Semiclassical methods for the two-dimensional Schrödinger operator with a strong magnetic field

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Abstract

Spectral properties of the two-dimensional Schrödinger operator with a two-periodic potential and a strong uniform magnetic field is studied with the help of semiclassical methods. The spectral asymptotics is described using the Reeb graph technique. In the case of the rational flux one constructs semiclassical magneto-Bloch functions and describes the asymptotics of the band spectrum on the physical level of proof.

Keywords:
magnetic Schrödinger operator, semiclassical analysis, Reeb graph, Bloch functions

Zusammenfassung

Es werden spektrale Eigenschaften des zweidimensionalen Schrödinger-Operators mit einem zweifach periodischen Potential und starkem magnetischem Feld untersucht mit Hilfe semiklassischer Methoden. Man beschreibt die spektrale Asymptotik durch Benutzung der Reeb-Graph-Technik. Im Falle des rationalen Flusses konstruiert man semiklassische Magneto-Bloch-Funktionen und beschreibt die Asymptotik des Spektrums auf dem physikalischen Beweisniveau.

Schlagwörter:
magnetischer Schrödinger-Operator, semiklassische Analysis, Reeb-Graph, Bloch-Funktionen

Contents

Introduction

The motion of a quantum charged particle in a uniform magnetic and periodic electric field is described by the quantum Hamiltonian $\hat{H}_{\boldsymbol{A},w}$ [56],

$$\hat{H}_{\boldsymbol{A},w} = \frac{1}{2m}\left(-i\hbar\nabla - \frac{e\boldsymbol{A}}{c}\right)^2 + w, \tag{0.1}$$

acting in $L^2(\mathbb{R}^2_z)$, $\boldsymbol{z} = (z_1, z_2)$, where

$\boldsymbol{A}(\boldsymbol{z}) = (-Bz_2, 0)$ is the vector potential of the magnetic field

(we use the so-called *Landau gauge*),

B is the strength of the magnetic field,

w is the potential of the electric field.

The function w is periodic with respect to some lattice Γ' spanned by two linearly independent vectors

$\boldsymbol{l}_1 = (l_{11}, l_{12})$ and $\boldsymbol{l}_2 = (l_{21}, l_{22})$.

Since $\hat{H}_{\boldsymbol{A},w}$ is invariant under gauge transformations, we assume without loss of generality that

$l_{12} = 0.$

Also without loss of generality we assume $B > 0$.

In the present thesis, we discuss some questions related to the spectral problem for $\hat{H}_{\boldsymbol{A},w}$:

$(\hat{H}_{\boldsymbol{A},w} - E)\Psi = 0.$

Let us reduce this operator to a certain normal form. Introducing new coordinates

$$\boldsymbol{x} = \frac{2\pi}{L_0}\boldsymbol{z},$$

where

$L_0 = l_{11}$ is the so-called *characteristic size* of the lattice,

we can rewrite the Hamiltonian in the form

$$\hat{H}_{\boldsymbol{A},w} = \frac{(eBL_0)^2}{m(2\pi c)^2}\hat{H}_{h,\epsilon},$$
$$\hat{H}_{h,\epsilon} := \frac{1}{2}\Big(-ih\frac{\partial}{\partial x_1} + x_2\Big)^2 + \frac{1}{2}\Big(-ih\frac{\partial}{\partial x_2}\Big)^2 + \epsilon v(x_1, x_2),$$

where

$$h = (2\pi)^2\frac{\hbar c}{|eB|L_0^2}, \quad \epsilon = \frac{(2\pi c)^2 mW}{(eL_0B)^2}, \tag{0.2}$$

and

$$W = \max|w|, \quad v(\boldsymbol{x}) = \frac{1}{W}w\Big(\frac{L_0}{2\pi}\boldsymbol{x}\Big).$$

The function v is periodic with respect to the vectors $\boldsymbol{a}^1 = (a_{11}, a_{12}) = (2\pi, 0)$ and $\boldsymbol{a}^2 = (a_{21}, a_{22})$.

Therefore, the spectral problem for $\hat{H}_{\boldsymbol{A},w}$ is reduced to the spectral problem for the operator $\hat{H}_{h,\epsilon}$; the spectra of $\hat{H}_{\boldsymbol{A},w}$ and $\hat{H}_{h,\epsilon}$ are connected by the relation

$$\operatorname{spec}\hat{H}_{\boldsymbol{A},w} = \frac{(eBL_0)^2}{(2\pi c)^2 m}\operatorname{spec}\hat{H}_{h,\epsilon}. \tag{0.3}$$

The study of the operator $\hat{H}_{h,\epsilon}$ has a very long history. If $v = 0$, then the spectrum of $\hat{H}_{h,\epsilon}$ consists of infinitely degenerate eigenvalues (*Landau levels*) [56]

$$E_\mu = h(\mu + \frac{1}{2}), \quad \mu \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$$

The appearance of the potential v leads to a "broadening" of these numbers into certain sets; these sets are called *Landau bands*. The notion of the Landau band makes sense only if these sets do not intersect (otherwise, it is impossible to separate them, see [70]); this holds, of course, if ϵ is smaller than h.

The spectral properties of $\hat{H}_{h,\epsilon}$ depend crucially on the number

$$\eta := (a_{11}a_{22} - a_{12}a_{21})/(2\pi h)$$

(in our case $\eta = a_{22}/h$), which measures the *number of the flux quanta through the unit cell of the lattice*. If η is rational, then the spectrum of $\hat{H}_{h,\epsilon}$ has band structure, and the singular spectral component is absent [71]; in this case the operator is invariant under the action of the magnetic translation group [89, 90], and the (non-commutative) Bloch theory is applicable [39]. If the flux is irrational, then the spectrum may contain some parts which are Cantor sets [45, 47, 46]; moreover, this is the generic situation: the operator $\hat{H}_{h,\epsilon}$ may be obtained as a limit of a sequence of operators with Cantor spectrum independent of the value η [39], but at present there is no general description of the spectrum in this case. Nevertheless, some particular potentials v were studied [21, 45, 47, 46]. In particular, in [45, 47, 46] Cantor structure of the spectrum of $\hat{H}_{h,\epsilon}$ was proved for a certain class of potentials and specific values of the flux. The presence of Cantor structure was discovered first in the physics literature [48]; recently, such spectrum was observed in experiments [55, 37, 40, 86].

We will study the asymptotic behavior of spectral characteristics of $\hat{H}_{h,\epsilon}$ as both parameters h and ϵ are small (and positive, of course). Let us try to understand the meaning of this condition from the physical point of view. Let us rewrite (0.2) in an alternative form:

$$h = (2\pi)^2 \left(\frac{l_M}{L_0}\right)^2, \quad \epsilon = h\frac{W}{\hbar\omega_c},$$

where

$$\omega_c = \frac{|eB|}{cm} \text{ is the } \textit{cyclotron frequency},$$

$$l_M = \sqrt{\frac{\hbar}{m\omega_c}} \text{ is the } \textit{magnetic length}.$$

Then, the smallness of h means that the magnetic length is much smaller than the characteristic size of the lattice, and ϵ is small if, for example, the electric energy W is comparable with the magnetic energy $\hbar\omega_c$ (this number is the doubled energy of the lowest Landau level for the operator $\hat{H}_{A,w}$). Such a situation is realized, for example, in arrays of quantum dots and antidots, see [1].

It is important to emphasize that these assumptions about the parameters ϵ and h are essential. In particular, this problem cannot be solved within the framework of the first order perturbation theory (like it was done, for example, in [61] for fixed h), because using such methods leads to the appearance of ϵ/h-like terms; analysis of such terms is impossible if we do not assume any relationship between ϵ and h. Another possible situation, when $l_M >> L_0$ (in our notation this means that the length of the vectors $\boldsymbol{a}^1$ and $\boldsymbol{a}^2$ is comparable with h), was studied in [72].

The key for studying the problem under consideration is given by the fundamental *Correspondence Principle* of quantum mechanics. According to this principle, the asymptotics of the spectrum of $\hat{H}_{h,\epsilon}$ as $h \to 0$ can be derived from certain properties of the corresponding classical Hamiltonian. This classical Hamiltonian, H, in our case has the form

$$H(\boldsymbol{p}, \boldsymbol{x}, \epsilon) = \frac{1}{2}(p_1 + x_2)^2 + \frac{1}{2}p_2^2 + \epsilon v(x_1, x_2),$$

such that $\hat{H}_{h,\epsilon} = H(-ih\nabla, \boldsymbol{x}, \epsilon)$.

Such a correspondence is very well known for problems with discrete spectrum. Let $\hat{H}$ be the quantum Hamiltonian corresponding (in the sense described above) to some classical Hamiltonian H. Assume that we know some family of compact invariant Lagrangian manifolds of H, and that these invariant manifolds $\Lambda(E)$ correspond to the energy interval $[E', E'']$ of H in the sense that

$$H|_{\Lambda(E)} = E \in [E', E''];$$

then the asymptotics of the spectrum of $\hat{H}$ as $h \to 0$ with error $O(h^L)$, $L > 0$, inside this energy interval can be obtained using the Bohr-Sommerfeld quantization rule for these manifolds:

$$\frac{1}{2\pi} \oint_{\gamma(E)} \langle \boldsymbol{p} | d\boldsymbol{x} \rangle = h\big(n + \frac{\operatorname{Ind} \gamma(E)}{4}\big),$$

where $\gamma(E)$ are non-contractible cycles on $\Lambda(E)$, and Ind denotes the Maslov index [66, §13].

It is easy to understand that we cannot apply directly this approach to $\hat{H}_{h,\epsilon}$. First of all, as we already mentioned, the spectrum of $\hat{H}_{h,\epsilon}$ is not discrete. Moreover, the condition of the existence of invariant Lagrangian manifolds in classical dynamical system is a very restrictive one. Usually this condition is equivalent to the integrability of the system, but the problem under study is non-integrable.

To avoid at least the second of these obstacles we exploit the smallness of ϵ. If $\epsilon = 0$, then the Hamiltonian system for H can be easily solved. Therefore, we have a classical problem with a small parameter. Using averaging methods we can reduce the Hamiltonian H to the form $H = H^0 + e^{-C/\epsilon} g$, where H^0 is a Hamiltonian defining an integrable system, C is a positive number, and the function g is bounded; the invariant manifolds of H^0 are "almost invariant" manifolds of H, and using them in semiclassical analysis gives an additional error $O(e^{-C/\epsilon})$. The classification of these manifolds is given using the topological theory of Hamiltonian systems [12, 13]. These manifolds are tori, cylinders, closed or open curves, and points. The spectral asymptotics can be constructed now using a combination of methods related to the Maslov canonical operator.

The thesis is organized as follows.

In Chapter 1, questions related to classical mechanics are discussed. Section 1.1 contains a short description of basic notions and facts of classical mechanics; this section is mainly necessary to fix the notation. In Section 1.2, a detailed description of the averaging procedure for the classical Hamiltonian is given; this section is based on the papers [17,36]. In Section 1.3, we describe the invariant manifolds of the averaged Hamiltonian; this classification is illustrated by Reeb graphs [12] and Reeb surfaces (see Section 1.4).

Chapter 2 contains an overview of certain semiclassical methods, in particular, a short description of Maslov's canonical operator. In Section 2.1, the main ideas behind semiclassical methods and the general structure of the asymptotics are described. Section 2.2 contains a description of the semiclassical methods with real phases, i.e. those related to invariant Lagrangian manifolds. Section 2.3 describes the construction of spectral asymptotics using rest points of a classical Hamiltonian. In Section 2.3, the simplest (but non-trivial) variant of Maslov's complex germ theory related to invariant curves of classical Hamiltonians is described.

In Chapter 3 we apply these methods to the invariant manifolds of the averaged Hamiltonian constructed in Chapter 1; as a result, we show that the spectral problem for $\hat{H}_{h,\epsilon}$ can be solved up to $O(h^L + \epsilon^K)$, where K and L are arbitrary positive numbers.

Chapter 4 describes possibilities of semiclassical methods in problems with band spectrum. We use approximate solutions of the spectral problem (constructed in the previous chapter) for constructing approximate solutions satisfying the magneto-Bloch conditions. This procedure improves a detalization of the approximation to the spectrum at least on physical level of proof.

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Chapter 1

A classical charged particle in a uniform magnetic field

1.1 Some notions of classical mechanics

In this section we fix terminology (which differs somewhat in the literature) and recall the necessary facts in a form suitable for us.

1.1.1 Isotropic and Lagrangian subspaces

Let us consider $\mathbb{R}^{2n}$ as the space of pairs $\boldsymbol{r} = (\boldsymbol{p}, \boldsymbol{x})$, $\boldsymbol{p}, \boldsymbol{x} \in \mathbb{R}^n$. This space is equipped with a Euclidean structure given by the inner product

$$\langle \boldsymbol{r}^1 | \boldsymbol{r}^2 \rangle = \langle \boldsymbol{p}^1 | \boldsymbol{p}^2 \rangle + \langle \boldsymbol{x}^1 | \boldsymbol{x}^2 \rangle, \quad \boldsymbol{r}^j = (\boldsymbol{p}^j, \boldsymbol{x}^j), \quad j \in \{1, 2\},$$

where $\langle \cdot | \cdot \rangle$ is the standard inner product in $\mathbb{R}^n$,

$$\langle \boldsymbol{p} | \boldsymbol{x} \rangle = \sum_{j=1}^{n} p_j x_j, \tag{1.1}$$

and with a symplectic structure, which is given by the non-degenerate two-form (the *skew-inner product*)

$$[\boldsymbol{r}^1 | \boldsymbol{r}^2] = \langle \boldsymbol{p}^1 | \boldsymbol{x}^2 \rangle - \langle \boldsymbol{p}^2 | \boldsymbol{x}^1 \rangle, \quad \boldsymbol{r}^j = (\boldsymbol{p}^j, \boldsymbol{x}^j), \quad j \in \{1, 2\}.$$

We use the same notation for the space $\mathbb{C}^{2n} = \mathbb{C}^n \times \mathbb{C}^n$. Let us emphasize that the form $\langle \cdot | \cdot \rangle$ defined by (1.1) is linear in both arguments, and it is *not* an inner product in $\mathbb{C}^n$.

In classical mechanics, the variables $\boldsymbol{p}$ are called *generalized momenta*, and the variables $\boldsymbol{x}$ are called *generalized positions*.

Let E_n denote the unit $n \times n$ matrix. Introduce the $2n \times 2n$ matrix J_{2n},

$$J_{2n} = \begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}.$$

In terms of this matrix one has $[\boldsymbol{r}^1|\boldsymbol{r}^2] = \langle \boldsymbol{r}^1 | J_{2n} \boldsymbol{r}^2 \rangle$.

Definition 1 (Isotropic subspace). A linear subspace $\lambda \subset \mathbb{R}^{2n}$ is called *isotropic* if $[\boldsymbol{r}|\boldsymbol{r}] = 0$ for any $\boldsymbol{r} \in \lambda$.

Proposition 1.1 (see [2]). *The dimension of an isotropic linear subspace in $\mathbb{R}^{2n}$ is not greater than n.*

Definition 2 (Lagrangian subspace). An isotropic subspace $\lambda \subset \mathbb{R}^{2n}$ of maximal dimension n is called *Lagrangian*.

Let us introduce some notation.

Definition 3 (I–coordinates). Let $\boldsymbol{r} = (\boldsymbol{p}, \boldsymbol{x})$ be a $2n$-dimensional vector, $\boldsymbol{p}, \boldsymbol{x} \in \mathbb{R}^n$. Let I denote a subset of $N = (1, \ldots, n)$, and $\bar{I} = N \backslash I$. Let us compose a matrix $J_{2n,I}$ in the following way:

$$\left(J_{2n,I}\right)_{jk} = \begin{cases} \left(E_{2n}\right)_{jk} & \text{if } [j \leq n \text{ and } j \in \bar{I}] \text{ or } [j > n \text{ and } j - n \in \bar{I}], \\ \left(J_{2n}\right)_{jk} & \text{if } [j \leq n \text{ and } j \in I] \text{ or } [j > n \text{ and } j - n \in I], \end{cases} \tag{1.2}$$
$$j, k = \in \{1, \ldots, 2n\}.$$

Set now

$$\boldsymbol{r}_I = \begin{pmatrix} \boldsymbol{p}_I \\ \boldsymbol{x}_I \end{pmatrix} = J_{2n,I} \begin{pmatrix} \boldsymbol{p} \\ \boldsymbol{x} \end{pmatrix} = J_{2n,I} \boldsymbol{r}. \tag{1.3}$$

In other words, each element $j \in I$ induces the transformation $(p_j, x_j) \mapsto (-x_j, p_j)$.

Definition 4 (Lagrangian coordinate subspace). Let I be a subset of $\{1, \ldots, n\}$, then the subspace

$$\lambda_I = \{\boldsymbol{p}_I = 0\}$$

is called a *Lagrangian coordinate subspace* or the *I-coordinate subspace*.

Proposition 1.2 (see [2]). *Any isotropic linear subspace is transversal to some Lagrangian coordinate subspace.*

1.1.2 Isotropic and Lagrangian manifolds

Definition 5 (Isotropic and Lagrangian manifolds). A manifold $\Lambda \subset \mathbb{R}^{2n}_{\boldsymbol{p},\boldsymbol{x}}$ is called *isotropic* if the form $d\boldsymbol{p} \wedge d\boldsymbol{x}$ vanishes on Λ. An isotropic manifold of dimension n is called *Lagrangian*. In other words, a manifold $\Lambda \subset \mathbb{R}^{2n}_{\boldsymbol{p},\boldsymbol{x}}$ is called isotropic (respectively, Lagrangian) if all its tangent spaces are isotropic (respectively, Lagrangian).

Proposition 1.3 (Corollary of Proposition 1.1). *The dimension of an isotropic manifold in $\mathbb{R}^{2n}$ is not greater than n.*

Proposition 1.4 (Corollary of Proposition 1.2). *Let $\Lambda \subset \mathbb{R}^{2n}$ be an isotropic manifold, then any point of Λ has a vicinity that can be diffeomorphically projected onto some Lagrangian coordinate subspace.*

Proposition 1.5 (Local structure of a Lagrangian manifold, Theorems 4.20 and 4.21 in [66]). *Let a manifold Λ be locally given by equations $\boldsymbol{p}_I = \boldsymbol{f}(\boldsymbol{x}_I)$, then Λ is a Lagrangian manifold if and only if there is a function $S(\boldsymbol{x}_I)$ such that*

$$\boldsymbol{p}_I = \partial S/\partial \boldsymbol{x}_I.$$

The function S in this case is called the generating function *of Λ in the I-coordinates.*

Proposition 1.6 (Corollary of Proposition 1.5). *A n-dimensional manifold Λ is Lagrangian if and only if the integral*

$$\int_{\boldsymbol{r}'}^{\boldsymbol{r}''} \langle \boldsymbol{p} | d\boldsymbol{x} \rangle$$

does not depend (locally) on the integration path $l(\boldsymbol{r}', \boldsymbol{r}'') \subset \Lambda$ between the points $\boldsymbol{r}', \boldsymbol{r}'' \in \Lambda$.

1.1.3 Canonical transformations

Definition 6 (Symplectic map). A linear map $g : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is called *symplectic* if $[g\boldsymbol{r}^1 | g\boldsymbol{r}^2] = [\boldsymbol{r}^1 | \boldsymbol{r}^2]$ for any $\boldsymbol{r}^{1,2} \in \mathbb{R}^{2n}$.

Definition 7. A $(2n) \times (2n)$ matrix A is called *symplectic* if $A^T J_{2n} A = J_{2n}$.

Proposition 1.7. *A linear map $g : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is symplectic if and only if its matrix in the standard basis of $\mathbb{R}^{2n}$ is symplectic.*

Definition 8 (Canonical transformation). A diffeomorphic map $g : (\boldsymbol{p}, \boldsymbol{x}) \to (\mathcal{P}, \mathcal{X})$ preserving the form $d\boldsymbol{p} \wedge d\boldsymbol{x}$ is called *canonical*. This means, in particular, that the linearization of g at any point is symplectic, i.e. the matrix (the Jacobian)

$$\frac{\partial(\boldsymbol{p}, \boldsymbol{x})}{\partial(\mathcal{P}, \mathcal{X})}$$

is symplectic.

Definition 9 (Canonical variables). If g is a canonical transformation and $(\mathcal{P}, \mathcal{X}) = g(\boldsymbol{p}, \boldsymbol{x})$, then the variables $(\mathcal{P}, \mathcal{X})$ are called *canonical variables*.

Proposition 1.8 (Lemma 2.9 in [64]). *Let Λ be an isotropic (respectively, Lagrangian) manifold and g be a canonical transformation, then $g\Lambda$ is also an isotropic (respectively, Lagrangian) manifold.*

1.1.4 Generating functions of a canonical transformation

Let $g : (\boldsymbol{p}, \boldsymbol{x}) \to (\mathcal{P}, \mathcal{X})$ be a canonical transformation in a certain domain $\Omega \subset \mathbb{R}^{2n}$. The function S defined locally by

$$dS = \langle \boldsymbol{p} | d\boldsymbol{x} \rangle - \langle \mathcal{P} | d\mathcal{X} \rangle.$$

is called a *generating function* of g.

The function S depends on $\boldsymbol{p}$ and $\boldsymbol{x}$. Nevertheless, we can sometimes express it through the new variables $\mathcal{P}$ and $\mathcal{X}$, or, at least, through some combination of the old and new variables. In such a case, the whole canonical transformation is uniquely defined by its generating function.

Proposition 1.9 (see §48.A in [4]). *Let $g : (\boldsymbol{p}, \boldsymbol{x}) \mapsto (\mathcal{P}, \mathcal{X})$ be a canonical transformation; assume that for some $\boldsymbol{r}^0$ one has*

$$\det \frac{\partial(\mathcal{P}, \boldsymbol{x})}{\partial(\boldsymbol{p}, \boldsymbol{x})}\Big|_{\boldsymbol{r}^0} \neq 0.$$

Introduce a function S by the relation

$$dS = \langle \boldsymbol{p} | d\boldsymbol{x} \rangle + \langle \mathcal{X} | d\mathcal{P} \rangle,$$

then in a certain neighborhhod of $\boldsymbol{r}^0$ the transformation g can be uniquely recovered from S by the formulas

$$\boldsymbol{p} = \frac{\partial S(\mathcal{P}, \boldsymbol{x})}{\partial \boldsymbol{x}}, \quad \mathcal{X} = \frac{\partial S(\mathcal{P}, \boldsymbol{x})}{\partial \mathcal{P}}. \tag{1.4}$$

The function S is also called a generating function *of g. From the other side, the transformation defined by (1.4) is always canonical.*

1.1.5 Invariant manifolds of Hamiltonian systems

Let $H(\boldsymbol{p}, \boldsymbol{x})$ be a smooth function (Hamiltonian) on $\mathbb{R}^{2n}$. Let us consider the corresponding Hamiltonian system:

$$\frac{d\boldsymbol{p}}{dt} = -\frac{\partial H}{\partial \boldsymbol{x}}, \quad \frac{d\boldsymbol{x}}{dt} = \frac{\partial H}{\partial \boldsymbol{p}}, \tag{1.5}$$

and denote by $g_H^t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ the corresponding phase flow, i. e., for any $(\boldsymbol{p}^0, \boldsymbol{x}^0)$ the function $\big(\boldsymbol{p}(t), \boldsymbol{x}(t)\big) = g_H^t(\boldsymbol{p}^0, \boldsymbol{x}^0)$ satisfies the system (1.5) together with the initial conditions $\big(\boldsymbol{p}(0), \boldsymbol{x}(0)\big) = \big(\boldsymbol{p}^0, \boldsymbol{x}^0\big)$.

Definition 10 (Invariant manifold of a Hamiltonian system). A manifold $\Lambda \subset \mathbb{R}^{2n}$ is called an *invariant manifold* of a Hamiltonian H if $g_H^t \Lambda \subset \Lambda$ for any $t \in \mathbb{R}$.

1.1.6 Hamiltonian systems and canonical transformations

Proposition 1.10 (see §45.A in [4]). *Let g be a canonical transformation, $g(\boldsymbol{p}, \boldsymbol{x}) = (\mathcal{P}, \mathcal{X})$, then in the new canonical coordinates $(\mathcal{P}, \mathcal{X})$ the Hamiltonian system (1.5) is written as*

$$\frac{d\mathcal{P}}{dt} = -\frac{\partial H}{\partial \mathcal{X}}, \quad \frac{d\mathcal{X}}{dt} = \frac{\partial H}{\partial \mathcal{P}}.$$

In the other words, a Hamiltonian system preserves its structure under canonical transformations.

Basing on this fact, the components $\mathcal{P}$ and $\mathcal{X}$ are usually called *generalized momenta* and *generalized positions*, respectively.

1.1.7 Action-angle variables

Let us suppose that a Hamiltonian $H(\boldsymbol{p}, \boldsymbol{x})$ can be reduced by a canonical change of variables

$$(\boldsymbol{p}, \boldsymbol{x}) \longrightarrow (\boldsymbol{I}, \boldsymbol{\varphi})$$

to the form

$$H(\boldsymbol{p}, \boldsymbol{x}) = \mathcal{H}(\boldsymbol{I}).$$

In this case the variables $\boldsymbol{I} = (I_1, \ldots, I_n)$ are called *action variables* and the variables $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_n)$ are called the corresponding *angle variables*. In these *action-angle variables* the Hamiltonian system takes the form

$$\dot{\boldsymbol{I}} = 0, \quad \dot{\boldsymbol{\varphi}} = \boldsymbol{\omega}(\boldsymbol{I}),$$

where

$$\boldsymbol{\omega}(\boldsymbol{I}) = \big(\omega_1(\boldsymbol{I}), \ldots, \omega_n(\boldsymbol{I})\big) = \frac{\partial \mathcal{H}(\boldsymbol{I})}{\partial \boldsymbol{I}}$$

is the *vector-frequency*, and the solutions of this system have the form

$$\begin{aligned} &\boldsymbol{I} = \boldsymbol{I}^0 = \text{const}, \\ &\boldsymbol{\varphi} = \boldsymbol{\varphi}^0 + \boldsymbol{\omega}(\boldsymbol{I})t, \quad \boldsymbol{\varphi}^0 = \text{const}. \end{aligned} \tag{1.6}$$

All the invariant manifolds of H are given by (1.6).

1.2 Averaging methods

1.2.1 The guiding center approach

As it was mentioned in the introduction, the classical motion of a classical charged particle in question is described by the Hamiltonian

$$H(\boldsymbol{p}, \boldsymbol{x}, \epsilon) = \frac{1}{2}(p_1 + x_2)^2 + \frac{1}{2}p_2^2 + \epsilon v(x_1, x_2),$$

where v is periodic with respect to the lattice Γ spanned by the vectors $\boldsymbol{a}^1 = (2\pi, 0)$ and $\boldsymbol{a}^2 = (a_{21}, a_{22})$.

The corresponding Hamiltonian system is

$$\begin{cases} \dot{p}_1 = -\epsilon \dfrac{\partial v}{\partial x_1}(x_1, x_2), & \dot{p}_2 = -(p_1 + x_2) - \epsilon \dfrac{\partial v}{\partial x_2}(x_1, x_2), \\ \dot{x}_1 = p_1 + x_2, & \dot{x}_2 = p_2. \end{cases} \tag{1.7}$$

This is a fourth order non-linear system, and it is unreasonable to expect that this system can be solved explicitly (even though some solutions can be found, as we will discuss later). But the presence of the small parameter ϵ makes it possible to use averaging methods.

Thus let us introduce new canonical variables, generalized momenta (P, y_1) and generalized positions (Q, y_2), as follows:

$$\begin{cases} p_1 = -y_2, & p_2 = -Q, \\ x_1 = Q + y_1, & x_2 = P + y_2. \end{cases} \tag{1.8}$$

It is also useful to introduce the corresponding polar coordinates (I_1, φ_1):

$$P = \sqrt{2I_1}\cos\varphi_1, \quad Q = \sqrt{2I_1}\sin\varphi_1. \tag{1.9}$$

It is easy to see that the variables (I_1, y_1) and (φ_1, y_2) can also be considered as generalized momenta and positions, respectively.

In these coordinates, the Hamiltonian H takes the form

$$H = \frac{1}{2}(P^2 + Q^2) + \epsilon v(Q + y_1, P + y_2), \tag{1.10}$$

or

$$H = I_1 + \epsilon v(\sqrt{2I_1}\sin\varphi_1 + y_1, \sqrt{2I_1}\cos\varphi_1 + y_2).$$

Such a choice of coordinates can be motivated as follows. If the potential v is 0 (or, equivalently, $\epsilon = 0$), then the projections of the trajectories of the system (1.7) onto the plane (x_1, x_2) are cyclotron orbits; denote the coordinates of their centers as (y_1, y_2), and their radiuses by $\sqrt{2I_1}$, see Fig. 1.1. Under the influence of the potential v the centers of these circles move and we have a superposition of a cyclotron motion, which is described by the variables (P, Q) or (I_1, φ_1), around a guiding center, see Fig. 1.2; the motion of the center is described by the coordinates $\boldsymbol{y}$, see [60].

The main idea of averaging is to remove the dependence of the classical Hamiltonian on the angle variable (φ_1 in our case) at least with some accuracy. It is easy to see, that independence of φ_1 reduces the order of the Hamiltonian system.

1.2.2 One step of the averaging procedure

Proposition 1.11. *Let a real-analytic function H, $H = H(P, Q, y_1, y_2, \epsilon)$, admit for some $K \in \mathbb{N}$ the representation*

$$H(P, Q, y_1, y_2, \epsilon) = I_1 + \epsilon u(I_1, y_1, y_2, \epsilon) + \epsilon^K g(P, Q, y_1, y_2, \epsilon), \tag{1.11}$$

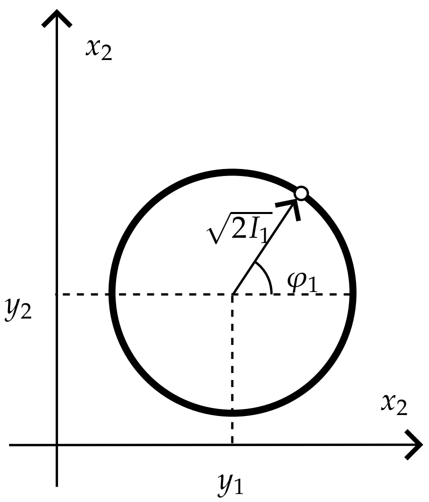

Figure 1.1: Cyclotron circles

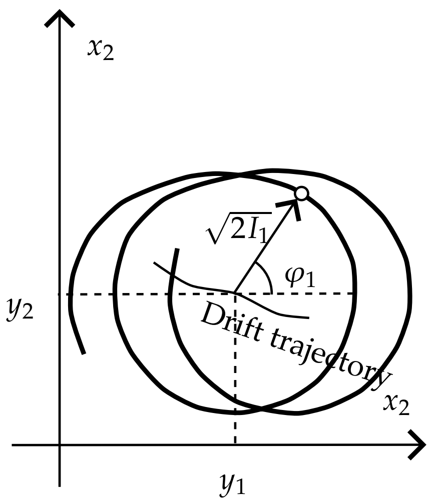

Figure 1.2: A guiding center

where u and g are real-analytic functions of all their arguments, periodic in $\boldsymbol{y}$ *with respect to* Γ*, and* $I_1 = \frac{1}{2}(P^2 + Q^2)$*. Then, for any* $\kappa > 0$*, there exists* $\epsilon_0(\kappa) > 0$ *such that for* $I_1 \leq \kappa$ *and* $0 \leq \epsilon < \epsilon_0$ *there exists a canonical change of variables*

$$\begin{cases} P = \mathcal{P} + \epsilon^K U_1(\mathcal{P}, \mathcal{Q}, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon), & Q = \mathcal{Q} + \epsilon^K U_2(\mathcal{P}, \mathcal{Q}, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon), \\ y_1 = \mathcal{Y}_1 + \epsilon^K W_1(\mathcal{P}, \mathcal{Q}, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon), & y_2 = \mathcal{Y}_2 + \epsilon^K W_2(\mathcal{P}, \mathcal{Q}, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon), \end{cases} \tag{1.12}$$

reducing H to the form

$$H(P, Q, y_1, y_2, \epsilon) = \mathcal{H}(\mathcal{I}_1, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon) + \epsilon^{K+1}\mathcal{G}(\mathcal{P}, \mathcal{Q}, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon). \tag{1.13}$$

Here

$$\mathcal{I}_1 = \frac{1}{2}(\mathcal{P}^2 + \mathcal{Q}^2),$$

$U_{1,2}$*,* $W_{1,2}$*,* $\mathcal{U}$*, and* $\mathcal{G}$ *are real-analytic functions of all their arguments, and all periodic in* $\boldsymbol{\mathcal{Y}}$ *with respecto to* Γ*.*

One can put, in particular,

$$\mathcal{H}(\mathcal{I}_1, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon) = \mathcal{I}_1 + \epsilon u(\mathcal{I}_1, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon) + \epsilon^K \overline{g}(\mathcal{I}_1, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon), \tag{1.14}$$

where

$$\overline{g}(I_1, y_1, y_2, \epsilon) = \frac{1}{2\pi}\int_0^{2\pi} g(\sqrt{2I_1}\cos\varphi, \sqrt{2I_1}\sin\varphi, y_1, y_2, \epsilon)d\varphi, \tag{1.15}$$

then the change of variables (1.12) *can be defined by the equations*

$$
\begin{aligned}
P &= \mathcal{P} + \epsilon^K \frac{\partial s(\mathcal{P}, Q, \mathcal{Y}_1, y_2, \epsilon)}{\partial Q}, \qquad & \mathcal{Q} &= Q + \epsilon^K \frac{\partial s(\mathcal{P}, Q, \mathcal{Y}_1, y_2, \epsilon)}{\partial \mathcal{P}}, \\
y_1 &= \mathcal{Y}_1 + \epsilon^K \frac{\partial s(\mathcal{P}, Q, \mathcal{Y}_1, y_2, \epsilon)}{\partial y_2}, \qquad & \mathcal{Y}_2 &= y_2 + \epsilon^K \frac{\partial s(\mathcal{P}, Q, \mathcal{Y}_1, y_2, \epsilon)}{\partial \mathcal{Y}_1},
\end{aligned}
\tag{1.16}
$$

where

$$
\left[
\begin{aligned}
s(P, Q, y_1, y_2, \epsilon) = {} & \frac{1}{2\Big(1 + \epsilon \frac{\partial u}{\partial I_1}(I_1, y_1, y_2, \epsilon)\Big)} \\
& \times \Big(\int_0^{\varphi} \tilde{g}(\sqrt{2I}\cos\phi, \sqrt{2I}\sin\phi, y_1, y_2, \epsilon) d\phi \\
& + \int_{\pi}^{\varphi} \tilde{g}(\sqrt{2I}\cos\phi, \sqrt{2I}\sin\phi, y_1, y_2, \epsilon) d\phi \Big) \Big|_{\substack{P=\sqrt{2I}\cos\varphi, \\ Q=\sqrt{2I}\sin\varphi}} \\
\tilde{g}(P, Q, y_1, y_2, \epsilon) = {} & \bar{g}\big(\tfrac{1}{2}(P^2 + Q^2), y_1, y_2, \epsilon\big) - g(P, Q, y_1, y_2, \epsilon).
\end{aligned}
\right.
\tag{1.17}
$$

Proof. If a canonical change of the form (1.12) exists, then, for ϵ small enough, this change can be determined by a generating function $S(\mathcal{P}, Q, \mathcal{Y}_1, y_2, \epsilon)$ (see Proposition 1.9) from the equations

$$
\begin{aligned}
P &= \frac{\partial S(\mathcal{P}, Q, \mathcal{Y}_1, y_2, \epsilon)}{\partial Q}, \qquad & \mathcal{Q} &= \frac{\partial S(\mathcal{P}, Q, \mathcal{Y}_1, y_2, \epsilon)}{\partial \mathcal{P}}, \\
y_1 &= \frac{\partial S(\mathcal{P}, Q, \mathcal{Y}_1, y_2, \epsilon)}{\partial y_2}, \qquad & \mathcal{Y}_2 &= \frac{\partial S(\mathcal{P}, Q, \mathcal{Y}_1, y_2, \epsilon)}{\partial \mathcal{Y}_1}.
\end{aligned}
\tag{1.18}
$$

Let us try to find the generating function in the form

$$
S(\mathcal{P}, Q, \mathcal{Y}_1, y_2, \epsilon) = \mathcal{P}Q + \mathcal{Y}_1 y_2 + \epsilon^K s(\mathcal{P}, Q, \mathcal{Y}_1, y_2, \epsilon),
$$

then (1.18) is reduced to (1.16). Let us show that (1.16) can be solved for Q and y_2 globally in any domain $I_1 \leq \kappa$ can be solved as $\epsilon \leq \epsilon_0(\kappa)$.

