String Representation of Gauge Theories

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Contents

1 Introduction

1.1 General Ideas and Motivations

Nowadays, there is no doubt that strong interactions of elementary particles are adequately described by Quantum Chromodynamics (QCD) [1] (see Ref. [2] for recent monographs). Unfortunately, usual field-theoretical methods are not adequate to this theory itself. That is because in the infrared (IR) region, the QCD coupling constant becomes large, which makes the standard Feynman diagrammatic technique in this region unapplicable. However, it is the region of the strong coupling, which deals with the physically observable colourless objects (hadrons), whereas the standard perturbation theory is formulated in terms of coloured (unphysical) objects: quarks, gluons, and ghosts. This makes it necessary to develop special techniques, applicable for the evaluation of effects beyond the scope of perturbation theory. The latter are usually referred to as *nonperturbative phenomena*. Up to now, those are best of all studied in the framework of the approach based on lattice gauge theory [3], which provides us with a natural nonperturbative regularization scheme. Various ideas and methods elaborated on in the lattice field theories during the two last decades, together with the development of algorithms for numerical calculations and progress in the computer technology, have made these theories one of the most powerful tools for evaluation of nonperturbative characteristica of QCD (see Ref. [4] for a recent review). However, despite obvious progress of this approach, there still remain several problems. Those include e.g. the problem of simultaneous reaching the continuum and thermodynamic limits. Indeed, physically relevant length scales lie deeply inside the region between the lattice spacing and the size of the lattice. However, due to the asymptotic freedom of QCD, in the weak coupling limit, not only the lattice spacing, but also the size of the lattice (for a fixed number of sites) becomes small, as well as the region between them. However, in order to achieve the thermodynamic limit, the size of the lattice should increase. This makes it necessary to construct large lattices, which in particular leads to the technical problem of critical slowing down of simulations on them. As far as the problem of reaching the continuum limit alone is concerned, recently some progress in the solution of this problem has been achieved by making use of the conception of improved lattice actions [5] in Ref. [6]. Another problem of the lattice formulation of QCD is the appearance of so-called fermion doublers (i.e. additional modes appearing as relevant dynamical degrees of freedom) in the definition of the fermionic action on the lattice due to the Nielsen-Ninomiya theorem [7]. According to this No-Go theorem, would we demand simultaneously hermiticity, locality, and chiral symmetry (which will be discussed later on) of the lattice fermionic action, the doublers unavoidably appear, which means that all these three physical requirements cannot be achieved together. This makes it necessary to introduce fermionic actions which violate one of these properties (e.g. Wilson fermions [8], violating chiral symmetry for a finite lattice or staggered (Kogut-Susskind) fermions [9], violating locality for a single flavour fermion), checking afterwards lattice artefacts associated with a particular choice of the action. Notice however, that recently a significant progress in the solution of this problem has been achieved (for a review see [10] and Refs. therein). Finally, there remains the important problem of reaching the chiral limit, which becomes especially hard if one accounts for dynamical fermions. That was just one of the reasons why the main QCD calculations on large lattices have been performed in the quenched approximation, i.e. when the creation/annihilation of dynamical quark pairs is neglected.

All these problems together with the necessity of getting deeper theoretical insights into nonperturbative phenomena require to develop *analytical* nonperturbative techniques in QCD and other theories displaying such phenomena. This is the main motivation for the present work.

It is worth realizing, that up to now a systematic way of analytical investigation of nonperturbative phenomena in QCD is still lacking. Instead of that, there exist various approaches, enabling one to take them into account phenomenologically for describing hadron interactions. Those approaches include e.g. the potential and bag models [11, 12], the large-N expansion methods [13], the effective Lagrangian approach [14], the QCD sum rule approach [15], and later on its generalization, the so-called Stochastic Vacuum Model [16, 17, 18, 19]. It is natural to compare the situation in QCD appearing due to the absence of a systematic approach for the investigation of nonperturbative phenomena with nonrelativistic Quantum Mechanics. One convinces himself that the latter is the correct theory of atomic spectra by studying simple objects, like the hydrogen atom, though the practical calculation of the spectrum of a certain composite atom within this theory is quite complicated. Unfortunately, in QCD there exists no analogy of an isolated "hydrogen atom", which forces us to study the theory of strong interactions at quite sophisticated examples.

The most fundamental problem associated with the IR dynamics of QCD, which is known to be one of the most important problems of modern Quantum Field Theory, is the problem of explanation and description of *confinement* (for a review see e.g. [20, 19]). In general, by confinement one implies the phenomenon of absence in the spectrum of a certain field theory of the physical $|\mathrm{in}\rangle$ and $|\mathrm{out}\rangle$ states of some particles, whose fields are however present in the fundamental Lagrangian. It is this phenomenon in QCD, which forces quarks and antiquarks to combine into colourless hadronic bound states. The important characteristic of confinement is the existence of string-like field configurations leading to a linearly rising $q\bar{q}$ (qq)-potential as expressed by the Wilson's area law (see the next Subsection). Such a field configuration emerging between external quarks is usually referred to as QCD string. In the present Dissertation, we shall demonstrate that the properties of this string can be naturally studied in the framework of the Stochastic Vacuum Model and derive by virtue of this model the corresponding string Lagrangian. Notice, that the advantage of the Stochastic Vacuum Model is that it deals directly with QCD, and therefore enables us to express the coupling constants of this Lagrangian in terms of the fundamental QCD quantities, which are the gluonic condensate and the so-called correlation length of the vacuum. This approach will also allow us to incorporate quarks and derive a Hamiltonian of the QCD string with quarks in the confining QCD vacuum.

Another fundamental phenomenon of nonperturbative QCD is the spontaneous breaking of chiral symmetry, i.e. the $U(N_f) \times U(N_f)$-symmetry of the massless QCD action. Indeed, though one could expect this symmetry to be observed on the level of a few MeV, it does not exhibit itself in the hadronic spectra. Were this symmetry exact, one would expect parity degeneracy of all hadrons, whereas in reality parity partners are generally split by a few hundred MeV. Such a spontaneous symmetry breaking has far-reaching consequences. In particular, it implies that there exist massless Goldstone bosons, which are identified with pions. The signal for chiral symmetry breaking is the appearance of a nonvanishing quark condensate, which plays an important role in many nonperturbative approaches [21, 22].

The non-Abelian character of the gauge group $SU(3)$ makes it especially difficult to study the problems of confinement and chiral symmetry breaking in the QCD case. To explain the mechanisms of these phenomena microscopically, a vast amount of models of the QCD vacuum have been proposed (see [19] for a recent review). Those are based either on an ensemble of classical field configurations (e.g. instantons [23, 24], see [25] for recent reviews) or on quantum background fields [26, 27]. The most general demand made on all of them was to reproduce two characteristic quantities of the QCD vacuum, which are nonzero quark [21] and gluon [15] condensates, related to the chiral symmetry breaking and confinement, respectively. However, at least the semiclassical scenario possesses several weak points. First of all, since the topological charge of the QCD vacuum as a whole is known to vanish, this vacuum cannot be described by a certain unique classical configuration, but should be rather built out of a superposition of various configurations, e.g. instantons and antiinstantons. However, such a superposition already does not satisfy classical equations of motion and is, moreover, unstable w.r.t. annihilation of the objects with the opposite topological charge. Secondly, in order to reproduce the phenomenological gluon condensate [15], classical configurations should be dense packed (about one configuration per fm^4), which leads to a significant distorsion of the solutions corresponding to these configurations within the original superposition ansatz [28] [1]. And last but not least, a further counterargument against semiclassical models of the QCD vacuum is that quite not all of them, once being simulated in the lattice experiments, yield the property of confinement (see discussion in Ref. [19]).

Another natural way of investigation of the nonperturbative phenomena in QCD might lie in the simplification of the problem under study by considering some solvable theories displaying the same type of phenomena. In this way, the problem of chiral symmetry breaking is best of all analytically studied in the so-called Nambu-Jona-Lasinio (NJL) type models, which are models containing local four-quark interactions [30, 31, 32, 33] (see Ref. [34] for a recent review). These models lead to a gap equation for the dynamical quark mass, signalling spontaneous breaking of chiral symmetry. After applying the so-called bosonization procedure (which can be performed either by making use of the standard Hubbard-Stratonovich transformation or within the field strength approach [35]) as well as a derivative expansion of the resulting quark determinant at low energies, this leads to the construction of nonlinear chiral meson Lagrangians [14]. The advantage of the latter ones is that they summarize QCD low-energy theorems, which is the reason why these Lagrangians are intensively used in the modern hadronic physics [36, 34]. The techniques developed for NJL type models have been in particular applied to the evaluation of higher-order derivative terms in meson fields [32, 33], which enabled one to estimate the structure constants of the effective chiral Lagrangians introduced in Ref. [36]. Furthermore, in this way in Refs. [31, 32, 33, 34] it has been demonstrated that the low-energy properties of light pseudoscalar, vector, and axial-vector mesons are well described by effective chiral Lagrangians following from the QCD-motivated NJL models. In addition, the path-integral bosonization of an extended NJL model including chiral symmetry breaking of light quarks and heavy quark symmetries of heavy quarks has been performed [37] (see the second paper of Ref. [38] for a review), which yielded the effective Lagrangians of pseudoscalar, vector, and axial-vector D or B mesons, interacting with light π, ρ, and a_1 mesons.

As far as the theories possessing the property of confinement are concerned, those firstly include compact QED and the 3D Georgi-Glashow model [20] and, secondly, the so-called Abelian-projected theories [39] [2]. In the present Dissertation, we shall concentrate ourselves on the confining properties of the above mentioned non-supersymmetric theories. In all of them, confinement occurs due to the expected condensation of Abelian magnetic monopoles, after which the vacuum structure of these theories becomes similar to that of the dual superconductor (the so-called dual Meissner scenario of confinement) [41, 42]. Such a vacuum then leads to the formation of strings (flux tubes) connecting external electric charges, immersed into it. These strings are

[1] Recently, some progress in the solution of this problem has been achieved in Ref. [29].

[2] In what follows, we shall not consider recently discovered supersymmetric theories, also possessing the property of confinement [40].

dual to the (magnetic) Abrikosov-Nielsen-Olesen strings [43]. The latter ones emerge as classical field configurations in the Abelian Higgs Model, which is the standard relativistic version of the Ginzburg-Landau theory of superconductivity. It turns out that the properties of electric strings in the dual superconductor are similar to the ones of the realistic strings in QCD, which connect quarks with antiquarks and ensure confinement. It is worth noting, that this analogy based on the 't Hooft-Mandelstam scenario led to several phenomenological dual models of QCD (see Ref. [44] for a review). Thus, the properties of the QCD string can be naturally studied in the framework of the Abelian projection method. Moreover, this approach turns out to provide us with the representations for the partition functions of effective dual models of Abelian-projected $SU(2)$- and $SU(3)$-gluodynamics in terms of the integrals over string world-sheets. Such an integration, which is absent in the approach to the QCD string based on the Stochastic Vacuum Model, appears now from the integration over the singular part of the phase of the magnetic Higgs field. The reformulation of the integral over the singularities of this field into the integral over string world-sheets is possible due to the fact that such singularities just take place at the world-sheets. In particular, an interesting string picture emerges in the $SU(3)$-gluodynamics, where after the Abelian projection there arise three types of magnetic Higgs fields, leading to three types of strings, which (self)interact via the exchanges of two massive dual gauge bosons. An exact procedure of the derivation of the string representations for the partition functions of Abelian-projected theories in the language of the path-integral, the so-called path-integral duality transformation, will be described in details below. In the framework of this approach, we shall also investigate field correlators in the Abelian-projected theories and find them to parallel those of QCD, predicted by the Stochastic Vacuum Model and measured in the lattice simulations. After that, we shall study the string representation and field correlators in 3D compact QED. As it will be demonstrated, this theory can be considered as the limiting case of a 3D Abelian Higgs Model with external monopoles for vanishing gauge boson mass. Notice that due to the absence of the Higgs field in 3D compact QED, the integration over the string world-sheets is realized in this theory in another way than in Abelian-projected theories. Namely, it results from the summation over the branches of the multivalued effective monopole potential. Finally, similar forms of the string effective actions in QCD, Abelian-projected theories, and compact QED will enable us to elaborate for all these theories a unified method of description of the string world-sheet excitations, based on the methods of nonlinear sigma-models, known from the standard string theory.

It is worth realizing, that the string theories, to be derived below, should be treated as effective, rather than fundamental ones. The actions of all of them turn out to have the form of an interaction between the elements of the string world-sheet, mediated by certain nonperturbative gauge field propagators. Being expanded in powers of the derivatives w.r.t. world-sheet coordinates, these actions yield as a first term of such an expansion the usual Nambu-Goto action. The latter one is known to suffer from the problem of conformal anomaly in $D \neq 26$ appearing during its quantization, which will not be discussed below. It this sense, throughout the present Dissertation, we shall treat the obtained string theories as effective four-dimensional ones. It is also worth noting, that within our approach only pure bosonic strings without supersymmetric extensions appear. As far as superstrings are concerned, during the last fifteen years, a great progress has been achieved in their development (see e.g. [45] for comprehensive monographs). Among the achievements of the superstring theory it is worth mentioning such ones as the calculation of the critical dimension of the space-time, inclusion of gravity in a common scheme, and, presumably, the absence of divergences for some of these theories. The aim of all the superstring theories is the unification of all the four fundamental interactions. In another language, one should eventually be able to derive from them both the Standard Model and gravity, whereas all the auxiliary heavy modes should become irrelevant. Therefore, the final strategy of superstring theories is a derivation of the known field theories out of them. Contrary to this ideology, the aim of the present Dissertation is the derivation of effective string theories from gauge field theories possessing string-like excitations. As it has been discussed above, such string-like field configurations naturally appear in the confining phases of gauge theories.

Another possible direction of investigation of confinement and chiral symmetry breaking in QCD is based on a derivation of self-coupled equations for gauge-invariant vacuum amplitudes starting directly from the QCD Lagrangian and seeking for solutions allowing for these properties [46, 47, 48, 49]. Recently, this approach turned our to be quite useful for the investigation of the problem of interrelation between these two phenomena [49]. Once such an interrelation takes place, there should exist a relation between quark and gluon condensates as well, which has just been established in Ref. [49]. We shall briefly demonstrate the method of derivation of such a relation later on.

The organization of the Dissertation is as follows. In the next Subsection of the Introduction, we shall introduce the main quantitative parameters for the description of confinement in gauge theories and quote the criterion of confinement in the sense of Wilson's area law. This criterion will then serve as our starting point in a derivation of certain string effective actions in various gauge theories. In the last Subsection, we shall consider theoretical foundations of the Stochastic Vacuum Model. Section 2 is devoted to a derivation and investigation of the QCD string effective action within this model. In Section 3, we investigate the problem of string representation of QCD from the point of view of Abelian-projected theories and demonstrate a correspondence between the Abelian projection method and Stochastic Vacuum Model. In Section 4, we study the string representation and vacuum correlators of 3D compact QED and show how this theory is related to the Abelian Higgs Model. This brings us to the conclusion that both QCD within the Stochastic Vacuum Model, Abelian-projected theories, and compact QED have similar string representations. Such an observation then enables us to consider strings in these theories from the same point of view and elaborate for them a unified mechanism of description of string excitations. Finally, we summarize the main results of the Dissertation and discuss possible future developments in the Conclusion. In three Appendices, some technical details of transformations performed in the main text are outlined.

1.2 Wilson's Criterion of Confinement and the Problem of String Representation of Gauge Theories

As a most natural characteristic quantity for the description of confinement one usually considers the so-called Wilson loop. For example in the case of QCD, this object has the following form

$$\langle W(C)\rangle = \frac{1}{N_c}\left\langle \operatorname{tr} P\,\exp\left(ig\oint\limits_C A_\mu dx_\mu\right)\right\rangle, \qquad (1)$$

which is nothing else, but an averaged amplitude of the process of creation, propagation, and annihilation of a quark-antiquark pair. In Eq. (1), A_μ stands for the vector-potential of the gluonic field [3], g is the QCD coupling constant, C is a closed contour, along which the quark-

[3]From now on in the non-Abelian case, $A_\mu \equiv A_\mu^a t^a, a = 1, ..., N_c^2 - 1$, where $t^a = (t^a)^{ij}$ is the Hermitean generator of the colour group in the fundamental representation, whereas in the Abelian case A_μ is simply a vector

antiquark pair propagates, P stands for the path-ordering prescription, which is present only in the non-Abelian case, and the average on both sides is performed with the QCD action [4].

In order to understand why this object really serves as a characteristic quantity in QCD, let us consider the case, when the contour C is a rectangular one and lies for concreteness in the (x_1, t)-plane. Let us also denote the size of C along the t-axis as T, and its size along the x_1-axis as R. Then such a Wilson loop in the case $T \gg R$ is related to the energy of the static (i.e. infinitely heavy) quark and antiquark, which are separated from each other by the distance R, by the formula

$$\langle W_{R\times T}\rangle \sim \mathrm{e}^{-E_0(R)\cdot T}, \; T \gg R. \tag{2}$$

In order to get Eq. (2), let us fix the axial gauge $A_4 = 0$ [5], so that only the segments of the rectangular contour C parallel to the x_1-axis, contribute to $\langle W_{R\times T}\rangle$. Denoting

$$\Psi_{ij}(t) \equiv \left[P \exp\left(ig\int\limits_0^R dx_1 A_1\left(\vec{x}, t\right)\right)\right]_{ij}, \tag{3}$$

where we have omitted for shortness the dependence of Ψ_{ij} of x_2 and x_3, we are not interested in, we get

$$\langle W_{R\times T}\rangle = \frac{1}{N_c}\left\langle \Psi_{ij}(0)\Psi^\dagger_{ji}(T)\right\rangle. \tag{4}$$

Inserting into Eq. (4) a sum over a complete set of intermediate states $\sum\limits_n |n\rangle\langle n| = 1$, we get

$$\langle W_{R\times T}\rangle = \frac{1}{N_c}\sum_n \left\langle \Psi_{ij}(0)\,|n\rangle\langle n|\,\Psi^\dagger_{ji}(T)\right\rangle = \frac{1}{N_c}\sum_n |\langle\Psi_{ij}(0)|n\rangle|^2\, \mathrm{e}^{-E_nT}, \tag{5}$$

where E_n is the energy of the state $|n\rangle$. At $T \to \infty$, only the ground state with the lowest energy survives in the sum over states standing in Eq. (5), and we finally arrive at Eq. (2).

The energy $E_0(R)$ in Eq. (2) includes a R-independent renormalization of the mass of a heavy (anti)quark due to its interaction with the gauge field. To the first order in g^2, up to a colour factor, it is the same as in QED [50] and reads

$$\Delta E_{\mathrm{mass}} = C_2\frac{g^2}{4\pi a}, \tag{6}$$

where $C_2 = \frac{N_c^2-1}{2N_c}$ stands for the Casimir operator of the fundamental representation, and $a \to 0$ is a cutoff parameter (e.g. lattice spacing). The difference $E(R) = E_0(R) - \Delta E_{\mathrm{mass}}$ therefore defines the potential energy of the interaction between a static quark and antiquark. In particular, the exponential dependence of the Wilson loop on the area of the minimal surface encircled by the contour C

$$\langle W(C)\rangle \to \mathrm{e}^{-\sigma\cdot \mathrm{Area}_{\mathrm{min.}}(C)} \tag{7}$$

potential.

[4]In what follows, we call the object defined by Eq. (1) for shortness a "Wilson loop", whereas in the literature it is sometimes referred to as a "Wilson loop average".

[5]Throughout the present Dissertation, we shall work in the Euclidean space-time.

(the so-called *area law* behaviour of the Wilson loop) corresponds to the linearly rising potential between a quark and an antiquark,

$$E(R) = \sigma R. \tag{8}$$

This is the essence of the Wilson's *criterion of confinement* [51].

In Eqs. (7) and (8), the coefficient σ is called *string tension*. This is because the gluonic field between a quark and an antiquark is contracted to a tube or a string (the so-called *QCD string*), whose energy is proportional to its length, and σ is the energy of such a string per unit length. This string plays the central role in the Wilson's picture of confinement, since with the distance R between a quark and an antiquark it stretches and prevents them from moving apart to macroscopic distances.

In order to get an idea of numbers, notice that according to the lattice data [52] the distance R, at which Wilson's criterion of confinement becomes valid, is of the order of 1.0 fm, and the string tension is of the order of $0.2\,\mathrm{GeV}^2$ (see e.g. [19]). It is worth realizing, that the *classical* QCD Lagrangian does not contain a dimensional parameter of such an order (i.e. of hundreds MeV) [6]. However, in quantum theory, there always exists a dimensional cutoff (like the lattice spacing a in Eq. (6)), which is related to the QCD coupling constant g through the Gell-Mann–Low equation

$$-a^2 \frac{dg^2\left(\frac{1}{a^2}\right)}{da^2} = g^2 \beta_{\mathrm{QCD}}\left(g^2\right). \tag{9}$$

Here $\beta_{\mathrm{QCD}}\left(g^2\right)$ stands for the QCD Gell-Mann–Low function, which at the one-loop level reads

$$\beta_{\mathrm{QCD}}\left(g^2\right) = -\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right)\frac{g^2}{16\pi^2}, \tag{10}$$

where N_f is the number of light quarks flavours, whose masses are smaller than $1/a$. It is Eq. (10), which tells us that QCD is an asymptotically free theory, provided that for $N_c = 3$, $N_f \leq 16$, which indeed holds in the real world. Consequently, the high-energy limit of QCD (the so-called perturbative QCD) is similar to the low-energy limit of QED, and the scale parameter following from the integration of Eq. (9),

$$\Lambda_{\mathrm{QCD}}^2 = \frac{1}{a^2}\exp\left[-\int^{g^2\left(\frac{1}{a^2}\right)} \frac{dg'^2}{g'^2\beta\left(g'^2\right)}\right], \tag{11}$$

is measurable in QCD as well as the QED fine-structure constant (=1/137) with the result $100\,\mathrm{MeV} \leq \Lambda_{\mathrm{QCD}} \leq 300\,\mathrm{MeV}$ [53]. The phenomenon of the appearance of a dimensional parameter in QCD, which remains finite in the limit of vanishing cutoff, is usually referred to as *dimensional transmutation*. All observable dimensional quantities in QCD (e.g. hadron masses), and, in particular, the string tension, are proportional to the corresponding power of Λ_{QCD} [7]. Then according to Eqs. (10) and (11), we get

[6] E.g. the masses of the light quarks are of the order of a few MeV.

[7] This means that the dimensionless ratios of these quantities, e.g. the ratio of $\sqrt{\sigma}$ to hadron masses, are universal (i.e. g-independent) numbers. The aim of all the nonperturbative phenomenological approaches to QCD, as e.g. the so-called Stochastic Vacuum Model, which will be described in the next Subsection, is to calculate these numbers, but not Λ_{QCD} itself.

$$\sigma \propto \Lambda_{\rm QCD}^2 = \frac{1}{a^2} \exp \left[-\frac{16\pi^2}{\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right) g^2 \left(\frac{1}{a^2}\right)} \right], \tag{12}$$

which means that all the coefficients in the expansion of the string tension in powers of g^2 vanish. This conclusion tells us that the QCD string has a pure nonperturbative origin, as well as the phenomenon of confinement, which leads to the process of formation of such strings in the vacuum itself.

Throughout the present Dissertation, we shall be mostly interested in the models and properties of the QCD string and strings in other gauge theories, possessing a confining phase. In the literature, this problem is usually referred to as a *problem of string representation of gauge theories.*

Contrary to the linearly rising quark-antiquark potential, the Coulomb potential, $E(R) \propto -\frac{g^2}{R}$, cannot confine quarks, since in this case the gauge field between them is distributed over the whole space. For such a potential, the Wilson loop for the large contour C has the following asymptotic behaviour

$$\langle W(C) \rangle \to {\rm e}^{-{\rm const}\cdot L(C)}, \tag{13}$$

where $L(C) \equiv \int\limits_0^1 ds \sqrt{\dot{x}^2(s)}$ stands for the length of the contour C, parametrized by the vector-function $x_\mu(s)$, $0 \le s \le 1$, $x_\mu(0) = x_\mu(1)$. Such a behaviour is called the *perimeter law*. It is dominant at small distances $R \le 0.25\,{\rm fm}$, where quarks can with a good accuracy be considered as separate particles not connected by strings, i.e. in the framework of perturbative QCD.

It is worth noting, that to each order of perturbation theory, it is the perimeter law (13), rather than the area law (7), that holds for the Wilson loops. Because of the ultraviolet divergencies, for such a perturbative expansion of Eq. (1) in powers of g, one needs a (gauge invariant) regularization. When such a regularization is introduced, the Wilson loop for a smooth contour C (i.e. a contour without cusps) takes the form

$$\langle W(C) \rangle = \exp \left[-C_2 \frac{g^2}{4\pi} \frac{L(C)}{a} \right] \langle W_{\rm ren.}(C) \rangle \,, \tag{14}$$

where $\langle W_{\rm ren.}(C) \rangle$ is finite when expressed via the renormalized charge $g_{\rm ren.}$. The exponential factor in Eq. (14) is due to the renormalization of the mass of a heavy (anti)quark, described by Eq. (6). The multiplicative renormalization of the smooth Wilson loop has been shown in Refs. [54],[55], and [56]. Notice also, that if the contour C has a cusp(s) but no self-intersections, then $\langle W(C) \rangle$ is still multiplicatively renormalizable [57], namely $\langle W(C) \rangle = Z(\gamma) \langle W_{\rm ren.}(C) \rangle$, where the diverging factor $Z(\gamma)$ depends on the cusp angle (or angles) γ (or γ's), and $\langle W_{\rm ren.}(C) \rangle$ is again finite when expressed via the renormalized charge $g_{\rm ren.}$.

Thus, we conclude that the Wilson loop indeed plays the role of the quantity relevant for the description of confinement. Its area law behaviour means that the gauge theory under study (e.g. QCD) is in the confining phase, whereas the perimeter law behaviour means that the theory is in the Coulomb phase[8]. We also see that due to Eq. (2), the change of $-\ln \langle W(C) \rangle$ determines the change of the action due to the interaction of an (anti)quark with the gauge field.

[8] In order to distinguish between these two laws, Creutz [58] proposed to consider the following ratio of the Wilson loops defined on rectangular contours $\chi(I,J) = -\ln \frac{\langle W_{I\times J}\rangle \langle W_{(I-1)\times(J-1)}\rangle}{\langle W_{(I-1)\times J}\rangle \langle W_{I\times(J-1)}\rangle}$ (the so-called Creutz ratio). One can see that the exponentials of the perimeter cancel out in this ratio (in particular, the mass renormalization (6) cancels out). If the area law (7) holds with $A_{\rm min.} = IJa^2$, then $\chi(I,J) = a^2\sigma$, i.e. it does not depend on I

1.3 Stochastic Vacuum Model of QCD: Theoretical Foundations

The nonlinearity of the QCD action makes it impossible to calculate in this theory the vacuum averages of gauge-invariant quantities in a closed form, e.g. by making use of the path-integral techniques. On the other hand, the standard perturbation theory also becomes unapplicable at large distances, where the running coupling constant is around 0.5-1. This makes it necessary to develop special approaches to nonperturbative QCD. In the present Subsection, we shall briefly describe one of them, the so-called Stochastic Vacuum Model (SVM) of QCD. The idea, which lies behind this approach, is that a large class of observables in QCD can be expressed in terms of the Wilson loops [51], [59]. In this respect, the calculation of some QCD observable consists of two steps: the calculation of the Wilson loop for an arbitrary contour and summation of the Wilson loops over contours with a certain weight, which is determined by the observable[9]. For example, the Green function of a system consisting of a scalar quark and antiquark with equal masses, m, which propagates in the QCD vacuum from the initial state $(y, \bar{y})$ to the final state $(x, \bar{x})$,

$$G(x, \bar{x}; y, \bar{y}) \equiv \left\langle \mathrm{tr}\, \left(\bar{\psi}(\bar{x})\, \Phi(\bar{x}, x)\, \psi(x)\right) \left(\bar{\psi}(y) \Phi(y, \bar{y})\, \psi(\bar{y})\right) \right\rangle, \tag{15}$$

where

$$\Phi(x, y) = \frac{1}{N_c} P \exp\left[ig \int\limits_y^x A_\mu(u) du_\mu\right] \tag{16}$$

stands for the parallel transporter factor along the straingt line[10], in the quenched approximation reads [61],[62],[63]

$$G(x, \bar{x}; y, \bar{y}) = \int\limits_0^{+\infty} ds \int\limits_0^{+\infty} d\bar{s} \mathrm{e}^{-m^2(s+\bar{s})} \int Dz \int D\bar{z} \exp\left(-\frac{1}{4}\int\limits_0^s \dot{z}^2 d\lambda - \frac{1}{4}\int\limits_0^{\bar{s}} \dot{\bar{z}}^2 d\lambda\right) \langle W(C) \rangle\, . \tag{17}$$

In Eq. (17), the dot stands for $\frac{d}{d\lambda}$, and the closed contour C consists of the trajectories z_μ and $\bar{z}_\mu$ of a quark and antiquark, $z(0) = y$, $z(s) = x$, $\bar{z}(0) = \bar{y}$, $\bar{z}(\bar{s}) = \bar{x}$, and straight-line pieces, which form the initial and final states. One can see that the weight factor in the path integral standing in Eq. (17) is completely determined by the free theory, and all the dependence of the gauge field factors out in the form of the Wilson loop.

After that, the main idea of SVM is not to directly evaluate the Wilson loop, but to express it via gauge-invariant irreducible correlators (the so-called cumulants) of the gauge field strength tensors. Such correlators have been measured in lattice experiments both at large and small distances [64, 65], which enables one to use them in practical calculations of various physical quantities. In order to express the Wilson loop (1) via cumulants, one should first make use of the non-Abelian Stokes theorem, derived in Refs. [66] and [67], which yields [11]

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and J and is proportional to the string tension. This property of the Creutz ratio is usually used for numerical calculations of the string tension.

[9]It should be mentioned, that contrary to the classical theory (where only the electric and magnetic field strengths are observable, and the vector potential plays only an auxiliary role for their determination), in the quantum theory Wilson loops are observable themselves. This can be proved e.g. in the Aharonov-Bohm experiment [60].

[10]From now on, for quantities including P-ordering along open paths, we adopt the convention that the matrices are ordered from the second argument to the first one.

[11]Here, the summation is performed over $\mu < \nu$ (*cf.* also Eq. (28) below).

$$\langle W(C)\rangle = \frac{1}{N_c}\left\langle \mathrm{tr}\, P\exp\left(ig\int\limits_{\Sigma} d\sigma_{\mu\nu}(x(\xi))F_{\mu\nu}(x(\xi),x_0)\right)\right\rangle . \tag{18}$$

On the R.H.S. of Eq. (18), the integration is performed over an arbitrary surface Σ bounded by the contour C and parametrized by the vector-function $x_\mu(\xi)$. Next, $\xi\equiv(\xi^1,\xi^2)$ is a two-dimensional coordinate, and

$$d\sigma_{\mu\nu}(x(\xi)) = \sqrt{g(\xi)}t_{\mu\nu}(\xi)d^2\xi \tag{19}$$

stands for the infinitesimal surface element, where $g(\xi)$ is the determinant of the so-called induced metric tensor of the surface, defined as

$$g_{ab}(\xi) = (\partial_a x_\mu(\xi))(\partial_b x_\mu(\xi)), \tag{20}$$

where $\partial_a \equiv \frac{\partial}{\partial\xi^a}$, $a,b=1,2$. Next,

$$t_{\mu\nu}(\xi) = \frac{1}{\sqrt{g(\xi)}}\varepsilon^{ab}(\partial_a x_\mu(\xi))(\partial_b x_\nu(\xi)) \tag{21}$$

is the extrinsic curvature tensor of the surface, $t^2_{\mu\nu}(\xi)=2$, $F_{\mu\nu}(x,x_0)\equiv\Phi(x_0,x)F_{\mu\nu}(x)\Phi(x,x_0)$ is the covariantly shifted non-Abelian field strength tensor,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right],$$

and x_0 is an arbitrary but fixed reference point, the dependence of which actually drops out.

The second step in expressing the Wilson loop via cumulants is to rewrite Eq. (18) via the *cumulant expansion theorem* [68]. The aim of this theorem is to express cumulants, which are usually denoted as

$$\langle\langle F_{\mu_1\nu_1}(x_1,x_0)...F_{\mu_n\nu_n}(x_n,x_0)\rangle\rangle\,,$$

through the usual averages

$$\langle F_{\mu_1\nu_1}(x_1,x_0)...F_{\mu_n\nu_n}(x_n,x_0)\rangle\,.$$

To this end, let us first consider for simplicity the Abelian case, where

$$\langle F_{\mu_1\nu_1}(x_1,x_0)...F_{\mu_n\nu_n}(x_n,x_0)\rangle = \langle F_{\mu_1\nu_1}(x_1)...F_{\mu_n\nu_n}(x_n)\rangle \equiv \int DA_\mu \eta\left(A_\mu\right) F_{\mu_1\nu_1}(x_1)...F_{\mu_n\nu_n}(x_n), \tag{22}$$

with $\eta\left(A_\mu\right)$ standing for a certain $O(4)$-invariant integration measure, and introduce the following generating functional of the field correlators

$$\mathcal{Z}\left[J_{\mu\nu}\right] = \left\langle \exp\left(ig\int d^4x\, J_{\mu\nu}(x)F_{\mu\nu}(x)\right)\right\rangle . \tag{23}$$

The field correlators then obviously have the form

$$\langle F_{\mu_1\nu_1}(x_1)...F_{\mu_n\nu_n}(x_n)\rangle = (ig)^{-n}\left.\frac{\delta^n \mathcal{Z}\left[J_{\mu\nu}\right]}{\delta J_{\mu_1\nu_1}(x_1)...\delta J_{\mu_n\nu_n}(x_n)}\right|_{J=0} . \tag{24}$$

By virtue of the Taylor expansion,

$$\mathcal{Z}\left[J_{\mu\nu}\right] = 1 + \sum_{n=1}^{+\infty} \frac{1}{n!} \int d^4x_1 ... \int d^4x_n J_{\mu_1\nu_1}(x_1) ... J_{\mu_n\nu_n}(x_n) \left(\left. \frac{\delta^n \mathcal{Z}\left[J_{\mu\nu}\right]}{\delta J_{\mu_1\nu_1}(x_1) ... \delta J_{\mu_n\nu_n}(x_n)} \right|_{J=0} \right), \quad (25)$$

where we have taken into account that $\mathcal{Z}[0] = 1$, we obtain for the generating functional

$$\mathcal{Z}\left[J_{\mu\nu}\right] = 1 + \sum_{n=1}^{+\infty} \frac{(ig)^n}{n!} \int d^4x_1 ... \int d^4x_n J_{\mu_1\nu_1}(x_1) ... J_{\mu_n\nu_n}(x_n) \left\langle F_{\mu_1\nu_1}(x_1) ... F_{\mu_n\nu_n}(x_n) \right\rangle . \quad (26)$$

By definition, an n-th order cumulant is defined as follows

$$\left\langle\left\langle F_{\mu_1\nu_1}(x_1) ... F_{\mu_n\nu_n}(x_n) \right\rangle\right\rangle = (ig)^{-n} \left. \frac{\delta^n \ln \mathcal{Z}\left[J_{\mu\nu}\right]}{\delta J_{\mu_1\nu_1}(x_1) ... \delta J_{\mu_n\nu_n}(x_n)} \right|_{J=0} . \quad (27)$$

In particular, if we specify a current to the form

$$J_{\mu\nu}(x) = \int\limits_{\Sigma} d\sigma_{\mu\nu}(x(\xi)) \delta(x - x(\xi)),$$

where Σ is a certain surface bounded by the contour C, then, by virtue of the Stokes theorem, the partition function (23) (where the summation is performed over $\mu < \nu$) is nothing else, but the Wilson loop. The definition (27) means that the cumulants of the field strength tensor are the analogues of usual irreducible (connected) Green functions, which is quite in line with the interpretation of $-\ln \langle W(C) \rangle$ as a correction to the free energy due to the interaction of the gauge field with an external charged particle (*cf.* the previous Subsection). Next, following the same steps which led from Eq. (24) to Eq. (26), we obtain from Eq. (27)

$$\langle W(C) \rangle = \exp\left(\sum_{k=1}^{+\infty} \frac{(ig)^k}{k!} \int\limits_{\Sigma} d\sigma_{\mu_1\nu_1}(x_1) ... \int\limits_{\Sigma} d\sigma_{\mu_k\nu_k}(x_k) \left\langle\left\langle F_{\mu_1\nu_1}(x_1) ... F_{\mu_k\nu_k}(x_k) \right\rangle\right\rangle \right). \quad (28)$$

Eq. (28) is usually referred to as a cumulant expansion.

