Market Completion and Robust Utility Maximization

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von
Dipl. math. oec. Matthias Müller
geboren am 12.12.1972 in Dresden

Präsident der Humboldt-Universität zu Berlin:
Prof. Dr. Jürgen Mlynek

Dekan der Mathematisch-Naturwissenschaftlichen Fakultät II:
Prof. Dr. Uwe Küchler

Gutachter:

1. Prof. Dr. Peter Imkeller
2. Prof. Dr. Alexander Schied
3. Prof. Dr. Said Hamadène

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Abstract

In this thesis we study two problems of financial mathematics that are closely related. The first part proposes a method to find prices and hedging strategies for risky claims exposed to a risk factor that is not hedgeable on a financial market. In the second part we calculate the maximal utility and optimal trading strategies on incomplete markets using Backward Stochastic Differential Equations.

We consider agents with incomes exposed to a non–hedgeable *external source of risk* who *complete the market* by creating either a bond or by signing contracts. Another possibility is a risk bond issued by an insurance company. The sources of risk we think of may be insurance, weather or climate risk. Stock prices are seen as exogenuosly given. We calculate prices for the additional securities such that supply is equal to demand, the market clears *partially.* The preferences of the agents are described by expected utility. In Chapter 2 through Chapter 4 the agents use exponential utility functions, the model is placed in a Brownian filtration. In order to find the equilibrium price, we use Backward Stochastic Differential Equations. Chapter 5 provides a one–period model where the agents use utility functions satisfying the Inada condition.

The second part of this thesis considers the robust utility maximization problem of a small agent on a incomplete financial market. The model is placed in a Brownian filtration. Either the probability measure or drift and volatility of the stock price process are uncertain. The trading strategies are constrained to closed convex sets. We apply a martingale argument and solve a saddle point problem. The solution of a Backward Stochastic Differential Equation describes the maximizing trading strategy as well as the probability measure that is used in the evaluation of the robust utility. We consider the exponential, the power and the logarithmic utility functions. For the exponential utility function we calculate utility indifference prices of not perfectly hedgeable claims.

Finally, we apply those techniques to the maximization of the expected utility with respect to a single probability measure. We apply a martingale argument and solve maximization problems instead of saddle point problems. This allows us to consider closed, in general non–convex constraints on the values of trading strategies.
Keywords:
market completion, incomplete financial market, utility maximization, backward stochastic differential equations

Zusammenfassung

In dieser Arbeit studieren wir zwei Probleme der Finanzmathematik, die eng zusammenhängen. Der erste Teil beschreibt eine Methode, Auszahlungen zu bewerten, die einem auf dem Finanzmarkt nicht absicherbaren Risiken ausgesetzt sind. Im zweiten Teil berechnen wir den maximalen Nutzen und optimale Handelsstrategien auf unvollständigen Märkten mit Hilfe von stochastischen Rückwärtsgleichungen.

Wir betrachten Händler, deren Einkommen einer externen Risikoquelle ausgesetzt sind. Diese vervollständigen den Markt, indem sie entweder einen Bond schaffen oder gegenseitig Verträge schließen. Eine andere Möglichkeit ist eine Anleihe, die von einer Versicherung herausgegeben wird. Die Risikoquellen, die wir in Betracht ziehen, können Versicherungs-, Wetter– oder Klimarisiko sein. Aktienpreise sind exogen gegeben. Wir berechnen Preise für die zusätzlichen Anlagen so dass Angebot und Nachfrage dafür gleich sind. Wir haben partielle Markträumung. Die Präferenzen der Händler sind durch erwarteten Nutzen gegeben. In Kapitel 2 bis Kapitel 4 haben die Händler exponentielle Nutzenfunktionen. Um den Gleichgewichtspreis zu finden, wenden wir stochastische Rückwärtsgleichungen an. In Kapitel 5 beschreiben wir ein Einperiodenmodell, wobei die Händler Nutzenfunktionen verwenden, die die Inada-Bedingungen erfüllen.

Der zweite Teil dieser Arbeit beschäftigt sich mit dem robusten Nutzenmaximierungsproblem eines kleinen Händlers auf einem unvollständigen Finanzmarkt. Entweder das Wahrscheinlichkeitsmaß oder die Koeffizienten des Aktienmarktes sind ungewiss. Die Handelsstrategien sind auf abgeschlossene konvexe Mengen beschränkt.Wir wenden ein Martingalargument an und lösen Sattelpunktprobleme. Die Lösung der Rückwärtsgleichung beschreibt die nutzenmaximierende Handelsstrategie und das Wahrscheinlichkeitsmaß, das in der Auswertung des robusten Nutzens benutzt wird. Für die exponentielle Nutzenfunktion berechnen wir Nutzenindifferenzpreise für nicht absicherbare Auszahlungen.

Ausserdem wenden wir diese Techniken auf die Maximierung des erwarteten Nutzens bezüglich eines Wahrscheinlichkeitsmaßes an. Wir nutzen ein Martingalargument und lösen Maximierungsprobleme anstelle von Sattelpunktproblemen. Dies erlaubt uns, abgeschlossene, im allgemeinen nicht konvexe zulässige Mengen für die Handelsstrategien zu betrachten.

Schlagwörter:
Marktvervollständigung, unvollständige Finanzmärkte,
Nutzenmaximierung, stochastische Rückwärtsgleichungen

Contents

Chapter 1

Introduction

1.1 Part I Market completion

Pricing and hedging of options on stocks is well understood. The famous Black– Scholes formula gives the price of a call option. Using martingale methods relying on the reqirement that the option does not create an opportunity of arbitrage, price processes and the hedging strategy of options written on stocks in a complete market can be calculated.

In recent years, new types of financial products have appeared. Insurances aim at transferring insurance risk to financial markets. This is done by *securitization*, a security, e.g. a bond, is created that depends on a non financial risk factor. Those securities are often called *CAT bonds*. The best known example are earthquake bonds for California. No insurance company is willing to take a large part of the earthquake risk because the loss potential is too high. Instead, earthquake bonds are sold to large banks or hedge funds. If an earthquake occurs the investors are not repayed. They even loose the principal of the bond. There are also CAT bonds covering the risk of hurricanes. Many examples are given in the article "Economic aspects of securitization of risk" by Cox et al., (CFP00).

An example of a security on weather risk is the Heating Degree Day (HDD) swap. This paper is traded at the Chicago Merkantile Exchange. The payoff of a HDD swap depends on the temperature during a heating period. If the temperature is higher than usual, the buyer has to pay to the seller. If the temperature is lower, the seller pays. The swap is ideal for energy producers. They can hedge volume risk, the volume of energy sold depends on the temperature. The Winthertur insurance issued a bond that transfers the risk of hailstorms. Structure and pricing of this bond are described in Schmock, (Sch99).

Another example is given by the risk a reinsurance company faces due to big accumulative losses for example in farming or fishing caused by the most well known short term climate event of the El Niño Southern Oscillation (ENSO).

All those securities have in common that their payoff depends on non–financial, i.e. *external risk factors*. Those risks cannot be hedged on a financial market. How should a claim be priced that depends on external risk? How does the price process of a security on an extenal risk factor evolve? Here we sketch some pricing methods described in the literature of financial mathematics.

One technique to construct prices and hedging strategies in incomplete markets comes from a *utility indifference* argument. The trader uses the trading strategy that maximizes the expected utility of the terminal wealth attained with the trading strategy minus the claim he has to pay at a certain time. The utility indifference price is given by the adjustment of the initial capital such that the maximal utility is the same as with the not adjusted initial capital without the liability. This means the trader is indifferent between either getting the price and accepting the obligation to pay or doing nothing. The utility indifference argument also yields a hedging strategy. For exponential utility functions, utility indifference prices are calculated in Becherer, (Bec01) and Delbaen et el. (Del03).

A very closely related pricing principle is the result of an infinitesimal indifference argument. The price of the nonhedgeable claim is chosen such that the trader is indifferent between either accepting an infinitesimal small part of the claim or doing nothing. Davis (Dav01) used this argument to price a temperature bond.

Both types of utility indifference arguments take either the point of view of a buyer or a seller. The preferences of only one trader are taken into account. The *quadratic hedging* approach sees the price from the perspective of the buyer and the seller simultanuosly. The expectation of the square of the difference between the terminal value of a trading strategy and the claim is taken. This quantity is minimized over all trading strategies. One can compare this functional over different initial capitals. The initial capital such that this functional is minimized is the price for the claim. Since gains and losses both are punished, the price can be seen as a compromise between a buyer and a seller. A survey can be found in the article of Schweizer, (Sch01). Møller (Møl01) uses this approach in order to price insurance contracts.

A fundamentally different approach are *equilibrium prices*. Karatzas et al. (KL90) consider agents who obtain a random income. The model is placed in a Brownian filtration. The agents construct securities in zero net supply such that they have a complete market. They trade those securities in order to find the trading strategy that maximizes their utility from consumption. Then the prices of the securities and the interest rate are chosen such that all trading strategies add up to zero. This equilibrium is called Arrow– Debreu equilibrium.

Barrieu (Bar02) considers the problem of *security design*. An insurance company intends to transfer some of its insured risk to an investor. The security is constructed such that the utility of the insurance is maximal under the constraint that the investor buys it. This means the structure and the price of the security is chosen such that the utility of the investor does not grow smaller if he buys the security. However, the role of the investor is passive. Institutional investors who are aiming at maximizing their profit might not be content with this situation. The investor should also have the possibility to maximize his utility.

A survey article about security design is Duffie, Rahi (DR95).

We aim at finding methods that allow pricing and hedging of claims that depend on both financial and external risk. The techniques for incomplete markets we have seen so far might lead to results that are not very useful for us because the stock market and the external risk factor are independent or not closely related. Hedging the external risk on the stock market alone is not enough. The Arrow– Debreu equilibrium on the other hand sees all securities as equal. All agents with their risk exposure have to be modeled and the price for every security is the result of the equilibrium of supply and demand. This approach is not perfectly suitable for our problem. The size of the stock market and the market for securities on external risk is very different. Furthermore, our goal is to explain prices for the external risk whereas stock prices are the result of trading at the stock markets that we consider as exogenuosly given. So the first task in this thesis is the choice of an appropriate economic model.

We propose an *equilibrium with partial market clearing*. Our model considers a group of agents with incomes affected by both financial and external, non– financial risk. Since the external risk is not tradeable on the stock market, the agents interested in trading this risk create a market for it. The agents may trade this risk among themselves. In our model, they *complete the market*. This is done either by creating an additional security (risk bond) or by signing mutual contracts. Given the stock price and a price of external risk, the agents choose the claim that maximizes their expected utility among all claims they can afford. This is done by trading with the stock and either by buying and selling the risk security or by contracts.

In order to achieve our equilibrium with partial market clearing, the price for the risk bonds and contracts on claims containing external risk is adjusted such that supply and demand are equal. The difference to the usual equilibrium is that we don't change the stock price. *The market clears only partially*, there is no clearing condition on the trading strategies with the stock. The reason is that our agents are considered as small trader on the stock market. This means, their demand is small compared to the overall volume of the stock market. The agents cannot change the stock price and they are assumed to find other traders to buy from or sell to who might not belong to the group of agents considered here. Thus stock prices are exogenuosly given and we don't require market clearing for trading with the stock within our group of agents.

In Chapter 2 through Chapter 4, our agents use the exponential utility function with an individual coefficient of relative risk aversion. We place ourselves in a Brownian framework. Equilibrium prices are obtained by the solution of a Backward Stochastic Differential Equation (BSDE).

In Chapter 2, the market is completed by a security in zero net supply that is traded continuously during the whole trading time. Since the external risk is described by a one dimensional Brownian motion, one additional security is enough to complete the market. We find a condition on drift and volatility of the price process of the risk security such that the market clears partially.

Chapter 3 considers the case of a more complicated external risk described by a finite dimensional Brownian motion. In that case the investors sign mutual contracts. The price of such a contract is calculated using a probability measure that is equivalent to the reference measure. Such a measure is called *pricing measure*. Since the price of financial risk cannot be changed, a pricing measure has to be choosen from the set of equivalent martingale measures for the stock. The equilibrium is atteined by adjusting the pricing measure.

In Chapter 4 an insurance company sells a *risk bond* in order to transfer some of its insured risk to the agents who are willing to trade it. We use the term risk bond because this security is not in zero net supply. The insurance company is interested in selling a claim to the agents on the market. A feedback of the interest rate payed by the insurance from the price of external risk on the market as well as a dependence on the external risk factor are possible. Partial market clearing means here that the demand for the bond is equal to the supply provided by the insurance. In contrast to Chapter 2, the terminal value, i.e. the payout of the bond, is specified. A candidate of a price process is given by the successive conditional expectations of the terminal value with respect to a martingale measure of the stock. We provide a criterion that characterizes the completeness of the market under the equilibrium price as well as a simple example for a risk bond completing the market.

Chapter 5 considers an abstract one– period model where the probability space $(\Omega, \mathcal{F}, P)$ is placed in a Borel space. The utility functions of the agents are allowed to be other than exponential. The incomes are modeled as random variables. An abstract stock market is represented by a sub $\sigma-$ algebra $\mathcal{G}$. All random variables measurable with respect to $\mathcal{G}$ are tradeable. On the other hand, the pricing measure is already fixed on $\mathcal{G}$. The agents complete the market using contracts. Partial market clearing is defined as in Chapter 3.

Chapter 6 finally considers an equilibrium model in an incomplete market. In contrast to the previous chapters, the traders do not complete the market. They are only willing to trade claims that are measurable with respect to a $\sigma-$ algebra $\mathcal{T}$, whereas the incomes might depend on a larger $\sigma-$ algebra. An interpretation for this fact is that the agents only trade claims that depend on observable factors in order to exclude moral hazard. In this chapter we use the usual equilibrium idea without the additional stock market.

Chapter 2 and Chapter 3 are published in Hu, Imkeller and Müller (HIM04a).

In the paper (CIM04), the pricing method presented here is applied to a simple model of climate risk, a particularly interesting external risk source. Numerical methods are developed based on the well known correspondence between non-linear BSDE and viscosity solutions of quasi-linear PDE to simulate optimal wealth and strategies of individual agents participating in the market. We focus on two or three agents exposed to the climate phenomenon of ENSO.

1.2 Part II Robust utility maximization

An investor on a financial market is interested in having an optimal wealth at a fixed time T. The investor may represent a company that has to report to its shareholders at that time. Which criterion describes optimality? This depends on the preferences of the investor. We use two concepts of preferences on random claims in this thesis: the expected utility with respect to a fixed probability and on the other hand the robust utility. The latter is the infimum of the expected utilities of a random claim over a whole set of probability measures.

In this thesis we calculate the optimal self financing trading strategy in an incomplete market for both types of preferences. Self financing means that the investor does not take money out or invests new money within the trading interval. He invests some initial capital. The wealth of the investor changes only due to gains or losses by trading with the stock. We consider the exponential, the power and the logarithmic utility functions. In the case of the exponential utility, the investor may hedge a liability that he has to pay out at the end of the trading time.

Here we describe and compare the robust and the usual utility maximization. We follow closely Section 2.5 in the book of F"ollmer and Schied (FS02) in our presentation. A random variable that represents the terminal wealth of a trading strategy is interpreted as a function which associates a real number to each scenario, i. e. a measurable function X_T on some measure space $(\Omega, \mathcal{F})$. Denote with $\mathcal{X}$ the set of all claims considered. A preference can be seen as a binary relation that is asymmetric and negatively transitive (see Definition 2.1 in (FS02)).

L. J. Savage (Sav54) introduced a set of axioms which guarantees that the preference relation can be represented in the form

$$U(X_T) = E_Q[u(X_T)] = \int u(X_T(\omega))Q(d\omega), \quad X_T \in \mathcal{X} \tag{1.1}$$

with a probability measure Q on $(\Omega, \mathcal{F})$ and a function $u : \mathbb{R} \to \mathbb{R}$. Of course, if $U(X_T^1) > U(X_T^2)$ for $X_T^1, X_T^2 \in \mathcal{X}$, then X_T^1 is preferred. The probability Q is determined by the preference relation and can differ from an "objective" probability measure. Thus, a "real world" measure might be distorted towards a more pessimistic or optimistic view. Usually, investors prefer higher claims and are risk averse. This leads to a growing and concave function u that is called *utility function*.

However, some very intuitive preferences cannot be written in a Savage representation. Investors are not only averse against risk but also against *uncertainty*. A very instructive example for uncertainty is the *Ellsberg paradox* (see e.g. Example 2.81 in (FS02)). A player is faced with the following problem: there are two urns, each containing 100 balls which are either red or black. The player knows that in the first urn there are 51 red and 49 black balls. The proportion of red and black balls in the second urn is unknown. Suppose that the player gets 1000 $ if he draws a red ball and 0 $ for a black ball. The player may choose between two random claims, one with a known and one with a completely unknown distribution. The typical decision is to draw from the first urn. On the other hand, if the player gets 1000 $ for a black ball and nothing for a red ball, he usually also draws from the first urn. If the player draws from the first urn, he is exposed to risk. A probability measure is fixed that describes the model. The second urn is different. The player has no information. It is impossible to find an "objective" probability measure for this urn. Such a situation is called *uncertainty*. Choosing the first urn even if the probability to win is less than $0,5$ is due to *uncertainty aversion*. The choices of the player define a preference relation. Describing this relation with a Savage representation would mean that we have to find one subjective probability measure for the second urn such that in both cases drawing from the first urn yields a higher expected utility. This is impossible.

Instead of taking only a single measure Q, the *robust Savage representation* considers a whole set $\mathcal{Q}$ of probability measures on $(\Omega, \mathcal{F})$. The representation is

$$U(X_T) = \inf_{Q \in \mathcal{Q}} E_Q[u(X_T)], \quad X_T \in \mathcal{X}. \tag{1.2}$$

The investor sees a whole set of probabilistic views as reasonable and takes a worst case approach in evaluating the expected utility of a given claim. The preference relation in the Ellsberg paradox can be represented in this form. Let p_r be the lowest probability to draw a red ball in the first urn for which the player chooses the first urn in both games. The set $\mathcal{Q}$ consists of all probability measures that agree with the information about the first urn and assigns the probability for a red ball in the second urn between p_r and $1 - p_r$.

Another type of uncertainty appears if coefficients of a stock price process are not exactly known. Drift and volatility might be the result of a statistic estimate that yields only a confidence interval. The robust utility of the terminal wealth of a trading strategy is calculated in the following way: compare the expected utilities for all possible processes of coefficients. The infimum is the robust utility. The expectation is taken with respect to a reference probability measure.

Schied (Sch04b) considers the robust utility maximization problem on a complete market. The price process of the stocks is assumed to be a semimartingale with respect to a probability P. Completeness means that there exists a unique probability $P^* \sim P$ under which S is a local martingale. The investor has an initial capital but no terminal liability. Schied proves a duality result under the assumption that a so called *least favorable measure* $Q_0 \sim P^*$ exists. The least favorable measure with respect to P^* is defined as the probability Q_0 in $\mathcal{Q}$ that satisfies

$$Q_0 \left[\frac{dP^*}{dQ_0} \leq x \right] = \inf_{Q \in \mathcal{Q}} Q \left[\frac{dP^*}{dQ_0} \leq x \right] \quad \text{for all } x > 0.$$

If this least favorable measure exists, (Sch04b) shows that for every growing, strictly concave utility function $u : (0, \infty) \to \mathbb{R}$, the robust utility maximization is equivalent to the utility maximization with respect to Q_0. Schied gives examples and characterizations of the least favorable measure. The model in (Sch04b) that is the most interesting for this thesis is the following: the stock prices are driven by a m–dimensional Brownian motion W under a reference probability measure:

$$dS_t^i = S_t^i(\sum_{j=1}^{d} \sigma_t^{i,j} dW_t^j + b_t^i dt), \quad i = 1, \dots, m.$$

The investor is uncertain about the drift b: any drift is possible that is adapted to the filtration generated by W and satisfies $b_t \in C_t$, where C_t is a nonrandom time– dependent bounded closed subset of $\mathbb{R}^m$. Then the set $\mathcal{Q}$ of probability measures in the robust Savage representation are all probability measures such that S has a drift with this properties. The volatility matrix is deterministic and has full rank. Let α_t^0 be the element in C_t that minimizes the norm $|\sigma_t^{-1} b_t|$. If both α_t^0 and σ_t are continuous, Proposition 3.2 in(Sch04b) states that the least favorable measure is the one under which the drift is equal to α_t^0. Of course, our method gives the same result under the assumptions of (Sch04b) for the utility functions we consider.

We find a simple result in a case where the least favorable measure does not exist. Let the market be complete. We use an exponential utility function. The investor has a terminal liability F, the uncertainty lies in the probability measures, the drift is known. Then the optimal trading strategy consists of two parts: the hedging strategy for the sum of F and an additional explicitly given random variable, and the utility maximizing trading strategy under the measure in $\mathcal{Q}$ under which the drift of the stock price is minimal (see Theorem 58 on page 105).

Gundel (Gun03) provides a duality result for robust utility maximization in complete and incomplete markets using reverse f–projections. She provides a duality result in the following problem:

$$\text{maximize} \quad \inf_{Q \in \mathcal{Q}} E_Q[u(X)] \quad \text{over all} \quad X \quad \text{with} \quad \sup_{P \in \tilde{P}} E_P[X] \leq x$$

for a convex set $\tilde{P}$ of equivalent local martingale measures for the stock price process.

We consider two types of uncertainty. For the first one we use an explicitly described set of probability measures $\mathcal{Q}$ in the robust savage representation defined in (1.2). In the second approach, the coefficients of the stock price process are uncertain.

Our model is placed in the filtration generated by an $m-$ dimensional Brownian motion with respect to a probability measure P. The densities of the probability measures in $\mathcal{Q}$ with respect to P are stochastic exponentials of stochastic integrands with respect to the Brownian motion. The integrands are restricted to time dependent random predictable closed convex sets $C_t(\omega)$ of $\mathbb{R}^m$, $t \in [0, T]$. Predictability for set– valued processes is explained in

Delbaen, (Del03) page 5, or in our thesis in Remark 46 on page 92. All sets $C_t(\omega)$, $\omega \in \Omega, t \in [0,T]$ have to lie in a bounded ball around the origin. Our setup covers some *multiplicatively stable* (m-stable) sets of probability measures in the sense of Definition 1.2 in Delbaen, (Del03). Multiplicatively stable means that we take the density of a probability measure in $\mathcal{Q}$ up to a stopping time. Then we continue with the density of another probability measure in $\mathcal{Q}$ that is equivalent to the reference measure. The probability measure with the density composed in this way has also to belong to $\mathcal{Q}$. Theorem 1.4 in Delbaen (Del03) applied to a Brownian filtration states that m–stable sets of densities have the same structure as our set $\mathcal{Q}$. However, we use the additional assumption that the constraints on the integrands have to be in a bounded ball around the origin.

The stock price process in our model is the solution of a stochastic differential equation driven by a Brownian motion. In Chapter 8 the uncertainty lies in the drift and volatility of the stock price. The investor has to take into account all stock price processes where the drift and volatility process take values within a convex set during the whole trading time.

In fact, the robust utility maximization problem in Chapter 7 and Chapter 8 can be seen as a saddle point problem. The saddle point consists of the optimal trading strategy and on the other hand on a probability measure or drift of the stock price. We find the saddle point using a martingale argument. This leads to a Backward Stochastic Differential Equation (BSDE). The solution of the BSDE enables us to construct the optimal trading strategy as well as the probability measure or the drift.

The powerful tool of BSDE has been introduced to stochastic control theory by Bismut (Bis76). Its mathematical treatment in terms of stochastic analysis was initiated by Pardoux and Peng (PP90), and its particular significance for the field of utility maximization in financial stochastics clarified in El Karoui, Peng and Quenez (EKPQ97). In (Pen90), Peng proves a maximum principle for stochastic control problems that is based on BSDE.

The method we use to calculate the saddle point is a generalization of the approach used in Hu, Imkeller, Müller, (HIM04b). In this paper, we solved the problem of maximizing the expected utilty with respect to a single probability measure.

El Karoui and Hamadène (EKH03) relates the solution of a saddle point of an expectation of an exponential cost functional to a BSDE. Our saddle point problem doe not satisfy their boundedness assumptions on the cost functional.

Quenez (Que04) considers the robust utility maximization if the stock price is given by a semimartingale. Using duality methods she proves existence of a saddle point. For a Brownian filtration and a logarithmic resp. a power utility function she finds Backward Stochastic Differential Equations that describe the optimal trading strategy as well as the probability measure used in the evaluation of the robust utility. However, the coefficients of the stock price process have to be constant for the power utility. We use a direct approach that does not rely on duality methods.

Peng (Pen90) proves a maximum principle for stochastic control problems.

In Chapter 9 we consider the utility maximization with respect to one single probability measure for the exponential, power and logarithmic utility functions. In the section 9.1 about the exponential utility, the investor may have a terminal liability. We summarize the results of Hu, Imkeller and Müller (HIM04b), where the method we use has been developed. In contrast to the chapters about the robust utility maximization, we simply solve a maximization problem instead of a saddle point problem. So the constraints to the values of the trading strategy are assumed to be closed, but in general *not convex*. This direct approach allows us to find the maximizing trading strategy without duality arguments.

In a related paper, El Karoui and Rouge (EKR00) compute the value function and the optimal strategy for exponential utility by means of BSDE, assuming more restrictively that the strategies be confined to a convex cone. Sekine (Sek02) relies on a duality result obtained by Cvitanic and Karatzas (CK92), also describing constraints through convex cones. He studies the maximization problem for the exponential and power utility functions, and uses an attainability condition which solves the primal and dual problems, finally writing this condition as a BSDE. In contrast to these papers, we do not use duality, and directly characterize the solution of the primal problem. This allows us to pass from convex to closed constraints.

Utility maximization is one of the most frequent problems in financial mathematics and has been considered by numerous authors. Here are some of the milestones viewed from our perspective of maximization under constraints using the tools of BSDEs. For a complete market, utility maximization has been considered in Karatzas et al. (KL87). Cvitanic and Karatzas (CK92) prove existence and uniqueness of the solution for the utility maximization problem in a Brownian filtration constraining strategies to convex sets. There are numerous papers considering general semimartingales as stock price processes. Delbaen et al. (DGR$^+$02) give a duality result between the optimal strategy for the maximization of the exponential utility and the martingale measure minimizing the relative entropy with respect to the real world measure P. This duality can be used to characterize the utility indifference price for an option. Also relying upon duality theory, Kramkov and Schachermayer (KS99) and Cvitanic et al. (CSW01) give a fairly complete solution of the utility optimization problem on incomplete markets for a class of general utility functions not containing the exponential one. See also the review paper by Schachermayer (Sch02) for a more complete account and further references.

Part II of this thesis is organized as follows:

In Chapter 7 we solve the robust utility maximizing problem for the exponential and power utilities. The uncertainty lies in the choice of probability measures.

Chapter 8 explains the utility maximization for an uncertain drift for exponential, power utilities and logarithmic utility.

Chapter 9 gives the solution for the utility maximization problem where the expectation is taken with a single probability measure. In this case, we allow nonconvex constraints on the trading strategies.

In all three chapters, the agent may have a terminal liability if he uses the exponential utility function. In this case, we calculate the utility indifference price of the liability.

Notations

We shall use the following notations. Let Q be a probability measure on $\mathcal{F}$, $k \in \mathbb{N}, p \geq 1$. Then $L^p(Q)$ or $L^p(\Omega, \mathcal{F}, Q)$ stands for the set of equivalence classes of Q–a.s. equal $\mathcal{F}_T$–measurable random variables which are p–integrable with respect to Q.

$L^0(\Omega, \mathcal{F}, Q)$ denotes all random variables that are measurable with respect to $\mathcal{F}$ whereas $L^\infty(\Omega, \mathcal{F}, Q)$ is the set of random variables that are bounded $Q-$a.s.

$\mathcal{H}^k(Q, \mathbb{R}^d)$ denotes the set of all $\mathbb{R}^d$–valued stochastic processes ϑ that are predictable and such that $E^Q[\int_0^T \|\vartheta_t\|^k dt] < \infty$. Here and in the sequel E^Q denotes the expectation with respect to Q.

We write λ for the Lebesgue measure on $[0, T]$ or $\mathbb{R}$. $\mathcal{H}^\infty(Q, \mathbb{R}^d)$ is the set of all predictable $\mathbb{R}^d$–valued processes that are $l \otimes Q$–a.e. bounded on $[0, T] \times \Omega$.

For a continuous semimartingale M with quadratic variation $\langle M \rangle$ the stochastic exponential $\mathcal{E}(M)$ (for an adapted continuous stochastic process M) is given by

$$\mathcal{E}(M)_t = \exp(M_t - \frac{1}{2}\langle M \rangle_t), \qquad t \in [0, T].$$

Let $C \subset \mathbb{R}^n$ be closed and $x \in \mathbb{R}^n$. The distance $\text{dist}_C(x)$ is

$$\text{dist}_C(x) = \min_{y \in C} \|x - y\|,$$

where $\| \cdot \|$ denotes the Euklidian norm. The projection of x on C is the set $\Pi_C(x)$ that satisfies

$$\Pi_C(x) = \{y \in C \mid \|x - y\| = \min_{a \in C} \|x - a\|\}. \tag{1.3}$$

If C is convex, $\Pi_C(x)$ consists of one element.

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Part I

Market completion, hedging external risk factors

Chapter 2

Equilibrium with risk security

In this chapter we calculate an equilibrium with partial market clearing in a model where the randomness comes from a two dimensional Brownian motion with respect to a probability measure P. One component of the Brownian motion drives a stock price process X^S with a quotient of drift and volatility θ^S. The other component describes the external risk. Our method works also if the stock depends on both components of the Brownian motion. Every agent within a finite group obtains incomes depending on both types of risk. In order to hedge the external risk, they create a risk security that completes the market. Given a candidate of the price process, the agents trade with both stock and risk security in order to maximize the expected utility of the wealth at the end of the trading period. The agents use exponential utility functions.

In order to obtain partial market clearing, we adjust the drift and the volatility of the risk security X^E such that the trading strategies for this asset add up to zero. We consider a whole set of quotients θ^E of drift and volatility for X^E. For every $\theta = (\theta^S, \theta^E)$ we find a unique probability measure Q^θ equivalent to P such that (X^S, X^E) is a $Q^\theta-$ martingale.

Since the agents maximize the utility of the wealth at the terminal time, we may transform our equilibrium condition on the strategies into a condition on the wealth: the sum of the incomes minus the preferred terminal wealth is a payoff that is replicable at the stock market. This difference is simply the sum of the trading strategies with the stock. The problem is simplified because we don't need to calculate with the only implicitly known optimal strategies anymore.

We apply utility maximization techniques for complete markets using martingale and BSDE methods. Martingale methods are treated in (KL87), (CH89) and (Pli86). The completeness of the market leads to a budget condition: every payoff that is not more expensive than the income of an agent can be replicated. The price of a payoff is calculated as its expectation under the martingale measure Q^θ. Using the Legendre transform, the payoff maximizing the expected utility within the budget set is calculated. For the exponential utility function, this payoff depends explicitly on θ^S and θ^E.

This explicit structure of the utility maximizing terminal wealth of the agents allows us to write down a Backward Stochastic Differential Equation that characterizes the quotient of drift and volatility θ^{E*} of the equilibrium price.

This chapter is organized as follows: in section 2.1 we explain our stock market, the external risk factor and the incomes of the agents. Section 2.2 defines the set of price processes for the risk security. Additionally, admissible trading strategies for both the stock and the risk security are defined. Section 2.3 recalls the solution of the utility maximization problem in a complete market. Finally, in Section 2.4 we define our equilibrium with partial market clearing and construct the price process of the risk security that attains partial market clearing.

2.1 The stock, external risk

The mathematical frame is given by a probability space $(\Omega, \mathcal{F}, P)$ carrying a two–dimensional Brownian motion $W = (W_1, W_2)$ indexed by the time interval $[0, T]$, where $T > 0$ is a deterministic time horizon. Note here that stochastic processes indexed by $[0, T]$ will be written $X = (X_t)_{t\in[0,T]}$. The filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is the completion of the natural filtration of W.

Let us now explain the first version of our model in more formal details. The stock market is represented by an exogenous $\mathbb{F}$–adapted index or stock price process X^S indexed by the trading interval $[0, T]$. The dynamics of this price process evolves according to the stochastic integral equation

$$X_t^S = X_0^S + \int_0^t X_s^S (b_s^S\, ds + \sigma_s^S\, dW_s^1), \quad t \in [0, T],$$

where X_0^S is a positive constant, so that we have

$$X_t^S = X_0^S\, \mathcal{E}\left(\int (b_s^S ds + \sigma_s^S\, dW_s^1)\right)_t. \tag{2.1}$$

Throughout the paper we shall work with the following assumption concerning the drift b^S and volatility σ^S of the stock price process X^S :

Assumption 1

$$b^S \in \mathcal{H}^\infty(P,\mathbb{R}),$$
$$\sigma^S \in \mathcal{H}^\infty(P,\mathbb{R}),$$
$$\textit{there is } \varepsilon > 0 \textit{ such that } \sigma^S > \varepsilon.$$

Observe that due to this assumption the process

$$\theta^S := \frac{b^S}{\sigma^S} \tag{2.2}$$

is also contained in $\mathcal{H}^\infty(P,\mathbb{R})$ and $P[X_t^S > 0 \text{ for all}\, t \in [0,T]] = 1$.

Our analysis relies on the fact that the integral equation describing the stock price is driven by only one component of the Brownian motion. If this is not the case, we have to construct a new Brownian motion that satisfies this condition. Observe that the coefficients b^S and σ^S may depend on the whole filtration $\mathbb{F}$. The following remark considers only the components of the Brownian motion in the integral in (2.1).

Remark 2 Let the stock price be described by

$$X_t^S = X_0^S + \int_0^t X_s^S (b_s^S\, ds + \sigma_s^{S,1}\, dW_s^1 + \sigma_s^{S,2}\, dW_s^2), \quad t \in [0,T],$$

where $b^S, \sigma^{S,1}$ and $\sigma^{S,2}$ satisfy Assumption 1. Then define

$$\tilde{W}_t^1 = \frac{\sigma_t^{S,1} W_t^1 + \sigma_t^{S,2} W_t^2}{\sqrt{(\sigma_t^{S,1})^2 + (\sigma_t^{S,2})^2}}, \quad t \in [0,T],$$

and

$$\tilde{W}_t^2 = \frac{-\sigma_t^{S,2} W_t^1 + \sigma_t^{S,1} W_t^2}{\sqrt{(\sigma_t^{S,1})^2 + (\sigma_t^{S,2})^2}}, \quad t \in [0,T].$$

With the well known characteritzation theorem of Lévy we see that $(\tilde{W}^1, \tilde{W}^2)$ is a Brownian motion. Furthermore, the integral equation for the stock price process S is driven only by $\tilde{W}^1$.

The external risk component enters our model through an $\mathbb{F}$–adapted stochastic process K, indexed by the trading interval as well. As an example, one might think of a climate process, such as the temperature process in the Eastern South Pacific which gives rise to the climate phenomenon of ENSO which largely affects the national economies of the neighboring states. See (CIM04), where the effects of this phenomenon and risk transfer strategies based on the concepts of which are developed in this thesis are captured by numerical simulations.

Agents on the market are symbolized by the elements a of a finite set $\mathcal{I}$. They can use a bank account with interest rate zero. Every agent $a \in \mathcal{I}$ is supposed to be endowed with an initial capital $v_0^a \geq 0$. At the end of the trading interval at time T he receives a stochastic income H_a which describes the profits that this agent or the company he represents obtains from his usual business. The income H_a is supposed to be a real valued bounded $\mathcal{F}_T$–measurable random variable, function of the processes X^S and K, i.e.

$$H_a = g_a(X^S, K).$$

A typical example covered by these assumptions is the following. Think of two agents, say a company c and a bank b. c could for example possess an income $H^c = g^c(K)$ purely dependent on the exterior risk. The bank has an income $H^b = g^b(X^S)$ which only depends on the stock market. c wants to hedge fluctuations caused by the external factor and signs a contract with b to transfer part of this risk. b's interest in the contract could be based on the wish to diversify its portfolio. For concrete numerically investigated toy examples in the context of ENSO risks see (CIM04).

2.2 Prices for the risk security, trading

In this section we describe the set of price processes we consider for the risk security. Then we define trading strategies using both the stock and the risk security and the wealth process gained by trading.

In order to complete the market, we want to construct a second security through which external risk can be traded with price process X^E of a form given by the following stochastic integral equation

$$X_t^E = X_0^E + \int_0^t X_s^E (b_s^E ds + \sigma_s^E dW_s^2), \quad t \in [0,T], \tag{2.3}$$

with coefficient processes b^E and $\sigma^E \in \mathcal{H}^2(P, \mathbb{R})$, and such that for some $\varepsilon > 0$ we have $\sigma^E > \varepsilon$. Let

$$\theta^E := \frac{b^E}{\sigma^E}. \tag{2.4}$$

The processes θ^S, θ^E are called market price of risk of the stock and the insurance security. Every market price of risk θ^E of the second security is supposed to belong to the following set:

$$\mathcal{V} = \left\{ \theta^E \in \mathcal{H}^2(P, \mathbb{R}) \,\middle|\, \int_0^\cdot \theta_s^E dW_s^2 \text{ is a } (P, \mathbb{F}) - \text{BMO martingale} \right\}. \tag{2.5}$$

The definition of BMO martingales as well as important results are explained in the appendix. We will use the fact that stochastic exponentials of BMO martingales are uniformly integrable martingales. The market price of risk vector θ time parametrizes a class of probability measures Q^θ for which the price processes (X^S, X^E) are martingales. More formally, denote

$$X := \begin{pmatrix} X^S \\ X^E \end{pmatrix}, \quad \theta := \begin{pmatrix} \theta^S \\ \theta^E \end{pmatrix} \quad \text{and} \quad \sigma := \begin{pmatrix} \sigma^S & 0 \\ 0 & \sigma^E \end{pmatrix}. \tag{2.6}$$

The matrix valued process σ is invertible for all $t \in [0,T]$ P–a.s. With $\theta^E \in \mathcal{V}$ and θ^S according to Assumption 1 it is seen by using (A.2) (Appendix) that the process $(\int_0^t \theta_s dW_s)_{t\in[0,T]}$ is a P–BMO martingale. This property in turn guarantees that the change of measure obtained by drifting W by θ induces an equivalent probability.

Lemma 3 *Suppose that $\theta = (\theta^S, \theta^E)$ with θ^S satisfying Assumption 1 and $\theta^E \in \mathcal{V}$. Then the process $Z^\theta := \mathcal{E}(-\int_0^\cdot \theta_t dW_t)$ defines the density process of an equivalent change of probability.*

Proof The process Z^θ is the stochastic exponential of a BMO–martingale. By Theorem 2.3 in (Kaz94) it is a uniformly integrable $(P, \mathbb{F})$–martingale.

□

According to Lemma 3 we may define the measure Q^θ with Radon–Nikodym density with respect to P given by

$$\frac{dQ^\theta}{dP} = Z_T^\theta = \mathcal{E}\left(-\int_0^T \theta_t dW_t\right)_T = \exp\left(-\int_0^T \theta_t dW_t - \frac{1}{2}\int_0^T \|\theta_t\|^2 dt\right). \tag{2.7}$$

This provides the unique probability for which the price process $X = (X^S, X^E)$ given by (2.1) and (2.3) is a martingale. Hence the choice of a particular insurance asset completing the market leads to a class of equivalent martingale measures for the price dynamics parametrized by the price of risk processes. By the well known Lévy characterization $W^\theta = W + \int_0^\cdot \theta_s ds$ is a Q^θ–Brownian motion.

The market being equipped with this structure, each agent $a \in \mathcal{I}$ will maximize the terminal wealth obtained from his portfolio in the securities (X^S, X^E) and his random risky income subject to the exterior risk H_a, according to his individual preferences. Thereby he will be allowed to follow trading strategies to be specified in the following. A trading strategy is given by a 2–dimensional $\mathbb{F}$–predictable process $\pi = (\pi_t)_{0\le t\le T}$ such that

$\int_0^T \|\pi_t\sigma_t\|^2 dt < \infty$ P–a.s., hence $\int_0^\cdot (\frac{\pi_{1,s}}{X_s^S}, \frac{\pi_{2,s}}{X_s^E}) dX_s$ is well–defined. This notation of a trading strategy describes the number of currency units invested in each security. The wealth process $V = V(\pi) = V(c, \pi)$ of a trading strategy π with initial capital c is given by

$$V_t = c + \int_0^t \left(\frac{\pi_{1,s}}{X_s^S}, \frac{\pi_{2,s}}{X_s^E} \right) d \begin{pmatrix} X_s^S \\ X_s^E \end{pmatrix}, \quad t \in [0, T].$$

The number of shares of security i is $\frac{\pi_{i,t}}{X_t^i}$, $i = S, E$. For the ease of notation, we shall write in the sequel $\frac{dX}{X}$ for the vector increment $(\frac{dX^S}{X^S}, \frac{dX^E}{X^E})$. Trading strategies are self–financing. This means that those parts of the wealth not invested into X^S or X^E are kept in the bond. Gains or losses are only caused by trading with the securities. The wealth process can equivalently be written as

$$V_t(c, \pi) = c + \int_0^t \pi_s \sigma_s (dW_s + \theta_s ds) = c + \int_0^t \pi_s \sigma_s dW_s^\theta, \quad t \in [0, T]. \quad (2.8)$$

A set Φ of strategies is called free of arbitrage if there exists no trading strategy $\pi \in \Phi$ such that

$$V_0(\pi) = 0, \quad V_T(\pi) \geq 0 \quad \text{and} \quad P[V_T(\pi) > 0] > 0.$$

We have to restrict the set of trading strategies by defining the set of admissible strategies in order to exclude opportunities of arbitrage.

Definition 4 (Admissible Strategies) *The set of admissible trading strategies $\mathcal{A}$ is given by the collection of the 2–dimensional predictable processes π with $\int_0^T \|\pi_t\sigma_t\|^2 dt < \infty$ Q^θ–a.s. such that the wealth process $V(c, \pi)$ is a $(Q^\theta, \mathbb{F})$–supermartingale.*

The set of admissible strategies $\mathcal{A}$ is free of arbitrage. In fact, we get from $V_0(0, \pi) = 0$ and $V_T(0, \pi) \geq 0$ that $V_T(0, \pi) = 0$ Q^θ– and thus P– a.s. Examples are strategies π with initial capital v_0 such that $V(v_0, \pi)$ is bounded from below uniformly on $[0, T] \times \Omega$. In this case, $V(v_0, \pi)$ is a local Q^θ–martingale bounded from below, hence a Q^θ–supermartingale.

