On the Interplay between String Theory and Field Theory

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Dipl.-Phys. Ilka Brunner
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Präsident der Humboldt-Universität zu Berlin
Prof. Dr. H. Meyer

Dekan der Mathematisch-Naturwissenschaftlichen Fakultät I
Prof. Dr. J.P. Rabe

Gutachter :
1. Prof. Lüst Humboldt-Universität zu Berlin
2. Dr. Dorn Humboldt-Universität zu Berlin
3. Prof. Theisen LMU München

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Contents

Why String Theory?

In the present thesis, we will discuss various aspects of recent developments in string theory. Before we address some specialized topics, we would like to give a general discussion about the present state of elementary particle physics and the aims of string theory.

There are four fundamental forces known in nature: The strong, weak, electromagnetic and gravitational interaction. The first three of these forces are well described in the framework of quantum field theory, or – to be more precise – by gauge field theories. This implies the picture that the corresponding forces are carried by particles (the gauge bosons). The theory of elementary particles and the forces that govern their interaction is given by the standard model. The predictions of the standard model are in excellent agreement with measurements at microscopic scales. The gravitational interaction is described by general relativity where the gravitational force is encoded in a non-trivial space-time geometry. This theory has been very successful in explaining the physics of macroscopic structures like the solar system or even cosmological scales. General relativity is a classical theory not including quantum effects. On the level of the classical theories, many parallels can be found between gauge theories and general relativity. The local symmetry group in general relativity is the group of diffeomorphisms of the space-time manifold. In the comparison, the role of charge in gauge theories is played by the mass of an object in general relativity. We can also find gravitational waves (as solutions of the linearized Einstein equations) on the classical level. This suggests that there might be a particle, the graviton, which mediates the interaction between massive particles. There might be the hope to include the graviton in a consistent theory of quantum gravity, which connects the principles of quantum physics to gravity. However, it is not easy to find such a quantum theory of gravity. The unification of quantum field theory and general relativity into one fundamental theory is one of the most challenging questions in theoretical physics.

Conventional quantum field theories are symmetric under the Lorentz group but do not include general relativity. Space time is treated as a non-dynamical background. One may try to unify gravity with gauge field theory by modifying the action such that it is invariant under general coordinate transformations and adding the Einstein-Hilbert action. However, the quantum theory of gravity is non-renormalizable. We cannot extract finite quantities from calculations using standard techniques of perturbative quantum field theory. This indicates, that we have to go to a framework which goes beyond the usual local quantum field theories of point particles. One suggestion to obtain a unified theory is string theory. Here, one considers one-dimensional extended objects, the strings, rather than point particles. A string propagating in space-time sweeps out a two-dimensional world sheet, in contrast to the one-dimensional world line of a particle. The particles we observe at low energies are interpreted as excitation modes of the string. From the point of view of the two-dimensional world sheet, the space-time coordinates give rise to scalar fields in a two-dimensional field theory. The space-time metric G appears as a coupling constant in this theory. Consistency conditions for the two-dimensional field theory have far-reaching consequences. In particular, the theories are invariant under reparametrizations of the world sheet. Classically, the two-dimensional theories are conformally invariant. Imposing the condition that this symmetry is preserved at the quantum level leads to a restriction of the possible space-time dimensions. The consistent string theories can be divided into two groups: First, there are string theories which also contain fermionic degrees of freedom and there are bosonic string theories with purely bosonic degrees of freedom. Conformal invariance of the supersymmetric string theories living in trivial backgrounds requires space time to be 10-dimensional, whereas the critical dimension for the bosonic string is 26. All these string theories automatically include gravity because they have a massless spin two particle – the graviton – in their spectrum. If we consider curved backgrounds, the Einstein-equations of general relativity can be recovered as the condition, that the β-function for the "coupling" G, the space-time metric, vanishes. The vanishing of the β-function is again a consequence of the requirement that conformal invariance is preserved at the quantum level.