Notice, that in Eqs. (26) and (28), it has been assumed that the series on their R.H.S. converge, which is true for non-pathological models, to which belongs e.g. QCD, even at large distances, i.e. at large g (see discussion in Ref. [19]). This fact means that the cumulant expansion in QCD is indeed a nonperturbative expansion. The important property of the cumulants in such models, which distinguishes them from the usual correlators (thus associated with the usual Green functions), is that any cumulant decreases vs. the distance between any two points, in which the fields in the cumulant are defined. An example of the model where the cumulant expansion diverges is the instanton gas [69].

One can now write down the following generating equation for cumulants

$$1 + \sum_{n=1}^{+\infty} \frac{(ig)^n}{n!} \int d^4x_1 ... \int d^4x_n J_{\mu_1\nu_1}(x_1) ... J_{\mu_n\nu_n}(x_n) \left\langle F_{\mu_1\nu_1}(x_1) ... F_{\mu_n\nu_n}(x_n) \right\rangle =$$

$$= \exp\left(\sum_{k=1}^{+\infty} \frac{(ig)^k}{k!} \int d^4x_1 ... \int d^4x_k J_{\mu_1\nu_1}(x_1) ... J_{\mu_k\nu_k}(x_k) \left\langle\left\langle F_{\mu_1\nu_1}(x_1) ... F_{\mu_k\nu_k}(x_k) \right\rangle\right\rangle \right), \quad (29)$$

which means that varying Eq. (29) several times w.r.t. $J_{\mu\nu}$ and setting then $J_{\mu\nu} = 0$, one can get relations between the cumulants and correlators of various orders.

In this way, the one-fold variation and setting $J_{\mu\nu} = 0$ yield $\langle\langle F_{\mu\nu}(x)\rangle\rangle = \langle F_{\mu\nu}(x)\rangle$. The two- and three-fold variations and setting $J_{\mu\nu} = 0$ yield the following relations

$$\left<F_{\mu_1\nu_1}(x_1)F_{\mu_2\nu_2}(x_2)\right> = \left<\left<F_{\mu_1\nu_1}(x_1)F_{\mu_2\nu_2}(x_2)\right>\right> + \left<F_{\mu_1}(x_1)\right>\left<F_{\mu_2\nu_2}(x_2)\right> \tag{30}$$

and

$$\left<F_{\mu_1\nu_1}(x_1)F_{\mu_2\nu_2}(x_2)F_{\mu_3\nu_3}(x_3)\right> = \left<\left<F_{\mu_1\nu_1}(x_1)F_{\mu_2\nu_2}(x_2)F_{\mu_3\nu_3}(x_3)\right>\right> +$$

$$+\left<\left<F_{\mu_1\nu_1}(x_1)F_{\mu_2\nu_2}(x_2)\right>\right>\left<F_{\mu_3\nu_3}(x_3)\right> + \left<\left<F_{\mu_1\nu_1}(x_1)F_{\mu_3\nu_3}(x_3)\right>\right>\left<F_{\mu_2\nu_2}(x_2)\right> +$$

$$+\left<F_{\mu_1\nu_1}(x_1)\right>\left<\left<F_{\mu_2\nu_2}(x_2)F_{\mu_3\nu_3}(x_3)\right>\right> + \left<F_{\mu_1\nu_1}(x_1)\right>\left<F_{\mu_2\nu_2}(x_2)\right>\left<F_{\mu_3\nu_3}(x_3)\right>, \tag{31}$$

respectively. Eq. (31) can be symbolically written as follows

$$\left<F_{\mu_1\nu_1}(x_1)F_{\mu_2\nu_2}(x_2)F_{\mu_3\nu_3}(x_3)\right> = \left<\left<F_{\mu_1\nu_1}(x_1)F_{\mu_2\nu_2}(x_2)F_{\mu_3\nu_3}(x_3)\right>\right> +$$

$$+(3)\left<\left<F_{\mu_1\nu_1}(x_1)F_{\mu_2\nu_2}(x_2)\right>\right>\left<F_{\mu_3\nu_3}(x_3)\right> + \left<F_{\mu_1\nu_1}(x_1)\right>\left<F_{\mu_2\nu_2}(x_2)\right>\left<F_{\mu_3\nu_3}(x_3)\right>, \tag{32}$$

where the coefficient in curly brackets denotes the number of the terms of the same type, which differ from each other only by the order of the arguments (and indices, respectively). Making use of this notation, we can next write e.g. the following equation

$$\left<F_{\mu_1\nu_1}(x_1)F_{\mu_2\nu_2}(x_2)F_{\mu_3\nu_3}(x_3)F_{\mu_4\nu_4}(x_4)\right> = \left<\left<F_{\mu_1\nu_1}(x_1)F_{\mu_2\nu_2}(x_2)F_{\mu_3\nu_3}(x_3)F_{\mu_4\nu_4}(x_4)\right>\right> +$$

$$+(3)\left<\left<F_{\mu_1\nu_1}(x_1)F_{\mu_2\nu_2}(x_2)\right>\right>\left<\left<F_{\mu_3\nu_3}(x_3)F_{\mu_4\nu_4}(x_4)\right>\right> +$$

$$+(4)\left<\left<F_{\mu_1\nu_1}(x_1)F_{\mu_2\nu_2}(x_2)F_{\mu_3\nu_3}(x_3)\right>\right>\left<F_{\mu_4\nu_4}(x_4)\right> +$$

$$+(6)\left<\left<F_{\mu_1\nu_1}(x_1)F_{\mu_2\nu_2}(x_2)\right>\right>\left<F_{\mu_3\nu_3}(x_3)\right>\left<F_{\mu_4\nu_4}(x_4)\right> + \left<F_{\mu_1\nu_1}(x_1)\right>\left<F_{\mu_2\nu_2}(x_2)\right>\left<F_{\mu_3\nu_3}(x_3)\right>\left<F_{\mu_4\nu_4}(x_4)\right>. \tag{33}$$

Now one can see that on the R.H.S. of Eqs. (32) and (33) stand with the coefficient equal to unity all possible terms, which correspond to various splittings of the set $\{F_{\mu_1\nu_1}(x_1), ..., F_{\mu_n\nu_n}(x_n)\}$ into subsets, so that to every subset corresponds a cumulant. This property holds also for the higher correlators, which gives a simple mnemonic rule for calculation of the higher cumulants without using Eq. (29). This rule is the essence of the cumulant expansion theorem.

Notice, that due to the $O(4)$-invariance of the Euclidean integration measure $\eta(A_\mu)$, the average $\left<F_{\mu\nu}(x)\right>$ vanishes.

For the non-Abelian case, the cumulant expansion theorem remains the same except for excluding of the terms which violate the path ordering prescription [67]. E.g. in the non-Abelian version of Eq. (31), the term

$$\langle\langle F_{\mu_1\nu_1}(x_1,x_0)F_{\mu_3\nu_3}(x_3,x_0)\rangle\rangle\,\langle F_{\mu_2\nu_2}(x_2,x_0)\rangle$$

will be absent for a given order of points x_1, x_2, x_3.

It looks natural to address the question on whether cumulants of various orders are independent of each other or there exist any relations between them. Such equations relating cumulants of different orders indeed take place. They have been derived by making use of the stochastic quantization method [70, 50] in Ref. [47] and further investigated in Refs. [71], [72], and [73]. Alternative equations for cumulants following from the non-Abelian Bianchi identities, have been proposed in Refs. [67] and [47] and further studied in Ref. [74]. Notice that quite recently, there has been proposed one more approach for a derivation of a system of self-coupled equations for cumulants. This has been done in Ref. [48] by considering a field of a certain colour in the gluodynamics Lagrangian as the one in the background of all the other fields and making use of the background field methods. Finally, besides equations for cumulants, there have recently been proposed new equations for Wilson loops [75] derived by virtue of the stochastic quantization method. Several techniques of the loop space approach, exploited in a derivation of these equations, have been then applied to the solution of the Cauchy problem for the loop equation in 3D turbulence [76].

By definition adopted in the theory of random processes, a set of random quantities is called Gaussian, provided that all the cumulants of these quantities higher than the quadratic one vanish. According to the lattice data [64, 65], a stochastic ensemble of fields in QCD can with a good accuracy be considered as a Gaussian one, i.e. the bilocal cumulant

$$\langle\langle F_{\mu\nu}(x,x_0)F_{\lambda\rho}(y,x_0)\rangle\rangle \qquad (34)$$

is much larger than all the higher cumulants, so that the cumulant expansion converges fastly. Due to this property of the QCD vacuum, one can disregard all the cumulants except the bilocal one, which leads to the *bilocal* or *Gaussian approximation* in the SVM of QCD. However, one can immediately see, that the neglection of all the cumulants higher than the bilocal one in the non-Abelian analogue of Eq. (28) leads to the appearance of the artificial dependences of the R.H.S. of this equation on the reference point x_0 and on the shape of the surface Σ.

The first problem could be solved by noting that the bilocal cumulant (34) decreases fastly when $|x-y| \simeq T_g$, where T_g is the so-called *correlation length of the vacuum*, $T_g \simeq 0.13\,\mathrm{fm}$ in the $SU(2)$-case [77], and $T_g \simeq 0.22\,\mathrm{fm}$ in the $SU(3)$-case [64]. On the other hand, as it has been already discussed in the previous Subsection, according to Ref. [52], the area law of the Wilson loop takes place when its size R is of the order of $1.0\,\mathrm{fm}$. This means that in the confining regime, i.e. for the loops of such size, one can in the general case (i.e. for the dominant number of cumulants) write down the following inequality

$$|x-x_0| \simeq |y-x_0| \simeq R \gg T_g \simeq |x-y|\,.$$

According to it, we can with a good accuracy neglect the dependence on the point x_0 in the bilocal cumulant (34), i.e. approximate the latter by the gauge- and translation-invariant cumulant as follows

$$\mathrm{tr}\,\langle\langle F_{\mu\nu}(x,x_0)F_{\lambda\rho}(y,x_0)\rangle\rangle \simeq \mathrm{tr}\,\langle\langle F_{\mu\nu}(x)\Phi(x,y)F_{\lambda\rho}(y)\Phi(y,x)\rangle\rangle\,.$$

As far as the problem of the artificial dependence on the shape of the surface Σ in the bilocal approximation is concerned, it cannot be solved on the basis of simple theoretical arguments, but one can choose Σ to be the surface of the minimal area for a given contour C [12]. As we shall see in Subsection 2.1, this enables one to reproduce the area law behaviour of the Wilson loop. Since such a surface $\Sigma_{\rm min.}$ is uniquely defined by the contour C, we finally arrive at the following expression for the Wilson loop in the bilocal approximation

$$\langle W(C)\rangle \equiv \langle W\left(\Sigma_{\rm min.}\right)\rangle \simeq$$

$$\simeq \frac{1}{N_c}{\rm tr}\ \exp\left(-\frac{g^2}{2}\int\limits_{\Sigma_{\rm min.}} d\sigma_{\mu\nu}(x)\int\limits_{\Sigma_{\rm min.}} d\sigma_{\lambda\rho}(x')\left\langle\left\langle F_{\mu\nu}(x)\Phi(x,x')F_{\lambda\rho}(x')\Phi(x',x)\right\rangle\right\rangle\right), \qquad (35)$$

where we have denoted for shortness $x\equiv x(\xi)$ and $x'\equiv x(\xi')$. Due to the colour invariance of the Euclidean integration measure in QCD, the bilocal cumulant on the R.H.S. of Eq. (35) is proportional to the unity matrix in the fundamental representation, $\hat 1_{N_c\times N_c}$, and therefore the P-ordering is not necessary any more.

The bilocal cumulant standing in the exponent on the R.H.S. of Eq. (35) can be now parametrized by two renormalization group-invariant coefficient functions D and D_1 as follows

$$\frac{g^2}{2}\left\langle\left\langle F_{\mu\nu}(x)\Phi(x,x')F_{\lambda\rho}(x')\Phi(x',x)\right\rangle\right\rangle = \hat 1_{N_c\times N_c}\Biggl\{(\delta_{\mu\lambda}\delta_{\nu\rho}-\delta_{\mu\rho}\delta_{\nu\lambda})D\left((x-x')^2\right)+$$

$$+\frac12\left[\frac{\partial}{\partial x_\mu}((x-x')_\lambda\delta_{\nu\rho}-(x-x')_\rho\delta_{\nu\lambda})+\frac{\partial}{\partial x_\nu}((x-x')_\rho\delta_{\mu\lambda}-(x-x')_\lambda\delta_{\mu\rho})\right]D_1\left((x-x')^2\right)\Biggr\}. \qquad (36)$$

The parametrization (36) of the bilocal cumulant is chosen in such a way, that the term containing the function D_1 yields a perimeter type contribution to the Wilson loop (35). Namely, by making use of the (ordinary) Stokes theorem, one can prove that [13]

$$\frac12\int\limits_{\Sigma_{\rm min.}} d\sigma_{\mu\nu}(x)\int\limits_{\Sigma_{\rm min.}} d\sigma_{\lambda\rho}(x')\left[\frac{\partial}{\partial x_\mu}((x-x')_\lambda\delta_{\nu\rho}-(x-x')_\rho\delta_{\nu\lambda})+\right.$$

$$\left.+\frac{\partial}{\partial x_\nu}((x-x')_\rho\delta_{\mu\lambda}-(x-x')_\lambda\delta_{\mu\rho})\right]D_1\left((x-x')^2\right)=\oint\limits_C dx_\mu\oint\limits_C dy_\mu G\left((x-y)^2\right), \qquad (37)$$

where

$$G\left(x^2\right)\equiv\int\limits_{x^2}^{+\infty} d\lambda D_1(\lambda). \qquad (38)$$

In particular, we see that the one-gluon-exchange diagram contribution to the Wilson loop, equal to

[12] In a complete string picture of QCD including quantum fluctuations above confining background, there should appear an integration over all world-sheets bounded by the contour C. It is natural to expect that the dominant contribution to this integral is brought about by the surface of the minimal area.

[13] From now on, for simplicity, we restore in Eq. (35) the usual agreement on the summation over all μ and ν (λ and ρ), not only $\mu<\nu$. This can always be done by the appropriate normalization of the functions D and D_1.

$$\exp\left(-C_2\frac{g^2}{4\pi^2}\oint\limits_C dx_\mu \oint\limits_C dy_\mu \frac{1}{(x-y)^2}\right)$$

(which for the contour C without cusps yields the renormalization factor standing on the R.H.S. of Eq. (14)) is contained due to Eq. (37) in the function D_1. According to Eq. (38), it is equal to

$$D_1^{\rm OGE}\left(x^2\right) = C_2\frac{g^2}{4\pi^2}\frac{1}{|x|^4}. \tag{39}$$

Eq. (39) is the leading contribution to the function D_1. Therefore, the effects of renormalization of the Wilson loop, discussed at the end of the previous Subsection, are taken into account in SVM by virtue of the function D_1.

Thus, the function D_1 is nonvanishing already in the order g^2, whereas the function D in this order of perturbation theory vanishes. The nonvanishing contributions to the function D emerging in the higher orders have been calculated in Ref. [78]. ¿From now on, we shall not be interested any more in the perturbative contributions to these functions. The nonperturbative parts of the functions D and D_1 have been calculated in the lattice experiments in Ref. [64], where it has been shown that they are related to each other as $D_1 \simeq \frac{1}{3}D$.

Due to the Lorentz structure standing at the function D in Eq. (36), its contribution to the Wilson loop (35) cannot be reduced to that of a perimeter type. This function is responsible for the area law behaviour of the Wilson loop and gives rise to the QCD string effective action. The problems of derivation of this action from Eq. (35) and its further investigation will be the topic of the next Section.

2 String Representation of QCD in the Framework of the Stochastic Vacuum Model

In this Section, we shall demonstrate the usefulness of SVM for the derivation and investigation of the local gluodynamics string effective action. Our interpretation of this topic in the next three Subsections will mainly follow the original papers [79], [80] in Subsection 2.1, [81] in Subsection 2.2, and [82] in Subsection 2.3. A short review of all these papers can be found in Ref. [83].

2.1 Gluodynamics String Effective Action from the Wilson Loop Expansion

Let us start with the problem of a derivation of the effective action of the gluodynamics string, generated by the strong vacuum background fields in QCD, which ensure confinement and yield a dominant contribution to the bilocal cumulant. To this end, we mention that, as it has already been discussed in the previous Section, the quantity $-\ln\langle W(C)\rangle$ is nothing else, but a correction to the gluodynamics free energy due to the interaction of a test quark, moving along the contour C, with these background fields. This observation enables us to treat this quantity with $\langle W(C)\rangle$ defined by Eq. (35) as a *nonlocal* background-induced gluodynamics string effective action associated with the minimal area surface $\Sigma_{\rm min.}$. In what follows, by the terms "local" and "nonlocal string effective action", we shall imply the actions depending on a single string world-sheet coordinate ξ or on two such coordinates, respectively.

Substituting Eq. (36) into Eq. (35), one can see that the coefficient function D plays the role of a propagator of the background field between the points $x(\xi)$ and $x(\xi')$, which lie on $\Sigma_{\rm min.}$. Though this propagator cannot be completely found analytically, its nonperturbative part can be parametrized with a good accuracy as

$$D(x) = \alpha_s \left\langle \left(F^a_{\mu\nu}(0)\right)^2 \right\rangle {\rm e}^{-|x|/T_g}, \tag{40}$$

where the gluonic condensate $\alpha_s \left\langle \left(F^a_{\mu\nu}(0)\right)^2 \right\rangle$ is of the order of $0.038\,{\rm GeV}^2$ [15] [14]. This will finally enable us to obtain the coupling constants of the few first terms of the resulting *local* string effective action expressed via the correlation length of the vacuum, T_g, and the gluonic condensate.

In order to derive the desired local effective action, let us proceed with Taylor expanding of the nonlocal action

$$S_{\rm eff.}\left(\Sigma_{\rm min.}\right) = 2 \int\limits_{\Sigma_{\rm min.}} d\sigma_{\mu\nu}(x) \int\limits_{\Sigma_{\rm min.}} d\sigma_{\mu\nu}(x') D\left(\frac{(x-x')^2}{T_g^2}\right) \tag{41}$$

in powers of the derivatives w.r.t. the string world-sheet coordinates ξ^a's. Obviously, the nonlocality of the initial action will then display itself in the appearance of higher derivatives w.r.t. ξ^a's in the final expression for the action. Notice also that in Eq. (41), we have explicitly emphasized the form of the dependence on T_g, since, as we shall eventually see, the parameter of this expansion will be proportional to T_g.

To perform an expansion, let us first rewrite infinitesimal surface elements on the R.H.S. of Eq. (41) by making use of Eqs. (19), (20), and (21). Next, we introduce instead of ξ'^a's new integration variables $\zeta^a \equiv \frac{(\xi'-\xi)^a}{T_g}$ and expand in power series of ζ^a's the quantities $\sqrt{g(\xi')}$, $t_{\mu\nu}(\xi')$, $x(\xi') - x(\xi)$, and finally $D\left(\frac{(x-x')^2}{T_g^2}\right)$. Such an expansion will automatically be a formal series in powers of T_g, e.g.

$$\left(x(\xi') - x(\xi)\right)^2 = T_g^2 \left[\zeta^a\zeta^b g_{ab}(\xi) + T_g \zeta^a\zeta^b\zeta^c (\partial_a x_\mu)(\partial_b\partial_c x_\mu) + {\cal O}\left(T_g^2\right)\right].$$

In what follows, we shall be interested only in the terms in this series not higher than of the fourth order in T_g. Then, taking into account that for an arbitrary odd n, $\int d^2\zeta \zeta^{i_1}...\zeta^{i_n} D(y) = 0$, where $y \equiv \zeta^a\zeta^b g_{ab}(\xi)$, we obtain

$$S_{\rm eff.} = 2T_g^2 \int d^2\xi \sqrt{g(\xi)} \Bigg\{ 2\sqrt{g(\xi)} \int d^2\zeta D(y) + T_g^2 \Bigg\{ \sqrt{g(\xi)} \Bigg[-\frac12 (\partial_a t_{\mu\nu}(\xi))(\partial_b t_{\mu\nu}(\xi)) \int d^2\zeta \zeta^a\zeta^b D(y) +$$

$$+ \left(\frac12 (\partial_a\partial_b x_\mu)(\partial_c\partial_d x_\mu) + \frac23 (\partial_a x_\mu)(\partial_b\partial_c\partial_d x_\mu)\right) \int d^2\zeta \zeta^a\zeta^b\zeta^c\zeta^d D'(y) +$$

$$+ (\partial_a x_\mu)(\partial_b\partial_c x_\mu)(\partial_d x_\nu)(\partial_e\partial_f x_\nu) \int d^2\zeta \zeta^a\zeta^b\zeta^c\zeta^d\zeta^e\zeta^f D''(y) \Bigg] +$$

[14] The colour index "a" should not be confused with the index of the two-vector ξ.

$$+2\left(\partial_a\sqrt{g(\xi)}\right)(\partial_b x_\mu)(\partial_c\partial_d x_\mu)\int d^2\zeta\zeta^a\zeta^b\zeta^c\zeta^d D'(y)+\left(\partial_a\partial_b\sqrt{g(\xi)}\right)\int d^2\zeta\zeta^a\zeta^b D(y)\Bigg\}\Bigg\},$$

where "$'$" stands for the derivative w.r.t. the argument, and we have arranged the terms in powers of the derivatives of $\sqrt{g(\xi)}$. In order to simplify the integrals over ζ^a's as much as possible, it is very useful to fix by reparametrization the conformal gauge for the induced metric, $g_{ab}(\xi)=\sqrt{g(\xi)}\delta_{ab}$ (see e.g. Ref. [20]). After that, by making use of the equality $\zeta^a\frac{\partial}{\partial y}=\frac{1}{2\sqrt{g(\xi)}}\frac{\partial}{\partial\zeta_a}$, it becomes possible to perform partial integrations over ζ^a's. It is also useful to replace the ordinary derivatives by the covariant ones by making use of the Gauss-Weingarten formulae

$$D_aD_bx_\mu=\partial_a\partial_bx_\mu-\Gamma^c_{ab}\partial_cx_\mu=K^i_{ab}n^i_\mu,$$

where n^i_μ's stand for the unit normal vectors to the string world-sheet, $n^i_\mu n^j_\mu=\delta^{ij}$, $n^i_\mu\partial_ax_\mu=0$, $i,j=1,2$, Γ^c_{ab} is a Christoffel symbol, and K^i_{ab} is the second fundamental form of the world-sheet.

Notice, that in the conformal gauge adopted, all the quantities are greatly simplified, e.g. $\Gamma^c_{ab}=\frac12\left(\delta^c_a\partial_b+\delta^c_b\partial_a-\delta_{ab}\partial^c\right)\ln\sqrt{g(\xi)}$. In particular, the expression for the scalar curvature $\mathcal{R}=\left(K^{i\,a}_{a}\right)^2-K^{i\,a}_{b}K^{i\,b}_{a}$ of the world-sheet in this gauge takes the form

$$\mathcal{R}=\frac{\partial^a\partial_a\ln\sqrt{g(\xi)}}{\sqrt{g(\xi)}},\tag{42}$$

and one can prove the validity of the following equations

$$T^{abcd}(\partial_a\partial_bx_\mu)(\partial_c\partial_dx_\mu)=T^{abcd}(D_aD_bx_\mu)(D_cD_dx_\mu)+2\frac{\left(\partial_a\ln\sqrt{g(\xi)}\right)^2}{\sqrt{g(\xi)}},$$

$$T^{abcd}(\partial_ax_\mu)(\partial_b\partial_c\partial_dx_\mu)=T^{abcd}(D_ax_\mu)(D_bD_cD_dx_\mu)+2\mathcal{R},$$

$$T^{abcd}(D_aD_bx_\mu)(D_cD_dx_\mu)=3(D^aD_ax_\mu)(D^bD_bx_\mu)-2\mathcal{R},$$

$$g^{ab}\frac{\partial_a\partial_b\sqrt{g(\xi)}}{\sqrt{g(\xi)}}=\mathcal{R}+\frac{\left(\partial_a\ln\sqrt{g(\xi)}\right)^2}{\sqrt{g(\xi)}},$$

and

$$T^{abcd}\left(\partial_a\sqrt{g(\xi)}\right)(\partial_bx_\mu)(\partial_c\partial_dx_\mu)=2\left(\partial_a\ln\sqrt{g(\xi)}\right)^2,$$

where $T^{abcd}\equiv g^{ab}g^{cd}+g^{ac}g^{bd}+g^{ad}g^{bc}$. Making use of all that, introducing a new integration variable $z^a\equiv g^{1/4}\zeta^a$, and recovering the metric dependence by virtue of the equality

$$(D^aD_ax_\mu)\left(D^bD_bx_\mu\right)=g^{ab}\left(\partial_at_{\mu\nu}\right)\left(\partial_bt_{\mu\nu}\right),$$

we arrive after some straightforward algebra at the following expression for the string effective action up to the order T_g^4

$$S_{\text{eff.}} = \sigma \int d^2\xi\sqrt{g} + \kappa \int d^2\xi\sqrt{g}\mathcal{R} + \frac{1}{\alpha_0}\int d^2\xi\sqrt{g}g^{ab}\left(\partial_a t_{\mu\nu}\right)\left(\partial_b t_{\mu\nu}\right) + \mathcal{O}\left(\frac{T_g^6\alpha_s\left\langle\left(F_{\mu\nu}^a(0)\right)^2\right\rangle}{R^2}\right), \quad (43)$$

where R is the size of the contour C in the confining regime (*cf.* Eq. (8)).

In Eq. (43), the coupling constants are completely determined via the zeroth and the first moments of the coefficient function D (as well as the omitted higher terms of the expansion are determined via the higher moments of the function D) as follows:

$$\sigma = 4T_g^2\int d^2zD\left(z^2\right), \quad (44)$$

$$\kappa = \frac{T_g^4}{6}\int d^2zz^2D\left(z^2\right),$$

and

$$\frac{1}{\alpha_0} = -\frac{T_g^4}{4}\int d^2zz^2D\left(z^2\right). \quad (45)$$

Making use of Eq. (40), one can estimate them as

$$\sigma = 4\pi T_g^2\alpha_s\left\langle\left(F_{\mu\nu}^a(0)\right)^2\right\rangle,\ \kappa = \frac{\pi}{6}T_g^4\alpha_s\left\langle\left(F_{\mu\nu}^a(0)\right)^2\right\rangle,\ \frac{1}{\alpha_0} = -\frac{\pi}{4}T_g^4\alpha_s\left\langle\left(F_{\mu\nu}^a(0)\right)^2\right\rangle.$$

Due to the lattice data quoted after Eq. (34), we thus get for the $SU(2)$-case the following values of the coupling constants $\sigma \simeq 0.2\,\text{GeV}^2$ (*cf.* the corresponding value before Eq. (9)), $\kappa \simeq 0.003$, and $\frac{1}{\alpha_0} = -0.005$. The obtained value of the Nambu-Goto term string tension demonstrates the agreement of SVM with the present lattice results, while the two other obtained coupling constants are simply small numbers, which confirms the validity of the performed expansion.

Before discussing various terms in Eq. (43), let us comment on the parameter of the performed expansion of the initial Eq. (41). First of all, as it has already been mentioned, this is an expansion in formal power series of T_g, which means that it is valid only when T_g is sufficiently small. That was a reason for the author of Ref. [49] to call the limit when T_g is small but σ is kept fixed as a "string limit of QCD". An n-th term of the expansion has the order of magnitude $\alpha_s\left\langle\left(F_{\mu\nu}^a(0)\right)^2\right\rangle R^4\left(\frac{T_g}{R}\right)^{2n}$, which means that the parameter of the expansion is $\left(T_g/R\right)^2$. In the confining regime, this parameter is of the order of 0.04 (see the lattice data after Eq. (34)), i.e. is indeed a small number. Therefore, in the "string limit", the operators of the lowest orders in the derivatives w.r.t. ξ^a's dominate in the expansion of the full nonlocal action (41) [15]. Contrary, in the QCD sum rule limit [15], the effects brought about by the nonlocality of the functions D and D_1 are disregarded, and these functions are simply replaced by the gluonic condensate. In the language of the parametrization (40), this means, that $T_g \to \infty$, and our expansion diverges. This observation clarifies once more the relevance of the string picture of QCD to the description

[15] Examples of the geometric structures, emerging in the expansion in the order T_g^6, are listed in the Appendix to the first paper of Ref. [79].

of confinement. We also see, that it is SVM (where the correlation length of the vacuum can be considered as a variable parameter), which provides us with such a picture.

Let us now proceed with the physical discussion of the obtained local string effective action (43). The first term on the R.H.S. of this equation is the celebrated Nambu-Goto term with the positive string tension σ, which ensures the area law behaviour of the Wilson loop (7), since $\int d^2\xi\sqrt{g(\xi)}$ is nothing else, but the area of the minimal surface, bounded by C. The obtained expression for the string tension coincides with the one obtained in Ref. [16]. Notice also, that though the Nambu-Goto term alone is known to suffer from the problem of appearing of a tachyon in its spectrum, this problem is absent for the full nonlocal string effective action (41).

The second term is known to be a full derivative in 2D, which can be most easily seen in the conformal gauge adopted, i.e. from Eq. (42). This term is a topological invariant proportional to the Euler character of the world-sheet

$$\chi = \frac{1}{4\pi}\int d^2\xi\sqrt{g(\xi)}\mathcal{R} = 2 - 2\times \text{number of handles} - \text{number of boundaries}.$$

In particular, for the surface $\Sigma_{\text{min.}}$ under study, $\chi = 1$.

The most interesting term in the obtained effective action (43) is the third one, which is usually referred to as a *rigidity term* [84], [85]. This term is not a full derivative, since it can be rewritten (just modulo full derivative terms) as

$$\frac{1}{\alpha_0}\int d^2\xi\sqrt{g(\xi)}K_b^{i\,a}K_a^{i\,b},$$

i.e. the integrand does not contain the complete expression for $\sqrt{g}\mathcal{R}$. The rigidity term has been introduced into string theory in the above mentioned papers from the general arguments, as the only possible one, which is invariant under the scale transformation $x_\mu(\xi) \to \lambda x_\mu(\xi)$.

As it has been for the first time mentioned in Ref. [85] and then confirmed by related calculations in Refs. [86] and [87], the negative sign of the coupling constant α_0 of the rigidity term is an important property, relevant to the stability of the string world-sheet. As it has been explained in these papers, this property (called there "anti-rigidity" or "negative stiffness") ensures vanishing of the imaginary part of the frequencies of small fluctuations of the world-sheet, which might lead to instabilities. This property has been further exploited in Ref. [88], where a new nonlocal string action, which manifests negative stiffness, has been proposed as a good candidate for modelling the gluodynamics string. However, contrary to our calculations, it remained unclear in that paper how the action introduced there can be derived either from the Wilson loop expansion or from some quantity relevant to gluodynamics (e.g. another vacuum amplitude or the gluodynamics Lagrangian).

Once being integrated out, the small transversal fluctuations of the world-sheet, mentioned above, produce a renormalization of the string tension, which at the two-loop level has been calculated in Ref. [89] and reads

$$\sigma_{\text{ren.}} = \sigma\left[1 + \alpha_0\frac{d-2}{16\pi}\left(1 + \ln\frac{4\Lambda^2}{\alpha_0^2\sigma}\right) + \alpha_0^2\frac{(d-2)(d-1)}{256\pi^2}\left(\ln\frac{4\Lambda^2}{\alpha_0^2\sigma}\right)^2\right],$$

where Λ is an UV momentum cutoff, and d is the dimension of the space-time. It is worth noting, that the appearance of the rigidity term yields also some modifications in the quark-antiquark potential, string tension behaviour at finite temperature, and thermal deconfinement properties, all of which were surveyed in Ref. [90].

However in the IR region, the rigidity term becomes irrelevant, and only the Nambu-Goto term in the string effective action (43) survives. For a certain Lorentz index λ, the propagator of this term is known to be

$$\langle x_\lambda(\xi)x_\lambda(\xi')\rangle_{\rm N.G.}=\int\frac{d^2p}{(2\pi)^2}\frac{{\rm e}^{ip^a(\xi-\xi')_a}}{p^2}=\frac{1}{2\pi}\ln\frac{1}{\mu\left|\xi-\xi'\right|},$$

where μ is an IR momentum cutoff. This means that the correlation length for the elements of the world-sheet (or, equivalently, for the normals to the world-sheet) is equal to $1/\mu$. Therefore, in the IR region, normals are very short-ranged, i.e. the world-sheet is terribly crumpled. That is the reason, why this problem is usually referred to as the *problem of crumpling of the string world-sheet*.

Let us now consider two possibilities of curing this problem. First of them has been put forward in Ref. [91] for the case of the effective string theory emerging from D-dimensional compact QED (the so-called confining string theory [92], which will be considered in details in Section 4). There, it has been demonstrated that in the weak-field limit of this theory, for the case $D\to\infty$, the correlation function of two transversal fluctuations of the string world-sheet has an oscillatory behaviour at large distances. Such a behaviour indicates that the world-sheet is smooth rather than crumpled. One might expect that the same mechanism works in all gauge theories, whose confining phases admit a representation in the form of some effective string theory with a non-local interaction between the world-sheet elements. However, it is not obvious whether this mechanism can be extended to the non-Abelian case of gluodynamics, where in the nonlocal string effective action (41) instead of the propagator of the massive vector field, appearing in the Abelian case, stands the coefficient function $D(x)$. Though, according to the lattice data [64, 65], the large distance asymptotic behaviour of the latter is indeed similar to the one of the massive vector field propagator, there nevertheless remain significant differences.

Let us therefore turn ourselves to the second possibility, proposed in Ref. [84] and elaborated on in Ref. [80], which is more applicable to gluodynamics. It is based on the introduction of the so-called topological term, which is equal to the algebraic number of self-intersections of the string world-sheet, into the string effective action. Then, by adjusting the coupling constant of this term, one can eventually arrange the cancellation of contributions into the string partition function coming from highly crumpled surfaces, whose intersection numbers differ by one from each other.

Thus it looks natural to address the problem of derivation of the topological term from the gluodynamics Lagrangian. Such a term has been recently derived in Ref. [93] for $4D$ compact QED with an additional θ-term. In the dual formulation of the Wilson loop in this theory (which is nothing else but the $4D$ confining string theory mentioned above), the latter one occurred to be crucial for the formation of the topological string term. However, such a mechanism of generation of a topological term is difficult to work out in gluodynamics, due to our inability to construct the exact dual formulation of the Wilson loop in this theory. Therefore, it looks suggestive to search for some model of the gluodynamics vacuum, which might lead to the appearance of the topological term in the string representation of the Wilson loop in this theory.

In Ref. [80], this idea has been realized by making use of recent results concerning the calculation of the field strength correlators in the dilute instanton gas model [94]. In the latter paper, it has been demonstrated that for the case of an instanton gas with broken CP invariance, the bilocal field strength correlator contains a term proportional to the tensor $\varepsilon_{\mu\nu\lambda\rho}$. This term is absent in the case of a CP-symmetric vacuum, since it is proportional to the topological charge of the system, $V(n_4 - \bar{n}_4)$, where V is the four-volume of observation, and within the notations of Ref. [94], n_4 and $\bar{n}_4$ stand for the densities of instantons and antiinstantons (I's and $\bar{I}$'s for shortness), respectively. Similarly, the paper [80] also dealt with the approximation of a dilute $I-\bar{I}$ gas with fixed equal sizes ρ of I's and $\bar{I}$'s. In the remaining part of this Subsection, we shall briefly consider the main points of this paper.