2.3 Utility maximization

Fixing a particular market price of risk $\theta^E \in \mathcal{V}$, in this section we describe the individual behavior of an agent $a \in \mathcal{I}$. In particular, the impact of the choice of θ^E determining the price process X^E of the insurance asset on his terminal wealth and trading strategy is clarified. Let us emphasize at this point that the introduction of X^E completes the market with price process X having components X^S and X^E. We use well known results about utility maximizing trading strategies and the associated terminal wealth in a complete market. They can be found e.g. in (KL87) for the maximization of an expected utility and in (Ame99) for the optimization of the conditional expected utility with respect to a non trivial sigma algebra.

Every agent $a \in \mathcal{I}$ has initial capital v_0^a at his disposal. At the terminal time T he receives a random income possibly depending on external risk and described by an $\mathcal{F}_T$–measurable bounded random variable H_a. The investor wants to hedge fluctuations in his income H_a or diversify his portfolio. His preferences are described by the expected utility using the utility function

$$u_a(x) = -\exp(-\alpha_a x) \qquad x \in \mathbb{R},$$

with an individual risk aversion coefficient $\alpha_a > 0$. The agents act as price takers.

The individual utility maximization problem for the traders acting on the whole time interval $[0, T]$ then takes the following mathematical form. Each one of them wants to find a trading strategy $\pi^a \in \mathcal{A}$ which attains

Problem 5 (Individual utility maximization, start at 0)

$$\begin{aligned} J^a(v_0^a, H_a, X^S, X^E) &= \sup_{\pi \in \mathcal{A}} E[-\exp(-\alpha_a(V_T(v_0^a, \pi) + H_a)] \\ &= \sup_{\pi \in \mathcal{A}} E\left[-\exp\left(-\alpha_a\left(v_0^a + \int_0^T \pi_s \frac{dX_s}{X_s} + H_a\right)\right)\right]. \end{aligned}$$

Since $x \mapsto -\exp(-\alpha x)$ is bounded from above, the expectations appearing in Problem 5 are well defined. It will be more convenient to reformulate our utility maximization problem using the martingale measure Q^θ with Brownian motion W^θ of our price process $X = (X^S, X^E)$. In particular, we aim for an alternative description of the budget set, described above as the set of final claims attained by admissible trading strategies, in terms of the martingale measure. This will turn out to be important in section 3 where we generalize our model to more complex situations: martingale measures will correspond to pricing rules there. At the end of the trading period, every agent has a claim of $\xi = V_T(v_0^a, \pi) + H_a$ based on his initial capital, his investments in X and external risk exposure. On the one hand, $V(v_0^a, \pi)$ being a $Q^\theta-$ supermartingale for each admissible trading strategy π this claim has to satisfy the inequality $E^\theta(\xi) \leq v_0^a + E^\theta(H_a)$. If it is even a $Q^\theta-$martingale, equality holds. On the other hand, the market being complete, every claim of this type can be replicated by appealing to the martingale representation theorem with respect to the Brownian motion W^θ under Q^θ. More precisely, H_a being bounded, for any $\xi \in L^1(Q^\theta)$ we may find an $\mathbb{F}-$predictable process ϕ satisfying $\int_0^T \|\phi_s\|^2 ds < \infty$ Q^θ–a.s. and

$$\begin{aligned} \xi - H_a &= E^\theta[\xi - H_a] + \int_0^T \phi_s dW_s^\theta \\ &= v_0^a + \int_0^T \phi_s \sigma_s^{-1} \frac{dX_s}{X_s} \\ &= V_T(v_0^a, \phi\,\sigma^{-1}). \end{aligned}$$

So we may set

$$\pi = \phi\,\sigma^{-1} \tag{2.9}$$

to obtain an admissible strategy. Here σ is defined by (2.6).

To summarize the result of our arguments in a slightly different manner: a random variable $\xi \in L^1(Q^\theta, \mathcal{F}_T)$ is the sum of the terminal value of the wealth process of an admissible trading strategy π with initial capital v_0 and a terminal income H_a if and only if $E^\theta[\xi] = v_0 + E^\theta[H_a]$.

This implies that our problem (5) boils down to the following maximization problem over random variables given by the claims. We collect claims ξ composed of final wealths of admissible strategies and final incomes H_a in the *budget set*

$$\mathcal{B}(v_0, H_a, \theta^S, \theta^E) := \{\xi \in L^1(Q^\theta, \mathcal{F}_T) \ : \ E^\theta[\xi] \le v_0 + E^\theta[H_a]\}, \tag{2.10}$$

and then have to find the random variable $\xi^a(\theta^S, \theta^E)$ that attains

$$J^a(v_0^a, H_a, \theta^S, \theta^E) := \sup_{\xi \in \mathcal{B}(v_0^a, H_a, \theta^S, \theta^E)} E[-\exp(-\alpha_a \xi)]. \tag{2.11}$$

The solution is obtained by well known methods via an application of the Fenchel–Legendre transform to the concave function $x \mapsto -\exp(-\alpha_a x)$.

Theorem 6 *Let H_a be a bounded $\mathbb{F}_T$–measurable random variable, $v_0^a \ge 0$. Define*

$$\xi^a(\theta^S, \theta^E) := \xi^a(v_0^a, H_a, \theta^S, \theta^E) = -\frac{1}{\alpha_a} \log(\frac{1}{\alpha_a} \lambda_a Z_T^\theta)$$

where λ_a is the unique real number such that

$$E^\theta[-\frac{1}{\alpha_a} \log(\frac{1}{\alpha_a} \lambda_a Z_T^\theta)] = v_0^a + E^\theta[H_a].$$

Then $\xi^a(\theta^S, \theta^E)$ is the solution of the utility maximization problem (2.11) for agent $a \in \mathcal{I}$.

Proof The main body of the proof is given by Theorem 2.3.2 of (KL90), stated for utility functions satisfying the Inada conditions, i.e. $U'(\infty) = 0$, $U'(0+) = \infty$, and under the hypothesis that the quadratic variation of $\int_0^\cdot \theta_s dW_s$ is bounded. In our setting, this process is a BMO–martingale for which the quadratic variation is not necessarily bounded. Therefore we have to show that for every $a \in \mathcal{I}, v \in \mathbb{R}$ there exists $\lambda_a > 0$ satisfying

$$E^\theta[-\frac{1}{\alpha_a}\log(\frac{1}{\alpha_a}\lambda_a Z_T^\theta)] = v. \tag{2.12}$$

A sufficient condition for this is that the relative entropy of Q^θ with respect to P is finite. We recall that for probability measures Q, R on $\mathbb{F}$ the relative entropy of Q with respect to R is defined by

$$H(Q|R) = \begin{cases} E^Q[\log \frac{dQ}{dR}], & \text{if } Q \ll R, \\ \infty, & \text{if not.} \end{cases}$$

Therefore we may finish the proof of the Theorem with an application of the following Lemma, stated in a more general setting. In fact, it implies that for θ of the type we have chosen the relative entropy $H(Q^\theta|P)$ is finite.

□

Lemma 7 *Let $\theta = (\theta^S, \theta^E)$, and suppose that θ^S satisfies Assumption 1 and $\theta^E \in \mathcal{V}$. Then $E^\theta[\log Z_T^\theta|\mathcal{F}_\tau]$ is finite $P-$a.s. for every stopping time $\tau \leq T$.*

Proof By Theorem 3.3 in (Kaz94), the process $M_t = -\int_0^t \theta_s dW_s^\theta$, $0 \leq t \leq T$, is a Q^θ–BMO martingale. Therefore there exists a constant c that does not depend on τ such that

$$E^\theta\left[\frac{1}{2}\int_\tau^T \|\theta_s\|^2 ds \middle| \mathcal{F}_\tau\right] \leq c.$$

The equation

$$-\int_\tau^T \theta_s dW_s - \frac{1}{2}\int_\tau^T |\theta_s|^2 ds = -\int_\tau^T \theta_s dW_s^\theta + \frac{1}{2}\int_\tau^T |\theta_s|^2 ds$$

yields

$$E^\theta[\log Z_T^\theta|\mathcal{F}_\tau] = E^\theta\left[\frac{1}{2}\int_\tau^T \|\theta_s\|^2 ds \middle| \mathcal{F}_\tau\right] < \infty.$$

□

So far we determined the individual utility maximizing investment strategy of an agent on our market, completed by the insurance asset X^E with parameter θ^E for the market price of external risk fixed, who starts trading at time 0. We now show that he might as well start acting at a stopping time τ that takes its values in $[0,T]$ without having to modify his optimal investment strategy. For this purpose, let us recall the results of (Ame99) for the maximization of a conditional expectation and apply them to our exponential utility function. Let $\tau \leq T$ denote a stopping time. We want to solve the following conditioned maximization problem:

Problem 8 (Individual utility maximization, start at τ)

$$\begin{aligned} J^a_\tau(v^a_\tau, H_a, \theta^S, \theta^E) &= \sup_{\pi\in\mathcal{A}} E[-\exp(-\alpha_a(V_T(v^a_\tau,\pi)+H_a))|\mathcal{F}_\tau] \\ &= \sup_{\pi\in\mathcal{A}} E\left[-\exp\left(-\alpha_a\left(v^a_\tau + \int_\tau^T \pi_s \frac{dX_s}{X_s} + H_a\right)\right)\middle|\mathcal{F}_\tau\right]. \end{aligned}$$

Hereby the initial capital v^a_τ is an $\mathcal{F}_\tau$–measurable random variable, the wealth process of an admissible trading strategy a Q^θ– supermartingale. Extending the arguments made above to reformulate the optimization problem in terms of maximization over a budget set, and in particular using Doob's optional stopping theorem, we find that the problem may be recast in the following way. Define the budget set $\mathcal{B}(\tau, v^a_\tau, H_a, \theta^S, \theta^E)$ using the conditional expectation with respect to $\mathcal{F}_\tau$ by

$$\mathcal{B}(\tau, v_\tau, H_a, \theta^S, \theta^E) := \{\xi \in L^1(Q^\theta, \mathcal{F}_T) \;:\; E^\theta[\xi|\mathcal{F}_\tau] \leq v_\tau + E^\theta[H_a|\mathcal{F}_\tau] \;P-a.s.\} \tag{2.13}$$

(see (Ame99) Proposition 4.3). Then we have to solve a maximization problem concerning random variables which represent the agents' individual claims:

$$J^a_\tau(v^a_\tau, H_a, \theta^S, \theta^E) = \sup_{\xi\in\mathcal{B}(\tau, v^a_\tau, H_a, \theta^S, \theta^E)} E[-\exp(-\alpha_a\xi)|\mathcal{F}_\tau]. \tag{2.14}$$

The exponential utility function does not satisfy the hypothesis made in (Ame99). But it is easy to apply the same method in our case. In fact, again an application of the Fenchel–Legendre transform will yield the result with the usual arguments.

Theorem 9 *Let H_a be a bounded $\mathcal{F}_T$–measurable random variable, v^a_τ an $\mathcal{F}_\tau$– measurable random variable. Define*

$$\xi^{a,\tau}(\theta^S, \theta^E) := \xi^{a,\tau}(v^a_0, H_a, \theta^S, \theta^E) = -\frac{1}{\alpha_a}\log(\frac{1}{\alpha_a}\Lambda_a Z^\theta_\tau),$$

where Λ_a is an $\mathcal{F}_\tau$–measurable random variable which satisfies

$$-\frac{1}{\alpha_a}\log\Lambda_a = v^a_\tau + E^\theta[H_a|\mathcal{F}_\tau] + \frac{1}{\alpha_a}\log\frac{1}{\alpha_a} + \frac{1}{\alpha_a}E^\theta[\log Z^\theta_T|\mathcal{F}_\tau].$$

Then $\xi^{a,\tau}(\theta^S,\theta^E)$ is the solution of the utility maximization problem (8) for agent $a \in \mathcal{I}$.

Proof Our reasoning via Theorem 2.3.2 of (KL90) this time leads us to the problem of finding an $\mathcal{F}_\tau$–measurable random variable which satisfies

$$-\frac{1}{\alpha_a}\log\Lambda_a = v^a_\tau + E^\theta[H_a|\mathcal{F}_\tau] + \frac{1}{\alpha_a}\log\frac{1}{\alpha_a} + \frac{1}{\alpha_a}E^\theta[\log Z^\theta_T|\mathcal{F}_\tau].$$

This again boils down to a finite relative entropy condition already covered by Lemma 7.

□

Let us summarize our findings of this section for ease of later reference by giving an explicit formula for the utility maximizing wealth at time T of agent $a \in \mathcal{I}$ if he uses his optimal strategy from a stopping time $\tau \leq T$ on with a Q^θ–integrable $\mathcal{F}_\tau$–measurable initial capital v^a_τ. We recall that the parameter θ determines uniquely the second security X^E on our market which is a possible candidate for making the external risk tradable. The formula we obtain from Theorem 9 by employing the explicit structure of the density Z^θ_τ reads

$$\begin{aligned}\xi^{a,\tau}(\theta^S,\theta^E) &= -\frac{1}{\alpha_a}\log\left(\frac{\Lambda_a}{\alpha_a}\right) + \frac{1}{\alpha_a}\int_\tau^T(\theta^S_t dW^1_t + \theta^E_t dW^2_t) \qquad (2.15)\\ &\quad + \frac{1}{2\alpha_a}\int_\tau^T(|\theta^S_t|^2 + |\theta^E_t|^2)dt.\end{aligned}$$

To emphasize its explicit dependence on the price of external risk, we further write $\pi^a(\theta^E)$ for the utility maximizing trading strategy attaining the claim

$$\xi^a(\theta^S,\theta^E) - H_a = V_T(v^a_0, \pi^a(\theta^E)) = v^a_0 + \int_0^T \pi^a(\theta^E)_s\frac{dX_s}{X_s}. \qquad (2.16)$$

The optimal trading strategy satisfies the principle of dynamic programming: if at time $t = 0$ an agent a chooses the optimal strategy $\pi^a(\theta^E)$ which provides the wealth $V_\tau(v^a_0, \pi^a(\theta^E))$ at a stopping time τ, he has to follow the same strategy if he starts acting at time τ with initial capital $V_\tau(v^a_0, \pi^a(\theta^E))$.

2.4 Equilibrium with partial market clearing

Let us now introduce our concept of equilibrium with partial market clearing for the market on which the external risk due to the risk process K is traded. Let us briefly recall the model components implemented so far. Every agent $a \in \mathcal{I}$ obtains an initial capital v_0^a and at time T a random risky income H_a that, besides the economic development described by the exogenous stock price process X^S, depends on the external risk process K. A second (insurance) security X^E is created to make individual risks immanent in the incomes H_a and caused by K tradable. It depends on the process parameter θ^E which describes a possible price of external risk in X^E. Given such a system of pricing risk every agent trades with X^S and X^E and calculates the trading strategy $\pi^a(\theta^E)$ that maximizes expected exponential utility with individual risk aversion α_a of the sum of his terminal wealth from trading and the income H_a. In order to reach a partial market clearing, we have to find a market price of external risk process $\theta^{E*} \in \mathcal{V}$ for which at any time t a market clearing condition for the second security is satisfied, i.e. $\sum_{a\in\mathcal{I}} \pi_{2,t}(\theta^{E*}) = 0$. This equilibrium is called partial since no market clearing for the stock X^S is required.

Definition 10 (equilibrium with partial market clearing) *Let the initial capitals $v_0^a \in \mathbb{R}$, the terminal incomes H_a, $a \in I$, and the stock price process X^S be given. A* equilibrium with partial market clearing *consists of a market price of external risk process $\theta^{E*} \in \mathcal{V}$ for the second security and trading strategies $\pi^a(\theta^{E*})$, $a \in \mathcal{I}$, which satisfy the following conditions:*

1. *for any $a \in \mathcal{I}$ the trading strategy $\pi^a(\theta^{E*})$ is the solution of the utility maximization problem 5 for the stock price process X^S and the price process of the second security associated with market price of risk θ^{E*},*

2. *the second component $\pi_2^a(\theta^{E*})$, $a \in \mathcal{I}$, satisfies the* partial market clearing condition
$$\sum_{a\in\mathcal{I}} \pi_2^a(\theta^{E*}) = 0 \qquad P \otimes \lambda - a.e.$$

The condition that the market clears partially puts a natural constraint on the set of processes of market price of risk for the second security. We shall now investigate the impact of this constraint. It will completely determine the structure of θ^{E*} and therefore also a unique martingale measure Q^{θ^*} obtained via (2.7) for $\theta^* = (\theta^S, \theta^{E*})$. So we shall have to compute θ^{E*} from the condition that the market be in equilibrium with respect to $X^E = \int_0^\cdot \sigma_s^E \,(dW_s^2 + \theta_s^E ds)$. Recall that Assumption 1 guarantees $\theta^S \in \mathcal{H}^\infty(P, \mathbb{R})$. In the following Lemma the overall effect of our equilibrium condition emerges. Plainly, if we take the sum of the terminal incomes and terminal wealth obtained by all agents from trading on the security market composed of X^S and X^E, the condition of partial market clearing just eliminates the contribution of X^E.

Lemma 11 *Let $\theta = (\theta^S, \theta^E)$ be such that θ^S satisfies Assumption 1, and $\theta^E \in \mathcal{V}$. The market is in an equilibrium with partial market clearing if and only if there exist an $\mathbb{F}$–predictable real valued stochastic process ϕ with $E^\theta\left[\left(\int_0^T (\phi_s)^2 ds\right)^{\frac{1}{2}}\right] < \infty$ such that the optimal claims $(\xi^a(\theta^S, \theta^E))_{a\in\mathcal{I}}$ and incomes $(H_a)_{a\in\mathcal{I}}$ satisfy the equation*

$$\sum_{a\in\mathcal{I}}(\xi^a(\theta^S,\theta^E) - H_a) = c_0 + \int_0^T \phi_s(dW_s^1 + \theta_s^S ds) \tag{2.17}$$

with some constant $c_0 \in \mathbb{R}$. Hence $\pi = (\pi_1, 0)$ with $\pi_1 = \phi(\sigma^S)^{-1} = \sum_{a\in I}\pi_1^a$, possesses the properties of an admissible trading strategy.

Proof First we apply the representation property (2.9) to the terminal wealth $\xi^a(\theta^S, \theta^E) - H_a$ of each individual agent $a \in \mathcal{I}$ with initial capital v_0^a, then sum over all $a \in \mathcal{I}$. Using linearity of the stochastic integral and recalling (2.8) we thus obtain

$$\begin{aligned}
&\sum_{a\in\mathcal{I}}(\xi^a(\theta^S,\theta^E) - H_a) \\
&= \sum_{a\in\mathcal{I}} v_0^a + \int_0^T (\sum_{a\in\mathcal{I}}\pi_{1,t}^a)\frac{dX_t^S}{X_t^S} + \int_0^T (\sum_{a\in\mathcal{I}}\pi_{2,t}^a)\frac{dX_t^E}{X_t^E} \qquad (2.18)\\
&= \sum_{a\in\mathcal{I}} v_0^a + \int_0^T (\sum_{a\in\mathcal{I}}\pi_{1,t}^a)\,\sigma_t^S(dW_t^1 + \theta_t^S dt) \\
&\quad + \int_0^T (\sum_{a\in\mathcal{I}}\pi_{2,t}^a)\,\sigma_t^E(dW_t^2 + \theta_t^E dt).
\end{aligned}$$

To prove the 'only if' part, write now $\pi_i = \sum_{a\in\mathcal{I}}\pi_i^a$, $i = 1, 2$. Since the market clears partially, we have $\pi_2 = 0$. Hence the desired equation (2.17) follows.

For the 'if' part, suppose that $\sum_{a\in\mathcal{I}}(\xi^a(\theta^S,\theta^E) - H_a)$ can be written as in (2.17). By comparison with (2.18) and uniqueness of integrands in stochastic integral representations we obtain $\pi_1 = \frac{\phi}{\sigma^S}$ and $\pi_2 = 0$. This establishes the equivalence. Finally, $\pi = (\pi_1, 0)$ is admissible, because $\sum_{a\in\mathcal{I}}(\xi^a(\theta^S,\theta^E) - H_a) \in L^1(Q^\theta)$, and the process $\int_0^\cdot \pi_{1,t}dX_t^S$ is even a Q^θ–martingale.

□

We now come to the main goal of this section, the construction of θ^E for which our equilibrium constraint is satisfied. At the same time, this will justify the existence of an equilibrium with partial market clearing. We use the characterization of the utility maximizing payoffs in our equilibrium described in Lemma 11 and the explicit formula (2.15). This will enable us to describe θ^{E*} and ϕ (or π) in terms of the solution of a BSDE. To abbreviate, we write

$$\bar{\alpha} = (\sum_{a\in\mathcal{I}} \frac{1}{\alpha_a})^{-1}, \quad \bar{H} = \sum_{a\in\mathcal{I}} H_a + \frac{1}{2\bar{\alpha}} \int_0^T |\theta_s^S|^2 ds. \tag{2.19}$$

We combine the two alternative descriptions of $\sum_{a\in\mathcal{I}}(\xi^a(\theta^S, \theta^{E*}) - H_a)$ provided by Lemma 11 and the equation

$$\begin{aligned}&\sum_{a\in\mathcal{I}}(\xi^a(\theta^S, \theta^{E*}) - H_a) \qquad (2.20)\\ &= c_1 + \frac{1}{\bar{\alpha}} \int_0^T (\theta_t^S\, dW_t^1 + \theta_t^{E*}\, dW_t^2) + \frac{1}{2\bar{\alpha}} \int_0^T (|\theta_t^S|^2 + |\theta_t^{E*}|^2) dt - \sum_{a\in\mathcal{I}} H_a\end{aligned}$$

which follows from (2.15) with a constant c_1 not specified further at this point, to obtain a condition determining θ^{E*} in the form of a BSDE. To keep to the habits of the literature on BSDE, set

$$\begin{aligned} z^S &= \theta^S - \bar{\alpha}\phi, \\ z^E &= \theta^{E*}. \end{aligned}$$

In this notation the comparison of (2.17) and (2.20) yields the equation

$$h_0 = \bar{\alpha}\bar{H} - \int_0^T (z_t^S dW_t^1 + z_t^E dW_t^2) - \int_0^T \frac{1}{2}|z_t^E|^2\, dt - \int_0^T \theta_t^S z_t^S\, dt. \tag{2.21}$$

Due to Assumption 1, $\bar{H}$ is bounded. By extending (2.21) from time 0 to any time $t \in [0, T]$ we obtain a BSDE whose solution uniquely determines $z^E = \theta^{E*}$. It defines backward in time a predictable stochastic process $(h_t)_{t\in[0,T]} \in \mathcal{H}^\infty(\mathbb{R}, P)$ with terminal value $h_T = \bar{\alpha}\bar{H}$ and an integrand $(z_t = (z_t^S, z_t^E))_{t\in[0,T]} \in \mathcal{H}^2(\mathbb{R}^2, P)$. The following Theorem provides an equilibrium solution by setting $\theta^{E*} := z^E$ which is obtained from known results on non-linear BSDE.

Theorem 12 *The backwards stochastic differential equation (BSDE)*

$$h_t = \bar{\alpha}\bar{H} - \int_t^T (z_s^S dW_s^1 + z_s^E dW_s^2) - \int_t^T \theta_s^S z_s^S ds - \int_t^T \frac{1}{2}|z_s^E|^2 ds, \tag{2.22}$$

$t \in [0,T]$, possesses a unique solution given by the triple of processes $(h,(z^S,z^E)) \in \mathcal{H}^\infty(P,\mathbb{R}) \times \mathcal{H}^2(P,\mathbb{R}^2)$. The choice $\theta^{E*} := z^E$ provides an equilibrium with partial market clearing for the market.

Proof $\bar{H}$ is $\mathcal{F}_T$–measurable and bounded. The process θ^S is $\mathbb{F}$–predictable and uniformly bounded in (ω,t). By Theorem 2.3 and Theorem 2.6 in (Kob00), equation (2.22) has a unique solution $(h,(z^S,z^E)) \in \mathcal{H}^\infty(P,\mathbb{R}) \times \mathcal{H}^2(P,\mathbb{R}^2)$. Let then $\theta^{E*} := z^E$ and $\phi := \frac{1}{\bar{\alpha}}(\theta^S - z^S)$. Then, thanks to Lemma 11 we get a equilibrium with partial market clearing, provided we can prove that $z^E \in \mathcal{V}$. This is done in Lemma 13 below. Given θ^{E*}, for the coefficients b^{E*} and σ^{E*} we are free to choose for example

$$b^{E*} = \theta^{E*}, \qquad \sigma^{E*} = 1.$$

□

Lemma 13 *Let z^E be the third component of the solution $(h,(z^S,z^E))$ of (2.22). Then the process $M = \int_0^\cdot z_s^E dW_s^2$ is a P–BMO martingale.*

Proof Without loss of generality, we may suppose $\bar{\alpha}\bar{H}$ nonnegative. To see this, recall that $\bar{\alpha}\bar{H}$ is bounded from below by a constant S. We may then solve the BSDE (2.22) for $\tilde{H} = \bar{\alpha}\bar{H} - S$ instead. By uniqueness its solution $(k,(y_1,y_2))$ satisfies $k = h - S, y_1 = z^S, y_2 = z^E$. If $\tilde{H} \geq 0$, the comparison theorem (Theorem 2.6 (Kob00)) gives $h \geq 0$. For every stopping time $\tau \leq T$, Itô's formula yields

$$E\left[\tilde{H}^2 - h_\tau^2 - \int_\tau^T (2h_s\theta_s^S z_s^S + |z_s^S|^2)ds \,\middle|\, \mathcal{F}_\tau\right]$$
$$= E\left[\int_\tau^T (h_s+1)|z_s^E|^2 ds \,\middle|\, \mathcal{F}_\tau\right] \geq E\left[\int_\tau^T |z_s^E|^2 ds \,\middle|\, \mathcal{F}_\tau\right].$$

To find also an upper bound for the left hand side in the inequality above we note

$$-2h_s\theta_s^S z_s^S - |z_s^S|^2 = |\theta_s^S|^2 h_s^2 - (\theta_s^S h_s + z_s^S)^2.$$

Let S_1 denote an upper bound for $\tilde{H}^2$ and S_2 an upper bound for $|\theta_s^S|^2 h_s^2$. Then we get for every stopping time $\tau \leq T$

$$\begin{aligned} S_1 + TS_2 &\geq E\left[\int_\tau^T |z_s^E|^2 ds \,\middle|\, \mathcal{F}_\tau\right] \\ &= E\left[\langle M\rangle_T - \langle M\rangle_\tau \middle| \mathcal{F}_\tau\right]. \end{aligned}$$

Therefore M is a P–BMO martingale.

□

Here we give an example where our equilibrium price of the external risk does not depend on the financial market. This is the case if the income of the agent is the sum of a payoff that depends only on financial risk and a payoff that depends on the external risk. Then our BSDE (2.22) decomposes into two BSDEs that can be solved seperately.

Example 14 Let the drift of the stock price θ^S be adapted to the filtration $\mathbb{F}^1 = (\mathcal{F}_t^1)$, the $P-$ augmentation of the filtration generated by W^1. Let $\mathbb{F}^2 = (\mathcal{F}_t^2)$ denote the $P-$ augmentation of the filtration generated by W^2. We assume that the sum of the incomes $H = \sum_{a \in \mathcal{I}} H_a$ can be decomposed in two parts:

$$H = H_1 + H_2,$$

where H_1 is measurable with respect to $\mathcal{F}_T^1$, H_2 is $\mathcal{F}_T^2-$ measurable and both random variables are bounded. Then we can decompose our BSDE (2.22) into a BSDE with respect to W^1 within $\mathbb{F}^1$ and a BSDE with respect to W^2 in the filtration $\mathbb{F}^2$. Here is the first BSDE:

$$Y_t^1 = (H_1 + \frac{1}{2\bar{\alpha}} \int_0^T |\theta_s^S|^2 ds) - \int_t^T z_s^S dW_s^1 - \int_t^T \theta_s^S z_s^S,$$

and the second one:

$$Y_t^2 = H_2 - \int_t^T z_s^E dW_s^2 - \int_t^T \frac{1}{2} |z_s^E|^2 ds.$$

Each BSDE can be solved separately within its filtration $\mathbb{F}^1$ and $\mathbb{F}^2$. The integrands z^S and z^E are equal to the integrands of the solution of (2.22). Furthermore, the proces Y in the solution of (2.22) satisfies $Y = Y^1 + Y^2$. The economic interpretation is simple: the income H_1 is hedged on the financial market. The income H_2 is distributed among the agents using the usual equilibrium approach: the market price of external risk θ^E is determined by the fact that supply and demand for the transfer of external risk is equal. In particular, under the assumptions in this example, $\theta^E = z^E$ does not depend on the market price of financial risk θ^E and the part of the income H_1 that is tradeable on the financial market.

In the following Theorem we shall show that the choice $\theta^{E*} = z^E$ made above provides the unique equilibrium price of external risk under the assumptions valid for the coefficient processes.

Theorem 15 *Suppose $\theta^{E*} = b^{E*}/\sigma^{E*}$ is such that we have an equilibrium with partial market clearing. Then $z^E = \theta^{E*}$ is the third component of the unique solution process $(h, (z^S, z^E))$ of (2.22).*

Proof We first apply Girsanov's Theorem to eliminate the known drift θ^S from our considerations. More formally, consider the probability measure $\tilde{Q}$ given by the density

$$\frac{d\tilde{Q}}{dP} = \mathcal{E}\left(-\int_0^T (\theta_t^S, 0) dW_t\right).$$

Let $\tilde{W} = W + \int_0^\cdot (\theta_s^S, 0) ds$ be the corresponding Brownian motion under $\tilde{Q}$.

Now define $z^S = \theta^S - \bar{\alpha}\phi$, $z^E = \theta^{E*}$ and $z_t = (z_t^S, z_t^E)^{tr}$. Since z^E guarantees that we have an equilibrium with partial market clearing, as for (2.21) we deduce with a constant c

$$\begin{aligned} c &= \bar{\alpha}\bar{H} - \int_0^T (z_t^S dW_t^1 + z_t^E dW_t^2) - \int_0^T \frac{1}{2}|z_t^E|^2\, dt - \int_0^T \theta_t^S z_t^S\, dt \quad (2.23) \\ &= \bar{\alpha}\bar{H} - \int_0^T z_t d\tilde{W}_t - \int_0^T \frac{1}{2}|z_t^E|^2\, dt. \end{aligned}$$

Hence we may further define the process h by

$$h_t = c + \int_0^t z_s d\tilde{W}_s + \frac{1}{2}\int_0^t (z_s^E)^2 ds,$$

with the alternative description

$$h_t = \bar{\alpha}\bar{H} - \int_t^T z_s d\tilde{W}_s - \int_t^T \frac{1}{2}|z_s^E|^2 ds, \quad t \in [0,T]. \qquad (2.24)$$

This yields that $(h, (z^S, z^E))$ solves (2.22). It remains to verify according to Theorem 2.6 in (Kob00) that

$(z^S, z^E) \in \mathcal{H}^2(P, \mathbb{R}^2)$,
h is uniformly bounded.

Let us first argue for the square integrability of (z^S, z^E). By the definition of our equilibrium, we have $\theta^E \in \mathcal{H}^2(P, \mathbb{R})$. θ^S being bounded, it remains to argue for P-square-integrability of ϕ, where ϕ is given by (2.17). By Burkholder-Davis-Gundy's inequality, we have $\sum_{a\in I}(\xi^a(\theta^S, \theta^E) - H_a) \in L^p(\tilde{Q})$ for $p \geq 1$, and this random variable can be represented as a stochastic integral with the integrand $(\phi, 0)$ with respect to the Brownian motion $\tilde{W}$. Hence,

$$E^{\tilde{Q}}([\int_0^T (\phi_s)^2 ds]^{\frac{p}{2}}) < \infty,$$

for $p \geq 1$. Therefore, due to Hölder's inequality and

$$E^P([\int_0^T (\phi_s)^2 ds]^{\frac{p}{2}}) = E^{\tilde{Q}}([\int_0^T (\phi_s)^2 ds]^{\frac{p}{2}} \mathcal{E}(\int_0^T (\theta_t^S, 0) d\tilde{W}_s))$$

we also obtain

$$E^P([\int_0^T (\phi_s)^2 ds]^{\frac{p}{2}}) < \infty$$

for all $p \geq 1$.

To prove the boundedness of h, we perform still another equivalent change of measure. Let $\hat{Q}$ be given by

$$\frac{d\hat{Q}}{dP} = \mathcal{E}(-\int_0^T (\theta_t^S, \frac{1}{2} z_t^E) dW_t).$$

Then by virtue of (2.24) we get

$$h_t = E^{\hat{Q}}[\bar{\alpha}\bar{H}|\mathcal{F}_t], \quad t \in [0, T].$$

Therefore h has a uniformly bounded version with the same bounds as $\bar{\alpha}\bar{H}$.

□

We conclude this section by showing that the unique equilibrium constructed persists if the individual utility maximization problems of the agents on the market start at some stopping time τ.

Remark 16 *The market price of risk θ^{E*} that attains partial market clearing satisfies a dynamic programming principle. Indeed, let θ^{E*} be the unique market price of risk process in $\mathcal{V}$ calculated for the individual utility maximization starting at time $t = 0$. Let $\tau \leq T$ be a stopping time and let the agents solve the conditioned maximization problem 8 beginning at time τ with terminal incomes H_a. Then the equilibrium is given by θ^{E*} as well.*

For the construction of an equilibrium with partial market clearing for trading after τ we proceed in the same way as in the case of the maximization of a conditioned expected utility. The definition of a partial equilibrium remains as in Definition 10. The starting point is Lemma 11 adapted to the sigma–algebra $\mathcal{F}_\tau$, where we have to replace the constant c_0 by an $\mathcal{F}_\tau$–measurable bounded random variable c_τ. Comparing the explicit solution of the utility maximization with respect to a candidate for an equilibrium market price of risk process θ^{E*} to (2.17) yields the following BSDE with $z = (\tilde{z}^S, \tilde{z}^E)$

$$\tilde{h}_t = \bar{\alpha} \sum_{a \in \mathcal{A}} H_a - \int_t^T \tilde{z}_s dW_s - \int_t^T \left(\frac{1}{2} |\tilde{z}_s^E|^2 + \theta^S \tilde{z}_s^S - \frac{1}{2} |\theta_s^S|^2 \right) ds, \quad t \in [\tau, T].$$

By uniqueness of the solution of the BSDE, we derive $\tilde{h}_t = h_t + \int_0^t \frac{1}{2}|\theta_s^S|^2 ds$ and for the integrands $(\tilde{z}^E, \tilde{z}^S) = (z^E, z^S)$. As for the utility maximization beginning at $t = 0$ we obtain $\theta^{E*} = z^E$ and $\phi = \frac{1}{\bar{\alpha}}(\theta^S - z^S)$. The market price of risk process $\theta^{E*} \in \mathcal{V}$ that attains the partial clearing is unique. The proof of Theorem 15 remains valid if we replace the constant c in (2.23) with an $\mathcal{F}_\tau-$ measurable bounded random variable.

Chapter 3

Market completion with contracts

In this section, we shall describe an alternative approach to the problem of transferring external risks by trading on a financial market in partial equilibrium. This approach is conceptually more flexible and therefore better appropriate for dealing with risk exposures too complicated to be tradable by just one security. The ingredients of the model are basically the same.

There is a stock market with a stock evolving according to an exogenous price process X^S. As in Chapter 2, we consider finitely many agents $a \in \mathcal{I}$ each one of which is endowed with an initial capital v_0^a and a random income H_a payed out at the terminal time T. H_a depends on the economic development described by X^S and a process K representing external risk which cannot be hedged by trading on the stock market. In this section we do not construct a second security to be traded together with X^S. Instead, the agents have the possibility to sign mutual or multilateral contracts in order to exchange *random payoffs* in addition to trading with the stock.

Let us first explain what corresponds to *market completion* in this version of the model. The agents' random payoffs are priced using one and the same pricing rule for the entire market. The value of a payoff that is replicable by a trading strategy must be equal to the initial capital of the trader. Therefore, a pricing rule that is consistent with the stock price is linear on the replicable payoffs. We only consider pricing rules which are linear on all payoffs. It is well known that pricing rules that are continuous linear functionals on an $L^p(P)$-space for some $p > 1$ and preserve constants can be described as expectations of a probability measure absolutely continuous with respect to P. Under the additional assumption that a nontrivial positive payoff has a positive price, these probability measures turn out to be equivalent to P. A pricing rule meeting all these claims and being consistent with the stock price is therefore given by the expectation under a probability measure equivalent to P for which X^S is a martingale. We call those measures *pricing measures.*

Given a particular pricing measure Q, every agent possesses a budget set which must contain those random payoffs that are cheaper than the sum of his initial capital and the value of his income H_a. The preferences of an agent a are described by the expected exponential utility with individual risk aversion α_a. Now every agent maximizes his utility by choosing the best priced payoff in his budget set under Q. He then has to replicate the difference between this payoff and his income H_a by trading with the stock, which is possible since the stock price process is a martingale under Q, and signing contracts with other agents.

And here is how we interpret the *equilibrium with partial market clearing* in this setting. Fix again a pricing measure Q for a moment. The random claim of each agent a may be decomposed into a part which is hedgeable under Q purely with X^S, and an additional part C^a which depends on Q and describes the remaining compound risk of his contracts with other agents. So we have to look for an *equilibrium pricing measure* Q^* for which the total compound risk $\sum_{a\in\mathcal{I}} C^a$ vanishes. In other terms, the difference of offers and demands of payoffs by the different agents creates a claim they are able to hedge on the financial market alone.

We use a version of the explicit formula (2.15) for the utility maximizing payoff and the partial market clearing condition to characterize the density of the pricing measure that attains the equilibrium in terms of the solution of a BSDE as before.

3.1 Stocks, prices of risk transfer

This time we work on a d–dimensional model with a Brownian motion $W = (W_1, \dots, W_d)$. The P–completion of the filtration generated by W is denoted by $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$. As in (2.1) the stock price process X is given by the stochastic equation

$$X_t^S = X_0^S + \int_0^t X_s^S(b_s^S ds + \sigma_s^S dW_s^1), \quad t \in [0,T]. \tag{3.1}$$

The basic facts about our model remain unchanged with respect to the previous sections. The coefficients b^S and σ^S satisfy Assumption 1 and therefore $\theta^S := b^S/\sigma^S$ is $\mathbb{F}$–predictable and uniformly bounded. If the integral in (3.1) depends on more than one component of the Brownian motion W, then we have to construct a new Brownian motion such that this integral is driven by only one component. This is explained in Remark 2.

The process K that describes the external risk is $\mathbb{F}$–adapted. For $a \in \mathcal{I}$ the income H_a that agent a receives at time T is again a real-valued bounded $\mathcal{F}_T$–measurable random variable of the form

$$H_a = g^a(X^S, K).$$

Every agent a is endowed at time $t = 0$ with an initial capital $v_0^a \geq 0$, and maximizes his expected utility with respect to the exponential utility function

$$u_a(x) = -\exp(-\alpha_a x), \quad x \in \mathbb{R},$$

with an individual risk aversion coefficient $\alpha_a > 0$.

According to the introductory remarks we next specify the system of prices admitted for pricing the claims of agents on our market. We aim at considering pricing measures which do not change prices for X^S. Hence we let $\mathbb{P}_e$ be the collection of all probability measures Q on $\mathcal{F}_T$ which are equivalent to P and such that X^S is a Q–martingale.

Remark 17 *The price of a claim ξ under $Q \in \mathbb{P}_e$ is described by the expectation*

$$E^Q[\xi] \tag{3.2}$$

which makes sense for all contingent claims such that this expectation is well defined, e.g. for ξ bounded from below. The set of equivalent martingale measures $\mathbb{P}_e$ parameterizes all linear pricing rules that are continuous in an $L^p(P)$– space for $p > 1$, strictly positive on $L^0_+(P)\backslash\{0\}$ and consistent with the stock price process X^S. These pricing systems do not allow arbitrage.

$\mathbb{P}_e$ can be described and thus parameterized explicitly. It consists of all probability measures Q^θ possessing density processes with respect to P of the following form

$$\left.\frac{dQ^\theta}{dP}\right|_{\mathcal{F}_t} = Z_t^\theta = \mathcal{E}\left(-\int(\theta_s^S, \theta_s^E)dW_s\right)_t, \quad t \in [0, T], \tag{3.3}$$

with a predictable $\mathbb{R}^{d-1}$–valued process θ^E such that the stochastic exponential is a uniformly integrable martingale. We denote $\theta = (\theta^S, \theta^E)$. The process θ^E plays the same part as in section 2.4. Using this parametrized set, the strategies agents are allowed to use can be formulated in the following way.

Definition 18 (admissible trading strategy, wealth process) *An admissible trading strategy with initial capital $v_0 \geq 0$ is a stochastic process*

π with $\int_0^T |\sigma_s^S \pi_s|^2\, ds < \infty$ P–a.s. and such that there exists a probability measure $Q^\theta \in \mathbb{P}_e$ such that the wealth process

$$V_t(v_0, \pi) = v_0 + \int_0^t \pi_s \frac{dX_s^S}{X_s^S}, \qquad t \in [0, T],$$

is a Q^θ–supermartingale.

The set of admissible trading strategies is free of arbitrage. A strategy π with a wealth process $V(v_0, \pi)$ that is bounded from below is admissible.

3.2 Utility maximization

For the purpose of utility maximization with respect to our exponential utility functions the set $\mathbb{P}_e$ has to be further restricted to the set $\mathbb{P}_f$ of equivalent martingale measures with finite relative entropy with respect to P (see section 2.3). Let $Q^\theta \in \mathbb{P}_f$ for $\theta = (\theta^S, \theta^E)$ be given. The condition under which agents maximize their expected utility is given by a *budget constraint*. An individual agent a can choose among all claims that are not more expensive than the sum of his initial capital v_0^a and the price of his income $E^\theta(H_a) = E^{Q^\theta}(H_a)$. The set of these claims is called the budget set for agent a, formally given by

$$\mathcal{B}^a := \mathcal{B}(v_0^a, H_a, Q^\theta) = \{D \in L^1(Q^\theta, \mathcal{F}_T) : E^\theta[D] \leq v_0^a + E^\theta[H_a]\}.$$

Every agent a chooses in his budget set the claim $\xi^a(Q^\theta)$ that maximizes his expected utility, i.e. the solution of the following maximization problem

$$J^a(v_0^a, H_a, Q^\theta) = \sup_{D \in \mathcal{B}(v_0^a, H_a, Q^\theta)} E[-\exp(-\alpha_a D)]. \tag{3.4}$$

According to the well known theory of utility maximization via Fenchel–Legendre transforms, the solution is given by the following Theorem. Here we put $I^a(y) = ((U^a)')^{-1}(y) = -\frac{1}{\alpha_a} \log \frac{y}{\alpha_a}$, for $Q^\theta \in \mathbb{P}_f$. Note that taking Q^θ from this set replaces an appeal to Lemma 7 in the proof.