Let us substitute (1.12) into (1.16) and define the functions $U_2(\mathcal{P}, \mathcal{Q}, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon)$ and $W_2(\mathcal{P}, \mathcal{Q}, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon)$ by the relations

$$
\begin{aligned}
\epsilon^K U_2 &= -\epsilon^K \frac{\partial s(\mathcal{P}, \mathcal{Q} + \epsilon^K U_2, \mathcal{Y}_1, \mathcal{Y}_2 + \epsilon^K W_2, \epsilon)}{\partial \mathcal{P}}, \\
\epsilon^K W_2 &= -\epsilon^K \frac{\partial s(\mathcal{P}, \mathcal{Q} + \epsilon^K U_2, \mathcal{Y}_1, \mathcal{Y}_2 + \epsilon^K W_2, \epsilon)}{\partial \mathcal{Y}_1}.
\end{aligned}
$$

The vector-operator in the right-hand side is a contracting one for ϵ small enough, therefore, the functions U_2 and W_2 are well-defined. The functions U_1 and W_1 can be defined now by the relations

$$\epsilon^K U_1 = \epsilon^K \frac{\partial s(\mathcal{P} + \epsilon^K U_1, \mathcal{Q} + \epsilon^K U_2, \mathcal{Y}_1 + \epsilon^K W_1, \mathcal{Y} + \epsilon^K W_2, \epsilon)}{\partial Q},$$
$$\epsilon^K W_1 = \epsilon^K \frac{\partial s(\mathcal{P}_1 + \epsilon^K U_1, \mathcal{Q} + \epsilon^K U_2, \mathcal{Y}_1 + \epsilon^K W_1, \mathcal{Y} + \epsilon^K W_2, \epsilon)}{\partial y_2}$$

in the same way.

Finally, we come to the equalities

$$\left[\begin{aligned} P &= \mathcal{P} + \epsilon^K \frac{\partial s(\mathcal{P}, \mathcal{Q}, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon)}{\partial \mathcal{Q}} + \epsilon^{K+1} z_1(\mathcal{P}, \mathcal{Q}, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon), \\ Q &= \mathcal{Q} - \epsilon^K \frac{\partial s(\mathcal{P}, \mathcal{Q}, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon)}{\partial \mathcal{P}} + \epsilon^{K+1} z_2(\mathcal{P}, \mathcal{Q}, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon), \\ y_1 &= \mathcal{Y}_1 + \epsilon^K \frac{\partial s(\mathcal{P}, \mathcal{Q}, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon)}{\partial \mathcal{Y}_2} + \epsilon^{K+1} z_3(\mathcal{P}, \mathcal{Q}, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon), \\ y_2 &= \mathcal{Y}_2 - \epsilon^K \frac{\partial s(\mathcal{P}, \mathcal{Q}, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon)}{\partial \mathcal{Y}_1} + \epsilon^{K+1} z_4(\mathcal{P}, \mathcal{Q}, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon), \end{aligned}\right. \tag{1.19}$$

where $z_{1,2,3,4}$ are real-analytic functions of all their arguments.

Let us substitute (1.19) into the Hamiltonian H:

$$\left[\begin{aligned} H(P, Q, y_1, y_2, \epsilon) &= I_1 + \epsilon u(I_1, y_1, y_2, \epsilon) + \epsilon^K g(P, Q, y_1, y_2, \epsilon) \\ &= \mathcal{I}_1 + \epsilon^K (\mathcal{P} \frac{\partial s(\mathcal{P}, \mathcal{Q}, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon)}{\partial \mathcal{Q}} \\ &\qquad - \mathcal{Q} \frac{\partial s(\mathcal{P}, \mathcal{Q}, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon)}{\partial \mathcal{P}}) \\ &\quad \cdot \Big(1 + \epsilon \frac{\partial u}{\partial I_1} (\frac{1}{2}(\mathcal{P}^2 + \mathcal{Q}^2), \mathcal{Y}_1, \mathcal{Y}_2, \epsilon)\Big) \\ &\quad + \epsilon^K g(\mathcal{P}, \mathcal{Q}, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon) + \epsilon^{K+1} z_5(\mathcal{P}, \mathcal{Q}, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon), \end{aligned}\right. \tag{1.20}$$

where z_5 is some real-analytic function.

Let us represent g as

$$g(P, Q, y_1, y_2, \epsilon) = \overline{g}(\frac{1}{2}(P^2 + Q^2), y_1, y_2) + \tilde{g}(P, Q, y_1, y_2, \epsilon),$$

where $\overline{g}$ is defined in (1.15). For obtaining the representation (1.13) with $\mathcal{G} = z_5$ by (1.14) and (1.15) the function s has to satisfy the equation

$$Q \frac{\partial s(P, Q, y_1, y_2, \epsilon)}{\partial P} - P \frac{\partial s(P, Q, y_1, y_2, \epsilon)}{\partial Q} = -\frac{1}{1 + \epsilon \frac{\partial u}{\partial I_1}(I_1, y_1, y_2, \epsilon)} \tilde{g}(P, Q, y_1, y_2, \epsilon), \tag{1.21}$$

and to be analytical with respect to all arguments.

Let us introduce a variable φ as follows:

$$P = \sqrt{2I_1}\cos\varphi, \quad Q = \sqrt{2I_1}\sin\varphi,$$

then (1.21) can be written as

$$\frac{\partial s(\sqrt{2I_1}\cos\varphi, \sqrt{2I_1}\sin\varphi, y_1, y_2, \epsilon)}{\partial\varphi} = -\frac{1}{1+\epsilon\frac{\partial u}{\partial I_1}(I_1, y_1, y_2, \epsilon)}\tilde{g}(\sqrt{2I}\cos\varphi, \sqrt{2I}\sin\varphi, y_1, y_2, \epsilon).$$

The general solution of this equation is

$$s(P, Q, y_1, y_2, \epsilon) = -\frac{1}{1+\epsilon\frac{\partial u}{\partial I_1}(I_1, y_1, y_2, \epsilon)}\int \tilde{g}(\sqrt{2I}\cos\varphi, \sqrt{2I}\sin\varphi, y_1, y_2, \epsilon)\, d\varphi,$$

which may have singularity like $\sqrt{I_1}$ near $I_1 = 0$. This can be avoided by a right choice of an integrating constant, one of such possibilities is given by (1.17). □

1.2.3 Averaging procedure for the original Hamiltonian

Now, applying Proposition 1.11 to (1.10), we arrive at the following fact.

Proposition 1.12. *Let H be given by (1.10). For any $\kappa > 0$ and $K > 0$ there is $\epsilon_0 = \epsilon_0(\kappa, K, v)$ and a canonical transformation in the domain $\{I_1 \le \kappa, \epsilon \le \epsilon_0\}$ given by*

$$\begin{cases} P = \mathcal{P} + \epsilon U_1(\mathcal{P}, \mathcal{Q}, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon), & Q = \mathcal{Q} + \epsilon U_2(\mathcal{P}, \mathcal{Q}, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon), \\ y_1 = \mathcal{Y}_1 + \epsilon W_1(\mathcal{P}, \mathcal{Q}, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon), & y_2 = \mathcal{Y}_2 + \epsilon W_2(\mathcal{P}, \mathcal{Q}, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon), \end{cases} \tag{1.22}$$

such that

$$H = \mathcal{H}(\mathcal{I}_1, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon) + \epsilon^K \mathcal{G}(\mathcal{P}, \mathcal{Q}, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon). \tag{1.23}$$

Here

$$\mathcal{I}_1 = \frac{1}{2}(\mathcal{P}^2 + \mathcal{Q}^2),$$

$U_{1,2}$, $W_{1,2}$, $\mathcal{U}$, and $\mathcal{G}$ are real-analytic functions of all their arguments, and periodic in $\boldsymbol{\mathcal{Y}}$ with respect to Γ.

Moreover, $\mathcal{H}$ is a polynomial in ϵ of degree K, and

$$\mathcal{H}(\mathcal{I}_1, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon) = \overline{H}(\mathcal{I}_1, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon) + O(\epsilon^2), \quad \overline{H} = \mathcal{I}_1 + \epsilon \mathcal{V}(\mathcal{I}_1, \mathcal{Y}_1, \mathcal{Y}_2) \tag{1.24}$$

where

$$\mathcal{V}(\mathcal{I}_1, \mathcal{Y}_1, \mathcal{Y}_2) = \frac{1}{2\pi} \int_0^{2\pi} v(\sqrt{2\mathcal{I}_1} \sin\varphi + \mathcal{Y}_1, \sqrt{2\mathcal{I}_1} \cos\varphi + \mathcal{Y}_2)\, d\varphi. \tag{1.25}$$

Remark. The expression (1.25) can be rewritten in a more elegant form [15, §3]: $\mathcal{V}(\mathcal{I}_1, \mathcal{Y}_1, \mathcal{Y}_2) = J_0(\sqrt{-2\mathcal{I}_1 \Delta}) v(\mathcal{Y}_1, \mathcal{Y}_2)$, $\Delta = \partial^2/\partial \mathcal{Y}_1^2 + \partial^2/\partial \mathcal{Y}_2^2$, where J_0 denotes the Bessel function of order zero.

Definition 11 (Averaged Hamiltonian). If by a canonical change of variables of the form (1.22) the Hamiltonian H is reduced to the form

$$H = \mathcal{H}(\frac{1}{2}(\mathcal{P}^2 + \mathcal{Q}^2), \mathcal{Y}_1, \mathcal{Y}_2, \epsilon) + f(\epsilon) \mathcal{G}(\mathcal{P}, \mathcal{Q}, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon),$$

where $\mathcal{H}$ and $\mathcal{G}$ are analytic and periodic in $\mathbf{\mathcal{Y}}$ with respect to Γ, and $f \to 0$ as $\epsilon \to 0$, then the function $\mathcal{H}$ is called an *averaged Hamiltonian up to $f(\epsilon)$*.

Proposition 1.12 gives an averaged Hamiltonian up to any power of ϵ. But is it possible to improve the remainder estimate for the averaged Hamiltonian? The answer turns out to be positive (under certain additional assumptions), but at the expense of losing the analyticity in ϵ (this plays no role in our further considerations, but may be important for other problems). More precisely, the following holds.

Proposition 1.13. *Let the function v be analytic in a certain complex δ–neighborhood of $\mathbb{R}^2 \subset \mathbb{C}^2$. For any $\kappa > 0$ there exists $\epsilon_0(\kappa) > 0$ and $C > 0$ such that for $0 \leq \epsilon < \epsilon_0$ and $I_1 \leq \kappa$ there exists a canonical transformation of the form (1.22) such that*

$$H(P, Q, y_1, y_2, \epsilon) = \mathcal{H}_1(\mathcal{I}_1, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon) + e^{-C/\epsilon} \mathcal{G}(\mathcal{P}, \mathcal{Q}, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon). \tag{1.26}$$

Here

$$\mathcal{I}_1 = \frac{1}{2}(\mathcal{P}^2 + \mathcal{Q}^2),$$

$U_{1,2}$, $W_{1,2}$, and $\mathcal{G}$ are real-analytic functions of $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{Y}_{1,2}$; $\mathcal{H}$ is a real-analytic function of $\mathcal{I}_1$, $\mathbf{\mathcal{Y}}$; $|\mathcal{G}| + |\nabla_{\mathcal{Y}} \mathcal{G}| \leq G$ for some number G independent of ϵ. All the functions $\mathcal{H}$, $\mathcal{G}$, $U_{1,2}$, and $W_{1,2}$ are periodic in $\mathbf{\mathcal{Y}}$ with respecto to Γ. Also the estimates (1.24) and (1.25) holds.

Proposition 1.13 is obtained independently in [17] and [36]. The proof is based on the procedure described above of consecatively defined changes of variables and uses so-called Neishtadt's inductive estimates [69]. Here we omit this proof because of technical complexity.

1.3 Invariant manifolds of the averaged Hamiltonian

The necessary accuracy of the averaging is determined by concrete needs. In what follows we will use the representation (1.26). Using the representation (1.23) instead will change all the estimates accordingly.

1.3.1 Reduction to a one-dimensional problem

The Hamiltonian system for $\mathcal{H}$ has the following form:

$$\frac{d\mathcal{I}_1}{dt} = 0, \qquad \frac{d\varphi_1}{dt} = \omega_1(\mathcal{I}_1, \epsilon), \tag{1.27}$$

$$\frac{d\mathcal{Y}_1}{dt} = -\epsilon\frac{\partial\mathcal{H}(\mathcal{I}_1, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon)}{\partial\mathcal{Y}_2}, \qquad \frac{d\mathcal{Y}_2}{dt} = \epsilon\frac{\partial\mathcal{H}(\mathcal{I}_1, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon)}{\partial\mathcal{Y}_1}, \tag{1.28}$$

where

$$\omega_1(\mathcal{I}_1, \epsilon) = \frac{\partial\mathcal{H}(\mathcal{I}_1, \boldsymbol{\mathcal{Y}}, \epsilon)}{\partial\mathcal{I}_1}. \tag{1.29}$$

From (1.27) we obtain $\mathcal{I}_1 = \text{const}$. This means, that the equations (1.28) form a Hamiltonian system depending also on $\mathcal{I}_1$ as a parameter. In particular, each trajectory of (1.28) belongs to some level set of $\mathcal{H}$, and ω_1 really does not depend on $\boldsymbol{\mathcal{Y}}$ along any trajectory.

1.3.2 One-dimensional Hamiltonian system on the torus and the Reeb graph

Periodicity of $\mathcal{H}$ may be used for an evident classification of the trajectories of the system (1.28). We consider the function $\mathcal{H}$ as defining a Hamiltonian system on the torus $\mathbb{T}^2 = \mathbb{R}^2/\Gamma$.

Proposition 1.14. *Each trajectory of the Hamiltonian system* (1.28) *belongs to one of the following classes of curves:*

- *extremum points of $\mathcal{H}$,*

- *saddle points of $\mathcal{H}$,*
- *separatrices,*
- *contractible closed smooth curves,*
- *non-contractible closed smooth curves.*

Proof. One only has to prove that all the trajectories except the separatrices are closed. It is well known that a trajectory of a Hamiltonian system on the torus is either closed or dense everywhere, see [4, §51.A]. In our situation, the latter case can not obtain in view of analyticity of the Hamiltonian. □

This classification may be illustrated by means of the *Reeb graph* of $\mathcal{H}$ on the torus.

Definition 12 (Reeb graph). Let w be a smooth function on a two-dimensional manifold M. Let us introduce an equivalence relation Θ on M:

$$(\boldsymbol{x}, \boldsymbol{y}) \in \Theta \Leftrightarrow \boldsymbol{x} \text{ and } \boldsymbol{y} \text{ lie in a connected component of a level set of } w.$$

The set

$$G = M/\Theta$$

is called the *Reeb graph* of the function w on M.

In what follows, we denote by $G(\mathcal{I}_1)$ the Reeb graph of the function $\mathcal{H}(\mathcal{I}_1, \cdot)$ on the torus $\mathbb{T}^2$.

The Reeb graphs $G(\mathcal{I}_1)$ can be constructed under very weak assumptions on $\mathcal{H}$, but, generally speaking, can have a very exotic form. To avoid such difficulties, we introduce additional assumptions.

Definition 13 (Morse function). A smooth function w on a smooth manifold M is called a *Morse function* if all its critical points are non-degenerate (i. e., the matrix $(\partial^2 w/\partial\xi_j\partial\xi_k)$, where $\xi_1, \dots, \xi_n$ are local coordinates on M, is non-degenerate at the critical points). A Morse function is called *simple* if its restriction to the set of critical points is injective, and is called *complex* otherwise.

It is known that the Reeb graph of a Morse function w on a smooth two-dimensional compact manifold realizes a finite graph [13, §2.3]. The extremum points and the connected singular level sets correspond to vertices of the graph.

Let us fix two vertices α and β corresponding to the values A and B of the function w. If $A = B$, then these vertices correspond to different edges and are not connected. If $A < B$, then α and β are connected by an edge iff there exists a continuous family C of a connected component of level sets $w = E$, $E \in (A, B)$, such that α and β intersect the closure of C. It is natural to distinguish between the *end vertices* (i. e. associated with a single edge) and the *branching vertices* (i. e. associated with more than one edge).

From now on we suppose that $\mathcal{H}$ is a Morse function for all $\mathcal{I}_1 \geq 0$, with the possible exception of some discrete subset of values.

Under these assumptions, extremum points of $\mathcal{H}$ correspond to the end vertices of the graph $G(\mathcal{I}_1)$, saddle points and separatrices lie on the same energy levels and correspond to the branching vertices of the graph, and contractible and non-contractible curves correspond to inner points of the edges of the graph; the construction of the Reeb graph becomes especially evident if one realizes the function $\mathcal{H}$ as the height function of a suitably deformed torus; examples of such a construction of the Reeb graph are given in Fig. 1.3.

We will enumerate the edges of the graph by index $r \in \mathbb{Z}$; the edge corresponding to the index value r will be denoted by $\mathcal{E}^r$.

There is an obvious correspondence between the trajectories of (1.28) on the torus and those on the plane $\mathbb{R}^2_{\mathfrak{y}}$, namely,

- extremum points on the torus correspond to extremum points on the plane,
- saddle points on the torus correspond to saddle points on the plane,
- separatrices on the torus correspond to separatrices on the plane,
- contractible closed smooth curves on the torus correspond to *closed* curves on the plane,
- non-contractible closed curves on the torus correspond to *open* curves on the plane.

Having in mind the above correspondence, we distinguish between *finite motion edges* (i. e. those corresponding to contractible trajectories on the torus) and *infinite motion edges* (i. e. those corresponding to non-contractible trajectories on the torus).

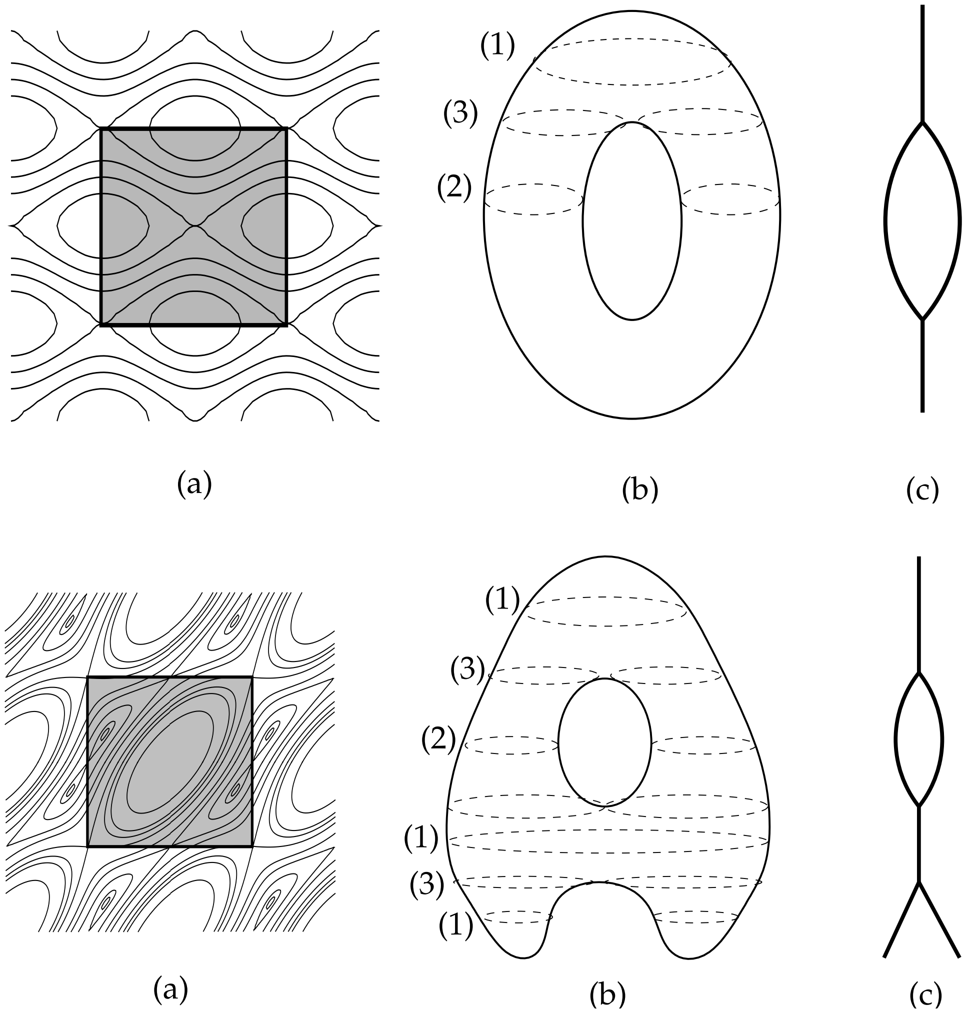

Figure 1.3: The construction of the Reeb graph: (a) Level curves on the plane; (b) their realization through the height function: (1) contractible closed curves, (2) non-contractible closed curves, (3) separatrices; (c) the Reeb graph

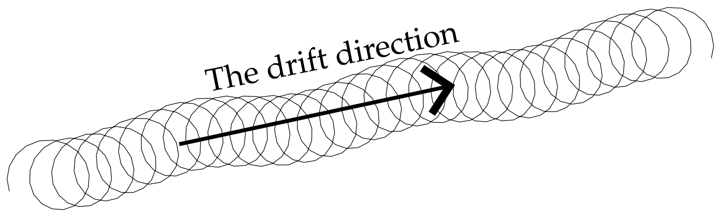

Figure 1.4: The drift direction

The structure of the Reeb graph can be very complicated, see the examples in [12, 13, 34]. Nevertheless, one can always say something about infinite motion edges. The following two cases are possible: (a) there is no infinite motion edge at all, or (b) all such edges form cycles [30].

Let us now describe the relationship between the points of the Reeb graph and the trajectories of the Hamiltonian system in greater detail.

First of all, as each non-separatrix trajectory $\tilde{\mathfrak{Y}}$ on the plane correspond to a periodic trajectory on the torus, there is a unique vector $\boldsymbol{d} = (d_1, d_2) \in \mathbb{Z}^2$ such that one has

$$\tilde{\mathfrak{Y}}(t+T) \equiv \tilde{\mathfrak{Y}}(t) + \boldsymbol{d} \cdot \boldsymbol{a}, \tag{1.30}$$

where T is the period of the trajectory on the torus. If both components d_1 and d_2 are non-zero, then they are mutually prime; if one of them is zero, then the other is equal to ± 1 or 0. We call $\boldsymbol{d}$ the *drift vector*. The drift vector is non-zero for infinite motion edges only. The meaning of this vector becomes especially clear if we consider the projection of the corresponding trajectory in the space $\mathbb{R}^4_{\boldsymbol{p},\boldsymbol{x}}$ onto the $\boldsymbol{x}$-plane, see Fig. 1.4.

Furthermore, for $\mathcal{I}_1$ given, at most two non-zero drift vectors with mutually opposite directions are possible. Denote these vectors by $\pm\boldsymbol{d}(\mathcal{I}_1, \epsilon)$. As the ratio $d_1 : d_2$ is nothing but the rotation number of a trajectory on the torus, trajectories corresponding to the same edge of the Reeb graph have equal drift vectors.

Let $\mathfrak{Y}^e$ be a trajectory of (1.28) on the torus; assume that this trajectory corresponds to a point e of the Reeb graph and introduce a numbering in the set of the corresponding trajectories on the plane.

Suppose that e is a point of a finite motion edge. Let us choose a closed trajectory $\tilde{\mathfrak{Y}}^e$ in the plane covering $\mathfrak{Y}^e$; give to $\tilde{\mathfrak{Y}}^e$ the index $(0,0)$ and denote this trajectory as $\tilde{\mathfrak{Y}}^{e,(0,0)}$, then the corresponding trajectory with the index $\boldsymbol{l} = (l_1, l_2) \in \mathbb{Z}^2$ is

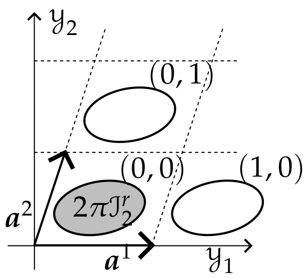

Figure 1.5: The action variable and the numbering for closed trajectories

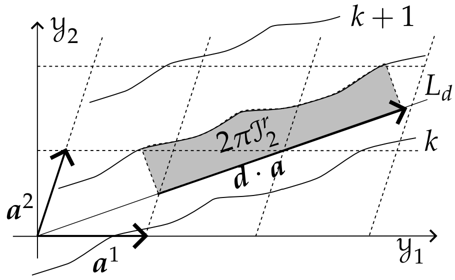

Figure 1.6: The action variable and the numbering for open trajectories

given by $\tilde{\boldsymbol{\mathcal{Y}}}^e + \boldsymbol{l} \cdot \boldsymbol{a}$ and will be denoted by $\tilde{\boldsymbol{\mathcal{Y}}}^{e,\boldsymbol{l}}$, see Fig 1.5. All the trajectories $\tilde{\boldsymbol{\mathcal{Y}}}^{e,\boldsymbol{l}}$ cover $\boldsymbol{\mathcal{Y}}^e$.

For infinite motion edges the numbering is different. Let us fix a certain vector $\boldsymbol{f} = (f_1, f_2) \in \mathbb{Z}^2$ dual to $\boldsymbol{d}$, i. e., $d_1 f_1 + d_2 f_2 = 1$. Let an open trajectory $\tilde{\boldsymbol{\mathcal{Y}}}^e$ on the plane cover $\boldsymbol{\mathcal{Y}}^e$. Give to $\tilde{\boldsymbol{\mathcal{Y}}}^e$ the index 0 and denote it as $\tilde{\boldsymbol{\mathcal{Y}}}^{e,0}$, then the corresponding trajectory $\tilde{\boldsymbol{\mathcal{Y}}}^{e,k}$ with the index $k \in \mathbb{Z}$ is given by $\tilde{\boldsymbol{\mathcal{Y}}}^e - k(J_2 \boldsymbol{f}) \cdot \boldsymbol{a}$, see Fig 1.6; all these trajectories cover $\boldsymbol{\mathcal{Y}}^e$.

1.3.3 The action variable on the torus

Consider a certain edge $\mathcal{E}^r$. To each point $e \in \mathcal{E}^r$ we assign a number $\mathcal{J}_2^r(E, \mathcal{J}_1, \epsilon)$ or $\mathcal{J}_2^r(e)$ defined as follows. Let $\boldsymbol{\mathcal{Y}}(t, E, \epsilon)$ be the corresponding trajectory on the torus lying on the energy level $\mathcal{H} = E$, and T be its period. Let $\tilde{\boldsymbol{\mathcal{Y}}}(\tau, E, \epsilon)$ be one of the corresponding trajectories on the plane, then we put

$$\mathcal{J}_2^r(E, \mathcal{J}_1, \epsilon) := \frac{1}{2\pi} \int_\sigma^{\sigma+T} \tilde{\mathcal{Y}}_1(\tau, E, \epsilon) d\tilde{\mathcal{Y}}_2(\tau, E, \epsilon) - \frac{1}{2\pi} \tilde{\mathcal{Y}}_2(\sigma, E, \epsilon)(\boldsymbol{d} \cdot \boldsymbol{a})_1 - \frac{1}{4\pi} (\boldsymbol{d} \cdot \boldsymbol{a})_1 (\boldsymbol{d} \cdot \boldsymbol{a})_2, \quad (1.31)$$

where σ is an arbitrary number.

For contractible trajectories on the torus (and closed trajectories on the plane), $2\pi\mathcal{J}_2^r$ is the oriented area of the domain bounded by the trajectory, see Fig. 1.5. It is easy to see that $\mathcal{J}_2^r = 0$ for the extremum points. One can also see that, for $\mathcal{J}_1$ and ϵ fixed, $\mathcal{J}_2$ is an increasing function of E. We require that the family of trajectories corresponding to the same edge and to the same index $\boldsymbol{l}$ depends on $\mathcal{J}_2^r$ continuously.

For non-contractible trajectories on the torus (and open trajectories on the plane), the variable $\mathcal{I}_2^r$ has a different interpretation. Denote by $L_{\boldsymbol{d}}$ the straight line given by

$$L_{\boldsymbol{d}} = \{\kappa(\boldsymbol{d}\cdot\boldsymbol{a}), \quad \kappa\in\mathbb{R}\}, \qquad \boldsymbol{d}\cdot\boldsymbol{a} := d_1\boldsymbol{a}^1 + d_1\boldsymbol{a}^2,$$

where $\boldsymbol{d} = \boldsymbol{d}(\mathcal{I}_1,\epsilon)$ is the corresponding drift vector. Let us fix $\sigma\in\mathbb{R}$, and denote by $\boldsymbol{\pi}(\sigma,E,\epsilon)$ and $\boldsymbol{\pi}(\sigma+T,E,\epsilon)$ the orthogonal projections of the points $\tilde{\boldsymbol{\mathfrak{Y}}}(\sigma,E,\epsilon)$ and $\tilde{\boldsymbol{\mathfrak{Y}}}(\sigma+T,E,\epsilon)$ onto $L_{\boldsymbol{d}}$ (here T is the period of the corresponding trajectory on the torus), then $2\pi\mathcal{I}_2^r$ is the oriented area of the curve-linear trapezium formed by the segments $[\tilde{\boldsymbol{\mathfrak{Y}}}(\sigma,E,\epsilon),\boldsymbol{\pi}(\sigma,E,\epsilon)]$, $[\boldsymbol{\pi}(\sigma,E,\epsilon),\boldsymbol{\pi}(\sigma+T,E,\epsilon)]$, $[\tilde{\boldsymbol{\mathfrak{Y}}}(\sigma+T,E,\epsilon),\boldsymbol{\pi}(\sigma+T,E,\epsilon)]$, and by the part of the trajectory between the points $\tilde{\boldsymbol{\mathfrak{Y}}}(\sigma,E,\epsilon)$ and $\tilde{\boldsymbol{\mathfrak{Y}}}(\sigma+T,E,\epsilon)$, see Fig. 1.6. It is clear, that $\mathcal{I}_2^r$ is defined only up to a_{22}; as we will see, this non-uniqueness plays no essential role in our further considerations. Sometimes, nevertheless, we indicate the dependence of $\mathcal{I}_2^r$ on k and write it as $\mathcal{I}_2^{r,k}$, such that

$$\mathcal{I}_2^{r,k} = \mathcal{I}_2^{r,0} + ka_{22}.$$

We require again that $\mathcal{I}_2^{r,k}$ changes continuously on each edge; then $\mathcal{I}_2^r$ becomes (like on the finite motion edges) a monotone function of the energy.

Finally, we see that any trajectory of (1.28) on the plane is uniquely defined by the value $\mathcal{I}_1$, the index r of the corresponding edge of the Reeb graph $G(\mathcal{I}_1)$, the corresponding value of $\mathcal{I}_2^r$, and by the index $\boldsymbol{l}$ or k of this trajectory among all the trajectories corresponding to the same edge and action; we write this as $\tilde{\boldsymbol{\mathfrak{Y}}}^{r,\boldsymbol{l}/k}(t,\mathcal{I}_1,\mathcal{I}_2^r,\epsilon)$.

Inverting the dependence of the action $\mathcal{I}_2^r$ on the energy, one obtains the expression of the averaged Hamiltonian $\mathcal{H}$ through the action variables $\mathcal{I}_1$ and $\mathcal{I}_2^r$:

$$\mathcal{H} = \mathcal{H}^r(\mathcal{I}_1,\mathcal{I}_2^r,\epsilon). \tag{1.32}$$

The analytic form of this expression depends on the corresponding edge of the Reeb graph.

1.3.4 Reeb surface

As the Reeb graph is defined as a topological space, it has no preferred direction or orientation. Let us realize this graph as a subspace in the three-dimensional

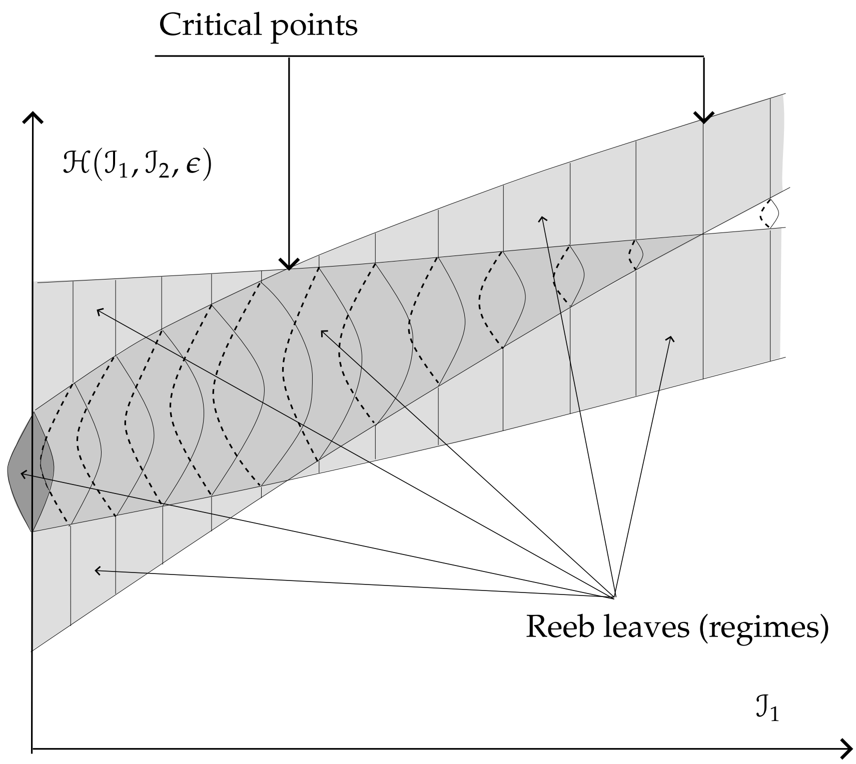

Figure 1.7: An example of the Reeb surface

space $\mathbb{R}^3_{w_1,w_2,w_2}$ in the following manner: we assume that $G(\mathcal{I}_1)$ belongs to the plane $w_1 = \mathcal{I}_1$, and the points of the graph corresponding to the level set $\mathcal{H} = E$ lie in the plane $w_2 = E$. There is still an arbitrariness in positions of the graph relative to the w_3-planes, we require only that $G(\mathcal{I}_1)$ changes continuously with respect to $\mathcal{I}_1$. For some values $\mathcal{I}_1 = \mathcal{I}_1^0$ the function $\mathcal{H}$ may not be a Morse function; in this case we define the Reeb graph as the limit of $G(\mathcal{I}_1)$ as $\mathcal{I}_1 \to \mathcal{I}_1^0$. Under these conditions the edges of the Reeb graphs are transformed continuously as $\mathcal{I}_1$ runs through $[0, +\infty)$; each edge traces out a surface, and we refer to all such surfaces as *Reeb leaves*. The set of the Reeb leaves is countable, we enumerate them also by the index $r \in \mathbb{N}$ and denote the leaf with the index r by $\mathcal{M}^r$. All these leaves are glued together along the lines formed by the branching points of the Reeb graphs (therefore, the set thus obtained has the structure of a stratified space [75]). We call this surface the *Reeb surface* of the Hamiltonian $\mathcal{H}$. A simple example of a Reeb surface is shown in Fig. 1.7 (more examples will be given below).

1.3.5 Description of invariant manifolds

We return to the Hamiltonian $\mathcal{H}$. Its invariant manifolds can be described as follows:

$$\mathcal{P} = \sqrt{2\mathcal{I}_1}\cos\Phi_1, \quad \mathcal{Q} = \sqrt{2\mathcal{I}_1}\sin\Phi_1, \quad \mathfrak{Y} = \tilde{\mathfrak{Y}}^{r,l/k}(\frac{\Phi_2}{2\pi}T, \mathcal{I}_1, \mathcal{I}_2^r, \epsilon), \\ \Phi_1,\ \Phi_2 \in \mathbb{R}. \tag{1.33}$$

Denote the manifold given by (1.33) as $\Lambda^r_{l/k}(\mathcal{I}_1, \mathcal{I}^r, \epsilon)$.

The type of this manifold depends on the type of the trajectory $\tilde{\mathfrak{Y}}^{r,l/k}$, more precisely,

- the manifold $\Lambda^r_l(\mathfrak{I}, \epsilon)$ is a point, if $\mathcal{I}_1 = 0$ and $\tilde{\mathfrak{Y}}^{r,l}$ is a point; we distinguish between the following two cases:
  - $\tilde{\mathfrak{Y}}$ is an extremum point (in other words, $\mathcal{I}_1 = 0$ and $\mathcal{I}_2^r = 0$),
  - $\tilde{\mathfrak{Y}}$ is a saddle point;
- the manifold Λ^r_l is a closed curve, if
  - $\mathcal{I}_1 = 0$ and $\tilde{\mathfrak{Y}}^{r,l}$ is a closed trajectory,

  or
  - $\mathcal{I}_1 \neq 0$ but $\tilde{\mathfrak{Y}}^{r,l}$ is a point;
- the manifold Λ^r_k is an open curve, if $\mathcal{I}_1 = 0$ and $\tilde{\mathfrak{Y}}^{r,k}$ is an open trajectory;
- the manifold Λ^r_l is a two-dimensional Lagrangian manifold diffeomorphic to the two-dimensional torus $\mathbb{T}^2$ if $\mathcal{I}_1 \neq 0$ and $\tilde{\mathfrak{Y}}^{r,l}$ is a closed trajectory;
- the manifold Λ^r_k is a two-dimensional Lagrangian manifold diffeomorphic to the two-dimensional cylinder $\mathbb{R} \times \mathbb{C}^1$ if $\mathcal{I}_1 \neq 0$ and $\tilde{\mathfrak{Y}}^{r,k}$ is an open trajectory;
- the manifold $\Lambda^r_{l/k}$ is a singular (non-closed) manifold otherwise.

This classification is illustrated in Fig. 1.8.

