right)^{(k-2)}} + c^\chi\right)C(1+T)^{\left(\frac{3}{2}\right)^{(k-2)}} + 2\hat{C}(1+T) \\
&\le \ C\left\{(1+T)^{\left(\frac{\chi}{2}+1\right)\left(\frac{3}{2}\right)^{(k-2)}} + (1+T)^{\left(\frac{3}{2}\right)^{(k-2)}} + (1+T)\right\} \\
&\le \ C(1+T)^{\left(\frac{3}{2}\right)^{(k-1)}}
\end{aligned}$$

and (4.34) is proven.

Finally, we get the following time-independent estimate

$$\begin{aligned}\|W\|^2_{Z_{0,T}} &\leq \sum_{k=1}^{s}\|W\|^2_{Z_{t_{k-1},t_k}} \leq \sum_{k=1}^{s} C(1+T)^{\left(\frac{3}{2}\right)^{(k-1)}}\\ &\leq \sum_{k=1}^{s} C(1+T)^{\left(\frac{3}{2}\right)^{(s-1)}} \leq sC(1+T)^{\left(\frac{3}{2}\right)^{(s-1)}}\end{aligned}$$

and our claim follows with $\gamma = \frac{1}{2}\left(\frac{3}{2}\right)^{(s-1)}$.

(ii) It now follows for the solution V of the second equation of system (4.28) that

$$\|V\|_{L^\infty(0,T;L^\infty(\Omega))} \leq C(1+T)^\gamma.$$

We have shown for the right hand side of the linear equation for V in (4.28) that

$$\|\delta W\|_{L^\infty(0,T;L^2(\Omega))} \leq \delta C(1+T)^\gamma$$

and it follows by standard arguments as in Ladyženskaja et al. [24] that $\|V\|_{L^\infty(0,T;L^\infty(\Omega))}$ depends Lipschitz continuously on $\|W\|_{L^\infty(0,T;L^2(\Omega))}$. □

Remark: If we had homogeneous Dirichlet boundary conditions, we would obtain time-independent estimates for the solution (U,V) of system (4.1).

Proof: In the case of homogeneous Dirichlet boundary conditions, we could use the plain L^2-norm of ∇W instead of the full H^1-norm of W in Gagliardo-Nirenberg's Inequality, so that (4.31) in the proof of Theorem 4.12 would become

$$\begin{aligned}\|W\|^4_{L^4(0,t;L^4(\Omega))} &\leq C_1\int_0^t \|W(s)\|^2_{L^2(\Omega)}\|\nabla W(s)\|^2_{L^2(\Omega)}ds\\ &\leq C_1\|W\|^2_{L^\infty(0,t;L^2(\Omega))}\|\nabla W\|^2_{L^2(0,t;L^2(\Omega))}\\ &\leq \frac{C_1}{2}\left(\|W\|^4_{L^\infty(0,t;L^2(\Omega))} + \|\nabla W\|^4_{L^2(0,t;L^2(\Omega))}\right)\\ &\leq \frac{C_1}{2}\|W\|^4_{Z_{0,t}}.\end{aligned}$$

It would follow that

$$\|W\|^2_{Z_{0,t_1}} \leq C_0\|W(0)\|^2_{L^2(\Omega)} + \tilde{C}\|\psi_t\|_{L^2(0,t;L^2(\Omega))}\|W\|^2_{Z_{0,t}}.$$

Partitioning now the interval $[0,T]$ into $[0,t_1],...,[t_{s-1},T]$ in such a manner that

$$\|\psi_t\|_{L^2(t_{k-1},t_k;L^2(\Omega))} \leq \frac{1}{2\tilde{C}} \quad \text{for } k=1,...,s$$

and

$$\|\psi_t\|_{L^2(t_{k-1},t_k;L^2(\Omega))} \geq \frac{1}{4\tilde{C}} \quad \text{for} \quad k=1,...,s-1,$$

we would finally obtain

$$\|W\|^2_{Z_{0,T}} \ \leq \ C,$$

and consequently $\|V\|_{L^\infty(0,T;L^\infty(\Omega))} \ \leq \ C$, where the constant C would not depend on the time T. □

Conclusion 4.13 By Proposition 4.10, claim (4.3) has been proven. Furthermore, we have shown that

$$e^{\chi S(V)} = (V+c)^\chi \ \leq \ (\|V\|_{L^\infty(0,T;L^\infty(\Omega))}+c)^\chi \ \leq \ (Ce^{CT^r}+c)^\chi =: C_e.$$

To bound U in $L^\infty(0,T;L^p(\Omega))$ for $2<p<\infty$, we can now apply the argument for bounded sensitivity functions (Section 4.1): As in the proof of Proposition 4.2, we obtain

$$\int_\Omega W^2(t)e^{\chi S(V(t))}dx \ \leq \ C_1e^{C_2(pC_e)^2}+C_3e^{C_2(pC_e)^2}C_eK^{\frac{p^2}{2}},$$

where $K^{\frac{p}{2}}$ is the bound for $\|U\|_{L^\infty(0,T;L^{\frac{p}{2}}(\Omega))}$. To obtain $\|U(t)\|^p_{L^p(\Omega)} \leq K^{p^2}$, we require that $\|U_0\|_{L^1(\Omega)} \leq K$, $C_1e^{C_2C_e^2}C_e \leq \frac{1}{2}K$ and $C_3e^{C_2C_e^2}C_e \leq \frac{1}{2}K^{\frac{1}{2}}$, which means that K has to behave like $e^{C_3C_e^2}$, i.e., $K = C\exp(Ce^{CT^r})$, and hence

$$\|U\|_{L^\infty(0,T;L^p(\Omega))} \leq C\exp(pCe^{CT^r})$$

for all $2<p<\infty$. Thus, estimate (4.4) holds, too. Proposition 4.12 shows that if $\chi \leq 1$, the estimates can be improved in the following way:

$$\|W\|_{L^\infty(0,T;L^2(\Omega))} \ + \ \|V\|_{L^\infty(0,T;L^\infty(\Omega))} \ \leq \ C(1+T)^\gamma,$$

from which we can also deduce

$$\|U\|_{L^\infty(0,T;L^p(\Omega))} \leq C\exp(pC(1+T)^{2\gamma})$$

for all $1<p<\infty$. We have thus proven Theorem 4.1 completely for the logarithmic sensitivity function.

4.4 Stationary Equation for V with a General Sensitivity Function

In this section we want to deal with the following system of equations:

$$\begin{aligned} U_t &= \Delta U - \chi\nabla(U\nabla S(V)) \\ 0 &= \alpha\Delta V - \beta V + \delta U S'(V) \end{aligned} \tag{4.35}$$

in $(0,T)\times\Omega \subset I\!R \times I\!R^2$. The solution (U,V) is subject to the homogeneous Neumann boundary conditions (4.2) and the initial condition $U(0,x) = U_0(x) \in L^2(\Omega)$.

We will demonstrate that the assumption of a stationary equation for the chemical substance V simplifies the mathematical treatment considerably, so that, in this situation, we can prove time-independent bounds for V in the space $L^\infty(0,T;L^\infty(\Omega))$ and for U in $L^\infty(0,T;L^2(\Omega))$ for a very general sensitivity function S. (Note that this suffices to prolong the solution to the whole half line $(0,+\infty)$ and thus obtain a global solution.) First of all, note that the proof of Theorem 2.1 can be modified without any problem so that we obtain existence and uniqueness of a pair of positive solutions (U,V) for system (4.35) satisfying

$$U \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \quad U_t \in L^2(0,T;(H^1(\Omega))^*),$$

$$V \in C([0,T];H^1(\Omega)) \cap L^\infty(0,T;W^{1,p}(\Omega)).$$

with a $p > 2$.

Secondly, the function

$$F(U(t),V(t)) = \int_\Omega \left\{U(t)\log U(t) - \chi U(t)S(V(t)) + \frac{\chi}{2\delta}(\alpha|\nabla V(t)|^2 + \beta(V(t))^2)\right\} dx$$

is a Lyapunov function for system (4.35), too, as we can easily verify that

$$\frac{d}{dt}F(U(t),V(t)) \;=\; -\int_\Omega U(t)|\nabla[\log U(t) - \chi S(V(t))]|^2 dx \;\leq\; 0,$$

for all $t > 0$.

Moreover, we obtain the same exponential condition as in Section 3.1, Lemma 3.4 to ensure boundedness of F and all its terms, respectively.

The next proposition shows that boundedness of all terms in F entails immediately the existence of a time-independent bound for the function V in $L^\infty(0,T;L^\infty(\Omega))$. The proof of the second part of the proposition follows the lines of a proof by Gajewski and Gröger [11] applied to the system of semiconductor equations.

Proposition 4.14 *Let $S \in \mathcal{S}$. If all the terms in the Lyapunov function F are bounded, then there exists a constant C, which is independent of the time $T > 0$, such that we have the following estimate for the solution (U, V) of system (4.35):*

$$\|V\|_{L^\infty(0,T;L^\infty(\Omega))} \; + \; \|U\|_{L^\infty(0,T;L^2(\Omega))} \;\leq\; C.$$

Proof:

(i) Since $U \log U$ is bounded independently of T in $L^\infty(0,T;L^1(\Omega))$, we have for the right hand side of the V-equation

$$US'(V)\log(US'(V)) \;\leq\; C'\left(|U\log U| + U\log C'\right)$$

and it follows that $\|US'(V)\log(US'(V))\|_{L^\infty(0,T;L^1(\Omega))} \;\leq\; C$, i.e., $US'(V)$ belongs to the Orlicz space $L^\Phi(\Omega)$. (See Appendix B, Section B.1.) We can therefore deduce from a result by Gröger [17] for linear elliptic equations that

$$\|V\|_{L^\infty(0,T;L^\infty(\Omega))} \;\leq\; C\|US'(V)\|_{L^\infty(0,T;L^\Phi(\Omega))} \;\leq\; C,$$

where the constant C does not depend on $T > 0$.