The new structure arising in the bilocal correlator in the $I-\bar{I}$ gas reads [94]

$$\Delta\,\mathrm{tr}\,\langle\langle F_{\mu\nu}(x,x_0)F_{\lambda\rho}(x',x_0)\rangle\rangle = 8\,(n_4-\bar{n}_4)\,I_r\left(\frac{(x-x')^2}{\rho^2}\right)\varepsilon_{\mu\nu\lambda\rho}. \tag{46}$$

In Eq. (46), the asymptotic behaviour of the function $I_r\left(z^2\right)$ at $z \ll 1$ and $z \gg 1$ has the following form

$$I_r\left(z^2\right) \longrightarrow \frac{\pi^2}{6}, \tag{47}$$

and

$$I_r\left(z^2\right) \longrightarrow \frac{2\pi^2}{|z|^4}\ln z^2, \tag{48}$$

respectively.

In what follows, we are going to present the leading term in the derivative expansion of the correction to the nonlocal string effective action (41), which has the form

$$\Delta S_{\mathrm{eff.}} = -\ln\Delta\left\langle W\left(\Sigma_{\mathrm{min.}}\right)\right\rangle. \tag{49}$$

Here, $\Delta\left\langle W\left(\Sigma_{\mathrm{min.}}\right)\right\rangle$ is the corresponding correction to the expression (35) for the Wilson loop, following from Eq. (46) in the CP-broken vacuum.

We shall not be interested in calculating corrections to the Nambu-Goto and rigidity terms arising due to additional contribution from the $I-\bar{I}$ gas to the function $D(x)$. This can be easily done by carrying out the corresponding integrals (44) and (45) of the function $D(x)$ in this gas. Notice only that, as it has already been mentioned in Ref. [80], due to the reasons discussed in details in Refs. [69] and [19], a correction to the string tension (44), obtained in such a way from the $I-\bar{I}$ gas contribution to the function $D(x)$, should be cancelled by the contributions coming from the higher cumulants in this gas. Clearly, this does not necessarily mean that the corresponding correction to the rigid string inverse bare coupling constant (45) vanishes. Indeed, one can imagine himself a function $D(x)$, for which $\int\limits_0^\infty dtD(t)=0$, whereas $\int\limits_0^\infty dttD(t)\neq 0$, where $t=z^2$. For example, this is true for the function $D(t)$, defined as

$$D(t)=\frac{1}{2c^3},\ 0<t<c;\ D(t)=-\frac{1}{t^3},\ t>c,$$

which can be obviously made continuous by smoothering of the jump at $t=c$ with some function, odd w.r.t. the line $t=c$.

One can now expand the correction (49), emerging from the term (46), in powers of the derivatives w.r.t. ξ^a's in the same manner, as it has been done above for the nonlocal string effective action (41). Noting that, since $\tilde{t}_{\mu\nu}t_{\mu\nu}=0$, an analogue of the Nambu-Goto term in this expansion vanishes, we get

$$\Delta S_{\text{eff.}} = \beta\nu + \mathcal{O}\left(\frac{\rho^6 (n_4 - \bar{n}_4)}{R^2}\right),$$

where

$$\nu \equiv \frac{1}{4\pi}\varepsilon_{\mu\nu\lambda\rho}\int d^2\xi\sqrt{g}g^{ab}(\partial_a t_{\mu\nu})(\partial_b t_{\lambda\rho})$$

is the algebraic number of self-intersections of the string world-sheet and

$$\beta = 16\pi\rho^4 (n_4 - \bar{n}_4)\int d^2 z z^2 I_r\left(z^2\right) \tag{50}$$

is the corresponding coupling constant.

Note that the averaged separation between the nearest neighbors in the $I - \bar{I}$ gas is given by $L = (n_4 + \bar{n}_4)^{-\frac{1}{4}}$. According to phenomenological considerations one obtains for the $SU(3)$-case, $\rho/L \simeq 1/3$ [95]; see also Ref. [96], where the ratio ρ/L has been obtained from direct lattice measurements to be $0.37 - 0.40$. L should then serve as a distance cutoff in the integral standing on the R.H.S. of Eq. (50). Taking this into account, we get from Eqs. (47), (48), and (50) the following approximate value of β

$$\beta \simeq (2\pi\rho)^4 (n_4 - \bar{n}_4)\left[\frac{1}{12} + \left(\ln\frac{L^2}{\rho^2}\right)^2\right], \tag{51}$$

where the second term in the square brackets on the R.H.S. of Eq. (51), emerging due to Eq. (48), is much larger than the first one, emerging due to Eq. (47). Making use of the value $\rho^{-1} \simeq 0.6\,\text{GeV}$ [24], one obtains $\beta \simeq 57680.4\text{GeV}^{-4} \cdot (n_4 - \bar{n}_4)$.

In conclusion of this Subsection, we have found that in the $I - \bar{I}$ gas with a nonzero topological charge, there appears a topological term in the string representation of the Wilson loop. The coupling constant of this term is given by Eq. (50). Together with the Nambu-Goto and rigidity terms (see Eq. (43)), this term forms the effective Lagrangian of the gluodynamics string following from the SVM.

2.2 Incorporation of Perturbative Corrections

As it has been discussed in the beginning of the previous Subsection, the origin of the nonlocal string effective action (41), which served as a starting point for the derivation of the local action (43), is essentially nonperturbative. This is because the dominant contribution to the coefficient function $D(x)$ is brought about by the strong background fields, which ensure confinement. However, as it has been argued in Ref. [97], in order to get the exponential growth of the multiplicity of states in the spectrum of the open bosonic string, one must account for the perturbative gluons, which interact with the string world-sheet. In this Subsection, we shall proceed with studying this interaction, by making use of the so-called perturbation theory in the nonperturbative QCD vacuum [26], [27], [72].

To this end, we shall split the total field A_μ^a as $A_\mu^a = B_\mu^a + a_\mu^a$, where $B_\mu^a \sim \frac{1}{g}$ is a strong nonperturbative background, and a_μ^a's are perturbative fluctuations around the latter, $a_\mu^a \sim gB_\mu^a$. Thus, our strategy should be to perform an integration over a_μ^a's in the expression (18) for the Wilson loop, where Σ is replaced by $\Sigma_{\text{min.}}$. However, due to the path-ordering, which remained after rewriting the contour integral as a surface one in the version of the non-Abelian Stokes theorem proposed in Refs. [66] and [67], such an integration is difficult to carry out starting with Eq. (18). That is why, we find it convenient to adopt another version of this theorem, proposed in Ref. [98], where the path-ordering is replaced by the integration over an auxiliary field from the $SU(N_c)/[U(1)]^{N_c-1}$ coset space. In what follows, we shall consider the $SU(2)$-case, where this field is a unit three-vector $\vec{n}$, which characterizes the instant orientation in colour space, and the non-Abelian Stokes theorem takes a remarkably simple form

$$\langle W(C)\rangle = \left\langle \int D\vec{n} \exp\left\{ \frac{iJ}{2} \left[-g \int\limits_{\Sigma_{\rm min.}} d\sigma_{\mu\nu}(x(\xi)) n^a(x(\xi)) \mathcal{F}^a_{\mu\nu}(x(\xi)) + \right.\right.\right.$$

$$\left.\left.\left. + \int\limits_{\Sigma_{\rm min.}} d\sigma_{\mu\nu} \varepsilon^{abc} n^a \left(\mathcal{D}_\mu \vec{n}\right)^b \left(\mathcal{D}_\nu \vec{n}\right)^c \right]\right\}\right\rangle. \qquad (52)$$

Here, $\mathcal{F}^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\varepsilon^{abc} A^b_\mu A^c_\nu$ is a strength tensor of the gauge field, $\mathcal{D}^{ab}_\mu = \partial_\mu \delta^{ab} - g\varepsilon^{abc} A^c_\mu$ is the covariant derivative, and $J = \frac{1}{2}, 1, \frac{3}{2}, \ldots$ is the colour "spin" of representation of the $SU(2)$ group under consideration, defined via its generators T^a's as $T^a T^a = J(J+1)\hat{1}$. The last term in the exponent on the R.H.S. of Eq. (52) is usually referred to as a gauged Wess-Zumino term.

As we shall see below, such a version of the non-Abelian Stokes theorem will indeed enable us to carry out the one-loop integration over perturbative fluctuations, which will then lead to a new (w.r.t. Eq. (41)) type of interaction between the string world-sheet elements. Once being expanded in powers of the derivatives w.r.t. ξ^a's, this interaction will finally yield a correction to the rigidity term, keeping the Nambu-Goto term untouched.

Let us start with the above mentioned splitting of the total gauge field A^a_μ in Eq. (52) and adopting the background field formalism [99], [26], [27], which yields

$$\langle W(C)\rangle = N \int DB^a_\mu \eta\left(B^b_\alpha\right) D\vec{n} \exp\left\{ \int d^4x \left[-\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 + \frac{i}{2} \int\limits_{\Sigma_{\rm min.}} d\sigma_{\mu\nu} \varepsilon^{abc} n^a \left(D_\mu \vec{n}\right)^b \left(D_\nu \vec{n}\right)^c \right]\right\} \cdot$$

$$\cdot \exp\left(-\frac{ig}{2} \int\limits_{\Sigma_{\rm min.}} d\sigma_{\mu\nu} n^a F^a_{\mu\nu} \right) \cdot$$

$$\cdot \int Da^a_\mu \exp\left\{ \int d^4x \left[a^a_\nu D^{ab}_\mu F^b_{\mu\nu} + \frac{1}{2} a^a_\mu \left(D^{ac}_\rho D^{cb}_\rho \delta_{\mu\nu} - 2ig\hat{F}^{ab}_{\mu\nu} \right) a^b_\nu - g\varepsilon^{acd} \left(D^{ab}_\mu a^b_\nu \right) a^c_\mu a^d_\nu + \right.\right.$$

$$\left.\left. + \frac{g^2}{4} \left(a^a_\mu a^a_\nu a^b_\mu a^b_\nu - \left(a^a_\mu\right)^2 \left(a^b_\nu\right)^2 \right) \right] - ig \int\limits_{\Sigma_{\rm min.}} d\sigma_{\mu\nu} \left(n^a D^{ab}_\mu a^b_\nu + a^a_\nu \left(D_\mu \vec{n}\right)^a \right) \right\} \cdot$$

$$\cdot \int D\bar{\theta}^a D\theta^a \exp\left(-\int d^4x \bar{\theta}^a D^{ac}_\mu \mathcal{D}^{cb}_\mu \theta^b \right). \qquad (53)$$

Here, $F^a_{\mu\nu}$ and D^{ab}_μ are the background field strength tensor and the corresponding covariant derivative, defined identically to $\mathcal{F}^a_{\mu\nu}$ and $\mathcal{D}^{ab}_\mu$ with the replacement $A^a_\mu \to B^a_\mu$. Secondly, in order to avoid double counting of fields during the integration and perform the averages over the background fields and quantum fluctuations separately, we have used the so-called 't Hooft identity [27]

$$\int DA_\mu^a f\left(A_\alpha^b\right) = \frac{\int DB_\mu^a \eta\left(B_\alpha^b\right) \int Da_\mu^a f\left(B_\alpha^b + a_\alpha^b\right)}{\int DB_\mu^a \eta\left(B_\alpha^b\right)},$$

valid for an arbitrary functional f. Here, an integration weight $\eta\left(B_\alpha^b\right)$ should be fixed by the demand that all the cumulants and the string tension of the Nambu-Goto term acquire their observed values. Notice also, that in what follows we shall work in the adjoint representation, so that $(T^a)^{bc} = -i\varepsilon^{abc}$, and $J = 1$. Finally, in the derivation of Eq. (53) we have adopted the background Feynman gauge, $\mathcal{L}_{\text{gauge fix.}} = -\frac{1}{2}\left(D_\mu^{ab} a_\mu^b\right)^2$, and denoted $\hat{F}_{\mu\nu}^{ab} \equiv F_{\mu\nu}^c (T^c)^{ab}$.

Notice, that the term

$$-ig \int\limits_{\Sigma_{\text{min.}}} d\sigma_{\mu\nu} a_\nu^a (D_\mu \vec{n})^a \tag{54}$$

in Eq. (53) emerged from the expansion of the Wess-Zumino term as a result of the following transformations

$$-\frac{ig}{2} \int\limits_{\Sigma_{\text{min.}}} d\sigma_{\mu\nu} \varepsilon^{abc} n^a n^d \left[\varepsilon^{bde} a_\mu^e (D_\nu \vec{n})^c + \varepsilon^{cde} a_\nu^e (D_\mu \vec{n})^b\right] =$$

$$= ig \int\limits_{\Sigma_{\text{min.}}} d\sigma_{\mu\nu} a_\nu^a \left[n^a n^b (D_\mu \vec{n})^b - (D_\mu \vec{n})^a\right] = -ig \int\limits_{\Sigma_{\text{min.}}} d\sigma_{\mu\nu} a_\nu^a (D_\mu \vec{n})^a,$$

where in the last equality we have used the facts that $\vec{n}^{\,2} = 1$ and $\varepsilon^{bcd} n^b n^c = 0$. It is also worth mentioning that the terms quadratic in quantum fluctuations, which emerge from the expansion of the field strength tensor $\mathcal{F}_{\mu\nu}^a$ and the Wess-Zumino term, cancel each other.

In what follows, we shall work in the one-loop approximation, and thus disregard in Eq. (53) the terms cubic and quartic in quantum fluctuations, as well as the ghost term. For simplicity, we shall also neglect the interaction of two perturbative gluons with the field strength tensor $F_{\mu\nu}^a$ (gluon spin term). Notice that, within the Feynman-Schwinger proper time path-integral representation for the perturbative gluon propagator, which will be used immediately below, such a term leads to insertions of the colour magnetic moment into the contour of integration (see Ref. [26]).

Bringing now together the term (54) and the term $-ig \int\limits_{\Sigma_{\text{min.}}} d\sigma_{\mu\nu} n^a D_\mu^{ab} a_\nu^b$ from Eq. (53) and performing Gaussian integration over perturbative fluctuations, we obtain

$$\langle W(C)\rangle = \left\langle\left\langle \exp\left(-\frac{ig}{2}\int\limits_{\Sigma_{\text{min.}}} d\sigma_{\mu\nu} n^a F_{\mu\nu}^a\right) \exp\left[-\frac{g^2}{2}\int\limits_{\Sigma_{\text{min.}}} d\sigma_{\mu\nu}(x) \int\limits_{\Sigma_{\text{min.}}} d\sigma_{\rho\nu}(x') K_{\mu\rho}(x,x')\right]\right\rangle_{\vec{n}}\right\rangle_{B_\mu^a}. \tag{55}$$

Here,

$$\langle\ldots\rangle_{\vec{n}} \equiv \int D\vec{n}\,(\ldots) \exp\left[\frac{i}{2}\int\limits_{\Sigma_{\text{min.}}} d\sigma_{\mu\nu} \varepsilon^{abc} n^a (D_\mu \vec{n})^b (D_\nu \vec{n})^c\right],$$

$$\langle ... \rangle_{B_\mu^a} \equiv N \int DB_\mu^a (...) \, \eta\left(B_\alpha^b\right) \exp\left[-\frac{1}{4}\int d^4x \left(F_{\mu\nu}^a\right)^2\right],$$

and the (generally speaking, non-translation invariant) interaction kernel $K_{\mu\rho}(x,x')$ is expressed via the perturbative gluon propagator as follows

$$K_{\mu\rho}(x,x') = \frac{\partial^2}{\partial x_\mu \partial x'_\rho} n^b(x) n^c(x') \int\limits_0^{+\infty} ds \int Dz \mathrm{e}^{-\frac{1}{4}\int\limits_0^s \dot{z}^2 d\lambda} \left[P \exp\left(ig \int\limits_0^s d\lambda \dot{z}_\alpha B_\alpha\right)\right]^{bc}, \tag{56}$$

where $z(0) = x'$, $z(s) = x$. In the derivation of Eq. (55), we have neglected the interaction of the string world-sheet with the background sources of the type $D_\mu^{ab} F_{\mu\nu}^b(y)$, where y is an arbitrary space-time point outside the world-sheet, which should be finally integrated over.

In order to derive a correction, emerging due to exchanges by perturbative gluons, to the background-induced gluodynamics string effective action (41), let us apply to Eq. (55) the following formula [68]

$$\left\langle \mathrm{e}^A B \right\rangle = \left\langle \mathrm{e}^A \right\rangle \left(\langle B \rangle + \sum\limits_{n=1}^{+\infty} \frac{1}{n!} \langle\langle A^n B \rangle\rangle \right),$$

where A and B stand for two commuting operators, and $\langle ... \rangle$ is an arbitrary average. Then, the leading correction, we are interested in, corresponds to the complete neglection of correlations between the arguments of the first and second exponential factors standing on the R.H.S. of Eq. (55). Secondly, it corresponds to putting the $\vec{n}$- and B_μ^a-averages of the second exponential factor inside it. Taking all this into account and following our definition of the string effective action $S_{\text{eff.}}$ as $-\ln \langle W(C) \rangle$, we obtain

$$S_{\text{eff.}} = -\ln \left\langle \left\langle \exp\left(-\frac{ig}{2} \int\limits_{\Sigma_{\text{min.}}} d\sigma_{\mu\nu} n^a F_{\mu\nu}^a \right) \right\rangle_{\vec{n}} \right\rangle_{B_\mu^a} + \Delta S_{\text{eff.}}, \tag{57}$$

where the desired leading correction to the string effective action reads

$$\Delta S_{\text{eff.}} = -\frac{g^2}{2} \int\limits_{\Sigma_{\text{min.}}} d\sigma_{\mu\nu}(x) \int\limits_{\Sigma_{\text{min.}}} d\sigma_{\rho\nu}(x') \left\langle \langle K_{\mu\rho}(x,x') \rangle_{\vec{n}} \right\rangle_{B_\mu^a}. \tag{58}$$

Clearly, in the bilocal approximation, the first term on the R.H.S. of Eq. (57) yields pure background part of the string effective action (41) (*cf.* Eq. (52) with A_μ^a replaced by B_μ^a).

The obtained correction (58) to the pure background string effective action (41) corresponds to a new type of interaction between the string world-sheet elements. Namely, instead of the propagator of the background gluon, represented by the function $D(x)$, in Eq. (58) stands a propagator of the perturbative gluon in the nonperturbative gluodynamics vacuum. Due to the statistical weight $\mathrm{e}^{-\frac{1}{4}\int\limits_0^s \dot{z}^2 d\lambda} P \exp\left(ig \int\limits_0^s d\lambda \dot{z}_\alpha B_\alpha\right)$ of this gluon, it is the region where s is small, which mainly contributes to the interaction kernel (56). This means that the dominant contribution to the obtained correction (58) comes from those points $x_\mu(\xi)$ and $x_\mu(\xi')$ of the world-sheet, which are very close to each other. That is in line with the performed derivative expansion of the nonlocal string effective action (41), where $|x-x'|\le T_g \ll R$. Taking this into account, we can adopt the simplest, local, approximation for the propagator $\left\langle n^b(x)n^c(x')\right\rangle_{\vec n}$, i.e. replace it by

$$\frac{\delta^{bc}}{3}\int D\vec n \exp\left[\frac{i}{2}\int\limits_{\Sigma_{\rm min.}} d\sigma_{\mu\nu}\varepsilon^{def}n^d\left(D_\mu\vec n\right)^e\left(D_\nu\vec n\right)^f\right].$$

This expression is a functional of the world-sheet as a whole (i.e. it is independent of $x_\mu(\xi)$) and therefore can be absorbed into the irrelevant normalization constant N.

Finally, in order to perform an expansion of the nonlocal correction (58) in powers of the derivatives w.r.t. ξ^a's and derive from it the first few local terms, it is convenient to pass to the integration over the trajectories $u_\mu(\lambda)=z_\mu(\lambda)+\frac{\lambda}{s}(x'-x)_\mu-x'_\mu$. This enables one to extract explicitly the dependence on the points x_μ and x'_μ from the integral over trajectories (necessary for the differentiation w.r.t. these points), which yields

$$K_{\mu\rho}(x,x')=\frac{\partial^2}{\partial x_\mu\partial x'_\rho}\int\limits_0^{+\infty}ds{\rm e}^{-\frac{(x-x')^2}{4s}}\int Du\,{\rm e}^{-\frac14\int\limits_0^s\dot u^2d\lambda}\cdot$$

$$\cdot{\rm tr}\,P\,\exp\left[ig\int\limits_0^s d\lambda\left(\frac{x-x'}{s}+\dot u\right)_\alpha B_\alpha\left(u+x'+\frac{\lambda}{s}(x-x')\right)\right]$$

with $u(0)=u(s)=0$. Then, in the bilocal approximation, the dominant contribution to the Nambu-Goto and rigidity terms comes out from taking the derivatives of the free propagation factor ${\rm e}^{-\frac{(x-x')^2}{4s}}$ and replacing the parallel transporter factor by the one over the closed path, which has the form ${\rm tr}\,P\,\exp\left[ig\int\limits_0^s d\lambda\dot u_\alpha B_\alpha(u)\right]$. Finally, substituting the so-obtained expression for $K_{\mu\rho}(x,x')$ into Eq. (58) and performing an expansion of this nonlocal correction in powers of the derivatives w.r.t. ξ^a's similarly to the previous Subsection, we arrive at the desired correction to the local effective action (43). In this way, it turns out that the Nambu-Goto term string tension acquires no correction due to perturbative gluonic exchanges, whereas the correction to the inverse bare coupling constant of the rigidity term reads

$$\Delta\frac{1}{\alpha_0}=-\frac{\pi g^2}{3}\int\limits_0^{+\infty}dss\int Du\,{\rm e}^{-\frac14\int\limits_0^s\dot u^2d\lambda}\left\langle{\rm tr}\,P\,\exp\left[ig\int\limits_0^s d\lambda\dot u_\alpha B_\alpha(u)\right]\right\rangle_{B_\mu^a}. \qquad (59)$$

It is worth noting that since $B_\mu^a\sim\frac{1}{g}$, the parallel transporter factor on the R.H.S. of Eq. (59) cannot be expanded in powers of g and should be considered as a whole. The path-integral on the R.H.S. of Eq. (59) is not simply the perturbative gluon propagator, since the integral over the proper time contains an additional power of s, which makes the whole quantity dimensionless, as it should be. Notice in conclusion, that the sign of the obtained correction (59) to the inverse bare coupling constant (45) depends on the form of background B_μ^a entering the Wilson loop (which is the most nontrivial content of this correction) on the R.H.S. of Eq. (59). However, since the derived correction is a pure perturbative effect (due to the factor g^2 present in Eq. (59)), even in the case when it is positive, it cannot change the negative sign of the leading term (45).

2.3 A Hamiltonian of the Straight-Line QCD String with Spinless Quarks

In this Subsection we shall derive a Hamiltonian corresponding to the quark-antiquark Green function (15) in the confining QCD vacuum. To this end, we shall first write it down in the Feynman-Schwinger proper time path-integral representation (17), after which we substitute for $\langle W(C)\rangle$ the above obtained expression $\exp(-S_{\rm eff.})$ with $S_{\rm eff.}$ defined by Eq. (43) (and inverse bare coupling constant of the rigidity term (45) modified by Eq. (59), i.e. $\frac{1}{\alpha_0}\to\frac{1}{\alpha_0}+\Delta\frac{1}{\alpha_0}$). It is worth noting, that for the first time such a Hamiltonian has been derived in Ref. [100], where, however, only the Nambu-Goto term in the string effective action (43) has been accounted for. Our aim below will be the derivation of a correction to this result due to the rigidity term. As a byproduct, we shall also rederive the leading terms, obtained in Ref. [100]. However, we shall see that the rigid string theory, being the theory with higher derivatives, leads to the interesting and phenomenologically relevant modifications of the Nambu-Goto–induced part of the Hamiltonian. We shall also generalize the result of Ref. [100] by considering the case of different masses of a quark and antiquark.

The two main approximations, under which we shall consider the Green function (17), are the same as the ones used in Ref. [100]. First, we shall neglect quark trajectories with backward motion in the proper time, which might lead to creation of additional $q\bar q$-pairs. Secondly, we shall use the straight-line approximation for the minimal surface $\Sigma_{\rm min.}$, which, as it has been argued in Ref. [100], corresponds to the valence quark approximation. Such a "minimal" string may rotate and oscillate longitudinally. This approximation is inspired by two limiting cases: $l=0$ and $l\to\infty$, where l is the orbital quantum number of the system.

The first case will be investigated below in more details, and the correction to the Hamiltonian of the relativistic quark model [101] due to the rigidity term in the limit of large masses of a quark and antiquark will be derived.

Let us now proceed with the derivation of the desired Hamiltonian. To this end, we shall start with the expression for the Green function (17) with $\langle W(C)\rangle$ defined via Eq. (43) and

$$K\equiv m_1^2 s+\frac14\int\limits_0^s d\lambda\dot z^2,\ \ \bar K\equiv m_2^2\bar s+\frac14\int\limits_0^{\bar s} d\lambda\dot{\bar z}^2$$

standing for the kinematical factors of a quark and antiquark, respectively. Then, by making use of the auxiliary field formalism [20], one can represent it in the following way

$$G(x,\bar x;y,\bar y)=\int\limits_0^{+\infty}dT\int D\vec z D\vec{\bar z}D\mu_1 D\mu_2 Dh_{ab}\exp\left(-K'-\bar K'\right)\exp\left[(-\sigma+2\bar\alpha)\int d^2\xi\sqrt{h}\right]\cdot$$

$$\cdot\exp\left[-\bar\alpha\int d^2\xi\sqrt{h}h^{ab}\left(\partial_a w_\mu\right)\left(\partial_b w_\mu\right)-\frac{1}{\alpha_0}\int d^2\xi\sqrt{h}h^{ab}\left(\partial_a t_{\mu\nu}\right)\left(\partial_b t_{\mu\nu}\right)\right], \qquad (60)$$

where we have integrated over the Lagrange multiplier $\lambda^{ab}(\xi)=\alpha(\xi)h^{ab}(\xi)+f^{ab}(\xi), f^{ab}h_{ab}=0$, and $\bar\alpha$ is the mean value of $\alpha(\xi)$. Here $t_{\mu\nu}=\frac{1}{\sqrt{h}}\varepsilon^{ab}\left(\partial_a w_\mu\right)\left(\partial_b w_\nu\right)$,

$$K'+\bar K'=\frac12\int\limits_0^T d\tau\left[\frac{m_1^2}{\mu_1(\tau)}+\mu_1(\tau)\left(1+\dot{\vec z}^2(\tau)\right)+\frac{m_2^2}{\mu_2(\tau)}+\mu_2(\tau)\left(1+\dot{\vec{\bar z}}^2(\tau)\right)\right], \qquad (61)$$

$$T = \frac{1}{2}\left(x_0 + \bar{x}_0 - y_0 - \bar{y}_0\right), \mu_1(\tau) = \frac{T}{2s}\dot{z}_0(\tau), \mu_2(\tau) = \frac{T}{2\bar{s}}\dot{\bar{z}}_0(\tau),$$

and the no-backtracking time approximation [100]

$$\mu_1(\tau) > 0, \;\; \mu_2(\tau) > 0 \tag{62}$$

has been used. Similarly to Ref. [100], we exploit in the valence quark sector (62) the approximation that the minimal surface $\Sigma_{\rm min.}$ may be parametrized by the straight lines, connecting points $z_\mu(\tau)$ and $\bar{z}_\mu(\tau)$ with the same τ, i.e. the trajectories of a quark and antiquark are synchronized: $z_\mu = (\tau, \vec{z}), \bar{z}_\mu = (\tau, \vec{\bar{z}}), w_\mu(\tau,\beta) = \beta z_\mu(\tau) + (1-\beta)\bar{z}_\mu(\tau), 0 \le \beta \le 1$.

Introducing auxiliary fields [100] $\nu(\tau,\beta) = T\sigma\frac{h_{22}}{\sqrt{h}}, \eta(\tau,\beta) = \frac{1}{T}\frac{h_{12}}{h_{22}}$ and making a rescaling $z_\mu \to \sqrt{\frac{\sigma}{2\bar{\alpha}}}z_\mu, \bar{z}_\mu \to \sqrt{\frac{\sigma}{2\bar{\alpha}}}\bar{z}_\mu$, one gets from the last exponent on the R.H.S. of Eq. (60) the following action of the string without quarks

$$S_{\rm eff.} = \int\limits_0^T d\tau \int\limits_0^1 d\beta \frac{\nu}{2}\Biggl\{\dot{w}^2 + \left(\left(\frac{\sigma}{\nu}\right)^2 + \eta^2\right)r^2 - 2\eta(\dot{w}r) + \frac{\sigma T^2}{\alpha_0\bar{\alpha}^2}\frac{1}{h}\Biggl[\ddot{w}^2 r^2 - (\ddot{w}r)^2 + \dot{w}^2\dot{r}^2 - (\dot{w}\dot{r})^2 +$$

$$+ 2((\ddot{w}\dot{w})(\dot{r}r) - (\ddot{w}\dot{r})(\dot{w}r)) + \left(\left(\frac{\sigma}{\nu}\right)^2 + \eta^2\right)\left(\dot{r}^2 r^2 - (\dot{r}r)^2\right) -$$

$$-2\eta\left((\ddot{w}\dot{r})r^2 - (\ddot{w}r)(\dot{r}r) + (\dot{w}\dot{r})(\dot{r}r) - (\dot{w}r)\dot{r}^2\right)\Biggr]\Biggr\}, \tag{63}$$

where a dot stands for $\frac{\partial}{\partial\tau}, r_\mu(\tau) = z_\mu(\tau) - \bar{z}_\mu(\tau)$ is the relative coordinate, and we have taken into account that h_{ab} is a smooth function.

Let us now introduce the centre of masses coordinate $R_\mu(\tau) = \zeta(\tau)z_\mu(\tau) + (1-\zeta(\tau))\bar{z}_\mu(\tau)$, where $\zeta(\tau) \equiv \zeta_1(\tau) + \frac{1}{\alpha_0}\zeta_2(\tau), 0 \le \zeta(\tau) \le 1$, should be determined from the requirement that $\dot{R}_\mu$ decouples from $\dot{r}_\mu$. These extremal values of ζ_1 and ζ_2 can be obtained from the corresponding saddle-point equation. Referring the reader for the details to Appendix 7.1, we shall present here only the final result for the path-integral Hamiltonian of the straight-line QCD string with quarks. It has the form

$$H = H^{(0)} + \frac{1}{\alpha_0}H^{(1)}. \tag{64}$$

Here

$$H^{(0)} = \frac{1}{2}\left[\frac{(\vec{p}_r^{\;2} + m_1^2)}{\mu_1} + \frac{(\vec{p}_r^{\;2} + m_2^2)}{\mu_2} + \mu_1 + \mu_2 + \sigma^2\vec{r}^{\;2}\int\limits_0^1 \frac{d\beta}{\nu} + \nu_0 + \frac{\vec{L}^{\;2}}{\rho\vec{r}^{\;2}}\right] \tag{65}$$

with

$$\rho = \mu_1 + \nu_2 - \frac{(\mu_1+\nu_1)^2}{\mu_1+\mu_2+\nu_0}, \;\; \nu_i \equiv \int\limits_0^1 d\beta\beta^i\nu, \;\; \vec{p}_r^{\;2} \equiv \frac{(\vec{p}\vec{r})^2}{\vec{r}^{\;2}}, \;\; \vec{L} \equiv [\vec{r},\vec{p}]$$

is the Hamiltonian of the "minimal" Nambu-Goto string with quarks, which for the case of equal masses of a quark and an antiquark has been derived and investigated in Ref. [100], while the new Hamiltonian $H^{(1)}$ has the form

$$H^{(1)} = \frac{a_1}{\rho^2}\vec{L}^{\,2} + \frac{a_2}{2\tilde{\mu}^3}|\vec{r}|\left(\vec{p}_r^{\;2}\right)^{\frac{3}{2}} + \frac{a_3}{\tilde{\mu}^4}\left(\vec{p}_r^{\;2}\right)^2 + \frac{a_4}{2\tilde{\mu}\rho^2}\frac{\sqrt{\vec{p}_r^{\;2}}\vec{L}^{\,2}}{|\vec{r}|} + \frac{a_5}{2\tilde{\mu}^2\rho^2}\frac{\vec{p}_r^{\;2}\vec{L}^{\,2}}{\vec{r}^{\,2}}, \tag{66}$$

where $\tilde{\mu} = \frac{\mu_1\mu_2}{\mu_1+\mu_2}$, and the coefficients $a_k,\ k = 1, ..., 5$, are listed in the Appendix 7.1.

We see that the obtained Hamiltonian (66) contains not only corrections to the orbital momentum of the system, but also several operators higher than of the second order in the momentum. The latter ones emerge as a consequence of the fact that the rigid string theory is a theory with higher derivatives.

The obtained (path-integral) Hamiltonian (64)-(66) contains auxiliary fields μ_1, μ_2 and ν. In order to construct from it the operator Hamiltonian, which acts upon the wave functions, one should integrate these fields out. This implies the substitution of their extremal values, which could be obtained from the corresponding saddle point equations, into Eqs. (64)-(66) and performing the Weil ordering [102].

In conclusion of this Subsection, let us apply Hamiltonian (66) to the derivation of the rigid string correction to the Hamiltonian of the so-called relativistic quark model [101], i.e. consider the case when the orbital momentum is equal to zero. Let us also put for simplicity the mass of a quark being equal to the mass of an antiquark $m_1 = m_2 \equiv m$. In order to get $H^{(1)}$, one should substitute the extremal values of the fields μ_1, μ_2 and ν of the zeroth order in $\frac{1}{\alpha_0}$, $\mu_1^{\rm ext.} = \mu_2^{\rm ext.} = \sqrt{\vec{p}^{\,2} + m^2}$, and $\nu_{\rm ext.} = \sigma|\vec{r}|$, into Eq. (66). The limit of large masses of a quark and antiquark means that $m \gg \sqrt{\sigma}$. In this case we obtain from Eq. (66) the rigid string Hamiltonian $H^{(1)} = -\frac{4\sigma|\vec{r}|}{m^4}\left(\vec{p}_r^{\;2}\right)^2$ and then from Eqs. (64) and (65) the following expression for the total Hamiltonian H

$$H = 2m + \sigma|\vec{r}| + \frac{\vec{p}^{\,2}}{m} - \left(\frac{1}{4m^3} + \frac{4\sigma|\vec{r}|}{\alpha_0 m^4}\right)\left(\vec{p}^{\,2}\right)^2. \tag{67}$$

One can see, that the new rigid string-inspired term, quartic in the relative momentum of the quark-antiquark pair, may cause a sufficient influence to the dynamics of the system.

3 String Representation of Abelian(-Projected) Theories

The SVM, exploited in the previous Section, enabled us to investigate the string effective action associated with a certain (e.g. minimal) world-sheet, but gave us no mechanism of getting the full string partition function in the form of an integral over all world-sheets. The reason for that is that within SVM one looses the meaning of the path-integral average over the gluodynamics vacuum, substituting this average by the phenomenological one. As a consequence, it looks difficult to extract singularities corresponding to QCD strings out of the resulting vacuum correlation functions.

One's intuitive feeling is that such an integral over world-sheets once being written down and then carried out in the saddle-point approximation should eventually yield the nonlocal string effective action (41), we started with within SVM. In this Section, we shall see that this is indeed the case and find an exact field-theoretical analogue of the phenomenological background gluon coefficient function $D(x)$. We shall argue that the general features of the gluodynamics vacuum are in line with the ones predicted by the so-called dual Meissner picture of confinement, first put forward by 't Hooft and Mandelstam [41, 42]. According to the 't Hooft-Mandelstam scenario, the properties of the QCD string should be similar to those of the electric vortex, which emerges between two electrically charged particles immersed into a superconducting medium filled with a condensate of Cooper pairs of magnetic monopoles. In the case of the usual Abelian Higgs Model (AHM), which is a relativistic version of the Ginzburg-Landau theory of superconductivity, such vortices are referred to as Abrikosov-Nielsen-Olesen strings [43].