Therefore, the type of an invariant manifold depends on the index r of the Reeb leaf and on the action variables $\mathcal{I}_1$ and $\mathcal{I}_2^r$. Manifolds corresponding to the same leaf, with index r, depend on the action variables $\mathcal{I}_1$ and $\mathcal{I}_2$ smoothly, and their

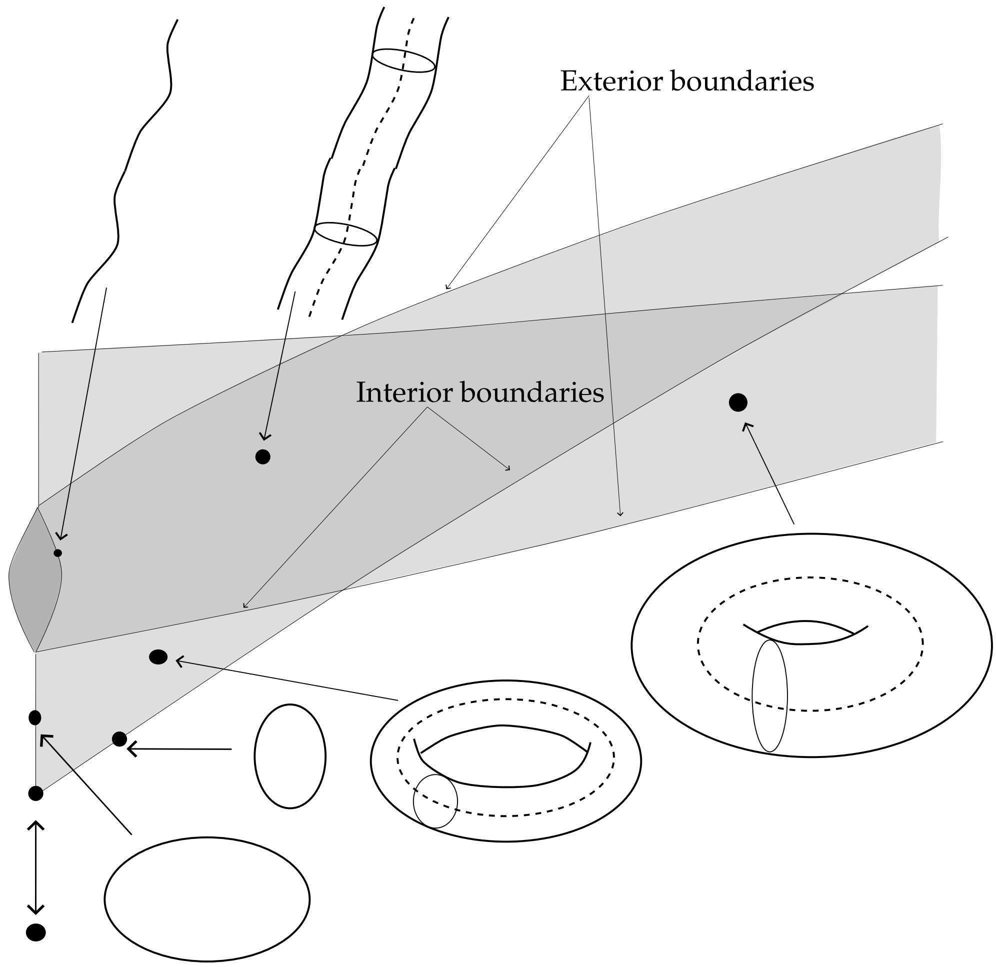

Figure 1.8: An example of the correspondence between the Reeb surface and invariant manifolds of the averaged Hamiltonian

type does not change in the interior of each Reeb leaf. In particular, such manifolds have equal drift vectors; we denote these drift vector now as $\pm \boldsymbol{d}^r$. Based on this observation, the Reeb leaves will also be called *regimes*. The regimes fall naturally into two classes: those of *finite motion* and those of *infinite motion* (their definition is obvious).

The curves formed by the end and branching points of the graphs $G(\mathcal{I}_1)$ will be called *boundaries*. We distinguish between *exterior* and *interior* boundaries: exterior boundaries are formed by the end points of the Reeb graphs (i.e., correspond to local extremum points of $\mathcal{H}$), interior boundaries are formed by branching points of the Reeb graphs (i. e., correspond to saddle points of $\mathcal{H}$). It is also reasonable to consider the graph $G(0)$ as the *left boundary* of the Reeb surface.

Let us return now to the four-dimensional phase space $\mathbb{R}^4_{\boldsymbol{p},\boldsymbol{x}}$. Substituting (1.33) into (1.12) and then into (1.8) we obtain the expression for $\Lambda^r_{\boldsymbol{l}/k}$. In the coordinates $(\boldsymbol{p}, \boldsymbol{x})$ their numbering looks as follows. If a two-dimensional torus / closed curve / point $\Lambda^r_{(0,0)}(\boldsymbol{\mathfrak{I}}, \epsilon)$ is given by the equations

$$\boldsymbol{p} = \boldsymbol{P}^r(\boldsymbol{\Phi}, \boldsymbol{\mathfrak{I}}, \epsilon), \quad \boldsymbol{x} = \boldsymbol{X}^r(\boldsymbol{\Phi}, \boldsymbol{\mathfrak{I}}, \epsilon), \quad \boldsymbol{\Phi} = (\Phi_1, \Phi_2) \in \mathbb{R}^2, \tag{1.34}$$

then the corresponding torus / curve/ point $\Lambda^r_{\boldsymbol{l}}(\boldsymbol{\mathfrak{I}}, \epsilon)$ is given by

$$\begin{aligned} p_1 &= P_1^{r,\boldsymbol{l}}(\boldsymbol{\Phi}, \boldsymbol{\mathfrak{I}}, \epsilon) := P_1^r(\boldsymbol{\Phi}, \boldsymbol{\mathfrak{I}}, \epsilon) - (\boldsymbol{l} \cdot \boldsymbol{a})_2, \\ p_2 &= P_2^{r,\boldsymbol{l}}(\boldsymbol{\Phi}, \boldsymbol{\mathfrak{I}}, \epsilon) := P_2^r(\boldsymbol{\Phi}, \boldsymbol{\mathfrak{I}}, \epsilon), \\ \boldsymbol{x} &= \boldsymbol{X}^{r,\boldsymbol{l}}(\boldsymbol{\Phi}, \boldsymbol{\mathfrak{I}}, \epsilon) := \boldsymbol{X}^r(\boldsymbol{\Phi}, \boldsymbol{\mathfrak{I}}, \epsilon) + \boldsymbol{l} \cdot \boldsymbol{a}. \end{aligned} \tag{1.35}$$

If a two-dimensional cylinder / open curve $\Lambda^r_0(\boldsymbol{\mathfrak{I}}, \epsilon)$ is given by (1.34), then the corresponding cylinder / curve $\Lambda^r_k(\boldsymbol{\mathfrak{I}}, \epsilon)$ is given by

$$\begin{aligned} p_1 &= P_1^{r,k}(\boldsymbol{\Phi}, \boldsymbol{\mathfrak{I}}, \epsilon) := P_1^r(\boldsymbol{\Phi}, \boldsymbol{\mathfrak{I}}, \epsilon) + k\big((J_2 \boldsymbol{f}) \cdot \boldsymbol{a}\big)_2, \\ p_2 &= P_2^{r,k}(\boldsymbol{\Phi}, \boldsymbol{\mathfrak{I}}, \epsilon) := P_2^r(\boldsymbol{\Phi}, \boldsymbol{\mathfrak{I}}, \epsilon), \\ \boldsymbol{x} &= \boldsymbol{X}^{r,k}(\boldsymbol{\Phi}, \boldsymbol{\mathfrak{I}}, \epsilon) := \boldsymbol{X}^r(\boldsymbol{\Phi}, \boldsymbol{\mathfrak{I}}, \epsilon) - k(J_2 \boldsymbol{f}) \cdot \boldsymbol{a}. \end{aligned} \tag{1.36}$$

The variables $\mathcal{I}_1$ and $\mathcal{I}_2^r$ can be calculated directly on $\Lambda^r_{\boldsymbol{l}/k}$:

Proposition 1.15. *Fix some point*

$$\boldsymbol{r}^0 = (\boldsymbol{p}, \boldsymbol{x}) = \big(\boldsymbol{P}^{r,\boldsymbol{l}/k}(\boldsymbol{\Phi}^0, \boldsymbol{\mathfrak{I}}, \epsilon), \boldsymbol{X}^{r,\boldsymbol{l}/k}(\boldsymbol{\Phi}^0, \boldsymbol{\mathfrak{I}}, \epsilon)\big) \in \Lambda^r_{\boldsymbol{l}/k}(\mathcal{I}_1, \mathcal{I}_2, \epsilon),$$

and put

$$\begin{aligned} \gamma_1 &= \Big\{\big(\boldsymbol{P}^{r,\boldsymbol{l}/k}(\Phi_1, \Phi_2^0, \boldsymbol{\mathfrak{I}}, \epsilon), \boldsymbol{X}^{r,\boldsymbol{l}/k}(\Phi_1, \Phi_2^0, \boldsymbol{\mathfrak{I}}, \epsilon)\big),\ \Phi_1 \in [0, 2\pi]\Big\}, \\ \gamma_2 &= \Big\{\big(\boldsymbol{P}^{r,\boldsymbol{l}/k}(\Phi_1^0, \Phi_2, \boldsymbol{\mathfrak{I}}, \epsilon), \boldsymbol{X}^{r,\boldsymbol{l}/k}(\Phi_1^0, \Phi_2, \boldsymbol{\mathfrak{I}}, \epsilon)\big),\ \Phi_2 \in [\Phi_2^0, \Phi_2^0 + 2\pi]\Big\}; \end{aligned}$$

then

$$\mathcal{I}_1 = \frac{1}{2}\int_{\gamma_1} \langle \boldsymbol{p} | d\boldsymbol{x} \rangle, \tag{1.37}$$

$$\mathcal{I}_2^r = \frac{1}{2\pi}\Big(\int_{\gamma_2} \langle \boldsymbol{p} | d\boldsymbol{x} \rangle + (\boldsymbol{d}^r \cdot \boldsymbol{a})_2 x_1 + \frac{1}{2}(\boldsymbol{d}^r \cdot \boldsymbol{a})_1 (\boldsymbol{d}^r \cdot \boldsymbol{a})_2 \Big). \tag{1.38}$$

Proof. Eq. (1.37) for any $\boldsymbol{d}$ and (1.38) for $\boldsymbol{d}^r = 0$ follow directly from the following well-known fact (the Poincaré-Cartan integral invariant): the integral

$$\oint_\gamma \langle \boldsymbol{p} | d\boldsymbol{x} \rangle,$$

where γ is any cycle, is preserved under canonical transformations, see, for example, [4, §44]. Let us prove (1.38) for $\boldsymbol{d}^r \neq 0$.

Denote by $S = \mathcal{P}Q + \mathcal{Y}_1 y_2 + \epsilon s(\mathcal{P}, Q, \mathcal{Y}_1, y_2, \epsilon)$ the generating function of the transformation $(P, y_1, Q, y_2) \mapsto (\mathcal{P}, \mathcal{Y}_1, \mathcal{Q}, \mathcal{Y}_2)$, such that (1.18) holds. Then we have the following equalities:

$$\begin{aligned}\int_{\gamma_2} \langle \boldsymbol{p} | d\boldsymbol{x} \rangle &= -\int_{\gamma_2} (Q\,dy_2 + y_2 dQ + y_2 dy_1 + Q\,dP) \\ &= -(PQ + Qy_2 + y_1 y_2)\Big|_{\gamma_2} + \int_{\gamma_2} \Big(y_1 dy_2 + P\,dQ\Big)\end{aligned}$$

(now we use (1.18) and the equality $PQ|_{\gamma_2} = 0$)

$$= -(Qy_2 + y_1 y_2)\Big|_{\gamma_2} + \int_{\gamma_2} \Big(\mathcal{P} + \epsilon \frac{\partial s}{\partial Q}\Big) dQ + \int_{\gamma_2} \Big(\mathcal{Y}_1 + \epsilon \frac{\partial s}{\partial y_2}\Big) dy_2$$

(use $\mathcal{P}Q|_{\gamma_2} = 0$)

$$= -(Qy_2 + y_1 y_2)\Big|_{\gamma_2} + \epsilon \int_{\gamma_2} \Big(\frac{\partial s}{\partial Q} dQ + \frac{\partial s}{\partial y_2} dy_2\Big) - \int_{\gamma_2} Q\,d\mathcal{P} - \int_{\gamma_2} y_2 d\mathcal{Y}_1$$

(again use (1.18))

$$\begin{aligned}&= -(Qy_2 + y_1 y_2 - \mathcal{Y}_1 y_2)\Big|_{\gamma_2} + \epsilon \int_{\gamma_2} \Big(\frac{\partial s}{\partial Q} dQ + \frac{\partial s}{\partial y_2} dy_2 + \frac{\partial s}{\partial \mathcal{P}} d\mathcal{P} + \frac{\partial s}{\partial \mathcal{Y}_1} d\mathcal{Y}_1\Big) \\ &\quad + \int_{\gamma_2} \mathcal{Q}\,d\mathcal{P} + \int_{\gamma_2} \mathcal{Y}_2 d\mathcal{Y}_1\end{aligned}$$

($\int_{\gamma_2} \mathcal{Q}\,d\mathcal{P} = 0$ because $\mathcal{P}|_{\gamma_2} = \text{const}$)

$$= -(Qy_2 + y_1 y_2 - \mathcal{Y}_1 y_2 + \mathcal{Y}_1 \mathcal{Y}_2 + s)\Big|_{\gamma_2} + \int_{\gamma_2} \mathcal{Y}_1 d\mathcal{Y}_2$$

(s is periodical in Φ_2, therefore, $s|_{\gamma_2} = 0$)

$$
\begin{aligned}
&= -\left(Qy_2 + y_1y_2 - \mathcal{Y}_1y_2 + \mathcal{Y}_1\mathcal{Y}_2\right)\Big|_{\gamma_2} + 2\pi\mathcal{I}_2 + (\boldsymbol{d}\cdot\boldsymbol{a})_1\mathcal{Y}_2(\Phi_2^0) \\
&\qquad + \frac{1}{2}(\boldsymbol{d}\cdot\boldsymbol{a})_1(\boldsymbol{d}\cdot\boldsymbol{a})_2 \\
&= 2\pi\mathcal{I}_2 - Q(\Phi_2^0)(\boldsymbol{d}\cdot\boldsymbol{a})_2 - (\boldsymbol{d}\cdot\boldsymbol{a})_2y_1(\Phi_2^0) - (\boldsymbol{d}\cdot\boldsymbol{a})_1y_2(\Phi_2^0) - (\boldsymbol{d}\cdot\boldsymbol{a})_1(\boldsymbol{d}\cdot\boldsymbol{a})_2 \\
&\quad + (\boldsymbol{d}\cdot\boldsymbol{a})_1y_2(\Phi_2^0) + \frac{1}{2}(\boldsymbol{d}\cdot\boldsymbol{a})_1(\boldsymbol{d}\cdot\boldsymbol{a})_2 \\
&= 2\pi\mathcal{I}_2 - (\boldsymbol{d}\cdot\boldsymbol{a})_2(Q + y_1)(\Phi_2^0) - \frac{1}{2}(\boldsymbol{d}\cdot\boldsymbol{a})_1(\boldsymbol{d}\cdot\boldsymbol{a})_2 \\
&= 2\pi\mathcal{I}_2 - (\boldsymbol{d}\cdot\boldsymbol{a})_2x_1 - \frac{1}{2}(\boldsymbol{d}\cdot\boldsymbol{a})_1(\boldsymbol{d}\cdot\boldsymbol{a})_2.
\end{aligned}
$$

The proposition is proved. □

1.4 Examples

In this section, we describe the Reeb surfaces for some particular potentials v. We perform only the averaging up to $O(\epsilon^2)$ to obtain explicit formulas, i. e., we construct only the Hamiltonian $\bar{H}$ in Propositions 1.12 and 1.13. Clearly, $\bar{H}$ can be obtained from (1.10) by applying Proposition 1.11.

1.4.1 The Harper potential

In this subsection, we consider the case $v(x_1, x_2) = A\cos x_1 + B\cos\beta x_1$, where A, B, and β are positive numbers. (The relation to the Harper equation will be explained later, see Section 3.10.)

The averaged Hamiltonian $\bar{H}$ has the following form:

$$
\bar{H}(\mathcal{I}_1, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon) = \mathcal{I}_1 + \epsilon\Big(AJ_0(\sqrt{2\mathcal{I}_1})\cos x_1 + BJ_0(\beta\sqrt{2\mathcal{I}_1})\cos\beta x_2\Big), \quad (1.39)
$$

where J_0 denotes the Bessel function of order zero. $\bar{H}$ is a simple Morse function if $|AJ_0(\sqrt{2\mathcal{I}_1})| \neq |BJ_0(\beta\sqrt{2\mathcal{I}_1})|$, $AJ_0(\sqrt{2\mathcal{I}_1}) \neq 0$, and $BJ_0(\beta\sqrt{2\mathcal{I}_1}) \neq 0$. In this case the level curves and the Reeb graph have the structure shown in the upper part of Fig. 1.3 (the level curves can be rotated by $\pi/2$). This function has one maximum, one minimum, and two saddle points; such functions are called *minimal* Morse functions. The drift vector is equal to $(\pm 1, 0)$ if $|AJ_0(\sqrt{2\mathcal{I}_1})| > |BJ_0(\beta\sqrt{2\mathcal{I}_1})|$ and is equal to $(0, \pm 1)$ if $|AJ_0(\sqrt{2\mathcal{I}_1})| < |BJ_0(\beta\sqrt{2\mathcal{I}_1})|$.

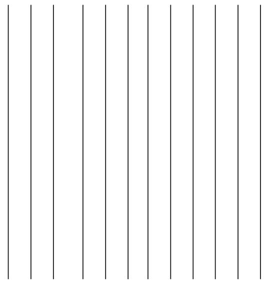

Figure 1.9: Level curves: degenerate case I

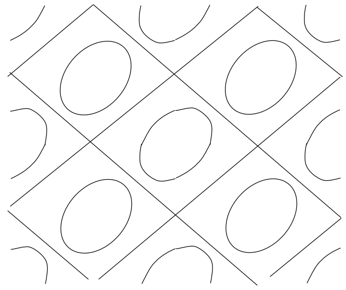

Figure 1.10: Level curves: degenerate case II

If one of the coefficients $AJ_0(\sqrt{2\mathcal{I}_1})$ or $BJ_0(\beta\sqrt{2\mathcal{I}_1})$ is equal to zero (degenerate case I), the function $\bar{H}$ depends only on $\mathcal{Y}_1$ or on $\mathcal{Y}_2$; it is not a Morse function, all its level curves are straight lines, see Fig. 1.9.

If both coefficients are non-zero, but their absolute values are equal (degenerate case II), then $\bar{H}$ is a complex Morse function; all its trajectories except separatrices are closed, see Fig. 1.10.

Therefore, we have the Reeb surface shown in Fig. 1.7. The points $\mathcal{I}_1$ in which $\mathcal{H}$ is not a simple Morse function we call *critical*; as $\mathcal{I}_1$ crosses a critical value, the topological type of $\mathcal{H}$ is changed; in the present case only the drift vector changes, but in more complicated situations a more fundamental reorganization of the Reeb graph is possible (see below).

Note that if $\beta = 1$, then the Reeb graph always has the same shape, anf the drift vector does not change.

1.4.2 Example of a non-separable potential

The Harper potential provides an example of a separable potentials of $v(x_1, x_2) = v_1(x_1) + v_2(x_2)$. Consider now a simple example of a non-separable potential, $v(x_1, x_2) = A\cos x_1 + B\cos\beta x_2 + C\cos(x_1 + \beta x_2)$, then

$$\bar{H}(\mathcal{I}_1, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon) = \mathcal{I}_1 + \epsilon\big(AJ_0(\sqrt{2\mathcal{I}_1})\cos\mathcal{Y}_1 + BJ_0(\beta\sqrt{2\mathcal{I}_1})\cos\mathcal{Y}_1 + CJ_0(\sqrt{2(1+\beta^2)\mathcal{I}_1})\cos(\mathcal{Y}_1 + \beta\mathcal{Y}_2)\big).$$

Here much more variants of Reeb graphs are possible; the Reeb surface is shown in Fig. 1.11 We see that in this case there are many types of critical points. Obviously, more complicated potentials will display the Reeb surface of higher com-

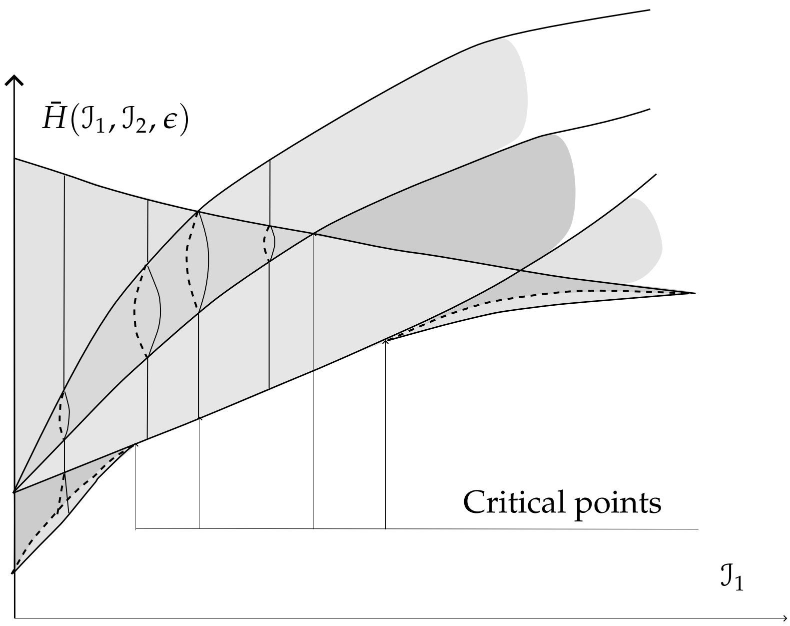

Figure 1.11: The Reeb surface for the potential $v(x_1, x_2) = A\cos x_1 + B\cos\beta x_2 + C\cos(x_1 + \beta x_2)$

plexity.

1.4.3 Square lattice of dots or antidots

In this subsection we consider the case

$$v(x_1, x_2) = A\cos^2\frac{x_1}{2}\cos^2\frac{x_2}{2}. \tag{1.40}$$

This potential is essentially localized near the points $(2\pi m, 2\pi n)$, $m, n \in \mathbb{Z}$; such potentials are used for modelling periodic arrays of quantum dots (if $A < 0$) or antidots (if $A > 0$). To perform averaging, it is more convenient to rewrite v as

$$v(x_1, x_2) = \frac{1}{4}A\big(1 + \cos x_1 + \cos x_2 + \cos x_1 \cos x_2\big), \tag{1.41}$$

then

$$\bar{H}(\mathcal{I}_1, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon) = \mathcal{I}_1 + \frac{1}{4}\epsilon A\Big(1 + J_0(\sqrt{2\mathcal{I}_1})\big(\cos x_1 + \cos x_2\big) + J_0(\sqrt{4\mathcal{I}_1})\cos x_1 \cos x_2\Big), \tag{1.42}$$

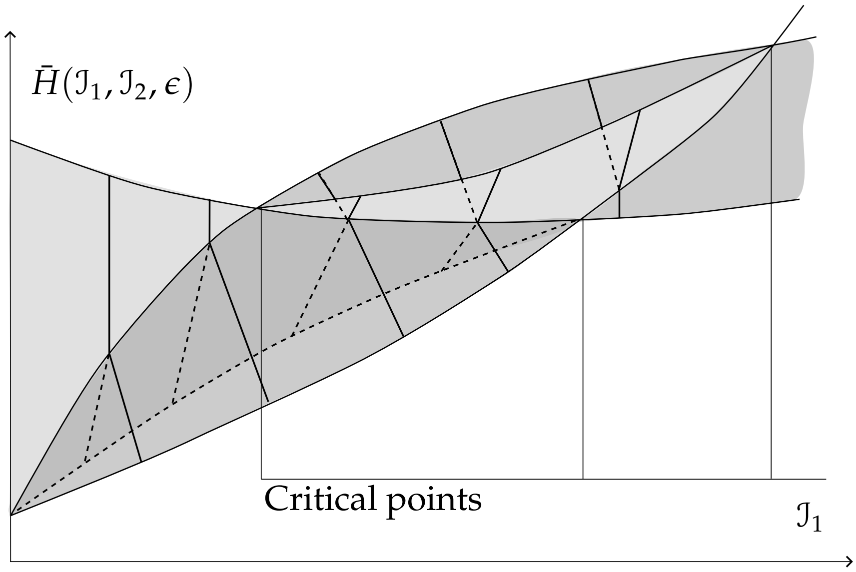

Figure 1.12: The Reeb surface for the dot/antidot potential

and the Reeb surface has the form shown in Fig. 1.12. We have now only finite motion regimes. It is easy to understand that this globally finite motion is provided by the symmetry property of $\mathcal{H}$:

$$\bar{H}(\mathcal{I}_1, \mathcal{Y}_1, \mathcal{Y}_2, \epsilon) \equiv \bar{H}(\mathcal{I}_1, -\mathcal{Y}_2, \mathcal{Y}_1, \epsilon). \tag{1.43}$$

Proposition 1.16. *The Reeb graph corresponding to a function $\bar{H}$ satisfying* (1.43) *contains only finite motion edges.*

Proof. It is enough to prove that all the drift vectors are zero. Let $\boldsymbol{d} = (d_1, d_2)$ be the drift vector of a certain trajectory of $\bar{H}$, then $(-d_2, d_1)$ is also the drift vector of some trajectory. If these vectors are non-zero, then they have to coincide up to sign; this situation is, of course, impossible. □

The property (1.43) can be easily verified without averaging:

Proposition 1.17. *Let the potential v be invariant under rotation: $v(x_1, x_2) \equiv v(-x_2, x_1)$, then the Hamiltonian $\bar{H}$ satisfies* (1.43).

Proof. As it was noted above (see Section 1.2), the Hamiltonian $\bar{H}$ is obtained from the Hamiltonian H (1.10) using Proposition 1.11. The Hamiltonian H satisfies the property $H(P, Q, y_1, y_2, \epsilon) \equiv H(Q, -P, -y_2, y_1, \epsilon)$. From the formulas of Proposition 1.11 it is easy to see that $\bar{H}$ preserves this property. □

Chapter 2

A brief survey of the complex WKB method

In this chapter, we describe very briefly some basic notions and facts of the complex WKB method.

2.1 The Correspondence Principle

2.1.1 The notion of a quasimode

Definition 14 (Quasimode). For a self-adjoint operator $\hat{A}$ acting in a Hilbert space $\mathfrak{H}$ a pair (ψ, E), $\psi \in \mathfrak{H}$, $E \in \mathbb{R}$, is called a *quasimode with error* α if $\|(\hat{A} - E)\psi\| / \|\psi\| \leq \alpha$.

The following proposition explains the importance of quasimodes.

Proposition 2.1 (Lemma 13.1 in [66]). *Let* $\hat{A}$ *be a self-adjoint operator acting in a Hilbert space* $\mathfrak{H}$*, and* (ψ, E) *be a quasimode of* $\hat{A}$ *with error* α*, then*

$$\operatorname{dist}(\operatorname{spec} \hat{A}, E) \leq \alpha. \tag{2.1}$$

Now let H be a classical Hamiltonian; it generates a self-adjoint operator in the following sense:

Definition 15 (The Weyl quantization, see §18.5 in [49]). Let $H = H(\boldsymbol{p}, \boldsymbol{x})$ be a real-valued smooth function semibounded from below. For $h > 0$, the operator

$\hat{H}_h$,

$$\hat{H}_h f(\boldsymbol{x}) = \Big(\frac{1}{2\pi h}\Big)^n \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}\langle x-y|z\rangle} H\big(\boldsymbol{z}, \frac{\boldsymbol{x}+\boldsymbol{y}}{2}\big) f(\boldsymbol{y})\, d\boldsymbol{y}\, d\boldsymbol{z}, \tag{2.2}$$

acting in $L^2(\mathbb{R}^n)$, is called the *Weyl quantization* of H.

Remark. Sometimes the Feynman notation is used:

$$\hat{H}_h = H\Big(-ih \overset{2}{\nabla}, \frac{1}{2}(\overset{1}{\boldsymbol{x}} + \overset{3}{\boldsymbol{x}})\Big),$$

where the over-line indices indicate the order of the action of the corresponding operations [63].

In the context of many problems of physics it is interesting to study the asymptotics of the spectrum for $\hat{H}_h$ as h is small. It follows from Proposition 2.1 that we can construct a $O(h^L)$–approximation to the spectrum if we can solve (in a suitable space) the equation

$$\hat{H}_h \psi(\boldsymbol{x}, h) = E(h)\psi(\boldsymbol{x}, h) + O(h^L). \tag{2.3}$$

In this case the pair $\big(\psi(\boldsymbol{x}, h), E(h)\big)$ is usually called a *spectral series* of the operator $\hat{H}_h$ (roughly speaking, a spectral series is a quasimode depending on the small parameter h); the function ψ is called an *asymptotic eigenfunction* and the number E is called an *asymptotic eigenvalue*. Proposition 2.1 also explains, what kind of results can be obtained using such methods: we can only prove that in some $O(h^L)$-neighborhood of the set of values E in (2.3) there are points of the spectrum. The asymptotic eigenfunctions, however, do not provide similar information about the true eigenfunctions [3].

2.1.2 Structure of semiclassical asymptotics

Semiclassical methods are based on the fundamental *Correspondence Principle* which claims that asymptotic properties of the quantum dynamics defined by the operator $\hat{H}_h$ as $h \to 0$ can be described in terms of the classical dynamics which is described by a Hamiltonian H. Such a correspondence is studied intensively during the whole period of developping quantum theory, see the review In particular, invariant manifolds of H (with some additional properties) can be used for solving (2.3). The corresponding method is known as the method of the *canonical operator*. We now describe the structure of those asymptotic eigenfunctions which can be obtained using the canonical operator method.

Consider first the asymptotic eigenfunctions ψ of $\hat{H}_h$ corresponding to invariant manifolds Λ^n of H of maximal dimension (i. e., to invariant Lagrangian manifolds). Denote by $\pi_x\Lambda^n$ the projection of Λ^n onto the x-subspace, then in $\pi_x\Lambda^n$ one has (locally)

$$\psi = \sum_j \mathcal{F}_j \Big[\varphi_j \exp \frac{i}{h} S_j\Big], \tag{2.4}$$

where the functions φ_j are regular with respect to h, the phase functions S_j are real-valued and do not depend on h, and $\mathcal{F}_j$ are unitary operators (more precisely, it can be the unit operator or a partial Fourier transform, see below). Such asymptotics are usually referred to as *asymptotics with real phases*. The method of the canonical operator does *not* give asymptotics of ψ outside $\pi_x\Lambda^n$. Usually one constructs some smooth continuation of the expression (2.4), such that $\psi = O(h^\infty)$ outside $\pi_x\Lambda^n$, and multiplies it by a smoothing function $e \in C^\infty$ such that $e(x) = 1$ as $x \in \pi_x\Lambda^n$ and $e = 0$ outside some neighborhood of $\pi_x\Lambda^n$. We will *always* have in mind this smoothing procedure.

The asymptotic eigenfunctions corresponding to invariant isotropic (but non-Lagrangian) manifolds Λ^k also have the form (2.4), but the phases S_j are complex-valued functions; nevertheless, one always has $\Im S_j \geq 0$ and $\Im S_j(x) = 0$ iff $x \in \pi_x\Lambda^k$. The functions $\varphi_j(x,h)$ also can be irregular with respect to h. Such asymptotics are called *asymptotics with complex phases*. In particular, the asymptotic eigenfunctions corresponding to invariant manifolds of minimal dimension (i. e., to the rest points of the Hamiltonian) are defined by a purely imagine phase function S. The smoothing procedure described above is also applicable to asymptotics with complex phases.

Therefore, in all these cases the asymptotic eigenfunctions are localized near the projections of the corresponding invariant manifolds onto the x-subspace (these projections are called *classically allowed regions*) in the following sense. Let $\psi(x,h)$ be an asymptotic eigenfunction corresponding to an isotropic manifold Λ, then

$$\psi(x,h) \to 0 \text{ as } h \to 0 \text{ for any } x \notin \pi_x\Lambda.$$

The latter equality is sometimes written as

$$\operatorname{supp}\psi \to \pi_x\Lambda \text{ as } h \to 0.$$

We discuss here only that part of semiclassical methods related to the problem under consideration (the two-dimensional magnetic Schrödinger operator). A more detailed description can be found, for example, in [4,3,2,6,9,8,23,24,58,62,65,64,66,67].

2.2 The real canonical operator

Throughout this section we denote by Λ a closed Lagrangian manifold in $\mathbb{R}^{2n}_{\boldsymbol{p},\boldsymbol{x}}$ without boundary.

2.2.1 Some preliminary constructions

Definition 16 (Canonical charts and focal coordinates). A simply connected domain $\Omega \subset \Lambda$ is called a *canonical chart* in Λ if it can be diffeomorphically projected onto some Lagrangian coordinate subspace λ_I (see Definition 4). If $I \neq \emptyset$, then the coordinates $(\boldsymbol{p}_I, \boldsymbol{x}_{\bar{I}})$ are called *I–focal coordinates* in Ω.

Definition 17 (Canonical atlas). A countable locally finite covering of Λ by canonical charts is called a *canonical atlas* on Λ.

Definition 18 (Critical and non-critical points and charts). A point $\boldsymbol{r}^0 \in \Lambda$ is called *non-critical* if some its neighborhood can be diffeomorphically projected onto $\mathbb{R}^n_{\boldsymbol{x}}$ and is called *critical* otherwise. A canonical chart $\Omega \in \Lambda$ is called *non-critical* if all its points are non-critical and is called *critical* otherwise.

Definition 19 (Inertial index). Let A be a real-valued symmetric matrix. The number of negative eigenvalues of A is called the *inertial index* of A and is denoted as inerdex A.

2.2.2 The pre-canonical operator

Let $d\sigma$ denote the volume on Λ (i. e. a non-vanishing n-form on Λ) and $\boldsymbol{r}^0 \in \Lambda$. Denote

$$S(\boldsymbol{r}^0, \boldsymbol{r}) = \int_{l(\boldsymbol{r}^0, \boldsymbol{r})} \langle \boldsymbol{p} | d\boldsymbol{x} \rangle,$$

where $l(\boldsymbol{r}^0, \boldsymbol{r}) \subset \Lambda$ is a curve between $\boldsymbol{r}^0$ and $\boldsymbol{r}$ (this function is well-defined at least locally, see Proposition 1.6).

Let Ω be a non-critical canonical chart on Λ, then any its $\boldsymbol{r} \in \Lambda$ is uniquely defined by the $\boldsymbol{x}$-component: $\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{x})$. Denote

$$S_{\boldsymbol{r}^0}(\boldsymbol{x}) = S\Big(\boldsymbol{r}^0, \boldsymbol{r}(\boldsymbol{x})\Big).$$

Now we introduce the pre-canonical operator $\mathcal{K}^{\boldsymbol{r}^0}_{h,\Omega} : C^\infty(\Omega) \mapsto C^\infty(\mathbb{R}^n)$ associated with this chart and the point $\boldsymbol{r}^0$ by the formula

$$(\mathcal{K}^{\boldsymbol{r}^0}_{h,\Omega}\psi)(\boldsymbol{x}) = \sqrt{\left|\frac{d\sigma}{d\boldsymbol{x}}\right|}\exp\Big(\frac{i}{h}S_{\boldsymbol{r}^0}(\boldsymbol{x})\Big)\psi(\boldsymbol{r}(\boldsymbol{x})).$$

Let Ω be a critical canonical chart on Λ. Fix some I–focal coordinates $(\boldsymbol{p}_I, \boldsymbol{x}_I)$, $I \neq \emptyset$, then we have $\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{x}_I)$.

In addition to the I-coordinates, let us introduce their "critical" parts. For any n-dimensional vector $\boldsymbol{y}$ and $I = (i_1, \dots, i_l) \subset (1, \dots, n)$, $l \neq 0$, $i_j < i_{j+1}$, we put

$$\boldsymbol{y}^I = (y_{i_1}, \dots, y_{i_l}) \in \mathbb{R}^l.$$

Define

$$S_{\boldsymbol{r}^0,I}(\boldsymbol{x}_I) := S\big(\boldsymbol{r}^0, \boldsymbol{r}(\boldsymbol{x}_I)\big) - \langle \boldsymbol{x}^I(\boldsymbol{x}_I) | \boldsymbol{p}^I\rangle.$$

For $I \neq \emptyset$ we denote by $\mathcal{F}^{-1}_{h,I}$ the I–partial inverse h-Fourier transform:

$$(\mathcal{F}^{-1}_{h,\gamma,I}\psi)(\boldsymbol{x}) = \Big(-\frac{1}{2\pi i h}\Big)^{|I|/2}\int_{\mathbb{R}^{|I|}}\exp\Big(\frac{i}{h}\langle \boldsymbol{p}^I | \boldsymbol{x}^I\rangle\Big)\psi(\boldsymbol{p}_I)d\boldsymbol{p}^I.$$

Introduce the pre-canonical operator $\mathcal{K}^{\boldsymbol{r}^0,\gamma}_{h,\Omega}$ associated with the point $\boldsymbol{r}^0$, the chart Ω, and the I–focal coordinates by the formula [66, §1.6]

$$(\mathcal{K}^{\boldsymbol{r}^0,I}_{h,\Omega}\psi)(\boldsymbol{x}) = \mathcal{F}^{-1}_{h,I}\sqrt{\left|\frac{d\sigma}{d\boldsymbol{x}_I}\right|}\exp\Big(\frac{i}{h}S_{\boldsymbol{r}^0,I}(\boldsymbol{x}_I)\Big)\psi(\boldsymbol{r}(\boldsymbol{x}_I)).$$

Proposition 2.2 (Unitarity of the pre-canonical operator). *The pre-canonical operator is unitary:* $\|\mathcal{K}^{\boldsymbol{r}^0,I}_{h,\Omega}\psi\|_{L^2(\mathbb{R}^n)} = \|\psi\|_{L^2(\Lambda,d\sigma)}$.