(ii) Let $Z := (U - K)^+$ with a constant $K > e$, which will be chosen later on. Testing the first equation in (4.35) with the function $e^{2t}Z$, we obtain

$$\begin{aligned}
\frac{1}{2}\left(e^{2t}\|Z(t)\|^2_{L^2(\Omega)} \;-\; \|Z(0)\|^2_{L^2(\Omega)}\right) \;-\; \int_0^t e^{2s}\int_\Omega Z^2(s)dxds \\
&= \int_0^t\int_\Omega \frac{d}{ds}\left(\frac{1}{2}Z^2(s)e^{2s}\right)dxds \;-\; \int_0^t e^{2s}\int_\Omega Z^2(s)dxds \\
&= \int_0^t e^{2s}\langle Z_t, Z\rangle ds \;=\; \int_0^t e^{2s}\langle U_t, Z\rangle ds \\
&= \int_0^t e^{2s}\int_\Omega[\chi U\nabla S(V)\nabla Z - \nabla U\nabla Z]dxds \\
&= \int_0^t e^{2s}\int_\Omega[\chi U\nabla S(V)\nabla Z - |\nabla Z|^2]dxds. \qquad (4.36)
\end{aligned}$$

(Note that once again we can only formally deal with Z_t. See the footnote in part *(i)* of the proof of existence in Chapter 2.)

We first want to treat the first term on the right hand side of (4.36), which originates from the chemotactic drift term in the equation. By an elliptic regularity result by Gröger [15] we have

$$\|\nabla V\|_{L^p(\Omega)} \;\leq\; C\|U\|_{(W^{1,p'}(\Omega))^*}.$$

We can apply Hölder's Inequality to the term we are concerned with, where the coefficients fulfill

$$\frac{1}{r} + \frac{1}{p} = \frac{1}{2} \; .$$

We are here in the same situation as in the proof of Proposition 4.10. Remember that $L^{r'}(\Omega) \hookrightarrow (W^{1,p'}(\Omega))^*$.

$$\begin{aligned}
\chi \int_\Omega U \nabla S(V) \nabla Z dx &\leq \chi C' \|U\|_{L^r} \|\nabla V\|_{L^p(\Omega)} \|\nabla Z\|_{L^2(\Omega)} \\
&\leq C \|U\|_{L^r} \|U\|_{(W^{1,p'}(\Omega))^*} \|\nabla Z\|_{L^2(\Omega)} \\
&\leq C \|U\|_{L^r} \|U\|_{L^{r'}(\Omega)} \|\nabla Z\|_{L^2(\Omega)} \\
&\leq C \left(\|Z\|_{L^r} + K|\Omega|\right) \left(\|Z\|_{L^{r'}(\Omega)} + K|\Omega|\right) \|\nabla Z\|_{L^2(\Omega)} \\
&\leq C \left(\|Z\|_{L^r} + K\right) \left(\|Z\|_{L^{r'}(\Omega)} + K\right) \|\nabla Z\|_{L^2(\Omega)} \qquad (4.37)
\end{aligned}$$

By Gagliardo-Nirenberg's Inequality, we have

$$\begin{aligned}
\|Z(t)\|_{L^r(\Omega)} &\leq C \|Z(t)\|_{L^1(\Omega)}^{\frac{1}{r}} \|Z(t)\|_{H^1(\Omega)}^{\frac{1}{r'}} \\
&\leq \tilde{C} \|Z(t)\|_{L^1(\Omega)}^{\frac{1}{r}} \left(\|Z(t)\|_{L^1(\Omega)}^{\frac{1}{r'}} + \|\nabla Z(t)\|_{L^2(\Omega)}^{\frac{1}{r'}} \right) \\
&\leq \tilde{C} \|Z(t)\|_{L^1(\Omega)} + \tilde{C} \|Z(t)\|_{L^1(\Omega)}^{\frac{1}{r}} \|\nabla Z(t)\|_{L^2(\Omega)}^{\frac{1}{r'}}
\end{aligned}$$

and

$$\|Z(t)\|_{L^{r'}(\Omega)} \leq C \left(\|Z(t)\|_{L^1(\Omega)} + \|Z(t)\|_{L^1(\Omega)}^{\frac{1}{r'}} \|\nabla Z(t)\|_{L^2(\Omega)}^{\frac{1}{r}} \right),$$

respectively. Inserting these estimates into (4.37), we can go on calculating as follows

$$\begin{aligned}
\chi \int_\Omega U \nabla S(V) \nabla Z dx &\leq \bar{C} \left(\|Z(t)\|_{L^1(\Omega)} + \|Z(t)\|_{L^1(\Omega)}^{\frac{1}{r}} \|\nabla Z(t)\|_{L^2(\Omega)}^{\frac{1}{r'}} + K \right) \\
&\quad \times \left(\|Z(t)\|_{L^1(\Omega)} + \|Z(t)\|_{L^1(\Omega)}^{\frac{1}{r'}} \|\nabla Z(t)\|_{L^2(\Omega)}^{\frac{1}{r}} + K \right) \|\nabla Z\|_{L^2(\Omega)} \\
&= \bar{C} \Big\{ \|Z(t)\|_{L^1(\Omega)}^2 + 2K \|Z(t)\|_{L^1(\Omega)} + K^2 \\
&\quad + \|Z(t)\|_{L^1(\Omega)}^{1+\frac{1}{r'}} \|\nabla Z(t)\|_{L^2(\Omega)}^{\frac{1}{r}} + K \|Z(t)\|_{L^1(\Omega)}^{\frac{1}{r'}} \|\nabla Z(t)\|_{L^2(\Omega)}^{\frac{1}{r}} \\
&\quad + \|Z(t)\|_{L^1(\Omega)}^{1+\frac{1}{r}} \|\nabla Z(t)\|_{L^2(\Omega)}^{\frac{1}{r'}} + K \|Z(t)\|_{L^1(\Omega)}^{\frac{1}{r}} \|\nabla Z(t)\|_{L^2(\Omega)}^{\frac{1}{r'}} \\
&\quad + \|Z(t)\|_{L^1(\Omega)} \|\nabla Z(t)\|_{L^2(\Omega)} \Big\} \|\nabla Z(t)\|_{L^2(\Omega)}
\end{aligned}$$

Using Young's Inequality three times, we obtain

$$
\begin{aligned}
\chi \int_\Omega U \nabla S(V) \nabla Z dx \;\; \leq \;\; & \left(\bar{C}\|Z(t)\|_{L^1(\Omega)} + \varepsilon_1 + \varepsilon_2 + \varepsilon_3\right) \|\nabla Z(t)\|^2_{L^2(\Omega)} \\
& + \frac{\bar{C}^2}{\varepsilon_1} \left(\|Z(t)\|^2_{L^1(\Omega)} + 2K\|Z(t)\|_{L^1(\Omega)} + K^2\right)^2 \\
& + \frac{\bar{C}^{\frac{2r}{r-1}}}{\varepsilon_2^{\frac{r+1}{r-1}}} \left(\|Z(t)\|^{1+\frac{1}{r'}}_{L^1(\Omega)} + K\|Z(t)\|^{\frac{1}{r'}}_{L^1(\Omega)}\right)^{\frac{2r}{r-1}} \\
& + \frac{\bar{C}^{2r}}{\varepsilon_3^{2r-1}} \left(\|Z(t)\|^{1+\frac{1}{r}}_{L^1(\Omega)} + K\|Z(t)\|^{\frac{1}{r}}_{L^1(\Omega)}\right)^{2r} \\
\leq \;\; & \left(\bar{C}\|Z(t)\|_{L^1(\Omega)} + \varepsilon_1 + \varepsilon_2 + \varepsilon_3\right) \|\nabla Z(t)\|^2_{L^2(\Omega)} \\
& + C(K), \qquad (4.38)
\end{aligned}
$$

where we used in the last step that

$$
\|Z(t)\|_{L^1(\Omega)} \;\leq\; \|U(t)\|_{L^1(\Omega)} \;+\; K|\Omega| \;\leq\; C(K).
$$

Gagliardo-Nirenberg's Inequality can be applied once more to the second term on the left hand side of inequality (4.36):

$$
\begin{aligned}
\int_0^t e^{2s}\|Z(s)\|^2_{L^2(\Omega)} ds \;\; \leq \;\; & C \int_0^t e^{2s}\|Z(t)\|_{L^1(\Omega)}\|Z(t)\|_{H^1(\Omega)} ds \\
\leq \;\; & \int_0^t e^{2s} \left(\tilde{C}\|Z(t)\|^2_{L^1(\Omega)} + \tilde{C}\|Z(t)\|_{L^1(\Omega)}\|\nabla Z(t)\|_{L^2(\Omega)}\right) ds \\
\leq \;\; & \int_0^t e^{2s} \left\{ \left(\tilde{C} + \frac{\tilde{C}^2}{\varepsilon_4}\right) \|Z(t)\|^2_{L^1(\Omega)} + \varepsilon_4 \|\nabla Z(t)\|^2_{L^2(\Omega)} \right\} ds \\
\leq \;\; & \int_0^t e^{2s} \left\{ C(\varepsilon_4, K) + \varepsilon_4 \|\nabla Z(t)\|^2_{L^2(\Omega)} \right\} ds. \qquad (4.39)
\end{aligned}
$$

Applying now (4.38) and (4.39) to equality (4.36), choosing the ε_i such that the sum $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 \leq \frac{1}{2}$, we obtain

$$
\begin{aligned}
& \frac{1}{2}\left(e^{2t}\|Z(t)\|^2_{L^2(\Omega)} \;-\; \|Z(0)\|^2_{L^2(\Omega)}\right) \\
& \quad \leq \int_0^t e^{2s} \Big\{ \left(\bar{C}\|Z(t)\|_{L^1(\Omega)} + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 - 1\right) \|\nabla Z(t)\|^2_{L^2(\Omega)} \\
& \qquad + \; C(K) \Big\} ds \\
& \quad \leq \int_0^t e^{2s} \left\{ \left(\bar{C}\|Z(t)\|_{L^1(\Omega)} - \frac{1}{2}\right) \|\nabla Z(t)\|^2_{L^2(\Omega)} + C(K) \right\} ds \qquad (4.40)
\end{aligned}
$$

In order to determine the value of K, we first look at the following estimate. (Remember that $\log K > 1$.)

$$
\begin{aligned}
\int_\Omega |U(t)\log U(t)|dx \;&\geq \int_{\{x\in\Omega:U(t,x)\geq K\}} U(t)\log U(t)dx \;\geq \int_{\{x\in\Omega:U(t,x)\geq K\}} U(t)(\log U(t)-1)dx \\
&\geq \int_{\{x\in\Omega:U(t,x)\geq K\}} U(t)(\log K-1)dx \;\geq\; (\log K-1)\int_{\{x\in\Omega:U(t,x)\geq K\}}(U(t)-K)dx \\
&= (\log K-1)\int_{\{x\in\Omega:U(t,x)\geq K\}}(U(t)-K)^+dx \;=\; (\log K-1)\|Z(t)\|_{L^1(\Omega)}. \quad (4.41)
\end{aligned}
$$