Up to now, there does not exist any analytical proof of the existence of the condensate of Abelian monopoles in QCD, though for lattice QCD there are a lot of numerical data in favour of this conjecture (see e.g. Refs. [103, 104, 105]). However, in all theories allowing for an analytical description of confinement, the latter one is due to the monopole condensation. These examples include compact QED and 3D Georgi-Glashow model [106, 20], and supersymmetric Yang-Mills theory [40]. As far as the origin of magnetic monopoles in QCD is concerned, they appear in the so-called Abelian projection method [39, 107]. The essence of this method is based on a certain partial gauge fixing procedure, which reduces the original gauge group $SU(N_c)$ to the maximal Abelian (or Cartan) subgroup $[U(1)]^{N_c-1}$, i.e. leaves Abelian degrees of freedom unfixed. Then, since the original $SU(N_c)$ group is compact, the resulting Abelian subgroup is compact as well, which is just the origin of Abelian magnetic monopoles in the original non-Abelian theory.

To perform such a gauge fixing, one chooses a certain composite operator X, transforming by the adjoint representation of $SU(N_c)$, $X \to X' = V^\dagger X V$, and diagonalizes it, i.e. finds such a gauge (unitary matrix V), that

$$X' = \text{diag}\,(\lambda_1, ..., \lambda_{N_c})\,.$$

As an example of such an operator may serve e.g. $F_{12}(x)$ or, at finite temperatures, the so-called Polyakov line [108]

$$P\,\exp\left[ig\int\limits_0^\beta dx_4 A_4\,(x_4, \vec{x})\right],$$

where $\beta = 1/T$ is an inverse temperature. The most interesting numerical results (see e.g. Refs. in [105]) were obtained for the $SU(2)$-case in the so-called Maximal Abelian gauge, defined by the condition of maximization of the functional

$$R\,[A_\mu] = -\int d^4x\left(\left(A_\mu^1\right)^2 + \left(A_\mu^2\right)^2\right),$$

which reads $\max\limits_V R\left[A^V\right]$. The condition of the local extremum for this functional,

$$\left(\partial_\mu \pm igA_\mu^3\right)A_\mu^\pm = 0,$$

where $A_\mu^\pm = A_\mu^1 \pm iA_\mu^2$, means that in this gauge we make the field A_μ as diagonal as possible.

The matrix V, which diagonalizes the operator X is obviously defined up to a left multiplication by the diagonal $SU(N_c)$ matrix, which belongs to the subgroup $[U(1)]^{N_c-1}$. This is just the way how the original non-Abelian group reduces to the maximal Abelian subgroup. Let us now perform a certain Abelian projection, i.e. consider the transformed vector potential

$$A'_\mu = V^\dagger\left(A_\mu + \frac{i}{g}\partial_\mu\right)V.$$

One can now see, that under the remained maximal Abelian subgroup, the diagonal field components of the matrix-valued vector potential $a^i_\mu \equiv \left(A'_\mu\right)^{ii}$, $i = 1, ..., N_c$, transform as Abelian gauge fields,

$$a^i_\mu \longrightarrow a^i_\mu + \frac{1}{g}\partial_\mu \alpha_i,$$

whereas the off-diagonal components $c^{ij}_\mu = \left(A'_\mu\right)^{ij}$ transform as charged matter fields,

$$c^{ij}_\mu \longrightarrow \exp\left[i\left(\alpha_i - \alpha_j\right)\right] c^{ij}_\mu.$$

It turnes out, that Abelian monopoles in non-Abelian theories naturally emerge within the Abelian projection method. The role of the monopoles (w.r.t. the "photon" fields of diagonal elements of the vector potential) is played by the singularities, which might occur if some of the eigenvalues $\lambda_1, ..., \lambda_{N_c}$ of the operator X coincide. This may happen in some 3D point $\vec{x}_0$, which becomes a world-line of magnetic monopole in 4D. In this case, the matrix of the gauge transformation V contains singularities, which makes the non-Abelian field strength tensor transform inhomogeneously under such singular gauge transformations,

$$F_{\mu\nu}[A] \longrightarrow F_{\mu\nu}\left[A^V\right] = V^\dagger F_{\mu\nu}[A]\,V + \frac{i}{g}V^\dagger(x)\left(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu\right)V(x).$$

The last term on the R.H.S. of this equation is nonvanishing due to the singular character of the matrix V. It gives rise to the singular part $f^{i\,\text{sing.}}_{\mu\nu}$ of the Abelian field strength tensor $f^{i\,\text{reg.}}_{\mu\nu} = \partial_\mu a^i_\nu - \partial_\nu a^i_\mu$, so that

$$f^{i\,\text{full}}_{\mu\nu} = f^{i\,\text{reg.}}_{\mu\nu} + f^{i\,\text{sing.}}_{\mu\nu},$$

where $\partial_\mu \tilde{f}^{i\,\text{sing.}}_{\mu\nu} = j^i_\nu (\neq 0)$ is the monopole current of the i-th kind with $\tilde{f}^{i\,\text{sing.}}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\lambda\rho} f^{i\,\text{sing.}}_{\lambda\rho}$. In particular for the $SU(2)$-case, one can define the monopole charge (of the one type of monopoles for this gauge group) as

$$m = \frac{1}{4\pi}\int\limits_{S_\varepsilon(\vec{x}_0)} d\sigma_{\mu\nu}\tilde{f}^{\text{sing.}}_{\mu\nu},$$

where $S_\varepsilon(\vec{x}_0)$ stands for a sphere of an infinitesimal radius ε, surrounding the 3D point $\vec{x}_0$, in which $\lambda_1 = \lambda_2$. This charge can be calculated by making use of the Gauss law [103], and the result has the form $m = 0, \pm\frac{1}{2g}, \pm\frac{1}{g}, ...$, which is just the condition of quantization of the Abelian monopole charge. This result is nothing else, but the winding number of $SU(2)$ over $S_\varepsilon(\vec{x}_0)$.

Notice, that the appearance of Abelian monopoles within the Abelian projection method might serve as a starting point for the complete analytical proof of confinement in QCD. Recently, some progress in this direction has been achieved in Refs. [109, 110].

Actually, for the gauge group $SU(N_c)$, the number of different monopoles is equal to the number of possibilities for the eigenvalues $\lambda_1, ..., \lambda_{N_c}$ to coincide, i.e. $\frac{N_c(N_c-1)}{2}$. In particular, as we have just mentioned, for the $SU(2)$-case, there exists only one type of monopoles, and therefore the Abelian projected $SU(2)$-gluodynamics in the continuum formulation should be quite similar to the dual AHM (DAHM) with monopoles, whereas in the $SU(3)$-case there emerge three types of monopoles, and the Abelian projection leads to a more sophisticated dual model. Thus, we shall begin our investigations of the dual models of confinement with the simplest Abelian(-projected) theory, i.e. DAHM extended by introduction of the gauge fields generated by external electrically charged particles called “quarks”. The Abelian projected version of the $SU(3)$-gluodynamics will be studied in Subsection 3.2.

3.1 Nonperturbative Field Correlators and String Representation of the Dual Abelian Higgs Model

3.1.1 London Limit

Double Gauge Invariance and Dual Formulation of the Wilson Loops We shall start with the following expression for the partition function of the extended DAHM

$$\mathcal{Z} = \int |\Phi| D|\Phi| DB_\mu D\theta \exp\left\{-\int d^4x \left[\frac{1}{4}\left(F_{\mu\nu} - F_{\mu\nu}^E\right)^2 + \frac{1}{2}|D_\mu\Phi|^2 + \lambda\left(|\Phi|^2 - \eta^2\right)^2\right]\right\}, \tag{68}$$

where $\Phi(x) = |\Phi(x)| e^{i\theta(x)}$ is an effective Higgs field of “Cooper pairs” of magnetic monopoles, B_μ and $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ are the dual (“magnetic”) gauge field and its field strength tensor [16], $D_\mu = \partial_\mu - 2ig_m B_\mu$ is the covariant derivative with g_m standing for the magnetic coupling constant. Notice that in Eq. (68), $F_{\mu\nu}^E$ denotes the field strength tensor generated by external “quarks”, defined according to the equation

$$\partial_\nu \tilde{F}_{\mu\nu}^E \equiv \frac{1}{2}\varepsilon_{\mu\nu\lambda\rho}\partial_\nu F_{\lambda\rho}^E = 4\pi j_\mu^E \tag{69}$$

with

$$j_\mu^E(x) \equiv e \oint\limits_C dx_\mu(s)\delta(x - x(s))$$

standing for the conserved electric current of a quark, which moves along the closed contour C (*cf.* notations to Eq. (13)). The electric coupling constant e is related to the magnetic one via Dirac's quantization condition $eg_m = \frac{n}{2}$, where n is an integer [17]. In what follows, we shall for concreteness restrict ourselves to the case of monopoles possessing the minimal charge, i.e. put $n = 1$.

The solution to Eq. (69) reads $F_{\mu\nu}^E = 4\pi e\tilde{\Sigma}_{\mu\nu}$, where $\Sigma_{\mu\nu}(x) \equiv \int\limits_\Sigma d\sigma_{\mu\nu}(x(\xi))\delta(x - x(\xi))$ is the so-called vorticity tensor current [112] defined on the string world-sheet Σ. Due to the Stokes theorem, the vorticity tensor current is related to the quark current according to the equation $e\partial_\nu\Sigma_{\mu\nu} = j_\mu^E$. In particular, this equation means, that in the case, when there are no external quarks, the vorticity tensor current is conserved, i.e. due to the conservation of electric flux all the strings in this case are closed. Notice, that when external quarks are introduced into the system, some amount of closed strings might survive. In our investigations of the extended DAHM in the next three Paragraphs, we shall restrict ourselves to the sector of the theory with open strings ending at quarks and antiquarks only.

[16] From now on in our investigations of DAHM, it is everywhere implicitly implied that the gauge-fixing factor is included in the integration measure of the field B_μ.

[17] Here, we have adopted the notations of Ref. [111].

Notice, that the gauge fields $F^E_{\mu\nu}$ are of quite a new type w.r.t. the standard (dual) gauge fields with the field strength tensor $F_{\mu\nu}$. Namely, if we approximate the space-time continuum by a simple hypercublic lattice of the spacing a, then superposition of $F^E_{\mu\nu}$'s can be written as [113] $4\pi e N_{\mu\nu}(x)/a^2$, where $N_{\mu\nu}(x)$ is an arbitrary integer-valued antisymmetric tensor field. This field can be imagined as living on the plaquettes of the lattice.

Notice also, that the transition from one surface Σ to another surface Σ', both of which are bounded by the contour C, is described as follows [113]

$$\tilde{\Sigma}'_{\mu\nu} = \tilde{\Sigma}_{\mu\nu} + \partial_\mu \delta_\nu \left(x, V^{(3)}\right) - \partial_\nu \delta_\mu \left(x, V^{(3)}\right). \tag{70}$$

Here,

$$\delta_\mu \left(x, V^{(3)}\right) \equiv \varepsilon_{\mu\nu\lambda\rho} \int d^3\zeta \frac{\partial x_\nu(\zeta)}{\partial \zeta^1} \frac{\partial x_\lambda(\zeta)}{\partial \zeta^2} \frac{\partial x_\rho(\zeta)}{\partial \zeta^3} \delta(x - x(\zeta))$$

stands for the δ-function defined on a 3D volume $V^{(3)}$, parametrized by the vector $x_\mu(\zeta)$, $\zeta = (\zeta^1, \zeta^2, \zeta^3)$, which is swept out by the surface Σ moving through the 4D space-time.

Let us also make a remark on the double gauge invariance of the action of "magnetic" gauge fields standing in the exponent of the R.H.S. of Eq. (68) in the presence of the quark world-lines, but without the Higgs field, i.e. the action

$$S^{\text{m.}} = \frac{1}{4} \int d^4x \left(F_{\mu\nu} - F^E_{\mu\nu}\right)^2. \tag{71}$$

Indeed, this action is two-fold invariant. First, it is invariant under "magnetic" gauge transformations, which transform $B_\mu \to B_\mu + \partial_\mu \Lambda$ (with the gauge function Λ satisfying the integrability condition $(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\Lambda = 0$) and leaves $F^E_{\mu\nu}$ unchanged. Besides that, there exists the second gauge invariance under "electric" gauge transformations

$$F^E_{\mu\nu} \to F^E_{\mu\nu} + \partial_\mu \Lambda^E_\nu - \partial_\nu \Lambda^E_\mu, \tag{72}$$

with integrable vector functions Λ^E_μ, which by virtue of Eq. (70) have the general form

$$\Lambda^E_\mu \left(x, V^{(3)}\right) = 4\pi e \delta_\mu \left(x, V^{(3)}\right)$$

with an arbitrary choice of the 3D volume $V^{(3)}$. Generally speaking, Λ^E_μ can be a superposition of such functions with various $V^{(3)}$'s,

$$\Lambda^E_\mu(x) = 4\pi e \sum_{V^{(3)}} \delta_\mu \left(x, V^{(3)}\right).$$

In this way, we obtain all functions of x, whose values are integers multiplied by $4\pi e$. The invariance of the action (71) then holds, provided that we accompany the transformation (72) by the shift $B_\mu \to B_\mu + \Lambda^E_\mu$. We see, that the physical significance of the "electric" gauge transformations is to change the string world-sheet without changing its boundary, the quark trajectory.

It is also worth remarking about the dual formulation of the partition function corresponding to the action (71). To this end, let us first rewrite the statistical weight associated with a certain electric current in the first-order formalism as follows

$$\int DB_\mu \mathrm{e}^{-S^{\text{m.}}} = \int DB_\mu Df_{\mu\nu} \exp\left[-\int d^4x \left(\frac{1}{4} f^2_{\mu\nu} + \frac{i}{2} f_{\mu\nu}\left(F_{\mu\nu} - F^E_{\mu\nu}\right)\right)\right].$$

Integrating now the B_μ-field out, one gets a constraint $\partial_\mu f_{\mu\nu} = 0$, which can be resolved by representing $f_{\mu\nu}$ in the form $f_{\mu\nu} = \varepsilon_{\mu\nu\lambda\rho}\partial_\lambda A_\rho$, where A_ρ is the usual "electric" vector potential. Finally, by making use of Eq. (69), we obtain

$$\int DB_\mu \mathrm{e}^{-S^{\mathrm{m.}}} = \int DA_\mu \exp\left[-\int d^4x \left(\frac{1}{4}F_{\mu\nu}^2(A) - 4\pi i A_\mu j_\mu^E\right)\right], \tag{73}$$

where $F_{\mu\nu}(A) = \partial_\mu A_\nu - \partial_\nu A_\mu$ [18]. The obtained action is obviously invariant under the standard gauge transformations $A_\mu \to A_\mu + \partial_\mu \Lambda'$ with an arbitrary integrable function Λ'.

Notice also that, as it has been shown in Ref. [113], under the above transformation, the monopole Wilson loop averaged with the action (71),

$$\left\langle \exp\left(i\int d^4x j_\mu^M B_\mu\right)\right\rangle_{S^{\mathrm{m.}}},$$

where $j_\mu^M(x) = g_m \oint\limits_\Gamma dy_\mu(s)\delta(x - y(s))$ stands for the monopole current, transforms into the usual Wilson loop of the field A_μ, averaged with the action $S^{\mathrm{el.}} = \frac{1}{4}\int d^4x \left(F_{\mu\nu}(A) - F_{\mu\nu}^M\right)^2$,

$$\left\langle \exp\left(i\int d^4x j_\mu^E A_\mu\right)\right\rangle_{S^{\mathrm{el.}}}.$$

Here, $F_{\mu\nu}^M = 4\pi g_m \tilde{\Sigma}_{\mu\nu}^M$ is the monopole field defined via the vorticity tensor current $\Sigma_{\mu\nu}^M(x) = \int\limits_{\Sigma^M} d\sigma_{\mu\nu}(x(\xi))\delta(x - x(\xi))$, where Σ^M is an arbitrary surface encircled by the contour Γ.

Now, in order to get the complete expression for the partition function, one should supplement Eq. (73) by a certain prescription of integration over the electric currents j_μ^E's. This integration can be most naturally understood in a sense of a grand canonical ensemble of fluctuating random oriented closed loops of arbitrary length and shape (the so-called loop gas [114]). The integration over such an ensemble is defined as follows [115, 114]

$$\int Dj_\mu^E \equiv \sum_{N=0}^{+\infty}\frac{1}{N!}\left[\prod_{n=0}^{N}\int_0^{+\infty}\frac{ds_n}{s_n}\int\limits_{x(0)=x(s_n)} Dx\left(s_n'\right)\right]\exp\left[-\frac{1}{4}\sum_{k=0}^{N}\int_0^{s_k} ds_k' \dot{x}^2\left(s_k'\right)\right], \tag{74}$$

due to which one has

$$\int Dj_\mu^E \exp\left(i\int d^4x A_\mu j_\mu^E\right) = \frac{1}{\det(-D_\mu^2(A))},$$

where the covariant derivative of the A_μ-field reads $D_\mu(A) = \partial_\mu - ieA_\mu$. By making use of this observation, one can finally write down the expression for the partition function in the form of an integral over the electric Higgs field (of the electric charge e) as follows [115]

$$\mathcal{Z}^{\mathrm{m.}} = \int DA_\mu D\Phi^E\left[-\int d^4x\left(\frac{1}{4}F_{\mu\nu}^2(A) + \frac{1}{2}\left|D_\mu(A)\Phi^E\right|^2\right)\right].$$

[18] In the rest part of this Paragraph, we include the factor 4π in the definition of j_μ^E.

String Representation for the Partition Function Let us now proceed with the string representation of the partition function (68) in the so-called London limit, $\lambda \to \infty$. In this limit, the radial part of the monopole field becomes fixed to its v.e.v., $|\Phi| \to \eta$, and the partition function (68) takes the form

$$\mathcal{Z} = \int DB_\mu D\theta^{\rm sing.} D\theta^{\rm reg.} \exp\left\{-\int d^4x \left[\frac{1}{4}\left(F_{\mu\nu} - F_{\mu\nu}^E\right)^2 + \frac{\eta^2}{2}\left(\partial_\mu\theta - 2g_m B_\mu\right)^2\right]\right\}, \qquad (75)$$

where from now on constant normalization factors will be omitted. In our further interpretation of the topic of the string representation for the partition function and field correlators of the extended DAHM in the London limit, we shall mainly follow Refs. [116, 117, 118] (for related investigations see Refs. [42, 87, 111, 112, 114, 119, 120]). In Eq. (75), we have performed a decomposition of the phase of the magnetic Higgs field $\theta = \theta^{\rm sing.} + \theta^{\rm reg.}$, where $\theta^{\rm sing.}(x)$ obeys the equation (see e.g. [121])

$$\varepsilon_{\mu\nu\lambda\rho}\partial_\lambda\partial_\rho\theta^{\rm sing.}(x) = 2\pi\Sigma_{\mu\nu}(x) \qquad (76)$$

and describes a given electric string configuration, whereas $\theta^{\rm reg.}(x)$ stands for a single-valued fluctuation around this configuration. Notice also, that as it has been shown in Ref. [120], the integration measure over the field θ factorizes into the product of measures over the fields $\theta^{\rm sing.}$ and $\theta^{\rm reg.}$.

Performing the path-integral duality transformation of Eq. (75) along the lines described in Ref. [121], we get

$$\mathcal{Z} = \int DB_\mu Dx_\mu(\xi) Dh_{\mu\nu} \exp\left\{\int d^4x \left[-\frac{1}{12\eta^2}H_{\mu\nu\lambda}^2 + \right.\right.$$

$$\left.\left. + i\pi h_{\mu\nu}\Sigma_{\mu\nu} - (2\pi e)^2\Sigma_{\mu\nu}^2 - \frac{1}{4}F_{\mu\nu}^2 - i\tilde{F}_{\mu\nu}\left(g_m h_{\mu\nu} + 2\pi i e\Sigma_{\mu\nu}\right)\right]\right\}, \qquad (77)$$

where $H_{\mu\nu\lambda} \equiv \partial_\mu h_{\nu\lambda} + \partial_\lambda h_{\mu\nu} + \partial_\nu h_{\lambda\mu}$ is the field strength tensor of a massive antisymmetric tensor field $h_{\mu\nu}$ (the so-called Kalb-Ramond field [122]). This antisymmetric spin-1 tensor field describes a massive dual gauge boson. Next, by carrying out the integration over the field B_μ in Eq. (77), we obtain

$$\mathcal{Z} = \int Dx_\mu(\xi) Dh_{\mu\nu} \exp\left\{-\int d^4x \left[\frac{1}{12\eta^2}H_{\mu\nu\lambda}^2 + \frac{1}{4e^2}h_{\mu\nu}^2 + i\pi h_{\mu\nu}\Sigma_{\mu\nu}\right]\right\}. \qquad (78)$$

The details of the derivation of Eqs. (77) and (78) are outlined in the Appendix 7.2. Thus, the path-integral duality transformation is just a way of getting a coupling of the gauge boson, described now by the field $h_{\mu\nu}$, to a string world-sheet, rather than to a world-line (as it takes place in the usual case of the Wilson loop).

Finally, the Gaussian integration over the field $h_{\mu\nu}$ in Eq. (78) (see Appendix 7.3) leads to the following expression for the partition function (75)

$$\mathcal{Z} = \int Dx_\mu(\xi) \exp\left\{-\pi^2 \int\limits_\Sigma d\sigma_{\lambda\nu}(x) \int\limits_\Sigma d\sigma_{\mu\rho}(y) D_{\lambda\nu,\mu\rho}(x-y)\right\}. \qquad (79)$$

In Eq. (79), the propagator of the field $h_{\mu\nu}$ has the following form

$$D_{\lambda\nu,\mu\rho}(x) \equiv D^{(1)}_{\lambda\nu,\mu\rho}(x) + D^{(2)}_{\lambda\nu,\mu\rho}(x),$$

where

$$D^{(1)}_{\lambda\nu,\mu\rho}(x) = \frac{\eta^3}{8\pi^2 e}\frac{K_1}{|x|}\left(\delta_{\lambda\mu}\delta_{\nu\rho} - \delta_{\mu\nu}\delta_{\lambda\rho}\right), \tag{80}$$

$$D^{(2)}_{\lambda\nu,\mu\rho}(x) = \frac{e\eta}{4\pi^2 x^2}\left\{\left[\frac{K_1}{|x|} + \frac{m}{2}(K_0 + K_2)\right]\left(\delta_{\lambda\mu}\delta_{\nu\rho} - \delta_{\mu\nu}\delta_{\lambda\rho}\right) + \right.$$

$$+\frac{1}{2|x|}\left[3\left(\frac{m^2}{4} + \frac{1}{x^2}\right)K_1 + \frac{3m}{2|x|}(K_0 + K_2) + \frac{m^2}{4}K_3\right]\cdot$$

$$\left.\cdot\left(\delta_{\lambda\rho}x_\mu x_\nu + \delta_{\mu\nu}x_\lambda x_\rho - \delta_{\mu\lambda}x_\nu x_\rho - \delta_{\nu\rho}x_\mu x_\lambda\right)\right\}. \tag{81}$$

¿From now on, $K_i \equiv K_i(m|x|)$, $i = 0, 1, 2, 3$, stand for the modified Bessel functions, and $m \equiv \frac{\eta}{e}$ is the mass of the dual gauge boson generated by the Higgs mechanism. Due to the Stokes theorem, the term

$$\int\limits_\Sigma d\sigma_{\lambda\nu}(x) \int\limits_\Sigma d\sigma_{\mu\rho}(y) D^{(2)}_{\lambda\nu,\mu\rho}(x-y)$$

can be rewritten as a boundary one (see Appendix 7.3), which finally leads to the following representation for the partition function of extended DAHM in the London limit

$$\mathcal{Z} = \exp\left[-\frac{e\eta}{2}\oint\limits_C dx_\mu \oint\limits_C dy_\mu \frac{K_1(m|x-y|)}{|x-y|}\right]\cdot$$

$$\cdot\int Dx_\mu(\xi)\exp\left[-\frac{\eta^3}{4e}\int\limits_\Sigma d\sigma_{\mu\nu}(x)\int\limits_\Sigma d\sigma_{\mu\nu}(y)\frac{K_1(m|x-y|)}{|x-y|}\right]. \tag{82}$$

The first exponent on the R.H.S. of Eq. (82) leads to the short-range Yukawa potential, $V_{\text{Yuk.}}(R) \propto \mathrm{e}^{-mR}/R$. Notice that since quarks and antiquarks were from the very beginning considered as classical particles, the external contour C explicitly enters the final result. Would one consider them on the quantum level, Eq. (82) must be supplied by a certain prescription of the summation over the contours (*cf.* Eq. (74)).

The integral over string world-sheets on the R.H.S. of Eq. (82) is the essence of the string representation of the partition function. The dominant contribution to it is obviously brought about by the surface $\Sigma_{\text{min.}}$. This yields a new nonlocal string effective action

$$S_{\text{eff.}} = \frac{\eta^3}{4e}\int\limits_{\Sigma_{\text{min.}}} d\sigma_{\mu\nu}(x)\int\limits_{\Sigma_{\text{min.}}} d\sigma_{\mu\nu}(x')\frac{K_1(m|x-x'|)}{|x-x'|}$$

and a rising confining quark-antiquark potential (8).

Comparing now Eq. (41) with Eq. (82), we obtain, by virtue of Eqs. (44) and (45), the following values of the string tension of the Nambu-Goto term and the inverse bare coupling constant of the rigidity term

$$\sigma = \pi\eta^2 K_0(c) \simeq \pi\eta^2 \ln\frac{2}{\gamma c}, \tag{83}$$

$$\frac{1}{\alpha_0} = -\frac{\pi e^2}{8}.$$

Here, $\gamma = 1.781...$ is the Euler's constant, and c stands for a characteristic small dimensionless parameter. In the London limit, we get $c \sim \frac{m}{M}$, where $M = 2\sqrt{2\lambda}\eta$ is the magnetic Cooper pair mass following from Eq. (68). This mass plays the role of the UV momentum cutoff somehow analogous to the inverse lattice spacing $1/a$ (*cf.* Eq. (9)). Notice, that the logarithmic divergency of the string tension in the Ginzburg-Landau model and AHM is a well known result, which can be obtained directly from the definition of this quantity as a free energy per unit length of the string (see e.g. [123]). The physical origin of this result is that on the string world-sheet (or in the centre of vortex in 3D), the condensate is destroyed, and dual gauge bosons are massless. This effect is sometimes taken into account explicitly by an extraction of the divergent part out of the full string tension (83) [121, 124]. Notice also, that as it has been shown in Ref. [124], accounting for the finite thickness of the string in the vicinity of the London limit leads to the replacement of the Yukawa interaction in the quark-antiquark potential by the Coulomb one. The effect of the finite thickness results also in the appearance of a new term in the potential proportional to $\left(e^{-R/R_0} - 1\right)$, where R_0 is a certain distance.

Clearly, both the string tension and the inverse bare coupling constant of the rigidity term are nonanalytic in g_m, which means that these quantities are essentially nonperturbative similarly to the QCD case (*cf.* Eq. (12)). Notice also, that the finite temperature behaviour of the magnetic Higgs field v.e.v. [42], $\eta(T) \propto \sqrt{1 - \frac{T}{T_c}}$ with T_c standing for the critical temperature, obviously governes the corresponding behaviour of the string tension (83) and, in particular, the deconfinement phase transition.

String Representation for the Generating Functional of Field Strength Correlators
In this Paragraph, we shall derive the string representation for the generating functional of field strength correlators in the London limit of extended DAHM. From this we shall then obtain an expression for the bilocal correlator and compare it with the one in QCD. Our starting expression for the generating functional reads as follows

$$\mathcal{Z}\left[S_{\alpha\beta}\right] = \int DB_\mu D\theta^{\text{sing.}} D\theta^{\text{reg.}} \exp\left\{-\int d^4x \left[\frac{1}{4}\left(F_{\mu\nu} - F_{\mu\nu}^E\right)^2 + \right.\right.$$
$$\left.\left. + \frac{\eta^2}{2}\left(\partial_\mu\theta - 2g_m B_\mu\right)^2 + iS_{\mu\nu}\tilde{F}_{\mu\nu}\right]\right\}, \tag{84}$$

where $S_{\mu\nu}$ is a source of the field strength tensor $\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\lambda\rho}F_{\lambda\rho}$, which obviously corresponds to the field strength of the usual gauge field A_μ in the AHM. Performing the same transformations which led from Eq. (75) to Eq. (79), we obtain from Eq. (84)

$$\mathcal{Z}\left[S_{\alpha\beta}\right] = \exp\left(-\int d^4x S_{\mu\nu}^2\right) \int Dx_\mu(\xi) \exp\left(-4\pi i e \int d^4x S_{\mu\nu}\Sigma_{\mu\nu}\right) \cdot$$
$$\cdot \exp\left\{-\int d^4x d^4y \left(\pi\Sigma_{\lambda\nu}(x) - \frac{1}{e}S_{\lambda\nu}(x)\right) D_{\lambda\nu,\mu\rho}(x-y) \left(\pi\Sigma_{\mu\rho}(y) - \frac{1}{e}S_{\mu\rho}(y)\right)\right\}. \tag{85}$$

Let us now derive from the general form (85) of the generating functional the expression for the bilocal correlator of the field strength tensors. The result reads

$$\left\langle\tilde F_{\lambda\nu}(x)\tilde F_{\mu\rho}(y)\right\rangle=-\frac{1}{\mathcal{Z}[0]}\frac{\delta^2\mathcal{Z}\left[S_{\alpha\beta}\right]}{\delta S_{\lambda\nu}(x)\delta S_{\mu\rho}(y)}\bigg|_{S_{\alpha\beta}=0}=\left(\delta_{\lambda\mu}\delta_{\nu\rho}-\delta_{\lambda\rho}\delta_{\mu\nu}\right)\delta(x-y)+\frac{2}{e^2}D_{\lambda\nu,\mu\rho}(x-y)+$$

$$+4\pi^2\left\langle\left(2e\Sigma_{\lambda\nu}(x)+\frac{i}{e}\int\limits_{\Sigma}d\sigma_{\alpha\beta}(z)D_{\alpha\beta,\lambda\nu}(z-x)\right)\left(2e\Sigma_{\mu\rho}(y)+\frac{i}{e}\int\limits_{\Sigma}d\sigma_{\gamma\zeta}(u)D_{\gamma\zeta,\mu\rho}(u-y)\right)\right\rangle_{x_\mu(\xi)}, \tag{86}$$

where

$$\langle\ldots\rangle_{x_\mu(\xi)}\equiv\frac{\int Dx_\mu(\xi)(\ldots)\exp\left[-\pi^2\int\limits_\Sigma d\sigma_{\alpha\beta}(z)\int\limits_\Sigma d\sigma_{\gamma\zeta}(u)D_{\alpha\beta,\gamma\zeta}(z-u)\right]}{\int Dx_\mu(\xi)\exp\left[-\pi^2\int\limits_\Sigma d\sigma_{\alpha\beta}(z)\int\limits_\Sigma d\sigma_{\gamma\zeta}(u)D_{\alpha\beta,\gamma\zeta}(z-u)\right]}$$

is the average over the string world-sheets, and the term with the δ-function on the R.H.S. of Eq. (86) corresponds to the free contribution to the correlator. In what follows, we shall consider the case $x\neq y$, so that this local term with the δ-function drops out.

One can now see that in the sense of the cumulant expansion (see Eq. (30)), the second term on the R.H.S. of Eq. (86) corresponds to the bilocal cumulant of the field strength tensors $\left\langle\left\langle\tilde F_{\lambda\nu}(x)\tilde F_{\mu\rho}(y)\right\rangle\right\rangle$. Following the SVM, let us parametrize it by the two Lorentz structures similarly to Eq. (36)

$$\left\langle\left\langle\tilde F_{\lambda\nu}(x)\tilde F_{\mu\rho}(0)\right\rangle\right\rangle=\left(\delta_{\lambda\mu}\delta_{\nu\rho}-\delta_{\lambda\rho}\delta_{\nu\mu}\right)\mathcal{D}\left(x^2\right)+$$

$$+\frac12\left[\partial_\lambda\left(x_\mu\delta_{\nu\rho}-x_\rho\delta_{\nu\mu}\right)+\partial_\nu\left(x_\rho\delta_{\lambda\mu}-x_\mu\delta_{\lambda\rho}\right)\right]\mathcal{D}_1\left(x^2\right). \tag{87}$$

Then by virtue of Eqs. (80) and (81), we arrive at the following expressions for the functions $\mathcal{D}$ and $\mathcal{D}_1$

$$\mathcal{D}\left(x^2\right)=\frac{m^3}{4\pi^2}\frac{K_1}{|x|}, \tag{88}$$

and

$$\mathcal{D}_1\left(x^2\right)=\frac{m}{2\pi^2x^2}\left[\frac{K_1}{|x|}+\frac{m}{2}\left(K_0+K_2\right)\right]. \tag{89}$$

We see, that in the limit $|x|\gg\frac{1}{m}$, the asymptotic behaviour of the coefficient functions (88) and (89) is given by

$$\mathcal{D}\longrightarrow\frac{m^4}{4\sqrt2\pi^{\frac32}}\frac{\mathrm{e}^{-m|x|}}{(m|x|)^{\frac32}}, \tag{90}$$

and

$$\mathcal{D}_1 \longrightarrow \frac{m^4}{2\sqrt{2}\pi^{\frac{3}{2}}} \frac{\mathrm{e}^{-m|x|}}{(m\,|x|)^{\frac{5}{2}}}. \tag{91}$$

For bookkeeping purposes, let us also list here the asymptotic behaviours of the functions (88) and (89) in the opposite case, $|x| \ll \frac{1}{m}$. Those read

$$\mathcal{D} \longrightarrow \frac{m^2}{4\pi^2 x^2}, \tag{92}$$

and

$$\mathcal{D}_1 \longrightarrow \frac{1}{\pi^2\,|x|^4}. \tag{93}$$

One can now see that according to the lattice data [64, 65], the asymptotic behaviours (90) and (91) are very similar to the large distance ones of the nonperturbative parts of the functions D and D_1 (*cf.* Eq. (40)). In particular, both functions decrease exponentially, and the function $\mathcal{D}$ is much larger then the function $\mathcal{D}_1$ due to the preexponential power-like behaviour. We also see that the dual gauge boson mass m indeed corresponds to the inverse correlation length of the vacuum T_g^{-1}. In particular, in the string limit of QCD, when $T_g \to 0$ while the value of the string tension is kept fixed, m corresponds to $\sqrt{\frac{D(0)}{\sigma}}$.

Moreover, the short distance asymptotic behaviours (92) and (93) also parallel the results obtained within the SVM of QCD in the lowest order of perturbation theory. Namely, at such distances the function D_1 to the lowest order also behaves as $\frac{1}{|x|^4}$ (see Eq. (39)) and is much larger than the function D to the same order. However, it should be realized that the effects of asymptotic freedom (which is the most important UV feature of QCD, distinguishing it from all the Abelian gauge theories) obviously cannot be obtained from DAHM, which is essentially a Gaussian theory.

In conclusion of this Paragraph, the found similarity in the large distance asymptotic behaviours of the functions $\mathcal{D}$ and $\mathcal{D}_1$, which parametrize the bilocal cumulant of the field strength tensors in DAHM and the gauge-invariant cumulant in QCD, supports the original conjecture by 't Hooft and Mandelstam concerning the dual Meissner nature of confinement.

String Representation for the Generating Functional of the Monopole Current Correlators In this Paragraph, we shall present the string representation for the generating functional of the monopole current correlators in the London limit of extended DAHM. Such a representation can be derived by virtue of the same path-integral duality transformation studied above. After that, we shall get from the obtained generating functional the correlator of two monopole currents and by making use of it rederive via the equations of motion the coefficient function $\mathcal{D}$ in the bilocal cumulant of the field strength tensors.

In the London limit, the generating functional of the monopole currents reads

$$\hat{\mathcal{Z}}\left[J_\mu\right] = \int DB_\mu D\theta^{\mathrm{sing.}} D\theta^{\mathrm{reg.}} \exp\left\{-\int d^4x \left[\frac{1}{4}\left(F_{\mu\nu} - F_{\mu\nu}^E\right)^2 + \frac{\eta^2}{2}\left(\partial_\mu\theta - 2g_m B_\mu\right)^2 + J_\mu j_\mu\right]\right\},$$

where $j_\mu \equiv -2g_m\eta^2(\partial_\mu\theta - 2g_m B_\mu)$ is just the magnetic monopole current [19].