Theorem 19 *Let H_a be a bounded $\mathbb{F}_T$–measurable random variable, $v_0^a \geq 0$. Define*

$$\xi^a(Q^\theta) = I(\lambda_a Z_T^\theta) = -\frac{1}{\alpha_a} \log(\frac{1}{\alpha_a} \lambda_a Z_T^\theta), \tag{3.5}$$

where λ_a is the unique real number such that

$$E^\theta[I^a(\lambda_a\, Z_T^\theta)] = v_0^a + E^\theta[H_a].$$

Then $\xi^a(Q^\theta)$ is the solution of the utility maximization problem (3.4) for agent $a \in \mathcal{I}$.

3.3 Equilibrium with contracts

Let us now describe more formally what we mean by an equilibrium with partial market clearing. We want to construct a stochastic process θ^{E*} and with $\theta^* = (\theta^S, \theta^{E*})$ via (3.3) a measure $Q^* = Q^{\theta*} \in \mathbb{P}_f$ under which the overall difference between demands and offers of agents' claims is replicable on the financial market, i.e. can be hedged with the security X^S. In different terms, we look for a price measure Q^* such that $\sum_{a\in I}(\xi^a(Q^*) - H_a)$ can be represented as a stochastic integral with respect to the stock price process X with an integrand given by an admissible trading strategy. Under Q^θ, agent a knows the claim $\xi^a(Q^\theta)$ which maximizes his expected utility. He covers the difference $\xi^a(Q^\theta) - H_a$ between his preferred payoff and his income by two components: the terminal wealth of a trading strategy $\pi^a(Q^\theta)$, and the payoff $C^a(Q^\theta)$ from the mutual contracts with the other participants in the market. Formally,

$$\xi^a(Q^\theta) - H_a = C^a(Q^\theta) + v_0^a + \int_0^T \pi^a(Q^\theta)_s \frac{dX_s^S}{X_s^S}.$$

We now define the equilibrium measure Q^* by claiming that

$$\sum_{a\in\mathcal{I}} C^a(Q^*) = 0.$$

Definition 20 (equilibrium with partial market clearing) *Let $(H_a)_{a\in\mathcal{I}}$ be a family of bounded $\mathcal{F}_T$–measurable incomes, $(v_0^a)_{a\in\mathcal{I}}$ a family of initial capitals of the agents, X^S the exogenous stock price process according to (3.1), $(U^a)_{a\in\mathcal{I}}$ a family of exponential utility functions with risk aversion coefficients $(\alpha_a)_{a\in\mathcal{I}}$, and $(\xi^a(Q^\theta))_{a\in\mathcal{I}}$ the family of utility maximizing claims according to (3.5) for $Q \in \mathbb{P}_f$. A probability measure $Q^* \in \mathbb{P}_f$ attains the* equilibrium with partial market clearing *if there exists an admissible trading strategy π^* such that we have*

$$\sum_{a\in\mathcal{I}}(\xi^a(Q^*) - H_a) = \sum_{a\in\mathcal{I}} v_0^a + \int_0^T \pi_s^* \frac{dX_s^S}{X_s^S}.$$

In view of the preceding remarks, to obtain the admissible trading strategy π^* of Definition 20 we have to sum all the individual strategies $\pi^a(Q^*)$ of agents a over $a \in \mathcal{I}$. Given the equilibrium measure, the existence of π^* is equivalent to the existence of an $\mathbb{F}$–predictable real valued stochastic process ϕ^* satisfying

$$\sum_{a\in\mathcal{I}}(\xi^a(Q^*) - H_a) = \sum_{a\in\mathcal{I}} v_0^a + \int_0^T \phi_t^*(dW_t^1 + \theta_t^S dt).$$

The process ϕ^* and the admissible trading strategy π^* are related by the equation

$$\pi^* = \frac{\phi^*}{\sigma^S}.$$

To construct Q^*, we just have to find an appropriate process θ^{E*} appearing in the exponential of an equivalent measure change and take $Q^* = Q^{(\theta^S, \theta^{E*})}$. But this just means that we can proceed as in section 2.4 and use the technology of BSDE. The process θ^{E*} will just be the higher dimensional version of the process θ^{E*} constructed there. Since we are in a d–dimensional model here, we shall give a few details of the analogous construction. Let

$$\bar{H} = \sum_{a \in \mathcal{I}} H_a + \frac{1}{2\bar{\alpha}} \int_0^T |\theta_t^S|^2 dt,$$

$z_1 = \theta_t^S - \bar{\alpha}\phi_t^*$, $z_i = \theta_{i-1}^E$, $i = 2, \dots d$. We obtain the following BSDE

$$h_t = \bar{\alpha}\bar{H} - \int_t^T (z_{1,s}, \dots, z_{d,s}) dW_s - \int_t^T \theta_s^S z_{1,s} ds - \frac{1}{2} \sum_{i=2}^d \int_t^T (z_{i,s})^2 ds,$$

$t \in [0, T]$. The process θ^S is uniformly bounded by Assumption 1 and $\bar{H}$ is also bounded. In this setting the following existence result for an equilibrium with partial market clearing holds.

Theorem 21 *The Backward Stochastic Differential Equation (BSDE)*

$$h_t = \bar{\alpha}\bar{H} - \int_t^T (z_{1,s}, \dots, z_{d,s}) dW_s - \int_t^T \theta_s^S z_{1,s} ds - \frac{1}{2} \sum_{i=2}^d \int_t^T (z_{i,s})^2 ds, \quad (3.6)$$

$t \in [0, T]$, *possesses a unique solution given by the triple of processes* $(h, (z^S, z^E)) \in \mathcal{H}^\infty(P, \mathbb{R}) \times \mathcal{H}^2(P, \mathbb{R}^d)$. *The choice* $\theta^{E*} = (z_2, \dots, z_d)$ *and* Q^* *defined via (3.3) with* (θ^S, θ^{E*}) *gives a pricing measure for which an equilibrium with partial market clearing is attained.*

Proof Due to Theorem 2.3 and Theorem 2.6 in (Kob00), (3.6) possesses a unique solution $(h, z) \in \mathcal{H}^\infty(P, \mathbb{R}) \times \mathcal{H}^2(P, \mathbb{R}^d)$. Now set

$$\theta^{E*} = (z_2, \cdots, z_d). \quad (3.7)$$

As in Lemma 13 it follows that $\int_0^\cdot (\theta_s^S, \theta_s^{E*}) dW_s$ is a P–BMO martingale. The stochastic exponential $\mathcal{E}(-\int (\theta_s^S, \theta_s^{E*}) dW_s)$ is a uniformly integrable martingale and the Radon–Nikodym density of a probability measure $Q^* \in \mathbb{P}_e$ with respect to P. As in Lemma 7, we get $H(Q^*|P) < \infty$ and by (2.15) and (2.12) the maximal utility for every agent is finite. By virtue of $\phi^* = \frac{1}{\bar{\alpha}}(\theta^S - z_1)$, $(\int_0^t \phi_s^* (dW_s^1 + \theta_s^S ds))_{t \in [0,T]})$ is a Q^*–martingale. Hence, Q^* defines via (3.3) a pricing measure that attains the partial market clearing.

□

For the corresponding uniqueness result, we need the technical condition that the stochastic integral process associated with (θ^S, θ^{E*}) belongs to BMO.

Theorem 22 *Let $Q^{(\theta^S, \theta^{E*})} \in \mathbb{P}_f$ attain the equilibrium with partial market clearing and suppose that $(\int_0^t (\theta^S_s, \theta^{E*}_s) dW_s)_{t\in[0,T]}$ is a P–BMO martingale. Then we have $\theta^{E*} = (z_2, \ldots, z_d)$ and $\phi^* = \frac{1}{\bar{\alpha}}(\theta^S - z_1)$ where $z = (z_1, \ldots, z_d)$ is given by the solution of (3.6).*

Proof The proof of this statement is quite similar to the one of Theorem 15.

□

We conclude this section by noting that as in section 2.4, θ^{E*} satisfies a dynamic programming principle.

Remark 23 *Let the probability measure Q^* be given through (3.7) and (3.3). If the agents solve the conditioned optimization problem 8 for a stopping time $\tau \leq T$ with the same incomes (H_a), then Q^* attains also an equilibrium with partial market clearing.*

The arguments needed to prove this are as for Remark 16.

Chapter 4

A risk bond

In this chapter we construct the price process of a bond that is issued by an insurance company. The insurance tranfers some of its insured risk to the financial market. In contrast to Chapter 2, the terminal value of the bond is specified, in fact, it is chosen by the insurance. Also, the risk bond is not in zero net supply. The insurance is interested in selling it completely to the agents present at the market.

The agents with their incomes and preferences are modeled as in Chapter 2. They receive at the end of the trading time an income that depends on financial and external risk. Using the stock and the risk bond, the agents maximize the expected utility of their risky income and the terminal wealth of the trading strategy. They apply the exponential utility function. The solutions of the utility maximization problems determine the demand of the risk bond and the stock.

The insurance sells the bond at the beginning of the trading time. During the trading period, the agents trade the risk bond among themselves. At the terminal time, the insurance pays out the bond. This payout consists of two parts. The payout at the end of the trading time is described by a random variable that may depend on external and financial risk. During the trading period, the insurance may continuously pay out an interest with a rate that depends on the external risk and also on the market price for the external risk. Thus, a feedback of the opinion of market about the external risk to the structure as well as the volume of the risk bond is possible. Since the interest rate for the bank account is equal to zero, there is no difference, if the interest is payed as a lump sum at the end of the trading period. Ulrich Horst pointed out the importance of an interest rate that depends on the external risk as well as on the market. The methods developed in Chapter 2 can be easily adapted to this situation.

The equilibrium condition in this chapter is straightforward: the price of the risk bond has to be adjusted such that the trading strategies add up to one, since we assume that the insurance has issued exactly one share. After selling the bond, the insurance does not trade anymore. Thus, our equilibrium gives exactly the price such that the bond is completely sold.

As in Chapter 2 and Chapter 3, the agents considered in our model are assumed to be small traders at the stock market. Thus, the stock price is exogenuosly given and there is no market clearing for trading with the stock required within our group of agents.

What are possible price processes for the risk bond? One constraint is imposed by the absence of arbitrage condition. Since the stock price process is already fixed, we can only choose among the martingale measures for the stock price process. A price process of the risk bond is then the successive conditional expectation of its terminal value. We construct via a BSDE the density of a martingale measure for the stock price process and then a risk bond price process such that market clears for the bond.

A problem is to show that our bond completes the market. Using Malliavin calculus we give an abstract criterion as well as an example. The fact that the property of completness depends on the equilibrium price makes the problem very difficult, since this price is only implicitly described by the solution of a BSDE.

4.1 The risk bond

In this section we describe the structure and the set of price processes that we consider for the bond. Furthermore, we prove a criterion that characterizes market completion. Finally, for every possible price process of the risk bond, the utility maximizing payoffs of the agents are calculated. The stock market consists of a bank account with interest rate zero. The stock defined in (2.1) satisfies Assumption 1. Prices for payoffs replicable with the stock are already fixed: the price is the initial capital needed to replicate the payoff with a trading strategy. This initial capital is equal to the expectation of the payoff under a martingale measure for the stock. We consider a subset of martingale measures, where every element Q^η, $\eta \in \mathcal{V}$, is defined by (3.3) in Chapter 2. The set $\mathcal{V}$ of possible market prices of external risk is defined in (2.5). In our approach, we fix the terminal value of the risk bond. A candidate of a price process for the risk bond is calculated by taking successive conditional expectations. The price for a payoff F under Q^η, i.e. the initial capital that is needed to replicate F with both the stock and the risk bond, is equal to

$$E^\eta[F].$$

The payout of the risk bond consists of two parts. The first one is a random terminal payment H_I,

$$H_I = g_I(K, X^S)$$

where g_I may depend on the whole path of K and X^S. We assume that H_I is bounded, but not necessarily positive.

The second part accumulates with a rate $r(t, \theta_t, \rho_t(K.), \eta_t)$ that depends on several factors. θ_t is the market price of risk for all contingent claims replicable with the stock. This process replaces θ^E that is used in Chapter 2. The insurer may want to adjust his payment according to the evolution of the external risk factor. He uses a predictable process $(\rho_t(K.))_{t\in[0,T]}$ that describes the risk caused by the external factor K from his point of view. The insurance might require more capital if the external risk is seen as more dangerous. In this case, $r(\cdot, \rho_t, \cdot)$ is negative. On the other hand, if the external risk evolves in less dangerous way, the insurance might pay a higher interest. For those ρ_t, $r(\cdot, \rho_t, \cdot)$ is positive.

The most important point is the possibility to let the payout depend on the market via the market price of external risk η. Here, η replaces θ^E. All factors are connected by a deterministic function $r : [0, T] \times (\mathbb{R}^d)^3 \to \mathbb{R}$ that is a priori chosen by the insurer. So the insurer pays at time T

$$H_I + \int_0^T r(t, \theta_t, \rho_t, \eta_t)dt.$$

The payout $\int_0^T r(t, \theta_t, \rho_t, \eta_t)dt$ might be interpreted as an interest. Since the interest rate for the bank account is zero, it does not matter whether the interest r for the risk bond is payed continuously or as a lump sum at time T. We assume that the whole sum is payed at time T. The price that the insurer gets for his bond under a pricing measure Q^η, $\eta \in \mathcal{V}$, is

$$h_I(\eta) = E^\eta \left[H_I + \int_0^T r(t, \theta_t, \rho_t, \eta_t)dt \right], \tag{4.1}$$

where Q^η is defined analoguosly to Q^θ in (3.3).

How should the feedback of the market price of the external risk on r be chosen by the insurer? The choice reflects a supply of the insurer depending on the market. One possibility is the following: under a favorable pricing system, the insurer is willing to provide a large volume of the bond. Contrary, if the price is not favorable for the insurer, the volume of the bond might be much less. This can be modeled by an interest rate r that is decreasing in η. Consider the following BSDE:

$$Y^\eta = H_I - \int_t^T z_s dW_s + \int_t^T (r(s, \theta_s, \rho_s, \eta_s) - z_s \eta_s)ds.$$

Of course, $E^\eta[H_I + \int_0^T r(t,\theta_t,\rho_t,\eta_t)dt] = Y_0^\eta$. Applying the comparison theorem for BSDE under appropriate assumptions on r, we see that $E^{\tilde{\eta}}[H_I] \leq E^{\hat{\eta}}[H_I]$ if $\tilde{\eta} \geq \hat{\eta}$ $P \otimes \lambda$- a.e.

We intend to use the results in Kobylanski (Kob00) in order to compute an equilibrium pricing density η^* and thus an equilibrium pricing measure $Q^{\eta*}$. So our assumptions include Assumption (H1) and (H2) in (Kob00).

Assumption 24 *Let the terminal payoff H_I be bounded and $\mathcal{F}_T$–measurable. Let $r : [0,T] \times (\mathbb{R}^m)^3 \to \mathbb{R}$ together with the predictable processes θ and ρ satisfy $P \otimes \lambda$ a.s. for all $z \in \mathbb{R}^m$*

1. *$|r(t,\theta_t,\rho_t,z)| \leq c|\|z\|^2$ for a constant $c < \frac{1}{2}$ and*
2. *r is differentiable in z and satisfies for a constant c_2 and a continuous function $k : [0,T] \to \mathbb{R}$:*

$$\frac{\partial r}{\partial z}(r(t,\theta_t,\rho_t,z) \leq k(t) + c_1|z|$$

.

The next step is to specify the set of price processes for the risk bond. Here we describe the case where the cumulative interest is payed at time T. Price processes for continuously payed interest are given in Remark 25. We consider only price processes such that there exists a probabilty measure equivalent to P that sees the stock price and the price of the risk bond as martingales. In fact, we fix a martingale measure for the stock price Q^η, where $\eta \in \mathcal{V}$, and $\mathcal{V}$ is defined in (2.5). Then the risk bond price process B^η is defined by

$$\begin{aligned} B_t^\eta &= E^\eta[H_I + \int_0^T r(s,\theta_s,\rho_s,\eta_s)ds \mid \mathcal{F}_t] \qquad (4.2)\\ &= E^\eta[H_I] + \int_0^t \kappa_s(\eta)(dW_s^1 + \theta_s ds) + \int_0^t \upsilon_s(\eta)(dW_s^2 + \eta_s ds), \end{aligned}$$

where $(\kappa(\eta), \upsilon(\eta))$ is the integrand in the representation of

$$E^\eta\left[H_I + \int_0^T r(s,\theta_s,\rho_t,\eta_t)ds \,\middle|\, \mathcal{F}_t\right] \qquad (4.3)$$

as a stochastic integral with respect to the Q^η–Brownian motion

$$W_t^\eta = \begin{pmatrix} W_t^1 + \int_0^t \theta_s ds \\ W^2 + \int_0^t \eta_s ds \end{pmatrix}, \quad t \in [0,T].$$

Since there are two risky securities, we have to define a two dimensional trading strategy

$$\bar{\pi}_t = \begin{pmatrix} \bar{\pi}_t^S \\ \bar{\pi}_t^B \end{pmatrix}, \quad t \in [0,T].$$

The number of shares of stock owned is denoted with $\bar{\pi}_t^S$, whereas $\bar{\pi}_t^B$ stands for the numbers of shares of the risk bond. Let v_0 denote the initial capital. The wealth process of a trading strategy for $0 \le t \le T$ is

$$\begin{aligned} V_t(\bar{\pi}) &= v_0 + \int_0^t \bar{\pi}_u^S dX_u^S + \int_0^t \bar{\pi}_u^B dB_u^\eta \\ &= v_0 + \int_0^t \bar{\pi}_u^S (X_u^S \sigma_u^S + \kappa_u(\eta))(dW_u^1 + \theta_u du) \\ &\quad + \int_0^t \bar{\pi}_u^B \upsilon_u(\eta)(dW_u^2 + \eta_u du) \end{aligned}$$

Remark 25 *The wealth process $V(\bar{\pi})$ is the same if the interest is payed out continuously during $[0,T]$. A price process $\tilde{B}^\eta$ for the bond with a terminal payout H_I and instantaneous interest payment $r(t,\theta_t,\rho_t,\eta_t)$ is given by the successive conditional expectation of the payout that is not yet payed:*

$$\begin{aligned} \tilde{B}_t^\eta &= E^\eta\left[H_I + \int_t^T r(s,\theta_s,\rho_s,\eta_s)ds \,\middle|\, \mathcal{F}_t \right] \\ &= v_0 + \int_0^t \kappa_s(\eta)(dW_u^1 + \theta_u du) + \int_0^t \upsilon_u(\eta)(dW_u^2 + \eta_u)du \\ &\quad - \int_0^t r(s,\theta_s,\rho_s,\eta_s)ds. \end{aligned}$$

The integrands κ and υ are the same as in (4.2). Since in the time interval $[0,t]$ the interest $\int_0^t r(s,\theta_s,\rho_s,\eta_s)ds$ is already payed, this payout is not included in the conditioned expectation. However, this interest is part of the wealth process, the holder of $\bar{\pi}_u^B$ shares of the bond is entitled to get the payment rate $\bar{\pi}_u^B r(u,\theta_u,\rho_u,\eta_u)$:

$$V_t(\bar{\pi}) = v_0 + \int_0^t \bar{\pi}_u^S dX_u^S + \int_0^t \bar{\pi}_u^B d\tilde{B}_u^\eta + \int_0^t \bar{\pi}_u^B r(u,\theta_u,\rho_s\eta_u)du.$$

Thus, the interest cancels out:

$$V_t(\bar{\pi}) = v_0 + \int_0^t \bar{\pi}_u^S X_u^S \sigma_u^S (dW_u^1 + \theta_u du) + \int_0^t \bar{\pi}_u^B \upsilon_u(\eta)(dW_u^2 + \eta_u du).$$

So the investor get for both types of bonds the same wealth process if he uses the same trading strategy.

In the sequel we assume that $\int_0^T r(s,\theta_s,\rho_s,\eta_s)ds$ is payed at time T. The price process we use is B^η defined in (4.2).

The set of admissible trading strategies depends on the price process of the risk bond, i.e. on η. Similar to (4), those trading strategies are called *admissible* that lead to a wealth process that is a supermartingale under Q^η:

$$\mathcal{A}(\eta) = \{\bar{\pi} : V(\bar{\pi}) \text{ is } Q^\eta - \text{ supermartingale}\}.$$

An agent indexed with $a \in \mathcal{I}$ tries to find the optimal trading strategy:

$$\sup_{\bar{\pi}\in\mathcal{A}(\eta)} E\left[-\exp\left(-\alpha_a\left(v_0^a + \int_0^T \bar{\pi}_u d\begin{pmatrix} X_u^S \\ B_u^\eta \end{pmatrix}\right)\right)\right], \tag{4.4}$$

the initial capital of the agents being denoted by v_0^a.

We intend to use techniques relying on a complete market. So the next proposition gives a sufficient criterion on a risk bond process B^η to complete the market. A simple example of a risk bond completing the market is stated in Example 34 on page 61.

Proposition 26 (Complete market) *Let the terminal value H_I and the interest rate r of the bond be according to Assumption 24. Then a bond price process B^η completes the market if and only if*

$$v_t(\eta) \neq 0, \qquad P \otimes l \quad a.e. \tag{4.5}$$

Then to every $F \in L^1(Q^\eta,\mathcal{F})$, there exist a unique trading strategy $(\bar{\pi}^S,\bar{\pi}^B) \in \mathcal{A}(\eta)$ such that the wealth process is a $Q^\eta-$martingale that satisfies

$$F = E^\eta[F] + \int_0^T (\bar{\pi}_t^S,\bar{\pi}_t^B)d\begin{pmatrix} X_t^S \\ B_t^\eta \end{pmatrix}.$$

Proof. The proof consists of several steps. First we apply the martingale representation theorem under P (see e.g. (RY91) Th.V.3.5) to $M_t = E[Z_T^\eta F|\mathcal{F}_t]$, where Z^η is defined analogously to Z^θ in (2.7), and obtain an integrand $\alpha \in \mathcal{H}^1(P,\mathbb{R}^2)$ satisfying

$$M_t = E[Z_T^\eta F] + \int_0^t \alpha_t dW_t, \qquad t \in [0,T].$$

Itô's formula yields

$$d(M_t(Z^\eta)_t^{-1}) = [M_t(Z^\eta)_t^{-1}(\theta_t,\eta_t)^{tr} + (Z^\eta)_t^{-1}\alpha_t](dW_t + (\theta_t,\eta_t)^{tr}dt).$$

Thus, the integrand

$$\beta_t = M_t (Z^\eta)_t^{-1} (\theta_t, \eta_t)^{tr} + (Z^\eta)_t^{-1} \alpha_t, \quad t \in [0,T],$$

satisfies

$$F = E^\eta[F] + \int_0^T \beta_s dW_s^\eta,$$

and $M(Z^\eta)^{-1}$ is a $Q^\eta-$martingale. Let (4.5) be satisfied. Then we may set

$$\bar{\pi}_t^B = \frac{\beta_{2,t}}{\upsilon_t(\eta)}$$

and

$$\bar{\pi}_t^S = \frac{\beta_{1,t} - \bar{\pi}_t^B \kappa_t(\eta)}{\sigma_t^S X_t^S}.$$

This strategy $(\bar{\pi}_t^S, \bar{\pi}_t^B)$ is admissible since the wealth process is a $Q^\eta-$martingale.

In order to see uniqueness, assume (4.5). We apply a well known argument. Let $\bar{\pi}$ and $\tilde{\pi}$ be two admissible trading strategies attaining F such that their wealth processes are $Q^\eta-$martingales. Define $\delta = (\delta^S, \delta^B)$ by

$$\delta_s = \bar{\pi}_s - \tilde{\pi}_s, \quad s \in [0,T].$$

Then

$$\int (\delta_u^S \sigma_u^S X_u^S, \delta_u^B \upsilon_u(\eta)) dW_s^\eta$$

is a Q^η–martingale with terminal value zero. Thus the quadratic variation satisfies

$$\int_0^t (\delta_u^S \sigma_u^S X_u^S)^2 + (\delta_u^B \upsilon_u(\eta))^2 ds = 0, \qquad t \in [0,T].$$

This means, $\delta = 0$ within the equivalence class generated by the vector space $\mathcal{H}^2$.

It remains to show the necessity of (4.5). This part of the proof uses the Kunita–Watanabe decomposition: for every $(Q^\eta, \mathbb{F})-$martingale M with $\|M_T\|_{L^2(\Omega)} < \infty$ there exists a unique integrand ϕ and a unique martingale $N \in \mathcal{H}^2(Q^\eta)$ strongly orthogonal to (X^S, B^η) such that

$$M_t = \int_0^t \phi_u d(X_u^S, B_u^\eta)^{tr} + N_t.$$

So, if $\upsilon = 0$ on some set $A \in \Omega \times [0,T]$ with $P \otimes \lambda[A] \neq 0$, we can find a predictable process ϕ satisfying $\phi = 0$ $P \otimes \lambda-$a.e. on A^c and $F = M_T = \int_0^T \phi_s dW_s^\eta \in L^2(\Omega)$, $M_T \neq 0$ in $L^2(\Omega)$. Uniqueness in the Kunita–Watanabe decomposition yields that F can not be represented as a stochastic integral with respect to (X^S, B^η) where the integral is a B^η–martingale.

□

Here we describe the preferred payoffs of our agents. They maximize their utility with respect to an exponential utility function.

Proposition 26 leads to a budget constraint: having fixed a risk bond price process B^η, a contingent claim ξ is the sum of the income H_a and the terminal wealth of a replicable trading strategy $\bar{\pi}$, if ξ is in the budget set:

$$\mathcal{B}_a(\eta) = \{\, \xi \in L^1(\mathcal{F}_T, Q^\eta) \mid E^\eta[\xi] \leq E^\eta[H_a] + v_0^a \,\}.$$

Observe that we have the same budget sets as in (2.10) in Chapter 2.

The maximization problem concerning the set of admissible trading strategies is equivalent to a maximization problem considering the attainable claims in the budget set:

$$\sup_{\xi \in \mathcal{B}_a(\eta)} E\left[-\exp\left(-\alpha_a\, \xi\right)\right].$$

According to Theorem 6, the utility maximizing terminal wealth $\xi_a(\eta)$ for a risk bond price process B^η, $\eta \in \mathcal{V}$, is then

$$\xi_a(\eta) = c_a + \frac{1}{\alpha_a}\int_0^T \theta_u dW_u^1 + \frac{1}{\alpha_a}\int_0^T \eta_u dW_u^2 + \frac{1}{2\alpha_a}\int_0^T (\theta_u^2 + \eta_u^2)\, du, \quad (4.6)$$

where c_a is a constant choosen such that

$$E^\eta[\xi_a(\eta)] = E^\eta\left[H_a + \int_0^T r(t, \theta_t, \rho_t, \eta_t)dt + v_0^a\right].$$

The solution $(\bar{\pi}^{S,a}(\eta), \bar{\pi}^{B,a}(\eta))$ of (4.4) is called optimal trading strategy for the bond price B^η, $\eta \in \mathcal{V}$. Since a strategy with a wealth process that is a supermartingale can't be optimal, $(\bar{\pi}^{S,a}(\eta), \bar{\pi}^{B,a}(\eta))$ is the strategy that attains $\xi_a(\eta) - H_a$ with a wealth process that is a $Q^\eta-$martingale. Thus, $(\bar{\pi}^{S,a}(\eta), \bar{\pi}^{B,a}(\eta))$ satisfies

$$\begin{aligned}
\xi_a(\eta) - H_a &= v_0^a + \int_0^T \bar{\pi}_u^{S,a}(\eta) dX_u^S + \int_0^t \bar{\pi}_u^{B,a}(\eta) dB_u^\eta \\
&= v_0^a + \int_0^T \bar{\pi}_u^{S,a}(\eta) X_u^S \sigma_u^S (dW_u^1 + \theta_u du) \\
&\quad + \int_0^t \bar{\pi}_u^{B,a}(\eta) \upsilon_u(\eta)(dW_u^2 + \eta_u) du.
\end{aligned}$$

4.2 Partial market clearing

In this section we formulate our equilibrium with partial market clearing in presence of a risk bond and prove its existence. Partial market clearing simply says that the strategies using the bond have to sum up to one.

Definition 27 (equilibrium with risk bond) *Let $(H_a)_{a\in\mathcal{I}}$ be a family of bounded $\mathcal{F}_T$–measurable incomes, H_I the bounded $\mathcal{F}_T$–measurable terminal value of the risk bond and $(r(t,\theta_t,\rho_t,\eta_t))_{t\in[0,T]}$ the interest rate according to Assumption 24. Let X^S denote the exogenous stock price process according to (2.1), $(u_a)_{a\in I}$ a family of exponential utility functions with risk aversion coefficients $(\alpha_a)_{a\in\mathcal{I}}$, and $(\bar\pi^{S,a},\bar\pi^{B,a})(\eta)$ the utility maximizing trading strategies for the bond price B^η, $\eta\in\mathcal{V}$.*

A bond price process $B^{\eta*}$ with $\eta^*\in\mathcal{V}$ together with the utility maximizing trading strategies $(\bar\pi^{S,a},\bar\pi^{B,a})(\eta^*)$ is an equilibrium with partial market clearing, if

$$\sum_{a\in\mathcal{I}}\bar\pi_t^{B,a}(\eta^*)(\omega)=1 \qquad \text{for } P\otimes\lambda \text{ a.e.}(\omega,t). \tag{4.7}$$

In order to find the market price of risk satisfying condition (4.7) we state an equivalent condition on the sum of the individual utility maximizing terminal wealths: there exists an integrand $\phi\in\mathcal{H}^1(Q^{\eta*},\mathbb{R})$ satisfying

$$\begin{aligned}\sum_{a\in\mathcal{I}}\xi_a(\eta^*) &= -h_I(\eta^*)+H_I+\int_0^T r(t,\theta_t,\rho_t,\eta_t^*)dt+\sum_{a\in\mathcal{I}}H_a \qquad (4.8)\\ &\quad +\sum_{a\in\mathcal{I}}v_0^a+\int_0^T\phi_t(dW_t^1+\theta_t dt).\end{aligned}$$

This conditon on the market price of external risk η^* has the following meaning: on the left hand side we have the sum of the payoffs preferred by the investors. On the right hand side are price of the bond $-h_I(\eta^*)$ (defined in 4.1), the payoff of the risk bond, the sum of the incomes of the investors and the terminal wealth of the cumulative trading strategy $\frac{\phi}{\sigma^S X^S}$ with the stock.

The equation says that the price for the risk bond is chosen such that the investors buy it completely. The budget constraint for each investor yields that the cumulative price at time 0 of the risk bond is equal to the deterministic value $h_I(\eta^*)$. The investors redistribute their risky incomes among themselves. Additionally they use the stock in order to hedge financial risks. The market price of risk at the stock exchange θ is exogenuously given. Recall that there is no market clearing required for the stock within the group $\mathcal{I}$ of investors. In the next propsition we state the equivalence of (4.7), a condition on the trading strategies, and (4.8), a condition on the terminal values.

Proposition 28 *Let $\eta \in \mathcal{V}$. The optimal trading strategies for the bond $\bar{\pi}^{B,a}(\eta)$ satisfy condition (4.7) if and only if the sum of the utility maximizing terminal wealth satisfies condition (4.8).*

Proof. We proceed as in the proof of Lemma 11, where the equilibrium condition for a bond in zero net supply is related to a condition on the terminal wealth of the trading strategies.

First, we show the "only if" part. Let (4.7) be satisfied. Since $B_T^\eta = H_I + \int_0^T r(u, \theta_u, \rho_u, \eta_u)du$, we obtain with the linearity of the stochastic integral

$$\begin{aligned}\sum_{a\in\mathcal{A}} \xi_a(\eta) - \sum_{a\in\mathcal{I}} H_a &= \sum_{a\in\mathcal{I}} v_0^a + \int_0^T \sum_{a\in\mathcal{I}} \bar{\pi}_u^{S,a}(\eta) dX_u^S + \int_0^T \sum_{a\in\mathcal{I}} \bar{\pi}^{B,a}(\eta) dB_u^\eta \\ &= -h_I(\eta) + H_I + \int_0^T r(u, \theta_u, \rho_u, \eta_u)du \\ &\quad + \sum_{a\in\mathcal{I}} v_0^a + \int_0^T \sum_{a\in\mathcal{I}} \bar{\pi}_u^{S,a}(\eta) dX_u^S.\end{aligned}$$

Thus, if (4.7) is satisfied, then we obtain with

$$\phi_t = \sum_{a\in\mathcal{A}} \bar{\pi}_t^{S,a}(\eta) X_t^S \sigma_t^S$$

also (4.8). Since all $\xi_a(\eta)$ are integrable with respect to Q^η and the wealth process of an optimal trading strategy is a $Q^\eta-$martingale, $\phi \in \mathcal{H}^1(Q^\eta, \mathbb{R})$ is satisfied.

Now let (4.8) be satisfied. Let $\bar{\pi}^a(\eta) = (\bar{\pi}_t^{S,a}(\eta), \bar{\pi}_t^{B,a}(\eta))^{tr}$ be the hedging strategy that replicates $\xi_a(\eta) - H_a$ obtained by Proposition 26. The sum of the contingent claims hedged by the agents is equal to

$$\sum_{a\in\mathcal{A}} \xi_a(\eta) - H_a = \sum_{a\in\mathcal{A}} \left(x_a + \int_0^T \bar{\pi}_t^{S,a}(\eta) dX_t^S + \int_0^T \bar{\pi}_t^{B,a}(\eta) dB_t^\eta \right).$$

Recall that the utility maximizing trading strategies generate wealth processes that are $Q^\eta-$martingales. The only admissible trading strategy attaining the terminal value B_T^η of the bond with a martingale wealth process is $(\pi_t^S, \pi_t^B) = (0, 1)$, $t \in [0, T]$. Thus

$$H_I + \int_0^T r(t, \theta_t, \rho_t, \eta_t)dt - h_I(\eta) = \int_0^T 1 dB_t^\eta.$$

Furthermore, since

$$\sum_{a\in\mathcal{A}}(\xi_a(\eta) - H_a - v_0^a) = \int_0^T \phi_t(dW_t^1 + \theta_t dt) + \int_0^T 1dB_t^\eta,$$

the contingent claim on the left hand side is replicable by an admissible trading strategy $(\bar\pi^S, \bar\pi^B)(\eta)$ with $\bar\pi_t^B(\eta) = 1$ and $\bar\pi_t^S(\eta) = \frac{\phi_t}{X_t^S\sigma_t^S}$. On the other hand,

$$\sum_{a\in\mathcal{I}}(\xi_a(\eta) - H_a - v_0^a) = \sum_{a\in\mathcal{A}}\int_0^T \bar\pi_t^{S,a}(\eta)dX_t^S + \int_0^T \bar\pi_t^{B,a}dB_t^\eta.$$

The linearity of the stochastic integral and the uniqueness of the integrand yield that $(\sum_{a\in\mathcal{I}}\bar\pi_t^{S,a}(\eta), 1)$ is the sum of the individual utility maximizing trading strategies.

□

Our main result is a BSDE that characterizes the equilibrium market price of the external risk factor η^*. The BSDE can be obtained by combining (4.8) and the explicit structure of the utility maximizing terminal wealth ξ_a, $a\in\mathcal{I}$ of the investors. Before stating the BSDE, we do some preparations. Let $\bar\alpha$ satisfy

$$\frac{1}{\bar\alpha} = \sum_{a\in\mathcal{I}}\frac{1}{\alpha_a}.$$

Since all investors use exponential utility functions with different coefficients of risk aversion, the preferred payoffs differ only by a deterministic factor and a deterministic constant. So the sum has a simple structure:

$$\sum_{a\in\mathcal{I}}\xi_a(\eta) = c + \frac{1}{\bar\alpha}\int_0^T(\theta_t,\eta_t)^{tr}d\begin{pmatrix} W_t^1 \\ W_t^2\end{pmatrix} + \frac{1}{2\bar\alpha}\int_0^T(\theta_t^2+\eta_t^2)dt, \tag{4.9}$$

where $c = \sum_{a\in\mathcal{I}} c_a$ with the constants $c_a, a\in\mathcal{I}$, from (4.6). Denote

$$H = \bar\alpha\left(\sum_{a\in\mathcal{I}} H_a + H_I - h_I(\eta^*)\right) + \frac{1}{2}\int_0^T \theta_s^2 ds.$$

Now we plug (4.9) into (4.8) and rearrange the terms. Thus, η^* has to be chosen such that

$$\begin{aligned} H &= \bar\alpha c + \int_0^T(\theta_t - \bar\alpha\phi_t)dW_t^1 + \int_0^T \eta_t^* dW_t^2 + \\ &\quad + \int_0^T \frac{1}{2}(\eta_t^*)^2 + \theta_t^2 - \bar\alpha\phi_t\theta_t - \bar\alpha r(t,\theta_t,\rho_t,\eta_t^*)\, dt.\end{aligned}$$

A change of variables leads to the notation used in BSDEs:

$$z_t^S = \theta_t - \bar{\alpha}\phi_t, \quad z_t^B = \eta_t.$$

Now we are able to write down the BSDE that characterizes the equilibrium market price of risk η^*:

$$\begin{aligned} Y_t = H \quad &- \int_t^T (z_s^S, z_s^B)\, d\begin{pmatrix} W_s^1 \\ W_s^2 \end{pmatrix} \\ &- \int_t^T \left[\frac{1}{2}((z_s^B)^2 + \theta_s z_s^S - r(s, \theta_s, \rho_s, z_s^B)\right] ds \end{aligned} \tag{4.10}$$

In the following Theorem we show that the choice $\eta^* := z^B$ yields a partial equilibrium.

Theorem 29 *There exists a market price of risk process $\eta^* \in \mathcal{V}$ that leads to an equilibrium with partial market clearing. η^* can be constructed using the solution $(Y, (z^S, z^B))$ of the BSDE (4.10) by setting $\eta^* = z^B$.*

Proof. By Theorem 2.3 and Theorem 2.6 in (Kob00), equation (4.10) has a unique solution $(Y, (z^S, z^B)) \in \mathcal{H}^\infty(P, \mathbb{R}) \times \mathcal{H}^2(P, \mathbb{R}^2)$.

In Lemma 30 below we prove that $(\int_0^t z_s^B dW_s^2)_{t\in[0,T]}$ is a BMO–martingale, hence $z^B \in \mathcal{V}$. The choice $\eta^* := z^B$ and $\phi := \frac{1}{\bar{\alpha}}(\theta - z^S)$ yields that the utility maximizing wealths $\xi_a(\eta^*)$ satisfy the equilibrium condition on the terminal wealth (28). Lemma 11 leads to an equilibrium with partial market clearing. Via (4.2) we obtain a bond price process $B^{\eta*}$.

□

In the next lemma we prove the BMO property that we need to define the equilibrium prices.

Lemma 30 *Let $(Y, (z^S, z^B))$ be the solution of the BSDE (4.10). Then $\int_0^\cdot z_s^B dW_s^2$ is a $P-$BMO martingale.*

Proof. Let Q^0 be defined by $\frac{dQ^0}{dP} = \mathcal{E}(\int(\theta_s, 0)dW_s)$. We show that $\int_0^\cdot z_s^B dW_s^2$ is a Q^0-BMO martingale and apply then Theorem 3.6 in (Kaz94). This is possible because $\int_0^\cdot \theta_s dW_s^1$ is also a P- BMO martingale.

We have to show: there exists a constant $c > 0$ such that for all stoppping times $\tau \le T$

$$E^0[\int_\tau^T (z_s^B)^2 ds | \mathcal{F}_\tau] \le c.$$

Under the probability measure Q^0, the conditional expectations of the BSDE reads

$$E^0[H - Y_\tau|\mathcal{F}_\tau] = E^0[\int_\tau^T (\frac{1}{2}(z_s^B)^2 - r(s, \theta_s, \rho_s, z_s^B)ds|\mathcal{F}_\tau].$$

According to Assumption 24 on r and the a priori estimate Corollary 2.2 in (Kob00), the left hand side is bounded by a constant c_1 that does not depend on τ. Assumption 24 also yields

$$\frac{1}{2}(z_s^B)^2 - r(s, \theta_s, \rho_s z_s^B) \geq (\frac{1}{2} - a)(z_s^B)^2 - b$$

for constants $0 \leq a < \frac{1}{2}$ and $b > 0$. Thus we obtain for all stopping times $\tau \leq T$

$$E^0[\int_\tau^T (z_s^B)^2|\mathcal{F}_\tau] \leq \frac{c_1 + b}{\frac{1}{2} - a}.$$

□

4.3 Risk bond completing the market

In this section we give a criterion and an example for a risk bond that completes the market. We use Malliavin calculus. Let us explain the Clark– Ocone formula for a $d-$dimensional Brownian motion. This formula gives the integrand in a stochastic integral as the conditional expectation of the Malliavin derivative of the terminal value that the integral attains. In order to understand the $d-$dimensional Clark– Ocone formula, use the parameter space explained in Example 1.1.2 in Nualart (Nua95), this is $\mathbb{T} = [0, T] \times \{1, \dots, d\}$. The measure μ is the product of the Lebesgue measure and the measure that gives mass one to each point $1, \dots, d$. The space of all componentwise square integrable functions $L^2(\mathbb{R}_+; \mathbb{R}^d)$ is isomorphic to Let $W = (W_t^1, \dots W_t^d)$ be a d–dimensional Brownian motion. Then $W_{\bar{t}} = W_t^i$ for $\bar{t} = (t, i)$, $t \in [0, t], i \in \{1, \dots, d\}$. For any $h \in \mathbb{H}$, the random variables

$$W(h) = \sum_{i=1}^d \int_0^T h_t^i dW_t^i$$

are a centered Gaussian family of random variables satisfying

$$E[W(h)W(g)] = \int_{\mathbb{T}} h(\bar{t})g(\bar{t})d\mu(\bar{t}).$$

Let

$$D_{\bar{t}} = D_{(t,i)} \qquad t \in [0,T], \;\; i \in \{1,\ldots,d\},$$

denote the Malliavin derivative in the space $\mathbb{D}^{1,2}$ of Malliavin differentiable random variables (see Definition 1.2.1 on page 24 and page 27 in (Nua95)). Then the Clarke– Ocone formula for a d–dimensional Brownian motion reads for $F \in \mathbb{D}^{1,2}$:

$$F = E[F] + \sum_{i=1}^{d} \int_0^T E[\, D_{(t,i)}\, F \,|\, \mathcal{F}_t\,] dW_t^i.$$

The proof is analogous to the proof for the one– dimensional Clark– Ocone formula Proposition 1.3.5 in (Nua95).