This shows that if we choose K such that

$$
(\log K-1) \;=\; 2\bar{C}\sup_{t\in[0,T]}\int_\Omega |U(t)\log U(t)|dx, \tag{4.42}
$$

then it follows from (4.41) that

$$
(\log K-1)\|Z(t)\|_{L^1(\Omega)} \;\leq\; \int_\Omega |U(t)\log U(t)|dx \;\leq\; \frac{1}{2\bar{C}}(\log K-1),
$$

i.e.,

$$
\bar{C}\|Z(t)\|_{L^1(\Omega)} \;\leq\; \frac{1}{2}
$$

for all $t\in[0,T]$. Thus, (4.40) becomes

$$
\begin{aligned}
\frac{1}{2}\left(e^{2t}\|Z(t)\|^2_{L^2(\Omega)} \;-\; \|Z(0)\|^2_{L^2(\Omega)}\right) \;&\leq\; \int_0^t e^{2s}C(K)ds \\
&= \frac{1}{2}\left(e^{2t}-1\right)C(K) \;\leq\; \frac{1}{2}\,e^{2t}C(K). \quad (4.43)
\end{aligned}
$$

Finally, we obtain for the function U

$$
\begin{aligned}
\|U(t)\|^2_{L^2(\Omega)} \;&=\; \int_\Omega [U(t)-K+K]^2dx \;\leq\; \int_\Omega [(U(t)-K)^+ + K]^2dx \\
&\leq\; \|Z(t)\|^2_{L^2(\Omega)} + K^2|\Omega| \\
&\overset{(4.43)}{\leq}\; e^{-2t}\|Z(0)\|^2_{L^2(\Omega)} + C(K) \\
&=\; e^{-2t}\int_{\{x\in\Omega:U_0(x)\geq K\}}(U_0-K)^2dx \;+\; C(K) \\
&\leq\; e^{-2t}\int_{\{x\in\Omega:U_0(x)\geq K\}}U_0^2dx \;+\; C(K) \\
&\leq\; e^{-2t}\|U_0\|^2_{L^2(\Omega)} \;+\; C(K). \quad (4.44)
\end{aligned}
$$

Since $\|U\log U\|_{L^\infty(0,T;L^1(\Omega))}$ is bounded independently of $T>0$, it follows from definition (4.42) that K does not depend on the time T, either:

$$
K \;=\; \exp\left(2\bar{C}\sup_{t\in[0,T]}\int_\Omega |U(t)\log U(t)|dx + 1\right) \;\leq\; C,
$$

and the claimed estimate for U has thus been proved by estimate (4.44). $\square$

Chapter 5

Asymptotic Behaviour

5.1 Convergence to a Steady State

In this section, we are going to investigate the behaviour for $t \to +\infty$ of a weak solution $(U(t,x), V(t,x))$ of system (1.6), (1.7) in the sense of Definition 1.1. We will consider sensitivity functions $S \in \mathcal{S}$.

For $n = 2$, the existence of such a pair of weak solutions was proven in Chapter 2. For $n > 2$, we again refer to the existence theorem by Amann [1] for smooth domains, which we discuss in Appendix A.

We will show for the class $\mathcal{S}$ of sensitivity functions convergence of a subsequence $(U(t_k), V(t_k))$ with $t_k \to +\infty$ to a possibly non-trivial steady state (U^*, V^*).

Theorem 5.1 *Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz domain and $S \in \mathcal{S}$. If we have a global solution (U,V) of system (1.6), (1.7) for which all terms in the Lyapunov function $F(U,V)$ are bounded, then there exist a sequence $t_k \to +\infty$, a function V^* and a constant W^* such that*

$$V(t_k) \longrightarrow V^* \quad \text{in } H^1(\Omega), \qquad \frac{U(t_k)}{e^{\chi S(V(t_k))}} \longrightarrow W^* \quad \text{in } L^p(\Omega) \ \text{ for all } \ 1 \le p < \infty,$$

$$U(t_k) \longrightarrow U^* := W^* e^{\chi S(V^*)} \quad \text{in } L^p(\Omega) \ \text{ for all } \ 1 \le p < \infty$$

and

$$F(U(t_k), V(t_k)) \longrightarrow F(U^*, V^*) \quad \text{as} \ \ t_k \to +\infty.$$

Furthermore, the limit function U^* *has the form*

$$U^* = \frac{\|U_0\|_{L^1(\Omega)}\ e^{\chi S(V^*)}}{\int_\Omega e^{\chi S(V^*)}dx}$$

and V^* *solves the following boundary value problem*

$$-\alpha\Delta V^* + \beta V^* \;=\; \delta U^* S'(V^*) \;=\; \delta\frac{\|U_0\|_{L^1(\Omega)}\ e^{\chi S(V^*)}}{\int_\Omega e^{\chi S(V^*)}dx}S'(V^*) \tag{5.1}$$

in Ω *with* $\nu\cdot\nabla V^* = 0$ *on* $\partial\Omega$.

Proof: Let us define $W(t,x) := \dfrac{U(t,x)}{e^{\chi S(V(t,x))}}$.

(i) There exists a sequence $t_k \to +\infty$, $t_k \in I\!R_+$, *such that* $V_t(t_k) \to 0$ *in* $L^2(\Omega)$, $V(t_k) \rightharpoonup V^*$ *in* $H^1(\Omega)$ *and* $W(t_k) \to W^*$ *in* $L^p(\Omega)$ *for all* $1 \le p < \infty$.

We have

$$C|\nabla\sqrt{W}|^2 \le 4e^{\chi S(V)}|\nabla\sqrt{W}|^2 = \frac{(\nabla U - \chi U\nabla S(V))^2}{U} = U|\nabla(\log U - \chi S(V))|^2. \tag{5.2}$$

Let us define the functional $I(t) := t(\|V_t(t)\|^2_{L^2(\Omega)} + \|\nabla\sqrt{W(t)}\|^2_{L^2(\Omega)})$. From (5.2) and Corollary 3.5, we have

$$\int_0^t (\|V_t(s)\|^2_{L^2(\Omega)} + \|\nabla\sqrt{W(s)}\|^2_{L^2(\Omega)})ds \le C$$

for all $t \ge 0$, so that

$$\int_0^t I(s)ds \le Ct \quad \text{for all } t \ge 0.$$

Therefore, there exists a sequence $t_k \to +\infty$, such that $I(t_k) \le 2C$ for all t_k.[†] We therefore deduce that as $t_k \to +\infty$

$$\|V_t(t_k)\|^2_{L^2(\Omega)} + \|\nabla\sqrt{W(t_k)}\|^2_{L^2(\Omega)} \;\longrightarrow\; 0.$$

With the a-priori-estimates from the Lyapunov function (See Corollary 3.5 in Section 3.1.) we additionally know that $\|V(t_k)\|_{H^1(\Omega)} \le C$ for all t_k, so that there exists a V^* such that

[†] If there existed a $t_0 \ge 0$ with $I(t) > 2C$ for all $t \ge t_0$, then it would follow that

$$Ct \ge \int_0^t I(s)ds \ge \int_{t_0}^t I(s)ds > 2C(t - t_0),$$

i.e., $2t_0 > t$ for all $t \ge t_0$, which is impossible.

$V(t_k) \rightharpoonup V^*$ in $H^1(\Omega)$. (If necessary, here and later we pass to a subsequence of $\{V(t_k)\}$.) Also, we have

$$\begin{aligned} \|\sqrt{W(t_k)}\|^2_{H^1(\Omega)} &= \|\sqrt{W(t_k)}\|^2_{L^2(\Omega)} + \|\nabla\sqrt{W(t_k)}\|^2_{L^2(\Omega)} \\ &= \|W(t_k)\|_{L^1(\Omega)} + \|\nabla\sqrt{W(t_k)}\|^2_{L^2(\Omega)} \\ &\leq \|U(t_k)\|_{L^1(\Omega)} + \|\nabla\sqrt{W(t_k)}\|^2_{L^2(\Omega)} \\ &= \|U_0\|_{L^1(\Omega)} + \|\nabla\sqrt{W(t_k)}\|^2_{L^2(\Omega)} \leq C, \end{aligned}$$

and the existence of a W^* follows with $\sqrt{W(t_k)} \rightharpoonup \sqrt{W^*}$ in $H^1(\Omega)$.

Because of $\|\nabla\sqrt{W(t_k)}\|_{L^2(\Omega)} \to 0$, we even have strong convergence

$$\sqrt{W(t_k)} \longrightarrow \sqrt{W^*} \text{ in } H^1(\Omega) \text{ as } t_k \to +\infty \quad \text{with } \nabla\sqrt{W^*} = 0,$$

and by the compact embedding of $H^1(\Omega)$ into $L^p(\Omega)$,

$$W(t_k) \longrightarrow W^* = \text{const. in } L^p(\Omega) \text{ for all } 1 \leq p < \infty.$$

(ii) We also have $e^{\chi S(V(t_k))} \longrightarrow e^{\chi S(V^*)}$ in $L^p(\Omega)$ as $t_k \to +\infty$ for all $1 \leq p < \infty$.