Performing the duality transformation, we get the following string representation for $\hat{\mathcal{Z}}[J_\mu]$

$$\hat{\mathcal{Z}}[J_\mu] = \exp\left[\frac{m^2}{2}\int d^4x J_\mu^2(x)\right]\int Dx_\mu(\xi)\exp\left[-\pi^2\int\limits_\Sigma d\sigma_{\alpha\beta}(z)\int\limits_\Sigma d\sigma_{\gamma\zeta}(u)D_{\alpha\beta,\gamma\zeta}(z-u)\right]\cdot$$

$$\cdot\exp\left\{-2g_m\varepsilon_{\lambda\nu\alpha\beta}\int d^4xd^4y\left[\frac{g_m}{2}\varepsilon_{\mu\rho\gamma\delta}\left(\frac{\partial^2}{\partial x_\alpha\partial y_\gamma}D_{\lambda\nu,\mu\rho}(x-y)\right)J_\beta(x)J_\delta(y)+\right.\right.$$

$$\left.\left.+\pi\Sigma_{\mu\rho}(y)\left(\frac{\partial}{\partial x_\alpha}D_{\lambda\nu,\mu\rho}(x-y)\right)J_\beta(x)\right]\right\}. \tag{94}$$

Varying now Eq. (94) twice w.r.t. J_μ, setting then J_μ equal to zero, and dividing the result by $\hat{\mathcal{Z}}[0]$, we arrive at the following expression for the correlator of two monopole currents

$$\langle j_\beta(x)j_\sigma(y)\rangle = m^2\delta_{\beta\sigma}\delta(x-y) + 4g_m^2\varepsilon_{\lambda\nu\alpha\beta}\varepsilon_{\mu\rho\gamma\sigma}\left[-\frac{1}{2}\frac{\partial^2}{\partial x_\alpha\partial y_\gamma}D_{\lambda\nu,\mu\rho}(x-y)+\right.$$

$$\left.+\pi^2\left\langle\int\limits_\Sigma d\sigma_{\delta\zeta}(z)\int\limits_\Sigma d\sigma_{\chi\varphi}(u)\left(\frac{\partial}{\partial x_\alpha}D_{\lambda\nu,\delta\zeta}(x-z)\right)\left(\frac{\partial}{\partial y_\gamma}D_{\mu\rho,\chi\varphi}(y-u)\right)\right\rangle_{x_\mu(\xi)}\right]. \tag{95}$$

It is straightforward to see that the contribution of the term (81) to the R.H.S. of Eq. (95) vanishes, whereas the second term in the square brackets on the R.H.S. of Eq. (95) again corresponds to the non-cumulant (disconnected) type contribution to the correlator. Let us next make use of the following equation [19]

$$\langle j_\beta(x)j_\sigma(y)\rangle = \left(\frac{\partial^2}{\partial x_\mu\partial y_\mu}\delta_{\beta\sigma} - \frac{\partial^2}{\partial x_\beta\partial y_\sigma}\right)\mathcal{D}\left((x-y)^2\right), \tag{96}$$

which follows from Eq. (87) due to equations of motion. Then, neglecting the local term on the R.H.S. of Eq. (95), and substituting this equation together with Eq. (80) into Eq. (96), we recover the expression for the function $\mathcal{D}$ given by Eq. (88).

Notice in conclusion, that only the function $\mathcal{D}$ can be obtained from the correlator (95) due to the independence of the latter of the function $\mathcal{D}_1$.

3.1.2 Dual Abelian Higgs Model beyond the London Limit

According to the previous Subsection, in the London limit of DAHM, the bilocal correlator is much larger than the higher ones, which confirms the validity of the bilocal approximation to SVM in this limit. This can be seen from Eq. (85) by varying it more than twice w.r.t. $S_{\alpha\beta}$ and setting then $S_{\alpha\beta}$ equal to zero. Indeed, let us again consider the limit of large distances between any two points x_i and x_j, in which the fields in a certain order correlator are defined, i.e. $|x_i - x_j| \gg \frac{1}{m}$. Then, the next-to-leading order correlator (after the bilocal one) will be the quartic one

$$\left\langle\tilde{F}_{\mu_1\nu_1}(x_1)\tilde{F}_{\mu_2\nu_2}(x_2)\tilde{F}_{\mu_3\nu_3}(x_3)\tilde{F}_{\mu_4\nu_4}(x_4)\right\rangle,$$

[19] Rigorously speaking, this is a current of the monopole Cooper pairs.

the dominant contribution to which comes about from the four-fold variation of the following term in the expansion of the generating functional (85)

$$\frac{1}{2e^4}\int d^4xd^4yd^4zd^4uS_{\mu\nu}(x)D_{\mu\nu,\lambda\rho}(x-y)S_{\lambda\rho}(y)S_{\alpha\beta}(z)D_{\alpha\beta,\gamma\zeta}(z-u)S_{\gamma\zeta}(u). \tag{97}$$

The $4! = 24$ terms containing all possible combinations of indices and arguments, which one gets during this variation, are obvious, and we shall not list them here for shortness. Since at the distances under consideration $D_{\mu\nu,\lambda\rho}(x) \sim \frac{\eta^4}{e^2}$, each of these terms has the order of magnitude m^8. This means that the bilocal correlator, which has the order of magnitude m^4, is dominant in the London limit.

The aim of this Subsection is to address the question in which case the higher correlators become relevant. In the interpretation of this topic, we shall mainly follow Ref. [117]. Since in what follows we shall be interested only in the contributions to cumulants, rather than to correlators, our aim will be to pick up only the terms which do not contain the averages over string world-sheets (*cf.* Eq. (86)). Therefore, let us consider unextended DAHM, where the string world-sheets Σ are closed, and $\partial_\mu \Sigma_{\mu\nu} = 0$.

Let us start with considering DAHM in the vicinity of the London limit. To this end, we shall expand the radial part of the magnetic Higgs field, $|\Phi(x)|$, in Eq. (68) as $|\Phi(x)| = \eta + \tau\phi(x)$, where $\tau \equiv \frac{1}{\lambda} \to 0$, and $\phi(x)$ is an arbitrary quantum fluctuation. Then neglecting the (trivial) Jacobian emerging during the change of the integration variables, $|\Phi(x)| \to \phi(x)$, which will be eventually cancelled after division of the final expression for the field correlator by $\mathcal{Z}[0]$, we arrive at the new expression for the generating functional (84) (without external quark fields $F^E_{\mu\nu}$)

$$\mathcal{Z}\left[S_{\alpha\beta}\right] = \int DB_\mu D\theta^{\text{sing.}} D\theta^{\text{reg.}} D\phi \exp\left\{-\int d^4x\left[\frac{1}{4}F^2_{\mu\nu} + \frac{\tau^2}{2}\left(\partial_\mu\phi\right)^2 + 4\tau\eta^2\phi^2 + \right.\right.$$
$$\left.\left. + \eta\left(\frac{\eta}{2} + \tau\phi\right)\left(\partial_\mu\theta - 2g_m B_\mu\right)^2 + iS_{\mu\nu}\tilde{F}_{\mu\nu}\right]\right\}. \tag{98}$$

Following the same steps, which led from Eq. (75) to Eq. (79), we get from Eq. (98) an additional weight factor in the integral over string world-sheets standing in Eq. (85). This weight factor emerges due to the ϕ-integration and reads

$$\exp\left\{\frac{M}{288\pi^2\eta^6}\int d^4xd^4y H^{\text{s.p.}\,2}_{\mu\nu\lambda}(x)\frac{K_1(M|x-y|)}{|x-y|}H^{\text{s.p.}\,2}_{\alpha\beta\gamma}(y)\right\}. \tag{99}$$

Here, $H^{\text{s.p.}}_{\mu\nu\lambda}$ stands for the strength tensor of the saddle-point value of the field $h_{\mu\nu}$ following from the action (*cf.* Eq. (78))

$$S = \frac{1}{12\eta^2}H^2_{\mu\nu\lambda} + \frac{1}{4e^2}h^2_{\mu\nu} - ih_{\mu\nu}\left(\pi\Sigma_{\mu\nu} - \frac{1}{e}S_{\mu\nu}\right).$$

Taking into account that $\partial_\mu\Sigma_{\mu\nu} = 0$, we get the following result for this saddle-point value

$$h^{\text{s.p.}}_{\lambda\nu}(x) = \frac{im}{2\pi^2}\int d^4y\frac{K_1(m|x-y|)}{|x-y|}\left\{\eta^2\left[\pi\Sigma_{\lambda\nu}(y) - \frac{1}{e}S_{\lambda\nu}(y)\right] + \right.$$
$$\left. + e\partial_\rho\left[\partial_\nu S_{\lambda\rho}(y) - \partial_\lambda S_{\nu\rho}(y)\right]\right\}. \tag{100}$$

Let us prove that the terms with the derivatives of $S_{\mu\nu}$ on the R.H.S. of Eq. (100) yield zero. Due to the Hodge decomposition theorem, $S_{\mu\nu}$ can be always represented in the form $S_{\mu\nu} = \partial_\mu I_\nu - \partial_\nu I_\mu + \partial_\alpha L_{\alpha\mu\nu}$, where I_μ and $L_{\alpha\mu\nu}$ stand for some vector and an antisymmetric rank-3 tensor, respectively. In the product $S_{\mu\nu}\tilde{F}_{\mu\nu}$, the contribution of the vector I_μ obviously vanishes due to the partial integration, and we are left with the representation of $S_{\mu\nu}$ of the form $S_{\mu\nu} = \partial_\alpha L_{\alpha\mu\nu}$. Substituting it into Eq. (100) we see that the terms with the derivatives of $S_{\mu\nu}$ on the R.H.S. of this equation indeed vanish.

Finally, upon substitution of the rest of Eq. (100) into Eq. (99), we arrive at the following additional term in the expansion of the generating functional, whose contribution to the quartic cumulant is the leading one

$$\frac{Mm^2}{32\pi^2 e^2}\int d^4x d^4y d^4u d^4v d^4z d^4w \frac{K_1(M|x-y|)}{|x-y|}\mathcal{D}_1\left((x-u)^2\right)\mathcal{D}_1\left((x-v)^2\right)\mathcal{D}_1\left((y-z)^2\right)\cdot$$
$$\cdot\mathcal{D}_1\left((y-w)^2\right)(x-u)_\mu(y-z)_\alpha\Big[(x-v)_\mu(y-w)_\alpha S_{\nu\lambda}(u)S_{\nu\lambda}(v)S_{\beta\gamma}(z)S_{\beta\gamma}(w)+$$
$$+4(x-v)_\lambda(y-w)_\gamma S_{\nu\lambda}(u)S_{\mu\nu}(v)S_{\beta\gamma}(z)S_{\alpha\beta}(w)+$$
$$+4(x-v)_\mu(y-w)_\gamma S_{\nu\lambda}(u)S_{\nu\lambda}(v)S_{\beta\gamma}(z)S_{\alpha\beta}(w)\Big]. \quad (101)$$

In Eq. (101), $\mathcal{D}_1$ stands for the coefficient function (89) entering the bilocal cumulant in the London limit. Even without explicit writing down the $3\cdot 4! = 72$ terms following from Eq. (101) after its four-fold variation, we see that the resulting dominant contribution to the quartic cumulant distinguishes from that of the London limit, which follows from Eq. (97). The most crucial difference is due to the presence of the Higgs boson exchange in Eq. (101) (expressed by the propagator factor $K_1(M|x-y|)/|x-y|$), which is absent in Eq. (97).

The other outcome of Eq. (101) is that the leading $1/\lambda$-correction to the quartic cumulant can be *completely* described in terms of the coefficient function (89). This is an example of equation relating correlators of various orders to each other. It is worth noting that in the (non-Abelian-projected) $SU(2)$-gluodynamics, there exists an analogous albeit more nontrivial relation connecting the bilocal correlator with the threelocal one [67, 19], which reads

$$\left.\frac{dD(x)}{dx^2}\right|_{x=0} = \frac{g}{8}\varepsilon^{abc}\left\langle F^a_{\alpha\beta}(0)F^b_{\beta\gamma}(0)F^c_{\gamma\alpha}(0)\right\rangle. \quad (102)$$

This equation means that would the threelocal correlator vanish, the function D vanishes too. Indeed, in this case $D = \text{const}$ everywhere in space-time, and since $D(|x|\to\infty)\to 0$, this constant is equal to zero. Therefore, as it has been first mentioned in Ref. [74], the nonvanishing threelocal correlator is a necessary condition for the consistency of SVM. In particular, this means that the bilocal approximation is not fully self-consistent and should be extended by the demand that the threelocal correlator is nonvanishing as well. The resulting ensemble of fields, possessing in addition the property of factorization of higher correlators into the products of bilocal and threelocal ones, is thus fully self-consistent, and has been called in Ref. [74] "minimally extended Gaussian ensemble". In this respect, it looks very desirable to find a certain relation between the bilocal and threelocal correlators in DAHM as well. Such a relation turned out to take place in the London limit of DAHM with an additional term which describes an interaction of an axion with two dual gauge bosons [125]. However, such a term is known to vanish in the real world, since it violates CP-invariance, which makes such a model of only an academic interest.

3.1.3 Dynamical Chiral Symmetry Breaking within the Stochastic Vacuum Model

The QCD Lagrangian with N_f massless flavours is known to be invariant under the global symmetry transformations, which are the $U(N_f)\times U(N_f)$ independent rotations of left- and right-handed quark fields. This symmetry is referred to as chiral symmetry. The above mentioned rotations of the two-component Weyl spinors are equivalent to the independent vector and axial $U(N_f)$ rotations of the full four-component Dirac spinors, under which the QCD Lagrangian remains invariant as well. At the same time, the axial transformations mixes states with different P-parities. Therefore, we conclude that if the chiral symmetry remains unbroken, one would observe parity degeneracy of all the states, whose other quantum numbers are the same. The observed splittings between such states occur, however, to be too large to be explained by the small bare or current quark masses. Namely, this splitting is of the order of hundreds MeV, whereas the current masses of light u- and d-quarks are of the order of a few MeV [20]. This observation tells us that the chiral symmetry of the QCD Lagrangian is broken down spontaneously. This phenomenon of the *spontaneous chiral symmetry breaking (SCSB)* naturally leads to the appearance of light pseudoscalar Goldstone bosons, whose role is played by pions, which are indeed the lightest of all the hadrons. Besides confinement, the explanation of SCSB is known to be the other most fundamental problem of the modern theory of strong interactions.

The order parameter of SCSB is the so-called *quark condensate* (else called chiral condensate) $\left\langle\bar\psi\psi\right\rangle \simeq -(250\,\mathrm{MeV})^3$. This is nothing else, but the quark Green function taken at the origin or a closed quark loop in the momentum representation. Clearly, if the quarks are at the beginning massless (i.e. only the kinetic term is present in its propagator), the phenomenon of SCSB leads to the appearance of a non-zero *dynamical* quark mass, which in general is momentum-dependent. At zero momentum, the value of the dynamical quark mass is of the order of 350-400 MeV, which is just the value of the so-called constituent quark mass.

Let us now proceed to the quantitative description of SCSB. Integrating the quarks out of the QCD Lagrangian obviously yields

$$\mathcal{Z}_{\mathrm{QCD}} = \prod_{f=1}^{N_f}\left\langle \det\left(i\hat D^{\mathrm{fund.}} + im_f\right)\right\rangle_{A_\mu^a},$$

where $\hat D^{\mathrm{fund.}} \equiv \gamma_\mu D_\mu^{\mathrm{fund.}}$ with $D_\mu^{\mathrm{fund.}} = \partial_\mu - igA_\mu$ standing for the covariant derivative in the fundamental representation. As it follows from its definition, the chiral condensate for a given flavour f then has the form

$$\left\langle\bar\psi_f\psi_f\right\rangle = -\frac{1}{V}\left(\frac{\partial}{\partial m_f}\ln\mathcal{Z}_{\mathrm{QCD}}\right)_{m_f\to 0}, \tag{103}$$

where V stands for the four-volume of observation. In what follows, let us for simplicity restrict ourselves to the case of one flavour only. Next, let Ψ_n be an eigenfunction of the Dirac operator, corresponding to a nonvanishing eigenvalue λ_n, $i\hat D^{\mathrm{fund.}}\Psi_n = \lambda_n\Psi_n$. Then, since γ_5 anticommutes

[20]The current mass of the s-quark, which is around 150 MeV, is still smaller than the typical splitting values at least in a factor of three.

with $i\hat{D}^{\text{fund.}}$, the function $\Psi_{n'} = \gamma_5 \Psi_n$ is also an eigenfunction of the Dirac operator, corresponding to the eigenvalue $\lambda_{n'} = -\lambda_n$. Due to this observation, the determinant can be rewritten as follows

$$\det\left(i\hat{D}^{\text{fund.}} + im\right) = \prod_n (\lambda_n + im) \sim \sqrt{\prod_n (\lambda_n^2 + m^2)} = \exp\left[\frac{1}{2}\sum_n \ln\left(\lambda_n^2 + m^2\right)\right] =$$

$$= \exp\left[\frac{1}{2}\int\limits_{-\infty}^{+\infty} d\lambda \nu(\lambda) \ln\left(\lambda^2 + m^2\right)\right],$$

where $\nu(\lambda) \equiv \sum\limits_n \delta(\lambda - \lambda_n)$ stands for the so-called spectral density of the Dirac operator. By virtue of Eq. (103), we thus obtain for the chiral condensate

$$\left\langle \bar{\psi}\psi \right\rangle = -\frac{1}{V}\int\limits_{-\infty}^{+\infty} d\lambda \left\langle \nu(\lambda)\right\rangle \frac{m}{\lambda^2 + m^2}\Bigg|_{m\to 0}, \tag{104}$$

where $\langle \nu(\lambda)\rangle$ denotes the spectral density averaged over the full QCD partition function, including the weight given by the determinant itself. The latter can be disregarded in the quenched approximation (justified at large N_c), in which one neglects the backward influence of quarks to the dynamics.

For a finite-volume system, the R.H.S. of Eq. (104) obviously vanishes. However, for the case when the volume increases, the spectrum becomes continuous, and we should use the formula

$$\lim_{m\to 0} \frac{m}{\lambda^2 + m^2} = \text{sign}\,(m)\pi\delta(\lambda).$$

Taking this into account, one finally gets the celebrated Banks-Casher relation [126]

$$\left\langle \bar{\psi}\psi \right\rangle = -\frac{1}{V}\text{sign}\,(m)\pi \left\langle \nu(0)\right\rangle. \tag{105}$$

Thus, we see that the chiral condensate is proportional to the averaged spectral density of the Dirac operator at the origin. Notice, that the sign function in Eq. (105) means that at small m, the QCD partition function depends on m nonanalytically, which is typical for the situation when the symmetry is spontaneously broken.

Up to now, there exist several microscopic models of SCSB in QCD [19]. The most elaborated and therefore popular of them is the one due to instantons [23] [21]. It is based on the observation that in the background field of one (anti)instanton, the Dirac operator has an exact zero mode [128]. The estimate of $\langle \nu(0)\rangle$ for the case of a gas of I's and $\bar{I}$'s has been performed in Ref. [23] directly by calculating the overlap of the zero modes and has the form $\langle \nu(0)\rangle \sim \frac{V}{L^2\rho}$ [22]. This result then has been rederived in Ref. [129] by solving a closed equation for the averaged quark propagator as an expansion in powers of the so-called packing fraction parameter $\frac{N\rho^4}{VN_c}$, where N stands for the total number of I's and $\bar{I}$'s. By virtue of Eq. (105), the chiral condensate reads $\left\langle \bar{\psi}\psi \right\rangle \sim -\frac{1}{L^2\rho}$.

[21] Recently in Ref. [127], there has been proposed a model based on the so-called dyonic gas. The advantage of this model w.r.t. the instanton one is that, as it has been argued there, it yields not only SCSB, but also the confinement property.

[22] Here we adopt the same notations as the ones at the end of Subsection 2.1, where however ρ and L stand not for fixed, but for the averaged size and separation in the $I - \bar{I}$ ensemble, respectively.

Notice, that this result can be also derived if we first average over the $I - \bar{I}$ gas and only after that calculate the chiral condensate by making use of the so-obtained effective theory [24]. In this case, quark interactions are due to the scattering of two or more (anti)quarks over the same I or $\bar{I}$, i.e. there arise four- (or more) fermion interaction terms. The range of such interaction, which is usually referred to as 't Hooft interaction [128], is obviously of the order of ρ. In the case of two flavours, this interaction is the four-fermion one and yields an effective theory similar to the well known Nambu-Jona-Lasinio model [30, 34].

There have been also derived other types of nonlocal NJL models in an approximate way from QCD by making use of path-integral techniques in bilocal fields. In particular, exact bosonization of 2D QCD has been performed in Ref. [130], and several attempts to bosonization of 4D QCD have been proposed [131, 35]. The general strategy of these approaches includes several steps. Firstly, one casts the effective four-fermion interaction (which can be either local or nonlocal with a certain interaction kernel) into the form of a (non)local Yukawa interaction by introducing a set of collective bosonic fields, which is just the essence of the bosonization procedure. Secondly, one can integrate over the fermions and derive an effective action in terms of these collective fields. Next, the saddle-point of this effective action for large number of colours yields the so-called Schwinger-Dyson (or gap) equation, which determines the dynamical mass, responsible for SCSB. After that, an expansion of the resulting meson action in small field fluctuations around this stationary solution yields the so-called Bethe-Salpeter equation, which determines the spectrum of meson excitations. This finally leads to a description of the chiral sector of QCD in terms of Effective Chiral Hadron Lagrangians [32, 34, 38] containing higher order derivative terms with fixed structure constants L_i's. The form of these Lagrangians agrees with that of the phenomenological hadron Lagrangians postulated at the end of the Sixties on the basis of pure group-theoretical arguments when considering nonlinear realizations of chiral symmetry [14]. However, contrary to those Lagrangians, the new ones are not obtained by symmetry principles alone, but derived from an underlying microscopic quark (diquark) picture, which enables one to estimate masses and coupling constants of composite hadrons.

However it is worth noting, that though the QCD-motivated NJL type models model well the SCSB phenomenon in QCD and describe with a good accuracy the hadron spectrum and coupling constants, they do not reproduce the confinement property. Recently, an attempt of derivation of an effective Lagrangian for a light quark propagating in the confining QCD vacuum, which could account simultaneously for both phenomena, has been done [49]. Within this approach, one starts with an expression for the Green function of a system consisting of an infinitely heavy quark and light antiquark. The propagation of a light (anti)quark alone in the QCD vacuum then leads to SCSB, whereas the presence of a heavy quark enables one to take into account the QCD string, joining both objects, and thus to incorporate confinement. Averaging over gluonic fields by making use of the cumulant expansion in the bilocal approximation, it is then straightforward to derive an effective NJL-type Lagrangian for the light quark with a certain nonlocal kernel. However, the form of this kernel is now determined via the coefficient function $D(x)$ (*cf.* Eq. (36)), which, as it has been demonstrated above, yields the string picture (*cf.* Eqs. (43), (44), (45)) i.e. is responsible for confinement. Such an interpolation between confinement and SCSB within this approach leads to a relation between chiral and gluonic condensates, which reads

$$\left\langle \bar{\psi}\psi \right\rangle \propto -D(0)T_g. \tag{106}$$

Following Ref. [49], let us briefly demonstrate how this relation can be obtained in the Abelian(-

projected) theories. To this end, for simplicity consider the case of Abelian-projected $SU(2)$-gluodynamics, studied above. (Our analysis can be straightforwardly extended to the Abelian-projected $SU(3)$-gluodynamics, which will be investigated in the next Subsection.) To proceed with, let us assume for a while that a monopole has an effective infinitesimal size b (which is known to become finite for the 't Hooft-Polyakov monopole). Then, every monopole with the world line of the length T produces T/b quasizero modes [132]. The total number of quasizero modes produced by all the monopoles from the 3D volume $V^{(3)}$ in the interval of modes $\Delta\lambda$ thus has the form $\langle\nu(\lambda)\rangle\,\Delta\lambda = \frac{T}{b}V^{(3)}n_3$, where n_3 is the 3D density of monopoles. Since $V = TV^{(3)}$, one gets an estimate

$$\frac{\langle\nu(0)\rangle}{V} \sim \frac{n_3}{\Delta\lambda b}. \tag{107}$$

Since the quasizero modes of all the monopoles are mixed due to the interaction, the denominator of this relation is of the order of unity. On the other hand, the 3D density of monopoles can be estimated via Eq. (96) as follows

$$n_3 \sim \int d^3x \left\langle j_\mu\left(\vec{x}, x_4\right) j_\mu\left(\vec{0}, x_4\right)\right\rangle \sim \mathcal{D}(0)T_g. \tag{108}$$

Here, as it has been argued after Eq. (93), the correlation length of the vacuum, T_g, for the Abelian-projected $SU(2)$-gluodynamics is equal to the inverse mass of the dual gauge boson, m^{-1}. Notice also, that in the derivation of the first estimate in Eq. (108), one uses the following observation [49]. The correlator $\langle j_\mu(x)j_\mu(0)\rangle$ estimates the probability of finding a monopole at the 4D point x, if there is one at the origin. Integrating over d^3x, one finds a probability of having a monopole at the origin, while another one is anywhere. Fixing x_4 means that the probability refers to a given moment. It has been also assumed that one magnetic monopole yields one quasizero fermion mode per unit length of its world-line. This is true for an isolated monopole, and this result has been extrapolated to the QCD vacuum as a whole.

Finally, substituting Eq. (108) into Eq. (107) and making use of Eq. (105), we arrive at Eq. (106).

3.2 Nonperturbative Field Correlators and String Representation of $SU(3)$-Gluodynamics within the Abelian Projection Method

In this Subsection, we shall extend the results of Subsection 3.1.1 to the case of $SU(3)$-gluodynamics. Our interpretation of this subject will mainly follow Ref. [133]. We shall start with the string representation of the partition function of the infrared effective dual model of confinement found in Ref. [107], which is nothing else but the Abelian projected $SU(3)$-gluodynamics.

String Representation for the Partition Function of the Dual Infrared SU(3)-Gluodynamics In the absence of quarks, the partition function of the infrared effective model of confinement reads [107]

$$\mathcal{Z}_{SU(3)} = \int D\vec{B}_\mu D\chi_a \delta\left(\sum_{a=1}^{3}\theta_a\right)\cdot$$

$$\cdot\exp\left\{-\int d^4x\left[\frac{1}{4}\vec{F}_{\mu\nu}^2 + \sum_{a=1}^{3}\left[\frac{1}{2}\left|\left(\partial_\mu - ig_m\vec{\varepsilon}_a\vec{B}_\mu\right)\chi_a\right|^2 + \lambda\left(|\chi_a|^2 - \eta^2\right)^2\right]\right]\right\}, \tag{109}$$

where $\vec{F}_{\mu\nu} = \partial_\mu \vec{B}_\nu - \partial_\nu \vec{B}_\mu$ denotes the field strength tensor of the Abelian vector potential $\vec{B}_\mu \equiv \left(B^3_\mu, B^8_\mu\right)$. These two (magnetic) fields, which are dual to the usual gluonic fields A^3_μ and A^8_μ, acquire a mass $m_B = \sqrt{\frac{3}{2}} g_m \eta$ due to the Higgs mechanism.

Next, in Eq. (109), $\chi_a = |\chi_a| \, \mathrm{e}^{i\theta_a}$, $a = 1, 2, 3$, are three complex scalar fields of monopoles possessing magnetic charges $g_m \vec{\varepsilon}_a$, respectively (notice, that throughout this Subsection we adopt the notation of Ref. [107]). Here,

$$\vec{\varepsilon}_1 = (1, 0)\,,\ \vec{\varepsilon}_2 = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)\,,\ \vec{\varepsilon}_3 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

stand for the so-called root vectors, which define the lattice at which monopole charges $\vec{\mathrm{m}}$ are distributed. Namely, $\vec{\mathrm{m}} = g_m \sum\limits_{a=1}^{3} \zeta_a \vec{\varepsilon}_a$, where ζ_a's are some integers. Notice, that the partition function (109) has been derived under the assumption that the dominant contribution to it is brought about by the monopoles with the smallest magnetic charge, $\zeta_a = \pm 1$ (*cf.* discussions in Ref. [20] and the next Section). The meaning of the root vectors can be better understood if we mention that after singling out of the maximal Abelian subgroup $U(1) \times U(1)$ by the redefinition of the $SU(3)$-generators $T_i \equiv \frac{\lambda_i}{2}$, $i = 1, ..., 8$, in the following way

$$\vec{H} \equiv (H_1, H_2) = (T_3, T_8)\,,\ E_{\pm 1} = \frac{1}{\sqrt{2}}\,(T_1 \pm iT_2)\,,\ E_{\pm 2} = \frac{1}{\sqrt{2}}\,(T_4 \mp iT_5)\,,\ E_{\pm 3} = \frac{1}{\sqrt{2}}\,(T_6 \pm iT_7)\,,$$

they obey the following commutation relations

$$\left[\vec{H}, E_a\right] = \vec{\varepsilon}_a E_a, \ \ \left[\vec{H}, E_{-a}\right] = -\vec{\varepsilon}_a E_{-a}.$$

Thus, the root vectors can be interpreted as structural constants in the so-obtained algebra. Expanding now the vector potential $A_\mu \equiv A^i_\mu T_i$ over the new set of generators, we see that these vectors define the $U(1) \times U(1)$ charges of off-diagonal gluons.

As it has been argued in Ref. [107], due to the fact that the unitary group under study is special, there should exist a constraint $\sum\limits_{a=1}^{3} \theta_a = 0$, which we have imposed by the introduction of a corresponding δ-function into the R.H.S. of Eq. (109). Physically, this constraint means that the trajectories of the monopoles of all three kinds are not independent and relates the monopole fields to each other.

Let us briefly comment on the derivation of the partition function (109). This partition function can be derived starting with the $SU(3)$-gluodynamics partition function and decomposing the fields in a sense of the above introduced decomposition of the generators,

$$A_\mu = A^3_\mu H_1 + A^8_\mu H_2 + \sum_{a=1}^{3} \left(C^{*a}_\mu E_a + C^a_\mu E_{-a}\right).$$

After that, the gluodynamics Lagrangian takes the form [107]

$$\mathcal{L}_{\rm gluodyn.} = \frac{1}{8} \vec{f}_{\mu\nu}^{\ 2} + \frac{1}{4} \sum_{a=1}^{3} \left[\left| (\mathcal{D}^a \wedge C^a)_{\mu\nu} + \frac{ig}{\sqrt{2}} \varepsilon^{abc} C^{*b}_\mu C^{*c}_\nu \right|^2 - \right.$$
$$\left. - \frac{ig}{4} \left(\vec{f}_{\mu\nu} \vec{\varepsilon}_a\right) (C^{*a} \wedge C^a)_{\mu\nu} \right] - \frac{g^2}{8} \left[\sum_{a=1}^{3} \vec{\varepsilon}_a \, (C^{*a} \wedge C^a)_{\mu\nu} \right]^2,$$

where $(C\wedge D)_{\mu\nu}\equiv C_\mu D_\nu - C_\nu D_\mu$, $\vec{f}_{\mu\nu}=\partial_\mu\vec{A}_\nu-\partial_\nu\vec{A}_\mu$, $\mathcal{D}^a_\mu\equiv\partial_\mu+ig\vec{\varepsilon}_a\vec{A}_\mu$, and g is the QCD coupling constant. The monopoles are then introduced according to the modified Bianchi identities,

$$\partial_\mu\tilde{\vec{f}}_{\mu\nu}=\sum_{n=0}^{N}\vec{\mathrm{m}}_n\oint\limits_{\Gamma_n}dx_\nu^{(n)}\delta\left(x-x^{(n)}\right),$$

where $\vec{\mathrm{m}}_n$ is defined in the same way as $\vec{\mathrm{m}}$ with the replacement $\zeta_a\to\zeta_a^{(n)}$. In order to resolve such modified Bianchi identities with nonvanishing R.H.S., one needs to add to $\vec{f}_{\mu\nu}$ as usual a certain nonlocal term. The latter, however, can be made local by introducing an integration over an auxiliary field $\vec{B}_\mu$, whose coupling to the monopole currents reads [107] $ig_m\sum\limits_{n=0}^{N}\oint\limits_{\Gamma_n}dx_\mu^{(n)}\vec{\varepsilon}_a\vec{B}_\mu$. Then, the summation over the monopole currents again can be performed by making use of the $SU(3)$ version of the formula, derived in Ref. [115]

$$\sum_{N=0}^{+\infty}\frac{1}{N!}\left[\prod_{n=0}^{N}\int\limits_0^{+\infty}\frac{ds_n}{s_n}\int\limits_{x(0)=x(s_n)}Dx\left(s'_n\right)\right]\exp\left[\sum_{k=0}^{N}\int\limits_0^{s_k}ds'_k\left(-\frac14\dot{x}^2\left(s'_k\right)+ig_m\dot{x}_\mu\left(s'_k\right)\vec{\varepsilon}_a\vec{B}_\mu\right)\right]=$$

$$=\int D\chi_a\exp\left[-\frac12\int d^4x\left|\left(\partial_\mu-ig_m\vec{\varepsilon}_a\vec{B}_\mu\right)\chi_a\right|^2\right],$$

where the monopole trajectory Γ_n is parametrized by the function $x_\mu\left(s'_n\right)$, i.e. $x_\mu^{(n)}=x_\mu\left(s'_n\right)$. After that, the $\vec{A}_\mu$-field can be integrated out. Notice, that the above performed summation over the monopole currents takes into account also the backtracking and overlapping monopole trajectories with opposite magnetic charges, which should be excluded e.g. by making use of the constraint $\sum\limits_{a=1}^{3}\theta_a=0$. Also, there must exist a repulsive force of a δ-function type between monopole trajectories, which is known to lead to $\lambda\left|\chi\right|^4$, $\lambda>0$, interaction, as well as to the mass term [114, 134] [23]. Other higher orders self-interactions, if any, are infrared irrelevant. The exact summation over the monopole trajectories needs further investigations of the properties of monopoles in QCD (see Ref. [135] for last results). In the derivation of Eq. (109), such higher order interactions have been disregarded.

In what follows, we shall again consider the model (109) in the London limit, where the monopole fields become infinitely heavy, and their radial parts can be integrated out. After that, we are left with the following partition function

$$\mathcal{Z}_{SU(3)}=\int D\vec{B}_\mu D\theta_a^{\rm sing}D\theta_a^{\rm reg}Dk\,\delta\left(\sum_{a=1}^{3}\theta_a^{\rm sing}\right)\cdot$$

$$\cdot\exp\left\{\int d^4x\left[-\frac14\vec{F}_{\mu\nu}^2-\frac{\eta^2}{2}\sum_{a=1}^{3}\left(\partial_\mu\theta_a-g_m\vec{\varepsilon}_a\vec{B}_\mu\right)^2+ik\sum_{a=1}^{3}\theta_a^{\rm reg}\right]\right\}. \qquad (110)$$

Similarly to DAHM, in the model (109), there exist string-like singularities (closed vortices) of the Abrikosov-Nielsen-Olesen type. That is why, in Eq. (110) we have again decomposed the total phases of the monopole fields into a singular and regular part, $\theta_a=\theta_a^{\rm sing}+\theta_a^{\rm reg}$, and imposed

[23] Notice also, that the $\lambda\left|\chi\right|^4$-theory naturally describes the ensemble of fluctuating random lines in the framework of the so-called disorder field theory [42].

the constraint of vanishing of the sum of regular parts by introducing the integration over the Lagrange multiplier $k(x)$. Analogously to the DAHM, in the model (110), $\theta_a^{\rm sing}$'s describe a given electric string configuration and are related to the world-sheets Σ_a's of strings of three types via the equations

$$\varepsilon_{\mu\nu\lambda\rho}\partial_\lambda\partial_\rho\theta_a^{\rm sing}(x) = 2\pi\Sigma_{\mu\nu}^a(x) \equiv 2\pi\int\limits_{\Sigma_a} d\sigma_{\mu\nu}(x_a(\xi))\delta(x - x_a(\xi)), \tag{111}$$

where $x_a \equiv x_\mu^a(\xi)$ is a four-vector, which parametrizes the world-sheet Σ_a.