The notation of a $d-$dimensional Malliavin derivative and Clarke– Ocone formula are tools to describe the integrand υ in (4.2) more exactly. The terminal value of the bond $B_T^{\eta *}$ is equal to

$$B_T^{\eta *} = H_I + \int_0^T r(t, \theta_t, \rho_t, \eta_t^*) dt.$$

Let H_M denote the payoff that is distributed among the agents in the group $\mathcal{I}$ in the case of an equilibrium with partial market clearing with the price measure $Q^{\eta *}$ and the bond price process $B^{\eta *}$:

$$H_M = \sum_{a \in \mathcal{A}} H_a + H_I - h_I(\eta^*) + \int_0^T r(t, \theta_t, \rho_t, \eta_t^*) dt + \int_0^T \phi_t (dW_t^1 + \theta_t dt),$$

where ϕ is the integrand in the equilibrium condition (4.8). The following proposition gives the integrand υ depending on H_M.

Proposition 31 *Let η^* be the market price of risk process that attains an equilibrium with partial market clearing according to Theorem 29. Assume that $B_T^{\eta *}$ is bounded and $B_T^{\eta *} \in \mathbb{D}^{1,2}$. Let H_M be bounded and $H_M \in \mathbb{D}^{1,2}$. Then the integrand υ in (4.2) is equal to*

$$\begin{aligned}
\upsilon_t &= E^{\eta *}[D_{(t,2)} B_T^{\eta *} | \mathcal{F}_t] - cov_t^*(B_T^{\eta *}, D_{(t,2)} H_M) && (4.11)\\
&= Z_t^{-1} E[Z_T D_{(t,2)} B_T^{\eta *} | \mathcal{F}_T] && (4.12)\\
&\quad - Z_t^{-1} E\left[B_T^{\eta *} Z_T \left(\eta_t^* + \int_t^T (D_{(t,2)} \theta_s, D_{(t,2)} \eta_s^*) dW_s \right) \middle| \mathcal{F}_t \right] \\
&\quad + Z_t^{-1} E\left[B_T^{\eta *} Z_T \int_t^T (\theta_s^* D_{(t,2)} \theta_s^* + \eta_s^* D_{(t,2)} \eta_s^*) ds \middle| \mathcal{F}_t \right].
\end{aligned}$$

Here, cov_t^* denotes the conditional covariance under $Q^{\eta *}$ with respect to $\mathcal{F}_t$.

Proof. The integrand (4.12) is already stated in (KO91). Let η^* be the market price of risk process that attains an equilibrium with partial market clearing according to Theorem 29. In order to keep the notation simple, we denote in this proof

$$Z = Z^{\eta*}, \quad Q^* = Q^{\eta*},$$

and W^* the Q^*-Brownian motion constructed via Girsanov transform applied to W.

According to (4.8), we have for a constant $c > 0$

$$Z_T^* = c \exp(-\bar{\alpha} H_M).$$

The proof consists of two parts. First we apply Itô's formula and the martingale representation theorem under the probability measure P and the Brownian motion W. Also under P we apply the Clark–Ocone formula. With Itô's formula we can transform the integrals into integrals with respect to W^*. Set

$$M_t = E[Z_T B_T^{\eta*}|\mathcal{F}_t] = E[Z_T B_T^{\eta*}] + \int_0^t \zeta_t dW_t, t \in [0,T],$$

where $\zeta \in \mathcal{H}^2(\mathbb{R}^2)$ is the integrand in the representation of M. This supposes only $B_T^{\eta*} \in L^1(Q^*)$. Thus

$$B_t^{\eta*} = Z_t^{-1} E[Z_T B_T^{\eta*}|\mathcal{F}_t] = Z_t^{-1} M_t.$$

Itô's formula yields

$$B_t^{\eta*} = E^*[B_T^{\eta*}] + \int_0^t Z_s^{-1}(\zeta_s + M_s(\theta_s, \eta_s^*)) \begin{pmatrix} dW_s^1 + \theta_s ds \\ dW_s^2 + \eta_s^* ds \end{pmatrix}, \quad t \in [0,T].$$

Hence

$$\upsilon_t = Z_t^{-1}(\zeta_t^2 + M_t \eta_t^*). \tag{4.13}$$

ζ^2 denotes the second component of ζ. In order to get more information about υ, we apply the Clarke–Ocone formula to $Z_T B_T^{\eta*}$ to obtain ζ more explicitly:

$$\zeta_t^i = E[D_{(t,i)}(Z_T B_T^{\eta*})|\mathcal{F}_t], \quad i = 1, 2.$$

We have to use the product and chain rules for Malliavin differentiation. Since H_M is bounded, there exists a constant c_0 such that $c_0 \leq H_M$ and we define

$$\tilde{e}(x) = \begin{cases} \exp(-x), & x > \bar{\alpha} c_0, \\ \exp(-\bar{\alpha} c_0), & x \leq \bar{\alpha} c_0. \end{cases}$$

Thus, $\exp(-\bar{\alpha}H_M) = \tilde{e}(H_M)$ and $\tilde{e}$ is Lipschitz continuous. Since H_M is assumed to be bounded and in $\mathbb{D}^{1,2}$, we obtain with Proposition 1.2.3 in Nualart (Nua95)

$$D_{(t,i)}Z_T = D_{(t,i)}\tilde{e}(H_M) = -\bar{\alpha}Z_T D_{(t,i)}H_M.$$

Since $B_T^{\eta*}$ is in $\mathbb{D}^{1,2}$ and bounded, we obtain

$$D_{(t,i)}(Z_T B_T^{\eta*}) = Z_T D_{(t,i)}B_T^{\eta*} - \bar{\alpha}B_T^{\eta*}Z_T D_{(t,i)}H_M. \tag{4.14}$$

Applying the Clark– Ocone formula to $Z_T = 1 - \int_0^T Z_s(\theta_s, \eta_s^*)dW_s$ we can write

$$\eta_t^* = -Z_t^{-1}E[D_{(t,2)}Z_T|\mathcal{F}_t] = -Z_t^{-1}\bar{\alpha}E[-Z_T D_{(t,2)}H_M|\mathcal{F}_t].$$

Combining all these derivatives we get from (4.13)

$$\begin{aligned} \upsilon_t &= Z_t^{-1}\underbrace{E[-\bar{\alpha}Z_T B_T^{\eta*}D_{(t,2)}H_M + Z_T D_{(t,2)}B_T^{\eta*}|\mathcal{F}_t]}_{\zeta_t^2} \\ &\quad + Z_t^{-1}\underbrace{E[Z_T B_T^{\eta*}|\mathcal{F}_t]}_{M_t}\, Z_t^{-1}\underbrace{E[\bar{\alpha}Z_T D_{(t,2)}H_M|\mathcal{F}_t]}_{\eta_t^*} \\ &= E^*[D_{(t,2)}B_T^{\eta*}|\mathcal{F}_t] - \bar{\alpha}\,\mathrm{cov}_t^*(B_T^{\eta*}, D_{(t,2)}H_M), \qquad t\in[0,T]. \end{aligned} \tag{4.15}$$

In order to see (4.12), we use $Z_T = \mathcal{E}(\int(\theta,\eta^*)dW)$. The Malliavin derivative of a stochastic integral is stated in (1.46) on page 38 in (Nua95). This yields (4.12).

□

So far, two not explicitly known parameters appear in Proposition 31. The first one is the random variable $\int_0^T \phi_s(dW_s^1 + \theta_s ds)$ within H_M. The second one is η^* that changes the interest rate of the bond. Here we give an example where the conditions stated in Proposition 31 involve only the incomes and the terminal value of the bond. We use the idea of Example 14 in Chapter 2. Let $\mathbb{F}^i = (\mathcal{F}_t^i)$ be the P– augmentation of the filtration generated by W^i, $i = 1, 2$.

Lemma 32 *Assume that the drift of the stock price θ^S is adapted to $\mathbb{F}^1$. Furthermore let the sum of the incomes $H = \sum_{a\in\mathcal{I}} H_a$ be decomposed into two parts:*

$$H = H_1 + H_2,$$

where H_i is measurable with respect to $\mathcal{F}_T^i$, $i = 1, 2$, and both random variables are bounded. Furthermore, let the interest rate r of the bond be equal to zero and H_I be $\mathcal{F}_T^2$–measurable. Then (4.11) simplifies to

$$\upsilon_t = E^{\eta*}[D_{(t,2)}H_I|\mathcal{F}_t] - cov_t^*(H_I, D_{(t,2)}(H_I + H_2)). \tag{4.16}$$

Proof. With the same arguments as in Example 14, we see that the market price of the external risk η^* depends only on $(H_2 + H_I)$ and satisfies for a constant c

$$H_2 + H_I = c + \int_0^T \eta_s^* dW_s^2 + \frac{1}{2}\int_0^T (\eta_s^*)^2 ds.$$

Furthermore, the adjustment of the market portfolio $\int_0^T \phi_s(dW_s^1 + \theta_s ds)$ is measurable with respect to $\mathcal{F}_T^1$. Observe that $D_{(t,2)}F = 0$ for any Malliavin differentiable random variable F that is $\mathcal{F}_T^1-$ measurable.

□

In this situation one may ask if the representation property of $B(0)$ entails the representation property of $B(\eta)$ for every $\eta \in \mathcal{V}$. In terms of the integrands: $\upsilon(0) \neq 0$ $P \otimes \lambda-$a.e. yields $\upsilon(\eta) \neq 0$ $Q^\eta \otimes \lambda$ a.e. In general, this is not true. Here is a counterexample for a one– dimensional Brownian motion.

Example 33 *Let $\eta_t = 1$, $t \in [0,T]$. Let for a $0 < k < T$*

$$H_I = \frac{1}{2}(W_T^2 - W_k^2).$$

Then

$$D_t H_I = W_T 1_{t \le T} - W_s 1_{t \le k},$$

where $1_{t \le k} = 1$ for $t \le k$ and 0 otherwise. Thus

$$\upsilon_t(0) = (W_t - W_k)1_{t>k}.$$

So we have a random variable with an integral representation that is equal to zero for $t < k$. However, under the equivalent probability measure Q^η with $\eta = 1$, we obtain an integrand $\upsilon(\eta)$ satisfying $\upsilon(\eta) \neq 0$ $Q^\eta \otimes \lambda$ a.e. To see this, observe that $D_t \eta_s = 0$, $t, s \in [0,T]$. Hence

$$\upsilon_t(\eta) = \begin{cases} E^\eta[W_T - W_k|\mathcal{F}_t] = E^\eta[-\int_k^T \eta_s ds|\mathcal{F}_t] = k - T, & t \le k, \\ E^\eta[W_T|\mathcal{F}_t] = W_t - (T-t), & t > k. \end{cases}$$

Here we give a simple example of a market that is completed by a weather bond. We use the explicit formula (4.13).

Example 34 (Temperature bond) The trading interval consists of N days, i.e. $T = N$. The external risk factor is the temperature curve during a heating period, modelled by a mean reverting Ornstein- Uhlenbeck process

$$dK_t = a(m - K_t)dt + dW_t^2, K_0 = 0.$$

The agents represent energy deliverers. During the heating period they sell more volume if the temperature is low. On the other hand, if the demand on energy volume is too large, the retailer has to buy more himself and gets no further benefit. So the income of an agent may have the following form:

$$H_a = c_a \sum_{i=1}^{N} ((k_0 - K_i)^+ - (k_1 - K_i)^+)$$

for temperature tresholds $k_1 < k_0$ and weights $c_a > 0$, $a \in \mathcal{A}$. In order to keep formulas short, denote

$$H_s = \sum_{i=1}^{N} ((k_0 - K_i)^+ - (k_1 - K_i)^+)$$

and

$$c_A = \sum_{a \in \mathcal{A}} c_a.$$

With the notation of Lemma 32, we have

$$H = H_2 = c_A H_s, \quad H_1 = 0.$$

What is the structure of a risk bond that completes the market? We choose the same structure as in the incomes but with the opposite sign. Set the interest rate equal to zero:

$$r = 0.$$

and the terminal payoff H_I that is equal to the value of the bond $B_T^{\eta *}$ at time T as

$$H_I = B_T^{\eta *} = -c_I H_s$$

with a constant c_I to be determined later. We may apply Proposition 26 and Lemma 32. The market is complete if and only if the integrand υ in (4.2) satisfies $\upsilon_t \neq 0$ $P \otimes l$ a.e. (4.16) yields

$$\upsilon_t = E^{\eta *}[\, c_I D_{t,2} H_s \, (\, \bar{\alpha}(c_A - c_I) H_s - E^{\eta *}[H_s | \mathcal{F}_t] - 1 \,) \,|\, \mathcal{F}_t \,].$$

The choice

$$c_I \geq c_A$$

implies that

$$\upsilon_t \leq -E^{*}[\, c_I D_{t,2} H_s \,|\, \mathcal{F}_t \,].$$

Proposition 1.2.3 in Nualart (Nua95) yields

$$D_{(t,2)}H = \sum_{i=1}^{N} 1_{t\leq i} \exp(-a(N-i)) 1_{k_1 \leq K_i \leq k_0}.$$

Our temperate process K is an Ornstein– Uhlenbeck process. The conditional law of K_i given $\mathcal{F}_t$, $t < i < T$, under P is Gaussian, thus equivalent to the Lebesgue measure. Thus

$$P[k_1 \leq K_i \leq k_0 | \mathcal{F}_t] > 0, \quad P - a.s., t < i < T,$$

Since our equilibrium price measure $Q^{\eta *}$ is equivalent to P, we obtain for all $t \in [0, T]$

$$E^* \left[\sum_{i=1}^{N} 1_{t\leq i} \exp(-a(N-i)) 1_{k_1 \leq K_i \leq k_0} \middle| \mathcal{F}_t \right] > 0, \quad Q^* - a.s.$$

In every interval $(i-1, i)$, $i = 1, \ldots, N)$, we have the successive conditioned expectations of the same random variable. We may estimate the integrand υ by a piecewise continuous version of the conditional expectation. Thus, there exists a version of υ_t such that $\upsilon_t > 0$ for $P \otimes \lambda$- a.e. (ω, t).

Chapter 5

Equilibrium with general utility functions

In this chapter we consider a larger class of utilities than the exponential. The utility functions are defined for positive wealth and required to satisfy the Inada conditions. However, we have to pay a price. In contrast to the exponential utility, it is not possible to characterize the equilibrium price by a BSDE. We calculate our equilibrium with partial market clearing in a one period model in a probability space $(\Omega, \mathcal{F}, P)$ where $(\Omega, \mathcal{F})$ is a Borel space.

The concept of the equilibrium with partial market clearing in this chapter is the same as in Chapters 2 through 4. We have two sources of risk: financial and external risk. We have agents concerned by both risk factors. Financial risk can be hedged on a stock market. In order to transfer external risk, the agents *complete the market:* they sign mutual contracts.

Financial risk is represented by a $\sigma-$algebra $\mathcal{G} \subset \mathcal{F}$. An example illustrates the external risk factor: this might be described by a random variable K. Then $\mathcal{F} = \sigma(K) \vee \mathcal{G}$. The risky incomes of the agents are modeled as $\mathcal{F}-$measurable random variables. In the sequel we don't use the fact that the risk factor is described as a real valued random variable.

Prices are considered as linear and given by the expectation under a probability measure equivalent to P. This is explained in Remark 17 on page 39. The probability measures that we use here in order to calculate prices are called *pricing measures.*

The stock market is given by $\mathcal{G}$ and an exogenously fixed pricing measure $Q^{\mathcal{G}}$ on $\mathcal{G}$. Every $\mathcal{G}-$measurable $Q^{\mathcal{G}}-$integrable payoff R can be traded on the financial market. The price is of course equal to $E^{Q^{\mathcal{G}}}[R]$. We call those claims replicable at the financial market.

Reasonable pricing measures are free of arbitrage. This means, the pricing measures we consider here agree on $\mathcal{G}$ with $Q^{\mathcal{G}}$, analogously to the martingale measures for the stock in the previous chapters.

Given a pricing measure, the agents choose the payoffs that maximize their expected utility among all random variables that are not more expensive than their income. In order to do so they trade on the stock market and sign mutual contracts. The equilibrium with partial market clearing is defined as in Chapter 3.2: the difference between the sum of the incomes and the sum of the preferred incomes has to be replicable on the stock market. Then the mutual contracts add up to zero.

We adapt the techniques used in the book Föllmer and Schied, (FS02), Chapter 3.4 to our situation. In (FS02), an Arrow– Debreu equilibrium is constructed. In this classical model, there is no exogenuos financial market. The agents obtain risky incomes and trade them among themselves. A pricing measure is constructed such that the sum of the utility maximizing payoffs is equal to the sum of the incomes. The market clears totally, not only partially.

5.1 Income, preferences, the market

In this section we describe the income of the agents, prices of random payoffs, the preferences and the market. We place ourselves in a probability space $(\Omega, \mathcal{F}, P)$. Since we aim at constructing continuous versions of conditioned expectations that depend on a parameter, we assume that regular conditined distributions exists. This is the case if $(\Omega, \mathcal{F})$ is a Borel space.

The agents in our model are exposed to two sources of risk. The first one is economic or financial risk that can be hedged on a financial market. This risk is represented by a $\sigma-$algebra $\mathcal{G} \subset \mathcal{F}$. The effect on the income of an agent can e.g. be described by a $\mathcal{G}-$measurable random variable S_a, $a \in \mathcal{I}$. The second typ of risk is caused by an external factor modeled by a real valued $\mathcal{F}-$measurable random variable K. The income H_a of an agent depends on both sources of risk. An example is

$$H_a = g_a(S_a, K), \qquad a \in \mathcal{I},$$

where $g_a : \mathbb{R}^2 \to \mathbb{R}$. We use the fact that all incomes H_a, $a \in \mathcal{I}$, are $\mathcal{F}-$measurable non-negative bounded random variables. The functions g_a and the random variables S_a, K are introduced in order to give an example. The sum of all incomes

$$H = \sum_{a \in \mathcal{I}} H_a$$

is called the market portfolio.

The price of a random payoff is calculated by taking the expectation under a probability measure equivalent to P. We call a random variable

$\phi \in L^1(\Omega, \mathcal{F}, P)$ price density. With ϕ, we define a probability measure Q^ϕ by

$$\frac{dQ^\phi}{dP} = \frac{\phi}{E[\phi]}.$$

The normalized price of an $\mathcal{F}-$measurable random payoff ξ is equal to

$$\frac{E[\phi\xi]}{E[\phi]} = E^\phi[\xi], \tag{5.1}$$

where E^ϕ is the expectation with respect to Q^ϕ. If we compare the price of two contingent claims under the same pricing measure, the regularizing denominator can be ignored.

Here we describe the finanial market in detail. All payoffs containing only financial risk can be *replicated on the financial market.* This means they can be sold or bought at a certain price. In our model, those risks are given by $\mathcal{G}-$measurable random variables. On the other hand, prices on the financial market are exogenously fixed. Here, we have a $\mathcal{G}-$measurable price density $\phi_\mathcal{G}$. In Assumption 35 below we state a condition on $\phi_\mathcal{G}$ that we use to calculate our equilibrium price density.

Assumption 35 *The price density $\phi_\mathcal{G}$ is bounded from above and away from zero: there exists constants $0 < \delta_0 < \delta_1$ such that*

$$0 < \delta_0 \leq \phi_\mathcal{G} \leq \delta_1 \quad P-a.s.$$

The set of payoffs that are replicable on the financial market is

$$\mathcal{R} = \{R \in L^1(\Omega, \mathcal{G}, Q^\mathcal{G}) \,|\, E[\, \phi_\mathcal{G} R\,] < \infty\}. \tag{5.2}$$

Since the agents are supposed to be small traders, they can buy or sell any amount of replicable contingent claims at the price determined by (5.1) with the price density $\phi_\mathcal{G}$. There is no market clearing required on the stock market.

The next remark compares the set $\mathcal{R}$ of replicable payoffs to the stock market in Chapter 3.

Remark 36 Let $T > 0$ be the end of a trading time. Let the probability space $(\Omega, \mathcal{F}, P)$ carry a $d-$dimensional Brownian motion $W = (W^1, \ldots, W^d)$ and let $\mathcal{F} = \mathcal{F}_T$, where $(\mathcal{F}_t)_{t\in[0,T]}$ is the $P-$completion of the filtration generated by W. Now let $(\mathcal{F}^1_t)_{t\in[0,T]}$ be the completion of the filtration generated by W^1. X^S denotes the stock price process according to (3.1) in Chapter 3 with the additional assumption that θ^S is predictable with respect to $(\mathcal{F}^1_t)$.

In contrast to Chapter 3, we restrict here trading strategies to be predictable with respect to $(\mathcal{F}_t^1)$. The $\sigma-$algebra $\mathcal{G}$ in (5.2) is equal to $\mathcal{F}_T^1$. In Chapter 3, trading strategies where allowed to use the whole information $(\mathcal{F}_t)$ that is available to an agent.

Let us explain the reason for this restriction. We could set $\mathcal{R}$ as the set of random variables such that there exists an integrand π predictable to $\mathbb{F}$ that is an admissible trading strategy. Then $\mathcal{G}$ would be the $\sigma-$algebra generated by $\mathcal{R}$: $\mathcal{G} = \sigma(\mathcal{R})$. Unfortunatly, not all $\sigma(\mathcal{R})-$ measurable random variables are replicable.

Here is a simple example: let $W = (W^1, W^2)$. We consider stochastic integrals with respect to W^1. An $\mathcal{F}-$predictable integrand would be $f(W_t^2)$ for a bounded function $f : \mathbb{R} \to \mathbb{R}$, thus

$$F = \int_0^T f(W_t^2) dW_t^1 \in \mathcal{R}.$$

Now consider the square of F:

$$F^2 = \int_0^T \int_0^t f(W_s^2) dW_s^1 f(W_t^2) dW_t^1 + \int_0^T f^2(W_t^2) dt.$$

Thus, F^2 is measurable with respect to $\sigma(F)$, but it is it impossible to represent this random variable with a stochastic integral with respect to W^2.

In order to *avoid opportunities of arbitrage*, the agents within the group $\mathcal{I}$ assign the same price to a replicable payoff as the financial market does. A price density ϕ is free of arbitrage if and only if the normalized price for a replicable payoff is the same under both price densities:

$$\frac{E[\phi_{\mathcal{G}} R]}{E[\phi_{\mathcal{G}}]} = \frac{E[\phi R]}{E[\phi]} \quad \forall\, R \in \mathcal{R} \tag{5.3}$$

This is satisfied if and only if

$$\frac{E[\phi|\mathcal{G}]}{E[\phi]} = \frac{\phi_{\mathcal{G}}}{E[\phi_{\mathcal{G}}]} \qquad P - a.s. \tag{5.4}$$

Hence we define the set $\mathcal{C}$ of pricing densities consistent with the financial market as

$$\mathcal{C} = \{\ \phi \mid \phi > 0,\ E[\phi|\mathcal{G}] = c\phi_{\mathcal{G}} \quad \text{for a } c > 0 \ \ P - a.s.\}. \tag{5.5}$$

The set $\mathcal{C}$ has the same meaning as the set of martingale measures for the stock as explained in Remark 17 in Chapter 3. Our equilibrium price will be a price density in $\mathcal{C}$. For a given price density $\phi \in \mathcal{C}$ an agent can choose a payoff in his budget set, i.e. the set of payoffs that are under ϕ not more expensive than his income H_a:

$$\mathcal{B}_a(\phi) = \{\xi \in L^1(\Omega, \mathcal{F}, Q^\phi) \mid 0 \leq \xi,\ E[\phi\xi] \leq E[\phi H_a]\ \} \tag{5.6}$$

Every agent acts on a complete market. He chooses the contingent claim in his budget set that maximizes his expected utility and solves the maximization problem

$$\xi_a(\phi) = \ \operatorname{argmax}_{\xi\in\mathcal{B}_a(\phi)} E[u_a(\xi)] \tag{5.7}$$

where $u_a : [0, \infty] \to \mathbb{R}$ is strictly growing, strictly concave, continuously differentiable on $(0, \infty)$ and satisfies the Inada conditions

$$\lim_{x\searrow 0} u_a'(x) = \infty, \qquad \lim_{x\to\infty} u_a'(x) = 0. \tag{5.8}$$

Additionally we impose that there exists a $\kappa > 0$ such that

$$\limsup_{x\to 0} x u_a'(x) = \kappa < \infty. \tag{5.9}$$

Observe that the utility function is only defined on $\mathbb{R}_+$, a negative wealth is not allowed. In the budget set of the agents, there is replicable and nonreplicable income included. However, on the financial market, the agents can at most sell the replicable income

$$\bar{R}_a = \text{ess sup}\{R \mid R \in \mathcal{R}, R \leq H_a\}.$$

If an agent wants to buy a replicable payoff that is more expensive than the replicable part of his income $\bar{R}_a$, he has to buy it from other agents within the group $\mathcal{I}$ and to pay with some of his nonreplicable income. However, the accumulated purchases of the agents cannot be more expensive than the replicable part of their income $\sum_{a\in\mathcal{I}} \bar{R}_a$. We make the asumption that the group of agents has enough nonrandom income h_0 such that they can afford the payoffs that are the solution of the utility maximization problems (5.7) for the budget sets $\mathcal{B}_a(\phi)$ for a class of price densities that contains our equilibrium price density. This constant h_0 has to be large enough.This depends in a nontrivial way on the utility functions that will be specified in Remark 39 below.

Assumption 37 *The income H_a is positive, bounded and satisfies*

$$P[H_a] > 0 \quad \textit{for all} \ \ a \in \mathcal{A}.$$

The market portfolio H satisfies the following condition:

$$H = h_0 + \tilde{H}$$

for a constant $h_0 > 0$ that is specified in (5.26) within Remark 39) and a bounded random variable $\tilde{H}$. There exist constants $0 < \epsilon_0 < s_0$ such that

$$0 < \epsilon_0 \leq \tilde{H} \leq s_0.$$

Here we describe the solution of the utility maximization problem. Define $I_a : (0, \infty) \to (0, \infty)$ as the continuous, strictly decreasing inverse function of u'_a. Thus,

$$\lim_{y \searrow 0} I_a(y) = \infty, \qquad \lim_{y \to \infty} I_a(y) = 0. \tag{5.10}$$

Applying the Legendre– Fenchel transform, we see that a random variable $X_a(\phi)$ is the solution of the utility maximization problem (5.7) if and only if it satisfies

$$\xi_a(\phi) = I_a(c_0 \phi) \tag{5.11}$$

for a constant c_0 such that $E[\phi I_a(c_0\phi)] = E[\phi H_a]$. This maximizer $\xi_a(\phi)$ is unique.

5.2 Equilibrium with partial market clearing

First we explain the usual idea of an equilibrium where the agents may only trade among themselves, i.e. without financial market. This means, the agents redistribute the market portfolio. This is the sum of all incomes $H = \sum_a H_a$. An *Arrow– Debreu equilibrium* is a collection of nonnegative payoffs ξ_a^e, $a \in \mathcal{I}$, together with a pricing density ϕ^e that satisfy:

$$\sum_{a \in \mathcal{I}} \xi_a^e = H, \qquad \xi_a^e = \xi_a(\phi^e), a \in \mathcal{I},$$

i. e. ξ_a^e solves the utility maximization problem of agent $a \in \mathcal{I}$ with respect to the pricing density ϕ^e. The pricing density ϕ^e in a usual Arrow— Debreu equilbrium does not need to be in $\mathcal{C}$, because there is no exogenously fixed price, hence no arbitrage. On the other hand, supply and demand must be equal.

We use the same concept of an equilibrium with partial market clearing as in Chapter 3. Fix a price density $\phi \in \mathcal{C}$. The difference between utility maximizing wealth $\xi_a(\phi)$ and the endowment H_a of an agent is the sum of a replicable payoff $R_a(\phi) \in \mathcal{R}$ that he has traded on the financial market and and a random payoff $C_a(\phi)$ that he has bought from other agents, i.e.

$$\xi_a(\phi) - H_a = R_a(\phi) + C_a(\phi).$$

A price density ϕ^* attains an *equilibrium with partial market clearing*, if the trades among the agents add up to zero:

$$\sum_{a \in \mathcal{I}} C_a(\phi^*) = 0 \qquad \text{P–a.s.}$$

Of course the sum of the replicable payoffs purchased by the agents

$$R(\phi^*) = \sum_{a \in \mathcal{I}} R_a(\phi^*) \tag{5.12}$$

is in general not equal to zero, but due the budget constraints yield $E[\phi_{\mathcal{G}} R(\phi^*)] = 0$. We call $R(\phi^*)$ adjustment of the market portfolio with respect to ϕ^*. This leads to our definition of the equilibrium with partial market clearing for one period models.

Definition 38 (Equilibrium with partial market clearing) *Let $(H_a)_{a \in \mathcal{I}}$ be the bounded nonnegative $\mathcal{F}-$measurable incomes, u_a, $a \in \mathcal{I}$, utility functions according to (5.8), (5.9) and (5.10). An* equilibrium with partial market clearing *consists of a price density $\phi^* \in \mathcal{C}$, the solutions of the utility maximization problems (5.7) $\xi_a(\phi^*)$ for the agents $a \in \mathcal{I}$ and a replicable payoff $R(\phi^*) \in \mathcal{R}$ according to (5.12) satisfying*

$$\sum_{a \in \mathcal{I}} \xi_a(\phi^*) = \sum_{a \in \mathcal{A}} H_a + R(\phi^*). \tag{5.13}$$

There are two differences to the usual Arrow– Debreu equilibrium: on one part of the market we don't require that there is no market clearing, but on the other hand the price density on this part of the market is already fixed.

Existence of an Arrow– Debreu equilibrium

In order to construct an Arrow– Debreu equilibrium, it is useful to adopt the view of a *representative agent*. He takes all the income $H = \sum_{a \in \mathcal{I}} H_a$ and redistributes it among the agents. Let $\bar{H}$ be a nonnegative random variable. An *allocation* $\zeta = (\zeta_a)_{a \in \mathcal{I}}$ of $\bar{H}$ consists of nonnegative random variables ζ_a, $a \in \mathcal{I}$, satisfying

$$\sum_{a \in \mathcal{I}} \zeta_a = \bar{H}.$$

The set $A(\bar{H})$ of allocations of $\bar{H}$ describes all possibilities to distribute $\bar{H}$ among the agents in the group $\mathcal{I}$:

$$A(\bar{H}) = \left\{ \zeta = (\zeta_a)_{a\in\mathcal{I}}, \zeta_a \in L^0(\Omega, \mathcal{F}, P),\ \zeta_a \geq 0, \sum_a \zeta_a = \bar{H} \right\}.$$

According to Lemma 3.57 in (FS02), there exists a unique allocation ζ^λ that solves (5.14). Here we sketch the ideas and techniques used in (FS02) for the construction of an Arrow– Debreu equilibrium (see Föllmer/Schied (FS02), proof of Lemma 3.57 on page 149 and Theorem 3.55 on page 148 and 153). We use stricter assumptions than Föllmer and Schied (FS02) for the construction of our equilibrium. The incomes (in (FS02) called endowments) H_a have to be in $L^0_+(\Omega, \mathcal{F}, P)$. This means they have to be nonnegative, $\mathcal{F}-$measurable and are considered as equal if they are $P-$a.s. equal. The market portfolio $H = \sum_{a\in\mathcal{I}} H_a$ satisfies $E[H] < \infty$. A nonnegative random variable ϕ is called pricing density if $E[\phi H] < \infty$. The utility functions $u_a : [0, \infty) \to \mathbb{R}$ have to be continuosly differentiable on $(0, \infty)$ and to satisfy (5.9). In order to keep our notation simple, we describe the result of (FS02) under our additional assumtion (5.8).

The goal is to find an allocation of the market portfolio $\zeta \in A(H)$ such that all agents are satisfied, i.e. ζ together with a pricing density ϕ^e is an Arrow– Debreu equilibrium. The first step is to solve weighted utility maximization problems. Define

$$\Lambda = \{\lambda = (\lambda_a)_{a\in\mathcal{I}} \in [0, 1]^{|\mathcal{I}|} | \sum_{a\in\mathcal{I}} \lambda_a = 1\}.$$

The number λ_a describes the importance that the representative agent assigns to agent $a \in \mathcal{I}$. For every $\lambda \in \Lambda$, the market portfolio H is redistributed in order to solve the following optimization problem:

$$\sup_{\zeta\in A(H)} \sum_{a\in\mathcal{I}} \lambda_a E[u_a(\zeta_a)]. \tag{5.14}$$

According to Lemma 3.57 on page 149 in (FS02), there exists a unique allocation ζ^λ of H that solves (5.14). This allocation ζ^λ is called λ- *efficient*. Furthermore, Lemma 3.57 (FS02) states a *first order condition*, i. e. there exists a price density ϕ^λ such that

$$\lambda_a u_a'(\zeta_a^\lambda) \leq \phi^\lambda, \quad \text{with equality on } \{\zeta_a^\lambda > 0\}. \tag{5.15}$$

This first order condition yields

$$\zeta_a^\lambda = I_a\left(\frac{\phi^\lambda}{\lambda_a}\right).$$

Furthermore, ζ_a^λ maximizes $E[u_a(\xi)]$ over all nonnegative $\xi \in L^\infty(\Omega, \mathcal{F}, P)$ satisfying

$$E[\phi^\lambda \xi] \leq E[\phi^\lambda \zeta_a^\lambda].$$

The contingent claim ζ_a^λ is the solution of an individual utility maximization problem with respect to the price density ϕ^λ and a budget set defined with ζ_a^λ instead of H_a. This means, ϕ^λ, ζ_a^λ, $a \in \mathcal{I}$ is an Arrow– Debreu equilibrium if for all $a \in \mathcal{I}$

$$E[\phi^\lambda \zeta_a^\lambda] = E[\phi^\lambda H_a].$$

Then all budget conditions are met. Otherwise, the weights λ_a have to be adjusted. Observe that $I_a(\frac{x}{\lambda_a})$ is increasing in λ_a. This means, the weight λ_a of an agent that obtains a too expensive payoff ζ_a^λ has to decrease, the weight λ_a of an agent with a too cheap payoff has to increase. To this end, define $g(\lambda) = (g_a(\lambda))_{a \in \mathcal{I}}$ by

$$g_a(\lambda) = \lambda_a + \frac{1}{E[V]} E[\phi^\lambda (H_a - \zeta_a^\lambda)], \quad \lambda \in \Lambda,$$

where $V = \kappa(1 + H)$, and κ according to (5.9).

Brouwers fixed point theorem yields a fixed point $\lambda^e \in \Lambda$ satisfying $g(\lambda^e) = \lambda^e$. Thus, the payoffs $\zeta_a^{\lambda e}$, $a \in \mathcal{I}$, with the price density $\phi^{\lambda e}$ constitute our equilibrium.

Equilibrium with partial market clearing

Our problem to find an equilibrium with partial market clearing is closely related to the construction of an Arrow– Debreu equilibrium. However, there is a difference. Our equilibrium price density ϕ^* has to satisfy $\phi^* \in \mathcal{C}$. On the other hand, we can adjust the market portfolio, i.e. add to H a replicable random payoff R satisfying $E[\phi_{\mathcal{G}} R] = 0$.

The price density obtained in the first order condition (5.15) for a $\lambda \in \Lambda$ depends on the payoff H distributed among the agents. Our idea is to add a replicable contingent claim $R^\lambda \in \mathcal{R}$ with $E[\phi_{\mathcal{G}} R^\lambda] = 0$ to H . Then we solve the weighted utility maximization problem (5.14) over all allocations of $H + R^\lambda$. In order to distinguish between allocations of the market portfolio H and the allocation of an adjusted payoff $H + R^\lambda$, the solution is denoted $\bar{\zeta}^\lambda$. R^λ is chosen in Lemma 40 such that the first order condition (5.15) applied to $\bar{\zeta}^\lambda$ yields a pricing density $\bar{\phi}^\lambda \in \mathcal{C}$. According to Lemma 3.57 (c) in (FS02), $\bar{\zeta}_a^\lambda$ maximizes $E[u_a(\xi)]$ over all $\xi \in L^0(\Omega, \mathcal{F}, P)$ satisfying

$$E[\bar{\phi}_a^\lambda \xi] \leq E[\bar{\phi}_a^\lambda \bar{\zeta}_a^\lambda.]$$

It remains to find a $\lambda^* \in \Lambda$ and $\bar{\phi}^{\lambda *}$, $R^{\lambda *}$ such that the components $\bar{\zeta}_a^{\lambda^*}$, $a \in \mathcal{I}$, of the allocation $\bar{\zeta}^{\lambda *}$ of $H + R^{\lambda *}$ that solve (5.14) for λ^* satisfy the budget condition with equality. This is done using a fixed point argument: define the function $g = (g_a(\lambda))_{a \in \mathcal{A}}$ as

$$g_a(\lambda) = \lambda_a + \frac{1}{E[\kappa(1 + H + R^\lambda)]} E[\bar{\phi}^\lambda (H_a - \bar{\zeta}_a^\lambda)].$$

In Lemma 41 we show that our function g satisfies $g(\Lambda) \subseteq \Lambda$ and that g is continuous. Then Brouwers fixed point theorem yields a fixed point λ^* of g. Thus, the individual budget constraints are satisfied. Then the price density $\bar{\phi}^*$, the utility maximizing payoffs $\bar{\zeta}_a^*$, $a \in \mathcal{I}$ and the replicable payoff $R^{\lambda *}$ are an equilibrium with partial market clearing. This is stated in Theorem 42.

Let us first explain how a pricing density gained by the first order condition depends on the payoff that is distributed among the agents.

The function $f : \Lambda \times [0, \infty]$ defined as

$$f(\lambda, y) := \sum_{a \in \mathcal{A}} I_a \left(\frac{y}{\lambda_a} \right)$$

is for fixed $y > 0$ bounded from above and away from zero in $\lambda \in \Lambda$, jointly continuous in all $(\lambda, y) \in \Lambda \times [0, \infty)$ and strictly decreasing in y. (5.8) yields $\lim_{y \searrow 0} f(\lambda, y) = +\infty$ and $\lim_{y \to \infty} f(\lambda, y) = 0$. This function f is already used in (FS02) to find the solution of (5.14). Define $h : \Lambda \times [0, \infty]$ as the unique solution of

$$f(\lambda, h(\lambda, x)) = x. \tag{5.16}$$

The function $h(\lambda, x)$ is strictly decreasing in x and for fixed $x > 0$ bounded from above and away from zero in λ. Furthermore $h(\lambda, 0+) = +\infty$ and $h(\lambda, \infty) = 0$. In Föllmer/Schied (FS02) page 153 it is shown that h is continuous in λ using the continuity of f in (λ, y) and the compactness of $[0, \infty]$. Their argument shows in fact that h is jointly continuous in (λ, x).

A pricing density ϕ^λ resulting from the first order condition applied on a $\lambda-$efficient allocation ζ^λ of H satisfies

$$f(\lambda, \phi^\lambda) = \sum_{a \in \mathcal{A}} I_a \left(\frac{\phi^\lambda}{\lambda_a} \right) = H.$$

On the other hand, we can apply h to the coefficients $\lambda = (\lambda_a)$ and the market portfolio H:

$$h(\lambda, H) = \phi^\lambda \qquad \forall\, \lambda \in \Lambda. \tag{5.17}$$

Using (5.17) we construct the adjustment of the market portfolio $R^\lambda \in \mathcal{R}$ such that the price density $\bar{\phi}^\lambda = h(\lambda, H + R^\lambda)$ satisfies for a function $c : \Lambda \to \mathbb{R}^+ \backslash \{0\}$

$$E[h(\lambda, H + R^\lambda)|\mathcal{G}] = c(\lambda)\phi_{\mathcal{G}} \quad \text{P–a.s.,} \tag{5.18}$$

hence $\bar{\phi}^\lambda \in \mathcal{C}$, thus this price density is free of arbitrage. Since the utility functions u_a, $a \in \mathcal{I}$ are only defined on $[0, \infty)$, the adjusted market portfolio must be nonnegative. Furthermore, R^λ has price zero on the financial market. Thus, R^λ has to satisfy

$$H + R^\lambda > 0, \tag{5.19}$$

$$E[\phi_{\mathcal{G}} R^\lambda] = 0. \tag{5.20}$$

Construction of R^λ

Here we sketch the construction of R^λ and specify the constant h_0 stated in Assumption 37. Of course, we summarize the result in Lemma 40 on page 77 and prove it. In order to find R^λ satisfying $E[\phi_{\mathcal{G}} R^\lambda] = 0$, we aim at applying the intermediate value theorem for continuous functions. The first step is to find a constant $c_m > 0$ and for every $\lambda \in \Lambda$ a $R^\lambda_m \in \mathcal{R}$ satisfying

$$R^\lambda_m \leq 0, \quad H + R^\lambda_m \geq \varepsilon_0 > 0 \tag{5.21}$$

and

$$E[h(\lambda, H + R^\lambda_m)|\mathcal{G}] = c_m \phi_{\mathcal{G}} \quad P - a.s.$$

Then we show that there exists a constant $c_p > 0$ and for every $\lambda \in \Lambda$ a $R^\lambda_p \in \mathcal{R}$ satisfying

$$R^\lambda_p \geq 0 \quad \text{and} \quad E[h(\lambda, H + R^\lambda_p)|\mathcal{G}] = c_p \phi_{\mathcal{G}} \;\; P - a.s.$$

Since $h(\lambda, \cdot)$ is strictly decreasing, we have $c_p < c_m$. Next we show that for every $c \in [c_p, c_m]$ there exists R^λ_c satisfying

$$R^\lambda_m \leq R^\lambda_c \leq R^\lambda_p \quad \text{and} \quad E[h(\lambda, H + R^\lambda_c)|\mathcal{G}] = c\phi_{\mathcal{G}} \;\; P - a.s.$$

The key in our proof is that

$$c \mapsto E[\phi_{\mathcal{G}} R^\lambda_c], \quad c \in [c_p, c_m]$$

is continuous. Since $E[\phi_{\mathcal{G}} R^\lambda_m] \leq 0$ and $E[\phi_{\mathcal{G}} R^\lambda_p] \geq 0$, there exists a $c(\lambda)$ and a $R^\lambda := R^\lambda_{c(\lambda)}$ satisfying (5.18) and (5.20).