From the Trudinger-Moser Inequality (which can be shown to hold on a general Lipschitz domain by extension of the functions to a smooth set if necessary), we have an estimate of the form

$$\|e^{\chi S(V)}\|_{L^p(\Omega)} \leq C(\|V\|_{H^1(\Omega)}, p)$$

for all $V \in H^1(\Omega)$ and all $1 \leq p < \infty$. Since the sequence $\{V(t_k)\}$ is uniformly bounded in $H^1(\Omega)$, it therefore follows that $\{e^{\chi S(V(t_k))}\}$ is uniformly bounded in every $L^p(\Omega)$. We use

$$\begin{aligned} |e^{\chi S(V(t_k,x))} - e^{\chi S(V^*(x))}| &\leq \chi S'(\tilde{V}(x)) e^{\chi S(\tilde{V}(x))} |V(t_k,x) - V^*(x)| \\ &\leq \chi C' (e^{\chi S(V(t_k,x))} + e^{\chi S(V^*(x))}) |V(t_k,x) - V^*(x)|, \end{aligned}$$

where $\tilde{V}(x)$ is an intermediate value between $V(t_k,x)$ and $V^*(x)$, and we obtain

$$\begin{aligned} \|e^{\chi S(V(t_k))} - e^{\chi S(V^*)}\|_{L^p(\Omega)} &= \left(\int_\Omega |e^{\chi S(V(t_k,x))} - e^{\chi S(V^*(x))}|^p dx\right)^{\frac{1}{p}} \\ &\leq \chi C' \left(\int_\Omega (e^{\chi S(V(t_k,x))} + e^{\chi S(V^*(x))})^p |V(t_k,x) - V^*(x)|^p dx\right)^{\frac{1}{p}} \\ &\leq \chi C' C \left(\|e^{\chi S(V(t_k))}\|_{L^{2p}} + \|e^{\chi S(V^*)}\|_{L^{2p}}\right) \|V(t_k) - V^*\|_{L^{2p}(\Omega)} \\ &\leq C\|V(t_k) - V^*\|_{L^{2p}(\Omega)} \longrightarrow 0. \end{aligned}$$

Thus, $e^{\chi S(V(t_k))} \longrightarrow e^{\chi S(V^*)}$ in $L^p(\Omega)$ for all $1 \le p < \infty$.

(iii) The remaining assertions in the theorem hold.

The convergence of the $\{U(t_k)\}$ follows from

$$\begin{aligned}
\|U(t_k) - W^* e^{\chi S(V^*)}\|^p_{L^p(\Omega)} &\le C \int_\Omega \Big\{ |W(t_k,x) e^{\chi S(V(t_k,x))} - W(t_k,x) e^{\chi S(V^*(x))}|^p \\
&\quad + |W(t_k,x) e^{\chi S(V^*(x))} - W^*(x) e^{\chi S(V^*(x))}|^p \Big\} \, dx \\
&\le C \int_\Omega \Big(W(t_k,x)[e^{\chi S(V(t_k,x))} - e^{\chi S(V^*(x))}] \Big)^p \, dx \\
&\quad + C \int_\Omega \Big([W(t_k,x) - W^*(x)] e^{\chi S(V^*(x))} \Big)^p \, dx \\
&\le C \, \|W(t_k)\|^p_{L^{2p}(\Omega)} \|e^{\chi S(V(t_k))} - e^{\chi S(V^*)}\|^p_{L^{2p}(\Omega)} \\
&\quad + C \, \|W(t_k) - W^*\|^p_{L^{2p}(\Omega)} \|e^{\chi S(V^*)}\|^p_{L^{2p}(\Omega)} \\
&\le C \left(\|e^{\chi S(V(t_k))} - e^{\chi S(V^*)}\|^p_{L^{2p}(\Omega)} + \|W(t_k) - W^*\|^p_{L^{2p}(\Omega)} \right) \\
&\longrightarrow 0
\end{aligned}$$

for all $1 \le 2p < \infty$, so that $U(t_k) \longrightarrow U^*$ in every $L^p(\Omega)$.

Passing to the limit $t_k \to +\infty$ in the weak formulation of the V-equation,† one obtains the stationary equation (5.1). Since on one hand

$$\|U^*\|_{L^1(\Omega)} = \|W^* e^{\chi S(V^*)}\|_{L^1(\Omega)} = W^* \int_\Omega e^{\chi S(V^*(x))} dx$$

and on the other hand $\|U^*\|_{L^1(\Omega)} = \|U_0\|_{L^1(\Omega)}$, we obtain

$$W^* = \frac{\|U_0\|_{L^1(\Omega)}}{\int_\Omega e^{\chi S(V^*)} dx}, \qquad \text{i.e.,} \qquad U^* = \frac{\|U_0\|_{L^1(\Omega)} \, e^{\chi S(V^*)}}{\int_\Omega e^{\chi S(V^*)} dx}.$$

Finally, testing the difference of the equations for the $V(t_k)$ and V^* with $(V(t_k) - V^*)$ yields the strong convergence of the sequence $\{V(t_k)\}$ to V^* in $H^1(\Omega)$ and the values of F converge as well. □

Remark: We can generalize Theorem 5.1 to higher dimensions. Let us consider the case $n = 3$. By the continuous embedding of $H^1(\Omega)$ into $L^p(\Omega)$ for $1 \le p \le 6$, we obtain in the proof of the theorem strong convergence of the sequence $\{V(t_k)\}$ to V^* in $L^6(\Omega)$ and $W(t_k) \longrightarrow W^*$ in $L^3(\Omega)$.

†Note that by continuity of $S'(V)$ we have pointwise convergence of $S'(V(t_k)) \to S'(V^*)$ and it follows by the Dominated Convergence Theorem that $S'(V(t_k)) \to S'(V^*)$ in every $L^p(\Omega)$.

If the sensitivity function is bounded, $e^{\chi S(V)}$ is globally Lipschitz continuous so that we obtain $e^{\chi S(V(t_k))} \longrightarrow e^{\chi S(V^*)}$ in $L^p(\Omega)$ for $1 \le p \le 6$ and we still get

$$\begin{aligned}
\|U(t_k) - W^* e^{\chi S(V^*)}\|^2_{L^2(\Omega)} &\le C \int_\Omega \Big(W(t_k, x)[e^{\chi S(V(t_k,x))} - e^{\chi S(V^*(x))}] \Big)^2 \, dx \\
&\quad + C \int_\Omega \Big([W(t_k, x) - W^*(x)] e^{\chi S(V^*(x))} \Big)^2 \, dx \\
&\le C \, \|W(t_k)\|^2_{L^3(\Omega)} \|e^{\chi S(V(t_k))} - e^{\chi S(V^*)}\|^2_{L^6(\Omega)} \\
&\quad + C \, \|W(t_k) - W^*\|^2_{L^3(\Omega)} \|e^{\chi S(V^*)}\|^2_{L^6(\Omega)} \\
&\le C \Big(\|e^{\chi S(V(t_k))} - e^{\chi S(V^*)}\|^2_{L^6(\Omega)} + \|W(t_k) - W^*\|^2_{L^3(\Omega)} \Big) \\
&\longrightarrow 0,
\end{aligned}$$

i.e., we have $U(t_k) \longrightarrow U^*$ in $L^2(\Omega)$ as well as the other results of the theorem.

If $S(V) = S_3(V) = \log(V + c)$ (and $\chi < 6$), then $(V(t_k) + c)^\chi \longrightarrow (V^* + c)^\chi$ in the space $L^{\frac{6}{\chi}}(\Omega)$ and $W(t_k) \longrightarrow W^*$ in $L^3(\Omega)$, so that we have convergence of the $U(t_k) = W(t_k)(V(t_k) + c)^\chi$ for $p \le \frac{6}{2+\chi}$ in case $\chi \le 4$. The rest of the theorem follows, too.

5.2 Trivial and Non-trivial Steady States

In this section, we will, in analogy to results by Gajewski and Zacharias in [14] for the equations without sensitivity function, study examples of solutions tending to trivial and non-trivial steady states, respectively.

On one hand, we will find conditions on the data of the problem which ensure convergence of the solution to the trivial constant state $(1, C_V)$ where the constant C_V is such that $\beta C_V = \delta S'(C_V)$. (W.l.o.g. we will assume here that $\|U_0\|_{L^1(\Omega)} = |\Omega|$, so that $C_U = \frac{1}{|\Omega|}\|U_0\|_{L^1(\Omega)} = 1$.)

On the other hand, we will give an example for the logarithmic sensitivity function $\log(V + c)$, $c \geq 1$, where the limit steady state (U^*, V^*) found in Section 5.1 is non-constant, i.e., different from the trivial constant solution $(1, C_V)$, provided the chemotactic coefficient as well as the production rate are large.

Proposition 5.2 *Let $\Omega \subset I\!R^2$ be a Lipschitz domain. Let the sensitivity function $S \in \mathcal{S}$ be twice continuously differentiable and satisfy the conditions $-\chi S''(V) \leq 1$ and $S''(V) \leq -\gamma\chi\,(S'(V))^2$ for a $\gamma > 1 + \frac{\alpha}{4}$. If the chemotactic coefficient χ and the coefficient δ in the production term for V are sufficiently small, then we obtain convergence of the solution $(U(t), V(t))$ of system (1.6) with (1.7) to the trivial constant steady state $(1, C_V)$ with $\beta C_V = \delta S'(C_V)$ in the space $L^\Phi(\Omega) \times L^2(\Omega)$ as $t \to \infty$.*

Proof: Let us define the functional

$$F_*(U(t), V(t)) = \int_\Omega \left\{ U(t)\log U(t) - \chi(U(t) - 1)S(V(t)) + \frac{1}{2}(V(t) - C_V)^2 \right\} dx. \quad (5.3)$$

We will show that under (smallness) conditions on the parameters of the system there exists a constant $b > 0$, such that

$$\frac{d}{dt}F_*(U(t), V(t)) \;\leq\; -bF_*(U(t), V(t)).^\dagger \quad (5.4)$$

If we differentiate $F_*(U(t), V(t))$ with respect to t, using the equations (1.6) with (1.7) and the relation $\frac{|\nabla U|^2}{U} = 4|\nabla\sqrt{U}|^2$ as well as $\beta C_V = \delta S'(C_V)$, we obtain

†Note that as a consequence of this estimate, $F_*(U, V)$ is a second Lyapunov function for system (1.6) with (1.7).