The path-integral duality transformation of the partition function (110) is parallel to that of Subsection 3.1.1. The only nontriviality brought about by the additional integration over the Lagrange multiplier occurs to be apparent due to the explicit form of the root vectors. Indeed, let us first cast Eq. (110) into the following form

$$\mathcal{Z}_{SU(3)} = \int D\vec{B}_\mu {\rm e}^{-\frac14\int d^4x\vec{F}_{\mu\nu}^2} D\theta_a^{\rm sing}\delta\left(\sum\limits_{a=1}^3\theta_a^{\rm sing}\right)\cdot$$

$$\cdot\int DkD\theta_a^{\rm reg}DC_\mu^a\exp\left\{\int d^4x\left[-\frac{1}{2\eta^2}\left(C_\mu^a\right)^2 + iC_\mu^a\left(\partial_\mu\theta_a - g_m\vec{\varepsilon}_a\vec{B}_\mu\right) + ik\sum\limits_{a=1}^3\theta_a^{\rm reg}\right]\right\} \tag{112}$$

and carry out the integration over the $\theta_a^{\rm reg}$'s. In this way, one needs to solve the equation $\partial_\mu C_\mu^a = k$, which should hold for an arbitrary index a. The solution to this equation reads

$$C_\mu^a(x) = \frac12\varepsilon_{\mu\nu\lambda\rho}\partial_\nu h_{\lambda\rho}^a(x) - \frac{1}{4\pi^2}\frac{\partial}{\partial x_\mu}\int d^4y\frac{k(y)}{(x-y)^2},$$

where $h_{\lambda\rho}^a$ stands for the Kalb-Ramond field of the a-th type. Next, making use of the constraint $\sum\limits_{a=1}^3\theta_a^{\rm sing} = 0$, replacing then the integrals over $\theta_a^{\rm sing}$'s by the integrals over $x_\mu^a(\xi)$'s by virtue of Eq. (111) and omitting again for simplicity the Jacobians [120] emerging during such changes of the integration variables, we arrive at the following representation for the partition function

$$\mathcal{Z}_{SU(3)} = \int D\vec{B}_\mu {\rm e}^{-\frac14\int d^4x\vec{F}_{\mu\nu}^2}\cdot$$

$$\cdot\int Dk\exp\left\{\frac{1}{4\pi^2}\int d^4xd^4y\left[-\frac{3}{2\eta^2}\frac{k(x)k(y)}{(x-y)^2} + ig_m\left(\frac{\partial}{\partial x_\mu}\frac{k(y)}{(x-y)^2}\right)\sum\limits_{a=1}^3\vec{\varepsilon}_a\vec{B}_\mu(x)\right]\right\}\cdot$$

$$\cdot\int Dx_\mu^a(\xi)\delta\left(\sum\limits_{a=1}^3\Sigma_{\mu\nu}^a\right)Dh_{\mu\nu}^a\exp\left\{\int d^4x\left[-\frac{1}{12\eta^2}\left(H_{\mu\nu\lambda}^a\right)^2 + i\pi h_{\mu\nu}^a\Sigma_{\mu\nu}^a - \frac{ig_m}{2}\varepsilon_{\mu\nu\lambda\rho}\vec{\varepsilon}_a\vec{B}_\mu\partial_\nu h_{\lambda\rho}^a\right]\right\},$$

where $H_{\mu\nu\lambda}^a = \partial_\mu h_{\nu\lambda}^a + \partial_\lambda h_{\mu\nu}^a + \partial_\nu h_{\lambda\mu}^a$ stands for the field strength tensor of the Kalb-Ramond field $h_{\mu\nu}^a$. Clearly, due to the explicit form of the root vectors, the sum $\sum\limits_{a=1}^3\vec{\varepsilon}_a\vec{B}_\mu$ vanishes, and the integration over the Lagrange multiplier thus yields an inessential determinant factor. Notice also, that due to Eq. (111), the constraint $\sum\limits_{a=1}^3\theta_a^{\rm sing} = 0$ resulted into a constraint for the world-sheets of strings of three types $\sum\limits_{a=1}^3\Sigma_{\mu\nu}^a = 0$. This means that actually only the world-sheets of two types are independent of each other, whereas the third one is unambiguously fixed by the demand that the above constraint holds.

Integrations over the dual gauge field $\vec{B}_\mu$ as well as over the Kalb-Ramond fields are now straightforward. Taking into account that due to the closeness of the world-sheets in the case under study all the boundary terms vanish, we arrive at the following desired string representation for the partition function

$$\mathcal{Z}_{SU(3)} = \int Dx_\mu^a(\xi)\delta\left(\sum_{a=1}^{3}\Sigma_{\mu\nu}^a\right)\cdot$$

$$\cdot\exp\left[-\frac{g_m\eta^3}{4}\sqrt{\frac{3}{2}}\int\limits_{\Sigma_a} d\sigma_{\mu\nu}(x_a(\xi))\int\limits_{\Sigma_a} d\sigma_{\mu\nu}(x_a(\xi'))\frac{K_1\left(m_B\left|x_a(\xi)-x_a(\xi')\right|\right)}{\left|x_a(\xi)-x_a(\xi')\right|}\right], \tag{113}$$

Finally, it is possible to integrate out one of the three world-sheets, for concreteness $x_\mu^3(\xi)$. This yields the expression for the partition function in terms of the integral over two independent string world-sheets

$$\mathcal{Z}_{SU(3)} = \int Dx_\mu^1(\xi)Dx_\mu^2(\xi)\cdot$$

$$\cdot\exp\left\{-\frac{g_m\eta^3}{2}\sqrt{\frac{3}{2}}\left[\int\limits_{\Sigma_1} d\sigma_{\mu\nu}(x_1(\xi))\int\limits_{\Sigma_1} d\sigma_{\mu\nu}(x_1(\xi'))\frac{K_1\left(m_B\left|x_1(\xi)-x_1(\xi')\right|\right)}{\left|x_1(\xi)-x_1(\xi')\right|}+\right.\right.$$

$$+\int\limits_{\Sigma_1} d\sigma_{\mu\nu}(x_1(\xi))\int\limits_{\Sigma_2} d\sigma_{\mu\nu}(x_2(\xi'))\frac{K_1\left(m_B\left|x_1(\xi)-x_2(\xi')\right|\right)}{\left|x_1(\xi)-x_2(\xi')\right|}+$$

$$\left.\left.+\int\limits_{\Sigma_2} d\sigma_{\mu\nu}(x_2(\xi))\int\limits_{\Sigma_2} d\sigma_{\mu\nu}(x_2(\xi'))\frac{K_1\left(m_B\left|x_2(\xi)-x_2(\xi')\right|\right)}{\left|x_2(\xi)-x_2(\xi')\right|}\right]\right\}. \tag{114}$$

According to Eq. (114), in the language of the effective string theory, the partition function (110) has the form of two independent string world-sheets, which (self-)interact by the exchanges of massive dual gauge bosons. It is also worth noting, that as it follows from Eq. (114), the energy density corresponding to the obtained effective nonlocal string Lagrangian increases not only with the distance between two points lying on the same world-sheet, but also with the distance between two different world-sheets. This means that also the ensemble of strings as a whole displays confining properties.

String Representation of Field and Current Correlators In this Paragraph, we shall derive string representations for field correlators of gluonic field strength tensors and monopole currents. Those will occur to be similar to the ones of the $SU(2)$-case of Subsection 3.1.1, albeit possessing indices w.r.t. $U(1)\times U(1)$ maximal Abelian subgroup. Let us start with the string representation for the generating functional of the correlators of gluonic field strength tensors. In the London limit, this object reads

$$\mathcal{Z}_{SU(3)}\left[\vec{S}_{\alpha\beta}\right] = \int D\vec{B}_\mu D\theta_a^{\rm sing}D\theta_a^{\rm reg}Dk\delta\left(\sum_{a=1}^{3}\theta_a^{\rm sing}\right)\cdot$$

$$\cdot \exp\left\{\int d^4x\left[-\frac{1}{4}\vec{F}^2_{\mu\nu}-\frac{\eta^2}{2}\sum\limits_{a=1}^{3}\left(\partial_\mu\theta_a-g_m\vec{\varepsilon}_a\vec{B}_\mu\right)^2+ik\sum\limits_{a=1}^{3}\theta_a^{\rm reg}+i\vec{S}_{\mu\nu}\tilde{\vec{F}}_{\mu\nu}\right]\right\}, \qquad (115)$$

where $\vec{S}_{\mu\nu}$ stands for the source of the field strength tensor $\tilde{\vec{F}}_{\mu\nu} \equiv \frac{1}{2}\varepsilon_{\mu\nu\lambda\rho}\vec{F}_{\lambda\rho}$, which is nothing else but the field strength of the usual gluonic field $\vec{A}_\mu = \left(A^3_\mu, A^8_\mu\right)$. Performing with Eq. (115) the path-integral duality transformation, we arrive at the following string representation for this generating functional

$$\mathcal{Z}_{SU(3)}\left[\vec{S}_{\alpha\beta}\right]=\exp\left(-\int d^4x\vec{S}_{\mu\nu}{}^2\right)\int Dx^a_\mu(\xi)\delta\left(\sum\limits_{a=1}^{3}\Sigma^a_{\mu\nu}\right)\cdot$$

$$\cdot\exp\left[-\int d^4xd^4y\left(\pi\Sigma^a_{\lambda\nu}(x)-g_m\vec{\varepsilon}_a\vec{S}_{\lambda\nu}(x)\right)D^{ab}_{\lambda\nu,\mu\rho}(x-y)\left(\pi\Sigma^b_{\mu\rho}(y)-g_m\vec{\varepsilon}_b\vec{S}_{\mu\rho}(y)\right)\right], \qquad (116)$$

where $D^{ab}_{\lambda\nu,\mu\rho}(x)$ denotes the propagator of the Kalb-Ramond field $h^a_{\mu\nu}$, defined as (*cf.* Eqs. (80) and (81))

$$D^{ab}_{\lambda\nu,\mu\rho}(x)\equiv\delta^{ab}\left[D^{(1)}_{\lambda\nu,\mu\rho}(x)+D^{(2)}_{\lambda\nu,\mu\rho}(x)\right],$$

where

$$D^{(1)}_{\lambda\nu,\mu\rho}(x)=\frac{g_m\eta^3}{8\pi^2}\sqrt{\frac{3}{2}}\frac{K_1}{|x|}\left(\delta_{\lambda\mu}\delta_{\nu\rho}-\delta_{\mu\nu}\delta_{\lambda\rho}\right),$$

$$D^{(2)}_{\lambda\nu,\mu\rho}(x)=\frac{\eta}{4\pi^2g_mx^2}\sqrt{\frac{2}{3}}\left\{\left[\frac{K_1}{|x|}+\frac{m_B}{2}\left(K_0+K_2\right)\right]\left(\delta_{\lambda\mu}\delta_{\nu\rho}-\delta_{\mu\nu}\delta_{\lambda\rho}\right)+\right.$$

$$+\frac{1}{2|x|}\left[3\left(\frac{m_B^2}{4}+\frac{1}{x^2}\right)K_1+\frac{3m_B}{2|x|}\left(K_0+K_2\right)+\frac{m_B^2}{4}K_3\right]\cdot$$

$$\left.\cdot\left(\delta_{\lambda\rho}x_\mu x_\nu+\delta_{\mu\nu}x_\lambda x_\rho-\delta_{\mu\lambda}x_\nu x_\rho-\delta_{\nu\rho}x_\mu x_\lambda\right)\right\},$$

In the last exponent on the R.H.S. of Eq. (116), the term

$$-\pi^2\int\limits_{\Sigma_a}d\sigma_{\lambda\nu}\left(x_a(\xi)\right)\int\limits_{\Sigma_a}d\sigma_{\mu\rho}\left(x_a(\xi')\right)D^{(2)}_{\lambda\nu,\mu\rho}\left(x_a(\xi)-x_a(\xi')\right)$$

can be again rewritten as a boundary one and thus vanishes due to the closeness of the string world-sheets.

The string representation for the bilocal correlator of the field strength tensors can now immediately be read off from Eq. (116) by making use of the equality $\varepsilon^i_a\varepsilon^j_a=\frac{3}{2}\delta^{ij}$, where $i,j=1,2$ are the $U(1)\times U(1)$ indices referring to the generators T_3, T_8. By the two-fold variation of Eq. (116) w.r.t. $S^i_{\alpha\beta}$, we obtain

$$\left\langle\tilde{F}^i_{\lambda\nu}(x)\tilde{F}^j_{\mu\rho}(y)\right\rangle=\delta^{ij}\left[\left(\delta_{\lambda\mu}\delta_{\nu\rho}-\delta_{\lambda\rho}\delta_{\mu\nu}\right)\delta(x-y)+g_m^2D^{aa}_{\lambda\nu,\mu\rho}(x-y)\right]-$$

$$-4\pi^2 g_m^2 \varepsilon_a^i \varepsilon_b^j \left\langle \int\limits_{\Sigma_c} d\sigma_{\alpha\beta}(x_c(\xi)) \int\limits_{\Sigma_d} d\sigma_{\gamma\zeta}(x_d(\xi')) D^{ac}_{\alpha\beta,\lambda\nu}(x_c(\xi)-x) D^{bd}_{\gamma\zeta,\mu\rho}(x_d(\xi')-y) \right\rangle_{x_a(\xi)}, \qquad (117)$$

where

$$\langle ... \rangle_{x_a(\xi)} \equiv$$

$$\equiv \frac{\int Dx_\mu^a(\xi)\delta\left(\sum\limits_{a=1}^3 \Sigma^a_{\mu\nu}\right)(...)\exp\left[-\frac{g_m\eta^3}{4}\sqrt{\frac{3}{2}}\int\limits_{\Sigma_a} d\sigma_{\mu\nu}(x_a(\xi))\int\limits_{\Sigma_a} d\sigma_{\mu\nu}(x_a(\xi'))\frac{K_1(m_B|x_a(\xi)-x_a(\xi')|)}{|x_a(\xi)-x_a(\xi')|}\right]}{\int Dx_\mu^a(\xi)\delta\left(\sum\limits_{a=1}^3 \Sigma^a_{\mu\nu}\right)\exp\left[-\frac{g_m\eta^3}{4}\sqrt{\frac{3}{2}}\int\limits_{\Sigma_a} d\sigma_{\mu\nu}(x_a(\xi))\int\limits_{\Sigma_a} d\sigma_{\mu\nu}(x_a(\xi'))\frac{K_1(m_B|x_a(\xi)-x_a(\xi')|)}{|x_a(\xi)-x_a(\xi')|}\right]} \qquad (118)$$

defines the average over the string world-sheets, and the term with the δ-function on the R.H.S. of Eq. (117) corresponds to the free contribution to the correlator.

Similarly to the DAHM case, our aim below will be to compare the bilocal cumulant, given for $x \neq y$ by the second term on the R.H.S. of Eq. (117), with the one of SVM, parametrized by two Lorentz structures as follows

$$\left\langle\left\langle \tilde{F}^i_{\lambda\nu}(x)\tilde{F}^j_{\mu\rho}(0)\right\rangle\right\rangle = \delta^{ij}\left\{\left(\delta_{\lambda\mu}\delta_{\nu\rho}-\delta_{\lambda\rho}\delta_{\nu\mu}\right)\hat{D}\left(x^2\right)+\right.$$

$$\left.+\frac{1}{2}\left[\partial_\lambda\left(x_\mu\delta_{\nu\rho}-x_\rho\delta_{\nu\mu}\right)+\partial_\nu\left(x_\rho\delta_{\lambda\mu}-x_\mu\delta_{\lambda\rho}\right)\right]\hat{D}_1\left(x^2\right)\right\}, \qquad (119)$$

i.e. to find the coefficient functions $\hat{D}$ and $\hat{D}_1$. Let us point out once more, that Eq. (119) is nothing else, but the cumulant of two usual gluonic field strength tensors, $F^3_{\mu\nu}(A)$ and/or $F^8_{\mu\nu}(A)$.

Direct comparison of Eqs. (117) and (119) then yields $\hat{D} = \mathcal{D}$ and $\hat{D}_1 = \mathcal{D}_1$, where $\mathcal{D}$ and $\mathcal{D}_1$ are given by Eqs. (88) and (89) with the replacement $m \to m_B$. Besides that, we see that the bilocal cumulant (119) is nonvanishing only for the gluonic field strength tensors of the same kind, i.e. for $i = j = 1$ or $i = j = 2$. Hence, for these diagonal cumulants (whose large-distance asymptotic behaviours match those of the $SU(3)$-gluodynamics as well as Eqs. (90) and (91) correspond to the $SU(2)$-case), the vacuum of the model (109) in the London limit exhibits a nontrivial correlation length, $T_g = \frac{1}{m_B}$.

Finally, it is again quite instructive to rederive the coefficient function $\hat{D}$ from the string representation for the correlator of two monopole currents, $\vec{j}_\mu = -g\eta^2\vec{\varepsilon}_a\left(\partial_\mu\theta_a - g_m\left(\vec{\varepsilon}_a\vec{B}_\mu\right)\right)$. This can be done by virtue of the equation

$$\left\langle j^i_\beta(x) j^j_\sigma(y)\right\rangle = \delta^{ij}\left(\frac{\partial^2}{\partial x_\mu \partial y_\mu}\delta_{\beta\sigma} - \frac{\partial^2}{\partial x_\beta \partial y_\sigma}\right)\hat{D}\left((x-y)^2\right), \qquad (120)$$

which can be obtained from the equations of motion. Besides that, it is also useful to derive the string representation for the generating functional of the monopole current correlators itself, which can then be applied to a derivation of the bilocal correlator. Such a generating functional reads

$$\hat{\mathcal{Z}}_{SU(3)}\left[\vec{J}_\mu\right] = \int D\vec{B}_\mu D\theta_a^{\rm sing} D\theta_a^{\rm reg} Dk\delta\left(\sum\limits_{a=1}^{3}\theta_a^{\rm sing}\right)\cdot$$

$$\cdot\exp\left\{-\int d^4x\left[\frac14\vec{F}_{\mu\nu}^2+\frac{\eta^2}{2}\sum\limits_{a=1}^{3}\left(\partial_\mu\theta_a-g_m\vec{\varepsilon}_a\vec{B}_\mu\right)^2-ik\sum\limits_{a=1}^{3}\theta_a^{\rm reg}+\vec{J}_\mu\vec{j}_\mu\right]\right\}.$$

Once being applied to this object, path-integral duality transformation yields for it the following string representation (*cf.* Eq. (94))

$$\hat{\mathcal{Z}}_{SU(3)}\left[\vec{J}_\mu\right] = \hat{\mathcal{Z}}[0]\exp\left[\frac{m_B^2}{2}\int d^4x\vec{J}_\mu{}^2(x)\right]\cdot$$

$$\cdot\Biggl\langle\exp\Biggl\{-g_m\varepsilon_{\lambda\nu\alpha\beta}\int d^4xd^4y\Biggl[\frac{g_m}{8}\varepsilon_{\mu\rho\gamma\delta}\left(\frac{\partial^2}{\partial x_\alpha\partial y_\gamma}D^{aa}_{\lambda\nu,\mu\rho}(x-y)\right)\vec{J}_\beta(x)\vec{J}_\delta(y)+$$

$$+\pi\vec{\varepsilon}_a\Sigma^b_{\mu\rho}(y)\left(\frac{\partial}{\partial x_\alpha}D^{ab}_{\lambda\nu,\mu\rho}(x-y)\right)\vec{J}_\beta(x)\Biggr]\Biggr\}\Biggr\rangle_{x_a(\xi)}, \tag{121}$$

where $\hat{\mathcal{Z}}_{SU(3)}[0]$ is defined by Eq. (113). Then, the string representation for the bilocal correlator following from Eq. (121), reads

$$\left\langle j^i_\beta(x)j^j_\sigma(y)\right\rangle = m_B^2\delta^{ij}\delta_{\beta\sigma}\delta(x-y)+g_m^2\varepsilon_{\lambda\nu\alpha\beta}\varepsilon_{\mu\rho\gamma\sigma}\Biggl\{-\frac14\delta^{ij}\frac{\partial^2}{\partial x_\alpha\partial y_\gamma}D^{aa}_{\lambda\nu,\mu\rho}(x-y)+\pi^2\varepsilon^i_a\varepsilon^j_b\cdot$$

$$\cdot\left\langle\int\limits_{\Sigma_c}d\sigma_{\delta\zeta}(x_c(\xi))\int\limits_{\Sigma_d}d\sigma_{\chi\varphi}(x_d(\xi'))\left(\frac{\partial}{\partial x_\alpha}D^{ac}_{\lambda\nu,\delta\zeta}\left(x-x_c(\xi)\right)\right)\left(\frac{\partial}{\partial y_\gamma}D^{bd}_{\mu\rho,\chi\varphi}\left(y-x_d(\xi')\right)\right)\right\rangle_{x_a(\xi)}\Biggr\}. \tag{122}$$

By comparing of Eq. (120) and the second term on the R.H.S. of Eq. (122), we recover the coefficient function $\hat{D}$. This confirms the consistency of our calculations.

The above generalization of the $SU(2)$ Abelian-projected gluodynamics to the $SU(3)$-case make it straightforward to generalize in the same way all the other results of Subsection 3.1.

3.3 Representation of Abelian-Projected Theories in Terms of the Monopole Currents

Our investigations of the monopole current correlators in the previous Subsection make it now a challenge to derive a representation for the partition functions of the Abelian-projected theories directly in terms of the monopole currents. This so-called "current" representation will be constructed in the present Subsection, and we shall find it very useful for our studies of the monopole gas in the next Section. In our interpretation, we shall follow Ref. [136].

Let us for simplicity consider unextended DAHM, where intuitively the resulting monopole effective action should contain besides the free part, quadratic in the monopole currents, also the interaction of these currents with the closed string world-sheets Σ. As our starting point will serve

Eqs. (166) and (167) of Appendix 7.2, where, however, we shall not perform any gauge fixing for the field A_μ. Then, the partition function has the form [24]

$$\mathcal{Z}_{\rm 4D\,DAHM} = \int DA_\mu Dx_\mu(\xi) Dh_{\mu\nu} \exp\left\{-\int d^4x\left[\frac{1}{12\eta^2}H^2_{\mu\nu\lambda} - i\pi h_{\mu\nu}\Sigma_{\mu\nu} + (g_m h_{\mu\nu} + \partial_\mu A_\nu - \partial_\nu A_\mu)^2\right]\right\}. \tag{123}$$

Notice that according to the equation of motion for the field A_μ, the absence of external electric currents is expressed by the equation $\partial_\mu \mathcal{F}_{\mu\nu} = 0$, where $\mathcal{F}_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + g_m h_{\mu\nu}$. Regarding $\mathcal{F}_{\mu\nu}$ as a full electromagnetic field strength tensor, one can write for it the corresponding Bianchi identity modified by the monopoles, $\partial_\mu \tilde{\mathcal{F}}_{\mu\nu} = g_m \partial_\mu \tilde{h}_{\mu\nu}$. This identity means that the monopole current can be written in terms of the Kalb-Ramond field $h_{\mu\nu}$ as

$$j_\mu = g_m \partial_\nu \tilde{h}_{\nu\mu}, \tag{124}$$

which manifests its conservation. (Notice, that this current is related to the field C_μ from Eq. (165) of Appendix 7.2 as $j_\mu = -g_m C_\mu$. This means that the δ-function in the last equality on the R.H.S. of this equation just imposes once more the conservation of this current.)

It is also instructive to write down the equation of motion for the Kalb-Ramond field in terms of the introduced full electromagnetic field strength tensor. This equation has the form $\mathcal{F}_{\nu\lambda} = \frac{g_m}{m^2}\partial_\mu H_{\mu\nu\lambda} + \frac{i\pi}{2g_m}\Sigma_{\nu\lambda}$. By virtue of conservation of the vorticity tensor current for the closed string world-sheets, $\partial_\mu \Sigma_{\mu\nu} = 0$, this equation again yields the condition of absence of external electric currents, $\partial_\mu \mathcal{F}_{\mu\nu} = 0$.

Let us now turn ourselves to a derivation of the monopole current representation for the partition function of DAHM. To this end, we shall first resolve the equation $\frac{g_m}{2}\varepsilon_{\mu\nu\lambda\rho}\partial_\nu h_{\lambda\rho} = -j_\mu$ w.r.t. $h_{\mu\nu}$, which yields

$$h_{\mu\nu}(x) = \frac{1}{2\pi^2 g_m}\varepsilon_{\mu\nu\lambda\rho}\int d^4y \frac{(x-y)_\lambda}{|x-y|^4} j_\rho(y).$$

Next, we get the following expressions for various terms on the R.H.S. of Eq. (123)

$$H^2_{\mu\nu\lambda} = \frac{6}{g_m^2} j_\mu^2, \quad \int d^4x h^2_{\mu\nu} = \frac{1}{2\pi^2 g_m^2}\int d^4x d^4y j_\mu(x)\frac{1}{(x-y)^2} j_\mu(y).$$

Bringing all this together and performing in Eq. (123) again the hypergauge transformation $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$ with the gauge function $\Lambda_\mu = -\frac{1}{g_m}A_\mu$, which eliminates the field A_μ, we finally arrive at the desired monopole current representation, which has the form

$$\mathcal{Z}_{\rm 4D\,DAHM} = \int Dx_\mu(\xi) Dh_{\mu\nu} \exp\left\{-\left[\frac{1}{2\pi^2}\int d^4x d^4y j_\mu(x)\frac{1}{(x-y)^2} j_\mu(y) + \frac{2}{m^2}\int d^4x j_\mu^2 + \right.\right.$$
$$\left.\left. + \frac{2\pi i}{g_m} S_{\rm int.}(\Sigma, j_\mu)\right]\right\}. \tag{125}$$

[24] Notice that for the closed strings, there arises another sign of the $h_{\mu\nu}\Sigma_{\mu\nu}$-term in comparison with Eq. (78), which, however, can be absorbed by the field redefinition $h_{\mu\nu} \to -h_{\mu\nu}$.

The first term in the exponent on the R.H.S. of Eq. (125) has the form of the Biot-Savart energy of the electric field generated by monopole currents [114], the second term corresponds to the (gauged) kinetic energy of Cooper pairs, and the term

$$S_{\rm int.}(\Sigma, j_\mu) = \frac{1}{4\pi^2}\varepsilon_{\mu\nu\lambda\rho}\int d^4x d^4y j_\mu(x)\frac{(y-x)_\nu}{|y-x|^4}\Sigma_{\lambda\rho}(y) \tag{126}$$

describes the interaction of the string world-sheet with the monopole current j_μ. This interaction can obviously be rewritten in the form $S_{\rm int.} = \int d^4x j_\mu H_\mu^{\rm str.}$, where $H_\mu^{\rm str.}$ is the four-dimensional analogue of the magnetic induction, produced by the electric string according to the equation

$$\varepsilon_{\mu\nu\lambda\rho}\partial_\lambda H_\rho^{\rm str.} = \Sigma_{\mu\nu}. \tag{127}$$

Notice, that if one includes an additional current describing an external monopole,

$$j_\mu^{\rm ext.}(x) = g_m \oint\limits_\Gamma dx_\mu(\tau)\delta(x - x(\tau)), \tag{128}$$

there arises among others an interaction term (126), which in this case takes the form $S_{\rm int.} = g_m \hat{L}(\Sigma, \Gamma)$, where $\hat{L}(\Sigma, \Gamma)$ is simply the Gauss linking number of the world-sheet Σ with the contour Γ [25].

Clearly, the functional integral over the Kalb-Ramond field in Eq. (125) has to be evaluated at the saddle-point

$$h_{\mu\nu}^{\rm s.p.}(x) = \frac{ig_m\eta^3}{\pi}\int\limits_\Sigma d\sigma_{\mu\nu}(x(\xi))\frac{K_1(m|x - x(\xi)|)}{|x - x(\xi)|}.$$

By virtue of Eq. (124), the monopole current can then be expressed via the string world-sheet Σ as follows

$$j_\mu(x) = \frac{im^2\eta}{8\pi}\varepsilon_{\mu\nu\lambda\rho}\int\limits_\Sigma d\sigma_{\lambda\rho}(x(\xi))\frac{(x - x(\xi))_\nu}{(x - x(\xi))^2}\cdot$$

$$\cdot\left\{\frac{K_1(m|x - x(\xi)|)}{|x - x(\xi)|} + \frac{m}{2}\Big[K_0(m|x - x(\xi)|) + K_2(m|x - x(\xi)|)\Big]\right\}.$$

It is straightforward to extend the above analysis to the case of the Abelian-projected $SU(3)$-gluodynamics, where the string representation for the partition function (110) has the form

$$\mathcal{Z}_{SU(3)} = \int Dx_\mu^a(\xi)\delta\left(\sum_{a=1}^3 \Sigma_{\mu\nu}^a\right) DA_\mu^a Dh_{\mu\nu}^a \exp\left\{-\int d^4x\left[\frac{1}{12\eta^2}\left(H_{\mu\nu\lambda}^a\right)^2 - \right.\right.$$

$$\left.\left. -i\pi h_{\mu\nu}^a\Sigma_{\mu\nu}^a + \left(g_m\frac{\sqrt{3}}{2\sqrt{2}}h_{\mu\nu}^a + \partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)^2\right]\right\} \tag{129}$$

with $A_\mu^a \equiv \vec{\varepsilon}_a\vec{A}_\mu$. Analogously to the argumentation following after Eq. (123), Eq. (129) means that the arising three monopole currents can be expressed in terms of three Kalb-Ramond fields

[25] Topological interactions of this kind are sometimes interpreted as a 4D analogue of the Aharonov-Bohm effect. In particular, this interaction, albeit for the current of an external electrically charged particle with the string world-sheet, emerges in the string representation for the Wilson loop of this particle in AHM [120].

as $j^a_\mu = g_m \frac{\sqrt{3}}{2\sqrt{2}} \partial_\nu \tilde{h}^a_{\nu\mu}$. Finally, rewriting Eq. (129) via these currents and resolving the constraint $\sum\limits_{a=1}^{3} \Sigma^a_{\mu\nu} = 0$ by integrating over one of the world-sheets (for concreteness, again $x^3_\mu(\xi)$), we obtain

$$\mathcal{Z}_{SU(3)} = \int Dx^1_\mu(\xi) Dx^2_\mu(\xi) Dh^a_{\mu\nu} \exp\left\{-\left[\frac{1}{2\pi^2}\int d^4x d^4y j^a_\mu(x) \frac{1}{(x-y)^2} j^a_\mu(y) + \frac{2}{m_B^2}\int d^4x \left(j^a_\mu\right)^2 + \right.\right.$$

$$\left.\left. +4\pi i \sqrt{\frac{2}{3}} \frac{1}{g_m}\left[S_{\text{int.}}\left(\Sigma^1, j^1_\mu\right) + S_{\text{int.}}\left(\Sigma^2, j^2_\mu\right) - S_{\text{int.}}\left(\Sigma^1, j^3_\mu\right) - S_{\text{int.}}\left(\Sigma^2, j^3_\mu\right)\right]\right]\right\}. \tag{130}$$

Eq. (130) is the desired representation for the partition function of the Abelian-projected $SU(3)$-gluodynamics in terms of three monopole currents, which should be evaluated at the saddle-point. The last four terms on its R.H.S. yield an interference between various possibilities of the interaction between the string world-sheets and monopole currents in this model to occur.

For illustrations, let us establish a correspondence of the above results to the 3D ones. Namely, let us derive a 3D analogue of Eq. (125), i.e. find a representation in terms of the monopole currents of the dual Ginzburg-Landau model. There, Eq. (76) is replaced by [42]

$$\varepsilon_{\mu\nu\lambda} \partial_\nu \partial_\lambda \theta^{\text{sing.}}(\vec{x}\,) = 2\pi \delta_\mu(\vec{x}\,). \tag{131}$$

Here, on the R.H.S. stands the so-called vortex density with $\delta_\mu(\vec{x}\,) \equiv \int\limits_L dy_\mu(\tau) \delta(\vec{x} - \vec{y}(\tau))$ being the transverse δ-function defined w.r.t. the electric vortex line L, parametrized by the vector $\vec{y}(\tau)$. This line is closed in the case under study, i.e. in the absence of external quarks, which means that $\partial_\mu \delta_\mu = 0$. Performing by virtue of Eq. (131) the path-integral duality transformation of the partition function of 3D DAHM in the London limit,

$$\mathcal{Z}_{\text{3D DAHM}} = \int DB_\mu D\theta^{\text{sing.}} D\theta^{\text{reg.}} \exp\left\{-\int d^3x \left[\frac{1}{4}F^2_{\mu\nu} + \frac{\eta^2}{2}(\partial_\mu\theta - 2g_m B_\mu)^2\right]\right\},$$

we get for it the following representation

$$\mathcal{Z}_{\text{3D DAHM}} = \int D\varphi Dy_\mu(\tau) Dh_\mu \exp\left\{-\int d^3x \left[\frac{1}{4\eta^2}(\partial_\mu h_\nu - \partial_\nu h_\mu)^2 - \right.\right.$$

$$\left.\left. -2\pi i h_\mu \delta_\mu + \left(g_m\sqrt{2} h_\mu + \partial_\mu\varphi\right)^2\right]\right\}. \tag{132}$$

Notice, that the Kalb-Ramond field has now reduced to a massive one-form field h_μ with the mass $m = 2g_m\eta$, as well as the A_μ-field reduced to a scalar φ. Analogously to the 4D case, the field $\mathcal{E}_\mu \equiv g_m\sqrt{2}h_\mu + \partial_\mu\varphi$ can be regarded as a full electric field, defined via the full dual electromagnetic field strength tensor as $\mathcal{E}_\mu = \frac{1}{2}\varepsilon_{\mu\nu\lambda}\mathcal{F}_{\nu\lambda}$. The absence of external quarks is now expressed by the equation $\partial_\mu\mathcal{E}_\mu = 0$, following from the equation of motion for the field φ. Correspondingly, the monopole currents are defined as $j_\nu = \partial_\mu\mathcal{F}_{\mu\nu} = g_m\sqrt{2}\varepsilon_{\mu\nu\lambda}\partial_\mu h_\lambda$ and are manifestly conserved. Notice also, that the condition of closeness of the vortex lines, $\partial_\mu\delta_\mu = 0$, unambiguously exhibits itself as a condition of absence of external quarks, $\partial_\mu\mathcal{E}_\mu = 0$, by virtue of equation of motion for the field h_μ, which can be written in the form $\mathcal{E}_\mu = \frac{1}{g_m\sqrt{2}}\left[\frac{1}{2\eta^2}\partial_\nu(\partial_\nu h_\mu - \partial_\mu h_\nu) + i\pi\delta_\mu\right]$.