2.2.3 The Maslov index

Definition 20 (The Maslov index of a chain of canonical charts). (A) Let Ω' and Ω'' be canonical charts on Λ with focal coordinates I' and I'' respectively; assume that their intersection is non-empty, then the index of the pair (Ω', Ω'') is defined as

$$\operatorname{Ind}(\Omega', \Omega'') = \operatorname{inerdex} \frac{\partial \boldsymbol{x}^{I'}}{\partial \boldsymbol{p}^{I'}}(\boldsymbol{r}^0) - \operatorname{inerdex} \frac{\partial \boldsymbol{x}^{I''}}{\partial \boldsymbol{p}^{I''}}(\boldsymbol{r}^0),$$

where $\boldsymbol{r}^0$ is an arbitrary point from $\Omega' \cap \Omega''$; this number does not depend on $\boldsymbol{r}^0$ [66, §7.1].

(B) Let $\{\Omega_j,\ j \in \{0, \dots, s\}\}$ be a chain of canonical charts such that Ω_{j-1} and Ω_j for any j satisfy the conditions of (A), then the index of this chain is defined via additivity:

$$\operatorname{Ind}\{\Omega_j\} = \sum_{j=1}^{s} \operatorname{Ind}(\Omega_{j-1}, \Omega_j).$$

Definition 21 (The Maslov index of a cycle). Assume that Λ is given by

$$\Lambda = \Big\{ \boldsymbol{r}(\boldsymbol{\alpha}) = \big(\boldsymbol{p}(\boldsymbol{\alpha}), \boldsymbol{x}(\boldsymbol{\alpha})\big),\ \boldsymbol{\alpha} \in \mathbb{R}^n \Big\}.$$

For $\delta > 0$ denote

$$J_\delta(\boldsymbol{\alpha}) = \det\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\alpha}} - i\delta \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{\alpha}} \right).$$

The Maslov index $\operatorname{Ind}\gamma$ of a cycle γ on Λ is defined as

$$\operatorname{Ind}\gamma = \frac{1}{\pi} \lim_{\delta \to 0+} \arg J_\delta \Big|_{\gamma},$$

where arg denotes any continuous branch.

The meaning of the Maslov index becomes especially clear if we restrict our attention to Lagrangian manifolds of generic position.

Definition 22 (Lagrangian manifold of generic position and the cycle of singularities). Let $\Lambda \in \mathbb{R}^n$ be a Lagrangian manifold given by

$$\Lambda = \Big\{ \boldsymbol{r}(\boldsymbol{\alpha}) = \big(\boldsymbol{p}(\boldsymbol{\alpha}), \boldsymbol{x}(\boldsymbol{\alpha})\big),\ \boldsymbol{\alpha} \in \mathbb{R}^n \Big\}.$$

Denote by $\Sigma(\Lambda)$ the set of points in which the jacobian $\partial \boldsymbol{x}/\partial \boldsymbol{\alpha}$ vanishes (this set is called the *the cycle of singularities*). Λ is called *a lagragian manifold of generic position* if $\Sigma(\Lambda)$ is the union of a manifold $\Sigma'(\Lambda)$ of dimension $n-1$ and the dimension of the boundary $\Sigma(\Lambda) \backslash \Sigma'(\Lambda)$ is less than $n-2$.

It is known [2] that any Lagrangian manifold can be reduced to a Lagrangian manifold of generic position by a small perturbation.

Proposition 2.3 (see [2] and Proposition 7.5 in [66]). *The Maslov index* $\mod 4$ *of a cycle on a Lagrangian manifold of generic position is a homotopic invariant.*

On a Lagrangian manifold of generic position the Maslov index of a cycle can be defined as through the number of intersections of the cycle with the cycle of singularities $\Sigma(\Lambda)$ We do not discuss here this and similar questions (one can find discussions in [2], [66, §7]) and use our definition which is suitable for calculations.

Let us illustrate the calculation of the Maslov index for a circle on the plane (and hence for any smooth closed curve without self-intersections). Consider the circle γ given by $p + ix = e^{i\varphi}$ on the plane $\mathbb{R}^2_{p,x}$, then $J_\delta = \cos\varphi + i\delta\sin\varphi$, and

$$\arg J_\delta\Big|_\gamma = 2\pi;$$

therefore, $\operatorname{Ind}\gamma = 2$.

2.2.4 The canonical operator

Let $d\sigma$ denote the volume on Λ. Let us fix a certain non-critical chart Ω_0, a point $\boldsymbol{r}^0$, a canonical atlas (Ω_j) on Λ and choose focal coordinates $(\boldsymbol{p}_{I_j}, \boldsymbol{x}_{I_j})$ in each chart Ω_j. Assume additionally that all the charts of the canonical atlas are bounded. Let us choose some partition of unity (e_j):

$$e_j \in C_0^\infty(\Omega_j), \quad 0 \le e_j \le 1, \quad \sum_j e_j \equiv 1.$$

Let us introduce the canonical operator

$$\mathcal{K}_{h,\Lambda}^{\boldsymbol{r}^0,(\Omega_j),(I_j),(e_j)} : C^\infty(\Lambda) \mapsto C^\infty(\mathbb{R}^n)$$

associated with the above choice of a canonical atlas, focal coordinates, and a partition of unity by the formula

$$\left(\mathcal{K}_{h,\Lambda}^{\boldsymbol{r}^0,(\Omega_j),(I_j),(e_j)}\varphi\right)(\boldsymbol{x}) = \sum_j \exp\left(-\frac{i\pi}{2}\gamma_j\right)\mathcal{K}_{h,\Omega_j}^{\boldsymbol{r}^0,I_j}(e_j\varphi), \tag{2.5}$$

where γ_j is the index of a chain of charts joining Ω_0 and Ω_j.

Remark. If Λ is a non-compact Lagrangian manifold, then it may not be possible to define the canonical operator on the whole space $C^\infty(\Lambda)$. More precisely, the canonical operator can be defined on the whole Λ iff for each bounded subset $K \subset \pi_x\Lambda$ the set $\Lambda \cap \pi_x^{-1}K$ is also bounded [22, §2].

Remark. After a series of transformations the expression for the canonical operator can be given in terms of the Fourier integral operators, see [50, Chapter 29], [67, Chapter 5], [68].

It is easy to see that (2.5) defines, generally speaking, a multi-valued function, because, for example, the integral

$$\oint_{r^0}^{r} \langle \boldsymbol{p} | d\boldsymbol{x} \rangle$$

may depend on the choice of a path between $\boldsymbol{r}^0$ and $\boldsymbol{r}$; the numbers γ_j may also depend on the choice of a chain of charts.

Proposition 2.4 (Theorem 8.1 in [66]). *Up to $O(h)$, the operator $\mathcal{K}_{h,\Lambda}^{\boldsymbol{r}^0,(\Omega_j),(I_j),(e_j)}$ does not depend on the choice of a canonical atlas, focal coordinates and a partition of unity iff*

$$\oint_{\gamma} \langle \boldsymbol{p} | d\boldsymbol{x} \rangle = 0 \text{ and } \operatorname{Ind} \gamma = 0$$

for any cycle γ on Λ.

Let us emphasize another simple and useful property of the canonical operator:

Proposition 2.5 (Locality of the canonical operator). *Let Ω be a domain in $\mathbb{R}^n$, and $\phi, \psi \in C^\infty(\Lambda)$ such that $\phi|_{\Lambda\cap\pi_x^{-1}\Omega} = \psi|_{\Lambda\cap\pi_x^{-1}\Omega}$, then $\mathcal{K}_{h,\Lambda}\phi|_\Omega = \mathcal{K}_{h,\Lambda}\psi|_\Omega$.*

2.2.5 The commutation formula

Proposition 2.6 (Theorem 8.4 in [66]). *Fix a point $\boldsymbol{r}^0 \in \Lambda$, a canonical atlas (Ω_j), a partition of unity (e_j), and focal coordinates in each chart Ω_j; denote by $\mathcal{K}_{h,\Lambda}$ the corresponding canonical operator.*

Consider in addition a Hamiltonian $H(\boldsymbol{p}, \boldsymbol{x})$ and its Weyl quantization $\hat{H}_h$; assume that Λ is an invariant manifold of H and that on Λ there exists a volume $d\sigma$ invariant under the corresponding flow g_H^t. For any $\varphi \in C_0^\infty(\Lambda)$ there exists a function $\psi \in C_0^\infty(\Lambda)$, $\psi = O(1)$ as $h \to 0$, such that

$$\hat{H}_h \mathcal{K}_{h,\Lambda}\varphi = \mathcal{K}_{h,\Lambda}\Big((H|_\Lambda - ih\frac{d}{dt})\varphi + h^2\psi \Big), \tag{2.6}$$

where

$$\frac{d}{dt}\varphi(\boldsymbol{r}) = \Big\langle \frac{\partial H}{\partial \boldsymbol{p}} \Big| \frac{\partial \varphi}{\partial \boldsymbol{x}} \Big\rangle - \Big\langle \frac{\partial H}{\partial \boldsymbol{x}} \Big| \frac{\partial \varphi}{\partial \boldsymbol{p}} \Big\rangle, \quad \boldsymbol{r} = (\boldsymbol{p}, \boldsymbol{x}), \tag{2.7}$$

i. e., d/dt is the operator of the differentiation along the trajectories of H.

Therefore, up to $O(h^2)$, the canonical operator reduces the action of the operator $\hat{H}_h$ to a first order differential operator on Λ.

2.2.6 Action-angle variables and invariant volume

We see that the formulas for the canonical operator depend on the choice of the volume $d\sigma$ on a Lagrangian manifold. Let us show that there exists a natural choice of an invariant volume connected with action-angle variables.

Let H be a classical Hamiltonian and $(\boldsymbol{I}, \boldsymbol{\varphi})$ be action-angle variables for H, i.e., $H = H(\boldsymbol{I})$. Denote by $\Lambda(\boldsymbol{I}^0)$ the invariant manifold which is given by (1.6). Assume that these manifolds have full dimension n (and then they are Lagrangian), then an invariant volume $d\sigma$ on $\Lambda(\boldsymbol{I}^0)$ may be given by $d\sigma = d\boldsymbol{\varphi}$, and, respectively, in all the formulas for the canonical operator one can put

$$\left|\frac{d\sigma}{d\boldsymbol{x}_I}\right| = \left|\det \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{x}_I}\right|.$$

We *always* use this choice of invariant volume.

2.2.7 The Bohr-Sommerfeld quantization rule

The assumptions of Proposition 2.4 are usually not satisfied. For example, they even do not hold for the invariant manifolds of the harmonic oscillator $H(p, x) = p^2 + x^2$: these invariant manifolds are circles, and their index is not equal to zero.

Proposition 2.7 (The Bohr-Sommerfeld quantization rule, see Theorem 13.3 in [66]). *The canonical operator $\mathcal{K}_{h,\Lambda}$ defines a single-valued function if and only if*

$$\frac{1}{2\pi h} \oint_{\gamma} \langle \boldsymbol{p} | d\boldsymbol{x} \rangle - \frac{\operatorname{Ind} \gamma}{4} \in \mathbb{Z} \tag{2.8}$$

for all non-contractible cycles γ on Λ.

Remark. It is clearly enough to satisfy (2.8) only for basis cycles γ (i. e., those forming a basis in the first homology group of Λ).

Obviously, the condition (2.8) cannot be satisfied for a fixed Lagrangian manifold Λ (excepting the case of simply connected manifold). Usually one has a family of invariant Lagrangian manifolds, and for each h the condition (2.8) selects some their subset.

We illustrate now applications of the commutation formula and of the quantization rules to spectral estimates (cf. [41, §II.7]). Suppose that a Hamiltonian H has a family of invariant manifolds $\Lambda(\boldsymbol{\alpha})$ depending on the parameters $\boldsymbol{\alpha} \in \Omega \subset \mathbb{R}^n$, and that all these manifolds are diffeomorphic to the n-dimensional torus

$$\mathbb{T}^n = \underbrace{\mathbb{C}^1 \times \cdots \times \mathbb{C}^1}_{n \text{ times}}.$$

Suppose, for each fixed h, that there is $\Omega_h \in \Omega$ such that for $\boldsymbol{\alpha} \in \Omega_h$ the manifolds $\Lambda(\boldsymbol{\alpha})$ satisfy (2.8) (clearly, the family of manifolds $\Lambda(\boldsymbol{\alpha})$ has to be large enough), then one can construct one can construct

$$E(\boldsymbol{\alpha}, h) := H\Big|_{\Lambda(\boldsymbol{\alpha})}, \quad \psi(\boldsymbol{x}, \boldsymbol{\alpha}, h) := \mathcal{K}_{\Lambda(\boldsymbol{\alpha})} \cdot 1.$$

By Proposition 2.6,

$$\hat{H}_h \psi(\boldsymbol{x}, \boldsymbol{\alpha}, h) = E(\boldsymbol{\alpha}, h)\psi(\boldsymbol{x}, \boldsymbol{\alpha}, h) + h^2 \phi(\boldsymbol{x}, \boldsymbol{\alpha}, h),$$

where $\phi(\boldsymbol{x}, \boldsymbol{\alpha}, h) \in L^2(\mathbb{R}^n)$, $\|\phi\| = O(1)$.

Now, if

$$\|\psi(\boldsymbol{x}, \boldsymbol{\alpha}, h)\| \geq c > 0 \text{ as } h \to 0, \tag{2.9}$$

then we obtain the estimate

$$\operatorname{dist}\left(E(\boldsymbol{\alpha}, h), \operatorname{spec} \hat{H}_h\right) = O(h^2).$$

The condition (2.9) is satisfied if

$$\left|\oint_\gamma \langle \boldsymbol{p} | d\boldsymbol{x} \rangle\right| \geq a(\gamma) > 0 \text{ as } h \to 0 \text{ for any } \Lambda(\boldsymbol{\alpha}).$$

For example, for the harmonic oscillator $H = p^2 + x^2$ the latter condition means that we construct asymptotics of the spectrum in the domain $H \geq E > 0$, where E is an arbitrary but independent of h number. To study the spectral asymptotics in $O(h)$-neighborhood of 0 one has to use other methods.

2.3 The oscillatory approximation method

In the oscillatory approximation method, rest points of a classical Hamiltonian are used for constructing quasimodes of the corresponding operator. One replaces the Hamiltonian by its quadratic expansion near the rest point; the corresponding operator is a generalized harmonic oscillator; the eigenfunctions and the eigenvalues of such an operator are well known.

2.3.1 Stable points

Definition 23 (Stability of a rest point in the linear approximation). Let $\boldsymbol{r}^0 = (\boldsymbol{p}^0, \boldsymbol{x}^0)$ be a non-degenerate rest point of a Hamiltonian $H(\boldsymbol{p}, \boldsymbol{x})$. Consider the linearization of the Hamiltonian system generated by H near the point $\boldsymbol{r}^0$; this linear system has the form

$$\dot{\boldsymbol{\xi}} = H''\boldsymbol{\xi}, \quad \boldsymbol{\xi} \in \mathbb{R}^{2n}, \tag{2.10}$$

where

$$H'' = \begin{pmatrix} -H_{\boldsymbol{xp}} & -H_{\boldsymbol{xx}} \\ H_{\boldsymbol{pp}} & H_{\boldsymbol{px}} \end{pmatrix}\Bigg|_{\boldsymbol{r}^0}. \tag{2.11}$$

The point $\boldsymbol{r}^0$ is called *stable in the linear approximation* if all the solutions of (2.10) are bounded on the whole real axis.

An obvious criterion for a rest point to be stable in the linear approximation can be given:

Proposition 2.8. *A non-degenerate rest point $\boldsymbol{r}^0$ of a Hamiltonian system is stable in the linear approximation if and only if the matrix H'' is diagonalizable and has purely imagine spectrum.*

2.3.2 Complex germ

Definition 24 (Complex germ at a rest point). Let $\boldsymbol{r}^0$ be a non-degenerate rest point of a Hamiltonian H. By a *complex germ* associated with the point $\boldsymbol{r}^0$ and the Hamiltonian $H(\boldsymbol{p}, \boldsymbol{x})$ we mean a n-dimensional linear subspace $\lambda \in \mathbb{C}^{2n}$ satisfying the following conditions (the complex germ axioms):

$$\lambda \text{ is a complex Lagrangian subspace: } [\boldsymbol{a}|\boldsymbol{b}] = 0 \text{ for any } \boldsymbol{a}, \boldsymbol{b} \in \lambda, \tag{2.12}$$

$$\frac{1}{2i}[\boldsymbol{a}|\overline{\boldsymbol{a}}] > 0 \text{ for any } \boldsymbol{a} \in \lambda, \quad \boldsymbol{a} \neq 0, \tag{2.13}$$

$$H''\lambda = \lambda. \tag{2.14}$$

Proposition 2.9 (Theorem 6.1 in [64]). *A complex germ associated with a non-degenerate rest point $\boldsymbol{r}^0$ of a Hamiltonian H exists if and only if $\boldsymbol{r}^0$ is stable in the linear approximation.*

Remark. Clearly, a complex germ is unique if the eigenvalues of H'' are distinct. Otherwise, the set of complex germs may be quite large [78].

2.3.3 Spectral series

Let us consider a non-degenerate rest point $\boldsymbol{r}^0$ of a Hamiltonian H; assume that $\boldsymbol{r}^0$ is stable in the linear approximation. Construct the corresponding complex germ λ; as follows from the definition, λ contains n eigenvectors of H''. Denote these eigenvectors by $\boldsymbol{\rho}^j$, $j \in \{1, \dots, n\}$; their $\boldsymbol{p}$- and $\boldsymbol{x}$-components we denote by $\boldsymbol{w}^j$ and $\boldsymbol{z}^j$, respectively, i. e., $\boldsymbol{\rho}^j = (\boldsymbol{w}^j, \boldsymbol{z}^j)$, $\boldsymbol{w}^j, \boldsymbol{z}^j \in \mathbb{C}^n$, $j \in \{1, \dots, n\}$. Let $i\beta^j$ be the corresponding eigenvalues of H'':

$$H''\boldsymbol{\rho}^j = i\beta^j \boldsymbol{\rho}^j, \quad j \in \{1, \dots, n\}.$$

Introduce operators

$$\hat{\rho}^j = \frac{1}{\sqrt{h}} \left[\left\langle \overline{\boldsymbol{z}}^j \middle| -ih\frac{\partial}{\partial \boldsymbol{x}} - \boldsymbol{p}^0 \right\rangle - \left\langle \overline{\boldsymbol{w}}^j \middle| \boldsymbol{x} - \boldsymbol{x}^0 \right\rangle \right].$$

These operators obey the property

$$\hat{H}_h \hat{\rho}^j - \hat{\rho}^j \hat{H}_h = h\beta^j \hat{\rho}^j + O(h^{3/2})$$

and are called *creation operators*.

Introduce $n \times n$-matrices B, C, and Q in the following way:

$$B_{jk} = w_j^k, \quad C_{jk} = z_j^k, \quad j, k \in \{1, \dots, n\}, \quad Q = BC^{-1}. \tag{2.15}$$

Put

$$S(\boldsymbol{x}) = \left\langle \boldsymbol{p}^0 \middle| \boldsymbol{x} - \boldsymbol{x}^0 \right\rangle + \frac{1}{2} \left\langle \boldsymbol{x} - \boldsymbol{x}^0 \middle| Q(\boldsymbol{x} - \boldsymbol{x}^0) \right\rangle.$$

One always has $\Im S \geq 0$ [24, §2].

Proposition 2.10 (Theorem 6.2 in [64]). *Let* $\boldsymbol{m} = (m_j)_{j \in \{1, \dots, n\}} \in \mathbb{Z}_+^n$, *then the number*

$$E_{\boldsymbol{m}}(h) = H|_{\boldsymbol{r}^0} + h\omega_{\boldsymbol{m}}, \quad \omega_{\boldsymbol{m}} = \sum_{j=1}^{n} \beta^j (m_j + \frac{1}{2}) \tag{2.16}$$

and the function

$$\psi_{\boldsymbol{m}}(\boldsymbol{x}, h) = \varphi_{\boldsymbol{m}}(\boldsymbol{x}, h) e^{-iS/h} = \frac{1}{h^{n/4}} \left(\prod_{j=1}^{n} (\hat{\rho}^j)^{m_j} \right) e^{\frac{i}{h} S(\boldsymbol{x})} \tag{2.17}$$

compose a quasimode of $\hat{H}_h$ *with error* $O(h^{3/2})$.

Remark. It some cases it is possible to construct quasimodes up to any power of h [79], we will touch this question in the next chapter. Under certain additional assumptions one can construct also quasimodes up to $O(e^{-C/h})$ for $C > 0$, but this procedure can be really performed only in vary special cases, see, for example [22, 80].

2.3.4 Canonical transformations of a complex germ

It is sometimes difficult to find the eigenvalues and the eigenvectors of H'' directly. In these cases, one can use the following simple fact:

Proposition 2.11. *Assume that in some canonical variables* $(\mathcal{P}, \mathcal{X})$ *a Hamiltonian* H *has the form* $H = \mathcal{H}(\mathcal{P}, \mathcal{X})$*; denote*

$$\mathcal{H}'' = \begin{pmatrix} -\mathcal{H}_{\mathcal{X}\mathcal{P}} & -\mathcal{H}_{\mathcal{X}\mathcal{X}} \\ \mathcal{H}_{\mathcal{P}\mathcal{P}} & \mathcal{H}_{\mathcal{P}\mathcal{X}} \end{pmatrix} \Bigg|_{\boldsymbol{r}^0}.$$

Let $\boldsymbol{\eta}$ *be an eigenvector of* $\mathcal{H}''$ *with eigenvalue* α*, then the vector*

$$\boldsymbol{\xi} = \frac{\partial(\boldsymbol{p}, \boldsymbol{x})}{\partial(\mathcal{P}, \mathcal{X})} \boldsymbol{\eta}$$

is an eigenvector of H'' *with the same eigenvalue* α*.*

It follows from this proposition that the complex germ axioms (2.12), (2.13), and (2.14) can be verified in arbitrary canonical coordinates.

2.4 Quasimodes corresponding to invariant closed curves

In this section, we describe the procdure of constructing quasimodes corresponding to closed trajectories of the classical Hamiltonian. Our approach here is based on the complex germ formalism [64], other interpretations can be found in [6,27,76,42].

Definition 25 (Complex germ over a curve). By a *complex germ* (λ) *over a curve* $\gamma = \big(\boldsymbol{P}(t), \boldsymbol{X}(t)\big)$, $t \in \mathbb{R}$, we mean a set of complex subspaces $\lambda(t)$, $t \in \mathbb{R}$, $\lambda(t + T) = \lambda(t)$, satisfying the following conditions (the complex germ axioms):

- each fiber $\lambda(t)$ is a complex Lagrangian subspace in $\mathbb{C}^{2n}$: $[\boldsymbol{a}|\boldsymbol{b}] = 0\ \forall \boldsymbol{a}, \boldsymbol{b} \in \lambda(t)$,
- each subspace $\lambda(t)$ contains the complexified tangent space to γ at the corresponding point $\big(\boldsymbol{P}(t), \boldsymbol{X}(t)\big)$: $\big(\dot{\boldsymbol{P}}(t), \dot{\boldsymbol{X}}(t)\big) =: \boldsymbol{v}(t) \in \lambda(t)$,
- for each vector $\boldsymbol{a} \in \lambda(t)$, $\boldsymbol{a} \neq \alpha \boldsymbol{v}(t)$, $\alpha \in \mathbb{C}$, one has

$$\frac{1}{2i}[\boldsymbol{a}|\overline{\boldsymbol{a}}] > 0,$$

2.4.1 Invariant complex germ

Let γ be now a closed smooth invariant curve of a Hamiltonian H. Denote by T its period. The linearization of the Hamiltonian system for H near γ is defined by the matrix

$$H''(t) = \begin{pmatrix} -H_{xp} & -H_{xx} \\ H_{pp} & H_{px} \end{pmatrix}\Bigg|_{\substack{p=P(t),\\ x=X(t)}} \tag{2.18}$$

and has the form

$$\dot{\boldsymbol{\xi}} = H''(t)\boldsymbol{\xi}. \tag{2.19}$$

Denote by D_H^t the corresponding phase flow.

Definition 26 (Invariant complex germ). A complex germ (λ) is called *invariant* if $D_H^{t_1}\lambda(t_2) = \lambda(t_1 + t_2)$ for any $t_1, t_2 \in \mathbb{R}$.

Like in the oscillatory approximation method, the definition of an invariant complex germ is independent of the choice of the canonical coordinates in the phase space in the following sense:

Proposition 2.12 (see §2.1 in [23]). *Let γ be an invariant trajectory of a Hamiltonian H and $(\mathcal{P}, \mathcal{X})$ be new canonical coordinates, $H(\boldsymbol{p}, \boldsymbol{x}) = \mathcal{H}(\mathcal{P}, \mathcal{X})$, and*

$$\mathcal{H}''(t) = \begin{pmatrix} -\mathcal{H}_{\mathcal{X}\mathcal{P}} & -\mathcal{H}_{\mathcal{X}\mathcal{X}} \\ \mathcal{H}_{\mathcal{P}\mathcal{P}} & \mathcal{H}_{\mathcal{P}\mathcal{X}} \end{pmatrix}\Bigg|_{\gamma}.$$

The transformation

$$\boldsymbol{\xi} = \frac{\partial(\boldsymbol{p}, \boldsymbol{x})}{\partial(\mathcal{P}, \mathcal{X})}\Big|_{\gamma}\boldsymbol{\eta} \tag{2.20}$$

carries the system

$$\dot{\boldsymbol{\eta}} = \mathcal{H}''(t)\boldsymbol{\eta} \tag{2.21}$$

into the system (2.19) and preserves the skew-inner product of any two solutions.

2.4.2 Orbital stability of a trajectory

Like in the oscillatory approximation method, the conditions for the existence of an invariant complex germ over a closed curve γ can be given in terms of the system (2.19).

Definition 27 (Orbital stability of a closed invariant curve in the linear approximation). A closed invariant curve γ of a Hamiltonian H is called *orbitally stable in the linear approximation* if all the solutions of (2.19) skew-orthogonal to $\boldsymbol{v}(t)$ are bounded on the whole real axis.

Proposition 2.13 (Theorem 6.5 in [64]). *An invariant complex germ associated with an invariant closed curve exists if and only if this curve is orbitally stable in the linear approximation.*

Proposition 2.14 (Theorem 3.2 in [24]). *An invariant closed trajectory* $\gamma = \big(\boldsymbol{P}(t), \boldsymbol{X}(t)\big)$ *of a Hamiltonian H is orbitally stable in linear approximation if and only if the matrix of the reduced monodromy operator for* (2.19) *(i.e., of the restriction of the monodromy matrix onto the subspace* $\mathbb{C}^{2n}/\boldsymbol{v}(0)$*) is diagonalizable and its spectrum belongs to the unit circle.*

2.4.3 Invariant basis and quasimodes

Definition 28 (Invariant basis on an invariant complex germ). A basis $\boldsymbol{\rho}^j(t)$, $j \in \{1, \dots, n\}$, in the fibers $\lambda(t)$ of an invariant complex germ (λ) is called *invariant* if

(a) $\boldsymbol{\rho}^n(t) = \boldsymbol{v}(t)$,

(b) $D_H^{t_1} \boldsymbol{\rho}^j(t_2) = \boldsymbol{\rho}^j(t_1 + t_2), \quad j \in \{1, \dots, n-1\}$,

and $\boldsymbol{\rho}^j$ are eigenvectors of the monodromy operator of (2.19), i. e.

(c) $\boldsymbol{\rho}^j(t + T) = \exp\Big(i\beta^j T\Big)\boldsymbol{\rho}^j(t), \quad j \in \{1, \dots, n\}$.

(Obviously, we can put $\beta^n = 0$.) As follows from [64, Proposition 6.3], an invariant basis always exists.

Note that, generally speaking, the invariance of a basis can be destroyed when using Proposition 2.12.

In each fiber $\lambda(t)$ of a complex germ let us choose an invariant basis

$$\boldsymbol{\rho}^j(t) = \big(\boldsymbol{w}^j(t), \boldsymbol{z}^j(t)\big), \quad \boldsymbol{w}^j, \boldsymbol{z}^j : \mathbb{R} \mapsto \in \mathbb{C}^n, \quad j \in \{1, \dots, n\},$$
$$\boldsymbol{\rho}^n(t) = \boldsymbol{v}(t).$$

To each vector $\boldsymbol{\rho}^j(t)$, $j \in \{1, \dots, n-1\}$ we assign a *creation operator*,

$$\hat{\rho}^j = \frac{1}{\sqrt{h}}\left[\left\langle \overline{\boldsymbol{z}}^j(t) \middle| -ih\frac{\partial}{\partial \boldsymbol{x}} - \boldsymbol{P}(t) \right\rangle - \left\langle \overline{\boldsymbol{w}}^j(t) \middle| \boldsymbol{x} - \boldsymbol{X}(t) \right\rangle \right]\bigg|_{t=\tau(\boldsymbol{x})},$$

where $\tau(\boldsymbol{x})$ is the solution of the equation

$$\left\langle \boldsymbol{x} - \boldsymbol{X}(\tau) \middle| \dot{\boldsymbol{X}}(\tau) \right\rangle = 0$$

defined in a small neighborhood of the projection $\pi_{\boldsymbol{x}}\gamma$ of the curve γ onto the $\boldsymbol{x}$-subspace.

Let us compose $n \times n$ complex matrices $B(t)$ and $C(t)$:

$$B^{(\lambda)}_{jk}(t) = \left(\boldsymbol{w}^k(t)\right)_j, \quad C^{(\lambda)}_{jk} = \left(\boldsymbol{z}^k(t)\right)_j, \quad j,k \in \{1, \dots, n\}, \tag{2.22}$$

and introduce a phase function S:

$$S(\boldsymbol{x}) = \int_{\tau^0}^{t} \langle \boldsymbol{P}(t') | d\boldsymbol{X}(t') \rangle + \left\langle \boldsymbol{P}(t) \middle| \boldsymbol{x} - \boldsymbol{X}(t) \right\rangle + \frac{1}{2}\left\langle \boldsymbol{x} - \boldsymbol{X}(t) \middle| B^{(\lambda)}(t)\left(C^{(\lambda)}\right)^{-1}(t)\left(\boldsymbol{x} - \boldsymbol{X}(t)\right) \right\rangle \bigg|_{t=\tau(\boldsymbol{x})}. \tag{2.23}$$

In what follows we always assume that

$$\dot{\boldsymbol{X}} \neq 0 \text{ for all } t$$

(this condition is always fulfilled in the problem in question); then $\det C^{(\lambda)} \neq 0$ [64, §III.3].

Introduce a function φ:

$$\varphi(\boldsymbol{x}) = \frac{e^{i\omega t}}{\sqrt{|\det C^{(\lambda)}(t)|}}\bigg|_{t=\tau(\boldsymbol{x})}, \quad \omega = \frac{2\pi m_n}{T} + \frac{1}{2}\sum_{j=1}^{n-1} \beta^j. \tag{2.24}$$

Proposition 2.15 (Quantization condition for closed curves, see §3.1 in [24] or Lemma 6.3 in [64]). *The expression*

$$\psi(\boldsymbol{x}) = \varphi(\boldsymbol{x}) \exp\left(\frac{i}{h}S(\boldsymbol{x})\right) \tag{2.25}$$

defines a single-valued function if and only if

$$\frac{1}{2\pi h}\oint_{\gamma} \langle \boldsymbol{p} | d\boldsymbol{x} \rangle = m \in \mathbb{Z}. \tag{2.26}$$

Obviously, the condition (2.26) cannot be satisfied for a fixed closed curve γ. Usually one has a family of closed curves, and for each fixed h the condition (2.26) selects some discrete subset of them, i. e., this family of curves is *quantized* (The existence of such a family of curves can be guaranteed by certain conditions of non-degeneracy, see [76, §1]).

Proposition 2.16 (Theorem 6.6 and Theorem 6.7 in [64]). *Let $\boldsymbol{m} \in \mathbb{Z}_+^{n-1} \times \mathbb{Z}$ and the condition (2.26) be satisfied for $m = m_n$ and $|m_n h| \geq c > 0$ as $h \to 0$, then the function*

$$\psi_{\boldsymbol{m}}(\boldsymbol{x}, h) = \varphi_{\boldsymbol{m}}(\boldsymbol{x}, h) e^{iS/h} = \frac{1}{\sqrt{h}} \prod_{j=1}^{n-1} \left(\hat{\rho}^j\right)^{m_j} \psi(\boldsymbol{x}), \tag{2.27}$$

where ψ is given by (2.25), and the number

$$E_{\boldsymbol{m}}(h) = H|_\gamma + h\omega_{\boldsymbol{m}}, \quad \omega_{\boldsymbol{m}} = \sum_{j=1}^{n-1} \beta^j (m_j + \frac{1}{2}) + \frac{2\pi m_n}{T} \tag{2.28}$$

are a quasimode of $\hat{H}_h$ with error $O(h^{3/2})$.

Remark. The numbers β^j, $j \in \{1, \ldots, n-1\}$, are defined only up to $2\pi/T$; this arbitrariness is already included into the term m_n, but, strictly speaking, these numbers must satisfy the following normalization conditions:

$$i \sum_{j=1}^{n-1} \beta^j T = \int_0^T \operatorname{tr}\left(H_{\boldsymbol{pp}} S_{\boldsymbol{xx}} + H_{\boldsymbol{px}}\right)\Big|_{\substack{\boldsymbol{p}=\boldsymbol{P}(t') \\ \boldsymbol{x}=\boldsymbol{X}(t')}} dt'.$$

2.4.4 The Hamilton-Jacobi and the generalized transport equations

We will need a modification of the procedure described above, and in this subsection we explain the "nature" of Proposition 2.16. Introduce first some non-standard notation.

Definition 29 ($O_f(h^\alpha)$, see §I.3 in [64]). Let $f(\boldsymbol{x})$ be a smooth non-negative function, then $\varphi(\boldsymbol{x}, h) = O_f(h^\alpha)$ iff

$$\left(\frac{\partial^{|\boldsymbol{l}|}\varphi}{\partial \boldsymbol{x}^{\boldsymbol{l}}}\right) e^{-f/h} = O(h^{\alpha - |\boldsymbol{l}|/2})$$

for $\alpha - |\boldsymbol{l}|/2 \geq 0$ on any compact set $\Omega \subset \mathbb{R}^n$.

Definition 30 ($\mathcal{O}_f(h^\alpha)$, see §III.4 in [64]). Let f be a smooth non-negative function. By $\mathcal{O}_f(h^\alpha)$ we denote the class of functions $\varphi(\boldsymbol{x}, h)$ such that

$$\varphi(\boldsymbol{x}, h) = \sum_{k=-m}^{l} \varphi_k(\boldsymbol{x}, h) h^{k/2}, \quad m, l \geq 0,$$

and $\varphi_k = O(h^{-k/2+\alpha})$ for $k \leq \alpha$.

We are looking for solutions of the equation

$$\big(\hat{H}_{h,\epsilon} - E(h)\big)\psi(\boldsymbol{x}, h) = O(h^{3/2}), \tag{2.29}$$

where $E(h) = E_0 + h\omega$, $\psi(\boldsymbol{x}, h) = \varphi(\boldsymbol{x}, h)e^{iS(x,h)}$, and $\varphi(\boldsymbol{x}, h) = \mathcal{O}_{\mathfrak{I}S}(1)$. Substituting these expressions into (2.29), we obtain

$$\begin{aligned} \hat{H}_h\psi(\boldsymbol{x}, h) = e^{iS/h}\Big(&(H(\boldsymbol{p}, \boldsymbol{x}) - E_0)\varphi - ih(\hat{\Pi} - i\omega)\varphi \\ &+ g(\boldsymbol{x}, h)\Big) = O(h^{3/2}), \\ \boldsymbol{p} = \partial S/\partial \boldsymbol{x}, \end{aligned} \tag{2.30}$$

where $g = \mathcal{O}_{\mathfrak{I}S}(h^2)$ and

$$\begin{aligned} \hat{\Pi} :=& \langle H_{\boldsymbol{p}}(\boldsymbol{p}, \boldsymbol{x}) | \frac{\partial}{\partial \boldsymbol{x}} \rangle + \frac{1}{2}\operatorname{tr}\Big(H_{\boldsymbol{pp}}S_{\boldsymbol{xx}} + H_{\boldsymbol{px}}\Big) \\ & - ih\sum_{j,k=1}^{n} H_{p_j p_k}(\boldsymbol{p}, \boldsymbol{x}) \frac{\partial^2}{\partial x_j \partial x_k}, \quad \boldsymbol{p} = \frac{\partial S(\boldsymbol{x})}{\partial \boldsymbol{x}}. \end{aligned}$$

Therefore, we should solve the two following equations:

$$\Big(H(\frac{\partial S}{\partial \boldsymbol{x}}, \boldsymbol{x}) - E_0\Big) = O_{\mathfrak{I}S}(h^{3/2}), \tag{2.31}$$

$$(\hat{\Pi} - i\omega)\varphi = O_{\mathfrak{I}S}(\sqrt{h}). \tag{2.32}$$

The first of these equations is called the *Hamilton-Jacobi equation*, and the second one is called the *generalized transport equation*. We find $S(\boldsymbol{x})$ in the form $\sigma\big(\tau(\boldsymbol{x})\big)$, where

$$\sigma(t) = S_0(t) + \Big\langle \boldsymbol{q}(t) \big| \boldsymbol{x} - \boldsymbol{X}(t) \Big\rangle + \frac{1}{2}\Big\langle \boldsymbol{x} - \boldsymbol{X}(t) \big| Q(t)\big(\boldsymbol{x} - \boldsymbol{X}(t)\big) \Big\rangle. \tag{2.33}$$

The quantization condition (2.26) is nothing but the periodicity condition, $\sigma(t + T) = \sigma(t)$, which guarantees that S is a single-valued function. The solution of the generalized transport equations we find in the form

$$\varphi(\boldsymbol{x}) = \Phi(t)\Big|_{t=\tau(\boldsymbol{x})}, \tag{2.34}$$

and the function Φ also has to be periodic $\Phi(t) \equiv \Phi(t+T)$ (cf. [76, SS1,2]).