In order to construct R^λ we define random functions that are continuous in (λ, x) and versions of the conditioned expectation $E[h(\lambda, h_0 + \tilde{H} + x)|\mathcal{G}]$. Recall that we assume $H = h_0 + \tilde{H}$ for a constant h_0 and a random variable satisfying $\epsilon_0 < \tilde{H} < s_0$ for constants $0 < \epsilon_0 < s_0$ (see assumption 37). Since $(\Omega, \mathcal{F})$ is assumed to be a Borel space, we max fix a version of the regular conditioned distribution $P[\tilde{H} \in dw|\mathcal{G}](\omega)$ and define

$$\Psi_\omega(\lambda, x) = \int_0^\infty h(\lambda, h_0 + w + x) P[\tilde{H} \in dw|\mathcal{G}](\omega), \quad x \geq -h_0. \tag{5.22}$$

Since $h(\lambda, \cdot)$ is decreasing and $0 < \epsilon_0 < \tilde{H}$, there exists a set $N \in \mathcal{F}$, $P[N] = 0$ such that the functionals $\Psi_\omega(\lambda, x)$, $\omega \in \Omega \backslash N$ are uniformly bounded and continuous in (λ, x). For $\omega \in N$, we modify our functionals to $\Psi_\omega(\lambda, x) = h(\lambda, \epsilon_0 + h_0 + x)$.

In the next remark we specify the constant h_0 that we require in Assumption 37 as well as the constants c_m and c_p.

Remark 39 In Assumption 37, we stated that the sum of the incomes of the agents H satisfies

$$H = h_0 + \tilde{H}$$

for a bounded nonnegative random variable $\tilde{H}$ and a constant h_0. This constant must be large enough to allow the construction of R_m^λ such that $H + R_m^\lambda > \varepsilon_0$ for all $\lambda \in \Lambda$. Recall that δ_0 and δ_1 are the lower and upper bound of $\phi_{\mathcal{G}}$. In order to find R_m^λ, we aim at adjusting h_0 and $c_m > 0$ such that

$$\Psi_\omega(\lambda, 0) \leq c_m \delta_0 \leq c_m \delta_1 \leq \Psi_\omega(\lambda, -h_0 + \epsilon). \tag{5.23}$$

Here we estimate the functionals $\Psi_\omega(\lambda, x)$ simultanuosly for all $\omega \in \Omega$ with a deterministic function of x. Since $s_0 \geq \tilde{H} \geq \epsilon_0$ and $h(\lambda, \cdot)$ is strictly decreasing, we have for $x > -h_0$

$$h(\lambda, h_0 + s_0 + x) \leq \Psi_\omega(\lambda, x) \leq h(\lambda, h_0 + \epsilon_0 + x) \quad P - a.s. \; \forall\, \lambda \in \Lambda. \tag{5.24}$$

Thus, (5.23) is satisfied if we find h_0 and c_m such that

$$h(\lambda, h_0 + \epsilon_0) \leq c_m \delta_0 < c_m \delta_1 \leq h(\lambda, s_0).$$

The following choice fulfills our requirement:

$$c_m = \frac{1}{\delta_1} \min_{\lambda \in \Lambda} h(\lambda, s_0) > 0. \tag{5.25}$$

and

$$h_0 = \max_{\lambda \in \Lambda} f(\lambda, c_m \delta_0) - \epsilon_0. \tag{5.26}$$

In order to prepare the construction of R_p^λ, we specify c_p. Since $\Psi_\omega(\lambda,\cdot)$ is strictly decreasing and $\lim_{x\to\infty}\Psi_\omega(\lambda,\cdot)=0$ for all $\lambda\in\Lambda$ and $\omega\in\Omega$, a constant c_p satisfying

$$\Psi_\omega(\lambda,0)>c_p\delta_1$$

is apropriate. Recall that for fixed $\lambda\in\Lambda$, $h(\lambda,\cdot)$ is the inverse function of $f(\lambda,\cdot)$, see (5.16). Now we have to find c_p. Thus, (5.24) yields that

$$c_p=\frac{1}{\delta_1}\max_{\lambda\in\Lambda}h(\lambda,h_0+s_0) \tag{5.27}$$

is sufficient. Now let $y\in[\delta_0,\delta_1]$ and $\lambda\in\Lambda$. Then there exists an $x_m\in[-h_0+\epsilon_0,0]$ satisfying $h(\lambda,h_0+x_m)=c_my$ and an $x_p\geq 0$ such that $h(\lambda,h_0+x_p)=c_py$. Since $h(\lambda,\cdot)$ is decreasing, we have $c_p\leq c_m$, thus the interval $[c_p,c_m]$ is well defined.

Lemma 40 *Suppose that the market portfolio H satisfies assumption (37) with the constant h_0 stated in (5.26). For every $\lambda\in\Lambda$ there exists a $\mathcal{G}$- measurable random variable R^λ and a continuous function $c:\Lambda\to\mathbb{R}_+$ satisfying for all $\lambda\in\Lambda$*

$$\begin{aligned} E[h(\lambda,H+R^\lambda)|\mathcal{G}] &= c(\lambda)\phi_{\mathcal{G}} \qquad P\text{–a.s.}, & (5.28)\\ E[\phi_{\mathcal{G}}R^\lambda] &= 0 & (5.29)\\ H+R^\lambda &\geq \varepsilon_0>0 \qquad P\text{–a.s.} \end{aligned}$$

$R^\lambda(\omega)$ can be choosen for all $\omega\in\Omega$ as a continuous function of λ. Furthermore, there exists a constant b_0 such that $-h_0<R^\lambda\leq b_0$ for all $\omega\in\Omega$.

Proof. Let Y be a $\mathcal{G}-$measurable random variable satisfying

$$0\leq Y(\omega)\leq\Psi_\omega(\lambda,-h_0),\quad \omega\in\Omega.$$

Since $\Psi_\omega(\lambda,\cdot)$ is strictly decreasing and $\Psi_\omega(\lambda,x)$ is $\mathcal{G}-$measurable for all $x>h_0$, there exists a unique $\mathcal{G}-$measurable random variable $R=:\Psi_\omega^{-1}(\lambda,Y(\omega))$ satisfying

$$\Psi_\omega(\lambda,R)=Y(\omega).$$

Inequality (5.24) yields that

$$R_m^\lambda=\Psi_\omega^{-1}(\lambda,c_m\phi_{\mathcal{G}}(\omega))$$

satisfies $-h_0\leq R_m^\lambda\leq 0$, thus, $E[\phi_{\mathcal{G}}R_m^\lambda]\leq 0$. Furthermore, let

$$R_p^\lambda=\Psi_\omega^{-1}(\lambda,c_p\phi_{\mathcal{G}})\delta_1.$$

Applying (5.23), we see that R^λ_p satisfies

$$0 \le R^\lambda_p \le \max_{\lambda\in\Lambda} f(\lambda, c_p\delta_0) =: b_0, \quad \lambda \in \Lambda, P - a.s.$$

In order to find R^λ that satisfies (5.28) and (5.29) we define for every $c \in [c_p, c_m]$ a $\mathcal{G}$–measurable random variable R^λ_c that satisfies

$$\Psi_\omega(\lambda, R^\lambda_c) = c\phi_{\mathcal{G}} \qquad P - a.s.$$

For every $\omega \in \Omega$, the function $\Psi_\omega(\lambda, x)$ is jointly continuous in (λ, x). With the same argument as stated in (FS02) on page 153 for the equation $f(\lambda, y) = x$, using the compactness of $[0, \infty]$, we see that the solution x of

$$\Psi_\omega(\lambda, x) = cy$$

depends jointly continuously on (λ, c). Thus for every $\omega \in \Omega$, $R^\lambda_c(\omega)$ is continuous in (λ, c). Furthermore, R^λ_c is strictly decreasing in c and bounded:

$$-h_0 < R^\lambda_m < R^\lambda_c < R^\lambda_p \le b_0, \quad \omega \in \Omega,$$

for a $b_0 > 0$.

Thus

$$(\lambda, c) \mapsto E[\phi_{\mathcal{G}} R^\lambda_c]$$

defines a continuous function that is strictly decreasing (applying dominated convergence). For every $\lambda \in \Lambda$ there exists a $c(\lambda) \in [c_m, c_p]$ satisfying

$$E[\phi_{\mathcal{G}} R^\lambda_{c(\lambda)}] = 0.$$

The solution $c(\lambda)$ and $R^\lambda(\omega) := R^\lambda_{c(\lambda)}(\omega)$ for all $\omega \in \Omega$, depend continuously on λ.

□

In order to prepare the fixed point argument we use in the construction of our equilibrium, we prove that the function g that adjusts the weights of the agents, is continuous and the image of Λ is contained in Λ.

Lemma 41 *The function $g = (g_a(\lambda))_{a\in\mathcal{A}}$ defined as*

$$g_a(\lambda) = \lambda_a + \frac{1}{E[\kappa(1 + H + R^\lambda)]} E[\bar{\phi}^\lambda(H_a - \bar{\zeta}^\lambda_a)] \tag{5.30}$$

is continuous and $g(\Lambda) \subseteq \Lambda$.

Proof We may apply the theorem of bounded convergence. For all $\omega \in \Omega$, $R^\lambda(\omega)$ is continuous in λ. Recall that

$$\bar{\phi}^\lambda = h(\lambda, h_0 + \tilde{H} + R^\lambda)$$

with $-h_0 \leq R^\lambda \leq b_0$ and $\epsilon_0 \leq \tilde{H} \leq s_0$. Thus $\bar{\phi}^\lambda(\omega)$ is for all $\omega \in \Omega$ uniformly bounded above and away from zero and continuous in λ. Hence,

$$\bar{\zeta}_a^\lambda = I_a\left(\frac{\bar{\phi}^\lambda}{\lambda_a}\right).$$

is also continuous in λ and bounded. The theorem of Lebesgue yields that $g(\lambda)$ is continuous for all $\lambda \in \Lambda$.

As explained on p. 151 in Föllmer/Schied (FS02), κ defined in (5.9) satisfies

$$u_a'(\bar{\zeta}_a^\lambda)\bar{\zeta}_a^\lambda \leq \kappa(1 + H + R^\lambda) \in L^1(P). \tag{5.31}$$

The first order condition (5.15) yields $g_a(\lambda) \geq 0$. The sum $\sum_{a \in \mathcal{A}} g_a(\lambda)$ is equal to 1 because

$$\sum_{a \in \mathcal{A}} (H_a - \bar{\zeta}_a^\lambda) = R^\lambda,$$

this is a $\mathcal{G}-$measurable random variable with

$$E[\bar{\phi}^\lambda R^\lambda] = E[\phi_{\mathcal{G}} R^\lambda] = 0.$$

So

$$\sum_{a \in \mathcal{A}} g_a(\lambda) = \sum_{a \in \mathcal{A}} \left(\lambda_a + \frac{1}{E[\kappa(1 + H + R^\lambda)]} E[\bar{\phi}^\lambda (H_a - \bar{\zeta}_a^\lambda)]\right) = 1.$$

□

The following theorem summarizes the existence of a partial equilibrium.

Theorem 42 *Let the sum of the endowments H satisfy Assumption 37, the pricing density satisfy Assumption 35 and the utility functions be given according to (5.8), (5.9) and (5.10). Then there exists a price density $\phi^* \in \mathcal{C}$ that is consistent with the price density $\phi_{\mathcal{G}}$ on $\mathcal{G}$ such that the utility maximizing contingent claims $(\xi_a(\phi^*))_{a \in \mathcal{I}}$ with respect to ϕ^* satisfy the partial market clearing condition*

$$\sum_{a \in \mathcal{I}} \xi_a(\phi^*) = \sum_{a \in \mathcal{I}} H_a + R(\phi^*)$$

for a replicable payoff $R(\phi^*)$ satisfying $E[\phi_{\mathcal{G}} R(\phi^*)] = 0$. Thus, ϕ^*, $\xi_a(\phi^*)$, $a \in \mathcal{I}$ and $R(\phi^*)$ are an equilibrium with partial market clearing according to Definition 38.

Proof. The function $g : \Lambda \to \Lambda$ defined in (5.30) is continuous and the set Λ is convex and compact. Brouwer's fixed point theorem yields a $\lambda^* \in \Lambda$ satisfying $g(\lambda^*) = \lambda^*$. The price density

$$\phi^* = \bar{\phi}^{\lambda*} = h(\lambda, H + R^{\lambda*})$$

with a replicable payoff $R^{\lambda*} \in \mathcal{R}$ constructed in Lemma 40 is consistent with the financial market, because for a $c(\lambda^*) > 0$ we have

$$E[\bar{\phi}^{\lambda*}|\mathcal{G}] = E[h(\lambda, H + R^{\lambda*})|\mathcal{G}] = c(\lambda^*)\phi_{\mathcal{G}} \quad P - a.s.$$

Let $\bar{\zeta}^{\lambda*}$ be the solution of the weighted utility maximization problem (5.14) for λ^* and the adjusted market portfolio $H + R^{\lambda*}$. Of course,

$$\sum_{a \in \mathcal{I}} \bar{\zeta}_a^{\lambda*} = \sum_{a \in \mathcal{I}} H_a + R^{\lambda*}.$$

Since λ^* is a fixed point of g, $\bar{\zeta}_a^{\lambda*}$ satisfies

$$E[\bar{\phi}^{\lambda*}\bar{\zeta}_a^{\lambda*}] = E[\bar{\phi}^{\lambda*}H_a], \quad a \in \mathcal{I}.$$

Thus, the random payoff $\bar{\zeta}_a^{\lambda*}$ is in the budget set $\mathcal{B}_a(\bar{\phi}^{\lambda*})$ of agent a. According to Lemma 3.57 in (FS02),

$$\xi_a(\phi^*) = \bar{\zeta}_a^{\lambda*}, \quad a \in \mathcal{I}$$

solves the individual utility maximization problem of agent a with respect to $\phi^* = \bar{\phi}^{\lambda*}$. Thus, $\phi^* = \bar{\phi}^{\lambda*}$, $\xi_a(\phi^*) = \bar{\zeta}_a^{\lambda*}$, $a \in \mathcal{I}$ and $R(\phi^*) = R^{\lambda*}$ are an equilibrium with partial market clearing.

□

Chapter 6

An incomplete market

In this section we turn to an equilibrium in an incomplete market. Let P be a probability measure on a Borel space $(\Omega, \mathcal{F})$. We assume that the incompleteness is described in a special way: the agents may trade only random payoffs that are measurable with respect to some sub– σ algebra $\mathcal{T} \subseteq \mathcal{F}$. Chapter 3.4 in the book of Föllmer and Schied (FS02) constructs an Arrow– Debreu equilibrium in a complete market: in their setup, all $\mathcal{F}-$measurable payoffs are tradeable. Why is it reasonable to consider an incomplete market? The income of an agent may depend on an observable influence like the temperature and on other non– observable factors. Then the community of agents is willing to transfer risks only depending on the observable factor. The information that is observable is represented by $\mathcal{T}$. Since the agents fear moral hazard, the agents won't take risks caused by non–observable factors.

Any payoff that is not $\mathcal{T}-$ measurable can be decomposed in a tradeable and a non tradeable part: the tradeable part is simply the essential supremum of all positive $\mathcal{T}-$measurable random variables that are smaller than the payoff considered.

We prove the existence of an Arrow– Debreu equilibrium on $\mathcal{T}$. This consists of a $\mathcal{T}-$measurable price density ϕ and $\mathcal{T}-$measurable random payoffs representing the demand of tradeable risk transfer. The supply is given by the tradeable part of the income of the agents. In fact, we transform the utility maximization problem in an incomplete market into a maximization problem on a smaller complete market with random utility functions. Then the methods and arguments stated in (FS02) yields the existence of the equilibrium in our incomplete market.

Denote the set of *tradeable payoffs* or *contingent claims*

$$\mathcal{X}(\mathcal{T}) = L^0_+(\mathcal{T}, P).$$

We have a finite set $\mathcal{I}$ of traders each endowed with a $\mathcal{F}$–measurable payoff $\hat{H}_a \geq 0$, $a \in \mathcal{I}$. Which contingent claims can an agent sell on the market? Since a negative terminal wealth is not allowed, he can sell the essential supremum H_a of all tradeable contingent claims that are smaller or equal to his income

$$H_a = \operatorname{ess\,sup}\{X \,|0 \leq X \leq \hat{H}_a \text{ P–a.s.}, X \in \mathcal{X}(\mathcal{T})\,\}.$$

So the income $\hat{H}_a$ can be decomposed into the tradable component H_a and a nontradable component $\tilde{H}_a$:

$$\hat{H}_a = H_a + \tilde{H}_a \tag{6.1}$$

We assume that $\tilde{H}_a$ is bounded from above: there exists a constant s_0 such that

$$\tilde{H}_a \leq s_0, \qquad P - a.s.$$

The *market portfolio* H is the sum of all tradeable parts of the incomes of the agents:

$$H = \sum_{a \in \mathcal{I}} H_a.$$

In the model with the complete market, all the income of the agents is the market portfolio. This is the income that can be redistributed. In order to be consistent with this setup, we define here also the income as market portfolio that can be transferred. As in (FS02) page 144, we assume

$$P[H_a > 0] > 0 \qquad \text{for all } a \in \mathcal{I}$$

and

$$E[H] < \infty. \tag{6.2}$$

Let ϕ denote a price density on $\mathcal{T}$, i.e. a $\mathcal{T}$–measurable strictly positive integrable random variable satisfying $E[\phi H] < \infty$. The budget set $\mathcal{B}_a(\phi)$ of agent a consists of all tradeable ($\mathcal{T}$–measurable) random variables that are not more expensive than his income H_a. Furthermore, an agent can't buy more than the market portfolio.

$$\mathcal{B}_a(\phi) = \{\xi \in \mathcal{X}(\mathcal{T}) \,|\, 0 \leq \xi \leq H,\ E[\phi\xi] \leq E[\phi H_a]\}.$$

The agent aims at maximizing the expected utility of the sum of the non–tradable part of his income and the contingent claims in his budget set. He uses a utility function $u_a : [0, \infty] \to \mathbb{R}$, where u_a is continuously differentiable, strictly growing and strictly concave. So the agents wants to find

$$\xi_a(\phi) = \arg\max\{E[u_a(\tilde{H}_a + \xi)] \,|\xi \in \mathcal{B}_a(\phi)\}. \tag{6.3}$$

We use the following assumptions on the utility functions and incomes:

Assumption 43 *1. For every agent $a \in \mathcal{I}$, the utility function u_a, the non tradeable income $\tilde{H}_a$ and the market portfolio H satisfy*

$$E[u_a(\tilde{H}_a + H)] < \infty. \tag{6.4}$$

2. The non tradeable income $\tilde{H}_a$, the market portfolio H and the derivative u_a' of the utility function u_a satisfy

$$E\left[u_a'\left(\tilde{H}_a + \frac{H}{|\mathcal{I}|}\right)\right] < \infty. \tag{6.5}$$

3. The first derivative u_a' of the utility function u_a satisfies

$$\limsup_{x \to 0} x u_a'(x) < \infty. \tag{6.6}$$

So far, we face an incomplete utility maximization problem. In order to find an equilibrium, we transform this incomplete utility maximization into a problem in a complete market on the $\sigma-$algebra $\mathcal{T}$ with random $\mathcal{T}-$measurable preferences. The projectivity of the conditional expectation is the key. Since

$$E[u_a(\tilde{H}_a + \xi)] = E[\, E[u_a(\tilde{H}_a + \xi) \,|\, \mathcal{T}\,]\,]$$

for every $\xi \in \mathcal{X}(\mathcal{T})$, we interpret $E[u_a(\tilde{H}_a + \xi) \,|\, \mathcal{T}\,]$ as random preferences. Since our model is placed in a Borel space, we chose a version of the conditional probability $P[\tilde{H}_a \in \cdot\, |\mathcal{T}]$ and write

$$\Psi_a(\omega, x) := \int u_a(\tilde{w} + x) P[\tilde{H}_a \in d\tilde{w}|\mathcal{T}](\omega).$$

There exists a set $N \in \mathcal{F}$ with $P[N] = 0$ such that for all $\omega \in \Omega \backslash N$ the functions $\Psi_a(\omega, x)$ are strictly growing, strictly concave and continuously differentiable with derivative

$$\Psi'(\omega, x) = \int u_a'(\tilde{w} + x) P[\tilde{H}_a \in d\tilde{w}|\mathcal{T}](\omega),$$

and $\Psi(\omega, x) = E[u_a(\tilde{H}_a + x)|\mathcal{T}]$. For $\omega \in N$, we set $\Psi_a(\omega, x) := u_a(x)$. So, (6.3) is equivalent to finding

$$\xi_a(\phi) = \arg\max \{E[\, \Psi_a(\xi)\,]\,|\xi \in \mathcal{B}_a(\phi)\}. \tag{6.7}$$

Let us now describe the solution of (6.7). Ψ_a' is decreasing and

$$\Psi_a'(x) \leq u_a'(x), \quad \Psi_a'(x) \geq u_a'(x + s_0), \quad x > 0.$$

Denote as in (FS02) page 135

$$a(\omega) := \lim_{x\uparrow\infty} \Psi_a'(\omega, x) \geq 0, \quad b(\omega) := \lim_{x\downarrow\infty} \Psi_a'(\omega, x) \leq \infty.$$

Define $I_a^+(\omega, \cdot) : (a(\omega), b(\omega)) \to (0, \infty)$ as the continuous, bijective, strictly decreasing inverse function of Ψ_a' on (a, b). We max extend I^+ continuosly to the full half axis by setting

$$I_a^+(\omega, y) := \begin{cases} 0 & \text{for} \quad y \geq b, \\ +\infty & \text{for} \quad y \leq a \end{cases} \tag{6.8}$$

Corollary 3.45 in (FS02) states that the unique solution of the utility maximization problem $\xi_a(\phi)$ with a given price density ϕ is

$$\xi_a(\phi) = I_a^+(\omega, c_a\phi) \wedge H$$

for a constant $c_a > 0$. Observe that $I_a^+(\omega, c_a\phi)$ is $\mathcal{T}$–measurable.

Now we are able to give the main theorem of this section.

Theorem 44 *Let the risky incomes $(H_a)_{a\in\mathcal{I}}$ given as in (6.1) satisfying assumption 43. Then there exists an Arrow– Debreu equilibrium, i.e. a price density ϕ^* and an allocation (ξ_a^*) of the market portfolio such that for every $a \in \mathcal{I}$, ξ_a^* is the utility maximizing contingent claim of agent a for the price density ϕ^*.*

Proof. In fact, all arguments needed for this proof are already given in the proof of Theorem 3.59 and Theorem 3.55 in (FS02). The first step is the weighted utility problem

$$\max \left\{ \sum_{a\in\mathcal{I}} \lambda_a E[\Psi_a(\omega, \xi_a(\omega))] \middle| \xi_a \in \mathcal{X}(\mathcal{T}), \sum \xi_a = \sum W_a \right\}$$

For every $\lambda \in \Lambda$, this problem has a unique solution ξ_a^λ. This is a consequence of the more abstract Remark 3.39. According to Corollary 3.45 in (FS02), the contingent claim ξ_a^λ maximizes $E[\Psi_a(\xi)]$ under all contingent claims ξ satisfying $0 \leq \xi \leq \xi_a^\lambda$, $E[\phi^\lambda \xi] \leq E[\phi^\lambda \xi_a^\lambda]$. Furthermore, (ξ_a^λ) satisfy a first order condition: there exists a price density ϕ^λ satisfying

$$\lambda_a \Psi_a'(\xi_a^\lambda) \leq \phi^\lambda, \quad \text{with equality on } \{\xi_a^\lambda > 0\}.$$

Now define

$$g_a(\lambda) = \lambda_a + \frac{1}{E[\kappa(1 + W)]} E[\phi^\lambda (W_a - \xi_a^\lambda)].$$

If $g_a(\lambda) = \lambda_a$ for all $a \in \mathcal{I}$, then $E[\phi^\lambda \xi_a^\lambda] = E[\phi^\lambda H_a$ for all $a \in \mathcal{I}$, and thus (ξ_a) and ϕ^λ are an Arrow– Debreu equilibrium. If this is not the case, an agent gets too few or too much. His weight is increased or decreased by g. As in F"ollmer / Schied (FS02), we see that g is a continuous mapping from Λ to Λ. So Brouwer's fixed point theorem yields a $\lambda^* \in \Lambda$ satisfying $g(\lambda^*) = \lambda^*$. Thus, $(\xi_a^{\lambda*})$, $\phi^{\lambda*}$ is an Arrow– Debreu equilibrium.

□

Part II

Utility maximization

Chapter 7

Robust utility maximization

Introduction

In this chapter we consider the problem of finding the trading strategy that maximizes the robust utility of a small trader in an incomplete market. The model is placed in a Brownian filtration. Thus, we have to maximize a functional as defined in (1.2) over the terminal wealth of all possible trading strategies. The set $\mathcal{Q}$ that includes all probability measures we consider is an m–stable set of probabilities $\mathcal{Q}$ in the sens of Delbaen (Del03), Definition 1.2. We consider the exponential and the power utility functions. In the case of an exponential utility function we are able to solve a more general problem: the investor has a terminal liabiliy and tries to hedge it. He has sold an option and is obliged to pay a random sum at the terminal time. In general it is impossible to replicate every contingent claim in an incomplete market. The investor maximizes the robust utility of the terminal wealth gained by a trading strategy minus the liability.

The set of trading strategies a trader may use is restricted. For example, a negative number of shares is not possible or the investment in risky stocks is not allowed to exceed a certain treshold. Every trading strategy has to take its values in a convex set that can be stochastic and time dependend.

The method we use is a generalisation of the approach in Hu, Imkeller and Müller, (HIM04b). In order to find the optimal trading strategy, we compare the expected utility of all trading strategies under all probability measures in $\mathcal{Q}$. In fact, we have to solve a max min problem. The goal is now to find a saddle point. Since the model is placed in a Brownian framework, we may represent the density of every equivalent probability measure as the stochastic exponential of a stochastic integral with respect to the Brownian motion. To this end, we construct a family of stochastic processes $R(p, \nu)$ indexed with all possible trading strategies p and integrands ν in the representation of the densities of the probability measures. The terminal value $R_T(p,\nu)$ is the product of two factors: the first one is the density of an equivalent probability measure with integrand ν. The second factor is $u(X_T^p)$, where X_T^p is the terminal wealth of the trading strategy p and u the utility function we consider. The initial value R_0 is the same for all (p,ν). The key is the following property: there exists a special trading strategy p^* and probability measure indexed with ν^* such that $R(p^*,\nu^*)$ is a martingale. This special (p^*,ν^*) will turn out to be our saddle point, because we construct R such that $R(p^*,\nu)$ is a submartingale and $R(p,\nu^*)$ is a supermartingale for all admissible trading strategies p and possible measures changes indexed with ν. This means, $R_0(p^*,\nu^*)$ is the maximal attainable robust utility, p^* is the trading strategy attaining it and ν^* describes the probability measure attaining the minimum in the robust utility for the terminal wealth of p^*.

We find the processes $R(p,\nu)$ by constructing a quadratic BSDE. The driver of the BSDE depends on saddle values for finite dimensional saddle point problems. With the unique solution of this BSDE, one can calculate the optimal trading strategy p^* and the integrand ν^* for the measure Q^*. For every (ω,t), p^* and ν^* solve finite dimensional saddle points.

In a complete market we get a fairly explicit description of the optimal trading strategy stated in Theorem 58. This is possible in presence of a terminal liability F and random constraints on the integral in the presentation of the densities in $\mathcal{Q}$ as stochastic exponentials of integrals with respect to the Brownian motion. This is a case where the least favorable measure does not necessarily exists.

This chapter is organized as follows: Secion 7.1 explains the financial market and the robust utility, in section 7.2 we consider the exponential utility function, and in section 7.3 the power utility.

7.1 Stock market and robust utility

A probability space $(\Omega,\mathcal{F},P)$ carrying an m–dimensional Brownian motion $(W_t)_{t\in[0,T]}$ is given. The filtration $\mathbb{F}$ is the completion of the filtration generated by W.

The financial market consists of one bond with interest rate zero and $d \leq m$ stocks. In case $d < m$ we face an incomplete market. The price process of stock i evolves according to the equation

$$\frac{dS_t^i}{S_t^i} = b_t^i dt + \sigma_t^i dW_t, \quad i = 1,\dots,d, \tag{7.1}$$

where b^i (resp. σ^i) is an $\mathbb{R}-$valued (resp. $\mathbb{R}^{1\times m}$–valued) predictable uniformly bounded stochastic process. The lines of the $d \times m$–matrix σ are given by the vector σ_t^i, $i = 1, \ldots, d$. The volatility matrix $\sigma = (\sigma^i)_{i=1,\ldots,d}$ has full rank and we assume that $\sigma\sigma^{tr}$ is uniformly elliptic, i.e. $KI_d \geq \sigma\sigma^{tr} \geq \varepsilon I_d$, P–a.s. for constants $K > \varepsilon > 0$. The predictable $\mathbb{R}^m$–valued process

$$\theta_t = \sigma_t^{tr}(\sigma_t\sigma_t^{tr})^{-1}b_t, \qquad t \in [0, T], \tag{7.2}$$

is then also uniformly bounded. We will see later that only θ enters the solution of the optimization problem. For simplicity we will call θ drift.

There are several possibilities to define a trading strategy. One can write down the number of shares of each stock held by the investor, the amount of money invested or the part of the wealth. We will choose the notation that fits well to our maximization problem: for the exponential utility in section 7.2 we use the amount of money (see Definition 47) and for the power utility in section 7.3, we consider trading strategies that are written as part of the wealth (see Definition 59). The definition of a wealth process depends also on the choice of the description of a trading strategy.

Robust utility

The preferences of our investor on replicable contingent claims are described by robust utility as explained on page 7. In contrast to the usual expected utility with respect to a single "real world" probability, the investor considers a whole set $\mathcal{Q}$ of probability measures. In order to calculate the robust utility for a contingent claim, the investor chooses the measure within $\mathcal{Q}$ that minimizes the expected utility of this random variable.

The robust utility can be seen as a worst case approach. It takes into account, that the investor is averse against risk caused by the random stock price and the uncertainty since he doesn't have an "objective" probability measure. More information about this topic can be found in the book of Föllmer / Schied (FS02) in Chapter 2.5. So the investor has to solve a max min problem. Under our assumptions, this leads to a saddle point problem.

The robust utility of a payoff F is

$$\inf_{Q \in \mathcal{Q}} E_Q[u(F)]$$

where u is a convex increasing function called utility function.

The densities of the probability measures in $\mathcal{Q}$ with respect to P are stochastic exponentials of stochastic integrands with respect to the Brownian motion. The integrands are restricted to time dependend random predictable closed convex subsets $C_t(\omega)$ of $\mathbb{R}^m$, $t \in [0, T]$, $\omega \in \Omega$. Predictable means here the set $\{((\omega, t)), v) | v \in C_t(\omega)\} \subset \Omega \times [0, T] \times \mathbb{R}^m$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^m)-$ measurable. Here we summarize our assumptions on the set $\mathcal{Q}$:

Assumption 45 *Let the closed convex subsets $C_t(\omega)$, $t \in [0,T]$, $\omega \in \Omega$ be predictable and contained in a bounded ball around the origin. We assume that the Radon– Nikodym density of every $Q \in \mathcal{Q}$ can be written as*

$$\frac{dQ}{dP} = \mathcal{E}\left(\int \nu_s dW_s\right)_T \tag{7.3}$$

for a predictable, $\mathbb{R}^m-$valued process ν where $\nu_t \in C_t$.

We denote the set of integrands in the representation (7.3) with

$$\mathcal{V} = \{(\nu_t)_{t\in[0,T]} \text{ predictable} \mid \nu_t \in C_t \; P \otimes \lambda \text{ a.s.}\} \tag{7.4}$$

So the set of probability measures $\mathcal{Q}$ is parametrized by the set of integrands $\mathcal{V}$. For the expectation of a random variable F with respect to Q^ν we also write

$$E_{Q^\nu}[F] =: E^\nu[F].$$

Our set $\mathcal{Q}$ is closely related to sets of *multiplicatively stable* sets of probability measures as defined in Delbaen, (Del03).

Remark 46 Let $\mathcal{S}$ be a set of probability measures such that for a reference probability measure $Q^r \in \mathcal{S}$ for every $Q \in \mathcal{S}$ the density $\frac{dQ}{dQ^r}$ satisfies $\frac{dQ}{dQ^r} \in L^1(Q^r)$. Then the density process $Z^Q_t = E^r[\frac{dQ}{dQ^r}|\mathcal{F}_t]$ is well defined, where E^r denotes the expectation with respect to Q^r. Furthermore, let the set of those densities be closed in $L^1(Q^r)$. Let Z^0 denote the density process of a $Q^0 \ll Q^r$ and Z the density process of a $Q \sim Q^r$. For every stopping time $\tau \leq T$ define

$$L_t = \begin{cases} Z^0_t, & t \leq \tau, \\ Z^0_\tau \frac{Z_t}{Z_\tau}, & t > \tau. \end{cases}$$

Assume also that every nonnegative $\mathcal{F}_0-$measurable random variable Z_0 satisfying $E^r[Z_0] = 1$ defines by $dQ = Z_0 dQ^r$ a probability Q that is in $\mathcal{S}$. Then $\mathcal{S}$ is $m-$stable if every L defined as above is the density process of a probability measure $Q^L \in \mathcal{S}$.

Denote with S^e the subset of $\mathcal{S}$ that consists of measures equivalent to the reference measure Q^r. If $\mathcal{S}$ is be $m-$stable, $Q^r \in \mathcal{S}$ given by exponentials of integrands with respect to a continuous martingale M and $Q^r \in \mathcal{S}$, then Theorem 1.4 in (Del03) states that there exists a predictable, closed, convex multivalued mapping (C_t) such that $\mathcal{S}^e$ is equal to the set of processes $Z = \mathcal{E}(\int q dM)$ where Z is a strictly positive martingale and $q(t,\omega) \in C_t(\omega)$.

Our setup is the Brownian filtration generated by the Brownian motion W. The initial σ algebra $\mathcal{F}_0$ is trivial. The density of any probability measure equivalent to P can be written as the stochastic exponential of a stochastic integral with respect to W. In order to describe the preferences of our investor, we use an $m-$ stable set of probability measures that satisfies an additional assumption.

7.2 Robust exponential utility maximization

Suppose an investor has a liability F at time T. This random variable F is assumed to be $\mathcal{F}_T$–measurable and bounded, but not necessarily positive. The investor tries to find a trading strategy such that the terminal wealth of the trading strategy minus the liability F maximizes the robust utility. In this section we consider the utility function

$$U(x) = -\exp(-\alpha x), \qquad x \in \mathbb{R}$$

for a parameter $\alpha > 0$ that is called the absolute risk aversion.

Here we formally describe trading strategies as we use for the robust utility maximization problem with the exponential utility function. A d–dimensional $\mathbb{F}$–predictable process $\pi = (\pi_t)_{0\leq t\leq T}$ is called *trading strategy* if $\int \pi \frac{dS}{S}$ is well defined, e.g. $\int_0^T \|\pi_t\sigma_t\|^2 dt < \infty$ P–a.s. For $1 \leq i \leq d$, the process π_t^i describes the amount of money invested in stock i at time t. The number of shares is $\frac{\pi_t^i}{S_t^i}$. The wealth process X^π of a trading strategy π with initial capital x satisfies the equation

$$X_t^{x,\pi} = x + \sum_{i=1}^d \int_0^t \frac{\pi_{i,u}}{S_{i,u}} dS_{i,u} = x + \int_0^t \pi_u \sigma_u (dW_u + \theta_u du), \quad t \in [0,T].$$

In this notation π has to be taken as a vector in $\mathbb{R}^{1\times d}$.

Trading strategies are self–financing. Gains or losses are only obtained by trading with the stock. The conditions on the trading strategies of the following definition guarantee that there is no arbitrage. In addition, we allow constraints on the trading strategies. Formally, they are supposed to take their values in a closed convex set $\tilde{A} \subseteq \mathbb{R}^{1\times d}$, i.e. $\pi_t(\omega) \in \tilde{A}$ for $\lambda \otimes P$–a.e. $(\omega, t) \in \Omega \times [0,T]$. It is also possible to consider random predictable, closed, convex constraints, see Remark 57 for more details. For technical reasons we impose some further integrability conditions on our trading strategies.

Definition 47 (Admissible Strategies with constraints) *Let $\tilde{A}$ be a closed set in $\mathbb{R}^{1\times d}$. The set of admissible trading strategies $\tilde{\mathcal{A}}$ consists of all d–dimensional predictable processes $\pi = (\pi_t)_{0\leq t\leq T}$ which satisfy $\pi_t \in \tilde{A}$ for $\lambda \otimes P$–a.e. $(\omega, t) \in \Omega \times [0,T]$, $\int_0^\cdot \pi_s\sigma_s dW_s$ is a $P-$BMO–martingale, and $E(U^-(X_T^\pi)) > -\infty$.*

The definition and main results about BMO–martingales are stated in the appendix. We use BMO–martingales because stochastic exponentials of them are uniformly integrable martingales. Our time interval is restricted. According to (A.2) on page 141, every uniformly bounded trading strategy π is admissible, but this is not a necessary condition.

The boundedness of θ and Theorem 3.6 in (Kaz94) imply that the wealth process X^π is a BMO–martingale under the equivalent probability measure Q^0 with Radon-Nikodym density $\frac{dQ^0}{dP} = \mathcal{E}(-\int \theta dW)$. Therefore the set $\tilde{\mathcal{A}}$ is free of arbitrage, i.e. in this set there is no trading strategy π with initial capital $X_0^\pi = 0$, terminal wealth $X_T^\pi \geq 0$ P–a.s. and $P[X_T^\pi > 0] > 0$.

For $t \in [0,T], \omega \in \Omega$ define the set $A_t(\omega) \subseteq \mathbb{R}^m$ by

$$A_t(\omega) = \tilde{A}\sigma_t(\omega). \tag{7.5}$$

The entries of the matrix–valued process σ are uniformly bounded. Therefore we get for $\lambda \otimes P-$ a.e. (t,ω) and some constant $k_1 \geq 0$.

$$\min\{\ |a|\ :\ a \in A_t(\omega)\ \}\ \leq\ k_1. \tag{7.6}$$

Remark 48 *Writing*

$$p_t = \pi_t \sigma_t, \qquad t \in [0,T],$$

the set of admissible trading strategies $\tilde{\mathcal{A}}$ is equivalent to a set $\mathcal{A}$ of $\mathbb{R}^{1\times m}$–valued predictable stochastic processes p with $p \in \mathcal{A}$ if $p_t(\omega) \in A_t(\omega)$ $P \otimes \lambda$–a.e. and $\int_0^\cdot p_s dW_s$ is a P–BMO–martingale. Such a process $p \in \mathcal{A}$ will also be named strategy, and $X^{x,p}$ denotes its wealth process.

With this definition of a trading strategy we define our maximization problem:

Problem 49 (Robust utility maximization) *Let F be a bounded F_T-measurable random variable. A solution of the robust utility maximization problem consists of an admissible trading strategy $\bar{p} \in \mathcal{A}$ and a probability measure $Q^{\bar{\nu}} \in \mathcal{Q}$ (resp. $\bar{\nu} \in \mathcal{V}$) attaining*

$$V(x,F) := \sup_{p\in\mathcal{A}} \inf_{\nu\in\mathcal{V}} E_\nu \left[-\exp(-\alpha(X_T^{x,p} - F))\right]. \tag{7.7}$$

Under the reference measure P the expectation in (7.7) reads

$$K(p,\nu) = E\left[-\exp\left(-\alpha(X_T^{x,p} - F) + \int_0^T \nu_s dW_s - \frac{1}{2}\int_0^T |\nu_s|^2 ds\right)\right]. \tag{7.8}$$

So problem 49 consists in finding a $\bar{p} \in \mathcal{A}$ and $\bar{\nu} \in \mathcal{V}$ attaining

$$V(x, F) = \sup_{p \in \mathcal{A}} \inf_{\nu \in \mathcal{V}} K(p, \nu). \tag{7.9}$$

Before stating and proving the main theorem we sketch the ideas leading to the solution of the robust utility maximization problem. We aim to show that the functional K has a saddle point $(\bar{p}, \bar{\nu})$. This saddle point satisfies

$$K(p, \bar{\nu}) \leq K(\bar{p}, \bar{\nu}) \leq K(\bar{p}, \nu) \qquad \forall\, p \in \mathcal{A},\ \nu \in \mathcal{V},$$

hence $(\bar{p}, \bar{\nu})$ is a solution of problem 49.

We can exploit the exponential structure of the maximization problem and apply a generalisation of the martingale argument developed in Hu, Imkeller, Müller (HIM04b). In order to solve our optimization problem 49, we construct a family of processes $R(p, \nu) = (R_t(p, \nu))_{t \in [0,T]}$ indexed with the admissible trading strategies $p \in \mathcal{A}$ and all possible integrands for the change of measure $\nu \in \mathcal{V}$. Together with R we have to find a trading strategy $\bar{p} \in \mathcal{A}$ and a $\bar{\nu} \in \mathcal{V}$ such that R satisfies for all $p \in \mathcal{A}$ and $\nu \in \mathcal{V}$:

- $R_T(p, \nu) = -\exp\left(-\alpha(X_T^{x,p} - F) + \int_0^T \nu_s dW_s - \frac{1}{2}\int_0^T |\nu_s|^2 ds\right)$,
- $R_0(p, \nu) = R_0$ does not depend on p and ν,
- $R(p, \bar{\nu})$ is a $P-$supermartingale for all $p \in \mathcal{A}$,
- $R(\bar{p}, \nu)$ is a $P-$submartingale for all $\nu \in \mathcal{V}$,
- $R(\bar{p}, \bar{\nu})$ is a $P-$martingale.