Next, after performing the gauge transformation $h_\mu \to h_\mu + \partial_\mu\gamma$ with the gauge function $\gamma = -\frac{1}{g_m\sqrt{2}}\varphi$, the field φ drops out. Expressing h_μ via j_μ as follows

$$h_\mu(\vec{x}) = -\frac{1}{4\sqrt{2}\pi g_m}\varepsilon_{\mu\nu\lambda}\frac{\partial}{\partial x_\nu}\int d^3y \frac{j_\lambda(\vec{y})}{|\vec{x}-\vec{y}|}$$

and substituting this expression into the R.H.S. of Eq. (132), we finally arrive at the desired representation for the partition function of 3D DAHM in terms of the monopole currents

$$\mathcal{Z}_{\rm 3D\,DAHM} = \int Dy_\mu(\tau)Dh_\mu \exp\Biggl\{-\Biggl[\frac{1}{4\pi}\int d^3xd^3y j_\mu(\vec{x})\frac{1}{|\vec{x}-\vec{y}|}j_\mu(\vec{y}) + + \frac{1}{m^2}\int d^3x j_\mu^2 + \frac{\sqrt{2}\pi i}{g_m}S_{\rm int.}(L, j_\mu)\Biggr]\Biggr\}. \qquad (133)$$

The interaction term of the electric vortex line with the monopole current now takes the form

$$S_{\rm int.}(L, j_\mu) = \frac{1}{4\pi}\varepsilon_{\mu\nu\lambda}\int d^3xd^3y j_\mu(\vec{x})\frac{(\vec{y}-\vec{x})_\nu}{|\vec{y}-\vec{x}|^3}\delta_\lambda(\vec{y}).$$

This interaction term can be again rewritten as $S_{\rm int.} = \int d^3x j_\mu H_\mu^{\rm vor.}$, where the magnetic induction, generated by the electric vortex line, obeys the equation $\varepsilon_{\mu\nu\lambda}\partial_\nu H_\lambda^{\rm vor.} = \delta_\mu$. In the particular case, when one introduces an external current of the form (128), there emerges a term $S_{\rm int.} = g_m\hat{L}(L,\Gamma)$ with $\hat{L}(L,\Gamma)$ standing for the Gauss linking number of the contours L and Γ. The functional integral over the field h_μ in Eq. (133) should again be evaluated at the saddle-point $h_\mu^{\rm s.p.}$, which is determined by the classical equation of motion, following from Eq. (132) after gauging away the field φ. This saddle-point has the form

$$h_\mu^{\rm s.p.}(\vec{x}) = \frac{i\eta^2}{2}\oint_L dy_\mu(\tau)\frac{{\rm e}^{-m|\vec{x}-\vec{y}(\tau)|}}{|\vec{x}-\vec{y}(\tau)|},$$

which yields the following expression for the monopole current

$$j_\mu(\vec{x}) = \frac{ig_m\eta^2}{\sqrt{2}}\varepsilon_{\mu\nu\lambda}\oint_L dy_\lambda(\tau)\frac{(\vec{x}-\vec{y}(\tau))_\nu}{(\vec{x}-\vec{y}(\tau))^2}\left(m + \frac{1}{|\vec{x}-\vec{y}(\tau)|}\right){\rm e}^{-m|\vec{x}-\vec{y}(\tau)|}.$$

In conclusion of this Section, our results demonstrate the usefulness of the Abelian projection method and path-integral duality transformation to the solution of the problem of string representation of non-Abelian gauge theories and provide us with some new insights concerning the vacuum structure in these theories. They give a new field-theoretical status to the SVM as well as to the 't Hooft-Mandelstam scenario of confinement.

4 String Representation of Compact QED and Description of String Excitations in Gauge Theories

In the previous Section, we have demonstrated the relevance of the massive Kalb-Ramond field coupled to the string world-sheet to the construction of the string representation of DAHM. We have also argued that the large-distance asymptotic behaviours of the propagator of this field match those of the bilocal field strength correlator in SVM of QCD. All this tells us that the Kalb-Ramond field coupled to the string world-sheet is indeed quite adequate for modelling of the QCD string.

In the present Section, we shall find the dual formulation of one more model allowing for the analytical description of confinement, which is 3D compact QED [106, 20]. There, confinement also occurs due to the monopole condensation, as in the Abelian(-projected) theories. The resulting dual action also turns out to be some kind of a Kalb-Ramond field action, albeit quite nonlinear one. However, in the low-energy limit, it reduces to the linear form analogous to Eq. (78), which means that there exists a correspondence between the dual versions of 3D compact QED and 3D AHM with monopoles. Consequently, some correspondence should take place between these theories not only in the dual, but also in the standard formulations. We shall see that this is indeed the case, namely 3D compact QED corresponds to the limit of vanishing gauge boson mass in 3D AHM with monopoles. As a by-product, we shall also get the bilocal field strength correlator in 3D compact QED.

Finally, we shall apply the partition function of the type (78) to the description of the string world-sheet excitations in DAHM, compact QED, and QCD and derive the effective action quadratic in these excitations. In our interpretation of the above mentioned topics in this Section we shall mainly follow the papers [92, 137] and [136].

4.1 Vacuum Correlators and String Representation of 3D Compact QED

In this Section, we shall consider 3D compact QED and find its string representation arising from the integral over the monopole densities. Besides that, we shall investigate vacuum correlators in the weak-field limit, and demonstrate the relation of this theory to 3D AHM with monopoles.

The most important feature of 3D *compact* QED, which distinguishes it from the noncompact case, is the existence of magnetic monopoles. Their general configuration is the Coulomb gas with the action [106]

$$S_{\rm mon.} = g^2 \sum_{a<b} q_a q_b \left(\Delta^{-1}\right)(\vec{z}_a, \vec{z}_b) + S_0 \sum_a q_a^2, \tag{134}$$

where Δ is the 3D Laplace operator, and S_0 is the action of a single monopole, $S_0 = \frac{\rm const.}{e^2}$. Here, similarly to Ref. [106], we have adopted standard Dirac notations, where $eg = 2\pi n$, restricting ourselves to the monopoles of the minimal charge, i.e. setting $n = 1$. Then, the partition function of the grand canonical ensemble of monopoles associated with the action (134) reads

$$\mathcal{Z}_{\rm mon.} = \sum_{N=0}^{+\infty} \sum_{q_a=\pm 1} \frac{\zeta^N}{N!} \prod_{i=0}^{N} \int d^3 z_i \exp\left[-\frac{\pi}{2e^2} \int d^3x d^3y \rho_{\rm gas}(\vec{x}\,) \frac{1}{|\vec{x}-\vec{y}\,|} \rho_{\rm gas}(\vec{y}\,)\right], \tag{135}$$

where $\rho_{\rm gas}(\vec{x}\,) = \sum\limits_a q_a \delta(\vec{x}-\vec{z}_a)$ is the monopole density, corresponding to the gas configuration. Here, a single monopole weight $\zeta \propto \exp(-S_0)$ has the dimension of $({\rm mass})^3$ and is usually referred to as fugacity. Notice also that, as usual, we have restricted ourselves to the values $q_a = \pm 1$, since at large values of the magnetic coupling constant g, monopoles with $|q| > 1$ turn out to be unstable and tend to dissociate into the monopoles with $|q| = 1$. Below in this Section, it will be demonstrated that the limit of a small gauge boson mass (which takes place e.g. at large g) is just the case, when 3D compact QED follows from 3D AHM with external monopoles.

Next, Coulomb interaction can be made local, albeit nonlinear one, by introduction of an auxiliary scalar field [106]

$$\mathcal{Z}_{\rm mon.} = \int D\chi \exp\left\{-\int d^3x \left[\frac{1}{2}(\partial_\mu\chi)^2 - 2\zeta\cos(g\chi)\right]\right\}. \tag{136}$$

The magnetic mass $m = g\sqrt{2\zeta}$ of the field χ, following from the quadratic term in the expansion of the cosine on the R.H.S. of Eq. (136), is due to the Debye screening in the monopole plasma. The next, quartic, term of the expansion determines the coupling constant of the diagrammatic expansion for the monopole gas, which is therefore exponentially small and proportional to $g^4 \exp(-{\rm const.}g^2)$.

Let us now cast the partition function (135) into the form of an integral over the monopole densities. This can be done by introducing into Eq. (135) a unity of the form

$$\int D\rho\delta\left(\rho(\vec{x}) - \rho_{\rm gas}(\vec{x})\right) = \int D\rho D\mu \exp\left\{i\left[\sum_a q_a\mu(\vec{z}_a) - \int d^3x\mu\rho\right]\right\}.$$

Then, performing the summation over the monopole ensemble in the same way as that leading to the representation (136), we get

$$\mathcal{Z}_{\rm mon.} = \int D\rho D\mu \exp\left\{-\frac{\pi}{2e^2}\int d^3x d^3y \rho(\vec{x})\frac{1}{|\vec{x}-\vec{y}|}\rho(\vec{y}) + \int d^3x\left(2\zeta\cos\mu - i\mu\rho\right)\right\}. \tag{137}$$

Finally, integrating over the field μ by resolving the corresponding saddle-point equation,

$$\sin\mu = -\frac{i\rho}{2\zeta}, \tag{138}$$

we arrive at the desired representation for the partition function

$$\mathcal{Z}_{\rm mon.} = \int D\rho \exp\left\{-\left[\frac{\pi}{2e^2}\int d^3x d^3y \rho(\vec{x})\frac{1}{|\vec{x}-\vec{y}|}\rho(\vec{y}) + V[\rho]\right]\right\}, \tag{139}$$

where

$$V[\rho] = \int d^3x \left\{\rho \ln\left[\frac{\rho}{2\zeta} + \sqrt{1 + \left(\frac{\rho}{2\zeta}\right)^2}\right] - 2\zeta\sqrt{1 + \left(\frac{\rho}{2\zeta}\right)^2}\right\} \tag{140}$$

is the parabolic-type effective monopole potential, whose asymptotic behaviours at $\rho \ll \zeta$ and $\rho \gg \zeta$ read

$$V[\rho] \longrightarrow \int d^3x \left(-2\zeta + \frac{\rho^2}{4\zeta}\right) \tag{141}$$

and

$$V[\rho] \longrightarrow \int d^3x \left[\rho\left(\ln\frac{\rho}{\zeta} - 1\right)\right],$$

respectively. Notice, that during the integration over the field μ in Eq. (137), we have chosen only the real branch of the solution to the saddle-point equation (138) and disregarded the complex ones.

The obtained representation for the partition function in terms of the monopole densities can be immediately applied to the calculation of the coefficient function $\mathcal{D}^{\text{mon.}}(x^2)$, related to the bilocal correlator of the field strength tensors as follows

$$\left\langle \mathcal{F}_{\lambda\nu}(\vec{x}\,)\,\mathcal{F}_{\mu\rho}(0)\right\rangle_{A_\mu,\rho} = \Bigl(\delta_{\lambda\mu}\delta_{\nu\rho} - \delta_{\lambda\rho}\delta_{\nu\mu}\Bigr)\mathcal{D}^{\text{mon.}}\left(x^2\right) +$$

$$+\frac{1}{2}\left[\partial_\lambda\Bigl(x_\mu\delta_{\nu\rho} - x_\rho\delta_{\nu\mu}\Bigr) + \partial_\nu\Bigl(x_\rho\delta_{\lambda\mu} - x_\mu\delta_{\lambda\rho}\Bigr)\right]\mathcal{D}_1^{\text{full}}\left(x^2\right), \tag{142}$$

where the average over the monopole densities is defined by the partition function (139), whereas the A_μ-average is defined as

$$\langle ...\rangle_{A_\mu} \equiv \frac{\int DA_\mu\,(...)\exp\left(-\frac{1}{4e^2}\int d^3x F_{\mu\nu}^2\right)}{\int DA_\mu \exp\left(-\frac{1}{4e^2}\int d^3x F_{\mu\nu}^2\right)}.$$

In Eq. (142), $\mathcal{F}_{\mu\nu} = F_{\mu\nu} + F_{\mu\nu}^M$ stands for the full electromagnetic field strength tensor, which includes also the monopole part

$$F_{\mu\nu}^M(\vec{x}\,) = -\frac{1}{2}\varepsilon_{\mu\nu\lambda}\frac{\partial}{\partial x_\lambda}\int d^3y\frac{\rho(\vec{y}\,)}{|\vec{x}-\vec{y}\,|}.$$

This monopole part yields the R.H.S. of the Bianchi identities modified by the monopoles,

$$\partial_\mu\mathcal{H}_\mu = 2\pi\rho, \tag{143}$$

where $\mathcal{H}_\mu = \frac{1}{2}\varepsilon_{\mu\nu\lambda}\mathcal{F}_{\nu\lambda}$ stands for the full magnetic induction. Eqs. (142) and (143) then lead to the following equation for the function $\mathcal{D}^{\text{mon.}}$

$$\Delta\mathcal{D}^{\text{mon.}}\left(x^2\right) = -4\pi^2\left\langle\rho(\vec{x}\,)\,\rho(0)\right\rangle_\rho, \tag{144}$$

which in fact is a 3D analogue of the 4D equation (96). The correlator standing on the R.H.S. of Eq. (144) can be found in the limit of small monopole densities, $\rho \ll \zeta$. By making use of Eqs. (139) and (141), we obtain

$$\left\langle\rho(\vec{x}\,)\,\rho(0)\right\rangle_\rho = -\frac{\zeta}{2\pi}\Delta\frac{\mathrm{e}^{-m|\vec{x}\,|}}{|\vec{x}\,|}.$$

Then, demanding that $\mathcal{D}^{\text{mon.}}(x^2\to\infty)\to 0$, we get by the maximum principle for the harmonic functions the desired expression for the function $\mathcal{D}^{\text{mon.}}$ in the low-density limit

$$\mathcal{D}^{\text{mon.}}\left(x^2\right) = 2\pi\zeta\frac{\mathrm{e}^{-m|\vec{x}\,|}}{|\vec{x}\,|}. \tag{145}$$

We see that in the model under study, the correlation length of the vacuum T_g, i.e. the distance at which the function $\mathcal{D}^{\text{mon.}}$ decreases, corresponds to the inverse mass of the field χ, m^{-1} (*cf.* the case of Abelian-projected theories, studied above). The coefficient function $\mathcal{D}_1^{\text{full}}(x^2)$ will be derived later on.

Let us now proceed to the problem of string representation of 3D compact QED. To this end, let us consider an expression for the Wilson loop and try to represent it as an integral over the world-sheets Σ's, bounded by the contour C. By virtue of the Stokes theorem, the Wilson loop can be rewritten in the form

$$\langle W(C)\rangle = \left\langle \exp\left(\frac{i}{2}\int\limits_{\Sigma} d\sigma_{\mu\nu}\mathcal{F}_{\mu\nu}\right)\right\rangle_{A_\mu,\rho} = \left\langle \exp\left(i\int\limits_{\Sigma} d\sigma_{\mu}\mathcal{H}_{\mu}\right)\right\rangle_{A_\mu,\rho} =$$

$$= \langle W(C)\rangle_{A_\mu}\left\langle \exp\left(\frac{i}{2}\int d^3x\rho\left(\vec{x}\,\right)\eta\left(\vec{x}\,\right)\right)\right\rangle_{\rho}, \tag{146}$$

where the free photon contribution reads

$$\langle W(C)\rangle_{A_\mu} = \left\langle \exp\left(i\oint\limits_{C} A_\mu dx_\mu\right)\right\rangle_{A_\mu} = \exp\left(-\frac{e^2}{8\pi}\oint\limits_{C} dx_\mu\oint\limits_{C} dy_\mu\frac{1}{|\vec{x}-\vec{y}\,|}\right). \tag{147}$$

In Eq. (146), $d\sigma_\mu \equiv \frac{1}{2}\varepsilon_{\mu\nu\lambda}d\sigma_{\nu\lambda}$, and $\eta\left(\vec{x}\,\right) = \frac{\partial}{\partial x_\mu}\int\limits_{\Sigma} d\sigma_\mu\left(\vec{y}\,\right)\frac{1}{|\vec{x}-\vec{y}\,|}$ stands for the solid angle under which the surface Σ is seen by an observer at the point $\vec{x}$. Notice that due to the Gauss law, in the case when Σ is a closed surface surrounding the point $\vec{x}$, $\eta\left(\vec{x}\,\right) = 4\pi$, which is the standard result for the total solid angle in 3D.

Eq. (146) seems to contain some discrepancy, since its L.H.S. depends only on the contour C, whereas the R.H.S. depends on an arbitrary surface Σ, bounded by C. However, this actually occurs to be not a discrepancy, but a key point in the construction of the desired string representation. The resolution of the apparent paradox lies in the observation that during the derivation of the effective monopole potential (140), we have accounted only for the one, namely real, branch of the solution to the saddle-point equation (138). Actually, however, one should sum up over all the (complex-valued) branches of the integrand of the effective potential (140) at every space point $\vec{x}$. This requires to replace $V[\rho]$ by

$$V_{\rm total}[\rho] = \sum_{\rm branches}\int d^3x\left\{\pm\rho\ln\left[\frac{\rho}{2\zeta}+\sqrt{1+\left(\frac{\rho}{2\zeta}\right)^2}\right]\mp 2\zeta\sqrt{1+\left(\frac{\rho}{2\zeta}\right)^2}+\binom{0}{i\pi}+2\pi in\right\},$$

$n = 0, \pm 1, \pm 2, ...$, where adding of 0 or $i\pi$ corresponds to choosing of upper or lower sign, respectively. Such a summation over the branches of the multivalued potential in the expression for the Wilson loop

$$\langle W(C)\rangle = \langle W(C)\rangle_{A_\mu}\int D\rho\exp\left\{-\left[\frac{\pi}{2e^2}\int d^3xd^3y\rho\left(\vec{x}\,\right)\frac{1}{|\vec{x}-\vec{y}\,|}\rho\left(\vec{y}\,\right)+V_{\rm total}[\rho]-\right.\right.$$

$$\left.\left.-\frac{i}{2}\int d^3x\rho\left(\vec{x}\,\right)\eta\left(\vec{x}\,\right)\right]\right\} \tag{148}$$

thus restores the independence of the choice of the world-sheet. (Notice, that from now on we omit an inessential normalization factor, implying everywhere the normalization $\langle W(0)\rangle = 1$.)

It is worth noting that the obtained string representation (148) has been for the first time derived in another, more indirect, way in Ref. [92]. It is therefore instructive to establish a correspondence between our approach and the one of that paper.

The main idea of Ref. [92] was to calculate the Wilson loop starting with the direct definition of this average in a sense of the partition function (135) of the monopole gas. The corresponding expression has the form

$$\left\langle W(C)\right\rangle_{\text{mon.}} =$$

$$= \sum_{N=0}^{+\infty} \sum_{q_a=\pm 1} \frac{\zeta^N}{N!} \prod_{i=0}^{N} \int d^3 z_i \exp\left[-\frac{\pi}{2e^2}\int d^3x d^3y \rho_{\text{gas}}(\vec{x}\,) \frac{1}{|\vec{x}-\vec{y}\,|}\rho_{\text{gas}}(\vec{y}\,) + \frac{i}{2}\int d^3x \rho_{\text{gas}}(\vec{x}\,)\,\eta(\vec{x}\,)\right] =$$

$$= \int D\chi \exp\left\{-\int d^3x \left[\frac{1}{2}(\partial_\mu \chi)^2 - 2\zeta\cos\left(g\chi + \frac{\eta}{2}\right)\right]\right\} =$$

$$= \int D\varphi \exp\left\{-\int d^3x \left[\frac{e^2}{8\pi^2}\left(\partial_\mu\varphi - \frac{1}{2}\partial_\mu\eta\right)^2 - 2\zeta\cos\varphi\right]\right\}, \tag{149}$$

where $\varphi \equiv g\chi + \frac{\eta}{2}$.

Next, one can prove the following equality

$$\exp\left[-\frac{e^2}{8\pi}\oint_C dx_\mu \oint_C dy_\mu \frac{1}{|\vec{x}-\vec{y}\,|} - \frac{e^2}{8\pi^2}\int d^3x \left(\partial_\mu\varphi - \frac{1}{2}\partial_\mu\eta\right)^2\right] =$$

$$= \int Dh_{\mu\nu} \exp\left[-\int d^3x \left(i\varphi\varepsilon_{\mu\nu\lambda}\partial_\mu h_{\nu\lambda} + g^2 h_{\mu\nu}^2 - 2\pi i h_{\mu\nu}\Sigma_{\mu\nu}\right)\right], \tag{150}$$

which makes it possible to represent the contribution of the kinetic term on the R.H.S. of Eq. (149) and the free photon contribution (147) to the Wilson loop as an integral over the Kalb-Ramond field. The only nontrivial point necessary to prove this equality is an expression for the derivative of the solid angle. One has

$$\partial_\lambda \eta(\vec{x}\,) = \int_\Sigma \left(d\sigma_\mu(\vec{y}\,)\frac{\partial}{\partial y_\lambda} - d\sigma_\lambda(\vec{y}\,)\frac{\partial}{\partial y_\mu}\right)\frac{\partial}{\partial y_\mu}\frac{1}{|\vec{x}-\vec{y}\,|} + \int_\Sigma d\sigma_\lambda(\vec{y}\,)\,\Delta\frac{1}{|\vec{x}-\vec{y}\,|}. \tag{151}$$

Applying to the first integral on the R.H.S. of Eq. (151) Stokes theorem in the operator form,

$$d\sigma_\mu \frac{\partial}{\partial y_\lambda} - d\sigma_\lambda \frac{\partial}{\partial y_\mu} \longrightarrow \varepsilon_{\mu\lambda\nu} dy_\nu,$$

one finally obtains

$$\partial_\lambda \eta(\vec{x}\,) = \varepsilon_{\lambda\mu\nu}\frac{\partial}{\partial x_\mu}\oint_C dy_\nu \frac{1}{|\vec{x}-\vec{y}\,|} - 4\pi\int_\Sigma d\sigma_\lambda(\vec{y}\,)\,\delta(\vec{x}-\vec{y}\,).$$

Making use of this result and carrying out the Gaussian integral over the field $h_{\mu\nu}$, one can demonstrate that both sides of Eq. (150) are equal to

$$\exp\left\{-\frac{e^2}{2}\left[\frac{1}{4\pi^2}\int d^3x (\partial_\mu\varphi)^2 + \frac{1}{\pi}\int_\Sigma d\sigma_\mu \partial_\mu \varphi + \int_\Sigma d\sigma_\mu(\vec{x}\,)\int_\Sigma d\sigma_\mu(\vec{y}\,)\,\delta(\vec{x}-\vec{y}\,)\right]\right\}$$

thus proving the validity of this equation.

Substituting now Eq. (150) into Eq. (149), it is easy to carry out the integral over the field φ, which has no more kinetic term, in the saddle-point approximation. This equation has the same form as Eq. (138) with the replacement $\rho \to \varepsilon_{\mu\nu\lambda}\partial_\mu h_{\nu\lambda}$. The resulting expression for the full Wilson loop then takes the form

$$\langle W(C)\rangle = \langle W(C)\rangle_{A_\mu} \langle W(C)\rangle_{\rm mon.} =$$

$$= \int Dh_{\mu\nu} \exp\left\{-\int d^3x \left(g^2 h_{\mu\nu}^2 + V_{\rm total}\left[\varepsilon_{\mu\nu\lambda}\partial_\mu h_{\nu\lambda}\right]\right) + 2\pi i \int\limits_\Sigma d\sigma_{\mu\nu} h_{\mu\nu}\right\}, \tag{152}$$

where the world-sheet independence of the R.H.S. is again provided by the summation over the branches of the multivalued action, which is now the action of the Kalb-Ramond field.

Comparing now Eqs. (148) and (152), we see that the Kalb-Ramond field is indeed related to the monopole density via the equation $\varepsilon_{\mu\nu\lambda}\partial_\mu h_{\nu\lambda} = \rho$. Thus, a conclusion following from the representation of the full Wilson loop in terms of the integral over the Kalb-Ramond field is that this field is simply related to the sum of the photon and monopole field strength tensors as $h_{\mu\nu} = \frac{1}{4\pi}\mathcal{F}_{\mu\nu}$. In the formal language, such a decomposition of the Kalb-Ramond field is just the essence of the Hodge decomposition theorem.

Let us now consider the weak-field limit of Eq. (152) and again restrict ourselves to the real branch of the effective potential, i.e. replace $V_{\rm total}\left[\varepsilon_{\mu\nu\lambda}\partial_\mu h_{\nu\lambda}\right]$ by $V\left[\varepsilon_{\mu\nu\lambda}\partial_\mu h_{\nu\lambda}\right]$. This yields the following expression for the Wilson loop

$$\langle W(C)\rangle_{\rm weak-field} = \int Dh_{\mu\nu} \exp\left\{-\int d^3x \left[\frac{1}{6\zeta} H_{\mu\nu\lambda}^2 + g^2 h_{\mu\nu}^2 - 2\pi i h_{\mu\nu}\Sigma_{\mu\nu}\right]\right\}. \tag{153}$$

Notice, that the mass of the Kalb-Ramond field resulting from this equation is equal to the mass m of the field χ from Eq. (136).

One can now see that Eq. (153) is quite similar to the 3D version of Eq. (123) (with the A_μ-field gauged away) we had in the DAHM case. However, the important difference from DAHM is that restricting ourselves to the real branch of the potential, we have violated the surface independence of the R.H.S. of Eq. (153). This problem is similar to the one we met in Subsection 1.3 considering SVM, where in the expression for the Wilson loop, written via the non-Abelian Stokes theorem and cumulant expansion, one disregards all the cumulants higher than the bilocal one. There, the surface independence is restored by replacing Σ by the surface of the minimal area, $\Sigma_{\rm min.} = \Sigma_{\rm min.}[C]$, bounded by the contour C. Let us follow this recipe, after which the quantity

$$S_{\rm str.} = -\ln \langle W(C)\rangle_{\rm weak-field}\Big|_{\Sigma\to\Sigma_{\rm min.}} \tag{154}$$

can be considered as a weak-field string effective action of 3D compact QED.

The integration over the Kalb-Ramond field in Eq. (153) is now almost the same as the one of Appendix 3 and yields

$$\langle W(C)\rangle_{\rm weak-field}\Big|_{\Sigma\to\Sigma_{\rm min.}} = \exp\left\{-\frac{1}{8}\int\limits_{\Sigma_{\rm min.}} d\sigma_{\lambda\nu}(\vec{x}) \int\limits_{\Sigma_{\rm min.}} d\sigma_{\mu\rho}(\vec{y}) \left\langle \mathcal{F}_{\lambda\nu}(\vec{x})\,\mathcal{F}_{\mu\rho}(\vec{y})\right\rangle_{A_\mu,\rho}\right\},$$

which is consistent with the result following directly from the cumulant expansion of Eq. (146). Here, the bilocal correlator is defined by Eq. (142) with the function $\mathcal{D}^{\rm mon.}$ given by Eq. (145) and $\mathcal{D}_1^{\rm full} = \mathcal{D}_1^{\rm phot.} + \mathcal{D}_1^{\rm mon.}$, where the photon and monopole contributions read

$$\mathcal{D}_1^{\rm phot.}\left(x^2\right) = \frac{e^2}{2\pi\left|\vec{x}\,\right|^3}$$

and

$$\mathcal{D}_1^{\rm mon.}\left(x^2\right) = \frac{e^2}{4\pi x^2}\left(m + \frac{1}{|\vec{x}\,|}\right){\rm e}^{-m|\vec{x}\,|}, \tag{155}$$

respectively. Since the approximation $\rho \ll \zeta$, in which Eq. (145) has been derived, is just the weak-field limit, in which Eq. (153) follows from Eq. (152), coincidence of the function $\mathcal{D}^{\rm mon.}$, following from the propagator of the Kalb-Ramond field, with the one of Eq. (145) confirms the consistency of our calculations.

Notice that by performing an expansion of the nonlocal string effective action (154) in powers of the derivatives w.r.t. the world-sheet coordinates ξ, one gets the string tension of the Nambu-Goto term and the inverse bare coupling constant of the rigidity term, which now read

$$\sigma = \pi^2\frac{\sqrt{2\zeta}}{g} \quad \text{and} \quad \frac{1}{\alpha_0} = -\frac{\pi^2}{8\sqrt{2\zeta}g^3}, \tag{156}$$

respectively. Similarly to the corresponding quantities in the Abelian-projected $SU(2)$- and $SU(3)$-gluodynamics, both of them are nonanalytic in g, which manifests the nonperturbative nature of string representation of all the three theories. Notice also, that the signs of σ and α_0 again turned out to be positive and negative, respectively.

We see that the long- and short distance asymptotic behaviours of the functions (145) and (155) have the same properties as the ones of the corresponding functions in QCD within SVM [65]. Namely, at large distances both of the functions (145) and (155) decrease exponentially with the correlation length m^{-1}, and at such distances $\mathcal{D}_1^{\rm mon.} \ll \mathcal{D}^{\rm mon.}$ due to the preexponential factor. In the same time, in the opposite case $|\vec{x}\,| \ll m^{-1}$, the function $\mathcal{D}_1^{\rm mon.}$ is much larger than the function $\mathcal{D}^{\rm mon.}$, which also parallels the SVM results. Notice, however, that the short-distance similarity takes place only to the lowest order of perturbation theory in QCD, where its specific non-Abelian properties are not important.

It is also worth noting, that the above described asymptotic behaviours of the functions $\mathcal{D}^{\rm mon.}$ and $\mathcal{D}_1^{\rm mon.}$ match those of the corresponding functions, which parametrize the bilocal correlator of the dual field strength tensors in DAHM (*cf.* the previous Section). This similarity, as well as the similarity of Eqs. (123) and (153), tells us that there should exist some relation between 3D compact QED and 3D AHM. In what follows, we shall demonstrate that such a relation really exists, namely 3D compact QED corresponds to the case of small gauge boson mass in the London limit of 3D AHM with monopoles. Let us stress that in 3D, monopoles are considered as particles at rest, contrary to the 4D case, where they are generally treated as world-lines of moving particles. That is why, in order to end up with 3D compact QED (i.e. the partition function (135) of the monopole gas), one should start with 3D AHM with the scalar density $\rho_{\rm gas}$ of external monopoles at rest, rather than with DAHM. The corresponding partition function has the form

$$\mathcal{Z}_{\rm 3D\,AHM} = \int DA_\mu D\theta^{\rm sing.}D\theta^{\rm reg.}\exp\left\{-\int d^3x\left[\frac{1}{4e^2}\mathcal{F}_{\mu\nu}^2 + \frac{\eta^2}{2}\left(\partial_\mu\theta - A_\mu\right)^2\right]\right\}. \tag{157}$$

Here, the full field strength tensor again reads $\mathcal{F}_{\mu\nu} = F_{\mu\nu} + F^M_{\mu\nu}$, where the monopole part obeys the equation (143) with the replacement $\rho \to \rho_{\rm gas}$. Making use of the relation (131) (where L's are now open lines of magnetic vortices, ending at monopoles and antimonopoles), one can again perform the path-integral duality transformation of the partition function (157), which yields

$$\mathcal{Z}_{\rm 3D\,AHM} = \int D\varphi Dy_\mu(\tau) Dh_\mu \exp\Biggl\{-\int d^3x \Biggl[\frac{1}{4\eta^2}\left(\partial_\mu h_\nu - \partial_\nu h_\mu\right)^2 - 2\pi i h_\mu \delta_\mu +$$

$$+\left(\frac{e}{\sqrt{2}}h_\mu + \partial_\mu\varphi\right)^2 + \frac{i}{e\sqrt{2}}\left(\frac{e}{\sqrt{2}}h_\mu + \partial_\mu\varphi\right)\frac{\partial}{\partial x_\mu}\int d^3y \frac{\rho_{\rm gas}(\vec{y}\,)}{|\vec{x}-\vec{y}\,|}\Biggr]\Biggr\}$$

(*cf.* Eq. (132)). Performing further the gauge transformation, which eliminates the field φ, and integrating over the field h_μ, we obtain

$$\mathcal{Z}_{\rm 3D\,AHM} = \int Dy_\mu(\tau) \exp\Biggl\{-\frac{\pi\eta^2}{8}\int d^3x d^3y \frac{{\rm e}^{-m_A|\vec{x}-\vec{y}\,|}}{|\vec{x}-\vec{y}\,|}\Biggl[4\pi\left(\frac{1}{m_A^2}\rho_{\rm gas}(\vec{x}\,)\rho_{\rm gas}(\vec{y}\,) + \delta_\mu(\vec{x}\,)\delta_\mu(\vec{y}\,)\right) +$$

$$+\int d^3z \frac{\rho_{\rm gas}(\vec{x}\,)\rho_{\rm gas}(\vec{z}\,)}{|\vec{y}-\vec{z}\,|}\Biggr]\Biggr\},$$

where $m_A = e\eta$ stands for the gauge boson mass. The integral

$$\int d^3y \frac{{\rm e}^{-m_A|\vec{x}-\vec{y}\,|}}{|\vec{x}-\vec{y}\,|\,|\vec{y}-\vec{z}\,|} = \int d^3u \frac{{\rm e}^{-m_A|\vec{u}\,|}}{|\vec{u}\,|\,|\vec{x}-\vec{z}-\vec{u}\,|}$$

can easily be calculated by expanding $\frac{1}{|\vec{x}-\vec{z}-\vec{u}\,|}$ in Legendre polynomials, and the result reads

$$\frac{4\pi}{m_A^2|\vec{x}-\vec{z}\,|}\left(1-{\rm e}^{-m_A|\vec{x}-\vec{z}\,|}\right).$$

Taking this into account, we can write down the final expression for the partition function of 3D AHM with external monopoles, which has the following simple form

$$\mathcal{Z}_{\rm 3D\,AHM} = \int Dy_\mu(\tau) \exp\Biggl\{-\frac{\pi\eta^2}{2}\int d^3x d^3y \Biggl[\frac{{\rm e}^{-m_A|\vec{x}-\vec{y}\,|}}{|\vec{x}-\vec{y}\,|}\delta_\mu(\vec{x}\,)\delta_\mu(\vec{y}\,) + \frac{1}{m_A^2}\frac{\rho_{\rm gas}(\vec{x}\,)\rho_{\rm gas}(\vec{y}\,)}{|\vec{x}-\vec{y}\,|}\Biggr]\Biggr\}. \quad (158)$$

The first term in square brackets on the R.H.S. of Eq. (158) represents again the Biot-Savart interaction between the points of the magnetic vortex (and also interaction between vortices, if we included several ones), which is Yukawa-type, i.e. their Coulomb interaction is screened by the condensate of electric Cooper pairs. Contrary, the interaction between *external* monopoles, represented by the last term on the R.H.S. of Eq. (158) remains to be unscreened.

We now see, that when the gauge boson mass becomes small (i.e., for example, the magnetic coupling constant $g = \frac{2\pi}{e}$ becomes large), the Biot-Savart term can be disregarded w.r.t. the interaction of external monopoles. In this limit,

$$\mathcal{Z}_{\rm 3D\,AHM} \longrightarrow \exp\left[-\frac{\pi}{2e^2}\int d^3x d^3y \frac{\rho_{\rm gas}(\vec{x}\,)\rho_{\rm gas}(\vec{y}\,)}{|\vec{x}-\vec{y}\,|}\right],$$

which is just the statistical weight of the partition function of the monopole gas (135). Clearly, this result is in agreement with that of the corresponding limiting procedure applied directly to Eq. (157).

4.2 A Method of Description of the String World-Sheet Excitations

The results of the previous Sections tell us that strings in QCD within SVM, DAHM (or $SU(2)$-gluodynamics within the Abelian projection method), and 3D compact QED can be with a good accuracy described by the same action of the massive Kalb-Ramond field interacting with the string world-sheet. This makes it reasonable to develop a unified mechanism of description of the world-sheet excitations in all these theories, based on the background-field method proposed in Ref. [138] for the nonlinear sigma models.

Let us for concreteness work with the action standing in the exponent on the R.H.S. of Eq. (78) for the case when there are no external quarks, i.e. the string world-sheets are closed. This action has the form

$$S = \int d^4x \left(\frac{1}{12\eta^2} H_{\mu\nu\lambda}^2 + \frac{1}{4e^2} h_{\mu\nu}^2 - i\pi h_{\mu\nu} \Sigma_{\mu\nu} \right), \tag{159}$$

(*cf.* Eq. (123) where the A_μ-field has been absorbed by fixing the gauge of the field $h_{\mu\nu}$). In order to develop the background-field method, one should define a geodesics passing through the background world-sheet $y_\mu(\xi)$ and the excited one, $x_\mu(\xi) = y_\mu(\xi) + z_\mu(\xi)$, where $z_\mu(\xi)$ stands for the world-sheet fluctuation. Such a geodesics has the form $\rho_\mu(\xi, s) = y_\mu(\xi) + s z_\mu(\xi)$, where s denotes the arc-length parameter, $0 \le s \le 1$. The expansion of the string effective action (159) in powers of quantum fluctuations $z_\mu(\xi)$ can be performed by virtue of the arc-dependent term describing the interaction of the Kalb-Ramond field with the string, which reads

$$S[\rho(\xi, s)] = -i\pi \int d^2\xi h_{\mu\nu}[\rho(\xi, s)] \varepsilon^{ab} \left(\partial_a \rho_\mu(\xi, s)\right) \left(\partial_b \rho_\nu(\xi, s)\right).$$

Then the term containing n quantum fluctuations has the form

$$S^{(n)} = \frac{1}{n!} \frac{d^n}{ds^n} S[\rho(\xi, s)] \bigg|_{s=0},$$

and we get

$$S^{(0)} = -i\pi \int d\sigma_{\mu\nu}(y(\xi)) h_{\mu\nu}[y(\xi)], \tag{160}$$

$$S^{(1)} = -i\pi \int d\sigma_{\mu\nu}(y(\xi)) z_\lambda(\xi) H_{\mu\nu\lambda}[y(\xi)], \tag{161}$$

and

$$S^{(2)} = -i\pi \int d^2\xi z_\nu(\xi) \varepsilon^{ab} \left(\partial_a y_\mu(\xi)\right) \left(\left(\partial_b z_\lambda(\xi)\right) H_{\nu\mu\lambda}[y(\xi)] + \frac{1}{2} z_\alpha(\xi) \left(\partial_b y_\lambda(\xi)\right) \partial_\alpha H_{\nu\mu\lambda}[y(\xi)] \right), \tag{162}$$

where during the derivation of Eqs. (161) and (162) we have omitted several full derivative terms.