The construction of these solutions is described in the previous subsections.

It is possible to extend the procedure presented for constructing quasimodes up to any power of h; one can show that it can be done under stronger conditions about stability of the trajectory [76, §3] (see also the next chapter).

Chapter 3

Semiclassical spectral series for the magnetic Schrödinger operator

3.1 Averaging and corrections to the spectrum

Returning from the coordinates $(\mathcal{P}, \mathcal{Y}_1, \mathcal{Q}, \mathcal{Y}_2)$ to the original coordinates $(\boldsymbol{p}, \boldsymbol{x})$, we come from the representation (1.26) to the representation

$$H(\boldsymbol{p}, \boldsymbol{x}, \epsilon) = H^0(\boldsymbol{p}, \boldsymbol{x}, \epsilon) + e^{-C/\epsilon} G(\boldsymbol{p}, \boldsymbol{x}, \epsilon). \tag{3.1}$$

Strictly speaking, such a representation is local and is valid in any domain $I_1 \leq \kappa$, see Propositions 1.12 and 1.13; we fix some κ and extend both terms in (3.1) smoothly in the whole space $\mathbb{R}^4_{\boldsymbol{p},\boldsymbol{x}}$. Obviously, we can do it preserving the boundedness of G. The manifolds $\Lambda^r_{l/k}(\mathfrak{I}, \epsilon)$ (constructed in Chapter 1) are invariant manifolds of H^0 but not of H; we call them *almost invariant manifolds* of H.

Let $\hat{H}^0_{h,\epsilon}$ and $\hat{G}_{h,\epsilon}$ be the Weyl quantizations of H^0 and G respectively, then

$$\hat{H}_{h,\epsilon} = \hat{H}^0_{h,\epsilon} + e^{-C/\epsilon} \hat{G}_{h,\epsilon}. \tag{3.2}$$

In particular, if (ψ, E) is a quasimode of $\hat{H}^0_{h,\epsilon}$ with error α, then it is also a quasimode of $\hat{H}_{h,\epsilon}$ with error $\alpha + O(e^{-C/\epsilon})$.

Note that $\hat{H}^0_{h,\epsilon}$ and $\hat{G}_{h,\epsilon}$ are already not differential operators, but only pseudodifferential ones [63].

3.2 Spectral series for invariant points

3.2.1 Description of invariant points

A simple analysis of Proposition 1.11 shows that

$$\mathcal{H}(0, \boldsymbol{\mathcal{Y}}, \epsilon) = \epsilon v(\boldsymbol{\mathcal{Y}}), \tag{3.3}$$

independently on the degree of the averaging (because the equality (3.3) holds on each step of the averaging, and $\mathcal{H}$ is obtained as a result of subsequent using of Proposition 1.11). Let $\boldsymbol{\mathcal{Y}}^0$ be a certain rest point of the function v, i. e.

$$\nabla v(\boldsymbol{\mathcal{Y}}^0) = 0,$$

then the corresponding family $\Lambda_{\boldsymbol{l}}$ of rest points of $\mathcal{H}$ is defined as

$$\Lambda_{\boldsymbol{l}} = (\mathcal{P} = 0, \mathcal{Q} = 0, \boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{Y}}^0 + \boldsymbol{l} \cdot \boldsymbol{a}). \tag{3.4}$$

Denote the coordinates of $\Lambda_{\boldsymbol{l}}$ in the space $\mathbb{R}^4_{\boldsymbol{p},\boldsymbol{x}}$ by $\boldsymbol{P}^{\boldsymbol{l}}$ and $\boldsymbol{X}^{\boldsymbol{l}}$.

3.2.2 The construction of the complex germ

We construct the complex germ using Proposition 2.11. Take $\boldsymbol{\mathcal{P}} = (\mathcal{P}, \mathcal{Y}_1)$, $\boldsymbol{\mathcal{X}} = (\mathcal{Q}, \mathcal{Y}_2)$, then

$$\mathcal{H}'' = \begin{pmatrix} 0 & 0 & -\omega & 0 \\ 0 & -\mathcal{H}_{12} & 0 & -\mathcal{H}_{22} \\ \omega & 0 & 0 & 0 \\ 0 & \mathcal{H}_{11} & 0 & \mathcal{H}_{12} \end{pmatrix}, \quad \omega = \frac{\partial \mathcal{H}}{\partial \mathcal{I}_1}\Big|_{\Lambda_{\boldsymbol{l}}}, \quad \mathcal{H}_{jk} = \frac{\partial^2 \mathcal{H}}{\partial \mathcal{Y}_1 \partial \mathcal{Y}_2}\Big|_{\Lambda_{\boldsymbol{l}}}.$$

The equation for the eigenvalues has a simple form:

$$(\lambda^2 + \omega^2) \det \begin{pmatrix} \mathcal{H}_{12} + \lambda & \mathcal{H}_{22} \\ \mathcal{H}_{11} & \mathcal{H}_{12} - \lambda \end{pmatrix} = 0,$$

and can be easily solved; the solutions are

$$\lambda_1 = i\omega, \quad \lambda_2 = -i\omega, \quad \lambda_3 = i\sqrt{\det(\mathcal{H}_{jk})}, \quad \lambda_4 = -i\sqrt{\det(\mathcal{H}_{jk})}. \tag{3.5}$$

Therefore, we come to the following conclusion:

Proposition 3.1. *The points $\Lambda_{\boldsymbol{l}}$ are stable in the linear approximation iff $\det(\mathcal{H}_{jk}) > 0$. The latter condition means that the point $\boldsymbol{\mathcal{Y}}^0$ is an (local) extremum point of $\mathcal{H}(0, \mathcal{Y}, \epsilon)$ i. e. of v.*

Therefore, we can construct quasimodes corresponding to the points $\Lambda_l^r(0,0,\epsilon)$ only (see Subsection 1.3.5).

The eigenvectors corresponding to (3.5) can be also easily found (using the smallness of ϵ); they are

$$\boldsymbol{\xi}_1 = (1,0,-i,0)^T, \qquad \boldsymbol{\xi}_2 = (1,0,i,0)^T, \tag{3.6}$$

$$\boldsymbol{\xi}_3 = \begin{pmatrix} 0 \\ \mathcal{H}_{22} \\ 0 \\ \mathcal{H}_{12} - i\sqrt{\det(\mathcal{H}_{jk})} \end{pmatrix}, \qquad \boldsymbol{\xi}_4 = \begin{pmatrix} 0 \\ \mathcal{H}_{22} \\ 0 \\ \mathcal{H}_{12} + i\sqrt{\det(\mathcal{H}_{jk})} \end{pmatrix}. \tag{3.7}$$

Now we can construct the corresponding complex germ (it exists, because all the eigenvalues are purely imagine) in these coordinates $(\mathcal{P}, \mathcal{X})$.

Proposition 3.2. *If $\mathcal{H}_{22} > 0$ (and we have local minimum point), then the basis of the complex germ at Λ_l is given by the vectors $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_3$; if $\mathcal{H}_{22} < 0$ (local maximum point), the basis is given by $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_4$.*

3.2.3 Spectral series

Respectively, we have the following formula for asymptotic eigenvalues:

$$E^r_{\boldsymbol{m}}(h,\epsilon) = \mathcal{H}|_{\Lambda^r_l} + \frac{\partial \mathcal{H}^r}{\partial \mathcal{I}_1}(0,0,\epsilon)\Big(m_1 + \frac{1}{2}\Big)h \pm \epsilon\sqrt{\det(\mathcal{H}_{jk})}\Big(m_2 + \frac{1}{2}\Big)h, \tag{3.8}$$

$m_1, m_2 \in \mathbb{Z}_+$.

where the signs "+" and "−" correspond to minimum and maximum points of v, respectively.

This expression can be simplified using the following obvious observation:

Proposition 3.3. *Let $\boldsymbol{x}^0$ be a non-degenerate minimum point of a function $w(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^2$. Let $S(E)$ be the square of the connected domain $w(\boldsymbol{x}) \le E$ containing the point $\boldsymbol{x}^0$, then*

$$E = \frac{1}{2\pi} \det\Big(\frac{\partial^2 w}{\partial x_j \partial x_k}\Big)\Big|_{\boldsymbol{x}^0} S(E) + O\big(S^{3/2}(E)\big).$$

Applying this proposition to $\epsilon v = \mathcal{H}$ and $\boldsymbol{\mathcal{Y}}^0$, we obtain the equality

$$\pm\epsilon\sqrt{\det(\mathcal{H}_{jk})} = \frac{\partial \mathcal{H}}{\partial \mathcal{I}_2}(0,0,\epsilon);$$

Therefore, (3.8) can be rewritten as

$$E^r_{\boldsymbol{m}}(h) = \mathcal{H}^r\big(\mathcal{I}_1^{(m_1)}, \mathcal{I}_2^{(m_2)}, \epsilon\big) + O(h^2), \tag{3.9}$$

where

$$\mathcal{I}_k^{(m_k)}(h) := h\Big(\frac{1}{2} + m_k\Big), \quad m_k \in \mathbb{Z}_+, \quad k = 1, 2. \tag{3.10}$$

Returning now to the original coordinates $(\boldsymbol{p}, \boldsymbol{x})$ one can construct the asymptotic eigenfunctions using the scheme of Section 2.3.

Proposition 3.4. *The asymptotic functions $\psi^r_{\boldsymbol{m},\boldsymbol{l}}(\boldsymbol{x}, h, \epsilon)$ corresponding to the asymptotic eigenvalues $E^r_{\boldsymbol{m}}(h)$ and to the points $\Lambda^r_{\boldsymbol{l}}$ can be normalized such that*

$$\psi^r_{\boldsymbol{m},\boldsymbol{l}}(\boldsymbol{x}, h) = \psi^r_{\boldsymbol{m},(0,0)}(\boldsymbol{x} - \boldsymbol{l}\cdot\boldsymbol{a}, h, \epsilon)\exp\Big(-\frac{i}{h}a_{22}l_2x_1\Big). \tag{3.11}$$

Before proving this proposition, let us prove an important auxiliary fact.

Proposition 3.5. *Let functions $\psi_{\boldsymbol{m},\boldsymbol{l}}$ satisfy (3.11) for $|\boldsymbol{m}| \leq M$ for some $M \geq 0$, and $(\boldsymbol{w}, \boldsymbol{z}) \in \mathbb{C}^4$ be an arbitrary vector. Consider an operator*

$$\hat{a}_{\boldsymbol{l}} = \frac{1}{\sqrt{h}}\Big[\langle \boldsymbol{w}|\boldsymbol{x} - \boldsymbol{X}^{\boldsymbol{l}}\rangle - \langle \boldsymbol{z}| - ih\frac{\partial}{\partial \boldsymbol{x}} - \boldsymbol{P}^{\boldsymbol{l}}\rangle\Big].$$

and put

$$\psi'_{\boldsymbol{m},\boldsymbol{l}} = \hat{a}_{\boldsymbol{l}}\psi_{\boldsymbol{m},\boldsymbol{l}},$$

then the functions $\psi'_{\boldsymbol{m},\boldsymbol{l}}$ also satisfy (3.11).

Proof of Proposition 3.5. One has the following chain of equalities:

$$\begin{aligned}
\hat{a}_{\boldsymbol{l}}\psi_{\boldsymbol{m},\boldsymbol{l}}(\boldsymbol{x}, h, \epsilon) &= \hat{a}_{\boldsymbol{l}}\Big[\psi_{\boldsymbol{m},\boldsymbol{0}}(\boldsymbol{x} - \boldsymbol{l}\cdot\boldsymbol{a}, h)\exp\Big(\frac{i}{h}a_{22}l_2x_1\Big)\Big] \\
&= \frac{1}{\sqrt{h}}\Big[\langle \boldsymbol{w}|(\boldsymbol{x} - \boldsymbol{l}\cdot\boldsymbol{a}) - \boldsymbol{X}^{\boldsymbol{0}}\rangle - \langle \boldsymbol{z}| - ih\frac{\partial}{\partial \boldsymbol{x}} - \boldsymbol{P}^{\boldsymbol{0}}\rangle + z_1l_2a_{22}\Big] \\
&\quad \psi_{\boldsymbol{m},\boldsymbol{0}}(\boldsymbol{x} - \boldsymbol{l}\cdot\boldsymbol{a}, h)\exp\Big(\frac{i}{h}a_{22}l_2x_1\Big) \\
&= \Big\{\frac{1}{\sqrt{h}}\Big[\langle \boldsymbol{w}|(\boldsymbol{x} - \boldsymbol{l}\cdot\boldsymbol{a}) - \boldsymbol{X}^{\boldsymbol{0}}\rangle - \langle \boldsymbol{z}| - ih\frac{\partial}{\partial \boldsymbol{x}} - \boldsymbol{P}^{\boldsymbol{0}}\rangle \\
&\quad + z_1l_2a_{22}\Big]\psi_{\boldsymbol{m},\boldsymbol{0}}(\boldsymbol{x} - \boldsymbol{l}\cdot\boldsymbol{a}, h)\Big\}\exp\Big(\frac{i}{h}a_{22}l_2x_1\Big) \\
&\quad + \psi_{\boldsymbol{m},\boldsymbol{0}}(\boldsymbol{x} - \boldsymbol{l}\cdot\boldsymbol{a}, h)\langle \boldsymbol{z}| - ih\frac{\partial}{\partial \boldsymbol{x}}\rangle\exp\Big(\frac{i}{h}a_{22}l_xx_1\Big) \\
&= \Big[\big(\hat{a}_{\boldsymbol{0}}\psi_{\boldsymbol{m},\boldsymbol{0}}\big)(\boldsymbol{x} - \boldsymbol{l}\cdot\boldsymbol{a}, h, \epsilon)\Big]\exp\Big(\frac{i}{h}a_{22}l_2x_1\Big).
\end{aligned}$$

The proposition is proved. □

Proof of Proposition 3.4. Let us calculate the phase function $S_{\boldsymbol{l}}$, corresponding to the point $\Lambda_{\boldsymbol{l}}$. Recall that this function has the form

$$S_{\boldsymbol{l}}(x) = \langle \boldsymbol{P}^{\boldsymbol{l}} | x - \boldsymbol{X}^{\boldsymbol{l}} \rangle + \frac{1}{2}\Big\langle x - \boldsymbol{X}^{\boldsymbol{l}} \Big| Q(x - \boldsymbol{X}^{\boldsymbol{l}}) \Big\rangle.$$

In particular,

$$\begin{aligned} S_{\boldsymbol{l}}(x) &= \Big\langle \boldsymbol{P}^{\boldsymbol{0}} - (l_2 a_{22}, 0) | x - \boldsymbol{X}^{\boldsymbol{0}} - \boldsymbol{l} \cdot \boldsymbol{a} \Big\rangle \\ &\quad + \frac{1}{2}\Big\langle x - \boldsymbol{X}^{\boldsymbol{0}} - \boldsymbol{l} \cdot \boldsymbol{a} \Big| Q\big(x - \boldsymbol{X}^{\boldsymbol{0}} - \boldsymbol{l} \cdot \boldsymbol{a}\big) \Big\rangle \\ &= S_{\boldsymbol{0}}(x - \boldsymbol{l} \cdot \boldsymbol{a}) - l_2 a_{22}(x - \boldsymbol{X}^{\boldsymbol{0}} - \boldsymbol{l} \cdot \boldsymbol{a})_1. \end{aligned}$$

Choosing normalizing constant in an appropriate way, we obtain (3.11) for $\boldsymbol{m} = \boldsymbol{0}$. Let

$$\hat{a}_{\boldsymbol{l}} = \frac{1}{\sqrt{h}}\Big[\langle w | x - \boldsymbol{X}^{\boldsymbol{l}} \rangle - \langle z | -ih\frac{\partial}{\partial x} - \boldsymbol{P}^{\boldsymbol{l}} \rangle\Big].$$

be one of the creation operators $\hat{\rho}_{\boldsymbol{l}}^{1/2}$. Applying Proposition 3.5 to $\psi_{\boldsymbol{0},\boldsymbol{l}}$ and $\hat{a}_{\boldsymbol{l}}$ we obtain (3.11) for all $\boldsymbol{m}$. □

Remark. Of course, we can find also exact rest points of H; these points are given by

$$p_1 = -y_2^0, \quad p_2 = 0, \quad x_1 = y_1^0, \quad x_2 = y_2^0, \tag{3.12}$$

where $\boldsymbol{y}^0$ are the critical points of v. Simple analysis shows that these points do not give additinal information about the spectrum, because they are contained in $O(e^{-C/\epsilon})$-neighborhhods of the points $\Lambda_{\boldsymbol{l}}$ (see above). The criterion for their stability in the linear approximation is the same. From the other side, these rest points exist independently on the smallness of ϵ, but their stability depends on ϵ: only the point corresponding to local minimum points of v are stable for any ϵ.

3.3 Spectral series for closed curves. The left boundary

In this section, we construct spectral series corresponding to invariant closed curves lying in the plane $\mathcal{I}_1 = 0$.

Consider a certain finite motion edge $\mathcal{E}^r$ of the graph $G(0)$. Recall that the corresponding closed curves $\Lambda_l^r(0, \mathcal{I}_2^r, \epsilon)$ are given by

$$\mathcal{P} = \mathcal{Q} = 0, \quad \boldsymbol{\mathcal{Y}} = \tilde{\boldsymbol{\mathcal{Y}}}^r(t, 0, \mathcal{I}_2^r, \epsilon) + \boldsymbol{l} \cdot \boldsymbol{a}, \tag{3.13}$$

where $\tilde{\mathcal{Y}}^r$ are non-constant closed trajectories. Denote by $T = T(\mathcal{I}_2^r)$ their periods.

3.3.1 Invariant complex germ

Like in the previous case, we construct a complex germ in the coordinates $\boldsymbol{\mathcal{P}} = (\mathcal{P}, \mathcal{Y}_1), \boldsymbol{\mathcal{X}} = (\mathcal{Q}, \mathcal{Y}_2)$ using Proposition 2.12. The matrix $\mathcal{H}''(t)$ has the form

$$\mathcal{H}''(t) = \begin{pmatrix} 0 & 0 & -\omega & 0 \\ 0 & -\mathcal{H}_{12}(t) & 0 & -\mathcal{H}_{22}(t) \\ \omega & 0 & 0 & 0 \\ 0 & \mathcal{H}_{11}(t) & 0 & \mathcal{H}_{12}(t) \end{pmatrix},$$

$$\omega = \frac{\partial \mathcal{H}}{\partial \mathcal{I}_1}\bigg|_{\substack{\mathcal{I}_1=0,\\ \boldsymbol{\mathcal{Y}}=\tilde{\boldsymbol{\mathcal{Y}}}(t)}}, \quad \mathcal{H}_{jk} = \frac{\partial^2 \mathcal{H}}{\partial \mathcal{Y}_1 \partial \mathcal{Y}_2}\bigg|_{\substack{\mathcal{I}_1=0,\\ \boldsymbol{\mathcal{Y}}=\tilde{\boldsymbol{\mathcal{Y}}}(t)}}, \quad j,k = 1,2.$$

The matrix M of the monodromy operator for the system $\dot{\boldsymbol{\eta}} = \mathcal{H}''(t)\boldsymbol{\eta}$ can be found explicitly; it has the form

$$M = \begin{pmatrix} \cos \omega T & -\sin \omega T & 0 & 0 \\ \sin \omega T & \cos \omega T & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and its eigenvalues are 1 (twice) and $\exp(\pm i\omega T)$.

It is important to note that the Jacobian in (2.20) is periodic in t; this means that the monodromy operators of systems (2.19) and (2.21) have equal eigenvalues, and the transformation (2.20) maps eigenvectors of the monodromy operator of (2.21) into eigenvectors of the monodromy operator of (2.19) with the same eigenvalues.

Therefore, all these trajectories are orbitally stable in the linear approximation, and there exists an invariant complex germ, which we denote by (λ). Let us construct an invariant basis $(\boldsymbol{\rho}^1(t), \boldsymbol{\rho}^2(t))$ of a complex germ. The vector $\boldsymbol{\rho}^2(t)$ is always fixed:

$$\boldsymbol{\rho}^2(t) = (\dot{\boldsymbol{\mathcal{P}}}(t), \dot{\boldsymbol{\mathcal{X}}}(t)) = (0,\ d\tilde{\mathcal{Y}}_1/dt,\ 0,\ d\tilde{\mathcal{Y}}_2/dt).$$

The vector $\boldsymbol{\rho}^1(t)$ is defined by the invariant complex germ axioms and has the form

$$\boldsymbol{\rho}^1(t) = (1,\ 0,\ -i,\ 0) \exp i\omega t.$$

3.3.2 Quantization conditions and asymptotic eigenvalues

The quantization condition (2.26) for the family $\Lambda_{\boldsymbol{l}}^r(0, \mathcal{I}_2, \epsilon)$ reads as

$$\mathcal{I}_2^r = \tilde{\mathcal{I}}_2^{(m)}(h) := hm, \quad m \in \mathbb{Z}$$

and we have the following formula for asymptotic eigenvalues:

$$E_{\boldsymbol{m},\nu}^r(h, \epsilon) = \mathcal{H}^r(0, \tilde{\mathcal{I}}_2^{(m_2)}, \epsilon) + h\frac{\partial \mathcal{H}}{\partial \mathcal{I}_1}(0, \tilde{\mathcal{I}}_2^{(m_2)}, \epsilon)\Big(\frac{1}{2} + m_1\Big) + h\frac{2\pi}{T}\nu, \tag{3.14}$$

$$\boldsymbol{m} = (m_1, m_2) \in \mathbb{Z}_+ \times \mathbb{Z}, \quad \nu \in \mathbb{Z}, \quad |m_2 h| \geq c > 0 \text{ as } h \to 0. \tag{3.15}$$

Let us simplify (3.14). Recall [4, §50.A] that $2\pi/T = \partial\mathcal{H}/\partial\mathcal{I}_2$; this means that the numbers $E^r_{m_1,m_2',\nu'}(h, \epsilon)$ and $E^r_{m_1,m_2'',\nu''}(h, \epsilon)$ coincide up to $O(h^2)$ if $m_2' + \nu' = m_2'' + \nu''$. The corresponding asymptotic eigenfunctions $\psi_{m_1,m_2',\nu'}$ and $\psi_{m_1,m_2'',\nu''}$ coincide up to $O(h)$. This means that the triples (m_1, m_2', ν') and (m_1, m_2'', ν'') define the same quasimode from the point of view of this approximation. In what follows, in (3.14) we always put $\nu = 0$ and write

$$E_{\boldsymbol{m}}^r(h, \epsilon) = \mathcal{H}^r\big(0, \tilde{\mathcal{I}}_2^{(m_2)}(h), \epsilon\big) + h\frac{\partial \mathcal{H}}{\partial \mathcal{I}_1}(0, \tilde{\mathcal{I}}_2^{(m_2)}, \epsilon)\Big(\frac{1}{2} + m_1\Big). \tag{3.16}$$

Taking into account the smallness of h, we can rewrite (3.16) in a final form:

$$E_{\boldsymbol{m}}^r = \mathcal{H}^r\big(\mathcal{I}_1^{(m_1)}(h), \mathcal{I}_2^{(m_2)}(h), \epsilon\big) + O(h^2), \quad \boldsymbol{m} \in \mathbb{Z}_+^2,$$

where

$$\mathcal{I}_j^{(m_j)}(h) = h(m_j + \frac{1}{2}), \quad m_j \in \mathbb{Z}, \quad j = 1, 2,$$

with m_1 and m_2 satisfying (3.15).

3.3.3 Properties of asymptotic eigenfunctions

Denote by $\psi_{\boldsymbol{m},\boldsymbol{l}}^r$ the asymptotic eigenfunction corresponding to the asymptotic eigenvalues $E_{\boldsymbol{m}}$ and to the curve $\Lambda_{\boldsymbol{l}}^r$.

Proposition 3.6. *The functions $\psi_{\boldsymbol{m},\boldsymbol{l}}^r(\boldsymbol{x}, h)$ can be normalized in such a way that*

$$\psi_{\boldsymbol{m},\boldsymbol{l}}^r(\boldsymbol{x}, h) = \psi_{\boldsymbol{m},(0,0)}^r(\boldsymbol{x} - \boldsymbol{l}\cdot\boldsymbol{a}, h) \exp\Big(-\frac{i}{h} l_2 a_{22} x_1\Big). \tag{3.17}$$

Proof. We prove this proposition first for $\boldsymbol{m} = 0$.

Recall that the phase function $S_{\boldsymbol{l}}$ corresponding to $\Lambda^r_{\boldsymbol{l}}(0, \tilde{\mathcal{I}}_2^{(m_2)}, \epsilon)$ has the form

$$S_{\boldsymbol{l}}(\boldsymbol{x}) = \int_{\tau^0_{\boldsymbol{l}}}^{\tau_{\boldsymbol{l}}} \boldsymbol{P}^{\boldsymbol{l}}(t)\dot{\boldsymbol{X}}^{\boldsymbol{l}}(t)dt + \left\langle \boldsymbol{P}^{\boldsymbol{l}} \middle| \boldsymbol{x} - \boldsymbol{X}^{\boldsymbol{l}}(t) \right\rangle + \frac{1}{2}\left\langle \boldsymbol{x} - \boldsymbol{X}^{\boldsymbol{l}}(\tau) | Q(\tau)\left(\boldsymbol{x} - \boldsymbol{X}^{\boldsymbol{l}}(\tau)\right)\right\rangle,$$

where $\tau_{\boldsymbol{l}}$ is defined from the equation

$$\left\langle \boldsymbol{x} - \boldsymbol{X}^{\boldsymbol{l}}(\tau) \middle| \dot{\boldsymbol{X}}^{\boldsymbol{l}}(\tau) \right\rangle = 0,$$

and $\tau^0_{\boldsymbol{l}}$ are fixed numbers. An important fact here is that Q does not depend on $\boldsymbol{l}$.

It is clear that we can put

$$\tau_{\boldsymbol{l}}(\boldsymbol{x}) = \tau_{\boldsymbol{0}}(\boldsymbol{x} - \boldsymbol{l} \cdot \boldsymbol{a}) \text{ and } \tau^0_{\boldsymbol{l}} = \tau^0_{\boldsymbol{0}}. \tag{3.18}$$

Therefore,

$$\begin{aligned} S_{\boldsymbol{l}}(\boldsymbol{x}) - S_{(0,0)}(\boldsymbol{x} - \boldsymbol{l} \cdot \boldsymbol{a}) &= \int_{\tau^0_{(0,0)}}^{\tau_{(0,0)}(\boldsymbol{x}-\boldsymbol{l}\cdot\boldsymbol{a})} \left[\boldsymbol{P}^{\boldsymbol{l}}(t)\dot{\boldsymbol{X}}^{\boldsymbol{l}}(t) - \boldsymbol{P}^{\boldsymbol{l}}(t)\dot{\boldsymbol{X}}^{\boldsymbol{l}}(t)\right]dt \\ &= -l_{22}a_{22}X_1^{(0,0)}(t)\Big|_{\tau^0_{(0,0)}}^{\tau_{(0,0)}(\boldsymbol{x}-\boldsymbol{l}\cdot\boldsymbol{a})} \\ &\quad - l_2 a_{22}\left(x_1 - X_1^{(0,0)}\left(\tau_{(0,0)}(\boldsymbol{x} - \boldsymbol{l} \cdot \boldsymbol{a})\right)\right) \\ &= -l_2 a_{22}\left(x_1 + X_1^{(0,0)}(\tau^0_{\boldsymbol{0}}) + (\boldsymbol{l} \cdot \boldsymbol{a})_1\right), \end{aligned}$$

and (3.17) is proved for $\boldsymbol{m} = 0$.

The proof for $\boldsymbol{m} \neq 0$ follows from Proposition 3.5 (see also the proof of Proposition 3.11). □

3.4 Spectral series for closed curves. The exterior boundaries

3.4.1 The monodromy operator

Now let us consider the family of curves $\Lambda^r_{\boldsymbol{l}}(\mathcal{I}_1, 0, \epsilon)$.

To construct an invariant complex germ we again use Proposition 2.12, but now we put

$$\mathcal{P} = (\mathcal{I}_1, \mathcal{Y}_1), \quad \mathfrak{X} = (\Phi_1, \mathcal{Y}_2).$$

In these variables the curves $\Lambda^r_{\boldsymbol{l}}(\mathcal{I}_1, 0, \epsilon)$ are given by

$$\mathcal{I}_1 = \text{const}, \quad \Phi_1 = \omega t + \Phi_1^0, \quad \boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{Y}}^0 + \boldsymbol{l} \cdot \boldsymbol{a}, \quad \omega = \frac{\partial \mathcal{H}}{\partial \mathcal{I}_1}.$$

where $\boldsymbol{\mathcal{Y}}^0$ is an extremum point of $\mathcal{H}(\mathcal{I}_1, \cdot, \epsilon)$. Again by $T = T(\mathcal{I}_1)$ we denote the periods of these trajectories.

The corresponding matrix $\mathcal{H}''$ is constant and has the following form:

$$\mathcal{H}'' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\mathcal{H}_{02} & -\mathcal{H}_{12} & 0 & -\mathcal{H}_{22} \\ \mathcal{H}_{00} & \mathcal{H}_{01} & 0 & \mathcal{H}_{02} \\ \mathcal{H}_{01} & \mathcal{H}_{11} & 0 & \mathcal{H}_{12} \end{pmatrix},$$

where the subindex 0 means the differentiation in $\mathcal{I}_1$ and the subindices 1/2 mean the differentiation in $\mathcal{Y}_1/\mathcal{Y}_2$; for example, $\mathcal{H}_{01} = \partial^2\mathcal{H}/\partial\mathcal{I}_1\partial\mathcal{Y}_1$ etc.; all the functions are calculated at the point $\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{Y}}^0$.

The equation for the eigenvalues of $\mathcal{H}''$ looks as

$$\lambda^2(\lambda^2 + \omega^2) = 0, \quad \omega = \sqrt{\det(\mathcal{H}_{jk})_{j,k=1,2}}.$$

Therefore, the eigenvalues are 0 (twice) and $\pm i\omega$. Note that the Jacobian (2.20) is periodic in t; this means that the monodromy operator M of the system has the form

$$M = e^{\mathcal{H}''T}.$$

3.4.2 Invariant basis

We will not study the orbital stability of these trajectories, because it is much easier to obtain the complex germ directly. The solutions of the linearized system corresponding to the eigenvalues $\pm i\omega t$ are

$$\boldsymbol{\xi}^{\pm}(t) = \begin{pmatrix} 0 \\ -\mathcal{H}_{22} \\ \mathcal{H}_{02} \pm i(\mathcal{H}_{01}\mathcal{H}_{22} - \mathcal{H}_{02}\mathcal{H}_{12})/\omega \\ \mathcal{H}_{12} \pm i\omega \end{pmatrix} e^{\pm i\omega t},$$

and the corresponding eigenvalues of the monodromy operator are $\exp^{\pm i\omega T}$.

An invariant basis $(\boldsymbol{\eta}^1, \boldsymbol{\eta}^2)$ of the corresponding complex germ (λ) is defined now as follows. The second vector $\boldsymbol{\eta}^2$ is again fixed:

$$\boldsymbol{\eta}^2 = \big(\dot{\mathcal{P}}(t), \dot{\mathcal{X}}(t)\big) = (0,\ 0,\ 1,\ 0).$$

The first basis vector $\boldsymbol{\eta}^1$ is defined by the complex germ axioms; $\boldsymbol{\eta} = \boldsymbol{\xi}^{\pm}$, where the sign "+"/"−" corresponds to the case $\mathcal{H}_{22} > 0/\ \mathcal{H}_{22} < 0$, i. e., to a minimum/maximum point of $\mathcal{H}$.

3.4.3 Formulas for asymptotic eigenvalues

The quantization condition (2.26) reads as

$$\mathcal{I}_1 = \tilde{\mathcal{I}}_1^{(m_1)} = hm_1, \quad m_1 \in \mathbb{Z}_+.$$

Therefore, the formulas for asymptotic eigenvalues (2.28) take the form

$$E_{\boldsymbol{m},\nu}(h,\epsilon) = \mathcal{H}^r\big(\tilde{\mathcal{I}}_1^{(m_1)}(h), 0, \epsilon\big) \pm \omega\Big(\frac{1}{2} + m_2\Big)h + \frac{2\pi}{T}\nu, \qquad (3.19)$$
$$m_2 \in \mathbb{Z}_+, \quad |m_1 h| \geq c > 0 \text{ as } h \to 0.$$

Taking into account the equality $2\pi/T = \partial\mathcal{H}/\partial\mathcal{I}_1$ and using the arguments of Subsection 3.3.2 we put $\nu = 0$ and rewrite (3.19) in the form

$$E_{\boldsymbol{m},\nu}(h,\epsilon) = \mathcal{H}^r(\mathcal{I}_1^{(m_1)}, \mathcal{I}_2^{(m_2)}, \epsilon) + O(h^2),$$
$$\mathcal{I}_k^{(m_k)} = h\Big(\frac{1}{2} + m_k\Big), \qquad k = 1, 2,$$
$$m_1 \in \mathbb{Z}, \quad |m_1 h| \geq c > 0 \text{ as } h \to 0, \quad m_2 \in \mathbb{Z}_+.$$

For the corresponding asymptotic eigenfunctions $\psi_{\boldsymbol{m},l}$ Proposition 3.6 holds.

3.5 Spectral series for tori

3.5.1 Calculation of the Maslov indices for the basis cycles

Let us consider the family of the tori $\Lambda_l^r(\mathcal{I}_1, \mathcal{I}_2, \epsilon)$.

Recall that in the coordinates $(\mathcal{P}, \mathcal{Y}_1, \mathcal{Q}, \mathcal{Y}_2)$ these tori are given by the equations.

$$\mathcal{P} = \sqrt{2\mathcal{I}_1}\cos\Phi_1, \quad \mathcal{Q} = \sqrt{2\mathcal{I}_1}\sin\Phi_1,$$
$$\boldsymbol{\mathcal{Y}} = \tilde{\boldsymbol{\mathcal{Y}}}^{r,l}\big(\frac{T\Phi_2}{2\pi}, \mathcal{I}_1, \mathcal{I}_2, \epsilon\big)$$

Therefore, these tori have two basis cycles, $\Gamma_1 = \{\Phi_2 = \text{const}\}$ and $\Gamma_2 = \{\Phi_1 = \text{const}\}$.

Proposition 3.7. *The Maslov indices of the cycles Γ_1 and Γ_2 are equal to* 2.