With this family of processes $R(p, \nu)$, $p \in \mathcal{A}, \nu \in \mathcal{V}$, we obtain

$$E[R_T(p, \bar{\nu})] \leq R_0(\bar{p}, \bar{\nu}) = E[R_T(\bar{p}, \bar{\nu})] \leq E[R_T(\bar{p}, \nu)] \quad \forall\, p \in \mathcal{A}, \nu \in \mathcal{V}.$$

Thus $(\bar{p}, \bar{\nu})$ is a saddle point of K and the solution of problem 49. We set

$$R_t(p, \nu) = -\exp\left(-\alpha X_t + \alpha Y_t + \int_0^t \nu_s dW_s - \frac{1}{2}\int_0^t |\nu_s|^2 ds\right)$$

where Y is defined by a BSDE with terminal value F:

$$Y_t = F - \int_t^T Z_s dW_s - \int_t^T f(s, \theta_s, Z_s) ds, \quad t \in [0, T]. \tag{7.10}$$

Now we have to construct the driver f of the BSDE such that R satisfies the properties described above. We can find f by solving deterministic saddle point problems. The idea is to write $R(p,\nu)$ as the product of a martingale and an increasing or decreasing process depending on p and ν:

$$\begin{aligned} R_t(p,\nu) &= -\exp(-\alpha x + \alpha Y_0)\mathcal{E}\left(\int(-\alpha p_s + \alpha Z_s + \nu_s)dW_s\right)_t \\ &\quad \times \exp\left(\int_0^t g(p_s,\nu_s,\theta_s,Z_s) + \alpha f(s,\theta_s,Z_s)\right) \end{aligned}$$

where $g : \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is equal to

$$g(q,v,z,\theta_t) = \frac{1}{2}(-\alpha q + \alpha z + v)^2 - q\theta_t - \frac{1}{2}v^2 \tag{7.11}$$

$$\begin{aligned} &= \frac{\alpha^2}{2}\left(q - \left(z + \frac{1}{\alpha}v + \frac{1}{\alpha}\theta_t\right)\right)^2 \\ &\quad -\frac{1}{2}(v+\theta_t)^2 - \alpha z\theta_t \end{aligned} \tag{7.12}$$

$$= \alpha v(z-q) + \frac{1}{2}\alpha^2 q^2 - q(\alpha^2 z + \theta_t) + \frac{1}{2}\alpha^2 z^2.$$

We write the function g with variables $q, v, z \in \mathbb{R}^m$ in order to distinguish clearly between the saddle point analysis in $\mathbb{R}^m \times \mathbb{R}^m$ and on the sets of processes $\mathcal{A} \times \mathcal{V}$. Here, q takes the place of p_t and v replaces ν_t, z stands for Z_t.

In order to obtain the desired properties for $R(p,\nu)$, we have to choose f such that there exists $\bar{p} \in \mathcal{A}$ and $\bar{\nu} \in \mathcal{V}$ satisfying for all $t \in [0,T]$ P–a.s.

$$\begin{aligned} g(p_t,\bar{\nu}_t,Z_t,\theta_t) + \alpha f(t,\theta_t,Z_t) &\geq 0 \qquad \forall\, p \in \mathcal{A} \\ g(\bar{p}_t,\nu_t,Z_t,\theta_t) + \alpha f(t,\theta_t,Z_t) &\leq 0 \qquad \forall\, \nu \in \mathcal{A} \\ g(\bar{p}_t,\bar{\nu}_t,Z_t,\theta_t) + \alpha f(t,\theta_t,Z_t) &= 0 \end{aligned}$$

Then R is the product of a negative martingale with an in / decreasing / constant process, hence a super / sub / martingale. The processes p resp. ν are constraint to be in convex sets $A_t(\omega)$ resp. $C_t(\omega)$ for almost every (ω,t). The first step is to prove for fixed $(z,\theta_t) \in \mathbb{R}^m \times \mathbb{R}^m$ the existence of a saddle point of the function g i.e. $(\bar{q},\bar{v}) \in A_t \times C_t$ satisfying

$$g(\bar{q},v,z,\theta_t) \leq g(\bar{q},\bar{v},z,\theta_t) \leq g(q,\bar{v},z,\theta_t) \qquad \forall\, q \in A_t,\ v \in C_t \tag{7.13}$$

(see Lemma 52 below). The value of g on the saddle point is

$$\begin{aligned} \bar{g}(t,z,\theta_t) = g(\bar{q},\bar{v},z,\theta_t) &= \inf_{q\in A_t}\sup_{v\in C_t} g(q,r,z,\theta_t) \\ = \sup_{v\in C_t}&\left(\frac{\alpha^2}{2}\mathrm{dist}^2_{A_t}\left(z + \frac{1}{\alpha}(v+\theta_t)\right) - \frac{1}{2}(v+\theta_t)^2 - \alpha z\theta_t\right). \end{aligned} \tag{7.14}$$

Then we choose

$$f(t, Z_t, \theta_t) = -\frac{1}{\alpha}\bar{g}(t, Z_t, \theta_t).$$

We aim at applying Theorem 2.3 of Kobylanski (Kob00) to prove the existence of a solution of the BSDE (7.10) with this choice of f. In order to do so, we have to show an estimate of the saddle value (Lemma 53 below).

Using a solution (Y, Z) of the BSDE we can find the processes $(\bar{p}, \bar{\nu})$ that solves the original saddle point problem: a measurable selection theorem (Lemma 1 in (Ben70)) yields two predictable processes $\bar{p}$ and $\bar{\nu}$ such that $(\bar{p}, \bar{\nu})$ is a saddle point of $g(q, v, Z_t, \theta_t)$, where $q \in A_t$ and $v \in C_t$. Recall the construction of R, $(\bar{p}, \bar{\nu})$ is also a saddle point of the functional K and the solution of the utility maximization problem 49.

Theorem 4.3 in El Karoui, Hamadène (EKH03) relates the saddle point of a risk-sensitive zero-sum game to a BSDE. Their control problem has the following form:

$$\sup_{v\in V} \inf_{u\in U} E^{u,v}\left[\exp\left\{\int_0^1 h(s, x., u_s, v_s)ds\right\}\right]$$

where the expectation is taken under a probability measure $P^{u,v}$ with density

$$\frac{P^{u,v}}{dP} = \mathcal{E}\left(\int_0^{\cdot} \sigma^{-1}(s, x.)f(s, x., u_s, v_s)dB_s\right),$$

the controls u and v are predictable processes on some metric spaces and x satisfies the following SDE:

$$dx_t = f(t, x., u_t, v_t)dt + \sigma(t, x.)dB_t^{u,v}, \quad x_0 = x \in \mathbb{R}^d.$$

Assumption (A4.3) in (EKH03) states that h has to be bounded. Our control problem does not satisfy this assumption.

Having completed this overview we state the main theorem that gives the solution of the robust utility maximization problem in terms of a BSDE.

Theorem 50 *There exists a solution $(\bar{p}, \bar{\nu})$ of the robust utility maximization problem 49. This solution is a saddle point of the functional K defined in (7.8). For every $(\omega, t) \in \Omega \times [0, T]$, $(\bar{p}, \bar{\nu})$ is a saddle point of $g(\cdot, \cdot, Z_t, \theta_t)$, i.e. it satisfies*

$$g(\bar{p}_t, r, Z_t, \theta_t) \le g(\bar{p}_t, \bar{\nu}_t, Z_t, \theta_t) \le g(q, \bar{\nu}_t, Z_t, \theta_t) \qquad \forall\, q \in A_t,\ r \in C_t. \quad (7.15)$$

The pair (Y, Z) is the solution of the BSDE (7.10) and g is defined in (7.11). The value function is

$$V(x, F) = -\exp(-\alpha(x - Y_0). \quad (7.16)$$

Applying (7.16), we may calculate the utility indifference price x_F of F. This is the extra initial capital that the investor needs in order to get the same maximal utility in presence the liability F than without F. The investor is indifferent between getting x_F and accepting the obligation to pay F and on the other hand doing nothing.

Remark 51 (Utility indifference price) *Let $x > 0$ be the initial capital of an agent. The utility indifference price x_F of F is defined as the solution of the equation*

$$V(x + x_F, F) = V(x, 0).$$

Let (Y^F, Z^F) denote the solution of the BSDE (7.10) with terminal value F, whereas (Y^0, Z^0) is the solution of (7.10) with terminal value 0. According to (7.16),

$$x_F = Y_0^F - Y_0.$$

In order to prove Theorem 50, we use some Lemmata. The proof of the theorem is summarized on page 103. The first step is to show the existence of a saddle point of the function g in $\mathbb{R}^m \times \mathbb{R}^m$ in Lemma 52. An estimate of the value of g at the saddle point is stated in Lemma 53. This leads to the existence of a solution of the BSDE 7.10 proven in Lemma 54. The selection of an admissible trading strategy $\bar{p} \in \mathcal{A}$ and of an integrand $\bar{\nu} \in \mathcal{V}$ solving the robust utility maximization problem is proven in Lemma 56 and Lemma 55.

We start proving the different Lemmata leading to Theorem 50 while showing the existence of a saddle point of the function g.

Lemma 52 *Let (A_t) and (C_t) be the constraints according to (7.5) and Assumption 45. For every $z \in \mathbb{R}^m$ and $(\omega, t) \in \Omega \times [0, T]$, there exists a saddle point $(\bar{q}, \bar{v})$ of the function g defined in (7.11).*

Proof. Fix $z \in \mathbb{R}^m$. We aim to apply Theorem 37.3 in Rockafellar (Roc70). The function g is convex in q and linear in v, hence convex–concave. We have to show: the convex functions $g(\cdot, v)$, $v \in \text{ri}C_t$, have no common direction of recession neither does the functions $-g(q, \cdot)$ for $q \in \text{ri}A_t$. *ri* denotes the relative interior of a convex set (see Section 6 in (Roc70)). Here we have to describe the definition of a direction of recession. If a convex set A_t is not constrained, there exists $q_0 \in A_t$ and y such that $q_0 + \lambda y \in A_t$ for all $\lambda > 0$. Due to the convexity of A_t we have $q + \lambda y \in A_t$ for all $q \in A_t$ and $\lambda > 0$. Such a y is a direction of recession, if for every $q \in A_t$

$$\liminf_{\lambda \to \infty} g(q + \lambda y, r) < +\infty,$$

(see Theorem 8.6 in Rockafellar (Roc70)). If the set of constraints is bounded, then there exists no direction of recession.

Since g is quadratic in q (coercive), the functions $g(\cdot, v)$ have no direction of recession. The functions $-g(q, \cdot)$ are linear, but they are only defined on the bounded set C_t. Hence they also have no direction of recession. Theorem 37.3 in Rockafellar (Roc70) states the existence of a saddle point $(\bar{q}, \bar{v}) \in A_t \times C_t$ with $|g(\bar{q}, \bar{v}, z, \theta_t)| < \infty$.

□

The next lemma gives an estimate of the saddle value.

Lemma 53 *Let $(\bar{q}, \bar{v})(t, z, \theta_t)$ be the saddle point of g depending on (z, θ_t) and on time depending constraints A_t, C_t that exists according to Lemma 52 for g defined in (7.11). The saddle value function $\bar{g}$ defined in (7.14) satisfies for a constant $c > 0$*

$$|\bar{g}(t, z, \theta_t)| \leq c(\|z\|^2 + \|\theta_t\|^2 + 1) \tag{7.17}$$

and

$$\bar{g}(t, z, \theta_t) \geq -\frac{1}{2} dist^2(\theta_t, C_t) - \alpha \|z\| \, \|\theta_t\|, \quad \forall \, z \in \mathbb{R}^m. \tag{7.18}$$

Furthermore, the norm of $\bar{q}$ can be estimated with a constant c_1 by

$$|\bar{q}| \leq \sup_{v \in C_t} \left(dist_{A_t} (z + \frac{1}{\alpha}(v + \theta_t)) + |z| + |v| + |\theta_t| \right) \leq c_1(1 + |z| + |\theta_t|). \tag{7.19}$$

Proof. For every $v \in C_t$ we have

$$\inf_{q \in A} \frac{\alpha^2}{2} \left(q - \left(z + \frac{1}{\alpha} v + \frac{1}{\alpha} \theta_t \right) \right)^2 \leq \frac{\alpha^2}{2} \left(q_0 - \left(z + \frac{1}{\alpha} v + \frac{1}{\alpha} \theta_t \right) \right)^2,$$

where q_0 attains $\min_{q \in A_t} \|q\|$. Applying (7.6) gives

$$|\inf_{q \in A} g(q, v, z, \theta_t)| \leq c(\|q\|^2 + \|v\|^2 + \|\theta_t\|^2 + 1) \quad \forall \, v \in C_t$$

for a $c > 0$. The uniform boundedness of the sets C_t, $(\omega, t) \in \Omega \times [0, T]$, yields (7.17). Equation (7.18) follows from

$$g(q, v, z, \theta_t) \geq -\frac{1}{2}(v + \theta_t)^2 - \alpha z \theta_t \qquad \forall \, q \in A_t, \, v \in C_t.$$

In order to show (7.19), we see with (7.14) that $\bar{q}$ satisfies

$$|\bar{q} - (z + \frac{1}{\alpha}(\bar{v} + \theta_t))| = \text{dist}_{A_t}(z + \frac{1}{\alpha}(\bar{v} + \theta_t)).$$

Since the sets A_t satisfy (7.6) and the sets C_t are uniformly bounded, we obtain (7.19).

□

In order to define the BSDE, we have to show that our driver

$$f(t, \theta_t, z) = -\frac{1}{\alpha}\bar{g}(t, \theta_t, z), \quad t \in [0, T],$$

defines a predictable process. This is done in Lemma 56 below. The BSDE that we use to construct the optimal strategy has the following form:

$$Y_t = F - \int_t^T Z_s dW_s + \frac{1}{\alpha}\int_t^T \bar{g}(s, Z_s, \theta_s) ds. \tag{7.20}$$

Now we are able to state the existence of a solution of (7.20).

Lemma 54 *For a bounded $\mathcal{F}_T$–measurable terminal value F, the BSDE (7.20) has a unique solution $(Y, Z) \in \mathcal{H}^\infty(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^m)$.*

Proof. According to Lemma 56, $(\bar{g}(t, \theta_t, 0))_{t\in[0,T]}$ is predictable. In order to get existence of a solution, we apply Theorem 2.3 of Kobylanski (Kob00). Due to the boundednesss of θ, (7.6) and (7.17), condition (H1) in Kobylanski is satisfied, i.e.

$$|\bar{g}(t, \theta_s, z)| \leq c\|z\|^2 + b$$

for constants $b, c > 0$.

Now we prove uniqueness of the solution. Suppose that (Y^1, Z^1) and (Y^2, Z^2) are solutions of the BSDE (7.20). They satisfy

$$Y_t^1 - Y_t^2 = 0 - \int_t^T (Z_s^1 - Z_s^2) dW_s - \int_t^T (\bar{g}(s, \theta_s, Z_s^1) - \bar{g}(s, \theta_s, Z_s^2) ds.$$

In the first step we estimate the difference in the ds integral. In order to distinguish between stochastic processes and elements of $\mathbb{R}^m$, we write as on page 96 q for p_t, v for ν_t and z for Z_t. a According to (7.11), we can write

$$\bar{g}(t, \theta_t, z) = \sup_{v\in C_t}\left(\frac{\alpha^2}{2}\operatorname{dist}^2_{A_t}\left(z + \frac{1}{\alpha}(v + \theta_t)\right) - \frac{1}{2}(v + \theta_t)^2\right) - \alpha z\theta_t.$$

Fix $z_1, z_2 \in \mathbb{R}^m$. There exists $v_1, v_2 \in \mathbb{R}^m$ that attain the sup for z_1 resp. z_2. Thus

$$\begin{aligned}\bar{g}(t, \theta_t, z_1) - \bar{g}(t, \theta_t, z_2) \;&\leq\; \bar{g}(t, v_1, z_1) - \frac{\alpha^2}{2}\operatorname{dist}^2_{A_t}\left(z_2 + \frac{1}{\alpha}(v_1 + \theta_t)\right) + \\ &\quad + \frac{\alpha^2}{4}(v_1 + \theta_t)^2 + \alpha z_2\theta_t\end{aligned}$$

$$= \frac{\alpha^2}{2}\text{dist}^2_{A_t}\left(z_1 + \frac{1}{\alpha}(v_1+\theta_t)\right) - \qquad (7.21)$$
$$-\frac{\alpha^2}{2}\text{dist}^2_{A_t}\left(z_2 + \frac{1}{\alpha}(v_1+\theta_t)\right) + \alpha\theta_t(z_1 - z_2).$$

Using the uniform boundedness of C_t and the Lipschitz continuity of the distance function from a closed set we obtain with a constant $c > 0$

$$\bar{g}(t,\theta_t,z_1) - \bar{g}(t,\theta_t,z_2) \leq c(1+|z_1|+|z_2|+|\theta_t|)|z_1 - z_2|.$$

The same inequality is valid if we change z_1 and z_2, thus we have an estimate for the absolute value of this difference. Set

$$\beta_t = \begin{cases} \frac{\bar{g}(t,\theta_t,Z_t^1)-\bar{g}(t,\theta_t,Z_t^2)}{Z_t^1-Z_t^2}, & \text{if } Z_t^1 - Z_t^2 \neq 0, \\ 0 & \text{if } Z_t^1 - Z_t^2 = 0. \end{cases}$$

Since θ is uniformly bounded, we obtain for a constant $c_1 > 0$

$$\beta_t \leq c_1(1+|Z_t^1|+|Z_t^2|), \quad t \in [0,T].$$

As shown in Lemma 55, the martingales $\int_0^{\cdot} Z_s^i dW_s$, $i = 1,2$, are P–BMO martingales. So $\int_0^{\cdot} \beta_s dW_s$ is also a P–BMO martingale. Thus we may define an equivalent probability measure Q by

$$\frac{dQ}{dP} = \mathcal{E}\left(-\int \beta_s dW_s\right)$$

and a $Q-$Brownian motion $W^Q = W + \int_0^{\cdot}\beta_s ds$. Thus we can write the difference of the solutions in the following way:

$$\begin{aligned} Y_t^1 - Y_t^2 &= -\int_t^T (Z_s^1 - Z_s^2)dW_s - \int_t^T \beta_s(Z_s^1 - Z_s^2)ds \\ &= -\int_t^T (Z_s^1 - Z_s^2)dW_s^Q. \end{aligned}$$

Since $F = Y_T^1 = Y_T^2$ we conclude that $Y_t^1 = Y_t^2$ $P\otimes\lambda$ a.e. on $\Omega\times[0,T]$ and $Z_s^1 = Z_s^2$ in $\mathcal{H}^2$.

□

Lemma 55 *Let $(Y,Z) \in \mathcal{H}^\infty(\mathbb{R})\times\mathcal{H}^2(\mathbb{R}^m)$ be a solution of the BSDE (7.20). Then the process $\int_0^{\cdot} Z_s dW_s$ is a P–BMO martingale.*

Proof. According to Corollary 2.2 of Kobylanski (Kob00), the process Y is uniformly bounded. Let k denote the upper bound. For every stopping time $\tau \le T$ Itôs formula applied to the process $(Y - k)$ yields

$$\begin{aligned} E\left[\int_\tau^T \|Z_s\|^2 ds \middle| \mathcal{F}_\tau\right] &= E[(F-k)^2|\mathcal{F}_\tau] - (Y_\tau - k)^2 \\ &\quad + \frac{2}{\alpha} E\left[\int_\tau^T (Y_s - k)\bar{g}(s, Z_s, \theta_s) ds \middle| \mathcal{F}_\tau\right]. \end{aligned}$$

Using the boundedness of Y and F, the non positivity of $(Y - k)$, (7.18) and the fact that θ is uniformly bounded, we obtain for constants c_1, c_2, c_3

$$\begin{aligned} E\left[\int_\tau^T \|Z_s\|^2 ds \middle| \mathcal{F}_\tau\right] &\le c_1 + c_2 E\left[\int_\tau^T \|Z_s\| \, \|\theta_s\| ds \middle| \mathcal{F}_\tau\right] \\ &\le c_3 + \frac{1}{2} E\left[\int_\tau^T \|Z_s\|^2 ds \middle| \mathcal{F}_\tau\right]. \end{aligned}$$

The second inequality is a consequence of

$$|\, a\, b \,| \le \frac{1}{2} a^2 + 2\, b^2 \qquad \text{for all } a, b \in \mathbb{R},$$

and the fact that θ is uniformly bounded. Thus

$$E\left[\int_\tau^T \|Z_s\|^2 ds \middle| \mathcal{F}_\tau\right] \le c$$

for all stopping times $\tau \le T$ and a constant c that does not depend on τ.

□

In the next Lemma we prove the existence of predictable processes $(\bar{p}, \bar{\nu})$ attaining the saddle value $\bar{g}(t, Z_t, \theta_t)$ applying a measurable selection theorem proved in Beneš (Ben70).

Lemma 56 *Let $\bar{g}(t, \theta_t, z)$ be the saddle value as defined in (7.13) where θ is given in (7.2) and $z \in \mathbb{R}^m$. For every $z \in \mathbb{R}^m$, the process $(\bar{g}(t, \theta_t, z))_t$ is predictable. More exactly: $\bar{g}$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) - \mathcal{B}(\mathbb{R})$ measurable.*

Moreover, let Z be a predictable process, e.g. the integrand part of the solution of (7.20). There exist two predictable processes $\bar{p}$ and $\bar{\nu}$ satisfying

$$g(\bar{p}_t, \bar{\nu}_t, Z_t, \theta_t) = \bar{g}(t, Z_t, \theta_t) \quad P \otimes \lambda \text{ -a.s.},$$

i.e. for amost every (ω, t) $(\bar{p}_t, \bar{\nu}_t)$ is a saddle point of (7.13).

Proof. Define

$$I((\omega,t),q,r) = \begin{cases} 1 & \text{for} \quad q \in A_t(\omega), v \in C_t(\omega), \\ -\infty & \text{for} \quad q \in A_t(\omega), v \notin C_t(\omega), \\ +\infty & \text{for} \quad q \notin A_t(\omega). \end{cases}$$

Since the graphs of (A_t) and (C_t) are $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ measurable, the function I is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) - \mathcal{B}(\bar{\mathbb{R}})$– measurable, where $\bar{\mathbb{R}}^d = \mathbb{R}^d \cup \{-\infty\} \cup \{+\infty\}$. Furthermore,

$$\bar{g}(t,\theta_t,z) = \min_{q\in\mathbb{R}^d} \max_{r\in\mathbb{R}^d} I((\omega,t),q,r) g(t,q,r,\theta_t,z).$$

So $\bar{g}$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) - \mathcal{B}(\bar{\mathbb{R}})$–measurable. Since $\bar{g}$ is finite, we have also $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) - \mathcal{B}(\mathbb{R})$– measurability.

Now we turn to prove existence of the predictable processes $(\bar{p},\bar{\nu})$ that are saddle points for every (ω,t). In order to do so, we apply a measurable selection theorem stated in Benes, (Ben70) Lemma 1. It is useful to interpret g in the following way:

$$(q,r,(\omega,t)) \mapsto g(q,r,Z_t,\theta_t)$$

is continuos in (q,r) for every fixed (ω,t) and predictable for every fixed (q,r). The saddle value $\bar{g}(t,Z_t,\theta_t)$ is also predictable and finite. Thus, Lemma 1 in (Ben70) yields the result.

□

Using all these Lemmata we prove Theorem 50.

Proof of Theorem 50. The existence of a saddle point of the function $g(q,v,z,\theta_t)$ is shown in Lemma 52. Lemma 53 states an estimate of the value of g at the saddle point in terms of z and θ_t uniformly for all A_t, C_t, $(\omega,t) \in \Omega \times [0,T])$. This estimate is used to proof existence of a solution of the BSDE (7.10) in Lemma 54. Lemma 56 states that there exist predictable processes $\bar{p}$ and $\bar{\nu}$ such that for all $(\omega,t) \in \Omega \times [0,T]$, $(\bar{p}_t,\bar{\nu}_t)$ is a saddle point of $g(q,v,Z_t,\theta_t)$ with constraints $q \in A_t$ and $v \in C_t$. Lemma 55 together with Lemma 53 and the fact that θ is bounded yield that $\int_0^\cdot \bar{p}_t dW_t$ and $\int_0^\cdot \bar{\nu}_t dW_t$ are P-BMO martingales, hence $\bar{p} \in \mathcal{A}$ and $\bar{\nu} \in \mathcal{V}$. The construction of $R(p,\nu)$ shows that $(\bar{p},\bar{\nu})$ is indeed a saddle point of the functional K defined in (7.8). Since K is concave in p and convex in ν, the saddle value of K is unique. Thus $(\bar{p},\bar{\nu})$ is a solution of the robust utility maximization problem 49. However, K is linear in ν for fixed p. So the saddle point $(\bar{p},\bar{\nu})$ may not be unique.

□

The construction of the optimal trading strategy is still possible under slightly more general conditions.

Remark 57 The constraints on trading strategies can be formulated more general than in Definition 47: the constraints on the trading strategies π may be random and time dependent. We formulate our assumption for the notion of trading strategies according to Remark 48. The set $\{(q,(\omega,t))|q \in A_t\}$ is assumed to be $\mathcal{P}\otimes\mathcal{B}-$measurable. The sets A_t, $(\omega,t) \in \Omega\times[0,T]$ are closed, convex and satisfy (7.6). Then Theorem 50 remains valid. We don't need the assumption that the sets (A_t) are generated by a process of matrices applied to a deterministic set $\tilde{A}$ resp. C.

The complete market

In the situation of a complete market we get a more explicit result. On a complete market, every random payoff is hedgeable by a trading strategy. This is the case if the number of stocks d is equal to the number of dimensions m of the Brownian motion. The assumptions on the volatility matrix stated below (7.1) yields then that σ_t is regular for all $t \in [0,T]$. Furthermore, there are no restrictions on the values π_t that a trading strategy takes at a particular time $t \in [0,T]$. This means, Definition 47 describing the set $\mathcal{A}$ of admissible trading strategies remains valid with $\tilde{A} = \mathbb{R}^m$. Of course, the trading strategy p in the notation of Remark 48 also may take values in $\mathbb{R}^m$, hence $A_t = \mathbb{R}^m$ for all $t \in [0,T]$.

We find the saddle point of the function g explicitly and obtain a simple BSDE that reveals the structure of the optimal trading strategy. It is even possible to determine $\bar{\nu}$. We have to calculate

$$\min_{q\in\mathbb{R}^m}\max_{v\in C_t} g(q,v,z,\theta_t).$$

There exists a saddle point, so we can change min and max. For a fixed $v \in C_t$, the minimum is attained for

$$q(v) = z + \frac{1}{\alpha}(v+\theta_t),$$

and we have

$$g(q(v),v,z,\theta_t) = -\frac{1}{2}(\theta_t+v)^2 - \alpha z\theta_t.$$

Hence

$$\bar{v} = \Pi_{C_t}(-\theta_t),$$

and

$$\bar{g}(t, z, \theta_t) = -\frac{1}{2} dist^2_{C_t}(-\theta_t) - \alpha z \theta_t. \tag{7.22}$$

In this case the BSDE simplifies to

$$Y_t = F - \int_t^T Z_s dW_s + \int_t^T \frac{1}{2\alpha} dist^2_{C_t}(-\theta_s) + Z_s \theta_s ds.$$

In order to solve the BSDE we change the probability measure to $\tilde{Q}$ via

$$\frac{d\tilde{Q}}{dP} = \mathcal{E}\left(-\int \theta_s dW_s\right)_T$$

and obtain the $\tilde{Q}-$Brownian Motion $\tilde{W}_t = W_t + \int_0^t \theta_s ds$. The BSDE reads

$$Y_t = F - \int_t^T Z_s d\tilde{W}_s + \int_t^T \frac{1}{2\alpha} dist^2_{C_t}(-\theta_s) ds.$$

Then Y is given by successive conditioned expectations under $\tilde{Q}$:

$$Y_t = \tilde{E}\left[F + \int_t^T \frac{1}{2\alpha} dist^2_{C_t}(-\theta_s) ds \middle| \mathcal{F}_t \right].$$

Z is the integrand satisfying

$$Y_0 + \int_0^t Z_s d\tilde{W}_s = \tilde{E}\left[F + \int_0^T \frac{1}{2\alpha} dist^2_{C_t}(-\theta_s) ds | \mathcal{F}_t \right], \quad t \in [0, T]. \tag{7.23}$$

Now we state the result in a theorem.

Theorem 58 *Let the stock price process S be given according to (7.1), where the number of stocks d is equal to the number of dimensions of the Brownian motion m. Let the market be complete, i.e. the constraints in Definition 47 of admissible trading stategies given by $A_t = \mathbb{R}^m$ for all $(\omega, t) \in \Omega \times [0, T]$. Furthermore let the set of probability measures $\mathcal{Q}$ in the robust preferences be given by Assumption 45. Then the solution of the robust utility maximization problem is given in the following way: the optimal strategy $\bar{p}$ is*

$$\bar{p}_t = Z_t + \frac{1}{\alpha}\theta_t + \frac{1}{\alpha}\Pi_{C_t}(-\theta_t),$$

where Z is the integrand in (7.23). The integrand $\bar{\nu}$ for the probability measure $\bar{Q} = Q^{\bar{\nu}}$ attaining the minimum in (7.7) is

$$\bar{\nu}_t = \Pi_{C_t}(-\theta_t).$$

The projection $\Pi_C(x)$ of an $x \in \mathbb{R}^m$ on a closed convex set C is defined in (1.3) on page 12.

Let us take a closer look at the stucture of the optimal trading strategy in a complete market. The optimal strategy consists of two parts:

1. the unique hedging strategy Z_t for $F + \int_0^T \frac{1}{2\alpha} dist^2_{C_s}(-\theta_s)ds$ under stock prices with drift θ, and
2. the optimal strategy for the maximization of the expected utility with respect to $Q^{\bar{\nu}}$ for $F = 0$. Observe that the drift of the stock price process under $Q^{\bar{\nu}}$ is equal to $\theta + \Pi_{C_t}(-\theta_t)$.

If $\int_0^T \frac{1}{2\alpha} dist^2_{C_s}(-\theta_s)ds$ is deterministic, then this term does not affect the hedging strategy.

Schied (Sch04b) finds a so called *least favorable measure* under the assumption that $F = 0$, (C_t) is deterministic and the market is complete. This is the probability in $\mathcal{Q}$ under which the Euclidian norm of the drift of the stock price process is minimized. This probability does not depend on the utility function. Observe that the market price of risk for the stock under the measure Q^{ν} is equal to $\theta + \nu$. Of course, under the same assumptions our probability $Q^{\bar{\nu}}$ is the same as in (Sch04b), Proposition 3.2. The exponential utility function allows us to find the same structure of $Q^{\bar{\nu}}$ even if (C_t) is not deterministic and $F \neq 0$. $Q^{\bar{\nu}}$ neither depends on the terminal liability F nor on the risk aversion α in the exponential utility function.

7.3 Power Utility

Our goal is the characterization of the optimal trading strategy for the robust utility maximization problem. In contrast to the previous section, we use another type of utility functions. They are called power utility and have the following form:

$$U_\gamma(x) = \frac{1}{\gamma} x^\gamma, \qquad x \geq 0, \quad \gamma \in (0,1).$$

The set of probability measures in our robust utility maximization problem is the same as defined in Assumption 45. This time, the additional liability is equal to zero, i.e. $F = 0$. In this section, we use a notion of trading strategy that is better adapted to the problem: $\tilde{\rho} = (\tilde{\rho}^i)_{i=1,\dots,d}$ denotes the part of the wealth invested in stock i. The number of shares of stock i is given by $\frac{\tilde{\rho}^i_t X_t}{S^i_t}$. A d–dimensional $\mathbb{F}$–predictable process $\tilde{\rho} = (\tilde{\rho}_t)_{0 \leq t \leq T}$ is called trading strategy (part of wealth) if the following wealth process is well defined:

$$X_t^{(\tilde\rho)} = x + \int_0^t \sum_{i=1}^d \frac{X_s^{(\tilde\rho)} \tilde\rho_{i,s}}{S_{i,s}} dS_{i,s} = x + \int_0^t X_s^{(\tilde\rho)} \tilde\rho_s \sigma_s (dW_s + \theta_s ds), \qquad (7.24)$$

and the initial capital x is positive. The wealth process $X^{(\tilde\rho)}$ can be written as:

$$X_t^{(\tilde\rho)} = x\mathcal{E}\left(\int \tilde\rho_s\sigma_s(dW_s+\theta_s ds)\right)_t, \quad t\in[0,T].$$

The trading strategies are constrained to take values in a closed convex set $\bar A_2 \subseteq \mathbb{R}^d$. Observe that the part of wealth invested in the stocks is constrained. This is a difference to the constraints on the number of shares considered in section 7.2.

As before, it is more convenient to introduce

$$\rho_t = \tilde\rho_t\sigma_t, \qquad t\in[0,T].$$

Accordingly, ρ is constraint to take its values in

$$A_t(\omega) = \tilde A_2\sigma_t(\omega) \quad t\in[0,T], \omega\in\Omega.$$

The sets A_t are closed convex subsets of $\mathbb{R}^m$ and satisfy (7.6). Before stating the maximization problem, we define the set of admissible trading strategies.

Definition 59 *The set of admissible trading strategies $\tilde{\mathcal{A}}$ consists of all m-dimensional predictable processes $\rho = (\rho_t)_{0\le t\le T}$ satisfying $\rho_t \in A_t(\omega)$ $P\otimes\lambda$–a.s and $\int_0^\cdot |\rho_s\sigma_s|^2 ds < \infty$ P–a.s.*

Now we are able to state the robust utility maximization problem.

Problem 60 *Let $0<\gamma<1$. The solution of the robust utility maximization problem with utility function $U(x) = x^\gamma$ consists of a trading strategy $\bar\rho \in \mathcal{A}$ and a probability measure $Q^{\bar\nu} \in \mathcal{Q}$ with a $\bar\nu \in \mathcal{V}$ attaining*

$$V(x) = \sup_{\rho\in\mathcal{A}} \inf_{\nu\in\mathcal{V}} E^\nu[(X_T^{(\rho)})^\gamma].$$

Recall that $\mathcal{Q}$ and $\mathcal{V}$ are defined in Assumption 45 and in (7.4). Our aim is to prove existence of a saddle point $(\bar\rho,\bar\nu)$. Of course, this saddle point is then the solution of problem 60. We aim to use the same techniques as in section 7.2. Define for $\rho\in\mathcal{A}$, $\nu\in\mathcal{V}$

$$\begin{aligned} K(\rho,\nu) &= E^\nu[(X_T^{(\rho)})^\gamma] \\ &= E\left[x^\gamma \exp\left(\gamma\int_0^T (\rho_s dW_s + \rho_s\theta_s - \frac{1}{2}\rho_s^2 ds)\exp(\int_0^T \nu_s dW_s - \frac{1}{2}\nu_s^2 ds)\right)\right]. \end{aligned}$$

We have to to prove that K has a saddle point. This means, we have to find a $\bar{\rho} \in \mathcal{A}$ and a $\bar{\nu} \in \mathcal{V}$ satisfying

$$K(\rho, \bar{\nu}) \leq K(\bar{\rho}, \bar{\nu}) \leq K(\bar{\rho}, \nu) \quad \forall \, \rho \in \mathcal{A}, \nu \in \mathcal{V}.$$

In order to find the saddle point, we construct a family of stochastic processes $(R_t(\rho, \nu))_{t \in [0,T]}$ indexed with $\mathcal{A} \times \mathcal{V}$ with the following properties: the initial value $R_0(\rho, \nu)$ does not depend on ρ and ν and the terminal value $R_T(\rho, \nu)$ is equal to $\frac{dQ^\nu}{dP}(X_T^{(\rho)})^\gamma$, the random variable in the expectation in $K(\rho, \eta)$. Furthermore, there exists $\bar{\rho} \in \mathcal{A}$ and a $\bar{\nu} \in \mathcal{V}$ such that $R(\rho, \bar{\nu})$ is a supermartingale, $R(\bar{\rho}, \bar{\nu})$ is a martingale, and $R(\bar{\rho}, \nu)$ is a submartingale. Thus, for all $\rho \in \mathcal{A}$ and $\nu \in \mathcal{V}$

$$E[R_T(\rho, \bar{\nu})] \leq E[R_T(\bar{\rho}, \bar{\nu})] = R_0(\bar{\rho}, \bar{\nu})) \leq E[R_T(\bar{\rho}, \nu)].$$

Since $K(\rho, \nu) = E[R_T(\rho, \nu)]$, $(\bar{\rho}, \bar{\nu})$ is the solution of problem 60. We can exploit the exponential structure of the problem and set

$$\begin{aligned} R_t(\rho, \nu) &= x^\gamma \exp(\int_0^t \gamma \rho_s dW_s + \gamma \rho_s \theta_s - \frac{\gamma}{2} \rho_s^2 ds) \exp(\int_0^t \nu_s dW_s - \frac{1}{2} \nu_s^2 ds) \\ & \quad \times \exp(Y_0 + \int_0^t Z_s dW_s + f(s, \theta_s, Z_s) ds). \end{aligned}$$

The processes (Y, Z) are the solution of a BSDE in the form

$$Y_t = 0 - \int_t^T Z_s dW_s - \int_t^T f(s, \theta_s, Z_s) ds$$

where $f : [0, T] \times \mathbb{R}^m \times \mathbb{R}^m$ has to be constructed such that $R(\rho, \nu)$ has the properties mentioned above. Observe that the terminal value Y_T is equal to zero and f does not depend on Y. So the initial and terminal condition on R will be satisfied as soon as we find f.

Now define the function $g : \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ as

$$\begin{aligned} g(r, v, \theta_t, z) &= \frac{1}{2}(\gamma r + v + z)^2 + \gamma r \theta_t - \frac{\gamma}{2} r^2 - \frac{1}{2} v^2 \qquad (7.25) \\ &= -\frac{\gamma(1-\gamma)}{2} \left(r + \frac{1}{\gamma - 1}(v + \theta_t + z) \right)^2 + \\ & \quad + \frac{\gamma}{2(1-\gamma)} \left(v + \theta_t + \frac{z}{\gamma} \right)^2 - \frac{1}{2\gamma} z^2 - \theta_t z, \end{aligned}$$

where $r \in \mathbb{R}^m$ takes the place of ρ_t, $v \in \mathbb{R}^m$ the place of ν_t and $z \in \mathbb{R}^m$ stands for Z_t. In order to keep reading simple, we don't replace θ_t. With this notation we have

$$\begin{aligned} R_t(\rho,\nu) &= x^\gamma \exp(Y_0)\mathcal{E}\left(\int(\gamma\rho_s+\nu_s+Z_s)dW_s\right)_t \times \\ &\quad\times \exp\left(\int_0^t g(\rho_s,\nu_s,\theta_s,Z_s)+f(s,\theta_s,Z_s)ds\right). \end{aligned}$$

For every θ_t, z and (ω,t), there exists a saddle point $(\bar r,\bar v)$ of g with the following constraints: $r\in A_t$, $v\in C_t$, i.e. $\bar r\in A_t$ and $\bar v\in C_t$ satisfying

$$g(r,\bar v,\theta_t,z)\le g(\bar r,\bar v,\theta_t,z) := \bar g(t,\theta_t,z)\le g(\bar r,v,\theta_t,z)\quad \forall\, r\in A_t, v\in C_t.$$

Observe that the functions $-g(\cdot,v,\theta_t,z)$, $v\in C_t$, and $g(r,\cdot,\theta_t,z)$, $r\in A_t$, have no directions of recession, since g is negative quadratic in r and v is constraint in the bounded set C_t. This is even more than assumed in Theorem 37.3 in Rockafellar (Roc70) that states that the saddle point $(\bar r,\bar v)$ exists and that the saddle value $\bar g(t,\theta_t,z)$ is finite. Since the constraints depend on (ω,t), the saddle point and the saddle value also do. The saddle point $(\bar r,\bar v)$ may not be unique since g is linear in v. We have to choose

$$f(t,\theta_t,Z_t) = -\bar g(t,\theta_t,Z_t).$$

In order to find a solution of the BSDE we first have to estimate the saddle value $\bar g(t,\theta_t,z)$ from above and below. Let us start with the estimate from above. We have

$$\bar g(t,\theta_t,z)\le \sup_{r\in A_t} g(r,v,\theta_t,z)\quad \forall\, v\in C_t.$$

This supremum is atteint for

$$r(t,v,\theta_t,z) = \Pi_{A_t}\left(-\frac{1}{\gamma-1}(v+\theta_t+z)\right). \tag{7.26}$$

Due to (7.6) and since the sets (C_t) are uniformly bounded, we obtain for some constant $k>0$

$$\bar g(t,\theta_t,z)\le k(1+\|\theta_t\|^2+\|z\|^2)\quad \forall\, z\in\mathbb{R}^m \tag{7.27}$$

and for the Euklidian norm of the first part of the saddle point $\bar r$ depending on (t,θ_t,z)

$$\|\bar r(t,\theta_t,z)\|^2\le k(1+\|\theta_t\|^2+\|z\|^2). \tag{7.28}$$

For the estimate from below we use the uniform boundedness of the sets C_t, $t\in[0,T]$, and get an $r_0\in A_t$ with $\|r_0\|\le k_1$ uniformly for all (ω,t). This yields for a constant k_2

$$\bar g(t,\theta_t,z)\ \ge\ g(r_0,\bar v,\theta_t,z)\ge k_2\ \forall\, z\in\mathbb{R}^m. \tag{7.29}$$

Now we are able to state the BSDE that leads to the solution of the robust utility maximization problem:

$$Y_t = 0 - \int_t^T Z_s dW_s + \int_t^T \bar{g}(s, \theta_s, Z_s) ds \tag{7.30}$$

Lemma 61 *The BSDE (7.30) has a unique solution $(Y, Z) \in \mathcal{H}^\infty(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R})$. Furthermore, $\int_0^\cdot Z_s dW_s$ is a P–BMO martingale.*

Proof. As in the proof of Lemma 56, we see that $\bar{g}(t, \theta_t, 0)$ is predictable. Since θ is uniformly bounded, estimates (7.27) and (7.29) yield that $\bar{g}(t, \theta_t, z)$ satisfies condition H1 in (Kob00). So the BSDE (7.30) has a solution (Y, Z). The key to prove uniqueness of the solution for the BSDE (7.20) is estimate (7.21). Since there exists a unique saddle point of g, we may exchange sup and inf and obtain

$$\begin{aligned} \bar{g}(t, \theta_t, z) &= \inf_{v \in C_t} \left\{ -\frac{\gamma(1-\gamma)}{2} \operatorname{dist}^2_{A_t} \left(\frac{1}{1-\gamma}(v + \theta_t + z) \right) \right. \\ &\quad \left. + \frac{\gamma}{2(1-\gamma)} \left(v + \theta_t + \frac{1}{\gamma} z \right)^2 - \frac{1}{2\gamma} z^2 - \theta_t z \right\}. \end{aligned} \tag{7.31}$$

Using the same calculation as in the proof on Lemma 54 we obtain for the dependence on the saddle point $\bar{g}(t, \theta_t, z)$ on z the following estimate

$$|\bar{g}(t, \theta_t, z_1) - \bar{g}(t, \theta_t, z_2)| \le c(1 + |\theta_t| + |z_1| + |z_2|)|z_1 - z_2|.$$

The uniqueness of the solution (Y, Z) in $\mathcal{H}^\infty(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^m)$ follows also as in the proof of Lemma 54. Furthermore, since $|Y|$ is uniformly bounded and $\bar{g}$ satisfies (7.29), the integral $\int_0^\cdot Z_s dW_s$ is a P– BMO martingale. (see Lemma 55).