Notice that as it was discussed in Ref. [138], the terms (160)-(162) are necessary and sufficient to determine all one-loop ultraviolet divergences in the theory (78). That is why in what follows we shall restrict ourselves to the derivation of the effective action, quadratic in quantum fluctuations $z_\mu(\xi)$.

In order to get such an action, we shall first carry out the integral

$$\int Dh_{\mu\nu} \exp\left[-\int d^4x \left(\frac{1}{12\eta^2} H_{\mu\nu\lambda}^2 + \frac{1}{4e^2} h_{\mu\nu}^2\right) - S^{(0)} - S^{(1)}\right].$$

It turns out to be equal to

$$\exp\left[-\pi^2 \int d\sigma_{\lambda\nu}(y(\xi)) \int d\sigma_{\mu\rho}(y(\xi')) \left(D^{(1)}_{\lambda\nu,\mu\rho}\left(y(\xi) - y(\xi')\right) + \right.\right.$$

$$\left.\left. + 2z_\alpha(\xi) \frac{\partial}{\partial y_\alpha(\xi)} D^{(1)}_{\lambda\nu,\mu\rho}\left(y(\xi) - y(\xi')\right) + z_\alpha(\xi) z_\beta(\xi') \frac{\partial^2}{\partial y_\alpha(\xi) \partial y_\beta(\xi')} D^{(1)}_{\lambda\nu,\mu\rho}\left(y(\xi) - y(\xi')\right)\right)\right]. \quad (163)$$

Here, we have taken into account that the world-sheets under study are closed, so that the boundary terms vanish.

Secondly, one should substitute the saddle-point of the integral

$$\int Dh_{\mu\nu} \exp\left[-\int d^4x \left(\frac{1}{12\eta^2} H_{\mu\nu\lambda}^2 + \frac{1}{4e^2} h_{\mu\nu}^2\right) - S^{(0)}\right]$$

into Eq. (162). This saddle-point reads

$$h^{\rm s.p.}_{\mu\nu}[y(\xi)] = \frac{ie^2m^3}{2\pi} \int d\sigma_{\mu\nu}(y(\xi')) \frac{K_1\left(m\left|y(\xi) - y(\xi')\right|\right)}{\left|y(\xi) - y(\xi')\right|},$$

where $m = \frac{\eta}{e}$ is again the mass of the Kalb-Ramond field. Upon the substitution of the obtained saddle-point value into Eq. (162) and accounting for Eq. (163), we finally get the following action quadratic in quantum fluctuations

$$S_{\rm quadr.} = \frac{\pi^2 e^2 m^2}{2} \int d\sigma_{\mu\nu}(y) \int d\sigma_{\mu\nu}(y') \left\{ 2z_\alpha(\xi)(y' - y)_\alpha \mathcal{D}_1\left((y - y')^2\right) + \right.$$

$$+ z_\alpha(\xi) z_\beta(\xi') \left[\delta_{\alpha\beta} \mathcal{D}_1\left((y - y')^2\right) - \frac{(y - y')_\alpha (y - y')_\beta}{(y - y')^2} \left(3\mathcal{D}_1\left((y - y')^2\right) + \right.\right.$$

$$\left.\left.\left. + \frac{m^3}{8\pi^2 |y - y'|} \left(3K_1\left(m|y - y'|\right) + K_3\left(m|y - y'|\right)\right)\right)\right]\right\} -$$

$$-\pi^2 e^2 m^2 \left[\int d^2\xi \varepsilon^{ab} \left(\partial_a y_\mu\right) z_\nu(\xi) \partial_b z_\lambda(\xi) + \frac{1}{2} \int d\sigma_{\mu\lambda}(y) z_\nu(\xi) z_\alpha(\xi) \frac{\partial}{\partial y_\alpha}\right] \cdot$$

$$\cdot \left\{ \int d\sigma_{\mu\lambda}(y')(y - y')_\nu + (\nu \to \mu,\, \mu \to \lambda,\, \lambda \to \nu) + (\lambda \to \mu,\, \nu \to \lambda,\, \mu \to \nu) \right\} \mathcal{D}_1\left((y - y')^2\right),$$

where $y \equiv y(\xi)$, $y' \equiv y(\xi')$, and the function $\mathcal{D}_1$ is defined by Eq. (89). It is remarkable that though the interactions between the points lieing on the background world-sheet are completely described via the function $\mathcal{D}$, dynamics of the world-sheet fluctuations is governed by the function $\mathcal{D}_1$, which in the case of open world-sheets is responsible for the perimeter-type interactions. This phenomenon can be interpreted as an interpolation between the world-sheet and world-line dynamics, which is absent on the background level.

5 Conclusion and Outlook

The main problem addressed in the present Dissertation was an attempt of an analytical description of confinement in QCD and other gauge theories. As a guiding principle for our investigations served the so-called Wilson's picture of confinement, according to which this phenomenon can be described in terms of some effective theory of strings, joining coloured objects to each other and preventing them from moving apart to macroscopic distances. In this Dissertation, we have proceeded with a derivation of such string theories corresponding to various gauge ones, including QCD, i.e. with the solution of the problem of string representation of gauge theories. We have started our analysis with the nonlocal string effective action, arising within the so-called Stochastic Vacuum Model of QCD, where the interaction between the string world-sheet elements is mediated by the phenomenological background gluon propagator. By performing the derivative expansion of this action, we have derived the first few terms of a string Lagrangian. The first two nontrivial of them turned out to be the Nambu-Goto and rigidity terms with the coupling constants expressed completely via the gluonic condensate and correlation length of the QCD vacuum. The signs of these constants ensure the stability of strings in the so-obtained effective string theory. After that, we have investigated the problem of crumpling for the string world-sheets by derivation of the topological string term in the instanton gas model of the gluodynamics vacuum. Next, by making use of perturbation theory in the nonperturbative QCD vacuum, we have calculated perturbative corrections to the obtained string effective action. Those lead to a new form of the nonlocal string effective action with the propagator between the elements of the world-sheet being the one of a perturbative gluon in the confining background. By the derivative expansion of this action, we got a correction to the rigidity term coupling constant, whereas the string tension of the Nambu-Goto term occurs to get no corrections due to perturbative gluonic exchanges. Finally, we have derived the Hamiltonian of QCD string with spinless quarks at the ends, associated with the obtained string effective action including the rigidity term. In the particular case of vanishing orbital momentum of the system, this Hamiltonian reduces to that of the so-called relativistic quark model, albeit with some modifications due to the rigidity term, which might have some influence on the dynamics of the QCD string with quarks. All these topics have been elaborated on in Section 2, and form the essence of the string representation of QCD within the Stochastic Vacuum Model.

In Section 3, we have addressed the problem of string representation of Abelian-projected theories. In this way, we have started with the string representation for the partition function of the simplest model of this kind, namely the Abelian-projected $SU(2)$-QCD, which is argued to be the dual Abelian Higgs Model with external electrically charged particles. The advantage of this approach to the string representation of QCD w.r.t. the one based on the Stochastic Vacuum Model is a possibility to get an integration over the string world-sheets, resulting from the integration over the singular part of the phase of the Higgs field. After the string representation of the partition function in the London limit, we have proceeded with the string representation for the generating functionals of the field strength and monopole current correlators. Those enabled us to find the corresponding bilocal cumulants and demonstrate that the large-distance asymptotic behaviour of the bilocal field strength cumulant matches the one of the corresponding gauge-invariant cumulant in QCD, predicted by the Stochastic Vacuum Model and measured in the lattice experiments. This result supports the method of Abelian projection on the one hand and gives a new field-theoretical status to the Stochastic Vacuum Model on the other hand. After that, we have extended our analysis beyond the London limit, and studied the relation of the quartic cumulant, which appears there, with the bilocal one in the London limit. Next, by making use of the Abelian projection method, we have addressed the problem of string representation of the $SU(3)$-gluodynamics. Namely, we have casted the related dual model, containing three types of magnetic Higgs fields, into the string form. Consequently, the latter one turned out to contain three types of strings, among which only two ones were actually independent. As a result, we have found, that both the ensemble of strings as a whole and individual strings display confining properties in a sense that all types of strings (self)interact via the exchanges of the massive dual gauge bosons. We have also derived bilocal cumulants in the effective dual model of confinement, corresponding to the $SU(3)$-gluodynamics, and they turned out to be also in line with the ones predicted by the Stochastic Vacuum Model. In conclusion of this topic, we have obtained another useful representation for the partition functions of the Abelian-projected theories in the form of an integral over the monopole currents.

In Section 4, we have studied another model, allowing for an analytical description of confinement, which is 3D compact QED. In this way, by virtue of the integral over the monopole densities, we have derived string representation for the Wilson loop in this theory and demonstrated the correspondence of this representation to another recently found one, the so-called confining string theory. After that, we have calculated the bilocal cumulant of the field strength tensors in the weak-field limit of the model under study. It also turned out to be in line with the general concepts of the Stochastic Vacuum Model and therefore matches the corresponding results known from the lattice measurements in QCD and found analytically for the effective Abelian-projected theories in the previous Section. Besides that, string representations of the partition functions of the weak-field limit of 3D compact QED and of the dual Abelian Higgs Model turned out to be also quite similar. We have illustrated later on that this correspondence is not accidental. Namely, we have shown that 3D compact QED is nothing else, but the limiting case of 3D Abelian Higgs Model with external monopoles, corresponding to the vanishing gauge boson mass. Finally, we have elaborated on a unified method of description of the string world-sheet excitations in the Abelian-projected theories, compact QED, and QCD, based on the techniques of nonlinear sigma-models, and obtained the effective action, quadratic in the world-sheet fluctuations.

In conclusion, the proposed nonperturbative techniques provide us with some new information on the mechanisms of confinement in QCD and other gauge theories and shed some light on the vacuum structure of these theories. They also show the relevance of string theory to the description of this phenomenon and yield several prescriptions for the construction of the adequate string theories from the corresponding gauge ones.

Further investigations of the problems addressed in the present Dissertation are planned to follow in at least two directions. First of them is a derivation of the string representation for the last up to now known model allowing for an analytical description of confinement, which is the low-energy $SU(2)$ $\mathcal{N}=2$ supersymmetric Yang-Mills theory (the so-called Seiberg-Witten theory) [40]. The second line of investigations is devoted to a better understanding of interrelation between confinement and chiral symmetry breaking. In this way, it is planned to develop further the approach proposed in Ref. [49] by virtue of bosonization of the equations obtained there. This turned out to be a rather hard topic due to the non-translation-invariant character of the corresponding interaction kernel. However, it is this kernel, which is responsible for the confining effects in the system, and therefore it should not probably be reduced to some more simple translation-invariant one. Investigations of this problem together with an application of other well elaborated methods known from the theory of NJL models might finally enable one to understand completely the interrelations between the phenomena of confinement and SCSB in QCD.

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7 Appendices

7.1 Details of the Derivation of the Hamiltonian of the Straight-Line QCD String with Quarks

Assuming that a meson as a whole moves with a constant speed (which is true for a free meson), i.e. $\ddot{R} = 0$, and bringing together quark kinetic terms (61) and the pure string action (63), we arrive at the following action of the QCD string with quarks

$$S_{\rm tot.} = \int\limits_0^T d\tau \left\{ \frac{m_1^2}{2\mu_1} + \frac{m_2^2}{2\mu_2} + \frac{\mu_1}{2} + \frac{\mu_2}{2} + \frac{1}{2}\left(\mu_1 + \mu_2 + \int\limits_0^1 d\beta\nu\right)\dot{R}^2 + \right.$$

$$+\left(\mu_1(1-\zeta_1) - \mu_2\zeta_1 + \int\limits_0^1 d\beta(\beta-\zeta_1)\nu\right)\left(\dot{R}\dot{r}\right) - \int\limits_0^1 d\beta\nu\eta\left(\dot{R}r\right) + \int\limits_0^1 d\beta(\zeta_1-\beta)\eta\nu(\dot{r}r) +$$

$$+\frac{1}{2}\left(\mu_1(1-\zeta_1)^2 + \mu_2\zeta_1^2 + \int\limits_0^1 d\beta(\beta-\zeta_1)^2\nu\right)\dot{r}^2 + \frac{1}{2}\int\limits_0^1 d\beta\left(\frac{\sigma^2}{\nu} + \eta^2\nu\right)r^2 +$$

$$+\frac{1}{\alpha_0}\left[\zeta_2(\mu_1(\zeta_1-1)+\mu_2\zeta_1)\dot{r}^2 - \zeta_2(\mu_1+\mu_2)\left(\dot{R}\dot{r}\right) + \int\limits_0^1 d\beta\nu\left(\zeta_2(\zeta_1-\beta)\dot{r}^2 - \zeta_2\left(\dot{R}\dot{r}\right) + \zeta_2\eta(\dot{r}r) + \right.\right.$$

$$+\frac{1}{2}(\beta-\zeta_1)^2\left[\ddot{\vec{r}},\vec{r}\right]^2 + \frac{1}{2}\dot{R}^2\dot{r}^2 - \frac{1}{2}\left(\dot{R}\dot{r}\right)^2 + (\beta-\zeta_1)\left(\left(\ddot{r}\dot{R}\right)(\dot{r}r) - (\ddot{r}\dot{r})\left(\dot{R}r\right)\right) + \frac{1}{2}\left(\left(\frac{\sigma}{\nu}\right)^2 + \eta^2\right)\left[\dot{\vec{r}},\vec{r}\right]^2 +$$

$$\left.\left.\left.+\eta\left((\beta-\zeta_1)\left((\ddot{r}r)(\dot{r}r) - (\ddot{r}\dot{r})r^2\right) + \left(\dot{R}r\right)\dot{r}^2 - \left(\dot{R}\dot{r}\right)(\dot{r}r)\right)\right)\right]\right\}, \qquad (164)$$

where we have performed a rescaling $z_\mu \to \bar{\alpha}\sqrt{\frac{h}{\sigma T^2}}z_\mu$, $\bar{z}_\mu \to \bar{\alpha}\sqrt{\frac{h}{\sigma T^2}}\bar{z}_\mu$, $\nu \to \frac{\sigma T^2}{\bar{\alpha}^2 h}\nu$. Integrating then over η, one gets in the zeroth order in $\frac{1}{\alpha_0}$

$$\eta_{\rm ext.} = \frac{(\dot{r}r)}{r^2}\left(\beta - \frac{\mu_1}{\mu_1+\mu_2}\right),$$

which together with the condition $\dot{R}\dot{r} = 0$ yields

$$\zeta_1^{\rm ext.} = \frac{\mu_1 + \int\limits_0^1 d\beta\beta\nu}{\mu_1 + \mu_2 + \int\limits_0^1 d\beta\nu},$$

$$\zeta_2^{\rm ext.} = \frac{(\dot{r}r)^2}{r^2}\frac{\frac{\mu_1}{\mu_1+\mu_2}\int\limits_0^1 d\beta\nu - \int\limits_0^1 d\beta\beta\nu}{\mu_1 + \mu_2 + \int\limits_0^1 d\beta\nu}.$$

Finally, in order to obtain the desired Hamiltonian, we shall perform the usual canonical transformation from $\dot{\vec{R}}$ to the total momentum $\vec{P}$, which in the Minkowski space-time reads

$$\int D\vec{R}\exp\left[i\int L\left(\dot{\vec{R}},...\right)d\tau\right]=\int D\vec{R}D\vec{P}\exp\left[i\int\left(\vec{P}\dot{\vec{R}}-H\left(\vec{P},...\right)\right)d\tau\right],$$

with $H\left(\vec{P},...\right)=\vec{P}\dot{\vec{R}}-L\left(\dot{\vec{R}},...\right)$, and choose the meson rest frame as

$$\vec{P}=\frac{\partial L\left(\dot{\vec{R}},...\right)}{\partial\dot{\vec{R}}}=0.$$

After performing the transformation from $\dot{\vec{r}}$ to $\vec{p}$ we arrive at the Hamiltonian (64) with the coefficient functions a_k, which read as follows

$$a_1=\frac{\sigma^2}{2}\int\limits_0^1\frac{d\beta}{\nu},\ a_2=3\frac{\dot{\tilde{\mu}}}{\tilde{\mu}}B-\dot{B},$$

$$a_3=\frac{1}{2(\mu_1+\mu_2)(\mu_1+\mu_2+\nu_0)}\left[\frac{\nu_0(\mu_1\nu_0-\nu_1(\mu_1+\mu_2))^2}{(\mu_1+\mu_2)(\mu_1+\mu_2+\nu_0)}-\nu_1(\mu_1+\mu_2)(\nu_1-2\mu_2)-\mu_1\nu_0(\mu_1+2\mu_2)\right],$$

$$a_4=\frac{\dot{\tilde{\mu}}}{\tilde{\mu}}B-\dot{B},$$

$$a_5=\nu_2+\frac{\nu_1^2+2\mu_1\nu_0-2\mu_2\nu_1}{\mu_1+\mu_2+\nu_0}+\frac{1}{\mu_1+\mu_2}\left[\frac{1}{\mu_1+\mu_2}\left(\frac{(\mu_1\nu_0-\nu_1(\mu_1+\mu_2))^2(3\nu_0+2(\mu_1+\mu_2))}{(\mu_1+\mu_2+\nu_0)^2}+\right.\right.$$
$$\left.\left.+\mu_1(\mu_1\nu_0-2\nu_1(\mu_1+\mu_2))\right)-\frac{\mu_1^2\nu_0}{\mu_1+\mu_2+\nu_0}\right],$$

and

$$B\equiv\frac{\nu_1(\mu_1+\mu_2)(\nu_1-2\mu_2)+\mu_1\nu_0(\mu_1+2\mu_2)}{(\mu_1+\mu_2)(\mu_1+\mu_2+\nu_0)}.$$

Notice, that during the derivation of $H^{(1)}$ we have chosen the origin at the centre of masses of the initial state, so that $\dot{\vec{R}}\dot{\vec{r}}\ll 1$, and the term $-\frac{1}{2\alpha_0}\int\limits_0^T d\tau\nu_0\left(\dot{\vec{R}}\dot{\vec{r}}\right)^2$ on the R.H.S. of Eq. (164) has been disregarded.

7.2 Path-Integral Duality Transformation

In this Appendix, we shall outline some details of the derivation of Eqs. (77) and (78). Firstly, one can linearize the term $\frac{\eta^2}{2}\left(\partial_\mu\theta-2g_mB_\mu\right)^2$ in the exponent on the R.H.S. of Eq. (75) and carry out the integral over $\theta^{\rm reg.}$ as follows

$$\int D\theta^{\rm reg.}\exp\left\{-\frac{\eta^2}{2}\int d^4x\left(\partial_\mu\theta-2g_mB_\mu\right)^2\right\}=$$

$$= \int DC_\mu D\theta^{\rm reg.} \exp\left\{\int d^4x \left[-\frac{1}{2\eta^2}C_\mu^2 + iC_\mu\left(\partial_\mu\theta - 2g_m B_\mu\right)\right]\right\} =$$

$$= \int DC_\mu \delta\left(\partial_\mu C_\mu\right) \exp\left\{\int d^4x \left[-\frac{1}{2\eta^2}C_\mu^2 + iC_\mu\left(\partial_\mu\theta^{\rm sing.} - 2g_m B_\mu\right)\right]\right\}. \qquad (165)$$

The constraint $\partial_\mu C_\mu = 0$ can be uniquely resolved by representing C_μ in the form $C_\mu = \frac{1}{2}\varepsilon_{\mu\nu\lambda\rho}\partial_\nu h_{\lambda\rho}$, where $h_{\lambda\rho}$ stands for an antisymmetric tensor field. Notice, that the number of degrees of freedom during such a replacement is conserved, since both of the fields C_μ and $h_{\mu\nu}$ have three independent components.

Then, taking into account the relation (76) between $\theta^{\rm sing.}$ and $\Sigma_{\mu\nu}$, we get from Eq. (165)

$$\int D\theta^{\rm sing.} D\theta^{\rm reg.} \exp\left\{-\frac{\eta^2}{2}\int d^4x \left(\partial_\mu\theta - 2g_m B_\mu\right)^2\right\} =$$

$$= \int Dx_\mu(\xi) Dh_{\mu\nu} \exp\left\{\int d^4x \left[-\frac{1}{12\eta^2}H_{\mu\nu\lambda}^2 + i\pi h_{\mu\nu}\Sigma_{\mu\nu} - ig_m\varepsilon_{\mu\nu\lambda\rho}B_\mu\partial_\nu h_{\lambda\rho}\right]\right\}. \qquad (166)$$

In the derivation of Eq. (166), we have replaced $D\theta^{\rm sing.}$ by $Dx_\mu(\xi)$ (since the surface Σ, parametrized by $x_\mu(\xi)$, is just the surface, at which the field θ is singular) and, for simplicity, have discarded the Jacobian arising during such a change of the integration variable [26].

Bringing now together Eqs. (75) and (166), we arrive at Eq. (77). In the literature, the above described sequence of transformations of integration variables is usually referred to as a "path integral duality transformation". In particular, it has been applied in Ref. [121] to the model with a *global* $U(1)$-symmetry.

Let us now derive Eq. (78). To this end, we find it convenient to rewrite

$$\exp\left(-\frac{1}{4}\int d^4x F_{\mu\nu}^2\right) = \int DG_{\mu\nu} \exp\left\{\int d^4x \left[-G_{\mu\nu}^2 + i\tilde{F}_{\mu\nu}G_{\mu\nu}\right]\right\},$$

after which the B_μ-integration in Eq. (77) yields

$$\int DB_\mu \exp\left\{-\int d^4x \left[\frac{1}{4}F_{\mu\nu}^2 + i\tilde{F}_{\mu\nu}\left(g_m h_{\mu\nu} + 2\pi ie\Sigma_{\mu\nu}\right)\right]\right\} =$$

$$= \int DG_{\mu\nu} \exp\left(-\int d^4x G_{\mu\nu}^2\right) \delta\left(\varepsilon_{\mu\nu\lambda\rho}\partial_\mu\left(G_{\lambda\rho} - g_m h_{\lambda\rho} - 2\pi ie\Sigma_{\lambda\rho}\right)\right) =$$

$$= \int DA_\mu \exp\left[-\int d^4x \left(g_m h_{\mu\nu} + 2\pi ie\Sigma_{\mu\nu} + \partial_\mu A_\nu - \partial_\nu A_\mu\right)^2\right], \qquad (167)$$

In the last line of Eq. (167), the constraint

$$\varepsilon_{\mu\nu\lambda\rho}\partial_\mu\left(G_{\lambda\rho} - g_m h_{\lambda\rho} - 2\pi ie\Sigma_{\lambda\rho}\right) = 0$$

has been resolved by setting $G_{\lambda\rho} = g_m h_{\lambda\rho} + 2\pi ie\Sigma_{\lambda\rho} + \partial_\lambda A_\rho - \partial_\rho A_\lambda$. Here A_μ is just the usual gauge field, dual to the dual field B_μ.

Finally, by performing in Eq. (167) the hypergauge transformation $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\lambda_\nu - \partial_\nu\lambda_\mu$ and fixing the gauge by choosing $\lambda_\mu = -\frac{1}{g_m}A_\mu$ (see e.g. Ref. [87]), we arrive, omitting the measure factor DA_μ, at Eq. (78).

[26] For the case when the surface Σ has a spherical topology, this Jacobian has been calculated in Ref. [120].

7.3 Integration over the Kalb-Ramond Field

Let us carry out the following integration over the Kalb-Ramond field

$$\mathcal{Z} = \int Dh_{\mu\nu} \exp\left[-\int d^4x \left(\frac{1}{12\eta^2} H^2_{\mu\nu\lambda} + \frac{1}{4e^2} h^2_{\mu\nu} + i\pi h_{\mu\nu}\Sigma_{\mu\nu}\right)\right]. \tag{168}$$

To this end, it is necessary to substitute the saddle-point value of the integral (168) back into the integrand. The saddle-point equation in the momentum representation reads

$$\frac{1}{2\eta^2}\left(p^2 h^{\text{extr.}}_{\nu\lambda}(p) + p_\lambda p_\mu h^{\text{extr.}}_{\mu\nu}(p) + p_\mu p_\nu h^{\text{extr.}}_{\lambda\mu}(p)\right) + \frac{1}{2e^2} h^{\text{extr.}}_{\nu\lambda}(p) = -i\pi\Sigma_{\nu\lambda}(p).$$

This equation can be most easily solved by rewriting it in the following way

$$\left(p^2 \mathbf{P}_{\lambda\nu,\alpha\beta} + m^2 \mathbf{1}_{\lambda\nu,\alpha\beta}\right) h^{\text{extr.}}_{\alpha\beta}(p) = -2\pi i\eta^2 \Sigma_{\lambda\nu}(p), \tag{169}$$

where we have introduced the following projection operators

$$\mathbf{P}_{\mu\nu,\lambda\rho} \equiv \frac{1}{2}\left(\mathcal{P}_{\mu\lambda}\mathcal{P}_{\nu\rho} - \mathcal{P}_{\mu\rho}\mathcal{P}_{\nu\lambda}\right) \text{ and } \mathbf{1}_{\mu\nu,\lambda\rho} \equiv \frac{1}{2}\left(\delta_{\mu\lambda}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\lambda}\right)$$

with $\mathcal{P}_{\mu\nu} \equiv \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}$. These projection operators obey the following relations

$$\mathbf{1}_{\mu\nu,\lambda\rho} = -\mathbf{1}_{\nu\mu,\lambda\rho} = -\mathbf{1}_{\mu\nu,\rho\lambda} = \mathbf{1}_{\lambda\rho,\mu\nu}, \tag{170}$$

$$\mathbf{1}_{\mu\nu,\lambda\rho}\mathbf{1}_{\lambda\rho,\alpha\beta} = \mathbf{1}_{\mu\nu,\alpha\beta} \tag{171}$$

(the same relations hold for $\mathbf{P}_{\mu\nu,\lambda\rho}$), and

$$\mathbf{P}_{\mu\nu,\lambda\rho}\left(\mathbf{1} - \mathbf{P}\right)_{\lambda\rho,\alpha\beta} = 0. \tag{172}$$

By virtue of properties (170)-(172), the solution of Eq. (169) reads

$$h^{\text{extr.}}_{\lambda\nu}(p) = -\frac{2\pi i\eta^2}{p^2 + m^2}\left[\mathbf{1} + \frac{p^2}{m^2}\left(\mathbf{1} - \mathbf{P}\right)\right]_{\lambda\nu,\alpha\beta} \Sigma_{\alpha\beta}(p),$$

which, once being substituted back into partition function (168), yields for it the following expression

$$\mathcal{Z} = \exp\left\{-\pi^2\eta^2 \int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2 + m^2}\left[\mathbf{1} + \frac{p^2}{m^2}\left(\mathbf{1} - \mathbf{P}\right)\right]_{\mu\nu,\alpha\beta} \Sigma_{\mu\nu}(-p)\Sigma_{\alpha\beta}(p)\right\}. \tag{173}$$

Rewriting Eq. (173) in the coordinate representation we arrive at Eq. (79).

Let us now prove that the term proportional to the projection operator $\mathbf{1} - \mathbf{P}$ on the R.H.S. of Eq. (173) indeed yields in the coordinate representation Eq. (81), which can be further reduced to the boundary term. One has

$$p^2(\mathbf{1} - \mathbf{P})_{\lambda\nu,\alpha\beta} = \frac{1}{2}(\delta_{\nu\beta}p_\lambda p_\alpha + \delta_{\lambda\alpha}p_\nu p_\beta - \delta_{\nu\alpha}p_\lambda p_\beta - \delta_{\lambda\beta}p_\nu p_\alpha). \tag{174}$$

By making use of Eq. (174), the term

$$-\pi^2\eta^2 \int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2+m^2}\frac{p^2}{m^2}(\mathbf{1}-\mathbf{P})_{\mu\nu,\alpha\beta}\int d^4x \int d^4y \mathrm{e}^{ip(y-x)}\Sigma_{\mu\nu}(x)\Sigma_{\alpha\beta}(y)$$

under study, after carrying out the integration over p, reads

$$\frac{\eta^2}{2m}\int d^4x \Sigma_{\mu\nu}(x)\int d^4y \Sigma_{\nu\beta}(y)\frac{\partial^2}{\partial x_\mu \partial y_\beta}\frac{K_1(m\,|x-y|)}{|x-y|}. \qquad (175)$$

Acting in Eq. (175) straightforwardly with the derivatives, we arrive at Eq. (81). However, one can perform the partial integration, which yields the argument of the first exponent standing on the R.H.S. of Eq. (82), i.e. the boundary term.

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8 Zusammenfassung

Die vorliegende Dissertationarbeit ist dem Problem der analytischen Beschreibung des Confinement-Mechanismus in der QCD und in anderen Eichtheorien gewidment. Als Leitprinzip der Arbeit wurde das sogenannte Wilsonsche-Confinement-Kriterium gewählt, gemäss welchem diese Erscheinung durch eine effektive Stringtheorie beschrieben werden kann. Die entstehenden Strings des Eichfeldes verbinden farbige-Objekte (Quarks, Gluonen) miteinander und hindern ihr Auseinandergehen auf makroskopische Abstände. Es werden verschiedene Verfahren der Ableitung dieser Stringstheorien aus unterschiedlichen Eichtheorien, einschliesslich der QCD, vorgestellt.

Kapitel 2 enthält die Untersuchung der nichtlokalen effektiven Stringwirkung, die im Rahmen des sogenannten stochastischen Vakuum-Modells der QCD entsteht, wobei die Wechselwirkung zwischen den Elementen der String-Weltfläche durch den phänomenologischen Background-Gluon-Propagator vermittelt wird. Durch Entwicklung dieser Wirkung nach Ableitungen wurden die ersten Terme niedrigster Ordnung bestimmt. Die ersten beiden Terme dieser Entwicklung sind die Nambu-Goto- und Rigidity-Terme mit Kopplungskonstanten, die sich durch das Gluon-Kondensat und die Korrelationlänge des QCD-Vakuums ausdrücken lassen. Die Vorzeichen dieser Konstanten zeigen, dass die durch dieses Verfahren erhaltenen Strings stabil sind. Danach wurde eine mögliche Lösung des "Crumpling" Problems auf der Basis eines zusätzlichen topologischen Stringtermes im Instantongas-Modell des QCD-Vakuums vorgestellt. Mittels Störungstheorie im nicht-störungstheoretischen QCD-Background berechneten wir zusätzliche-Korrekturen zur ursprünglichen nicht-störungstheoretischen Stringwirkung. Diese Korrekturen führen zu neuen Formen der nichtlokalen effektiven Stringwirkung, die den störungstheoretischen Gluon-Propagator im Backgroundfeld zwischen den Elementen der Weltfläche enthalten. Durch Ableitungsentwicklung dieser Wirkung bekommen wir eine Korrektur zur Kopplungskonstante des Rigidity-Terms; die Stringsspannung des Nambu-Goto-Terms jedoch bleibt unverändert. Am Ende dieses Kapitels wurde der Hamilton-Operator des QCD-Strings mit spinlosen Quarks hergeleitet, der der effektiven Stringwirkung mit Rigidity-Term entspricht. Dieser Hamilton-Operator liefert einen Korrekturterm zur Wechselwirkung im relativistischen Quarkmodell-Operator.

Im Kapitel 3 untersuchten wir das Problem der Stringdarstellung von Abelsch-projezierten Eichtheorien. Als erstes wurde die Herleitung der Stringdarstellung der erzeugenden Funktion für das einfachste Modell dieser Art, d.h. die Abelsch-projezierte $SU(2)$-QCD gegeben, die einem dualen Abelschen Higgs-Modell mit äusseren elektrisch geladendenen Teilchen äquivalent ist. Der Vorteil dieses Stringszuganges im Vergleich zum Zugang des stochastischen Vakuum-Modells der QCD besteht in der Berücksichtigung der Integration über String-Weltflächen, die auf Grund der Integration über den Singulärteil der Higgsfeld-Phase entsteht. Zusätzlich zur Stringdarstellung der erzeugenden Funktion wurde im London-Limes die Stringdarstellung für die erzeugenden Funktionale der Feldstärke- und Monopolstromkorrelatoren hergeleitet. Dies gab uns die Möglichkeit, die entsprechenden bilokalen Kumulanten zu finden und zu zeigen, dass die bilokalen Kumulanten der Feldstärke für grosse Abstände das gleiche Verhalten wie die entsprechenden eichinvarianten Kumulanten der QCD zeigen. Das Letztere wurde durch das stochastische Vakuum-Modell vorhergesagt und durch Gitterexperimente berechnet. Dieses Ergebnis unterstützt einerseits die Methode der Abelschen Projektion und gibt anderseits dem stochastischen Vakuum-Modell der QCD einen neuen feldtheoretischen Status. Danach erweiterten wir unsere Analyse über den Rahmen des London-Limes hinaus untersuchten den Zusammenhang von quartischen Kumulanten und bilokalen Kumulanten. Anschliessend wurde die Stringdarstellung der $SU(3)$-Gluodynamik hergeleitet. Dabei wurde die Stringdarstellung für ein entsprechendes duales Modell formuliert, das drei Arten des magnetischen Higgs-Feldes enthält. Infolgedessen liefert das Modell drei Strings, von denen nur zwei wirklich unabhängig sind. Alle diese Strings wechselwirken untereinander durch Austausch zweier massiver dualer Eichbosonen. Ausserdem erhielten wir die bilokalen Kumulanten des effektiven dualen Modells der $SU(3)$-Gluodynamik. Die entsprechenden bilokalen Kumulanten zeigen für grosse Abstände ein Verhalten wie es durch das stochastische Vakuum-Modell vorhergesagt wurde. Zum Schluss dieses Kapitels geben wir eine nützliche Darstellung für erzeugende Funktionen von Abelsch-projezierten Theorien in Form von Integralen über Monopolströme an.

Im Kapitel 4 wurde ein weiteres Modell untersucht, das eine analytische Beschreibung des Confinement-Mechanismus zulässt, nämlich die 3D kompakte QED. Für den Wilson-Loop der entsprechenden Theorie mit Monopoldichten wurde die Äquivalenz zur sogenannten Confining-Stringtheorie demonstriert. Ausserdem wurde das Verhalten der bilokalen Kumulante der Feldstärke im Limes schwacher Felder untersucht. Dieses Verhalten befindet sich ebenfalls in Übereinstimmung mit den Voraussagen des stochastischen Vakuum-Modells. Erwartungsgemäss sind die Stringdarstellungen der erzeugenden Funktionen der 3D kompakten QED im Limes schwacher Felder und der dualen Abelschen Higgs-Modelle sehr ähnlich. Wir zeigten ausserdem, dass diese Entsprechung nicht zufällig ist. Die 3D kompakte QED ergibt sich nämlich im Limes verschwindender Eichbosonmasse aus dem 3D Abelschen Higgs-Modell mit äusseren Monopolen. Zum Schluss wurde ein allgemeines Verfahren der Beschreibung der Anregungen der Stringweltfläche in Abelsch-projezierten Theorien (kompakte QED und QCD) ausgearbeitet. Es ist auf der Methode der nicht-linearen Sigma-Modelle gegründet und gibt eine Möglichkeit, die in diesen Fluktuationen quadratische Effektive Wirkung zu erhalten.

In der Dissertation wurden analytische nicht-störungstheoretische Verfahren ausgearbeitet, die neue Informationen über den Confinement-Mechanismus in der QCD und anderen Eichtheorien liefern und zum besseren Verständnis der Vakuumstruktur dieser Theorien beitragen können. Sie sind insbesondere relevant für die Herleitung effektiver Stringtheorien aus Eichtheorien.

9 Selbständigkeitserklärung

Hiermit versichere ich, die vorliegende Dissertation *"String Representation of Gauge Theories"* selbständig angefertigt und keine anderen als die aufgeführten Hilfsmittel verwendet zu haben.

PUBLIKATIONSLISTE

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