Proof. Clearly, for each fixed Φ_1 or Φ_2 the jacobian J_δ becomes a periodic function of another variable. Therefore, like for the circle, $J_\delta\big|_{\Gamma_{1/2}} = 2\pi$. □

Therefore, the Bohr-Sommerfeld condition (2.8) for these tori looks as

$$\mathcal{I}_k = \mathcal{I}_k^{m_k}(h) := h\Big(m_k + \frac{1}{2}\Big), \quad m_k \in \mathbb{Z}, \quad k = 1, 2,$$

and the corresponding asymptotic eigenvalues are

$$E^r_{\boldsymbol{m}}(h, \epsilon) = \mathcal{H}^r\big(\mathcal{I}_1^{(m_1)}(h), \mathcal{I}_2^{(m_2)}(h), \epsilon\big), \quad |\mathcal{I}_k^{m_k}| \geq c_k > 0 \text{ as } h \to 0. \tag{3.20}$$

3.5.2 Construction of asymptotic eigenfunctions

The asymptotic eigenfunctions can be found with the help of the canonical operator. It is useful to agree on the choice of a canonical atlas, a partition of unity, focal coordinates, and a marked point on $\Lambda^r_{\boldsymbol{l}}\big(\boldsymbol{\mathcal{I}}^{\boldsymbol{m}}(h), \epsilon\big)$. Let us choose a canonical atlas $(\Omega^{\boldsymbol{0}}_j)$, focal coordinates $(I^{\boldsymbol{0}}_j)$, a partition of unity $(e^{\boldsymbol{0}}_j)$, and a marked point $\boldsymbol{r}^{0,\boldsymbol{0}}$ on $\Lambda^r_{\boldsymbol{0}}\big(\boldsymbol{\mathcal{I}}^{(\mathcal{M}}(h), \epsilon\big)$, then a canonical atlas $(\Omega^{\boldsymbol{l}}_j)$, focal coordinates $I^{\boldsymbol{l}}_j$, a partition of unity $(e^{\boldsymbol{m}}_j)$, and a marked point $\boldsymbol{r}^{0,\boldsymbol{l}}$ corresponding to $\Lambda^r_{\boldsymbol{l}}\big(\boldsymbol{\mathcal{I}}^{\boldsymbol{m}}(h), \epsilon\big)$ will be fixed as follows:

$$\begin{aligned}
&\Omega^{\boldsymbol{l}}_j = \Omega^{\boldsymbol{0}}_j + \big(-(\boldsymbol{l}\cdot\boldsymbol{a})_2, 0, \boldsymbol{l}\cdot\boldsymbol{a}\big),\\
&I^{\boldsymbol{l}}_j = I^{\boldsymbol{0}}_j,\\
&e^{\boldsymbol{l}}_j(\boldsymbol{r}) = e^{\boldsymbol{0}}_j\Big(\boldsymbol{r} - \big(-(\boldsymbol{l}\cdot\boldsymbol{a})_2, 0, \boldsymbol{l}\cdot\boldsymbol{a}\big)\Big),\\
&\boldsymbol{r}^{0,\boldsymbol{l}} = \boldsymbol{r}^{0,\boldsymbol{0}} + \big(-(\boldsymbol{l}\cdot\boldsymbol{a})_2, 0, \boldsymbol{l}\cdot\boldsymbol{a}\big).
\end{aligned}$$

Proposition 3.8. *Let the canonical operators $\mathcal{K}_{h,\Lambda^r_{\boldsymbol{l}}}$ on $\Lambda^r_{\boldsymbol{l}}$ be defined according to the rules described above, functions $u_{\boldsymbol{l}} \in C^\infty(\Lambda^r_{\boldsymbol{l}})$ satisfy*

$$u_{\boldsymbol{l}}(\boldsymbol{r}) = u_{\boldsymbol{0}}\Big(\boldsymbol{r} - \big(-(\boldsymbol{l}\cdot\boldsymbol{a})_2, 0, \boldsymbol{l}\cdot\boldsymbol{a}\big)\Big), \tag{3.21}$$

and $\psi_{\boldsymbol{l}} = \mathcal{K}_{h,\Lambda^r_{\boldsymbol{l}}} u_{\boldsymbol{l}}$, then up to normalizing constants we have

$$\psi_{\boldsymbol{l}}(\boldsymbol{x}, h, \epsilon) = \psi_{\boldsymbol{0}}(\boldsymbol{x} - \boldsymbol{l}\cdot\boldsymbol{a}, h, \epsilon)\exp\Big(-\frac{i}{h} l_2 a_{22} x_1\Big).$$

Proof. It is enough to prove for any j the equality

$$\phi_{\boldsymbol{l},j}(\boldsymbol{x}) = \phi_{\boldsymbol{0},j}(\boldsymbol{x} - \boldsymbol{l} \cdot \boldsymbol{a}) \exp\Big(-\frac{i}{h} l_2 a_{22} x_1 \Big),$$

where

$$\phi_{\boldsymbol{l},j} = \mathcal{K}_{h,\Omega_j^{\boldsymbol{l}}}^{\boldsymbol{r}^{0,\boldsymbol{l}}, I_j^{\boldsymbol{l}}} \big(f_j^{\boldsymbol{l}} \big), \quad f_j^{\boldsymbol{l}} = e_j^{\boldsymbol{l}} u_{\boldsymbol{l}}.$$

The functions $f_j^{\boldsymbol{l}}$ inherit the periodicity properties of $e_j^{\boldsymbol{l}}$ and $u_{\boldsymbol{m}}$.

Let us find the generating function $S^{\boldsymbol{l}}$ for $\Lambda_{\boldsymbol{l}}^r$. We have:

$$\left[\begin{aligned} S^{\boldsymbol{l}}(\boldsymbol{r}^{0,\boldsymbol{l}}, \boldsymbol{r}) &= \int_{\gamma_{\boldsymbol{l}}(\boldsymbol{r}^{0,\boldsymbol{l}},\boldsymbol{r})} \langle \boldsymbol{p} | d\boldsymbol{x} \rangle \\ &= \int_{\gamma_{\boldsymbol{l}}\left(\boldsymbol{r}^{0,\boldsymbol{0}} + \left(-(\boldsymbol{l}\cdot\boldsymbol{a})_2, 0, \boldsymbol{l}\cdot\boldsymbol{a}\right), \boldsymbol{r}\right)} \langle \boldsymbol{p} | d\boldsymbol{x} \rangle \\ &= \int_{\gamma_{\boldsymbol{0}}\left(\boldsymbol{r}^{0,\boldsymbol{0}}, \boldsymbol{r} - \left(-(\boldsymbol{l}\cdot\boldsymbol{a})_2, 0, \boldsymbol{l}\cdot\boldsymbol{a}\right)\right)} \Big\langle (\boldsymbol{p} - (\boldsymbol{l}\cdot\boldsymbol{a})_2, 0) \Big| d(\boldsymbol{x} + \boldsymbol{l}\cdot\boldsymbol{a}) \Big\rangle \\ &= \int_{\gamma_{\boldsymbol{0}}\left(\boldsymbol{r}^{0,\boldsymbol{0}}, \boldsymbol{r} - \left(-(\boldsymbol{l}\cdot\boldsymbol{a})_2, 0, \boldsymbol{l}\cdot\boldsymbol{a}\right)\right)} \langle \boldsymbol{p} | d\boldsymbol{x} \rangle - l_2 a_{22} \int_{\gamma_{\boldsymbol{0}}(\boldsymbol{r}^{0,\boldsymbol{0}},\boldsymbol{r})} dx_1 \\ &= S^{\boldsymbol{0}}\Big(\boldsymbol{r}^{0,\boldsymbol{0}}, \boldsymbol{r} - \big(-(\boldsymbol{l}\cdot\boldsymbol{a})_2, 0, \boldsymbol{l}\cdot\boldsymbol{a}\big)\Big) - l_2 a_{22}(x_1 - (\boldsymbol{l}\cdot\boldsymbol{a})_1 - x_1^{0,\boldsymbol{0}}), \end{aligned}\right. \tag{3.22}$$

where $\gamma_{\boldsymbol{l}}(\boldsymbol{r}', \boldsymbol{r}'') \subset \Lambda_{\boldsymbol{l}}^r$ denotes a curve between point $\boldsymbol{r}', \boldsymbol{r}'' \in \Lambda_{\boldsymbol{l}}^r$, $\boldsymbol{r} = (\boldsymbol{p}, \boldsymbol{x})$, $\boldsymbol{r}^{0,\boldsymbol{l}} = (\boldsymbol{p}^{0,\boldsymbol{l}}, \boldsymbol{x}^{0,\boldsymbol{l}})$.

Note also that the functions $\varphi_{\boldsymbol{l}}(\boldsymbol{r}) = \sqrt{|\partial\boldsymbol{\Phi}/\partial\boldsymbol{x}|}$ also satisfy (3.21).

Consider first the case $I_j^{\boldsymbol{0}} = I_j^{\boldsymbol{l}} = \emptyset$, i. e., non-critical charts $\Omega_j^{\boldsymbol{0}}$ and $\Omega_j^{\boldsymbol{l}}$.

Let us express the vectors $\boldsymbol{r} \in \Lambda_{\boldsymbol{l}}^r$ through their $\boldsymbol{x}$-components: $\boldsymbol{r} = \boldsymbol{r}^{\boldsymbol{l}}(\boldsymbol{x})$. The functions $\boldsymbol{r}^{\boldsymbol{l}}$ obey the property

$$\boldsymbol{r}^{\boldsymbol{l}}(\boldsymbol{x}) = \boldsymbol{r}^{\boldsymbol{0}}(\boldsymbol{x} - \boldsymbol{l}\cdot\boldsymbol{a}) + \big(-(\boldsymbol{l}\cdot\boldsymbol{a})_2, 0, \boldsymbol{l}\cdot\boldsymbol{a}\big). \tag{3.23}$$

Put $S_0^{\boldsymbol{l}} = S^{\boldsymbol{l}}\big(\boldsymbol{r}^{\boldsymbol{l}}(\boldsymbol{x})\big)$, $\varphi_{\boldsymbol{l}}^0(\boldsymbol{x}) = \varphi_{\boldsymbol{l}}\big(\boldsymbol{r}^{\boldsymbol{l}}(\boldsymbol{x})\big)$, then we have $\varphi_{\boldsymbol{l}}^0(\boldsymbol{x}) \equiv \varphi_{\boldsymbol{0}}^0(\boldsymbol{x} - \boldsymbol{l}\cdot\boldsymbol{a})$. Also, substituting (3.23) into (3.22) we come to the equality

$$S_0^{\boldsymbol{l}}(\boldsymbol{x}) = S_0^{\boldsymbol{0}}(\boldsymbol{x} - \boldsymbol{l}\cdot\boldsymbol{a}) - l_2 a_{22}\big(x_1 - (\boldsymbol{l}\cdot\boldsymbol{a})_1 - x_1^{0,\boldsymbol{0}}\big).$$

Now we only have to recall the formula for the pre-canonical operator in a non-critical chart.

Now consider the case $I_j^{\mathbf{0}} = I_j^{\boldsymbol{l}} = \{1\}$, i. e., focal coordinates $\boldsymbol{x}_{\{1\}} = (p_1, x_2)$. Let us express the vectors $\boldsymbol{r} \in \Lambda_{\boldsymbol{l}}^r$ through their p_1– and x_2–components: $\boldsymbol{r} = \boldsymbol{r}^{\boldsymbol{l}}(p_1, x_2)$, then

$$\boldsymbol{r}^{\boldsymbol{l}}(p_1, x_2) = \boldsymbol{r}^{\mathbf{0}}(p_1 + l_2 a_{22}, x_2 - l_2 a_{22}) + \big(-(\boldsymbol{l}\cdot\boldsymbol{a})_2, 0, \boldsymbol{l}\cdot\boldsymbol{a} \big), \tag{3.24}$$

and the corresponding functions

$$S^{\boldsymbol{l}}_{\{1\}}(p_1, x_2) := S^{\boldsymbol{l}}(\boldsymbol{r}^{0,\boldsymbol{l}}, \boldsymbol{r}^{\boldsymbol{l}}(p_1, x_2)) - x_1^{\boldsymbol{l}}(p_1, x_2) p_1$$

satisfy the property

$$S^{\boldsymbol{l}}_{\{1\}}(p_1, x_2) = S^{\mathbf{0}}_{\{1\}}(p_1 + l_2 a_{22}, x_2 - l_2 a_{22}) - \\ - l_{22} a_{22} \big(x_1^{\mathbf{0}}(p_1 + l_2 a_{22}, x_2 - l_2 a_{22}) - x^{0,\mathbf{0}} \big) - (\boldsymbol{l}\cdot\boldsymbol{a})_1 p_1.$$

Denote $\varphi^I_{\boldsymbol{l}}(p_1, x_2) = \varphi_{\boldsymbol{l}}\big(\boldsymbol{r}^{\boldsymbol{l}}(p_1, x_2)\big)$, then $\varphi^I_{\boldsymbol{l}}(p_1, x_2) \equiv \varphi^I_{\mathbf{0}}(p_1 + l_2 a_{22}, x_2 - l_2 a_{22})$. Now we have the following chain of equalities:

$$\begin{aligned}
\phi_{\boldsymbol{l},j}(\boldsymbol{x}) = {} & \sqrt{-\frac{1}{2\pi i h}} \int_{-\infty}^{+\infty} \varphi^I_{\boldsymbol{l}}(p_1, x_2) \exp\Big\{ \frac{i}{h} \big(S^{\boldsymbol{l}}_{\{1\}}(p_1, x_2) + x_1 p_1 \big) \Big\} \\
& \times f_j^{\boldsymbol{l}}\big(\boldsymbol{r}^{\boldsymbol{l}}(p_1, x_2)\big) dp_1 \\
= {} & \sqrt{-\frac{1}{2\pi i h}} \int_{-\infty}^{+\infty} \varphi^I_{\mathbf{0}}(p_1 + l_2 a_{22}, x_2 - l_2 a_{22}) \\
& \times \exp\Big\{ \frac{i}{h} \big[S^{\mathbf{0}}_{\{1\}}(p_1 + l_2 a_{22}, x_2 - l_2 a_{22}) - l_2 a_{22} \big(x_1^{\mathbf{0}}(p_1 + l_2 a_{22}, x_2 - l_2 a_{22}) \\
& \quad - x_1^{0,\mathbf{0}} \big) + \big(x_1 - (\boldsymbol{l}\cdot\boldsymbol{a})_1 \big) p_1 \big] \Big\} \\
& f_j^{\mathbf{0}}\big(p_1 + l_2 a_{22}, p_2^{\mathbf{0}}(p_1 + l_2 a_{22}, x_2 - l_2 a_{22}), \\
& \quad x_1^{\mathbf{0}}(p_1 + l_2 a_{22}, x_2 - l_2 a_{22}), x_2 - l_2 a_{22} \big) dp_1
\end{aligned}$$

(denote $p_1' = p_1 + l_2 a_{22}$)

$$\begin{aligned}
= {} & \sqrt{-\frac{1}{2\pi i h}} \int_{-\infty}^{+\infty} \varphi^I_{\mathbf{0}}(p_1', x_2 - l_2 a_{22}) \\
& \times \exp\Big\{ \frac{i}{h} \big[S^{\mathbf{0}}_{\{1\}}(p_1', x_2 - l_2 a_{22}) - l_2 a_{22} \big(x_1^{\mathbf{0}}(p_1', x_2 - l_2 a_{22}) \\
& \quad - x_1^{0,\mathbf{0}} \big) + \big(x_1 - (\boldsymbol{l}\cdot\boldsymbol{a})_1 \big)(p_1' - l_2 a_{22}) \big] \Big\} \\
& f_j^{\mathbf{0}}\big(p_1', p_2^{\mathbf{0}}(p_1', x_2 - l_2 a_{22}), x_1^{\mathbf{0}}(p_1', x_2 - l_2 a_{22}), x_2 - l_2 a_{22} \big) dp_1' \\
& \quad = \exp\Big(-\frac{i}{h} l_2 a_{22} \big(x_1 - (\boldsymbol{l}\cdot\boldsymbol{a})_1 - x_1^{0,\mathbf{0}} \big) \Big) \phi_{\mathbf{0},j}(\boldsymbol{x} - \boldsymbol{l}\cdot\boldsymbol{a}).
\end{aligned}$$

The cases $I_j^{0,\boldsymbol{l}} = \{2\}$ and $I_j^{0,\boldsymbol{l}} = \{1, 2\}$ can be considered in the same way. □

Remark. It is easy to see that the equalities (3.21) are equivalent to the following simple condition: the function $u_{\boldsymbol{l}}\big(\boldsymbol{P}^{r,\boldsymbol{l}}(\boldsymbol{\Phi},\boldsymbol{\mathfrak{I}},\epsilon),\boldsymbol{X}^{r,\boldsymbol{l}}(\boldsymbol{\Phi},\boldsymbol{\mathfrak{I}},\epsilon)\big)$ does not depend on $\boldsymbol{l}$.

Define the canonical operators on $\Lambda^r_{\boldsymbol{l}}(\boldsymbol{\mathfrak{I}}^{(\boldsymbol{m})}(h),\epsilon)$ according to the above choice and put

$$\psi_{\boldsymbol{m},\boldsymbol{l}}(\boldsymbol{x},h,\epsilon)=\mathcal{K}_{h,\Lambda^r_{\boldsymbol{l}}\left(\mathfrak{I}_1^{(m_1)}(h),\mathfrak{I}_2^{(m_2)}(h),\epsilon\right)}\cdot 1. \tag{3.25}$$

These functions are asymptotic eigenfunctions of $\hat{H}^0_{h,\epsilon}$ (up to $O(h^2)$) corresponding to the asymptotic eigenvalues $E_{\boldsymbol{m}}$ and, respectively, they are at the same time asymptotic eigenfunctions of $\hat{H}_{h,\epsilon}$ up to $O(h^2+e^{-C/\epsilon})$. Choosing right normalizing constants and applying Proposition 3.8 we obtain the equalities

$$\psi_{\boldsymbol{m},\boldsymbol{l}}(\boldsymbol{x},h,\epsilon)=\psi_{\boldsymbol{m},\boldsymbol{0}}(\boldsymbol{x}-\boldsymbol{l}\cdot\boldsymbol{a},h,\epsilon)\exp\Big(-\frac{i}{h}l_2a_{22}x_1\Big).$$

3.6 Spectral series for cylinders

3.6.1 Quantization conditions

Consider the family of the two-dimensional cylinders $\Lambda^r_k(\boldsymbol{\mathfrak{I}},\epsilon)$; they have the common drift vector $\boldsymbol{d}^r(\mathfrak{I}_1,\epsilon)$, i. e., each of these cylinders is invariant under the shift on the vector $\big(-(\boldsymbol{d}\cdot\boldsymbol{a})_2,0,\boldsymbol{d}\cdot\boldsymbol{a}\big)$. If $\Lambda^r_k(\boldsymbol{\mathfrak{I}},\epsilon)$ is given by the equations

$$\begin{aligned}
&p_1=P_1^{r,k}(\Phi_1,\Phi_2),\quad p_2=P_2^{r,k}(\Phi_1,\Phi_2),\\
&x_1=X_1^{r,k}(\Phi_1,\Phi_2),\quad x_2=X_2^{r,k}(\Phi_1,\Phi_2),\\
&(\Phi_1,\Phi_2)\in\mathbb{R}^2.
\end{aligned}$$

then the functions $P^{r,k}_{1/2}$, $X^{r,k}_{1/2}$ satisfy the property

$$\left[\begin{aligned}
&P_1^{r,k}(\Phi_1,\Phi_2+2\pi,\epsilon)=P_1^{r,k}(\Phi_1,\Phi_2,\epsilon)-(\boldsymbol{d}\cdot\boldsymbol{a})_2,\\
&P_2^{r,k}(\Phi_1,\Phi_2+2\pi,\epsilon)=P_2^{r,k}(\Phi_1,\Phi_2,\epsilon),\\
&\boldsymbol{X}^{r,k}(\Phi_1,\Phi_2+2\pi,\epsilon)=\boldsymbol{X}^{r,k}(\Phi_1,\epsilon)+\boldsymbol{d}\cdot\boldsymbol{a}.
\end{aligned}\right. \tag{3.26}$$

Each of these cylinders has only one basis cycle $\Gamma_1=\{\Phi_1\in[0,2\pi],\Phi_2=\text{const}\}$. Therefore, only the variable $\mathfrak{I}_1$ have to be quantized for construction of the canonical operator.

Proposition 3.9. Ind $\Gamma_1=2$.

Proof is similar to that for Proposition 3.7.

Therefore, we have the following quantization conditions:

$$\mathcal{I}_1 = \mathcal{I}_1^{(m)}(h) := h(\frac{1}{2} + m), \quad m \in \mathbb{Z}, \quad \mathcal{I}_1^m \geq c > 0 \text{ as } h \to 0.$$

On each of the cylinders $\Lambda_k^r(\mathcal{I}^{(m)}(h), \mathcal{I}_2, \epsilon)$ one can construct the canonical operator. Like for the tori, we choose a canonical atlas, a partition of unity, focal coordinates, and marked points on each of these cylinders in a special way. Fix some canonical atlas (Ω_j^0), focal coordinates (I_j^0), a partition of unity $(e_j)^0$, and a marked point $\boldsymbol{r}^{0,0}$ on $\Lambda_0^r(\mathcal{I}_1^{(m)}, \mathcal{I}_2, \epsilon)$; we require that the canonical atlas and the partition of unity are invariant under the shift onto the vector $\big(-(\boldsymbol{d}\cdot\boldsymbol{a})_2, 0, \boldsymbol{d}\cdot\boldsymbol{a}\big)$. Then a canonical atlas $(\Omega_j^{\boldsymbol{l}})$, focal coordinates I_j^k, a partition of unity (e_j^k), and a marked point $\boldsymbol{r}^{0,k}$ corresponding to $\Lambda_k^r(\mathcal{I}_1^{(m)}, \mathcal{I}_2, \epsilon)$ will be fixed as follows:

$$\begin{aligned}
&\Omega_j^k = \Omega_j^{\boldsymbol{0}} + \big(k(J\boldsymbol{f}\cdot\boldsymbol{a})_2, 0, -kJ\boldsymbol{f}\cdot\boldsymbol{a}\big),\\
&I_j^k = I_j^{\boldsymbol{0}},\\
&e_j^k(\boldsymbol{r}) = e_j^0\Big(\boldsymbol{r} - \big(k(J\boldsymbol{f}\cdot\boldsymbol{a})_2, 0, -kJ\boldsymbol{f}\cdot\boldsymbol{a}\big)\Big),\\
&\boldsymbol{r}^{0,k} = \boldsymbol{r}^{0,0} + \big(k(J\boldsymbol{f}\cdot\boldsymbol{a})_2, 0, -kJ\boldsymbol{f}\cdot\boldsymbol{a}\big).
\end{aligned}$$

Define the canonical operators according to this choice.

Proposition 3.10. *Let a function $u_k \in C^\infty(\Lambda_k^r)$ be periodic, i. e.,*

$$u_k(\boldsymbol{r}) = u_k\Big(\boldsymbol{r} + \big(-(\boldsymbol{d}\cdot\boldsymbol{a})_2, 0, \boldsymbol{d}\cdot\boldsymbol{a}\big)\Big), \tag{3.27}$$

then the functions $\psi_k^r = \mathcal{K}_{h,\Lambda_k^r} u_k$ enjoy the property

$$\psi_k^r(\boldsymbol{x} + \boldsymbol{d}\cdot\boldsymbol{a}) = \psi_k^r(\boldsymbol{x}) \exp\Big(\frac{i}{h}\big(2\pi\mathcal{I}_2^{r,k} - (\boldsymbol{d}\cdot\boldsymbol{a})_2 x_1 - \frac{1}{2}(\boldsymbol{d}\cdot\boldsymbol{a})_1(\boldsymbol{d}\cdot\boldsymbol{a})_2\big)\Big). \tag{3.28}$$

Proof. Proof for non-critical charts directly follows from Proposition 1.15. For critical charts, the procedure is similar to the proof of Proposition 3.8. □

Remark. The equality (3.27) means that the function $u_k\big(\boldsymbol{P}^{r,k}(\boldsymbol{\Phi}), \boldsymbol{X}^{r,k}(\boldsymbol{\Phi})\big)$ is periodic in Φ_2 with the period 2π.

Put

$$E_m^r(\mathcal{I}_2, h, \epsilon) = \mathcal{H}^r\big(\mathcal{I}_1^{(m)}, \mathcal{I}_2, \epsilon\big), \quad \psi^r_{m,\mathcal{I}_2,k}(\boldsymbol{x}, h, \epsilon) = \mathcal{K}_{\Lambda_k^r\left(\mathcal{I}^{(m)}(h), \mathcal{I}_2, \epsilon\right)} \cdot 1. \tag{3.29}$$

As follows from Propositions 3.8 and 3.10, the functions $\psi^r_{m,\mathcal{J}_2,k}$ obey the following properties:

$$\psi^r_{m,\mathcal{J}_2,k}(\boldsymbol{x},h,\epsilon) = \psi^r_{m,\mathcal{J}_2,0}\big(\boldsymbol{x}+k(J_2\boldsymbol{f}\cdot\boldsymbol{a}),h,\epsilon\big)\exp\Big(\frac{i}{h}kf_1a_{22}x_1\Big), \tag{3.30}$$

$$\psi^r_{m,\mathcal{J}_2,k}(\boldsymbol{x}+\boldsymbol{d}\cdot\boldsymbol{a},h,\epsilon) = \psi^r_{m,\mathcal{J}_2,k}(\boldsymbol{x},h,\epsilon) \cdot\exp\Big(\frac{i}{h}(2\pi\mathcal{J}_2^k - (\boldsymbol{d}\cdot\boldsymbol{a})_2x_1 - \frac{1}{2}(\boldsymbol{d}\cdot\boldsymbol{a})_1(\boldsymbol{d}\cdot\boldsymbol{a})_2)\Big). \tag{3.31}$$

Note that the existence of the numbers and functions (3.29) does not imply directly any relation between $E^r_{m,\mathcal{J}_2}$ and the spectrum of $\hat{H}_{h,\epsilon}$, because the functions $\psi_{m,\mathcal{J}_2}$ do not belongs to $L^2(\mathbb{R}^2)$ and Proposition 2.1 is not applicable. Nevertheless, in our case, there is such a relation.

Proposition 3.11. dist $\big(E^r_m(\mathcal{J}_2,h,\epsilon), \operatorname{spec}\hat{H}_{h,\epsilon}\big) = O(h^2)+O(e^{-C/\epsilon})$.

We divide the proof into several steps.

Put

$$\Pi_{\boldsymbol{d}} = \Big\{\tau_1\boldsymbol{d}\cdot\boldsymbol{a}+\tau_2J_2(\boldsymbol{d}\cdot\boldsymbol{a}),\quad \tau_1\in[0,1],\quad \tau_2\in\mathbb{R}.\Big\} \tag{3.32}$$

Proposition 3.12. *The numbers $E^r_m(\mathcal{J}_2,h,\epsilon)$ and the functions $\psi^r_{m,\mathcal{J}_2,k}$ compose quasi-modes of $\hat{H}_{h,\epsilon}$ in $L^2(\Pi_{\boldsymbol{d}})$ with error $O(h^2+e^{-C/\epsilon})$.*

Proof. Let us choose a function $e \in C_0^\infty(\Lambda^r_k(\mathcal{J}^{(m)}(h),\mathcal{J}_2,\epsilon))$ such that $e(p_1,p_2,x_1,x_2)=1$ as $(x_1,x_2)\in\Pi_{\boldsymbol{d}}$ and put

$$\varphi^r_{m,\mathcal{J}_2,k}(\boldsymbol{x},h,\epsilon) = \mathcal{K}_{\Lambda^r_k\left(\mathcal{J}^{(m)}(h),\mathcal{J}_2,\epsilon\right)}\cdot e$$

Then it follows from Proposition 2.5 that

$$\psi^r_{m,\mathcal{J}_2,k}(\boldsymbol{x},h,\epsilon) = \varphi^r_{m,\mathcal{J}_2,k}(\boldsymbol{x},h,\epsilon) \text{ as } \boldsymbol{x}\in\Pi_{\boldsymbol{d}}.$$

As follows from the commutation formula (see Proposition 2.6), there exists a function $\phi\in L^2(\mathbb{R}^2)$, $\phi=O(1)$,

$$\big(\hat{H}^0_{h,\epsilon} - E^r_m(\mathcal{J}_2,h,\epsilon)\big)\varphi^r_{m,\mathcal{J}_2,k}(\boldsymbol{x},h) = h^2\phi(\boldsymbol{x},h,\epsilon) \text{ as } \boldsymbol{x}\in\Pi_{\boldsymbol{d}}.$$

Taking into account the representation (3.2), we can reduce the latter equality to

$$\big(\hat{H}_{h,\epsilon} - E^r_m(\mathcal{J}_2,h,\epsilon)\big)\varphi^r_{m,\mathcal{J}_2,k}(\boldsymbol{x},h,\epsilon) = h^2\phi(\boldsymbol{x},h,\epsilon)+e^{-C/\epsilon}\chi(\boldsymbol{x},h,\epsilon) \text{ as } \boldsymbol{x}\in\Pi_{\boldsymbol{d}},\quad \chi\in L^2(\mathbb{R}^2). \tag{3.33}$$

Now note that the operator $\hat{H}_{h,\epsilon}$ is local, this means that in (3.33) one can replace $\varphi^r_{m,\mathcal{J}_2,k}(\boldsymbol{x},h,\epsilon)$ by $\psi^r_{m,\mathcal{J}_2,k}(\boldsymbol{x},h,\epsilon)$. The proposition is proved. □

3.6.2 Gauge-rotating transformations

Let us show now that by a gauge-rotating transformation we can direct the vector $\boldsymbol{d}^r$ along the x_1-axis (i. e., we can put $d^r_2 = 0$).

Denote

$$\alpha = \frac{(\boldsymbol{d}\cdot\boldsymbol{a})_1}{|\boldsymbol{d}\cdot\boldsymbol{a}|}, \quad \beta = \frac{(\boldsymbol{d}\cdot\boldsymbol{a})_2}{|\boldsymbol{d}\cdot\boldsymbol{a}|}.$$

Introduce new coordinates $\boldsymbol{y}$,

$$\boldsymbol{y} = A\boldsymbol{x}, \quad A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},$$

and a function S,

$$S(\boldsymbol{y}) = \frac{1}{2}(-\alpha\beta y_1^2 + \alpha\beta y_2^2 + 2\beta^2 y_1 y_2).$$

Introduce now an operator $\hat{U}$,

$$f(\boldsymbol{x}) \overset{\hat{U}}{\mapsto} g(\boldsymbol{y}) = e^{-\frac{i}{h}S(\boldsymbol{y})} f(A^{-1}\boldsymbol{y});$$

$\hat{U}$ is a unitary operator in $L^2(\mathbb{R}^2)$, and, moreover, it is well-defined in $L^2_{\text{loc}}(\mathbb{R}^2)$. Now put

$$\tilde{H}_{h,\epsilon} = \hat{U}\hat{H}_{h,\epsilon}\hat{U}^{-1},$$

i. e.

$$\tilde{H}_{h,\epsilon} = \frac{1}{2}\Big(-ih\frac{\partial}{\partial y_1} + y_2\Big)^2 + \frac{1}{2}\Big(-ih\frac{\partial}{\partial y_2}\Big)^2 + \epsilon w(\boldsymbol{y}), \quad w(\boldsymbol{y}) = v(A^{-1}\boldsymbol{y}).$$

The operators $\tilde{H}_{h,\epsilon}$ and $\hat{H}_{h,\epsilon}$ are unitary equivalent, therefore, their spectra coincide.

3.6.3 Spectral estimate

Proposition 3.13. *Let a number $E(\epsilon, h)$ and a function $\psi(\boldsymbol{x}, h, \epsilon) \in C^\infty$ be a quasimode of $\hat{H}_{h,\epsilon}$ with error $O\big(f(h,\epsilon)\big)$ in $L^2(\Pi_{\boldsymbol{d}})$, where $f(h,\epsilon) \to 0$ as $h, \epsilon \to 0$, and ψ satisfy* (3.31)*, then*

$$\operatorname{dist}\big(E(h,\epsilon), \operatorname{spec}\hat{H}_{h,\epsilon}\big) = O\big(f(h,\epsilon)\big). \tag{3.34}$$

Proof. First of all, without loss of generality we can assume

$$\|\psi\|_{L^2(\Pi_d)} \geq c > 0 \text{ as } h \to 0. \tag{3.35}$$

It is more convenient to deal with the rotated coordinates (y_1, y_2) constructed in the previous subsection. The function $\varphi(\boldsymbol{y}, h, \epsilon) := \hat{U}\psi(\boldsymbol{x}, h, \epsilon)$ satisfies the condition

$$\varphi(y_1 + |\boldsymbol{d}\cdot\boldsymbol{a}|, y_2, h, \epsilon) = \varphi(y_1, y_2, h, \epsilon) \exp\Big(2\pi \frac{i}{h}\mathfrak{I}_2\Big).$$

Denote

$$\tilde{\Pi}_s = \Big\{(y_1, y_2) \in \mathbb{R}^2 : \quad -s|\boldsymbol{d}\cdot\boldsymbol{a}| \leq y_1 \leq s|\boldsymbol{d}\cdot\boldsymbol{a}|\Big\}, \quad s \in \mathbb{Z},$$

then the pair $\big(E(h,\epsilon), \varphi\big)$ is a quasimode of $\tilde{H}_{h,\epsilon}$ in $L^2(\tilde{\Pi}_0)$. Note that

$$\|f\|_{L^2(\tilde{\Pi}_s)} = \sqrt{s}\|f\|_{L^2(\tilde{\Pi}_1)}$$

for any function f satisfying $f(y_1 + |\boldsymbol{d}\cdot\boldsymbol{a}|) = e^{i\alpha} f(y_1, y_2)$, $\alpha \in \mathbb{R}$; in particular, this holds for $f = \varphi$ and for $f = \Phi := \big(\tilde{H}_{h,\epsilon} - E(h,\epsilon)\big)\varphi$.

Choose now a smooth function $e(\xi)$ such that

$$\begin{aligned} 0 \leq & e(\xi) && \leq 1, \\ & e(\xi) && = 1 \quad \text{as } \xi \in (-|\boldsymbol{d}\cdot\boldsymbol{a}|, |\boldsymbol{d}\cdot\boldsymbol{a}|), \\ & e(\xi) && = 0 \quad \text{as } \xi \notin (-2|\boldsymbol{d}\cdot\boldsymbol{a}|, 2|\boldsymbol{d}\cdot\boldsymbol{a}|), \end{aligned}$$

and choose a constant C_0 such that

$$|e| + |e'| + |e''| \leq C_0.$$

Put $e_s(\boldsymbol{y}) := e(y_1/s)$.

Now we have the following chain of equalities and inequalities:

$$\begin{aligned} &\sqrt{s}\,\mathrm{dist}\,\big(E(h,\epsilon), \mathrm{spec}\,\hat{H}_{h,\epsilon}\big)\|\varphi\| \\ &\quad \leq \mathrm{dist}\,\big(E(h,\epsilon), \mathrm{spec}\,\hat{H}_{h,\epsilon}\big)\|e_s\varphi\|_{L^2(\tilde{\Pi}_s)} \\ &\quad \leq \Big\|\big(\tilde{H}_{h,\epsilon} - E(h,\epsilon)\big)(e_s\varphi)\Big\| \\ &\quad \leq \Big\|e_s\Phi - \frac{1}{2}h^2\Delta e_s\varphi - h^2\langle\nabla e|\nabla\varphi\rangle - ihx_2\frac{\partial e_s}{\partial x_1}\Big\| \\ &\quad \leq \|e_s\Phi\| + \frac{1}{2}h^2\|\Delta e_s\varphi\| + h^2\|\langle\nabla e_s|\nabla\varphi\rangle\| + h\Big\|x_2\frac{\partial e_s}{\partial y_1}\varphi\Big\| \\ &\quad \leq C_0\sqrt{2s}\|\Phi\|_{L^2(\tilde{\Pi}_1)} + \frac{h^2C_0\sqrt{s}}{2s^2}\|\varphi\|_{L^2(\tilde{\Pi}_1)} + \frac{h^2C_0\sqrt{s}}{s}\Big\|\frac{\partial\varphi}{\partial y_1}\Big\|_{L^2(\tilde{\Pi}_1)} \\ &\quad\quad + \frac{h^2C_0\sqrt{s}}{s}\Big\|\frac{\partial\varphi}{\partial y_2}\Big\|_{L^2(\tilde{\Pi}_1)} + \frac{hC_0\sqrt{s}}{s}\Big\|x_2\varphi\Big\|_{L^2(\tilde{\Pi}_1)}. \end{aligned}$$

As s tends to $+\infty$ we obtain the inequality

$$\operatorname{dist}\big(E(h,\epsilon),\operatorname{spec}\hat{H}_{h,\epsilon}\big)\|\varphi\|_{L^2(\tilde{\Pi}_1)} \le C_0\sqrt{2}\big\|\big(\tilde{H}_{h,\epsilon}-E(h,\epsilon)\big)\varphi\big\|_{L^2(\tilde{\Pi}_1)}.$$

Taking into account (3.35), we arrive at (3.34). □

Proof of Proposition 3.11. Proof follows now from Propositions 3.13 and 3.12. □

3.7 Spectral series for open curves

Consider a certain infinite motion edge $\mathcal{E}^r$ of the graph $G(0)$ and the corresponding family of open curves $\Lambda_k^r(0,\mathcal{I}_2^r,\epsilon)$; recall that they are given by (3.13), where $\tilde{\mathfrak{Y}}^r$ are open trajectories of $\mathcal{H}(0,\cdot)$.