□

Now everything is prepared for the result of the robust utility maximization problem with a power utility function.

Theorem 62 *Let (Y, Z) be the solution of the BSDE (7.30). There exists a solution $(\bar{\rho}, \bar{\nu})$ of the robust utility maximization problem 60. For almost every $(\omega, t) \in \Omega \times [0, T]$ this solution satisfies*

$$g(r, \bar{\nu}, \theta_t, Z_t) \le g(\bar{\rho}_t, \bar{\nu}_t, \theta_t, Z_t) = \bar{g}(t, \theta_t, Z_t) \le g(\bar{\rho}, v, \theta_t, Z_t) \quad \forall r \in A_t, v \in C_t. \tag{7.32}$$

The maximal utility $V(x)$ is equal to

$$V(x) = x^\gamma (Y_0)^\gamma, \qquad x > 0.$$

Proof. Let (Y, Z) be the solution of the BSDE (7.30). Lemma 1 in Beneš yields existence of two predictable processes $\bar{\rho}$ and $\bar{\nu}$ satisfying (7.32) (see the proof of Lemma 56). Since $\int_0^{\cdot} Z_s dW_s$ is a P–BMO martingale, (7.28) and the boundedness of the sets (C_t) yield that $\int_0^{\cdot} \bar{\rho}_s dW_s$ and $\int_0^{\cdot} \bar{\nu}_s dW_s$ are also $P-$BMO martingales. So, $\bar{\rho} \in \mathcal{A}$ and $\bar{\nu} \in \mathcal{V}$. Due to the construction of R, we see that $(\bar{\rho}, \bar{\nu})$ (resp. $\bar{\rho}$ and the probability measure $Q^{\bar{\nu}}$) is indeed the solution of the robust utility maximization problem 60. Furthermore, $V(x) = R_0$.

□

The Complete market

Let us now consider the situation of a complete market, i.e. the number of stocks d is equal to the dimension of the Brownian motion m and there are no constraint on the trading strategies: $\tilde{A}_2 = \mathbb{R}^m$. Due to the assumptions on σ below (7.1), the matrices σ_t are invertible. Hence $A_t = \mathbb{R}^m$. Since there exists a saddle point of $g(r, v, \theta_t, z)$ for every $z \in \mathbb{R}^m$ and every (ω, t) with constraints A_t, C_t, we may change sup and inf. So

$$\begin{aligned}\bar{g}(t, \theta_t, z) &= \inf_{v \in C_t} \sup_{r \in \mathbb{R}^m} g(r, v, \theta_t, z) \\ &= \frac{1}{2} \frac{\gamma}{1-\gamma} \mathrm{dist}^2_{C_t} \left(-\theta_t - \frac{z}{\gamma} \right) - \frac{1}{2\gamma} z^2 - \theta_t z.\end{aligned}$$

The BSDE reads for all $t \in [0, T]$

$$Y_t = 0 - \int_t^T Z_s dW_s - \int_t^T \frac{1}{2} \frac{\gamma}{1-\gamma} \mathrm{dist}^2_{C_t} \left(-\theta_s - \frac{1}{\gamma} Z_s \right) ds \quad (7.33)$$

$$- \int_t^T \left[\frac{1}{2\gamma} Z_s^2 + \theta_s Z_s \right] ds. \quad (7.34)$$

Of course, if (C_t) is deterministic, we have the least favorable measure in the saddle point.

Remark 63 Let the sets C_t, $t \in [0, T]$ be deterministic. Then the probability measure $Q^{\bar{\nu}}$ appearing in the saddle point is the least favorable measure stated in Schied, (Sch04b) Proposition 3.2 The second component of the solution (Y, Z) of (7.33) satisfies $Z = 0$. The integrand for the measure $Q^{\bar{\nu}}$ is

$$\bar{\nu}_t = \Pi_{C_t}(-\theta_t),$$

and the optimal trading strategy is

$$\bar{\rho}_t = -\frac{1}{\gamma - 1}(\Pi_{C_t}(-\theta_t) + \theta_t).$$

Observe that the measure $Q^{\bar{\nu}}$ does not depend on γ. For a maximization of the expected utility under Q^{ν}, the drift is equal to $\theta + \nu$. So the agent maximizes the utility under the probability measure that minimizes the drift of the stock.

Chapter 8

Uncertain stock price dynamics

Introduction

In this chapter we solve the robust utility maximization problem where the drift (b_t) and the volatility (σ_t) of the stock prices are not exactly known. In contrast to Chapter 7, the investor maximizes under a fixed probability measure P. The uncertainty lies in the coefficients of the stock price process. We assume that $\theta_t = \sigma_t^{tr}(\sigma_t \sigma_t^{tr})^{-1} b_t$ is contained in a closed bounded convex set for all $t \in [0, T]$, but we don't know the exact value. This knowledge of the stock price process might be the result of statistics where the coefficients are estimated to be in a certain confidence interval. In this setup, the robust utility of a trading strategy is defined in the following way: the investor compares the expected utility of the terminal wealth for every possible θ. Similar to the robust maximization with a set of probability measures, the investor takes the worst θ that is possible. He tries to find the trading strategy that maximizes this robust utility with uncertain drift.

We consider the exponential, power and logarithmic utility functions. If the investor uses the exponential utility function, he may hedge a liability he has to pay out at the end of the trading interval that is described by an $\mathcal{F}_T-$measurable bounded random variable F. The utility indifference price for F with respect to the robust utility maximization under uncertain drift is stated in (8.11).

8.1 Exponential utility

In this section we consider the exponential utility function. Let us first describe the parameters of the stock price process. Since the coefficients b and σ enter the maximization problem only via $\theta_t = b_t \sigma_t$, we specify only θ

and call this process drift. For every $(\omega, t) \in \Omega \times [0, T]$, θ_t must be contained in the set $C_t(\omega)$ defined in as follows:

Assumption 64 *Let $(C_t(\omega))$ denote a predictable multivariate mapping of convex sets in the following sense: for every $(\omega, t) \in \Omega \times [0, T]$, $C_t(\omega) \subset \mathbb{R}^m$ is closed and convex and the graph of (C_t) i.e. the set $\{((\omega, t)), v) | v \in C_t(\omega)\} \subset \Omega \times [0, T] \times \mathbb{R}^m$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^m)$ measurable. Furthermore, the sets $C_t(\omega)$ are contained in a bounded ball around the origin for all $(\omega, t) \in \Omega \times [0, T]$.*

So we define

$$\Theta = \{\theta \; \mathbb{R}^m\text{–valued, predictable}, \theta_t \in C_t \; \forall t \in [0, T] \; P\text{–a.s. }\}. \tag{8.1}$$

In this section, we use the set of admissible trading strategies as defined in Definition 47 and Remark 48 for the exponential utility function. The optimization problem is the following saddle point problem:

Problem 65 *Let F be an $\mathcal{F}_T$–measurable bounded random variable. A solution of the robust utility maximization problem consists of a trading strategy $\bar{p} \in \mathcal{A}$ and a $\bar{\theta} \in \Theta$ attaining*

$$V(x, F) = \sup_{p \in \mathcal{A}} \inf_{\theta \in \Theta} E\left[- \exp\left(-\alpha\left(x + \int_0^T p_t(\theta_t dt + dW_t) - F\right)\right)\right]. \tag{8.2}$$

We use the same martingale argument as in section 7.2 to find the saddle point in problem 65. The process θ takes now the role of ν. Set

$$\begin{aligned} R_t(p, \theta) &= - \exp(-\alpha x + \alpha Y_0)\mathcal{E}\left(\int (-\alpha p_s + \alpha Z_s) dW_s\right)_t \times \\ &\quad \times \exp\left(\int_0^t \left(\frac{1}{2}(-\alpha p_s + \alpha Z_s)^2 - \alpha p_s \theta_s + \alpha f(s, Z_s)\right) ds\right), \end{aligned} \tag{8.3}$$

where f still has to be determined. The pair of processes (Y, Z) is the solution of a BSDE:

$$Y_t = F - \int_t^T Z_s dW_s - \int_t^T f(s, Z_s) ds. \tag{8.4}$$

Now we prove the steps that lead to the solution of problem 65. By choosing f we construct a family of stochastic processes indexed with $\mathcal{A} \times \Theta$. For every (p, θ), the terminal value $R_T(p, \theta)$ is equal to the random variable in the expectation in (8.2), i.e. $E[R_T(p, \theta)] = E[- \exp(\alpha(X_T^{\theta,p} - F))]$, where $X^{\theta,p}$ is the wealth process for the trading strategy $p \in \mathcal{A}$ and drift $\theta \in \Theta$. The goal is to find a driver f for the BSDE such that there exists a $(\bar{p}, \bar{\theta}) \in \mathcal{A} \times \Theta$ satisfying the following conditions: $R(\bar{p}, \bar{\theta})$ is a martingale, $R(p, \bar{\theta})$ is a supermartingale for every $p \in \mathcal{A}$ and $R(\bar{p}, \theta)$ is a submartingale for every $\theta \in \Theta$.

Then $(\bar{p}, \bar{\theta})$ is our saddle point and the solution of the robust utility maximization problem 65 because

$$E[R(p, \bar{\theta})] \le R_0 = E[R(\bar{p}, \bar{\theta})] \le E[R(\bar{p}, \theta)] \quad \forall\, p \in \mathcal{A}, \theta \in \Theta.$$

In order to find the appropriate function f we solve a deterministic finite dimensional saddle point problem.

Our notation distinguishes between stochastic processes and elements of $\mathbb{R}^m$. We write $q \in \mathbb{R}^m$ for p_t, $u \in \mathbb{R}^m$ for θ_t and $z \in \mathbb{R}^m$ for Z_t. Define $g : \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ as

$$\begin{aligned} g(q,u,z) &= -\alpha u q + \frac{\alpha^2}{2}(q^2 - 2qz + z^2) \\ &= \frac{\alpha^2}{2}\left(q - \left(z + \frac{1}{\alpha}u\right)\right)^2 - \alpha u z - \frac{1}{2}u^2 \end{aligned}$$

The function $f : \Omega \times [0,T] \times \mathbb{R}^m$ is chosen such that there exists for every $(\omega, t, z) \in \Omega \times [0,T] \times \mathbb{R}^m$ a $(\bar{q}, \bar{u})$ depending on (ω, t, z) satisfying

$$\begin{aligned} g(q,\bar{u},z) &+ \alpha f(t,z) \ge 0 \quad \forall\, q \in A_t, z \in \mathbb{R}^m, \\ g(\bar{q},\bar{u},z) &+ \alpha f(t,z) = 0 \quad \forall z \in \mathbb{R}^m, \\ g(\bar{q},u,z) &+ \alpha f(t,z) \le 0 \quad \forall\, u \in C_t, z \in \mathbb{R}^m. \end{aligned} \tag{8.5}$$

So we take

$$f(t,z) = -\frac{1}{\alpha}\bar{g}(t,z), \quad t \in [0,T], z \in \mathbb{R}^n,$$

where $\bar{g}(t,z)$ is the saddle value of g satisfying

$$g(\bar{q},u,z) \le g(\bar{q},\bar{u},z) = g(\bar{q},\bar{u},z) \le g(q,\bar{u},z), \quad \forall q \in A_t, u \in C_t.$$

Observe that the dependence of f on (ω, t) is a result of the fact that the constraints A_t and C_t depend on (ω, t). There exists a saddle point $(\bar{q}, \bar{u})$ of g that depends on (ω, t) and z. The saddle value $\bar{g}(t,z)$ can be estimated as in Lemma 53. To this end observe that

$$\bar{g}(t,z) = \sup_{v \in C_t}\left(\frac{\alpha^2}{2}\operatorname{dist}^2_{A_t}(z + \frac{1}{\alpha}u)) - \alpha u z - \frac{1}{2}u^2\right). \tag{8.6}$$

Since all sets C_t, $t \in [0,T]$, are contained in a bounded ball around the origin and the sets A_t satisfy (7.6), we obtain for constants c_1, c_2 uniformly for $(\omega, t) \in \Omega \times [0,T]$ and all $z \in \mathbb{R}^m$

$$-\alpha c_1|z| \le \bar{g}(t,z) \le c_2(1 + |z|^2). \tag{8.7}$$

We obtain an estimate for the norm of the saddle point. Since the sets C_t are contained in a ball around the origin, there exists a constant $c > 0$ such that

$$\bar{u} \leq c \quad P - a.s.$$

For the first component of the saddle point we have

$$|\bar{q}| \leq c(1 + |z|), \quad u \in \mathbb{R}^m. \tag{8.8}$$

With this saddle value function, we have the driver of the BSDE (8.4)

$$f(t, z) = -\frac{1}{\alpha}\bar{g}(t, z), \quad t \in [0, T], z \in \mathbb{R}^m. \tag{8.9}$$

Analoguosly to Lemma 56, $(f(t, 0))_{t\in[0,T]}$ is a predictable process. Theorem 2.3 of Kobylanski (Kob00) states that a solution (Y, Z) of the BSDE (8.4) with driver (8.9) exists. As in Lemma 54, one can show that (Y, Z) is the unique solution of (8.4) in $\mathcal{H}^\infty \times \mathcal{H}^2$. Furthermore, the boundedness of F and (8.6) yields that $\int_0^\cdot Z_s dW_s$ is a P– BMO martingale. This leads to the solution of the robust utility maximization problem with unknown drift.

Theorem 66 *A solution $(\bar{p}, \bar{\theta})$ of Problem 65 exists. Let (Y, Z) be the solution of the BSDE (8.4) with driver (8.9). The optimal trading strategy $\bar{p}_t$ and the drift $\bar{\theta}_t$ satisfy for $P \otimes \lambda$ a. e. (ω, t),*

$$g(q, \bar{\theta}_t, Z_t) \geq g(\bar{p}_t, \bar{\theta}_t, Z_t) \geq g(\bar{p}_t, u, Z_t) \quad \forall\, q \in A_t, u \in C. \tag{8.10}$$

The value of the robust utility maximization problem 65 is

$$V(x, F) = -\exp(-\alpha x + \alpha Y_0).$$

Proof. First, there exist predictable processes $\bar{p}, \bar{\theta}$ satisfying (8.10). This is a consequence of Lemma 1 in Beneš (see Lemma 56). Furthermore, $\bar{p} \in \mathcal{A}$: this trading strategy satisfies $\bar{p}_t \in A_t$ $P \otimes \lambda$–a.e. The fact that $\int_0^\cdot Z_s dW_s$ is a P–BMO martingale and (8.8) implies that $\int_0^\cdot \bar{p}_s dW_s$ is also a P–BMO martingale. The optimality of $(\bar{p}, \bar{\theta})$ follows from (8.5) and (8.3), see the construction of the processes $R(p, \theta)$. So $(\bar{p}, \bar{\theta})$ is the solution of problem (65).

□

Of course, we can calculate the utility indifference price of F for the robust utility maximization with uncertain drift. This is the additional initial capital x_F that the investor needs to get the same maximal utility with the liability to pay out F than without this liability, see Remark 51. Let (Y^F, Z^F) denote the solution of (8.4) with terminal value F and (Y^0, Z^0) the solution of (8.4) with terminal value 0. The utility indifference price is

$$x_F = Y_0^F - Y_0^0. \tag{8.11}$$

The complete market

Let the market be complete without trading constraints, i.e. we have an m–dimensional Brownian motion, $d = m$ stocks and $\tilde{A} = \mathbb{R}^m$ in Definition 47 of admissible trading strategies. So we have also $A_t = \mathbb{R}^m$. Then

$$\bar{g}(t, z) = \max_{u \in C_t} \left(-\frac{1}{2}(u + \alpha z)^2 + \frac{1}{2} z^2 \right).$$

Using this expression, one can easily find the saddle point

$$\bar{u} = \Pi_{C_t}(-\alpha z) \quad \text{and} \quad \bar{q} = z + \frac{1}{\alpha}\Pi_C(-\alpha z).$$

The BSDE has the following form:

$$Y_t = F - \int_t^T Z_s dW_s - \int_t^T (-dist_C^2(-\alpha Z_s) + \frac{1}{2} Z_s^2) ds. \tag{8.12}$$

In contrast to (7.23), we cannot see the solution of the BSDE (8.12) so easily. However, in the situation of Schied, (Sch04b) Proposition 3.2, we obtain the least favorable measure that is defined there.

Remark 67 Let $F = 0$ and the sets C_t deterministic. Then we can write down the solution of the BSDE and the solution of the robust utility maximizing problem 65 explicitly. Since $dist_{C_t}^2(0)$ is also deterministic, (Y, Z) with

$$Y_t = \int_t^T dist_{C_t}^2(0) ds, \qquad Z_t = 0$$

is a solution of the BSDE. The optimal trading strategy $\bar{p}$ and the market price of risk $\bar{\theta}$ attaining the saddle point have a particularly simple structure:

$$\bar{\theta}_t = \Pi_{C_t}(0), \qquad \bar{p}_t = \frac{1}{\alpha}\Pi_{C_t}(0).$$

Thus, the Euclidian norm of the drift is minimized. The drift $\bar{\theta}$ is the same as the drift stated in Schied (Sch04b), Proposition 3.2.

8.2 Power utility

In this section we solve the robust utility maximization problem for the power utility function $U(x) = x^\gamma$ in a setup where the drift and volatility of the stock are uncertain. We use set $\tilde{A}$ of admissible trading strategies described in Definition 59 together with the set Θ of possible drift processes defined in (8.1). The drift θ satisfies $\theta_t \in C_t$ where (C_t) is given according to (64). The agent solves the worst case scenario written in the following optimization problem:

Problem 68 *The solution of the robust utility maximization problem with uncertain drift for a utility function $U(x) = x^\gamma$ with $\gamma \in (0,1)$ consists of an admissible trading strategy $\bar{\rho} \in \tilde{\mathcal{A}}$ and a drift $\bar{\theta} \in \Theta$ attaining*

$$V(x) = \sup_{\rho\in\mathcal{A}} \inf_{\theta\in\Theta} E\left[x^\gamma \exp\left(\gamma\int_0^T \rho_s dW_s + \gamma\int_0^T \left(\theta_s\rho_s - \frac{1}{2}\rho_s^2\right) ds\right)\right]. \tag{8.13}$$

Within the expectation is written the utility of the terminal wealth for a trading strategy ρ and a stock price process with drift θ.

We aim at finding a saddle point for this optimization problem. Our method to solve it is to construct a family of stochastic processes $R(\rho,\theta)$ such that $R_T(\rho,\theta)$ is equal to the random variable in the expectation in (8.13), the initial value R_0 does not depend on (ρ,θ), there exists $\bar{\rho}$ and $\bar{\theta}$ such that $R(\bar{\rho},\bar{\theta})$ is a martingale, $R(\bar{\rho},\theta)$ is a submartingale and $R(\rho,\bar{\theta})$ is a supermartingale for all $\rho \in \mathcal{A}$ and $\theta \in \Theta$. Similarly to (8.3) for the exponential utility function, we set

$$\begin{aligned} R_t(\rho,\theta) &= x^\gamma \exp(Y_0)\mathcal{E}\left(\int(\gamma\rho_s + Z_s)dW_s\right)_T \times \\ &\quad \times \exp\left(\int_0^T \frac{1}{2}(\gamma\rho_s + Z_s)^2 ds + \gamma\rho_s\theta_s - \frac{1}{2}\gamma\rho_s^2 + f(s,Z_s)ds\right). \end{aligned}$$

As in the case of the robust utility maximization problem for the exponential utility function, (Y,Z) is the solution of the BSDE

$$Y_t = 0 - \int_t^T Z_s dW_s - \int_t^T f(s,Z_s)ds. \tag{8.14}$$

We have to choose the driver f such that R has the desired properties. Define

$$\begin{aligned} g(r,u,z) &= \frac{1}{2}(\gamma r+z)^2+\gamma r u-\frac{1}{2}\gamma r^2, \quad r,u,z\in\mathbb{R}^m \\ &= \frac{1}{2}\gamma(\gamma-1)r^2+\gamma r(u+z)+\frac{1}{2}z^2 \\ &= -\frac{\gamma(1-\gamma)}{2}\left(r-\frac{1}{1-\gamma}(u+z)\right)^2-\frac{\gamma}{2(1-\gamma)}(u+z)^2+\frac{1}{2}z^2. \end{aligned}$$

Here we replaced ρ_t by $r\in\mathbb{R}^m$, Z_t by $z\in\mathbb{R}^m$ and θ_t by $u\in\mathbb{R}^m$. Theorem 37.3 in Rockafellar and the boundedness of the constraints (C_t) yields a saddle point, i.e. $(\bar{r},\bar{u})$ depending on z and A_t, C_t satisfying

$$g(r,\bar{u},z)\le g(\bar{r},\bar{u},z)\le g(\bar{r},u,z)\quad \forall\, r\in A_t, u\in C_t, z\in\mathbb{R}^m.$$

One can also estimate $\bar{g}$:

$$\bar{g}(t,z)\le k(1+\|z\|^2)$$

and

$$\bar{g}(t,z)\ge k_2$$

for constants $k_1,k_2>0$. We choose

$$f(t,Z_t)=\bar{g}(t,Z_t).$$

Since this driver f satisfies condition (H1) in (Kob00), the BSDE has a solution (Y,Z). This solution is unique in $\mathcal{H}^\infty(\mathbb{R})\times\mathcal{H}^2(\mathbb{R}^m)$.The process $\int_0^\cdot Z_s dW_s$ is a P– BMO martingale. So we have the solution of the utility maximization problem:

Theorem 69 *Let (Y,Z) be the solution of the BSDE with driver $-\bar{g}$. Then there exists a couple of predictable processes $(\bar{\rho},\bar{\theta})$ satisfying $\bar{\rho}\in\mathcal{A}$, $\bar{\theta}\in\Theta$ and*

$$g(r,\bar{\theta}_t,Z_t)\le g(\bar{\rho}_t,\bar{\theta}_t,Z_t)\le g(\bar{\rho}_t,u,Z_t)\quad \forall\, r\in A_t, u\in C_t.$$

So $(\bar{\rho},\bar{\theta})$ is the solution of the robust utility maximization problem 60. The maximal robust utility is equal to

$$V(x)=(x^\gamma)\exp(Y_0),\qquad x>0.$$

The complete market

Again, in the case of a complete market, the BSDE simplifies. Since $A_t = \mathbb{R}^m$, for $u \in C_t$, $z \in \mathbb{R}^m$

$$r = \frac{1}{1-\gamma}(u+z)$$

attains $\sup_{r\in\mathbb{R}^m} g(r,u,z)$. Thus

$$\bar{g}(t,z) = \inf_{u\in C_t} \frac{\gamma}{2(1-\gamma)}(u+z)^2 + \frac{1}{2}z^2.$$

This infimum is attained by $\bar{u} = \Pi_{C_t}(-z)$. So $\bar{r} = \frac{1}{1-\gamma}(\Pi_{C_t}(-z)+z)$. The corresponding BSDE is

$$Y_t = 0 - \int_t^T Z_s dW_s + \int_t^T \frac{\gamma}{2(1-\gamma)}(\Pi_{C_t}(-Z_s)+Z_s)^2 - \frac{1}{2}Z_s^2 ds.$$

If the constraints on the drift (C_t) are deterministic, we obtain the same drift as Schied, (Sch04b) Proposition 3.2.

Remark 70 If the constraints (C_t) are deterministic, then

$$Z = 0, \quad \text{and} \quad Y_t = \int_0^t \frac{\gamma}{2(1-\gamma)}(\Pi_{C_t}(0)$$

is the solution of the BSDE (8.14). The solution of the robust utility maximization problem 68 is

$$\bar{\theta}_t = \Pi_{C_t}(0), \quad \bar{\rho}_t = \frac{1}{1-\gamma}\Pi_{C_t}(0).$$

So the worst case drift is the same as in Remark 67 for the exponential utility function.

8.3 Logarithmic utility

To complete the spectrum of important utility functions, we consider the utility function $U(x) = \log x$. We solve the robust utility maximization problem for an uncertain drift of the stock price process. As in the preceding section, the investor has no terminal liability. The notion of trading strategies is as in Chapter 7.3 for the power utility, $\tilde{\rho}_t^i$, $i = 1,\ldots,d$, $t \in [0,T]$, describes the amount of money invested in stock i at time t. In order to simplify the calculations, we write $\rho_t = \tilde{\rho}_t\sigma_t$, where σ is the volatility matrix of the stock price process defined in 7.1. For every $(\omega,t) \in \Omega\times[0,T]$, the trading strategy ρ_t takes values in a closed, convex set $A_t(\omega)$ that is defined in (7.5). The set of admissible trading strategies $\mathcal{A}_l$ is given as in Definition 59 with an additional mild integrability condition:

Definition 71 *The set of admissible trading strategies $\mathcal{A}_l$ consists of all $\mathbb{R}^d-$valued predictable processes ρ satisfying $E[\int_0^T |\rho_s|^2 ds] < \infty$ and $\rho_t \in A_t$ $P \otimes \lambda-a.s.$*

The set of drift processes considered in the robust utility maximization Θ is defined in (8.1). Recall from (7.24) that the wealth process $X^{(\rho,\theta)}$ for a trading strategy $\rho \in \mathcal{A}_l$ and drift $\theta \in \Theta$ for the stock price process is

$$X_T^{(\rho,\theta)} = x \exp\left(\int_0^T \rho_s dW_s + \int_0^T \left[\rho_s\theta_s - \frac{1}{2}\rho_s^2\right] ds\right).$$

So the log of the wealth process is

$$\log X_T^{(\rho,\theta)} = x + \int_0^T \rho_s dW_s + \int_0^T \left[\rho_s\theta_s - \frac{1}{2}\rho_s^2\right] ds.$$

The investor tries to solve the robust utility maximizing problem with a not exactly known drift:

Problem 72 *The solution of the robust utility maximization problem with the logarithmic utility function is given by a trading strategy $\bar{\rho} \in \mathcal{A}_l$ and a drift $\bar{\theta} \in \Theta$ attaining*

$$\begin{aligned} V(x) &= \sup_{\rho \in \tilde{A}} \inf_{\theta \in \Theta} E[\log X^{\rho,\theta}] \\ &= \sup_{\rho \in \tilde{A}} \inf_{\theta \in \Theta} E\left[x + \int_0^T \rho_s dW_s + \int_0^T \left[\rho_s\theta_s - \frac{1}{2}\rho_s^2\right] ds\right]. \end{aligned}$$

Observe that we can write

$$\log X_T^{(\rho,\theta)} = x + \int_0^T \rho_s dW_s + \int_0^T \left[-\frac{1}{2}(\rho_s - \theta_s)^2 + \frac{1}{2}\theta_s^2\right] ds.$$

The expected utility of the terminal wealth of a trading strategy $\rho \in \mathcal{A}_l$ and a stock price with drift $\theta \in \Theta$ is

$$E\left[\int_0^T \left(-\frac{1}{2}(\rho_s - \theta_s)^2 + \frac{1}{2}\theta_s^2\right) ds\right].$$

Distinguishing stochastic processes and elements of $\mathbb{R}^m$, we write r for ρ_t and u for θ. So we define

$$g(r,u) = ru - \frac{1}{2}r^2, \quad r \in A_t, u \in C_t.$$

In order to solve Problem 72, we have to find for every $(\omega, t) \in \Omega \times [0, T]$ a saddle point of the function g. This saddle points $(\bar{r}, \bar{u})(\omega, t)$ exists. According to Lemma 56, there exist predictable processes $(\bar{\rho}, \bar{\theta})$ that are for each (ω, t) equal to $(\bar{r}, \bar{u})$. Since the sets (C_t) are contained in a bounded ball around the origin, the process $\bar{\rho}$ is also uniformly bounded and admissible. The following theorem summarizes this result.

Theorem 73 *There exist two predictable processes $\bar{\rho}$ and $\bar{\theta}$, $\bar{\rho} \in \mathcal{A}_l$ and $\bar{\theta} \in \Theta$, such that for every $(\omega, t) \in \Omega \times [0, T]$, $(\bar{\rho}, \bar{\theta})$ is a saddle point of the function g with constraints A_t, C_t. This couple $(\bar{\rho}, \bar{\theta})$ of processes solves the saddle point problem 72.*

Now let the market be complete, i.e. the number of stocks is equal to the dimension of the Bromnian motion and we have no trading constraint. This means, the constraints on the trading strategies satisfy $A_t = \mathbb{R}^m$. Then we see that

$$\bar{g}(t) = \min_{u \in C_t} \frac{1}{2} u^2,$$

and the solution $(\bar{\rho}, \bar{\theta})$ of Problem 72 is

$$\bar{\theta}_t = \Pi_{C_t}(0), \quad \bar{\rho}_t = -\Pi_{C_t}(0).$$

This minimal drift corresponds to the result stated in Schied, (Sch04b), Proposition 3.2. In contrast to the robust utility maximization with uncertain drift for the exponential and the power utility we don't need the assumption of (Sch04b), Proposition 3.2 that the constraints (C_t) are deterministic. Here we summarize the results of the robust utility maximizing problems with uncertain drift for the different utility functions that we have considered.

Remark 74 Let the constraints for the drift (C_t) be deterministic. Then we have for the utility functions $U(x) = -\exp(-\alpha x)$, $\alpha > 0$, for $U(x) = x^\gamma$, $\gamma \in (0, 1)$ and for $U(x) = \log x$ that the drift in the solution of the robust utility maximization problem is

$$\bar{\theta}_t = \Pi_{C_t}(0), \quad t \in [0, T].$$

So we find that the drift that is used in the robust utility maximization for our utility functions is the same as the drift stated in Proposition 3.2 in Schied (Sch04b).

Chapter 9

Utility maximization with nonconvex constraints

This chapter is a summary of the results proven in Hu, Imkeller and Müller, (HIM04b). We consider the utility maximization problem with respect to one probability measure. An investor tries to find a trading strategy that maximizes the expected utility of his wealth at the end of a finite time interval $[0, T]$. In contrast to the previous chapter, he considers only a single probability measure that he sees as "objective" or "real world measure". He maximizes a concave functional, because he is risk averse. However, he does not take uncertainty into account since the parameters of the stock price process as well as the probability measure he uses are assumed to be known. We consider three types of utility functions: the exponential, the power and the logarithmic utility. In the case of an exponential utility, the investor may have a terminal liability: this is the obligation to pay out a random amount of money described by a bounded random variable F. Then the expected utility of the terminal wealth of the trading strategy minus the liability has to be maximized.

The model is placed in a Brownian filtration. So we may work with Backward Stochastic Differential Equations (BSDE). The maximal expected utility and an optimal trading strategy are obtained by the solution of a BSDE. The presence of only one probability measure simplifies our analysis. In order to construct our BSDE, we have to solve maximization problems instead of a saddle point problems. The assumptions on the constraints of the trading strategies can be relaxed, for every $(\omega, t) \in \Omega \times [0, T]$ the trading strategy is restricted to be in a closed set. The assumption that the set is convex can be dropped.

The method that we apply in order to obtain value function and optimal strategy is simple. We propose to construct a stochastic process R^ρ depending on the investor's trading strategy ρ, and such that its terminal value equals the utility of the trader's terminal wealth. As mentioned above, to model the constraint, trading strategies are supposed to take their values in a closed set. We don't assume that this set is convex. In our market, the absence of completeness is not explicitly described by a set of martingale measures equivalent to the historical probability. Instead, we choose R^ρ such that that for every trading strategy ρ, R^ρ is a supermartingale. Moreover, there exists at least one particular trading strategy ρ^* such that R^{ρ^*} is a martingale. Hereby, the initial value is supposed not to depend on the strategy. Evidently, the strategy ρ^* related to the martingale has to be the optimal one. Then the value function of the optimization problem is just given by the initial value of R^{ρ^*}. We obtain the particular control process ρ^* by the solution of a BSDE. Our direct approach does not use any duality, but characterizes directly the solution of our optimization problem.

As in Chapter 7, we consider three types of utility functions: the exponential, the power and the logarithmic utility.

This chapter is organized as follows: in section 9.1 we solve our utility maximzation problem for an exponential utility in presence of a terminal liability. We give in (9.9) the utility indifference price of F. In section 9.2 we consider the power utility functions and in section 9.3 the logarithmic utility. In both cases, the investor has no terminal liability.

9.1 The exponential utility

The exponential utility maximization allows the presence of a terminal liability that the investor wants to hedge. This liability is represented by a bounded $\mathcal{F}_T-$measurable random variable F:

$$F \in L^\infty(P, \mathcal{F}_T).$$

The stock price process is as in (3.2) with a drift θ described in (7.2). In this subsection, a trading strategy is a $d-$dimensional predictable process π that describes the amount of the currency invested in the stocks. The wealth process for a trading strategy π is

$$X_t^\pi = x + \sum_{i=1}^d \int_0^t \frac{\pi_{i,u}}{S_{i,u}} dS_{i,u} = x + \int_0^t \pi_u \sigma_u (dW_u + \theta_u du), \quad t \in [0,T].$$

The definition of admissibility formalises the constraints and guarantees that there is no arbitrage. Formally, they are supposed to take their values in a closed set, i.e. $\pi_t(\omega) \in \tilde{A}$, with $\tilde{A} \subseteq \mathbb{R}^{1\times d}$. We emphasize that $\tilde{A}$ is not assumed to be convex. We also impose a different integrability condition.

Definition 75 (Admissible Strategies with constraints) *Let $\tilde{A}$ be a closed set in $\mathbb{R}^{1\times d}$. The set of admissible trading strategies $\tilde{\mathcal{A}}$ consists of all d–dimensional predictable processes $\pi = (\pi_t)_{0\le t\le T}$ which satisfy $E[\int_0^T |\pi_t\sigma_t|^2 dt] < \infty$ and $\pi_t \in \tilde{A}$ $\lambda \otimes P$–a.s., as well as*

$$\{\exp(-\alpha X_\tau^\pi): \quad \tau \quad \text{stopping time with values in} \quad [0,T]\}$$

is a uniformly integrable family.

The investor faces the following optimization problem:

$$\sup_{\pi\in\tilde{\mathcal{A}}} E\left[-\exp(-\alpha(X_T^\pi - F))\right]. \tag{9.1}$$

This means, he tries to find the trading strategy that maximizes the sum of the terminal wealth of the trading strategy and the liability.

Remark 76 *We shall show below that the* sup *is taken by a particular strategy π^* which is admissible in the sense of our definition. Note that this process might not lead to a wealth process which is bounded from below, and therefore not admissible in this sense. For further details see Schachermayer (Sch04a) and Merton (Mer71).*

Remark 77 The condition of square integrability in Definition 75 guarantees that there is no arbitrage. In fact, the square integrability condition on π and the boundedness of θ yields that $E[\sup_{0\le t\le T}(X_t^\pi)^2] < \infty$. According to Theorem 2.1 in Pardoux, Peng (PP90), $(X_t, \pi_t\sigma_t)$ is the unique solution of the BSDE

$$X_t = X_T - \int_t^T (\pi_s\sigma_s)dW_s - \int_t^T (\pi_s\sigma_s)\theta_s ds,$$

with $E[\int_0^T (X_s^\pi)^2 ds] < \infty$, $E[\int_0^T (\pi_s\sigma_s)^2 ds] < \infty$. So the initial capital X_0^π needed to attain X_T^π is uniquely determined. In particular, Theorem 2.2 in El Karoui, Peng, Quenez (EKPQ97) yields if $X_0^\pi = 0$ and $X_T^\pi \ge 0$ P–a.s., then $X_T^\pi = 0$ P–a.s.

Remark 78 *In accordance with the classical literature (see Dellacherie, Meyer (DM75)) the uniform integrability condition in Definition 1 coincides with the notion of* class D.

Remark 79 *If X^π is square integrable and $\pi_t \in \tilde{A}$ $\lambda \otimes P$–a.s., as well as X^π is bounded from below on $[0,T]$, it is obvious that $\pi \in \tilde{\mathcal{A}}$.*

For $t \in [0,T], \omega \in \Omega$ define the set $A_t(\omega) \subseteq \mathbb{R}^m$ by

$$A_t(\omega) = \tilde{A}\sigma_t(\omega). \tag{9.2}$$

The entries of the matrix–valued process σ are uniformly bounded. Therefore we get

$$\min\{\ |a|\ :\ a \in A_t(\omega)\ \} \ \leq\ k_1 \qquad \text{for} \qquad \lambda \otimes P - \text{a.e. } (t,\omega) \tag{9.3}$$

with a constant $k_1 \geq 0$. Furthermore, for every (ω, t), the set $A_t(\omega)$ is closed. This is crucial for our analysis.

Remark 80 *Writing*

$$p_t = \pi_t \sigma_t, \qquad t \in [0,T],$$

the set of admissible trading strategies $\tilde{\mathcal{A}}$ is equivalent to a set $\mathcal{A}$ of $\mathbb{R}^{1\times m}$–valued predictable stochastic processes p with $p \in \mathcal{A}$ iff $E[\int_0^T |p(t)|^2 dt] < \infty$ and $p_t(\omega) \in A_t(\omega)$ P–a.s., as well as

$$\{\exp(-\alpha X^p_\tau) : \quad \tau \quad \textit{stopping time with values in} \quad [0,T]\}$$

is a uniformly integrable family.

Such a process $p \in \mathcal{A}$ will also be named strategy, and $X^{(p)}$ denotes its wealth process.

The maximization problem (9.1) is evidently equivalent to

$$V(x) = \sup_{p \in \mathcal{A}} E\left[-\exp\left(-\alpha\left(x + \int_0^T p_t(dW_t + \theta_t dt) - F\right)\right)\right]. \tag{9.4}$$

In order to find the value function and an optimal strategy we construct a family of stochastic processes $R^{(p)}$ with the following properties:

- $R^{(p)}_T = -\exp(-\alpha(X^p_T - F))$ for all $p \in \mathcal{A}$,
- $R^{(p)}_0 = R_0$ is constant for all $p \in \mathcal{A}$,
- $R^{(p)}$ is a supermartingale for all $p \in \mathcal{A}$ and there exists a $p^* \in \mathcal{A}$ such that $R^{(p*)}$ is a martingale.

The process $R^{(p)}$ and its initial value R_0 depend of course on the initial capital x. Given processes possessing these properties we can compare the expected utilities of the strategies $p \in \mathcal{A}$ and $p^* \in \mathcal{A}$ by

$$E[-\exp(-\alpha(X^p_T - F))] \leq R_0(x) = E[-\exp(-\alpha(X^{p*}_T - F))] = V(x), \tag{9.5}$$

whence p^* is the desired optimal strategy. To construct this family, we set

$$R_t^{(p)} := -\exp(-\alpha(X_t^{(p)} - Y_t)), \qquad t \in [0,T],\ p \in \mathcal{A},$$

where (Y, Z) is a solution of the BSDE

$$Y_t = F - \int_t^T Z_s dW_s - \int_t^T f(s, Z_s) ds, \quad t \in [0,T].$$

In these terms we are bound to choose a function f for which $R^{(p)}$ is a supermartingale for all $p \in \mathcal{A}$ and there exists a $p^* \in \mathcal{A}$ such that $R^{(p*)}$ is a martingale. This function f also depends on the constraint set (A_t) where (p_t) takes its values (see (9.2)). We get

$$V(x) = R_0^{(p,x)} = -\exp(-\alpha(x - Y_0)), \qquad \text{for all } p \in \mathcal{A}.$$

In order to calculate f, we write R as the product of a (local) martingale $M^{(p)}$ and a (not strictly) decreasing process $\tilde{A}^{(p)}$ that is constant for some $p^* \in \mathcal{A}$. For $t \in [0,T]$ define

$$M_t^{(p)} = \exp(-\alpha(x - Y_0)) \exp\left(-\int_0^t \alpha(p_s - Z_s) dW_s - \frac{1}{2}\int_0^t \alpha^2 (p_s - Z_s)^2 ds\right).$$

Comparing $R^{(p)}$ and $M^{(p)}\tilde{A}^{(p)}$ yields

$$\tilde{A}_t^{(p)} = -\exp\left(\int_0^t v(s, p_s, Z_s) ds\right), \quad t \in [0,T],$$

where v is defined as

$$v(t, p_t, z) \;=\; -\alpha p_t \theta_t + \alpha f(t,z) + \frac{1}{2}\alpha^2 |p_t - z|^2$$

for $t \in [0,T]$ and $z \in \mathbb{R}^m$. In order to obtain a decreasing process $\tilde{A}^{(p)}$ evidently f has to satisfy

$$v(t, p_t, Z_t) \geq 0 \qquad \text{for all } p \in \mathcal{A}$$

and

$$v(t, p_t^*, Z_t) = 0$$

for some particular $p^* \in \mathcal{A}$. For $t \in [0,T]$ we have

$$\begin{aligned}
\frac{1}{\alpha} v(t, p_t, Z_t) &= \frac{\alpha}{2}|p_t|^2 - \alpha p_t (Z_t + \frac{1}{\alpha}\theta_t) + \frac{\alpha}{2}|Z_t|^2 + f(t, Z_t) \\
&= \frac{\alpha}{2}|p_t - (Z_t + \frac{1}{\alpha}\theta_t)|^2 - \frac{\alpha}{2}|Z_t + \frac{1}{\alpha}\theta_t|^2 + \frac{\alpha}{2} Z_t^2 + f(t, Z_t) \\
&= \frac{\alpha}{2}|p_t - (Z_t + \frac{1}{\alpha}\theta_t)|^2 - Z_t \theta_t - \frac{1}{2\alpha}|\theta_t|^2 + f(t, Z_t).
\end{aligned}$$

Now set

$$f(t,z) = -\frac{\alpha}{2}\mathrm{dist}^2_{A_t}\left(z + \frac{1}{\alpha}\theta_t\right) + z\theta_t + \frac{1}{2\alpha}|\theta_t|^2.$$

For this choice we get $v(t,p_t,z) \geq 0$ and for

$$p_t^* \in \Pi_{A_t(\omega)}\left(Z_t + \frac{1}{\alpha}\theta_t\right), \quad t \in [0,T],$$

we obtain $v(\cdot, p_t^*, Z) = 0$.