$$
\begin{aligned}
\frac{d}{dt}F_*(U,V) \;&=\; \int_\Omega \{U_t(\log U - \chi S(V)) \;+\; \chi V_t(1-U)S'(V) \;+\; (V-C_V)V_t\}\,dx \\
&\overset{(1.6)}{=} -\int_\Omega \Big\{U|\nabla(\log U - \chi S(V))|^2 \;+\; \alpha\nabla V\nabla[\chi(1-U)S'(V)+V]\Big\}\,dx \\
&\quad - \int_\Omega (\beta V - \delta U S'(V))(\chi(1-U)S'(V) + (V-C_V))dx \\
&= -4\int_\Omega |\nabla\sqrt{U}|^2 dx \;+\; 2\chi\int_\Omega \nabla U\nabla S(V)dx - \chi^2\int_\Omega U|\nabla S(V)|^2 dx \\
&\quad + \chi\alpha\int_\Omega U|\nabla V|^2 S''(V)dx \;+\; \chi\alpha\int_\Omega \nabla U\nabla S(V)dx \\
&\quad - \alpha\int_\Omega |\nabla V|^2(\chi S''(V)+1)dx \;-\; \beta\int_\Omega (V-C_V)^2 dx \\
&\quad - \delta\int_\Omega (S'(C_V) - US'(V))(V-C_V)dx \\
&\quad - \chi\int_\Omega (\beta V - \beta C_V + \delta S'(C_V) - \delta U S'(V))(1-U)S'(V)dx \\
&= -4\int_\Omega |\nabla\sqrt{U}|^2 dx \;+\; \chi(2+\alpha)\int_\Omega \nabla U\nabla S(V)dx \\
&\quad + \chi\int_\Omega U|\nabla V|^2(\alpha S''(V) - \chi(S'(V))^2)dx \\
&\quad -\alpha\int_\Omega |\nabla V|^2(\chi S''(V)+1)dx \;-\; \beta\int_\Omega (V-C_V)^2 dx \\
&\quad - \delta\int_\Omega [(S'(C_V) - S'(V))(V-C_V) \;+\; S'(V)(1-U)(V-C_V)]\,dx \\
&\quad -\chi\int_\Omega \beta(V-C_V)(1-U)S'(V)dx \\
&\quad -\chi\delta\int_\Omega [S'(C_V) - S'(V) + S'(V)(1-U)](1-U)S'(V)dx,
\end{aligned}
$$

We are going to use the conditions required of the sensitivity function $-\chi S''(V) \leq 1$ and $S''(V) \leq -\chi\gamma(S'(V))^2$. Moreover, we apply Young's Inequality to the second term on the right hand side where we write $\nabla U\nabla S(V)$ as $\frac{\nabla U}{2\sqrt{U}}2\sqrt{U}\nabla S(V)$ with an $\varepsilon_1 > 0$.

$$
\begin{aligned}
\frac{d}{dt}F_*(U,V) \;&\leq\; -4\int_\Omega |\nabla\sqrt{U}|^2 dx \;+\; \int_\Omega \left(\frac{\varepsilon_1}{2}|\nabla\sqrt{U}|^2 \;+\; \frac{2(2+\alpha)^2\chi^2 U|\nabla S(V)|^2}{\varepsilon_1}\right)dx \\
&\quad - \chi^2\int_\Omega U|\nabla S(V)|^2(1+\alpha\gamma)dx \;-\; \beta\int_\Omega (V-C_V)^2 dx \\
&\quad - \delta\int_\Omega [(S'(C_V) - S'(V))(V-C_V) \;+\; S'(V)(1-U)(V-C_V)]\,dx \\
&\quad -\chi\int_\Omega \beta(V-C_V)(1-U)S'(V)dx \\
&\quad -\chi\delta\int_\Omega [S'(C_V) - S'(V) + S'(V)(1-U)](1-U)S'(V)dx,
\end{aligned}
$$

Note that with $\gamma > 1 + \frac{\alpha}{4}$, it is possible to choose $\varepsilon_1 < 8$ such that the coefficient

$1 + \alpha\gamma - \dfrac{2(2+\alpha)^2}{\varepsilon_1} \geq 0$, and it follows that $4 - \dfrac{\varepsilon_1}{2} =: \varepsilon_2 > 0$. Using Young's Inequality three more times, we obtain

$$\begin{aligned}
\frac{d}{dt}F_*(U,V) \;&\leq\; -\varepsilon_2 \int_\Omega |\nabla\sqrt{U}|^2 dx \\
&- \left(\beta - \delta C'' - \frac{\delta C'}{2} - \frac{\chi\beta C'}{2} - \frac{\chi\delta C'C''}{2}\right)\int_\Omega (V - C_V)^2 dx \\
&+ \left(\frac{\delta C'}{2} + \frac{\chi\beta C'}{2} + \frac{\chi\delta C'C''}{2}\right)\int_\Omega (U-1)^2 dx. \qquad (5.5)
\end{aligned}$$

We also need the following estimate:

$$\|U - 1\|^2_{L^2(\Omega)} \;\leq\; \frac{1}{k}\|\nabla\sqrt{U}\|^2_{L^2(\Omega)} \qquad (5.6)$$

with a positive constant k. In order to prove (5.6), we remind that, by the continuous Sobolev Embedding $W^{1,1}(\Omega) \hookrightarrow L^2(\Omega)$, there exists a $\bar{C} > 0$ such that

$$\|U\|^2_{L^2(\Omega)} \;\leq\; \bar{C}\left(\|U_x\|^2_{L^1(\Omega)} \;+\; \|U_y\|^2_{L^1(\Omega)}\right)$$

for all $U \in W^{1,1}(\Omega)$ with vanishing spatial mean value. Therefore, we can calculate for our function U

$$\begin{aligned}
\|U-1\|^2_{L^2(\Omega)} \;&\leq\; \bar{C}\left(\|U_x\|^2_{L^1(\Omega)} \;+\; \|U_y\|^2_{L^1(\Omega)}\right) \\
&=\; 4\bar{C}\left(\|\sqrt{U}(\sqrt{U})_x\|^2_{L^1(\Omega)} \;+\; \|\sqrt{U}(\sqrt{U})_y\|^2_{L^1(\Omega)}\right) \\
&\leq\; 4\bar{C}\|U\|_{L^1(\Omega)}\|\nabla\sqrt{U}\|^2_{L^2(\Omega)} \;\leq\; 4\bar{C}|\Omega|\; \|\nabla\sqrt{U}\|^2_{L^2(\Omega)},
\end{aligned}$$

which is (5.6) with $\dfrac{1}{k} = 4\bar{C}|\Omega|$. Using (5.6) in estimate (5.5), we finally obtain

$$\begin{aligned}
\frac{d}{dt}F_*(U,V) \;&\leq\; -\left(\varepsilon_2 k - \frac{\delta C'}{2} - \frac{\chi\beta C'}{2} - \frac{\chi\delta C'C''}{2}\right)\int_\Omega (U-1)^2 dx \\
&- \left(\beta - \delta C'' - \frac{\delta C'}{2} - \frac{\chi\beta C'}{2} - \frac{\chi\delta C'C''}{2}\right)\int_\Omega (V - C_V)^2 dx
\end{aligned}$$

It is obvious that we can choose χ and δ so small that

$$\left(\varepsilon_2 k - \frac{\delta C'}{2} - \frac{\chi\beta C'}{2} - \frac{\chi\delta C'C''}{2}\right) \;=\; b_1 \;> 0 \qquad (5.7)$$

and

$$\left(\beta - \delta C'' - \frac{\delta C'}{2} - \frac{\chi\beta C'}{2} - \frac{\chi\delta C'C''}{2}\right) \;=\; b_2 \;> 0. \qquad (5.8)$$

Besides, because of $\int_\Omega U dx = |\Omega|$,

$$
\begin{aligned}
-\chi \int_\Omega (U-1)S(V)dx &= -\chi \int_\Omega (U-1)(S(V)-S(C_V))dx \\
&\leq \chi^2 \int_\Omega (U-1)^2 dx + (C')^2 \int_\Omega (V-C_V)^2 dx \\
&\leq \chi^2 \int_\Omega (U-1)^2 dx + K \int_\Omega (V-C_V)^2 dx, \qquad (5.9)
\end{aligned}
$$

with any $K \geq (C')^2$. For the $\int_\Omega (U-1)^2 dx$-term we need that

$$
\int_\Omega U \log U dx \leq \int_\Omega (U-1)^2 dx, \qquad (5.10)
$$

which follows from

$$
h(U) := (U-1)^2 - U(\log U - 1) - 1 \geq 0 \qquad (5.11)
$$

for all $U \geq 0$. To prove estimate (5.11), consider that $h(1) = h(0) = \lim_{U\to 0} h(U) = 0$.

For $U \geq 1$, we have

$$
\begin{aligned}
h'(U) &= 2(U-1) - \log U = 2(U-1) - \int_1^U \frac{dt}{t} \\
&\geq 2(U-1) - \int_1^U dt = U - 1 \geq 0,
\end{aligned}
$$

so that $h(U) \geq h(1) = 0$ for all $U \geq 1$.

For $0 < U < 1$ we set $W = \frac{1}{U}$ and see that

$$
h(U) = h\left(\frac{1}{W}\right) = \frac{1}{W^2}(W \log W - W + 1) = \frac{1}{W^2} \int_1^W \log s \, ds > 0
$$

for $W > 1$ and we have shown (5.11) and thus (5.10) for all $U \geq 0$.

With (5.9) and (5.10) we can go on calculating as follows

$$
\begin{aligned}
\frac{d}{dt} F_*(U,V) &\leq -\frac{b_1}{2} \int_\Omega (U-1)^2 dx - \frac{b_2}{2\beta} \beta \int_\Omega (V-C_V)^2 dx \\
&\quad - \frac{b_1}{2\chi^2} \chi^2 \int_\Omega (U-1)^2 dx - \frac{b_2}{2K} K \int_\Omega (V-C_V)^2 dx \\
&\leq -\frac{b_1}{2} \int_\Omega U \log U dx - \frac{b_2}{2\beta} \beta \int_\Omega (V-C_V)^2 dx \\
&\quad - \frac{b_2}{2K} \left(\chi^2 \int_\Omega (U-1)^2 dx + K \int_\Omega (V-C_V)^2 dx \right) \\
&\leq \frac{b_2}{2K} \int_\Omega \left\{ -U \log U + \chi(U-1)S(V) - \beta(V-C_V)^2 \right\} dx \\
&\leq -\frac{b_2}{2K} F_*(U,V)
\end{aligned}
$$

if K is chosen so large that $\frac{b_2}{K}$ is smaller than $\frac{b_1}{\chi^2}$, $\frac{b_2}{\beta}$ and b_1.

Thus, we have proven estimate (5.4) under the smallness conditions for χ and δ (5.7) and (5.8) with $b = \frac{b_2}{2K}$.