To construct the spectral series, let us analyze the procedure of the construction of spectral series for closed curves, see Section 2.4. Firstly, the solution S of the Hamilton-Jacobi equation (2.31) is always defined by (2.33) and (2.23). In the case at hand we do not need the periodicity of the function S, because the expression (2.23) always defines a single-valued function. From the other side, it follows from (2.23) and Proposition 1.15 that

$$S(\boldsymbol{x}+\boldsymbol{d}\cdot\boldsymbol{a})-S(\boldsymbol{x})=2\pi\mathcal{I}_2-(\boldsymbol{d}\cdot\boldsymbol{a})_2x_1-\frac{1}{2}(\boldsymbol{d}\cdot\boldsymbol{a})_1(\boldsymbol{d}\cdot\boldsymbol{a})_2. \tag{3.36}$$

Therefore, the function $\partial S/\partial\boldsymbol{x}$ is periodic. Now, we are going to find solutions of the generalized transport equation (2.32); let us represent these solutions in the form (2.34) and request that the function Φ is periodic, then the resulting function $\varphi(\boldsymbol{x},h)$ will also be periodic: $\varphi(\boldsymbol{x}+\boldsymbol{d}\cdot\boldsymbol{a},h)=\varphi(\boldsymbol{x},h)$. Now, we have the same problem as for closed curves; therefore the solution $\varphi(\boldsymbol{x},h)$ and the number ω can be found using the calculations of Section 2.4. Obviously, the resulting formulas literally coincide with those obtained in Section 3.3: we have asymptotic eigenvalues

$$E_m^r(\mathcal{I}_2,h,\epsilon)=\mathcal{H}^r\big(\mathcal{I}_1^{(m)}(h),\mathcal{I}_2,\epsilon\big)+O(h^2),$$
$$\mathcal{I}_1^{(m)}(h)=h\Big(\frac{1}{2}+m\Big),\quad m\in\mathbb{Z}_+,$$

and asymptotic eigenfunctions $\psi^r_{m,\mathcal{J}_2}, k$ satisfying the conditions

$$\begin{aligned}
&\psi^r_{m,\mathcal{J}_2,k}(\boldsymbol{x}+\boldsymbol{d}\cdot\boldsymbol{a},h,\epsilon)\\
&\qquad=\psi^r_{m,\mathcal{J}_2,k}(\boldsymbol{x},h,\epsilon)\exp\Big(\frac{i}{h}\big(2\pi\mathcal{J}_2^k-(\boldsymbol{d}\cdot\boldsymbol{a})_2x_1-\frac{1}{2}(\boldsymbol{d}\cdot\boldsymbol{a})_1(\boldsymbol{d}\cdot\boldsymbol{a})_2\big)\Big),\\
&\psi^r_{m,\mathcal{J}_2,k}(\boldsymbol{x},h,\epsilon)=\psi^r_{m,\mathcal{J}_2,0}(\boldsymbol{x}+k(J_2\boldsymbol{f})\cdot\boldsymbol{a},h,\epsilon)\exp\Big(\frac{i}{h}kf_1a_{22}x_1\Big).
\end{aligned}$$

Using Proposition 3.13 we obtain the estimate

$$\operatorname{dist}\big(E^r_m(\mathcal{J}_2,h,\epsilon),\operatorname{spec}\hat{H}_{h,\epsilon}\big)=O(h^{3/2}+e^{-C/\epsilon}). \tag{3.37}$$

3.8 Higher approximations

In the previous sections we have constructed the asymptotics of the spectrum with error $O(h^{3/2}+e^{-C/\epsilon})$ for the boundaries of the regimes and with error $O(h^2+e^{-C/\epsilon})$ in the interior parts of regimes. Now we are going to show that really one can construct in both cases the asymptotics up to $O(h^L+\epsilon^K)$, where K and L are arbitrary positive numbers; as we will see, this asymptotics has the same structure.

3.8.1 The commutation formula

First of all, let us give an improved version of Proposition 2.6.

Proposition 3.14 (Theorem 9.3 in [66]). *Assume that Λ is an invariant Lagrangian manifold of a classical Hamiltonian H, and that on Λ there exists a volume $d\sigma$ invariant with respect to the corresponding Hamiltonian flow g^t_H. Fix some canonical atlas, focal coordinates and a partition of unity on Λ; denote the corresponding canonical operator by $\mathcal{K}_{h,\Lambda}$. There exists a sequence of linear differential operators $\{R^j\}_{j=1}^{\infty}$,*

$$R^j : C^\infty(\Lambda)\mapsto C^\infty(\Lambda),$$

with smooth coefficients such that for any function $\varphi\in C^\infty(\Lambda)$ and any number $N\in\mathbb{N}$ there is a function $\psi\in C^\infty(\Lambda)$ satisfying the condition

$$\hat{H}_h\mathcal{K}_{h,\Lambda}\varphi=\mathcal{K}_{h,\Lambda}\Big(\Big(\sum_{j=0}^{N}(ih)^jR^j\varphi\Big)+h^{N+1}\psi\Big). \tag{3.38}$$

In particular, R^0 is the operator of multiplication by the scalar function $H|_\Lambda$, and $R^1=-d/dt$, see (2.7).

3.8.2 Higher approximations for tori

It is useful to consider all the functions on the tori $\Lambda_l^r\big(\mathbf{J}^{(m)}(h),\epsilon\big)$ as functions of the angle variables $\mathbf{\Phi} = (\Phi_1, \Phi_2)$; in these coordinates one has

$$\frac{d}{dt} = \omega_1 \frac{\partial}{\partial \Phi_1} + \omega_2 \frac{\partial}{\partial \Phi_2}, \quad \omega_k = \frac{\partial \mathcal{H}}{\partial \mathcal{J}_k}\bigg|_{\Lambda_l^r\left(\mathbf{J}^{(m)}(h),\epsilon\right)}, \quad k \in \{1,2\}. \tag{3.39}$$

Recall that $\omega_1 = 1 + O(\epsilon)$, $\omega_2 = O(\epsilon)$ (it follows from the expression (1.26) for the averaged Hamiltonian).

Before we constructed asymptotic eigenfunctions $\psi_{\boldsymbol{m}}^r$ and asymptotic eigenvalues $E_{\boldsymbol{m}}^r$ of $\hat{H}_{h,\epsilon}$ in the form

$$\psi_{\boldsymbol{m}}^r = \mathcal{K}_{\Lambda_l^r\left(\mathbf{J}^{(m)},\epsilon\right)} \cdot 1,$$
$$E_{\boldsymbol{m}}^r = \mathcal{H}^r\big(\mathbf{J}^{(m)}(h),\epsilon\big).$$

Now we put

$$\left[\begin{aligned} \psi_{\boldsymbol{m}}^{r,L} &= \mathcal{K}_{\Lambda_l^r\left(\mathbf{J}^{(m)},\epsilon\right)} u \\ E_{\boldsymbol{m}}^{r,L} &= \sum_{j=1}^{N} (ih)^j \lambda_j, \\ u(\mathbf{\Phi}) &= \Big(\sum_{j=0}^{N} (ih)^j u_j(\mathbf{\Phi})\Big). \end{aligned}\right. \tag{3.40}$$

Substituting (3.40) into (3.38), we will find that

$$\Big(\hat{H}^0_{h,\epsilon} - E_{\boldsymbol{m}}^{r,L}\Big)\psi_{\boldsymbol{m}}^{r,L} = \sum_{n=0}^{N} \Big\{ \sum_{s+j=n} (R^j - \lambda_j) u_s \Big\} + O(h^{N+1}).$$

We will require that the expressions in the curly brackets vanish. For $n = 0$ we obtain the equation $(\mathcal{H}(\mathbf{J}^{(m)},\epsilon) - \lambda_0)u_0 = 0$. We set $u_0 = 1$, $\lambda_0 = \mathcal{H}(\mathbf{J}^{(m)},\epsilon)$. For $n = 1$ one has the equation

$$\Big(\frac{d}{dt} + \lambda_1\Big)u_0 = 0,$$

and we can set $\lambda_1 = 0$.

The equations for $n \geq 2$ have the form

$$-\frac{d}{dt}u_{n-1} = \sum_{j=0}^{n-2} (R^{n-j} - \lambda_{n-j})u_j. \tag{3.41}$$

These equations are called *homological*. From this equation one has to find u_{n-1} and λ_{n-1}. We will solve all these equations only up to $O(\epsilon^K)$.

3.8.3 Approximate solutions of homological equations

Proposition 3.15 (On the solution of the homological equation). *Consider the equation*

$$\left(\omega_1 \frac{\partial}{\partial \Phi_1} + \omega_2 \frac{\partial}{\partial \Phi_2}\right)\varphi = E + f, \tag{3.42}$$

and assume that

$$\omega_1 = 1 + O(\epsilon) \text{ and } \omega_2 = O(\epsilon), \tag{3.43}$$

$$f \in C^\infty(\mathbb{T}^2), \tag{3.44}$$

then, for any $M > 0$, there is a number E and a function $\varphi \in C^\infty(\mathbb{T}^2)$ satisfying (3.42) up to $O(\epsilon^M)$.

Proof. Expand all the functions into their Fourier series:

$$f = \sum_{(k_1,k_2)\in\mathbb{Z}^2} f_{k_1,k_2} e^{i(k_1\Phi_1 + k_2\Phi_2)},$$

$$\varphi = \sum_{(k_1,k_2)\in\mathbb{Z}^2} \varphi_{k_1,k_2} e^{i(k_1\Phi_1 + k_2\Phi_2)}.$$

Substituting *formally* these series into (3.42) one obtains the equalities

$$\varphi_{k_1,k_2} = \frac{1}{k_1\omega_1 + k_2\omega_2} f_{k_1,k_2}, \tag{3.45}$$

$$E = -f_{0,0}. \tag{3.46}$$

In general, it may happen that the coefficients (3.45) do not define any function; the classical *small denominators problem* arises [5]. To avoid this obstacle let us use the conditions (3.43) and (3.44).

The condition (3.44) means that for any $\alpha > 0$ there are positive numbers $C(\alpha)$ and $N(\alpha) > 0$ such that

$$|g_{k_1,k_2}| \le \frac{C(\alpha)}{|\boldsymbol{k}|^\alpha} \text{ as } |\boldsymbol{k}| \ge N(\alpha). \tag{3.47}$$

Set $\alpha = 2M$. Introduce a set $K(\epsilon, \alpha)$ as follows:

$$K(\epsilon,\alpha) = \left\{\boldsymbol{k} = (k_1,k_2) \in \mathbb{Z}^2 : \ |\boldsymbol{k}| \le N(\alpha)\right\} \bigcup \left\{\boldsymbol{k} = (k_1,k_2) \in \mathbb{Z}^2 : \ |k_2| \le \frac{1}{\sqrt{\epsilon}}.\right\} \backslash \{(0,0)\}.$$

One can easily check that

$$|k_1\omega_1 + k_2\omega_2| \geq \frac{1}{2} \text{ for } \boldsymbol{k} \in K(\alpha,\epsilon) \text{ as } \epsilon \text{ is small enough.} \tag{3.48}$$

Put now

$$F = \sum_{(k_1,k_2)\in K(\epsilon,\alpha),\boldsymbol{k}\neq \boldsymbol{0}} f_{k_1,k_2} e^{i(k_1\Phi_1+k_2\Phi_2)}, \quad g = \sum_{(k_1,k_2)\notin K(\epsilon,\alpha)} f_{k_1,k_2} e^{i(k_1\Phi_1+k_2\Phi_2)},$$

then the equation (3.42) reads as

$$\Big(\omega_1\frac{\partial}{\partial\Phi_1} + \omega_2\frac{\partial}{\partial\Phi_2}\Big)\varphi = F + g. \tag{3.49}$$

The estimate (3.48) means that the equation $d\varphi/dt = F$ can be solved in Fourier series. We only have to prove that g gives a necessary accuracy.

For $\boldsymbol{k} \in \mathbb{Z}^2 \backslash K(\epsilon,\alpha)$ one has

$$f_{(k_1,k_2)} \leq \frac{C(\alpha)}{|\boldsymbol{k}|^\alpha} \leq C(\alpha)\epsilon^N.$$

This means that $g = O(\epsilon^N)$. □

Using Proposition 3.15 we can solve each of the equations (3.41) up to $O(\epsilon^K)$. Therefore, we construct quasimodes $\big(\psi^{r,K,L}_{\boldsymbol{m},\boldsymbol{l}}, E^{r,K,L}_{\boldsymbol{m}}\big)$ of $\hat{H}_{h,\epsilon}$ up to $O(h^L+\epsilon^K)$. Emphasize that our considerations do not depend on the index $\boldsymbol{l}$ of the tori (more precisely, the frequencies ω_1 and ω_2 are the same for all the tori $\Lambda^r_{\boldsymbol{l}}$, therefore, the function $u(\boldsymbol{\Phi})$ in (3.40) does not depend on $\boldsymbol{l}$, see Remark after Proposition 3.8); therefore the functions $\psi^{r,K,L}_{\boldsymbol{m},\boldsymbol{l}}$ can be expressed through each other as previously:

$$\psi^{r,K,L}_{\boldsymbol{m},\boldsymbol{l}}(\boldsymbol{x},h,\epsilon) = \psi^{r,K,L}_{\boldsymbol{m},\boldsymbol{0}}(\boldsymbol{x}-\boldsymbol{l}\cdot\boldsymbol{a},h,\epsilon)\exp\Big(-\frac{i}{h}l_2a_{22}x_1\Big) \tag{3.50}$$

(proof is similar to that for Proposition 3.8). We have also the estimate

$$E^{r,K,L}_{\boldsymbol{m}}(h,\epsilon) = E^r_{\boldsymbol{m}}(h,\epsilon) + O(h^2),$$

where $E^r_{\boldsymbol{m}}$ is defined in (3.20), for any $K, L \geq 2$.

3.8.4 Higher approximations for cylinders

The construction of the higher approximations for cylinders is similar to that for the tori. We again find a quasimode in the form (3.40). Applying the commutation formula we again come to the same homological equations (3.41). We again are going to solve them up to $O(\epsilon^K)$. To do this, we *a priori* require for the resulting functions $\psi^{r,K,L}_{m,\mathcal{J}_2,k}$ satisfying the condition

$$\psi^{r,K,L}_{m,\mathcal{J}_2,k}(\boldsymbol{x}+\boldsymbol{d}\cdot\boldsymbol{a},h,\epsilon) = \psi^{r,K,L}_{m,\mathcal{J}_2,k}(\boldsymbol{x},h,\epsilon)\exp\Big(\frac{i}{h}\big(2\pi\mathcal{J}_2^k-(\boldsymbol{d}\cdot\boldsymbol{a})_2x_1-\frac{1}{2}(\boldsymbol{d}\cdot\boldsymbol{a})_1(\boldsymbol{d}\cdot\boldsymbol{a})_2\big)\Big). \quad (3.51)$$

This means that all the functions u_j in (3.40) have to be periodic relative Φ_2 with the period 2π; they are always periodic in Φ_1 with the period 2π. Therefore, these functions can be considered as functions on the torus $\mathbb{T}^2$, and the homological equations can be solved using Proposition 3.15. As before (see Propositions 3.8 and 3.10 and Remarks after them), we also have the equalities

$$\psi^{r,K,L}_{m,\mathcal{J}_2,k}(\boldsymbol{x},h,\epsilon) = \psi^{r,K,L}_{m,\mathcal{J}_2,0}(\boldsymbol{x}+k(J_2\boldsymbol{f})\cdot\boldsymbol{a},h,\epsilon)\exp\Big(\frac{i}{h}kf_1a_{22}x_1\Big). \quad (3.52)$$

Using Proposition 3.13 we again obtain the estimate

$$E^{r,K,L}_m(\mathcal{J}_2,h,\epsilon) = E^r_m(\mathcal{J}_2,h,\epsilon)+O(h^2),$$

where E^r_m is defined in (3.29).

3.8.5 Higher approximations for points and curves

The construction of the higher approximations for point and curves follows essentially the same scheme.

Assume that we have already constructed a quasimode $\big(E(h),\psi(\boldsymbol{x},h)\big)$ with error $O(h^{(M+3)/2})$, where $M\in\mathbb{Z}_+$, $E(h)=E_{\boldsymbol{m}}(h,\epsilon)+O(h^{3/2})$, $\psi(\boldsymbol{x},h)=\Psi_{\boldsymbol{m}}(\boldsymbol{x},h,\epsilon)e^{iS(x/h)}$, and $\Psi_{\boldsymbol{m}}=\varphi_{\boldsymbol{m}}(\boldsymbol{x},h)+\mathcal{O}_{\mathfrak{J}S}(h^{(M+1)/2})$, where $E_{\boldsymbol{m}}$ and $\varphi_{\boldsymbol{m}}$ are defined in (2.16) and (2.17) or in (2.27) and (2.28). Let us try to find a quasimode (ψ',E') with error $O(h^{(M+4)/2})$ in the form

$$\psi'(\boldsymbol{x},h) = \big(\Psi(\boldsymbol{x},h)+\theta(\boldsymbol{x},h)\big)e^{iS(x)/h}, \quad \theta=\mathcal{O}_{\mathfrak{J}S}(h^{M/2}),$$
$$E'(h)=E(h)+\tilde{\omega}h^{M+3/2}$$

Substituting these expressions into (2.30) we come to the equation [64, §VI.10]:

$$(\hat{\Pi}-i\omega_{\boldsymbol{m}})\theta = F(\boldsymbol{x},h)+i\tilde{\omega}\varphi_{\boldsymbol{m}}+\mathcal{O}_{\mathfrak{J}S}(h^{(M+1)/2}), \quad (3.53)$$

where $F(\boldsymbol{x}, h) = \mathcal{O}(h^{M/2})$ is a certain known function. This equation is called the *generalized transport equation with a right-hand term*. We have to find from this equation θ and $\tilde{\omega}$.

Proposition 3.16 (On the solution of the generalized transport equation with a right-hand term, Proposition 6.11 in [64]). *As follows from the definition of $\mathcal{O}_{\mathfrak{IS}}(h^{M/2})$, the functions $F(\boldsymbol{x}, h)$ and $\varphi_{\boldsymbol{m}}(\boldsymbol{x}, h)$ admit the following representation:*

$$F(\boldsymbol{x}, h) = \sum_{j=-N_1}^{N_2} h^{M/2+j/2} F^j(\boldsymbol{x}, h), \quad F^j = O_{\mathfrak{IS}}(h^{-j/2}), \quad j \le 0,$$

$$\varphi_{\boldsymbol{m}}(\boldsymbol{x}, h) = \sum_{j=-|\boldsymbol{m}|}^{N_3} \varphi^j_{\boldsymbol{m}}, \quad \varphi^j_{\boldsymbol{m}} \in C^\infty, \quad \varphi^j_{\boldsymbol{m}} = O_{\mathfrak{IS}}(h^{-j/2}), \quad j \le 0,$$

where $N_{1/2/3} \in \mathbb{Z}_+$. Let functions

$$\theta^j = O_{\mathfrak{IS}}(h^{-j/2}), \quad j = -\max\{N_1, |\boldsymbol{m}|, \dots, -1, 0\}, \tag{3.54}$$

satisfy the system of equations

$$\left[\begin{array}{l} (\hat{\Pi}^0 - i\omega_{\boldsymbol{m}})\theta^j = i\tilde{\omega}\varphi^j_{\boldsymbol{m}} + \tilde{F}^j + O_{\mathfrak{IS}}(h^{-(j+1)/2}), \\ \tilde{F}^j = F^j + \dfrac{1}{2}\displaystyle\sum_{m,s=1}^{n} H_{p_m p_s} \dfrac{\partial\theta^2_{j-2}}{\partial x_m \partial x_s}, \\ \hat{\Pi}^0 = \langle H_{\boldsymbol{p}}(\boldsymbol{p}, \boldsymbol{x}) | \dfrac{\partial}{\partial \boldsymbol{x}} \rangle + \dfrac{1}{2}\operatorname{tr}\Big(H_{\boldsymbol{pp}} S_{\boldsymbol{xx}} + H_{\boldsymbol{px}}\Big), \quad \boldsymbol{p} = \dfrac{\partial S(\boldsymbol{x})}{\partial \boldsymbol{x}} \end{array}\right. \tag{3.55}$$

(*these equations are called the* transport equations with right-hand terms), *where $\varphi^j_{\boldsymbol{m}} \stackrel{\text{def}}{=} 0$ for $j \le |\boldsymbol{m}|$ if $|\boldsymbol{m}| > N_1$. Then the function*

$$\theta = \sum_{j=-\max\{N_1, \boldsymbol{m}\}}^{0} \theta^j h^{-j/2}$$

satisfies the equation (3.53).

Proposition 3.17 (Solution of the transport equations with right-hand terms for invariant points, Theorem 6.10 in [64]). *If the numbers β^j, $j \in \{1, \dots, n\}$, defined in subsection 2.3.3 are linearly independent over each bounded subset of $\mathbb{Z}$, then the system of equations* (3.55) *related to invariant points can be solved. The number $\tilde{\omega}$ is defined uniquely up to $O(\sqrt{h})$.*

Proposition 3.18 (Solution of the transport equations with right-hand terms for invariant curves, Theorem 6.11 in [64]). *If the numbers β^j, $j \in \{1, \dots, n\}$, defined in subsection 2.4.3, and 2π are linearly independent over each bounded subset of $\mathbb{Z}$, then the system of equations* (3.55) *related to invariant curves can be solved. The number $\tilde{\omega}$ is defined uniquely up to $O(\sqrt{h})$.*

Let us apply now this scheme to the invariant points and curves $\Lambda^r_{l/k}$.

Consider first the points $\Lambda^r_{\boldsymbol{l}}(0,0,\epsilon)$. To them we can apply Proposition 3.17, because the numbers $\beta^1 = \omega_1 = 1 + O(\epsilon)$ and $\beta^2 = \omega_2 = O(\epsilon)$ satisfy the necessary conditions. Therefore, we can construct quasimodes $\big(\psi^{r,L}_{\boldsymbol{m},\boldsymbol{l}}, E^{r,L}_{\boldsymbol{m}}\big)$ of $\hat{H}_{h,\epsilon}$ with error $O(h^L + e^{-C/\epsilon})$. Moreover, the equation (3.53) for all the indices $\boldsymbol{l}$ may be obtained from the corresponding equation for $\boldsymbol{l} = 0$ by the transformation $\boldsymbol{x} \mapsto \boldsymbol{x} + \boldsymbol{l} \cdot \boldsymbol{a}$. This means, that for any $\boldsymbol{m}$ and L we again have equalities

$$\psi^{r,L}_{\boldsymbol{m},\boldsymbol{l}}(\boldsymbol{x},h,\epsilon) = \psi^{r,L}_{\boldsymbol{m},\boldsymbol{0}}(\boldsymbol{x},h,\epsilon) e^{-\frac{i}{h} l_2 a_{22} x_1}.$$

Obviously, similar consideration can be applied to the curves $\Lambda^r_{l/k}(0,\mathcal{I}_2,\epsilon)$ and $\Lambda^r_{l/k}(\mathcal{I}_1,0,\epsilon)$, because Proposition 3.18 is also satisfied in these cases (see also Subsection 3.7).

3.9 Summary

Let us summarize all the considerations of this chapter.

3.9.1 Formulation of results

Denote

$$\mathcal{I}^{(m_k)}_k(h) = h\Big(\frac{1}{2} + m_k\Big), \qquad m_k \in \mathbb{Z}, \quad k \in \{1,2\}. \tag{3.56}$$

For any regime $\mathcal{M}^r$ by $\mathcal{D}(\mathcal{M}^r) \subset \mathbb{R}_+ \times \mathbb{R}$ we denote the domain of the corresponding action variables $(\mathcal{I}_1, \mathcal{I}_2)$. Fix now arbitrary positive numbers K and L.

Proposition 3.19 (Quasimodes in finite motion regimes). *For any finite motion regime $\mathcal{M}^r$ and any $(m_1,m_2) \in \mathbb{Z}_+ \times \mathbb{Z}$ such that $\big(\mathcal{I}^{(m_1)}_1(h), \mathcal{I}^{(m_2)}_2(h)\big) \in \mathcal{D}(\mathcal{M}^r)$ there exist a number*

$$E^{r,K,L}_{m_1,m_2}(h,\epsilon) = \mathcal{H}^r\big(\mathcal{I}^{(m_1)}_1(h), \mathcal{I}^{(m_2)}_2(h), \epsilon\big) + O(h^{3/2})$$

and a set of functions

$$\psi^{r,K,L}_{m_1,m_2,l_1,l_2}(\boldsymbol{x},h,\epsilon) \in L^2(\mathbb{R}^2), \quad (l_1,l_2) \in \mathbb{Z}^2,$$

constructed using the canonical operator and satisfying the following conditions:

- $\big(\psi^{r,K,L}_{m_1,m_2,l_1,l_2}(\boldsymbol{x},h,\epsilon), E^{r,K,L}_{m_1,m_2}(h,\epsilon)\big)$ *is a quasimode of $\hat{H}_{h,\epsilon}$ with error $O(h^L + \epsilon^K)$ in $L^2(\mathbb{R}^2)$,*

- *the functions* $\psi^{r,K,L}_{m_1,m_2,l_1,l_2}$ *satisfy* (3.50),
- *each of the functions* $\psi^{r,K,L}_{m_1,m_2,l_1,l_2}$ *has compact support and is asymptotically localized near the projections* $\pi_{\boldsymbol{x}}\Lambda\big(\mathcal{I}_1^{(m_1)}(h),\mathcal{I}_2^{(m_2)}(h),\epsilon\big)$ *as* h *tends to* 0,

and

$$\operatorname{dist}\left(E^{r,K,L}_{m_1,m_2}(h,\epsilon),\hat{H}_{h,\epsilon}\right)=O(h^L+\epsilon^K).$$

Proposition 3.20 (Quasimodes in infinite motion regimes). *For any infinite motion regime* $\mathcal{M}^r$ *and any* $(m,\mathcal{I}_2)\in\mathbb{Z}_+\times\mathbb{R}$ *such that* $\big(\mathcal{I}_1^{(m)}(h),\mathcal{I}_2\big)\in\mathcal{D}(\mathcal{M}^r)$ *there exist a number*

$$E^{r,K,L}_{m}(\mathcal{I}_2,h,\epsilon)=\mathcal{H}^r\big(\mathcal{I}_1^{(m)}(h),\mathcal{I}_2,\epsilon\big)+O(h^{3/2})$$

and a set of functions

$$\psi^{r,K,L}_{m,\mathcal{I}_2,k}(\boldsymbol{x},h,\epsilon)\in L^2_{\mathrm{loc}}(\mathbb{R}^2),\quad k\in\mathbb{Z},$$

constructed using the canonical operator and satisfying the following conditions:

- $\big(\psi^{r,K,L}_{m,\mathcal{I}_2,k}(\boldsymbol{x},h,\epsilon),E^{r,K,L}_{m}(\mathcal{I}_2,h,\epsilon)\big)$ *is a quasimode of* $\hat{H}_{h,\epsilon}$ *in* $L^2(\Pi_{\boldsymbol{d}^r})$ *with error* $O(h^L+\epsilon^K)$; $\Pi_{\boldsymbol{d}^r}$ *is defined in* (3.32),
- *the functions* $\psi^{r,K,L}_{m,\mathcal{I}_2,k}$ *satisfy* (3.51) *and* (3.52),
- *each of the functions* $\psi^{r,K,L}_{m_1,m_2,l_1,l_2}$ *and is asymptotically localized near the projections* $\pi_{\boldsymbol{x}}\Lambda\big(\mathcal{I}_1^{(m)}(h),\mathcal{I}_2,\epsilon\big)$ *as* h *tends to* 0,

and

$$\operatorname{dist}\left(E^{r,K,L}_{m}(\mathcal{I}_2,h,\epsilon),\hat{H}_{h,\epsilon}\right)=O(h^L+\epsilon^K).$$

3.9.2 General structure of the spectrum

Therefore, we have the following visual picture of the spectrum. We quantize first the action variable $\mathcal{I}_1$ using the quantization condition (3.56). Then on each finite edge of the Reeb graph $G\big(\mathcal{I}^{(m)}(h)\big)$ we quantize the second action variable $\mathcal{I}_2$ using (3.56) again. The set of the values of $\mathcal{H}$ in all these points (including the values on the whole infinite motion edges) gives a certain approximation to the spectrum of $\hat{H}_{h,\epsilon}$.

Definition 31 (Semiclassical Landau band). For any $m \in \mathbb{Z}_+$, the union of all the numbers $E^{r,K,L}_{m,m'}(h,\epsilon)$ and $E^{r,K,L}_{m}(\mathcal{I}_2,h,\epsilon)$ defined in Propositions 3.19 and 3.20 will be called the *mth semiclassical Landau band with error* $O(h^L+\epsilon^K)$ and will be denoted by $\mathcal{L}^{K,L}_m(h,\epsilon)$.

Definition 32 (Semiclassical spectrum). We call the set

$$\Sigma_{K,L}(h,\epsilon) = \bigcup_{m\in\mathbb{Z}_+} \mathcal{L}^{K,L}_m(h,\epsilon)$$

the *semiclassical spectrum of* $\hat{H}_{h,\epsilon}$ *with error* $O(h^L+\epsilon^K)$.

Propositions 3.19 and 3.20 state only that $\Sigma_{K,L}(h,\epsilon)$ is contained in a $O(h^L+\epsilon^K)$-neighborhood of spec $\hat{H}_{h,\epsilon}$. It is the most complete result that can be obtained using the method of the Maslov canonical operator, and we hope that really we have obtained more than we can prove, namely, we hope that the whole spectrum is contained in a $O(h^L+\epsilon^K)$-neighborhood of $\Sigma_{K,L}(h,\epsilon)$.

The definition of the semiclassical Landau band is purely empirical, because we cannot prove any relation between the semiclassical Landau band and the true one. From the other side, the sets $\mathcal{L}^{K,L}_m(h,\epsilon)$ have some properties of the true Landau bands. In particular, they do not intersect as ϵ is (much) smaller than h. (See also the discussion in the next section.) From the previous considerations we can derive the diameter diam $\mathcal{L}^{K,L}_m$ of each semiclassical Landau band:

$$\operatorname{diam}\mathcal{L}^{K,L}_m(h,\epsilon) = \max_{r,\mathcal{I}^r_2} \mathcal{H}^r\big(\mathcal{I}^{(m)}_1(h),\mathcal{I}^r_2\epsilon\big) - \min_{r,\mathcal{I}^r_2} \mathcal{H}^r\big(\mathcal{I}^{(m)}_1(h),\mathcal{I}^r_2\epsilon\big) = O(\epsilon).$$

We see that finite motion edges and infinite motion ones (ore, more globally, regimes) make different contributions to the semiclassical spectrum. Infinite motion edges imply intervals with length $O(\epsilon)$, while finite motion edges imply point sets; the points are separated by “gaps” of length $O(\epsilon h)$, see Fig. 3.1. Strictly speaking, the picture related to finite motion edges is distorted near the branching points of the Reeb graph (or, equivalently, interior boundaries), here the distances between the quantization points become indefinitely small.

Remark. As we see, the formulas for asymptotic eigenvalues of $\hat{H}_{h,\epsilon}$ corresponding to the curves and points (low-dimensional manifolds) can be obtained from

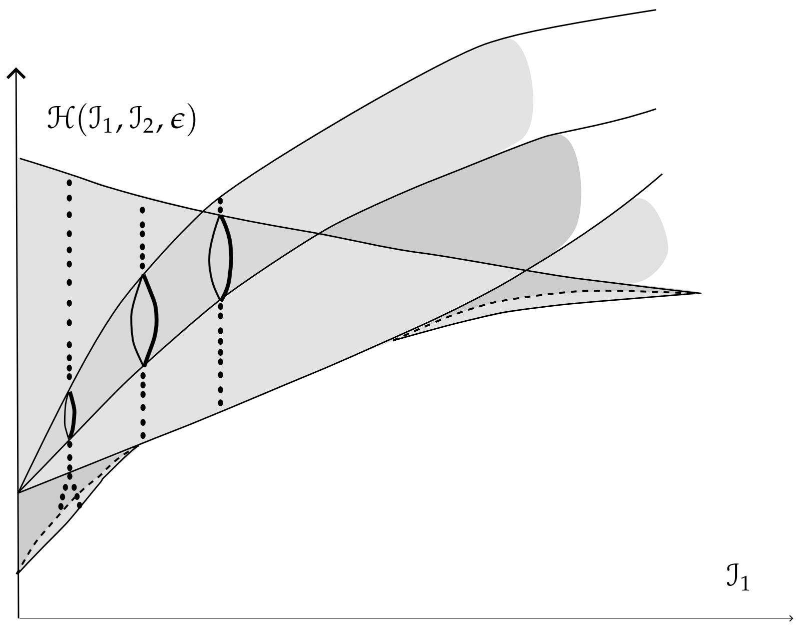

Figure 3.1: An example of a quantized Reeb surface

the formulas for asymptotic eigenvalues corresponding to Lagrangian manifolds (tori and cylinders) by passing to the limit. It is necessary to emphasize that the existence of such a limit is not a justification of these formulas, a rigorous justification is given by means of the complex germ theory. From the other side, such a limit exists in the wide class of multidimensional spectral problems [8].

3.10 The Landau bands and Harper-like equations

It is a well-known fact that a canonical transformation in classical mechanics corresponds to a unitary transformation in quantum mechanics. More precisely, let $H(\boldsymbol{p}, \boldsymbol{x})$ be a classical Hamiltonian, $g : (\boldsymbol{p}, \boldsymbol{x}) \mapsto (\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{X}})$ be a canonical transformation, and $H(\boldsymbol{p}, \boldsymbol{x}) = \mathcal{H}(\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{X}})$, then, from the physical point of view, the operators $\hat{H}_h$ and $\hat{\mathcal{H}}_h$ are "almost unitary equivalent" [28, 33, 50]. We will not try to give a rigorous formulation of this fact, it is rather difficult, because the transformation defined by Proposition 1.12 is local (see [51]), but we try to show what kind of results can be obtained using such an approach.

Consider the classical Hamiltonian $\mathcal{H}$ (1.26) and its Weyl quantization $\hat{\mathcal{H}}_{h,\epsilon}$ acting in the space $L^2(\mathbb{R}^2_{\mathcal{Q},\mathcal{Y}_2})$. The fact that $\mathcal{H}$ depends only on $\mathcal{I}_1 = \frac{1}{2}(\mathcal{P}^2 + \mathcal{Q}^2)$ means that $\hat{\mathcal{H}}_{h,\epsilon}$ commutes with the harmonic oscillator

$$\hat{\mathcal{I}}_{1h} = \frac{1}{2}\Big(-h^2\frac{\partial^2}{\partial \mathcal{Q}^2} + \mathcal{Q}^2\Big).$$

Let us suppose that $\Psi(\mathcal{Q}, \mathcal{Y}_2, h, \epsilon)$ is an eigenfunction of $\hat{\mathcal{H}}_{h,\epsilon}$,

$$\hat{\mathcal{H}}_{h,\epsilon}\Psi = \mathcal{E}(h,\epsilon)\Psi, \tag{3.57}$$

then it is also an eigenfunction of $\hat{\mathcal{I}}_{1h}$ and one can put

$$\Psi(\mathcal{Q}, \mathcal{Y}_2, h, \epsilon) = \psi_n(\mathcal{Q}, h)\Phi(\mathcal{Y}_2, h, \epsilon), \tag{3.58}$$

where ψ_n is the eigenfunction of $\hat{\mathcal{I}}_{1h}$ with the eigenvalue $\mathcal{I}_1^{(n)}(h) = h(\frac{1}{2}+n)$, $n \in \mathbb{Z}_+$; then the function $\Phi(\mathcal{Y}_2, h, \epsilon)$ has to satisfy the equation

$$\hat{\mathcal{W}}_h\big(\mathcal{I}_1^{(n)}(h), \epsilon\big)\Phi(\mathcal{Y}_2, h, \epsilon) = \mathcal{E}(h,\epsilon)\Phi(\mathcal{Y}_2, h, \epsilon), \tag{3.59}$$

where the operator in the left-hand side is the Weyl quantization of the function $\mathcal{H}\big(\mathcal{I}_1^{(n)}(h), \mathcal{Y}_1, \mathcal{Y}_2\big)$ which is considered as a function of $\mathcal{Y}_1$ and $\mathcal{Y}_2$; we will write this for simplicity as

$$\hat{\mathcal{W}}_h\big(\mathcal{I}_1^{(n)}(h), \epsilon\big) = \mathcal{H}\big(\mathcal{I}_1^{(n)}(h), -ih\frac{\partial}{\partial \mathcal{Y}_2}, \mathcal{Y}_2, \epsilon\big) \tag{3.60}$$

It is important to emphasize again that the function $\mathcal{H}$ is periodic in both $\mathcal{Y}_1$ and $\mathcal{Y}_2$; this means that $\hat{\mathcal{W}}_h$ is not a differential operator. If, for example,

$$\mathcal{H}(\mathcal{I}_1, \boldsymbol{\mathcal{Y}}, \epsilon) = \sum_{k\in K} A_k(\mathcal{I}_1, \epsilon)\cos k\mathcal{Y}_1 + B(\mathcal{I}_1, \epsilon)v(\mathcal{Y}_2),$$

where K is a finite subset of $\mathbb{Z}$, then the corresponding operator $\hat{\mathcal{W}}_h$ is a difference operator, and the equation (3.59) looks as

$$\frac{1}{2}\sum_{k\in K} A_k(\mathcal{I}_1, \epsilon)\big(\Phi(\mathcal{Y}_2 + kh, h, \epsilon) + \Phi(\mathcal{Y}_2 - kh, h, \epsilon)\big)$$
$$+ B(\mathcal{I}_1, \epsilon)v(\mathcal{Y}_2)\Phi(\mathcal{Y}_2, h, \epsilon) = \mathcal{E}(h,\epsilon)\Phi(\mathcal{Y}_2, h, \epsilon);$$

such equations (with Hamiltonian periodic in both variables) are usually called *Harper-like equations*. In particular, for the Harper potential and $\mathcal{H} = \bar{H}$ (see Subsection 1.4.1) we obtain the family of *Harper's equations*

$$AJ_0(\sqrt{2\mathcal{I}_1})\big(\Phi(\mathcal{Y}_2 + h) + \Phi(\mathcal{Y}_2 - h)\big) + 2BJ_0(\beta\sqrt{2\mathcal{I}_1})\cos\beta\mathcal{Y}_2\Phi(\mathcal{Y}_2, h)$$
$$= 2(\mathcal{E} - \mathcal{I}_1)\Phi(\mathcal{Y}_2), \quad \mathcal{I}_1 = \mathcal{I}_1^{(m)}(h).$$

The operator $\mathcal{W}_h\big(\mathcal{I}_1^{(n)}(h)\big)$ can now be studied using semiclassical methods [18,30, 29,31,87,88,57,82]. The invariant manifolds of the corresponding classical Hamiltonian $\mathcal{H}\big(\mathcal{I}^{(n)}(h), \mathcal{Y}, \epsilon\big)$ can be classified using the Reeb graph $G(\mathcal{I}_1)$ (see Subsection 1.3.2); these invariant manifolds are closed and open curves and points. The family of open trajectories is not quantized (no cycles), the family of closed curved is quantized as $\mathcal{I}_2 = \mathcal{I}_2^{(m)}(h)$, $m \in \mathbb{Z}$. Therefore, we obtain the same set of asymptotic eigenvalues [30].