Here we see why the set $\tilde{A}$ and hence A_t on which trading strategies are restricted is assumed to be closed. In order to find the value function we have to minimize the distance between a point and a set. Furthermore there must exist some element in A_t realizing the minimal distance. Both requirements are satisfied for closed sets. In a convex set the minimizer is unique. This would lead to a unique utility maximizing trading strategy. However, we prove existence of a possibly non–unique trading strategy solving the maximization problem *for closed but not necessarily convex constraints.*

Theorem 81 *The value function of the optimization problem (9.4) is given by*

$$V(x) = -\exp(-\alpha(x - Y_0)),$$

where Y_0 is defined by the unique solution $(Y,Z) \in \mathcal{H}^\infty(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^m)$ of the BSDE

$$Y_t = F - \int_t^T Z_s dW_s - \int_t^T f(s,Z_s)ds, \quad t \in [0,T], \tag{9.6}$$

with

$$f(\cdot,z) = -\frac{\alpha}{2}dist^2_{A_t}\left(z + \frac{1}{\alpha}\theta\right) + z\theta + \frac{1}{2\alpha}|\theta|^2.$$

There exists an optimal trading strategy $p^* \in \mathcal{A}$ with

$$p_t^* \in \Pi_{A_t(\omega)}(Z_t + \frac{1}{\alpha}\theta_t), \quad t \in [0,T],\ P - a.s. \tag{9.7}$$

The distance dist to and the projection Π on a closed subset of $\mathbb{R}^m$ are defined on page 12.

Proof. In order to get the existence of solutions of the BSDE (9.6) we apply Theorem 2.3 of (Kob00). As in Lemma 56, we see that $(f(t,z))_{t\in[0,T]}$ defines a predictable process for every $z \in \mathbb{R}^m$. A sufficient condition for the existence of a solution is condition (H1) in (Kob00): there are constants c_0, c_1 such that

$$|f(t,z)| \leq c_0 + c_1|z|^2 \qquad \text{for all} \quad z \in \mathbb{R}^n\ P - a.s. \tag{9.8}$$

By means of (7.6) we get for every $z \in \mathbb{R}^m, t \in [0,T]$

$$\mathrm{dist}^2_{A_t}\left(z + \frac{1}{\alpha}\theta_t\right) \le 2|z|^2 + 2(\frac{1}{\alpha}|\theta_t| + k_1)^2.$$

So (9.8) follows from the boundedness of θ. Theorem 2.3 in (Kob00) states that the BSDE (9.6) possesses at least one solution $(Y,Z) \in \mathcal{H}^\infty(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^m)$. The proof of uniqueness is similar to the proof of Lemma 54. To find the value function of our optimization problem, we proceed with the unique solution $(Y,Z) \in \mathcal{H}^\infty(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^m)$ of (9.6).

With the same approach as in Lemma 56 we can find a (maybe non unique) predictable process p^* that satisfies

$$p^*_t \in \Pi_{A_t}(Z_t + \frac{1}{\alpha}\theta_t)$$

This trading strategy p^* turns out to be optimal, because $\tilde{A}^{(p*)}_t(\omega) = -1$ for $\lambda \otimes P$ almost all (t,ω). Furthermore, $\int_0^\cdot (p^*_s - Z_s)dW_s$ is a P–BMO–martingale, thus $R^{(p^*)}$ is uniformly integrable (Theorem 2.3 in (Kaz94)). Since, moreover, Y is a bounded process, we obtain the uniform integrability of the family $\{\exp(-\alpha X^{(p^*)}_\tau) : \tau \text{ stopping time in } [0,T]\}$. Therefore $p^* \in \mathcal{A}$. Hence $R^{(p*)}$ is a martingale and

$$\begin{aligned} R^{(p*)}_0 &= E\left[-\exp\left(-\alpha\left(x + \int_0^T p^*_s(dW_s + \theta_s ds) - F\right)\right)\right] \\ &= -\exp(-\alpha(x - Y_0)). \end{aligned}$$

It remains to show that $R^{(p)}$ is a supermartingale for all $p \in \mathcal{A}$. Since $p \in \mathcal{A}$, the process $M = M_0\mathcal{E}(-\alpha\int(p_s - Z_s)dW_s)$ is a local martingale. Hence there exists a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ satisfying $\lim_{n\to\infty}\tau_n = T$ P–a.s. such that $(M_{t\wedge\tau_n})_t$ is a positive martingale for each $n \in \mathbb{N}$. The process $\tilde{A}^{(p)}$ is decreasing. Thus $R^{(p)}_{t\wedge\tau_n} = M_{t\wedge\tau_n}\tilde{A}^{(p)}_{t\wedge\tau_n}$ is a supermartingale, i.e. for $s \le t$

$$E[R^{(p)}_{t\wedge\tau_n}|\mathcal{F}_s] \le R^{(p)}_{s\wedge\tau_n}.$$

For any set $A \in \mathcal{F}_s$ we have

$$E[R^{(p)}_{t\wedge\tau_n} 1_A] \le E[R^{(p)}_{s\wedge\tau_n} 1_A].$$

Since $\{R^{(p)}_{t\wedge\tau_n}\}_n$ and $\{R^{(p)}_{s\wedge\tau_n}\}_n$ are uniformly integrable by the definition of admissibility and the boundedness of Y, we may let n tend to ∞ to obtain

$$E[R^{(p)}_t 1_A] \le E[R^{(p)}_s 1_A].$$

This implies the claimed supermartingale property of $R^{(p)}$.

□

Remark 82 *If the process* $\int_0^\cdot p_s dW_s$ *is a BMO martingale and* $E[\exp(-\alpha(X_T^{(p)} - F))] < \infty$, *a variant of an argument of the above proof can be used to see that* $p \in \mathcal{A}$. *In fact, we see that* $M^{(p)}$ *is a uniformly integrable martingale, while* $A^{(p)}$ *is decreasing. Hence* $R^{(p)}$ *is a supermartingale. This just states that for stopping times* τ

$$-\exp(-\alpha(X_\tau^{(p)} - Y_\tau)) \geq E[-\exp(-\alpha(X_T^{(p)} - F))|\mathcal{F}_\tau].$$

Consequently

$$\exp(-\alpha X_\tau^{(p)}) \leq \exp(-\alpha Y_\tau)\, E[\exp(-\alpha(X_T^{(p)} - F))|\mathcal{F}_\tau].$$

This clearly implies uniform integrability of $\{\exp(-\alpha X_\tau^{(p)}) : \tau \text{ stopping time in } [0,T]\}$.

The utility indifference price of F is the additional initial capital x_F that the investor needs to get the same maximal utility with the liability to pay out F than without this liability, see Remark 51. Let (Y^F, Z^F) denote the solution of (9.6) with terminal value F and (Y^0, Z^0) the solution of (9.6) with terminal value 0. The utility indifference price is

$$x_F = Y_0^F - Y_0^0. \tag{9.9}$$

Observe that the utility indifference price depends on the preferences.

We can show that the strategy p^* is optimal in a wider sense. In fact, an investor who has chosen at time 0 the strategy p^* will stick to this decision if he starts solving the optimization problem at some later time between 0 and T. For this purpose, let us formulate the optimization problem more generally for a stopping time $\tau \leq T$ and an $\mathcal{F}_\tau$–measurable random variable which describes the capital at time τ, i.e. $X_\tau = X_\tau^p$ for some $p \in \mathcal{A}$. So we consider the maximization problem

$$V(\tau, X_\tau) = \text{ess sup}_{p\in\mathcal{A}} E\left[-\exp\left(-\alpha\left(X_\tau + \int_\tau^T p_s(dW_s + \theta_s ds) - F\right)\right)\right]. \tag{9.10}$$

Proposition 83 (Dynamic Principle) *The value function* $V(x)$ *of the maximization problem (9.1) satisfies the dynamic programming principle, i.e.*

$$V(\tau, X_\tau) = -\exp(-\alpha(X_\tau - Y_\tau))$$

for all stopping times $\tau \leq T$ *where* Y_τ *belongs to a solution of the BSDE (9.6). An optimal strategy that attains the essential supremum in (9.10) is given by* p^*, *the optimal strategy constructed in Theorem 81.*

Proof. For $t \in [0,T]$, set

$$R_t = -\exp(-\alpha(X_t - Y_t))\mathcal{E}\left(-\int_t^T \alpha(p_s - Z_s)dW_s\right)\exp(\int_t^T v(s, p_s, Z_s)ds)$$

and apply the optional stopping theorem to the stochastic exponential. The claim follows as in Theorem 81.

□

Remark 84 *If the constraint $\tilde{A}$ on the strategies is a convex cone, the value function V and the optimal strategy p^* both constructed in Theorem 81 are equivalent to those determined in (Sek02) and (EKR00).*

Sekine considers the utility function $x \mapsto -\frac{1}{\alpha}\exp(-\alpha x)$. He obtains the value function

$$V(x) = -\frac{1}{\alpha}\exp(-\alpha x + \bar{Y}_0)$$

starting with the BSDE

$$\bar{Y}_t = \alpha F - \int_t^T \bar{z}_s dW_s - \int_t^T \bar{f}(s, \theta_s, \bar{z}_s)ds, \quad t \in [0,T],$$

where

$$\bar{f}(t, \theta_t, \bar{z}) = \theta_t \Pi_{A_t}(\bar{z} + \theta_t) - \frac{1}{2}|\bar{z} - \Pi_{A_t}(\bar{z} + \theta_t)|^2.$$

We evidently have to show that $\bar{Y}_t = \alpha Y_t$ for $t \in [0,T]$ or equivalently $\alpha f(t, \theta_t, \frac{z}{\alpha}) = \bar{f}(t, \theta_t, z)$. Note that for a convex set C, the projection $\Pi_C(a)$ is unique. If C is a convex cone and $\beta > 0$, then $\beta \Pi_C(a) = \Pi_C(\beta a)$. The equality for the functions f and $\bar{f}$ therefore follows. El Karoui and Rouge (EKR00) have obtained the same BSDE and value function before Sekine.

9.2 Power utility

In this section we calculate the value function and characterize the optimal strategy for the utility maximization problem with respect to

$$U_\gamma(x) = \frac{1}{\gamma}x^\gamma, \qquad x \geq 0, \quad \gamma \in (0,1).$$

This time, our investor maximizes the expected utility of his wealth at time T without an additional liability. The trading strategies are constrained to take values in a closed set $\tilde{A} \subseteq \mathbb{R}^d$. In this section, we shall use a somewhat different notion of trading strategy: $\tilde{\rho} = (\tilde{\rho}^i)_{i=1,\ldots,d}$ denotes the part of the wealth invested in stock i. The number of shares of stock i is given by $\frac{\tilde{\rho}^i_t X_t}{S^i_t}$. A d–dimensional $\mathbb{F}$–predictable process $\tilde{\rho} = (\tilde{\rho}_t)_{0\le t\le T}$ is called trading strategy (part of wealth) if the following wealth process is well defined:

$$X_t^{(\tilde{\rho})} = x + \int_0^t \sum_{i=1}^d \frac{X_s^{(\tilde{\rho})}\tilde{\rho}_{i,s}}{S_{i,s}} dS_{i,s} = x + \int_0^t X_s^{(\tilde{\rho})}\tilde{\rho}_s\sigma_s(dW_s + \theta_s ds), \qquad (9.11)$$

and the initial capital x is positive. The wealth process $X^{(\tilde{\rho})}$ can be written as:

$$X_t^{(\tilde{\rho})} = x\mathcal{E}\left(\int \tilde{\rho}_s\sigma_s(dW_s + \theta_s ds)\right)_t, \quad t \in [0,T].$$

It is more convenient to introduce

$$\rho_t = \tilde{\rho}_t\sigma_t, \qquad t \in [0,T].$$

Accordingly, ρ is constrained to take its values in

$$A_t(\omega) = \tilde{A}\sigma_t(\omega) \quad t \in [0,T], \omega \in \Omega.$$

The sets A_t satisfy (7.6). In order to formulate the optimization problem we first define the set of admissible trading strategies.

Definition 85 *The set of admissible trading strategies $\tilde{\mathcal{A}}$ consists of all d–dimensional predictable processes $\rho = (\rho_t)_{0\le t\le T}$ that satisfy $\rho_t \in A_t(\omega)$ $P\otimes\lambda$–a.s and $\int_0^T |\rho_s|^2 ds < \infty$ P–a.s.*

Define the probability measure $Q \sim P$ by

$$\frac{dQ}{dP} = \mathcal{E}\left(-\int \theta_s dW_s\right)_T.$$

The set of admissible trading strategies is free of arbitrage because for every $\rho \in \tilde{\mathcal{A}}$, the wealth process $X^{(\rho)}$ is a local Q–martingale bounded from below, hence a Q–supermartingale. Since Q is equivalent to P, the set of admissible trading strategies $\tilde{\mathcal{A}}$ is free of arbitrage.

The investor faces the maximization problem

$$\bar{V}(x) = \sup_{\tilde{\rho}\in\tilde{\mathcal{A}}} E\left[U\left(X_T^{(\tilde{\rho})}\right)\right]. \qquad (9.12)$$

In order to find the value function and an optimal strategy we apply the same method as for the exponential utility function. We therefore have to construct a stochastic process $\tilde{R}^{(\rho)}$ with terminal value

$$\tilde{R}_T^{(\rho)} = U\left(x + \int_0^T X_s\rho_s\frac{dS_s}{S_s}\right),$$

and an initial value $\tilde{R}_0^{(\rho)} = \tilde{R}_0^x$ that does not depend on ρ, $\tilde{R}^{(\rho)}$ is a supermartingale for all $\rho \in \tilde{\mathcal{A}}$ and a martingale for a $\rho^* \in \tilde{\mathcal{A}}$. Then ρ^* is the optimal strategy and the value function given by $\bar{V}(x) = \tilde{R}_0^x$. Applying the utility function to the wealth process yields

$$(X_t^{\rho,x})^\gamma = x^\gamma \exp\left(\int_0^t \gamma\rho_s dW_s + \int_0^t \gamma\rho_s\theta_s ds - \frac{1}{2}\int_0^t \gamma|\rho_s|^2 ds\right), \quad t \in [0,T].$$

This equation suggests the following choice:

$$\tilde{R}_t^{(\rho)} = x^\gamma \exp\left(\int_0^t \gamma\rho_s dW_s + \int_0^t \gamma\rho_s\theta_s ds - \frac{1}{2}\int_0^t \gamma|\rho_s|^2 ds + Y_t\right), \tag{9.13}$$

where (Y,Z) is a solution of the BSDE

$$Y_t = 0 - \int_t^T Z_s dW_s - \int_t^T f(s,Z_s)ds, \quad t \in [0,T].$$

In order to get the supermartingale property of $\tilde{R}^{(\rho)}$ we have to construct $f(t,z)$ such that for $t \in [0,T]$

$$\gamma\rho_t\theta_t - \frac{1}{2}\gamma|\rho_t|^2 + f(t,Z_t) \le -\frac{1}{2}|\gamma\rho_t + Z_t|^2 \quad \text{for all}\, \rho \in \tilde{\mathcal{A}}. \tag{9.14}$$

$\tilde{R}^{(\rho^*)}$ will even be a martingale if equality holds for $\rho^* \in \tilde{\mathcal{A}}$. This is equivalent to

$$f(t,Z_t) \le \frac{1}{2}\gamma(1-\gamma)\left|\rho_t - \frac{1}{1-\gamma}(Z_t+\theta_t)\right|^2 - \frac{1}{2}\frac{\gamma|Z_t+\theta_t|^2}{1-\gamma} - \frac{1}{2}|Z_t|^2.$$

Hence the appropriate choice for f is

$$f(t,z) = \frac{\gamma(1-\gamma)}{2}\text{dist}_{A_t}^2\left(\frac{1}{1-\gamma}(z+\theta_t)\right) - \frac{\gamma|z+\theta_t|^2}{2(1-\gamma)} - \frac{1}{2}|z|^2,$$

and a candidate for the optimal strategy must satisfy

$$\rho_t^* \in \Pi_{A_t(\omega)}\left(\frac{1}{1-\gamma}(Z_t+\theta_t)\right), \quad t \in [0,T].$$

In the following Theorem both value function and optimal strategy are described.

Theorem 86 *The value function of the optimization problem is given by*

$$V(x) = x^\gamma \exp(Y_0) \qquad \textit{for} \quad x > 0,$$

where Y_0 *is defined by the unique solution* $(Y, Z) \in \mathcal{H}^\infty(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^m)$ *of the BSDE*

$$Y_t = 0 - \int_t^T Z_s dW_s - \int_t^T f(s, Z_s) ds, \quad t \in [0, T], \tag{9.15}$$

with

$$f(t, z) = \frac{\gamma(1-\gamma)}{2} dist^2_{A_t}\left(\frac{1}{1-\gamma}(z + \theta_t)\right) - \frac{\gamma|z + \theta_t|^2}{2(1-\gamma)} - \frac{1}{2}|z|^2.$$

There exists an optimal trading strategy $\rho^* \in \tilde{\mathcal{A}}$ *with the property*

$$\rho^*_t \in \Pi_{A_t(\omega)}\left(\frac{1}{1-\gamma}(Z_t + \theta_t)\right). \tag{9.16}$$

Proof. As in Lemma 56, $(f(t,z))_{t\in[0,T]}$ is a predictable stochastic process which also depends on σ. Due to (7.6) and the boundedness of θ, Condition (H1) for Theorem 2.3 in (Kob00) is fulfilled. We obtain the existence of a solution $(Y, Z) \in \mathcal{H}^\infty(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^m)$ for the BSDE (9.15). Uniqueness follows from the comparison arguments in the uniqueness part of the proof of Theorem 81.

Let ρ^* denote the predictable process satisfying

$$\rho^*_t \in \Pi_{A_t}(\frac{1}{1-\gamma} Z_t + \theta_t).$$

(see Lemma 56) Lemma 89 below shows that $\rho^* \in \tilde{\mathcal{A}}$. By Theorem 2.3 in (Kaz94), the process $\tilde{R}^{(\rho*)}$ is a martingale with terminal value

$$\tilde{R}^{(\rho*)}_T = x^\gamma \exp\left(\int_0^T \gamma\rho^*_s dW_s + \int_0^T \gamma\rho^*_s\theta_s ds - \frac{1}{2}\int_0^T \gamma|\rho^*_s|^2 ds\right).$$

This is the power utility from terminal wealth of the trading strategy ρ^*. Therefore the expected utility of ρ^* is equal to $\tilde{R}^{(\rho*,x)}_0 = x^\gamma \exp(Y_0)$.

To show that this provides the value function let $\rho \in \tilde{\mathcal{A}}$. (9.14) yields

$$\tilde{R}^{(\rho)}_t = x^\gamma \exp(Y_0)\mathcal{E}\left(\int(\gamma\rho_s + Z_s)dW_s\right)_t \exp\left(\int_0^t v_s ds\right), \quad t \in [0, T],$$

for a process v with $v_s \leq 0$ $\lambda \otimes P$ a.s.

The stochastic exponential is a local martingale. There exists a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$, $\lim_{n\to\infty}\tau_n = T$ such that

$$E[\tilde{R}^{(\rho)}_{t\wedge\tau_n}|\mathcal{F}_s] \le \tilde{R}^{(\rho)}_{s\wedge\tau_n}, \qquad s \le t$$

for every $n \in \mathbb{N}$. Furthermore, $\tilde{R}^{(\rho)}$ is bounded from below by 0. Passing to the limit and applying Fatou's lemma yields that $\tilde{R}^{(\rho)}$ is a supermartingale. The terminal value $\tilde{R}^{(\rho)}_T$ is the utility of the terminal wealth of the trading strategy ρ. Consequently

$$E[U(X^{(\rho,x)}_T)] \le \tilde{R}^{(x)}_0 = x^\gamma \exp(Y_0) \quad \text{for all } \rho \in \tilde{\mathcal{A}}.$$

□

Again we can show that an investor starting to act at some stopping time in the trading interval $[0,T]$ will perceive the strategy ρ^* just constructed as optimal. Let $\tau \le T$ denote a stopping time and X_τ an $\mathcal{F}_\tau$–measurable random variable which describes the capital at time τ, i.e. $X_\tau = X^\rho_\tau$ for a $\rho \in \tilde{\mathcal{A}}$ and an initial capital $x > 0$. Consider the maximization problem

$$\bar{V}(\tau, X_\tau) = \text{ess sup}_{\rho\in\mathcal{A}_\tau} E\left[U\left(X_\tau + \int_\tau^T X_s\rho_s(dW_s + \theta_s ds)\right)\right]. \qquad (9.17)$$

Proposition 87 (Dynamic Principle) *The value function $x^\gamma \exp(y)$ satisfies the dynamic programming principle, i.e.*

$$\bar{V}(\tau, X_\tau) = (X_\tau)^\gamma \exp(Y_\tau)$$

for all stopping times $\tau \le T$, where Y_τ is given by the unique solution (Y, Z) of the BSDE (9.15). An optimal strategy which attains the essential supremum in (9.17) is given by ρ^* constructed in Theorem 86.

Proof. See Proposition 83.

Remark 88 *Suppose that the constraint set $\tilde{A}$ is a convex cone. Then the optimal strategy ρ^* constructed in Theorem 86 is the same as in (Sek02).*

Sekine uses the utility function $x \mapsto \frac{1}{\gamma}x^\gamma$ and obtains the value function

$$\tilde{V}(x) = \frac{1}{\gamma}x^\gamma \exp((1-\gamma)\tilde{Y}_0),$$

where $\tilde{Y}_0$ is defined by the unique solution $(\tilde{Y}, \tilde{Z}) \in \mathcal{H}^\infty(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^m)$ of the BSDE

$$\tilde{Y}_t = 0 - \int_t^T \tilde{Z}_s dW_s - \int_t^T g(s, \tilde{Z}_s) ds, \qquad t \in [0,T].$$

Here

$$g(t,\tilde{z}) = \frac{|\theta_t|^2}{2} - \frac{1}{2}\left|\theta_t - \Pi_{A_t}\left(\tilde{z} + \frac{\theta_t}{1-\gamma}\right)\right|^2 - \frac{1-\gamma}{2}\left|\tilde{z} - \Pi_{A_t}\left(\tilde{z} + \frac{\theta_t}{1-\gamma}\right)\right|^2.$$

As for the exponential utility function we have to show $(1-\gamma)\tilde{Y} = Y$ or equivalently $(1-\gamma)g(t, \frac{z}{1-\gamma}) = f(t,z)$. In fact, we have

$$\begin{aligned}
(1-\gamma)g\left(t, \frac{z}{1-\gamma}\right) &= (1-\gamma)\left[\frac{|\theta_t|^2}{2} - \frac{1}{2}\left|\theta_t - \Pi_{A_t}\left(\frac{z+\theta_t}{1-\gamma}\right)\right|^2\right] \\
&\quad - \frac{(1-\gamma)^2}{2}\left|\frac{z}{1-\gamma} - \Pi_{A_t}\left(\frac{z+\theta_t}{1-\gamma}\right)\right|^2 \\
&= \theta_t \Pi_{A_t}(z+\theta_t) - \frac{1}{2(1-\gamma)}|\Pi_{A_t}(z+\theta_t)|^2 \\
&\quad - \frac{1}{2}|z|^2 + z\Pi_{A_t}(z+\theta_t) - \frac{1}{2}|\Pi_{A_t}(z+\theta_t)|^2 \\
&= (z+\theta_t)\Pi_{A_t}(z+\theta_t) - \frac{2-\gamma}{2(1-\gamma)}|\Pi_{A_t}(z+\theta_t)|^2 - \frac{1}{2}|z|^2 \\
&= -\frac{\gamma}{2(1-\gamma)}|\Pi_{A_t}(z+\theta_t)|^2 - \frac{1}{2}|z|^2.
\end{aligned}$$

To obtain the last equality, we use

$$(z+\theta_t)\Pi_{A_t}(z+\theta_t) = |\Pi_{A_t}(z+\theta_t)|^2$$

(see (9.18) below).

For the function f we obtain

$$\begin{aligned}
f(t,z) &= \frac{\gamma(1-\gamma)}{2}\left|\frac{1}{1-\gamma}(z+\theta_t) - \Pi_{A_t}\left(\frac{1}{1-\gamma}(z+\theta_t)\right)\right|^2 \\
&\quad - \frac{\gamma}{2}\frac{(z+\theta_t)^2}{(1-\gamma)} - \frac{1}{2}|z|^2 \\
&= -\frac{\gamma}{1-\gamma}(z+\theta_t)\Pi_{A_t}(z+\theta_t) + \frac{\gamma}{2(1-\gamma)}|\Pi_{A_t}(z+\theta_t)|^2 - \frac{1}{2}|z|^2 \\
&= -\frac{\gamma}{2(1-\gamma)}|\Pi_{A_t}(z+\theta_t)|^2 - \frac{1}{2}|z|^2.
\end{aligned}$$

For $t \in [0,T], z \in \mathbb{R}^m$ we therefore have

$$(1-\gamma)g(t, \frac{z}{1-\gamma}) = f(t,z).$$

It remains to prove that for a convex cone C and $a \in \mathbb{R}^m$ the following equality holds:

$$\Pi_C(a)(a - \Pi_C(a)) = 0. \tag{9.18}$$

If $\Pi_C(a) = 0$ then the identity is satisfied. If not, consider the half line $\lambda\Pi_C(a)$, $\lambda \geq 0$. This half line is part of the cone C, so $\Pi_C(a)$ is also the projection of a on the half line.

□

Lemma 89 *Let $(Y,Z) \in \mathcal{H}^\infty(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^m)$ be a solution of the BSDE (9.15), and let ρ^* be given by (9.16). Then the processes*

$$\int_0^\cdot Z_s dW_s, \quad \int_0^\cdot \rho_s^* dW_s$$

are P–BMO martingales.

Proof. We may take a lower bound k for Y, and apply Itô's formula to $|Y-k|^2$, to conclude in the same manner as before.

□

9.3 Logarithmic Utility

To complete the spectrum of important utility functions, in this section we shall consider logarithmic utility. As in the preceding section, the agent has no liability at time T. Trading strategies and wealth process have the same meaning as in section 9.2 (see (9.11)). The trading strategies $\tilde{\rho}$ are constrained to take values in a closed set $\tilde{A} \subset \mathbb{R}^d$. For $\rho_t = \tilde{\rho}_t \sigma_t$ the constraints are described by $A_t = \tilde{A}\sigma_t, t \in [0,T]$. In order to compare the logarithmic utility of the terminal wealth of two trading strategies we have to impose a mild integrability condition on ρ. Recall that $\rho^i > 1$ means that the investor has to borrow money in order to buy stock i and if $\rho^i < 0$ then the investor has a negative number of stock i. In order to find the maximal utility, we impose an integrability condition on ρ that is not restrictive.

Definition 90 *The set of admissible trading strategies $\mathcal{A}_l$ consists of all $\mathbb{R}^d$–valued predictable processes ρ satisfying $E[\int_0^T |\rho_s|^2 ds] < \infty$ and $\rho_t \in A_t$ $P \otimes \lambda$–a.s.*

For the logarithmic utility function

$$U(x) = \log(x), \quad x > 0,$$

we obtain a particularly simple BSDE that leads to the value function and the optimal strategy. The optimization problem is given by

$$\begin{aligned} V(x) &= \sup_{\rho\in\mathcal{A}_l} E[\log(X_T^{(\rho)})] && (9.19)\\ &= \log(x) + \sup_{\rho\in\mathcal{A}_l} E\left[\int_0^T \rho_s dW_s + \int_0^T (\rho_s\theta_s - \frac{1}{2}|\rho_s|^2)ds\right], && (9.20)\end{aligned}$$

where the initial capital x is positive again. We aim to determine a process $R^{(\rho)}$ with $R_T^{(\rho)} = \log(X_T^{(\rho)})$, and an initial value that does not depend on ρ. Furthermore, $R^{(\rho)}$ is a supermartingale for all $\rho \in \mathcal{A}_l$, and there exists a $\rho^* \in \mathcal{A}_l$ such that $R^{(\rho*)}$ is a martingale. The strategy ρ^* is the optimal strategy and $R_0^{\rho^*}$ is the value function of the optimization problem (9.19).

We can choose for $t \in [0, T]$

$$R_t^{(\rho)} = \log x + Y_0 + \int_0^t (\rho_s + Z_s)dW_s + \int_0^t \left(-\frac{1}{2}|\rho_s - \theta_s|^2 + \frac{1}{2}\theta_s^2 + f(s)\right) ds,$$

where

$$f(t) = \frac{1}{2}\text{dist}^2_{A_t}(\theta_t) - \frac{1}{2}|\theta_t|^2, \qquad t \in [0, T],$$

and (Y_t, Z_t) is the unique solution of the following BSDE:

$$Y_t = 0 - \int_t^T Z_s dW_s - \int_t^T f(s)ds, \qquad t \in [0, T].$$

Due to definition 90, the boundedness of θ and (7.6), the stochastic integral in $R^{(\rho)}$ is a martingale for all $\rho \in \mathcal{A}_l$. Hence $R^{(\rho)}$ is a supermartingale for all $\rho \in \mathcal{A}_l$. An optimal trading strategy ρ^* which satisfies

$$\rho_t^* \in \Pi_{A_t}(\theta_t), \quad P \otimes \lambda \ a.e. \ (\omega, t) \tag{9.21}$$

can be constructed by means of Lemma 56. The initial value Y_0 satisfies

$$Y_0 = -E\left[\int_0^T f(s)ds\right].$$

We summarize our results in a theorem:

Theorem 91 *There exists a trading strategy $\rho^* \in \mathcal{A}$ attaining the supremum in 9.19. This trading strategy is stated in (9.21). The maximal utility for the initial capital $x > 0$ is*

$$V(x) = R_0^{\rho^*}(x) = \log(x) + E\left[-\int_0^T f(s)ds\right].$$

In particular ρ_t^* only depends on θ_t, σ_t and the set A_t describing the constraints on the trading strategies, i.e. the value that those processes take at time t.

Appendix A

BMO martingales

Here we recall and collect a few well known facts from the theory of martingales of *bounded mean oscillation*, briefly called BMO–martingales. We follow the exposition in (Kaz94). The statements will be made for infinite time horizon. In the text they will be applied to the simpler framework of finite horizon, replacing ∞ with T.

Definition 92 *Let $M = (M_t)_{t\geq 0}$ be a uniformly integrable martingale with respect to a probability measure P and a complete, right continuous filtration $\mathbb{F}$ satisfying $M_0 = 0$. For $1 \leq p < \infty$ set*

$$\|M\|_{BMO_p} := \sup_{\tau\ \mathbb{F}-stopping\ time} E[|M_\infty - M_\tau|^p|\mathcal{F}_\tau]^{1/p}. \tag{A.1}$$

The normed linear space $\{M : \|M\|_{BMO_p} < \infty\}$ with norm $\|M\|_{BMO_p}$ (taken with respect to P) is denoted by BMO_p (Kazamaki (Kaz94), p. 25).

By Corollary 2.1 in (Kaz94), p. 28, we have for all $1 \leq p < \infty$

$$M \in \mathrm{BMO}_1 \quad \text{iff} \quad M \in \mathrm{BMO}_p.$$

$BMO(P)$ denotes all uniformly integrable P–martingales such that $\|M\|_{\mathrm{BMO}_1} < \infty$. The norm in $BMO_2(P)$ can be alternatively expressed as

$$\|M\|_{\mathrm{BMO}_2} = \sup_{\tau\ \mathbb{F}-\text{stopping time}} E[\langle M\rangle_\infty - \langle M\rangle_\tau|\mathcal{F}_\tau]^{1/2}. \tag{A.2}$$

The combined inequalities of Doob and Burkholder–Davis–Gundy read for $p > 1$

$$(\frac{p}{p-1})^p E[|M_\infty|^p] \geq E[\sup_{0\leq t\leq\infty} |M_t|^p] \geq c_p\, E[\langle M\rangle_\infty^{p/2}] \tag{A.3}$$

with a universal positive constant c_p. Therefore for any BMO–martingale M we obtain $\langle M\rangle_t \in L^p(P)$ for all $p > 1, t \in [0, \infty]$.

BMO-martingales possess the convenient property of generating uniformly integrable exponentials according to the following Theorem.

Theorem 93 *(Theorem 2.3 (Kaz94)) If $M \in BMO$, then $\mathcal{E}(M)$ is a uniformly integrable martingale.*

According to the following Theorem, the BMO property is preserved by equivalent changes of measure. In fact, let $M \in BMO(P)$ and $\hat{P}$ given by the measure change $d\hat{P} = \mathcal{E}(M)_\infty dP$. Define $\phi : X \mapsto \hat{X} = \langle X, M\rangle - X$.

Theorem 94 *(Theorem 3.6 (Kaz94)) If $M \in BMO(P)$, then $\phi : X \mapsto \hat{X}$ is an isomorphism of $BMO(P)$ onto $BMO(\hat{P})$.*

Bibliography

[Ame99] Amendinger, J.: *Initial Enlargement of Filtrations and Additional Information in Financial Markets*. Ph.D. thesis, Technische Universität Berlin, 1999.

[Bar02] Barrieu, P.: *Structuration Optimale de Produits Financiers en Marché Illiquide et Trois Excursions dans d'autres Domaines des Probabilités*. Ph.D. thesis, Université Paris VI, 2002.

[Bec01] Becherer, D.: *Rational Hedging and Valuation with Utility-based Preferences*. Ph.D. thesis, Technische Universität Berlin, 2001.

[Ben70] Beneš, V. E.: Existence of Optimal Strategies Based on Specified Information, for a Class of Stochastic Decision Problems. In: *SIAM J. Control*, volume 8(2):pp. 179–188, 1970.

[Bis76] Bismut, J.-M.: Théorie probabiliste du contrôle des diffusions. In: *Mem. Amer. Math. Soc.*, 1976.

[CFP00] Cox, S.; Fairchild, J.; Pedersen, H.: Economic aspects of securitization of risk. In: *ASTIN Bull.*, volume 1:pp. 157–193, 2000.

[CH89] Cox, J.; Huang, C.F.: Optimal consumption and portfolio policies when asset prices follow a diffusion process. In: *J. Econom. Theory*, volume 49:pp. 33–83, 1989.

[CIM04] Chaumont, S.; Imkeller, P.; Müller, M.: Equilibrium trading of climate and weather risk and numerical simulation in a Markovian framework. Preprint, HU Berlin, 2004.

[CK92] Cvitanic, J.; Karatzas, I.: Convex duality in constrained portfolio optimization. In: *Ann. Appl. Probab.*, volume 2, 1992.

[CSW01] Cvitanic, J.; Schachermayer, W.; Wang, H.: Utility maximization in incomplete markets with random endowment. In: *Finance Stoch.*, volume 5, 2001.

[Dav01] Davis, M.: Pricing weather derivatives by marginal value. In: *Quant. Finance*, volume 1:pp. 305–308, 2001.

[Del03] Delbaen, F.: The structure of $m-$ stable sts and in particular of the set of risk neutral measures. Preprint, ETH Zürich, 2003.

[DGR+02] Delbaen, F.; Grandits, P.; Rheinländer, T.; Samperi, D.; Schweizer, M.; Stricker, C.: Exponential hedging and entropic penalties. In: *Math. Finance*, volume 12:pp. 99–123, 2002.

[DM75] Dellacherie, C.; Meyer, P.-A.: *Probabilités et potentiel. Chapitre I à IV.* Hermann, 1975.

[DR95] Duffie, D.; Rahi, R.: Financial market innovation and security design: an introduction. In: *Journ. Econ. Theor.*, volume 65:pp. 1–42, 1995.

[EKH03] El Karoui, N.; Hamadène, S.: BSDEs and risk-sensitive control, zero-sum and nonzero-sum game problems of stochastic functional differential equations. In: *Stoch. Proc. Appl.*, volume 107:pp. 145–169, 2003.

[EKPQ97] El Karoui, N.; Peng, S.; Quenez, M. C.: Backward stochastic differential equations in finance. In: *Math. Finance*, pp. 1–71, 1997.

[EKR00] El Karoui, N.; Rouge, R.: Pricing via utility maximization and entropy. In: *Math. Finance*, volume 10:pp. 259–276, 2000.

[FS02] Föllmer, H.; Schied, A.: *Stochastic Finance. An Introduction in Discrete Time.* de Gruyter, 2002.

[Gun03] Gundel, A.: Robust Utility Maximization for Complete and Incomplete Market Models. Preprint, HU Berlin, 2003.

[HIM04a] Hu, Y.; Imkeller, P.; Müller, M.: Partial equilibrium and market completion. In: *Int. Journ. Theor. Appl. Fin.*, 2004. To appear.

[HIM04b] Hu, Y.; Imkeller, P.; M"uller, M.: Utility maximization in incomplete markets. In: *Ann. Appl. Prob.*, 2004. To appear.

[Kal97] Kallenberg, O.: *Foundation of Modern Probability.* Springer, 1997.

[Kar97] Karatzas, I.: *Lectures on the Mathematics of Finance*, volume 8 of *CRM Monograph Series.* American Mathematical Society, 1997.

[Kaz94] Kazamaki, N.: *Continuous Exponential Martingales and BMO*, volume 1579 of *Lecture Notes in Mathematics.* Springer, 1994.

[KL87] Karatzas, I.; Lehoczky, S.E., J.P.and Shreve: Optimal portfolio and consumption decisions for a "small investor" on a finite horizon. In: *SIAM J. Control Optim.*, volume 25:pp. 1157–1586, 1987.

[KL90] Karatzas, I.; Lehoczky, S.E., J.P.and Shreve: Existence and uniqueness of multi–agent equilibrium in a stochastic, dynamic consumption/investment model. In: *Math. Oper. Res.*, volume 15:pp. 80–128, 1990.

[KO91] Karatzas, I.; Ocone, D.: A generalized Clark representation formula, with application to optimal portfolios. In: *Stochastics and Stochastics Reports*, volume 34:pp. 187–220, 1991.

[Kob00] Kobylanski, M.: Backward stochastic differential equations and partial differential equations with quadratic growth. In: *Ann. Probab.*, volume 28:pp. 558–602, 2000.

[KS99] Kramkov, D.; Schachermayer, W.: The asymptotic elasticity of utility functions and optimal investment in incomplete markets. In: *Ann. Appl. Probab.*, volume 9:pp. 904–950, 1999.

[Mer71] Merton, R. C.: Optimum consumption and portfolio rules in a continuous-time model. In: *Journal of Economic Theory*, volume 3:pp. 373–413, 1971.

[Møl01] Møller, T.: Risk-minimizing hedging strategies for insurance payment processes. In: *Finance and Stochastics*, volume 5:pp. 419–446, 2001.

[MZ01] Musiela, M.; Zariphopoulou, T.: Musiela, m. and zariphopoulou, t. Preprint, Univ. Texas, 2001.

[Nua95] Nualart, D.: *The Malliavin Calculus and Related Topics.* Springer, 1995.

[Pen90] Peng, S.: 'A gemeral stochastic maximum principle for optional control problems. In: *SIAM J. Contr. Opt.*, volume 28(4):pp. 966–979, 1990.

[Pli86] Pliska, S: A stochastic calculus model of continuous trading: optimal portfolio. In: *Math. Oper. Res.*, volume 11:pp. 370–382, 1986.

[PP90] Pardoux, E.; Peng, S.G.: Adapted solution of a backward stochastic differential equation. In: *Systems Control Lett.*, volume 14:pp. 55–61, 1990.

[Que04] Quenez, M. C.: Optimal portfolio in a multiple-proirs model. In: *Seminar on Stochastic Analysis, Random Fields and Applications IV*, volume 58 of *Progr. Probab.*, pp. 291–321. Birkhäuser, Basel, 2004.

[Roc70] Rockafellar, R. T.: *Convex Analysis*. Princeton Landmarks in Mathematics. Princeton, 1970.

[RY91] Revuz, D.; Yor, M.: *Continuous Martingales and Brownian Motion*. Springer, 1991.

[Sav54] Savage, L. J.: *The foundation of statistics*. Wiley Publ. Stat., New York, 1954.

[Sch99] Schmock, U.: Estimating the Value of the WinCAT Coupons of the Winterthur Insurance Convertible Bond: A Study of the Model Risk. In: *ASTIN Bull.*, volume 29(1):pp. 101–163, 1999.

[Sch01] Schweizer, M.: A guided tour through quadratic hedging approaches. In: Jouini, E.; Cvitanic, J.; Musiela, M., editors, *Option Pricing, Interest Rates and Risk Management*, pp. 538–574. Cambridge University Press: Cambridge, 2001.

[Sch02] Schachermayer, W.: Optimal investment in incomplete financial markets. In: Geman, H.; Madan, D.; Pliska, St. R.; Vorst, T., editors, *Mathematical Finance Bachelier Congress 2000*, pp. 427–462. Springer, Berlin, 2002.

[Sch04a] Schachermayer, W.: Utility optimisation in incomplete markets. Preprint, Vienna University of Technology, 2004.

[Sch04b] Schied, A.: Optimal Investments for Robust Utility Functionals in Complete Market Models. In: *Math. Oper. Res.*, 2004. To appear.

[Sek02] Sekine, J.: Exponential hedging by solving a backward stochastic differential equation: an alternative approach. Preprint, Osaka University, 2002.

Selbständigkeitserklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbst"andig ohne unerlaubte Hilfe verfaßt und nur die angegebene Literatur und Hilfsmittel verwendet habe.

Kapitel 2 und 3 beruhen auf der Arbeit "Partial equilibrium and market completion" mit Ying Hu und Peter Imkeller. In Kapitel 9 stehen die Ergebnisse der Arbeit "Utility maximization in incomplete markets", die ebenfalls mit Ying Hu und Peter Imkeller geschrieben wurde. Beide Arbeiten wurden von den Autoren jeweils zu gleichen Teilen geschrieben.

Matthias Müller
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