By Gronwall's Lemma, it now follows that

$$\begin{aligned}\int_\Omega U \log U dx \quad &- \quad \chi \int_\Omega (U(t)-1) S(V(t)) dx \; + \; \frac{1}{2}\|V(t) - C_V\|^2_{L^2(\Omega)} \\ &= \quad F_*(U(t), V(t)) \;\leq\; e^{-bt} F_*(U_0, V_0).\end{aligned}$$

Similarly to (5.9), we can estimate

$$\begin{aligned} -\chi \int_\Omega (U(t)-1) S(V) dx \quad &= \quad -\chi \int_\Omega (U(t)-1)(S(V) - S(C_V)) dx \\ &\geq \quad -\frac{1}{4}\,\|V(t) - C_V\|^2_{L^2(\Omega)} \;-\; \chi^2 (C')^2 \|U(t) - 1\|^2_{L^1(\Omega)} \\ &\geq \quad -\frac{1}{4}\,\|V(t) - C_V\|^2_{L^2(\Omega)} \;-\; \chi^2 (C')^2 \|U(t) \log U(t)\|_{L^1(\Omega)},\end{aligned}$$

where we applied again (5.10) in the last step, and we obtain under the additional smallness condition $\chi < \frac{1}{C'}$ the existence of a positive constant $\tilde{C}$ such that

$$\tilde{C} \|U(t) \log U(t)\|_{L^1(\Omega)} \; + \; \frac{1}{4}\|V(t) - C_V\|^2_{L^2(\Omega)} \;\leq\; e^{-bt} F_*(U_0, V_0). \tag{5.12}$$

In the next step we will need that

$$\|U - 1\|_{L^\Phi(\Omega)} \;\leq\; \|U \log U\|_{L^1(\Omega)}. \tag{5.13}$$

This follows from

$$\begin{aligned}\Phi(|U-1|) \quad &= \quad (1 + |U-1|)\log(1 + |U-1|) - |U-1| \\ &\leq \quad U(\log U - 1) + 1 =: f(U),\end{aligned} \tag{5.14}$$

since $\int_\Omega U dx = |\Omega|$. (See also Section B.1.)

For $U \geq 1$, we obviously have $\Phi(|U-1|) = \Phi(U-1) = f(U)$. Therefore, let us show (5.14) for $0 \leq U < 1$. In that case, we calculate

$$\Phi(|U-1|) \;=\; \Phi(1-U) \;=\; (2-U)\log(2-U) - 1 + U =: g(U).$$

Since $g(1) = f(1)$, (5.14) is true if $f'(U) \leq g'(U)$ for all $0 \leq U < 1$. Differentiating f and g yields that this condition is equivalent to

$$\log U \;\leq\; -\log(2-U) \quad\Longleftrightarrow\quad \log(U(2-U)) \;\leq\; 0,$$

which is always true since $U(2-U) = 1-(1-U)^2 < 1$ and (5.14) is proven.

Finally, inserting (5.13) into (5.12) gives

$$\tilde{C}\|U(t)-1\|_{L^\Phi(\Omega)} \;+\; \frac{1}{4}\|V(t)-C_V\|^2_{L^2(\Omega)} \;\le\; e^{-bt}F_*(U_0,V_0),$$

which yields the claimed convergence. □

Proposition 5.3 *Consider system (1.6), (1.7) with* $S(V) = \log(V+c),\ c \ge 1$, *in the two-dimensional domain* $\Omega = \{(x_1,x_2) : 0 < x_1 < a,\ 0 < x_2 < b\}$. *Let*

$$U_0(x_1,x_2) \;=\; V_0(x_1,x_2) \;=\; 1 \;+\; \cos\frac{\pi x_1}{a} \tag{5.15}$$

for $(x_1,x_2) \in \Omega$.

There exist (sufficiently large) coefficients χ, δ *and* β, *such that the solution* (U,V) *of system (1.6), (1.7) will tend to a non-constant steady state* (U^*,V^*) *in the sense of Theorem 5.1.*

Proof: From Theorem 4.1 we know that the solution $(U(t),V(t))$ of system (1.6) is global for the initial values given in (5.15). Furthermore, a subsequence $(U(t_k),V(t_k))$ convergences by Theorem 5.1 to a steady state (U^*,V^*) satisfying

$$U^* = \frac{\|U_0\|_{L^1(\Omega)}\; e^{\chi S(V^*)}}{\int_\Omega e^{\chi S(V^*)}dx} \tag{5.16}$$

and

$$-\alpha\Delta V^* + \beta V^* = \delta U^* S'(V^*). \tag{5.17}$$

Since

$$\|U_0\|_{L^1(\Omega)} = \int_0^b\int_0^a \left(1+\cos\frac{\pi x_1}{a}\right)dx_1 \;=\; ab + b\sin\frac{\pi x_1}{a}\bigg|_0^a \;=\; ab \;=\; |\Omega|,$$

we obtain by (5.16) and (5.17) for the trivial, i.e., constant steady state (C_U,C_V) that $C_U = 1$ and that C_V satisfies the relation

$$\beta C_V \;=\; \delta S'(C_V).$$

It follows for the logarithmic sensitivity function that

$$C_V^2 + c\,C_V \;=\; \frac{C_V}{S'(C_V)} \;=\; \frac{\delta}{\beta}, \tag{5.18}$$

so that we calculate

$$C_V = -\frac{c}{2} + \sqrt{\frac{c^2}{4}+\frac{\delta}{\beta}}. \tag{5.19}$$

Hence, the value of the Lyapunov function

$$F(U(t),V(t)) = \int_\Omega \left\{ U(t)\log U(t) - \chi U(t) S(V(t)) + \frac{\chi}{2\delta}(\alpha|\nabla V(t)|^2 + \beta (V(t))^2) \right\} dx$$

at the point $(1, C_V)$ is

$$\begin{aligned}
F(1,C_V) &= \chi \int_\Omega \left[\frac{\beta}{2\delta} C_V^2 - \log(C_V + c) \right] dx \\
&\overset{(5.18)}{=} \chi ab \left[\frac{\beta}{2\delta} \left(\frac{\delta}{\beta} - c\, C_V \right) - \log(C_V + c) \right] \\
&= \chi ab \left[\frac{1}{2} - \frac{\beta c}{2\delta}\, C_V - \log(C_V + c) \right] \\
&\overset{(5.19)}{=} \chi ab \left[\frac{1}{2} - \frac{\beta c}{2\delta} \left(-\frac{c}{2} + \sqrt{\frac{c^2}{4} + \frac{\delta}{\beta}} \right) - \log \left(\frac{c}{2} + \sqrt{\frac{c^2}{4} + \frac{\delta}{\beta}} \right) \right] \\
&= \chi ab \left[\frac{1}{2} + \frac{\beta c^2}{4\delta} - \frac{\beta c}{2\delta} \sqrt{\frac{c^2}{4} + \frac{\delta}{\beta}} - \log \left(\frac{c}{2} + \sqrt{\frac{c^2}{4} + \frac{\delta}{\beta}} \right) \right]. \qquad (5.20)
\end{aligned}$$

Now, we are going to determine the value of $F(U_0, V_0)$. Integral calculations, involving principally trigonometric relations, show that

$$\begin{aligned}
\int_\Omega U_0 \log U_0 dx &= \int_0^b \int_0^a \left(1 + \cos\frac{\pi x_1}{a} \right) \log \left(1 + \cos\frac{\pi x_1}{a} \right) dx_1 dx_2 \\
&= \frac{ab}{\pi} \int_0^\pi (1 + \cos y) \log(1 + \cos y)\ dy \ = \ ab(1 - \log 2)
\end{aligned}$$

and

$$\begin{aligned}
-\chi \int_\Omega U_0 S(V_0) dx &= -\chi \int_0^b \int_0^a \left(1 + \cos\frac{\pi x_1}{a} \right) \log \left(1 + c + \cos\frac{\pi x_1}{a} \right) dx_1 dx_2 \\
&= -\frac{\chi ab}{\pi} \int_0^\pi (1 + \cos y) \log(1 + c + \cos y) dy \\
&= -\chi ab \left(1 + c - \sqrt{c^2 + 2c} + \log \frac{1 + c + \sqrt{c^2 + 2c}}{2} \right).
\end{aligned}$$

(One can show that

$$\int_0^\pi (1 + \cos y) \log(b + \cos y) dy = \pi \left[b - \sqrt{b^2 - 1} + \log \left(\frac{b + \sqrt{b^2 - 1}}{2} \right) \right]$$

for $b \geq 1$.)

Finally

$$\begin{aligned}
\frac{\chi}{2\delta} \int_\Omega \left(\alpha |\nabla V_0|^2 + \beta V_0^2 \right) dx &= \frac{\chi ab}{2\delta\pi} \int_0^\pi \left(\alpha \sin^2 y + \beta \left(1 + 2\cos y + \cos^2 y \right) \right) dy \\
&= \frac{\chi ab}{4\delta} \left(\frac{\alpha \pi^2}{a^2} + 3\beta \right)
\end{aligned}$$

Inserting the last three equalities into $F(U_0, V_0)$, we obtain

$$\begin{aligned} F(U_0, V_0) &= \int_\Omega \left\{ U_0 \log U_0 - \chi U_0 S(V_0) + \frac{\chi}{2\delta}(\alpha|\nabla V_0|^2 + \beta V_0^2) \right\} dx \\ &= ab\chi \left(\frac{\alpha\pi^2}{4\delta a^2} + \frac{3\beta}{4\delta} - 1 - c + \sqrt{c^2+2c} - \log \frac{1+c+\sqrt{c^2+2c}}{2} \right) \\ &\quad + ab\,(1 - \log 2) \end{aligned} \tag{5.21}$$

We will now show, that for big values of χ, δ and β

$$F(U_0, V_0) \; < \; F(1, C_V). \tag{5.22}$$

By (5.20) and (5.21), inequality (5.22) is equivalent to

$$\begin{aligned} \frac{1-\log 2}{\chi} \; &+ \; \frac{\alpha\pi^2}{4\delta a^2} + \frac{3\beta}{4\delta} - \frac{\beta c^2}{4\delta} + \frac{\beta c}{2\delta}\sqrt{\frac{c^2}{4} + \frac{\delta}{\beta}} + \log\left(\frac{c}{2} + \sqrt{\frac{c^2}{4} + \frac{\delta}{\beta}} \right) \\ &< \frac{3}{2} + c - \sqrt{c^2+2c} + \log \frac{1+c+\sqrt{c^2+2c}}{2}. \end{aligned} \tag{5.23}$$

One can choose the parameters of the problem such that (5.23) is satisfied.