Therefore, from this points of view, each semiclassical Landau band described by a Harper-like equation; this equation depends on the index of the Landau band.

Chapter 4

The asymptotics of the band spectrum

4.1 The magneto-Bloch conditions

Let us consider now the case of rational flux. Assume that

$$\eta := \frac{a_{22}}{h} = \frac{N}{M},$$

where N and M are mutually prime integers and $M > 0$.

As it was noted in the introduction, the spectrum of $\hat{H}_{h,\epsilon}$ in this case has band structure. In what follows we assume that the Landau bands in the spectrum do not intersect. Let us number all the bands in the spectrum by the index $\mu \in \mathbb{Z}$; the band corresponding to the index μ will be denoted as $\mathcal{B}^{\mu}(h,\epsilon)$. It is useful to parameterize the points of the bands by *quasimomenta*

$$\boldsymbol{q} = (q_1, q_2) \in [0, \frac{1}{M}) \times [0, 1);$$

more precisely, for each band $\mathcal{B}^{\mu}$ there exists a real-analytic function $E^{\mu}(\boldsymbol{q}, h, \epsilon)$ (the dispersion law) and M functions $\Psi^{\mu}_{j}(\boldsymbol{x}, \boldsymbol{q}, h, \epsilon)$, $j \in \{0, \dots, M-1\}$, depending on $\boldsymbol{q}$ analytically, such that

$$\hat{H}_{h,\epsilon}\Psi^{\mu}_{j}(\boldsymbol{x}, \boldsymbol{q}, h, \epsilon) = E^{\mu}(\boldsymbol{q}, h, \epsilon)\Psi^{\mu}_{j}(\boldsymbol{x}, \boldsymbol{q}, h, \epsilon), \tag{4.1}$$

$$\begin{cases} \Psi_j^\mu(\boldsymbol{x}+\boldsymbol{a}^1,\boldsymbol{q},h,\epsilon) = \Psi_j^\mu(\boldsymbol{x},\boldsymbol{q},h,\epsilon)e^{-2\pi i\left(q_1-j\eta\right)}, \\ \qquad\qquad\qquad\qquad j\in\{0,\dots,M-1\}, \\ \Psi_j^\mu(\boldsymbol{x}+\boldsymbol{a}^2,\boldsymbol{q},h,\epsilon) = \Psi_{j+1}^\mu(\boldsymbol{x},\boldsymbol{q},h,\epsilon)e^{-i\eta(x_1+\frac{1}{2}a_{21})}, \\ \qquad\qquad\qquad\qquad j\in\{0,\dots,M-2\}, \\ \Psi_{M-1}^\mu(\boldsymbol{x}+\boldsymbol{a}^2,\boldsymbol{q},h,\epsilon) = \Psi_0^\mu(\boldsymbol{x},\boldsymbol{q},h,\epsilon)e^{-i\eta(x_1+\frac{1}{2}a_{21})-2\pi i q_2}, \end{cases} \tag{4.2}$$

and

$$\mathcal{B}^\mu(h,\epsilon) = \cup_{\boldsymbol{q}\in[0,1/M)\times[0,1)}E^\mu(\boldsymbol{q},h,\epsilon).$$

The equalities (4.2) are called the *magneto-Bloch conditions*. In this chapter, we are going to use the quasimodes previously constructed (see Propositions 3.19 and 3.20) for constructing asymptotic eigenfunctions satisfying these conditions.

Definition 33 (Magneto-Bloch quasimodes). Functions $\Psi_j(\boldsymbol{x},\boldsymbol{q},h,\epsilon)$, $j\in\{0,\dots,M-1\}$, depending analytically on $\boldsymbol{q}$ are called *magneto-Bloch quasimodes* of $\hat{H}_{h,\epsilon}$ if they satisfy the magneto-Bloch conditions (4.2) and all they are generalized eigenfunctions of $\hat{H}_{h,\epsilon}$ up to $O(h^L+\epsilon^K)$.

4.2 Magneto-Bloch quasimodes in finite motion regimes

Let us consider a certain finite motion regime $\mathcal{M}^r$; choose some $(m_1,m_2)\in\mathbb{Z}_+\times\mathbb{Z}$ such that $\left(\mathcal{I}_1^{(m_1)}(h),\mathcal{I}_2^{(m_2)}(h)\right)\in\mathcal{D}(\mathcal{M}^r)$ and consider the corresponding quasimodes $\left(\psi^{r,K,L}_{m_1,m_2,l_1,l_2},E^{r,K,L}_{m_1,m_2}(h,\epsilon)\right)$, $(l_1,l_2)\in\mathbb{Z}^2$, see Proposition 3.19. In the rest of this section we omit the dependence on r, m_1, m_2, K, L, and ϵ.

We find magneto-Bloch quasimodes in the form

$$\Psi_j(\boldsymbol{x},\boldsymbol{q},h) = \sum_{(l_1,l_2)\in\mathbb{Z}^2} C^j_{l_1,l_2}(\boldsymbol{q},h)\psi_{l_1,l_2}(\boldsymbol{x},h). \tag{4.3}$$

Using (3.50) we can rewrite (4.3) as

$$\Psi_j(\boldsymbol{x},\boldsymbol{q},h) = \sum_{(l_1,l_2)\in\mathbb{Z}^2} C^j_{l_1,l_2}(\boldsymbol{q},h)\psi_{0,0}(\boldsymbol{x}-\boldsymbol{l}\cdot\boldsymbol{a},h)\exp\Big(-\frac{i}{h}l_2a_22x_1\Big). \tag{4.4}$$

This representation is a natural generalization of the well-known *Lifshits-Gelfand-Zak* representation for Bloch functions.

Proposition 4.1 (Magneto-Bloch quasimodes in finite motion regimes). *There exist exactly M collections of functions of the form (4.4) satisfying the magneto-Bloch conditions (4.2). The function $\Psi_{s,j}$ (the jth member of the sth collection, $s,j \in \{0,\dots,M-1\}$) can be defined by the coefficients $C^{s,j}_{l_1,l_2}$ of the following form:*

$$C^{s,j}_{l_1,l_2}(\boldsymbol{q},h) = \begin{cases} e^{-2\pi i(q_1 l_1 + q_2 n) + 2\pi i \eta l_1 j - i\eta l_2 a_{21}/2} & \text{if } l_2 + j - s + nM = 0, n \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}$$

The functions $\Psi_{s,j}$, $s,j \in \{0,\dots,M-1\}$, are linearly independent.

Proof. Substituting (4.15) into (4.2) and taking into account the fact that the functions ψ_{l_1,l_2} are linearly independent, we come to the equalities

$$\begin{cases} C^{j}_{l_1+1,l_2} = C^{j}_{l_1,l_2} e^{-2\pi i(q_1 - j\eta)}, & j \in \{0,\dots,M-1\}, \\ C^{j}_{l_1,l_2+1} = C^{j+1}_{l_1,l_2} e^{-i\eta a_{21}/2}, & j \in \{0,\dots,M-2\}, \\ C^{M-1}_{l_1,l_2+1} = C^{0}_{l_1,l_2} e^{-i\eta a_{21}/2 - 2\pi i q_2}. \end{cases} \tag{4.6}$$

Therefore, all the numbers $C^{j}_{l_1,l_2}(\boldsymbol{q},h)$ are uniquely determined by arbitrary chosen M numbers $C^{j}_{0,0}(q,h)$, $j \in \{0,\dots,M-1\}$. Therefore, there exist at most M^2 solutions grouped into M families of magneto-Bloch quasimodes. Put

$$C^{s,j}_{0,0} = \delta_{sj}, \quad s,j \in \{0,\dots,M-1\}. \tag{4.7}$$

and define the corresponding functions $\Psi_{s,j}(\boldsymbol{x},\boldsymbol{q},h)$ by (4.4).

From the first equality in (4.6) one easily obtains

$$C^{s,j}_{l_1,l_2} = C^{s,j}_{0,l_2} e^{-2\pi l_1 i(q_1 - j\eta)}, \quad j \in \{0,\dots,M-1\}. \tag{4.8}$$

The numbers $C^{s,j}_{0,l_2}$ can be determined from the two last equalities in (4.6):

$$\left[\begin{array}{l} C^{s,j}_{0,l_2} = C^{s,(l_2+j) \mod M}_{0,0} e^{-il_2\eta a_{21}/2} \prod\limits_{k=\min\{0,l_2\}}^{\max\{0,l_2\}} \sigma_{(k+j) \mod M}, \\ \qquad j \in \{0,\dots,M-1\}, \quad \sigma_{0,\dots,M-2} = 1, \sigma_{M-1} = e^{-2\pi i q_2}. \end{array} \right. \tag{4.9}$$

Therefore, $C^{s,j}_{0,l_2}$ (and, therefore, all the coefficients $C^{s,j}_{l_1,l_2}$) is non-zero iff

$$l_2 + j - s + nM = 0, \quad n \in \mathbb{Z}. \tag{4.10}$$

If (4.10) is satisfied, then (4.9) can be rewritten as

$$C^{s,j}_{0,l_2} = C^{s,(l_2+j) \mod M}_{0,0} e^{-il_2\eta a_{21}/2} e^{2\pi i q_2 n}. \tag{4.11}$$

Now, combining (4.11) and (4.8), we come to (4.5).

Now let us prove that the functions $\Psi_{s,j}$ are linearly independent. To do this, it is enough to prove the linear independence of their M^2-parts

$$A^{s,j} = \big(C^{s,j}_{l_1,l_2}\big)_{l_1,l_2\in\{0,\dots,M-1\}}, \quad s,j\in\{0,\dots,M-1\}.$$

Let us calculate the Gram matrix

$$G_{(s_1,j_1),(s_2,j_2)} = \langle\overline{A^{s_1,j_1}}|A^{s_2,j_2}\rangle, \quad s_1,s_2,j_1,j_2\in\{0,\dots,M-1\}.$$

Let $j_1 \geq s_1$ and $j_2 \geq s_2$. As follows from (4.5), the product $\overline{C^{s_1,j_1}_{l_1,l_2}}C^{s_1,j_1}_{l_1,l_2}$ is non-zero if and only if $l_2 = M + s_1 - j_1$ and $s_1 - j_1 = s_2 - j_2$ (because $0 \leq l_1, l_2 < M$), and then

$$\begin{aligned}
G_{(s_1,j_1),(s_2,j_2)} &= \sum_{l_1,l_2=0}^{M-1} \overline{C^{s_1,j_1}_{l_1,l_2}}C^{s_1,j_1}_{l_1,l_2} \\
&= \sum_{l_1=0}^{M-1} \overline{C^{s_1,j_1}_{l_1,M+s_1-j_1}}C^{s_1,j_1}_{l_1,M+s_1-j_1} \\
&= \sum_{l_1=0}^{M-1} \Big[e^{2\pi i(q_1l_1-q_2)-2\pi i\eta l_1 j_1-i\eta(M+s_1-j_1)a_{21}/2} \\
&\qquad e^{-2\pi i(q_1l_1-q_2)+2\pi i\eta l_1 j_2+i\eta(M+s_1-j_1)a_{21}/2}\Big] \\
&= \sum_{l_1=0}^{M-1} e^{2\pi\eta l_1(j_2-j_1)} = \begin{cases} M & \text{if } j_1=j_2 \text{ (and then } s_1=s_2\text{)}, \\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

The consideration for $j_1 < s_1$ and $j_2 < s_2$ is similar.

Now let $j_1 \geq s_1$ but $j_2 < s_2$. Then the product $\overline{C^{s_1,j_1}_{l_1,l_2}}C^{s_1,j_1}_{l_1,l_2}$ is non-zero iff $l_2 = s_2 - j_2$ and $s_2 - s_1 + j_1 - j_2 = M$, then

$$\begin{aligned}
G_{(s_1,j_1),(s_2,j_2)} &= \sum_{l_1,l_2=0}^{M-1} \overline{C^{s_1,j_1}_{l_1,l_2}}C^{s_1,j_1}_{l_1,l_2} \\
&= \sum_{l_1=0}^{M-1} \overline{C^{s_1,j_1}_{l_1,s_2-j_2}}C^{s_1,j_1}_{l_1,s_2-j_2} \\
&= \sum_{l_1=0}^{M-1} \Big[e^{2\pi i(q_1l_1-q_2)-2\pi i\eta l_1 j_1-i\eta(s_2-j_2)a_{21}/2} \\
&\qquad e^{-2\pi i(q_1l_1)+2\pi i\eta l_1 j_2+i\eta(s_2-j_2)a_{21}/2}\Big] \\
&= e^{-2\pi i q_2} \sum_{l_1=0}^{M-1} e^{2\pi\eta l_1(j_2-j_1)}.
\end{aligned}$$

The last term in this chain is non-zero iff $j_1 = j_2$, but this equality implies $s_2 - s_1 = M$; this cannot hold, because $0 \le s_1, s_2 < M$. Therefore, in this case all the coefficients are zero. Similar considerations show that we have the same situation for $j_1 < s_1$ and $j_2 \ge s_2$.

Therefore, the matrix $G_{(s_1,j_2),(s_2,j_2)}$ has diagonal structure:

$$G_{(s_1,j_2),(s_2,j_2)} = \begin{cases} M & \text{if } s_1 = s_2 \text{ and } j_1 = j_2, \\ 0 & \text{otherwise.} \end{cases}$$

This finishes the proof. □

4.3 Magneto-Bloch quasimodes in infinite motion regimes

Let us consider a certain infinite motion regime $\mathcal{M}^r$; choose some $m \in \mathbb{Z}_+$ such that $\big(\mathcal{I}_1^{(m)}(h), \mathcal{I}_2\big) \in \mathcal{D}(\mathcal{M}^r)$ and consider the corresponding quasimodes $\big(\psi^{r,K,L}_{m,\mathcal{I}_2^r,k}, E^{r,K,L}_m(h,\epsilon)\big)$, $k \in \mathbb{Z}^2$, see Proposition 3.20. Our further considerations depend on the drift vector $\boldsymbol{d} = \boldsymbol{d}^r\big(\mathcal{I}_1^{(m)}(h), \epsilon\big) \neq \boldsymbol{0}$. As it was mentioned above, there exists also an infinite motion regime $\mathcal{M}^{r'}$ with drift vector $-\boldsymbol{d}^r\big(\mathcal{I}_1^{(m)}(h), \epsilon\big)$, corresponding to the same energies, and it is natural to consider these regimes together.

In this section we omit the dependence of all the functions on m, ϵ, r, K and L.

From (4.2) one easily obtains:

$$\begin{aligned} &\Psi_j(\boldsymbol{x} + m_1\boldsymbol{a}^1 + m_2\boldsymbol{a}^2, \boldsymbol{q}) = \Psi_{(j+m_2) \mod M}(\boldsymbol{x}, \boldsymbol{q}) \\ &\qquad \cdot \exp\Big[-2\pi i m_1(q_1 - j\eta) - i\eta m_2 x_1 - i\eta m_2^2 a_{21}/2 + 2\pi i L q_2 \Big], \\ &-L \le \frac{m_2 + j}{M} \le (-L+1) - \frac{1}{M}, \quad L \in \mathbb{Z}, \end{aligned}$$

in particular:

$$\left[\begin{aligned} &\Psi_j(\boldsymbol{x} + \boldsymbol{d}\cdot\boldsymbol{a}, \boldsymbol{q}) = \Psi_{(j+d_2) \mod M}(\boldsymbol{x}, \boldsymbol{q}) \\ &\qquad \cdot \exp\Big[-2\pi i d_1(q_1 - j\eta) - i\eta d_2 x_1 - i\eta d_2^2 a_{21}/2 + 2\pi i L' q_2 \Big], \\ &-L' \le \frac{d_2 + j}{M} \le (-L'+1) - \frac{1}{M}, \quad L' \in \mathbb{Z}, \end{aligned} \right. \tag{4.12}$$

and

$$\left[\begin{array}{l} \Psi_j(\boldsymbol{x}+J_2\boldsymbol{f}\cdot\boldsymbol{a},\boldsymbol{q}) = \Psi_{(j+f_1) \mod M}(\boldsymbol{x},\boldsymbol{q}) \\ \qquad \cdot\exp\left[2\pi i f_2(q_1-j\eta) - i\eta f_1 x_1 - i\eta f_1^2 a_{21}/2 + 2\pi i L'' q_2\right], \\ -L'' \le \dfrac{f_1+j}{M} \le (-L''+1) - \dfrac{1}{M}, \quad L'' \in \mathbb{Z}. \end{array}\right. \tag{4.13}$$

From the other side, the functions $\psi_{\mathrm{J}_2,k}$ satisfy

$$\left[\begin{array}{l} \psi_{\mathrm{J}_2,k}(\boldsymbol{x}+\boldsymbol{d}\cdot\boldsymbol{a}) = \psi_{\mathrm{J}_2,k}(\boldsymbol{x}) \\ \qquad \cdot\exp\left[2\pi\frac{i}{h}\mathrm{J}_2 + 2\pi i k\eta - i d_2\eta x_1 - \frac{1}{2}i(2\pi d_1 + a_{21}d_2)d_2\right], \\ \psi_{\mathrm{J}_2,k}(\boldsymbol{x}+J_2\boldsymbol{f}\cdot\boldsymbol{a}) = \psi_{\mathrm{J}_2,k+1}(\boldsymbol{x}) \\ \qquad \cdot\exp\left[-if_1\eta x_1 + ikf_1\eta(-2\pi f_2 + a_{21}f_1)\right]. \end{array}\right. \tag{4.14}$$

Let us try to find magneto-Bloch quasimodes in the form

$$\Psi_j(\boldsymbol{x},\boldsymbol{q}) = \sum_{k\in\mathbb{Z}} C_k^j(\boldsymbol{q})\psi_{\mathrm{J}_2,k}(\boldsymbol{x}). \tag{4.15}$$

Substituting (4.14) into (4.15) we obtain

$$\left[\begin{array}{l} \Psi_j(\boldsymbol{x}+\boldsymbol{d}\cdot\boldsymbol{a},\boldsymbol{q}) = \displaystyle\sum_{k\in\mathbb{Z}}\Big\{C_k^j\psi_{\mathrm{J}_2,k}(\boldsymbol{x}) \\ \quad \cdot\exp\left[2\pi\dfrac{i}{h}\mathrm{J}_2 + 2\pi i k\eta - i d_2\eta x_1 - \dfrac{1}{2}i(2\pi d_1 + a_{21}d_2)d_2\right]\Big\}, \\ \Psi_j(\boldsymbol{x}+J_2\boldsymbol{f}\cdot\boldsymbol{a},\boldsymbol{q}) \\ \quad = \displaystyle\sum_{k\in\mathbb{Z}} C_k^j\psi_{\mathrm{J}_2,k+1}\exp\left[-if_1\eta x_1 + ikf_1\eta(-2\pi f_2 + a_{21}f_1)\right], \\ \quad = \displaystyle\sum_{k\in\mathbb{Z}} C_{k-1}^j\psi_{\mathrm{J}_2,k}\exp\left[-if_1\eta x_1 + i(k-1)f_1\eta(-2\pi f_2 + a_{21}f_1)\right]. \end{array}\right. \tag{4.16}$$

Substituting now (4.16) into (4.12) and (4.13) and taking into account the linear independence of ψ_k, we come to the following system of equations for C_k^j:

$$\left[\begin{array}{l} C_k^j\exp\left[2\pi\dfrac{i}{h}\mathrm{J}_2 + 2\pi i k\eta - \dfrac{1}{2}i(2\pi d_1 + a_{21}d_2)d_2\right] \\ = C_k^{(j+d_2) \mod M}\exp\left[-2\pi i d_1(q_1-j\eta) - i\eta d_2^2 a_{21}/2 + 2\pi i L' q_2\right], \\ \qquad -L' \le \dfrac{d_2+j}{M} \le (-L'+1) - \dfrac{1}{M}, \quad L' \in \mathbb{Z}, \\ C_{k-1}^j\exp\left[i(k-1)f_1\eta(-2\pi f_2 + a_{21}f_1)\right] \\ = C_k^{(j+f_1) \mod M}\exp\left[2\pi i f_2(q_1-j\eta) - i\eta f_1^2 a_{21}/2 + 2\pi i L'' q_2\right], \\ \qquad -L'' \le \dfrac{f_1+j}{M} \le (-L''+1) - \dfrac{1}{M}, \quad L'' \in \mathbb{Z}. \end{array}\right. \tag{4.17}$$

Consider this system for $\boldsymbol{d} = (\pm 1, 0)$, then $\boldsymbol{f} = (\pm 1, 0)$.

Proposition 4.2 (Magneto-Bloch quasimodes in infinite motion regimes with drift vector $(\pm 1, 0)$). *Let $\boldsymbol{d} = (\pm 1, 0)$, then magneto-Bloch quasimodes of the form (4.15) exist if and only if*

$$\mathcal{I}_2^{\pm} = \mathcal{I}_2^{\pm}(n^{\pm}, q_1, h) = h\Big(\frac{n^{\pm}}{M} \mp q_1\Big), \quad n^{\pm} \in \mathbb{Z}. \tag{4.18}$$

These quasimodes are defined by the coefficients of the following form:

$$C_k^{j,\pm} = \begin{cases} e^{i\eta k^2 a_{21}/2 + 2\pi i n q_2} & \text{if } j \mp k + nM = 0, \quad n = n^{\pm}\tilde{N} + \tilde{n}M, \quad \tilde{n} \in \mathbb{Z}, \\ 0 & \text{otherwise}, \end{cases} \tag{4.19}$$

where $\tilde{N}$ is an integer number such that for some $\tilde{M} \in \mathbb{Z}$ one has $\tilde{N}N + \tilde{M}M = 1$.

Proof. Equations (4.17) take the following form:

$$C_k^{j,\pm} e^{\pm 2\pi \frac{i}{h}(\mathcal{I}_2^{\pm} + k a_{22})} = C_k^{j,\pm} e^{-2\pi i (q_1 - j\eta)}, \quad k \in \mathbb{Z}, \quad j \in \{0, \dots, M-1\}, \tag{4.20}$$

$$C_{k\pm 1}^{(j+1) \mod M, \pm} = C_k^{j,\pm} e^{i\eta a_{21}/2 \pm ik\eta a_{21}} \sigma_j, \tag{4.21}$$

$$k \in \mathbb{Z}, \quad j \in \{0, \dots, M-1\}, \quad \sigma_{M-1} = e^{2\pi i q_2}, \quad \sigma_j = 1, \quad j \neq M-1, \tag{4.22}$$

where the sign "+"/"−" corresponds to $(\pm 1, 0)$. It follows from (4.20) that for each k and j we have either $C_k^{j,\pm} = 0$ or

$$\pm\Big(\frac{\mathcal{I}_2^{\pm}}{h} + k\eta\Big) = -(q_1 - j\eta) + n_{k,j}, \quad n_{k,j} \in \mathbb{Z}. \tag{4.23}$$

From (4.22) we see that all the coefficients $C_k^{j,\pm}$ are uniquely determined by arbitrary chosen numbers $C_0^{j,\pm}$; therefore, we have at most M solutions. For the sth solution, $s \in \{0, \dots, M-1\}$, we put $C_0^{s,j,\pm} = \delta_{sj}$.

Consider now separately the cases "+" and "−".

Calculate the coefficients $C_k^{s,j,+}$. From (4.22) we obtain

$$\begin{aligned} C_k^{s,j,+} &= C_0^{s,(j-k) \mod M,+} \prod_{l=1}^{k} \Big(e^{i\frac{1}{2}\eta a_{21}} e^{i(k-l)\eta a_{21}} \sigma_{(j-l) \mod M}\Big) \\ &= C_0^{s,(j-k) \mod M,+} e^{i\eta k^2 a_{21}} \prod_{l=1}^{k} \sigma_{(j-l) \mod M}, \end{aligned} \tag{4.24}$$

and

$$\prod_{l=1}^{k} \sigma_{(j-l) \mod M} = e^{2\pi i m q_2}, \quad \frac{1}{M} + (m-1) \leq \frac{k-j}{M} \leq m, \quad m \in \mathbb{Z}, \tag{4.25}$$

for $k \geq 0$ and

$$\begin{aligned} C_k^{s,j,+} &= C_0^{s,(j-k) \mod M,+} \prod_{l=1}^{-k} \left(e^{-i\frac{1}{2}\eta a_{21}} e^{il\eta a_{21}} \frac{1}{\sigma_{(j+l-1) \mod M}} \right) \\ &= C_0^{s,(j-k) \mod M,+} e^{i\eta k^2 a_{21}/2} \prod_{l=1}^{-k} \frac{1}{\sigma_{(j+l-1) \mod M}}, \end{aligned} \tag{4.26}$$

and

$$\prod_{l=1}^{-k} \frac{1}{\sigma_{(j+l-1) \mod M}} = e^{2\pi i m q_2}, \quad -m \leq \frac{j-k}{M} \leq (-m+1) - \frac{1}{M}, \quad m \in \mathbb{Z}. \tag{4.27}$$

for $k < 0$. Therefore, the coefficient $C_k^{s,j,+}$ is non-zero if and only if

$$j - k - s + nM = 0, \quad n \in \mathbb{Z}. \tag{4.28}$$

Substituting (4.28) into (4.25) and (4.27) and then into (4.24) and (4.26) we obtain the following final expression for the coefficients:

$$C_k^{s,j,+} = \begin{cases} e^{i\frac{1}{2}\eta k^2 a_{21} + 2\pi i n q_2}, & \text{if (4.28) is fulfilled,} \\ 0 & \text{otherwise.} \end{cases} \tag{4.29}$$

Substituting now (4.28) into (4.23) we come to the dependence of $\mathcal{I}_2^+$ on q_1:

$$\begin{aligned} \mathcal{I}_2^+ &= \mathcal{I}_2^+(n^+, q_1, h) = h\Big(\frac{n^+}{M} - q_1\Big), \\ n^+ &= nN - n_{k,j}M. \end{aligned}$$

To obtain an analytic dependence on $\boldsymbol{q}$ we request $n^+ = \text{const}$. Introduce a vector $(\tilde{N}, \tilde{M}) \in \mathbb{Z}^2$ dual to (N, M), i. e.

$$\tilde{N}N + \tilde{M}M = 1,$$

then $n = n^+\tilde{N} + \tilde{n}M$ and $n_{j,k} = -n^+\tilde{M} - \tilde{n}M$, $\tilde{n} \in \mathbb{Z}$. It is easy to see that the functions $\Psi_j^{s,+}$ and $\Psi_{(j+1) \mod M}^{(s+1) \mod M,+}$ coincide; to simplify formulas we put $s = 0$.

Now let us calculate the indices $C_k^{s,j,-}$. For $k \geq 0$ we have

$$\begin{aligned} C_k^{s,j,-} &= C_0^{s,(j+k) \mod M,-} \prod_{l=1}^{k} \left(e^{-i\eta a_{21}/2 + il\eta a_{21}} \frac{1}{\sigma_{j+l-1}} \right) \\ &= C_0^{s,(j+k) \mod M,-} e^{i\eta k^2 a_{21}/2} e^{2\pi i m}, \end{aligned} \tag{4.30}$$

$$-m \leq \frac{k+j}{M} \leq (-m+1) - \frac{1}{M}, \quad m \in \mathbb{Z}. \tag{4.31}$$

For $k < 0$ we have

$$\begin{aligned} C_k^{s,j,-} &= C_0^{s,(j+k) \mod M,-} \prod_{l=1}^{-k} \left(e^{i\eta a_{21}/2 - i(-l+1)\eta a_{21}} \frac{1}{\sigma_{j-l}} \right) \\ &= C_0^{s,(j-k) \mod M,-} e^{i\eta k^2 a_{21}/2} e^{2\pi i m}, \end{aligned} \tag{4.32}$$

$$\frac{1}{M} + (-m-1) \leq -k-j \leq -m, \quad m \in \mathbb{Z}. \tag{4.33}$$

Therefore, again the coefficient $C_k^{s,j,-}$ is non-zero if and only if

$$j + k - s + nM = 0, \quad n \in \mathbb{Z}. \tag{4.34}$$

Again, substituting (4.34) into (4.31) and (4.33) and then into (4.30) and (4.32), we come to the following formula:

$$C_k^{s,j,-} = \begin{cases} e^{i\eta k^2 a_{21}/2 + 2\pi i n}, & \text{if (4.34) is fulfilled,} \\ 0 & \text{otherwise.} \end{cases} \tag{4.35}$$

Substituting now (4.34) into (4.23) we come to the dependence of $\mathcal{I}_2^-$ on q_1:

$$\begin{aligned} \mathcal{I}_2^- &= \mathcal{I}_2^-(n^-, q_1, h) = h\Big(\frac{n^-}{M} + q_1\Big), \\ n^- &= nN - n_{k,j}M. \end{aligned}$$

Like in the previous case we request $n^- = \text{const}$ and obtain $n = n^-\tilde{N} + \tilde{n}M$, $n_{k,j} = -n^-\tilde{M} - \tilde{n}M$, $\tilde{n} \in \mathbb{Z}$. The functions $\Psi_j^{s,-}$ and $\Psi_{(j+1) \mod M}^{(s-1) \mod M,-}$ again coincide and we also put $s = 0$.

Summarizing all these considerations we come to the conclusion of the proposition. □

It is rather difficult to write the expressions for the coefficients in the case $\boldsymbol{d} \neq (\pm 1, 0)$, the system (4.17) has in this case a very complicated form. Nevertheless, it is clear that all the coefficients C_k^j are non-zero and the action variable $\mathcal{I}_2$ depends on a certain linear combination of the quasimomenta q_1 and q_2; from this point of view the case we have considered above is the simplest one. From the other side, the case with an arbitrary drift vector $\boldsymbol{d}$ can be reduced to the previously considered one using the following procedure.

A basis on the lattice of periods of v is, of course, not unique; in particular, as a new basis we can choose the vectors $\boldsymbol{b}^1 = \boldsymbol{d} \cdot \boldsymbol{a}$ and $\boldsymbol{b}^2 = J_2 \boldsymbol{f} \cdot \boldsymbol{a}$. The drift vector with respect to this basis is equal to $(\pm 1, 0)$. Now, using the gauge-rotating transformation (we use the notation of Subsection 3.6.2) we can direct the first basis vector along the y_1-axis. Taking now the magneto-Bloch conditions in this $\boldsymbol{y}$-representation we have the required situation.

4.4 Separation of the bands

In the previous two sections we have constructed formal magneto-Bloch solutions. In the both cases we used only one edge of the corresponding Reeb graphs. Discuss now the situation when on some energy level there are points from different edges.

Discuss first the case of finite motion edges. As we have found, for each quantization point there are M families of magneto-Bloch quasimodes. We expect that in a small neighborhood of each quantization point there are M bands; these bands have the length $O(h^\infty + \epsilon^\infty)$. Their presence cannot be "caught" by the methods we use; nevertheless, it is possible to give additional (but non-rigorous) arguments. Let us enlarge the unit cell by the rule $\boldsymbol{a}^1 \mapsto M\boldsymbol{a}^1$, then we come to the case of integer flux. The structure of the Reeb graph changes, a finite motion edge splits into M edges. Let us construct magneto-Bloch quasimodes for each of these new edges using the procedure described above. The flux is integer, the denominator is equal to 1, and for each of the quantization points on each edge there is only one magneto-Bloch quasimode. Therefore, we have again M magneto-Bloch quasimodes corresponding to the same energy value, but now these quasimodes are not connected to each other. Therefore, very small variation of the potential v (in the class of periodic functions) changes these energy values, and we obtain M magneto-Bloch quasimodes corresponding to different energy values, see Fig. 4.1. The similar procedure can be applied also to the case when the Reeb graph of $\mathcal{H}$ has more than one finite motion edge on a certain energy interval originally (i. e. already in the case of rational flux).

We can expect that the functions Ψ_j^s constructed in section 4.2 form an "approximate" basis in the space of true magneto-Bloch function corresponding to these minibands. Then we can expect that the sum of the Chern classes of all the minibands corresponding to a fixed quantization point $E_{\boldsymbol{m}}^{K,L}(h,\epsilon)$ is equal to 0. This hypothesis is motivated by the procedure of enlarging the unit cell and by the fact that the bundle of the magneto-Bloch quasimodes has trivial Chern class in the case of integer flux.

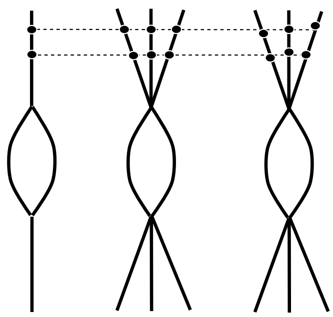

Figure 4.1: Separation of the minibands

Consider now the situation with two infinite motion edges on the same energy level. Consider two such infinite motion edges and suppose for simplicity that their drift vectors are $(1,0)$ and $(-1,0)$ (by a gauge-rotating transformation we can reduce any drift vectors to this situation). Denote by $\Psi_j^{\pm}$ the corresponding magneto-Bloch quasimodes. Each edge implies also a certain dependence ("dispersion laws") of the energy of these quasimodes on the quasimomenta:

$$E^{\pm}_{n^{\pm}}(q_1,h,\epsilon) = \mathcal{H}\big(\mathcal{I}_1^{(m)}, \mathcal{I}_2^{\pm}(\boldsymbol{q},h)\big),$$

where $\mathcal{I}_2^{\pm}$ are defined in (4.23) (the dependence on q_2 is absent at least up to $O(h^L + \epsilon^K)$). It is clear that one of these functions decreases with respect to q_1, and another one increases. This means that in some "critical" points $q_1 = q^*_{1,n^-,n^+}$ one has $E^-_{n^-} = E^+_{n^+}$, see Fig. 4.2. We expect that near such points $E^{\pm}_{n^{\pm}}(q_1^*,h,\epsilon)$ there are gaps in the spectrum of $\hat{H}_{h,\epsilon}$. Such a situation is typical in quantum physics. Obtaining true dispersion laws from their semiclassical approximations in such situations is well known in the physics literature as the *avoiding band crossing problem* (or, shortly, the *ABC problem*, see, for example, [54,87] and references there-in). At present, there is no rigorous justification of this procedure (more precisely, to give a rigorous justification one has to know the asymptotics of the true eigenfunctions outside the classically allowed regions; in this problem we do not have such information), and we use only the following (again non-rigorous) observation. Consider for simplicity the zase of integer flux; then the true dispersion laws are continuous and periodic in q_1 [38,70]; our heuristic dispersion laws are always monotone; therefore, we can expect that in the critical points q_1^* we have a jump from an increasing heuristic dispersion law to a decreasing one, because there are no other possibilities to make them be periodic. Therefore, we obtain a picture shown in Fig. 4.3.

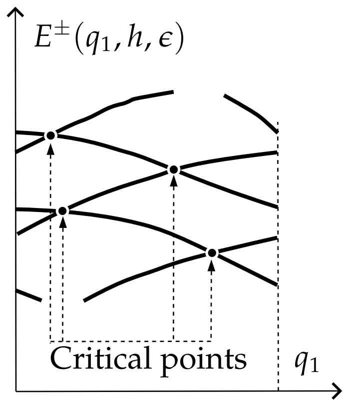

Figure 4.2: Heuristic dispersion laws

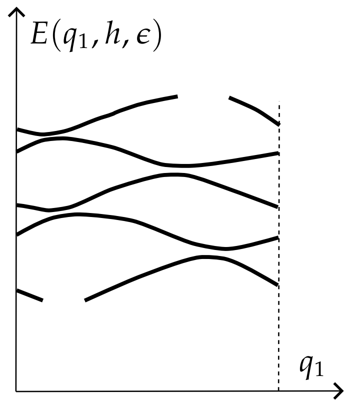

Figure 4.3: Expected true dispersion laws

Of course, more complicated situations are possible, for example, several finite motion edges and infinite motion edges on the same energy levels. Each of such cases needs a separate study.

Basing on the above described expectations, we can give a general description of the asymptotics of the band spectrum of $\hat{H}_{h,\epsilon}$ for the rational flux $\eta = N/M$. Let us use the notation of Propositions 3.19 and 3.20. Each quantization point $E_{\boldsymbol{m}}(h, \epsilon)$ splits into M subbands with the length $O(h^\infty + \epsilon^\infty)$. These groups of bands are separated by gaps with the length of order ϵh. The intervals $E^r_m(\mathcal{I}_2, h, \epsilon)$ are separated into bands with the length if order ϵh. These bands are separated by gaps of the length $O(h^\infty + \epsilon^\infty)$.

Having in mind the correspondence between the Harper-like equations and the problem in question (see Subsection 3.10), it is interesting to compare the hypotesises posed with the results concerning the rational Harper-like equations with flux $\eta = N/M$. Here we have the following situation [30,88]: the splitting of each quantization point into M minibands is confirmed by some examples [31,44]. The same estimate for the Chern class is also obtained [31]. The spectrum of the rational Harper-like model consists of exactly N bands [88, 57], therefore, we can expect that each (semiclassical) Landau band also consists of N subbands [14,15].

Under this assumption, we have a chaotic picture of the spectrum as η approaches irrational values, because both the nominator N and the denominator M of η change irregularly.

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Erklärung

Hiermit versichere ich, dass die vorliegende Dissertation selbständig und ohne unerlaubte Hilfe angefertigt wurde.

Berlin, den 27. Juni 2002

Konstantin Pankrachkine