Suppose for instance that $\delta = 3\beta$ and $c = 1$. Under these assumptions, (5.23) reduces to the condition

$$\frac{1-\log 2}{\chi} + \frac{\alpha\pi^2}{4\delta a^2} < \frac{7}{3} - \sqrt{3} + \log\frac{2+\sqrt{3}}{2} - \frac{1}{6}\sqrt{\frac{13}{4}} - \log\left(\frac{1}{2} + \sqrt{\frac{13}{4}}\right) \simeq 0.09,$$

so that (5.22) is fulfilled provided χ and δ are chosen sufficiently large.

Since we know that the values of the Lyapunov function F decrease along the evolution of the solution (U, V), this proves that the limit steady state (U^*, V^*) cannot be equal to the trivial stationary state $(1, C_V)$, so that we must have convergence of the solution to a non-constant stationary state. □

Remark: The example, which proves the existence of a non-trivial stationary solution, was chosen to be one-dimensional for simplicity. However, we know that by continuity we obtain the same situation in a neighbourhood of the initial values (5.15).

Appendix A

Existence Theorem in Higher Dimensions

As our proof of existence in Section 2.1 only holds in two space dimensions but we have also discussed properties of solutions for $n > 2$, we are going to demonstrate that our system fits into a more general set of equations, for which Amann [1] proves existence of solutions in $\mathbb{R}^n$.

So, let $\Omega \subset \mathbb{R}^n$ be a smooth domain with boundary $\partial\Omega$. Setting $w := (U, V) \in \mathbb{R}^2$, we can write our chemotaxis equations (1.6), (1.7) in the following form:

$$\begin{aligned} \partial_t w + \mathcal{A}(w)w &= f(w) \quad \text{in } (0,T)\times\Omega \\ \mathcal{B}w &= 0 \quad \text{on } (0,T)\times\partial\Omega \\ w(0,\cdot) &= (U_0, V_0) \quad \text{in } \Omega, \end{aligned} \tag{A.1}$$

where the operators $\mathcal{A}$ and $\mathcal{B}$ from D_0 (a non-empty open subset of $\mathbb{R}^2$) to $\mathcal{L}(\mathbb{R}^2)$ are defined as

$$\mathcal{A}(\eta) := -\partial_j(a_{jk}(\eta)\partial_k w)$$

with

$$\begin{aligned} a_{11}(\eta)(y_1, y_2) &= a_{22}(\eta)(y_1, y_2) := (y_1 - \eta_1 S'(\eta_2)y_2, \alpha y_2), \\ a_{12}(\eta)(y_1, y_2) &= a_{21}(\eta)(y_1, y_2) := (0, 0), \end{aligned}$$

for all $y \in \mathbb{R}^2$, $\eta \in D_0$ and

$$\mathcal{B}(\eta)w := \nu^j a_{jk}(\eta)\partial_k w.$$

The production rate is

$$f(\eta) := (0, \delta\eta_1 S'(\eta_2) - \beta\eta_2).$$

The operators $\mathcal{A}$ and $\mathcal{B}$ and the right hand side f possess the regularity required in Amann [1], Section 14 and we can apply Theorem 14.6: There exists a classical solution w of (A.1), that is,

$$w \in C([0,T) \times \bar{\Omega}, D_0) \cap C^{1,2}((0,T) \times \bar{\Omega}, I\!R^2)$$

and w satisfies (A.1) pointwise.

As all our sensitivity functions are infinitely often differentiable, the coefficients a_{jk} are in $C^\infty(D_0, \mathcal{L}(I\!R^2))$ and $f \in C^\infty(I\!R^2)$. It now follows from Corollary 14.7 in [1] that w is in $C^\infty((0,T) \times \bar{\Omega}, I\!R^2)$.

Appendix B

A Technical Lemma

Lemma B.1 *$\Omega \subset \mathbb{R}^2$ be a Lipschitz-domain. Let $r = 1$ or $r = 2$ and $1 + r \leq q < +\infty$. Then, for any $\varepsilon > 0$, there exists a positive constant k_1, depending on ε such that $k_1 \to +\infty$ as $\varepsilon \to 0$, and*

$$\begin{aligned} \|w\|^q_{L^q(\Omega)} \leq{}& \varepsilon\ \|\nabla w\|^{(q-r)}_{L^2(\Omega)}\ \|w^r \log|w|^r\ \|_{L^1(\Omega)} + \varepsilon\ \|w\|^{(q-r)}_{L^r(\Omega)}\ \|w^r \log|w|^r\ \|_{L^1(\Omega)} \\ &+k_1(\varepsilon)\ \|w\|^r_{L^r(\Omega)} \end{aligned}$$

holds for any $w \in H^1(\Omega)$.

Proof: Consider a number $N > 1$ and the function

$$h(s) = \begin{cases} 0 & \text{for} \quad |s| \leq N \\ 2(|s| - N) & \text{for} \quad N < |s| \leq 2N \\ |s| & \text{for} \quad 2N < |s| \end{cases}$$

Then, on one hand,

$$\begin{aligned} \|\ |w| - h(w)\|^q_{L^q(\Omega)} &= \int\limits_{\Omega\cap\{|w|\leq N\}} |w|^q dx\ + \int\limits_{\Omega\cap\{N<|w|\leq 2N\}} (2N - |w|)^q dx \\ &\leq \int\limits_{\Omega\cap\{|w|\leq 2N\}} |w|^q dx\ \leq \int\limits_{\Omega\cap\{|w|\leq 2N\}} (2N)^{(q-r)}|w|^r dx \\ &= (2N)^{(q-r)}\|w\|^r_{L^r(\Omega)}. \end{aligned} \tag{B.1}$$

and

$$\begin{aligned} \|h(w)\|^r_{L^r(\Omega)} &= \int\limits_{\Omega\cap\{N<|w|\leq 2N\}} 2^r(|w| - N)^r dx\ + \int\limits_{\Omega\cap\{|w|>2N\}} |w|^r dx \\ &\leq \int\limits_{\Omega\cap\{|w|>N\}} |w|^r dx\ \leq\ (\log N^r)^{-1} \int\limits_{\Omega} |w|^r \log|w|^r dx \\ &= (\log N^r)^{-1}\|w^r \log|w|^r\ \|_{L^1(\Omega)}. \end{aligned} \tag{B.2}$$

On the other hand, the H^1-norm of $h(w)$ can be estimated as follows:

$$\begin{aligned}\|h(w)\|_{H^1(\Omega)} &= \left(\|\nabla h(w)\|^2_{L^2(\Omega)} + \|h(w)\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} = \left(\|h'(w)\nabla w\|^2_{L^2(\Omega)} + \|h(w)\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} \\ &\leq \left(4\ \|\nabla w\|^2_{L^2(\Omega)} + \|w\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} \leq 2\ \|w\|_{H^1(\Omega)}. \end{aligned} \tag{B.3}$$

Using (B.1) and the Gagliardo-Nirenberg Inequality in the second estimation and (B.2) and (B.3) in the third, we obtain

$$\begin{aligned}\|w\|^q_{L^q(\Omega)} &\leq 2^{(q-1)}(\|\ |w| - h(w)\|^q_{L^q(\Omega)} + \|h(w)\|^q_{L^q(\Omega)}) \\ &\leq 2^{(q-1)}(2N)^{(q-r)}\|w\|^r_{L^r(\Omega)} + 2^{(q-1)}C\ \|h(w)\|^{(q-r)}_{H^1(\Omega)}\ \|h(w)\|^r_{L^r(\Omega)} \\ &\leq (4N)^{(q-1)}\|w\|^r_{L^r(\Omega)} + 2^{(q-1)}2^{(q-r)}C\|w\|^{(q-r)}_{H^1(\Omega)}(\log N^r)^{-1}\|w^r\log|w|^r\ \|_{L^1(\Omega)} \\ &\leq (4N)^{(q-1)}\|w\|^r_{L^r(\Omega)} \\ &\quad +8^{(q-1)}C\ (\|\nabla w\|^{(q-r)}_{L^2(\Omega)} + \|w\|^{(q-r)}_{L^r(\Omega)})\ (\log N^r)^{-1}\|w^r\log|w|^r\ \|_{L^1(\Omega)}.\end{aligned}$$

Choosing now $N^r := \exp\left(\dfrac{8^{(q-1)}C}{\varepsilon}\right)$ gives

$$\begin{aligned}\|w\|^q_{L^q(\Omega)} &\leq \left[4^r\exp\left(\frac{8^{(q-1)}C}{\varepsilon}\right)\right]^{\frac{(q-1)}{r}}\|w\|^r_{L^r(\Omega)} + \varepsilon\ \|\nabla w\|^{(q-r)}_{L^2(\Omega)}\ \|w^r\log|w|^r\ \|_{L^1(\Omega)} \\ &\quad +\varepsilon\ \|w\|^{(q-r)}_{L^r(\Omega)}\ \|w^r\log|w|^r\ \|_{L^1(\Omega)},\end{aligned}$$

which is the claim of the lemma with $k_1(\varepsilon) = \left[4^r\exp\left(\dfrac{8^{(q-1)}C}{\varepsilon}\right)\right]^{\frac{(q-1)}{r}}$. □

Remark: The Lemma is a generalization of a result by Biler et al. [2], where on a smooth $\Omega \subset I\!R^2$ it was shown that for every $\varepsilon > 0$ there exists a constant C_ε such that

$$\|w\|^3_{L^3(\Omega)} \leq \varepsilon\|w\|^2_{H^1(\Omega)}\|w\log|w|\|_{L^1(\Omega)} + C_\varepsilon\|w\|_{L^1(\Omega)},$$

for all $w \in H^1(\Omega)$.

B.1 Corollary for Functions in the Orlicz Space $L^\Phi(\Omega)$

We want to apply Lemma B.1 to a function U in the Orlicz space $L^\Phi(\Omega)$ for the Young function $\Phi(s) = (1+s)\log(1+s) - s$. Before doing so, we will sum up the definition of the space as well as its needed properties.

We will work with the following functions, which are complementary in the sense of Young:

$$\begin{aligned} \Phi(s) &= (1+s)\log(1+s) - s \quad \text{for } s \geq 0 \\ \Psi(t) &= e^t - t - 1 \quad \left(= \sum_{i=2}^{\infty} \frac{t^i}{i!} \right) \quad \text{for } t \geq 0. \end{aligned}$$

For $\Omega \subset I\!R^2$, a bounded Lipschitz domain, we define the Orlicz spaces corresponding to Φ and Ψ, respectively:

$$L^\Phi(\Omega) := \{g \in L^1(\Omega) : \|g\|_{L^\Phi(\Omega)} < +\infty\},$$

$$L^\Psi(\Omega) := \{h \in L^1(\Omega) : \|h\|_{L^\Psi(\Omega)} < +\infty\},$$

where the norms are defined in the following way

$$\begin{aligned} \|g\|_{L^\Phi(\Omega)} &:= \sup_{h \in L^\Psi(\Omega)} \left\{ \left| \int_\Omega g h dx \right| : \int_\Omega \Psi(|h(x)|) dx \leq 1 \right\}, \\ \|h\|_{L^\Psi(\Omega)} &:= \inf \left\{ k > 0 : \int_\Omega \Psi\left(\frac{1}{k} |h(x)| \right) \leq 1 \right\}. \end{aligned}$$

For $p > 1$, we have $L^p(\Omega) \overset{d}{\hookrightarrow} L^\Phi(\Omega) \overset{d}{\hookrightarrow} L^1(\Omega)$. Moreover, we want to use that

$$L^\Phi(\Omega) = \{g \in L^1(\Omega) : g \log|g| \in L^1(\Omega)\}.$$

In order to show this relation, we will need that

$$L^\Phi(\Omega) = \{g \in L^1(\Omega) : \Phi(|g(\cdot)|) \in L^1(\Omega)\},^\dagger$$

i.e., $L^\Phi(\Omega) = \{g \in L^1(\Omega) : g\log(1+|g|) \in L^1(\Omega)\}$. Suppose now that $g \in L^\Phi(\Omega)$. Since

$$|g| \log|g| < |g| \log(1+|g|),$$

by the monotonicity of the logarithm, and since $|g|\log|g|$ is bounded from below by $-\frac{1}{e}$, it follows that

$$|g \log|g|| < |g| \log(1+|g|) + \frac{1}{e},$$

†See Kufner, John and Fučík [23].

so that we obtain $g \log |g| \in L^1(\Omega)$.

Let now $g \in L^1(\Omega)$ with $g \log |g| \in L^1(\Omega)$. From

$$\log(1 + |g|) \; < \; \log |g| \; + \; 1,$$

if $|g| \geq 1$, and

$$|g| \log(|g| + 1) \; < \; \log 2,$$

if $0 \leq |g| \leq 1$, we obtain

$$|g| \log(1 + |g|) \; < \; |g \log |g|| \; + \; |g| + \log 2.$$

Hence $g \log(1 + |g|) \in L^1(\Omega)$ follows.

Furthermore, we are going to use the extended Hölder Inequality for general complementary Young functions $\tilde{\Phi}$ and $\tilde{\Psi}$:

$$\left| \int_\Omega g f \, dx \right| \; \leq \; \|g\|_{L^{\tilde{\Phi}}(\Omega)} \|f\|_{L^{\tilde{\Psi}}(\Omega)} \tag{B.4}$$

for all $g \in L^{\tilde{\Phi}}(\Omega), \; f \in L^{\tilde{\Psi}}(\Omega)$.

For more details on Orlicz spaces see Kufner, John and Fučík [23].

Corollary B.2 *If U belongs to the Orlicz space $L^\Phi(\Omega)$, then there exists for every $\kappa > 0$ a constant $C = C(\kappa, \|U \log U\|_{L^1(\Omega)})$, such that*

$$\|U\|^2_{L^2(\Omega)} \; \leq \; \kappa \, \|\nabla \sqrt{U}\|^2_{L^2(\Omega)} \; + \; C.$$

Proof: Applying Lemma B.1 with $r = 2$ and $q = 4$ yields for any $\varepsilon > 0$

$$\begin{aligned} \|w\|^4_{L^4(\Omega)} \; \leq \; & \varepsilon \, \|\nabla w\|^2_{L^2(\Omega)} \, \|w^2 \log w^2 \, \|_{L^1(\Omega)} + \varepsilon \, \|w\|^2_{L^2(\Omega)} \, \|w^2 \log w^2 \, \|_{L^1(\Omega)} \\ & + \, k_1(\varepsilon) \, \|w\|^2_{L^2(\Omega)}. \end{aligned}$$

Taking $w = \sqrt{U}$ and using that

$$\|U\|_{L^1(\Omega)} \; \leq \; C \left(\|U \log U\|_{L^1(\Omega)} + 1 \right) \; \leq \; C$$

by $U \in L^\Phi(\Omega)$, we obtain

$$\begin{aligned} \|U\|^2_{L^2(\Omega)} \; & \leq \; \varepsilon \, \|\nabla \sqrt{U}\|^2_{L^2(\Omega)} \|U \log U\|_{L^1(\Omega)} + \varepsilon \|U\|_{L^1(\Omega)} \|U \log U\|_{L^1(\Omega)} + k_1(\varepsilon) \, \|U\|_{L^1(\Omega)} \\ & \leq \; \varepsilon \, C(\|U \log U\|_{L^1}) \, \|\nabla \sqrt{U}\|^2_{L^2(\Omega)} + C(\varepsilon). \end{aligned}$$

Choosing $\varepsilon \; \leq \; \kappa \, C(\|U \log U\|_{L^1})^{-1}$, gives the assertion. □

Appendix C

Reduction of a Three-Species-System to a Two-Species-System

Inspired by an argument by Merz [27], which was applied to electro-reaction-diffusion equations as for instance the semiconductor equations (see also Hünlich and Glitzky [20]), we want to give another motivation for the form of the production term $\delta US'(V)$ in our system of equations (1.6). Starting from a three-species-system, distinguishing between two states of different sensitivity the amoebae can assume, we will derive system (1.6) under the condition of fast exchange between these states. Still being forced to make different assumptions in the line of the argument, we do not present the discussion as a rigorous explanation but simply as an offer of additional motivation. At any rate, the argument makes clear once more that the equations of chemotaxis and semiconductor equations are special cases of a more general system of equations, admitting positive *and* negative values for the coefficient in the drift term, and that both systems can be treated in a similar manner.

We will start from a three-species-system where we distinguish between two possible states U_1 and U_2 for the amoebae, in which the cells react less and more sensitively, respectively, to the concentration of the chemical substance V. Assuming that the amoebae can change from one state to the other, we have to add reaction terms to the equations, which will be chosen to be of the form

$$R_i := (-1)^{i+1} \left[k_1 U_1 Q_1(V) - k_2 U_2 Q_2(V)\right] \quad \text{for} \quad i = 1, 2.$$

The k_i are positive reaction coefficients and the V-dependence of the functions Q_i will be specified at a later stage.

Assuming that the differing sensitivity of the cells in the states U_1 and U_2 is reflected in their chemotactic behaviour as well as in their inclination to produce the chemical V, we model the phenomenon as follows

$$
\begin{aligned}
(U_1)_t &= \Delta U_1 - \nabla(\chi_1 U_1 \nabla V) + R_1 \\
(U_2)_t &= \Delta U_2 - \nabla(\chi_2 U_2 \nabla V) + R_2 \\
V_t &= \alpha \Delta V - \beta V + \delta(\chi_1 U_1 + \chi_2 U_2)
\end{aligned} \tag{C.1}
$$

in $(0, T) \times \Omega$, where $0 \leq \chi_1 < \chi_2$ are the constant chemotactic coefficients for the two possible states of sensitivity for the amoebae and all other coefficients are positive constants, too. We again take homogeneous Neumann boundary conditions:

$$
\nu \cdot \nabla U_1 \;=\; \nu \cdot \nabla U_2 \;=\; \nu \cdot \nabla V \;=\; 0 \;\text{ on } (0, T) \times \partial\Omega.
$$

Assuming now that the oscillation of the amoebae between the two states U_1 and U_2 is significantly faster than the remaining processes, this exchange reaction can be assumed to attain an equilibrium, so that the reaction terms R_i vanish for $i = 1, 2$, and we can consider the sum of the amoebae

$$
U := U_1 + U_2. \tag{C.2}
$$

From $R_i = 0$, we obtain the relation $U_2 = Q(V)U_1$, where we have set $Q(V) = \dfrac{k_1 Q_1(V)}{k_2 Q_2(V)}$, so that we obtain via (C.2) by $U = U_1 + U_2 = U_1(1 + Q(V))$ that

$$
U_1 \;=\; \frac{U}{1 + Q(V)} \quad \text{and} \quad U_2 \;=\; \frac{Q(V)U}{1 + Q(V)}. \tag{C.3}
$$

Adding the first two equations in system (C.1) and using the relations (C.2) and (C.3), we obtain

$$
\begin{aligned}
U_t &= \Delta U - \nabla\left(\left[\frac{\chi_1}{1 + Q(V)} + \frac{\chi_2 Q(V)}{1 + Q(V)}\right] U \nabla V\right) \\
V_t &= \alpha \Delta V - \beta V + \delta U \left(\frac{\chi_1}{1 + Q(V)} + \frac{\chi_2 Q(V)}{1 + Q(V)}\right).
\end{aligned} \tag{C.4}
$$

We see that we have obtained a system with the same function of V both in the chemotactic and in the V-production term. In our system (1.6) we wrote $\chi S'(V)$ in that place. To obtain here this form of the function for different sensitivity functions, where χ would have to be an intermediate chemotactic coefficient for the cells $\chi_1 < \chi < \chi_2$, we only have to choose $Q(V)$ appropriately.

Let us assume that the amoebae are insensitive to the V-concentration in the state U_1, that is, that $\chi_1 = 0$ and $Q_1(V) = k_1$. Taking the logarithmic sensitivity function

$S_3(V) = \log(V+1)$, we then would like to have

$$\frac{\chi}{V+1} = \frac{\chi_2 Q(V)}{1+Q(V)} = \frac{\chi \dfrac{k_1}{Q_2(V)}}{1+\dfrac{k_1}{Q_2(V)}} = \frac{k_1\chi_2}{k_1+Q_2(V)},$$

so that we need to require

$$Q_2(V) = k_1\left[\frac{\chi_2}{\chi}(1+V)-1\right].$$

For the bounded sensitivity function $S_1(V) = \dfrac{V}{1+V}$ we need

$$\frac{k_1\chi_2}{k_1+Q_2(V)} = \frac{\chi}{(1+V)^2}$$

and therefore obtain

$$Q_2(V) = k_1\left[\frac{\chi_2}{\chi}(1+V)^2-1\right].$$

Note that, in both cases, $Q_2(V) \geq 0$ for all $V \geq 0$.

Admitting additionally a positive χ_1, we are able to derive a greater variety of possible sensitivity functions.

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