The program of unifying quantum field theory and general relativity turned out to be of particular importance in the context of black hole physics. Black holes are classical solutions to Einstein's equations, which have a horizon: Anything which falls into the black hole will never come back. The information is completely lost. The work of Bekenstein and Hawking has shown that once we include quantum effects we can formulate thermodynamics of a black hole. Here, quantum effects are included in a semi-classical way. Gravity is considered as a classical background and one investigates quantum fields propagating in a curved non-dynamical background. Using the semi-classical approximation, it was found that a black hole emits thermal radiation of a temperature, which is proportional to the surface tension of the black hole. An empirical observation is that the following law holds

$$dM = TdS, \tag{1}$$

where M is the mass of the black hole and S is one quarter of the area of the event horizon. Analogy of equation (1) to the first law of thermodynamics suggests to give S the interpretation of an entropy (the Bekenstein-Hawking entropy). If the Bekenstein-

Hawking entropy S is really an entropy, there should be a microscopic interpretation in terms of the logarithm of the density of states. There have been attempts to clarify this since the discovery of Bekenstein and Hawking. Because string theory claims to provide us with a consistent theory of quantum gravity, it should be able to solve this puzzle. Indeed, recently for a certain class of black holes a statistical interpretation of S has been given using string theory.

The discussion so far focused on the physics in the presence of very strong gravitational fields, such as in the neighbourhood of black holes. However, these are not the situations which can be tested experimentally on earth. The energy regimes tested by present day accelerators are well below the Planck scale, which is about 10^{19} GeV. Let us therefore discuss which implications string theory has for the low-energy physics. We already pointed out that the low energy physics is described by the standard model of elementary particle physics, which so far has been able to explain the experiments performed in accelerators. However, it is not completely satisfactory. The standard model contains a set of parameters, including the particle masses and coupling constants. One might expect from this that there exists a more fundamental theory which explains the particular values of these parameters, which can be measured in experiments. If string theory is a fundamental and unique theory, it should finally be able to explain the parameters. But these philosophical reasons are not the only reasons to think that the standard model is not the final answer. In addition, there are some problems, which are not satisfactory solved within the framework of the standard model, like the so-called hierarchy-problem. In the standard model, the scale of electroweak symmetry breaking is set by the expectation value of a scalar particle. This scale is many orders of magnitude smaller than the Planck scale. But boson masses are affected by large radiative corrections. The quantum corrections to the mass are quadratically divergent. To maintain the hierarchy of the scales therefore leads to a fine-tuning problem. It is certainly possible to adjust the bare mass appropriately, but this does not seem to be natural. In supersymmetric theories the quadratic divergencies coming from bosons and fermions cancel exactly, which would solve the hierarchy problem. As super-string theory is supersymmetric, it would allow for such a scenario.

An introduction to more recent developments in string theory and an outline of the thesis can be found at the beginning of the next chapter.

Chapter 1

Recent Developments in String Theory

Our understanding of superstring theory [1, 2, 3] has improved tremendously during the past few years. For the first time it became possible to control also parts of the non-perturbative regime of the theories. This was made possible by the discovery of the so-called duality symmetries. These symmetries map the strong coupling regime of one theory to the better understood weak coupling region of another theory. Using these duality symmetries we can therefore obtain information about the strong coupling properties by doing calculations in a weakly coupled theory.

There is also another related reason why duality symmetries shed a new light on string theory: In 10 dimensions we have 5 different perturbative superstring theories: type IIA, type IIB, the heterotic string with gauge group $SO(32)$, the heterotic string with gauge group $E_8 \times E_8$ and the type I superstring. The first four theories are closed string theories, whereas the type I string is a theory of open strings. String theory claims to be in some sense unique, so the existence of 5 consistent theories in 10 dimensions is somewhat discouraging. The existence of the dualities gives rise to the compelling picture that all these different theories are only perturbative regimes of one fundamental underlying (eleven dimensional) M theory. It is not yet clear what this fundamental theory will finally be. But whatever it is, at low energies it should reduce to eleven dimensional supergravity. 11 dimensional supergravity has a membrane and a five brane in its spectrum, so M theory is most likely a theory of higher dimensional extended objects rather than strings. Another established property of M theory is that its compactification on a circle gives the IIA string [4] and the compactification of M theory on a circle modded out by $\mathbb{Z}_2$ yields the heterotic $E_8 \times E_8$ string [5, 6]. The coupling of these string theories is related to the radius of the circle on which we reduce. M theory on a big circle (or even decompactified) therefore provides the strong coupling regime of the IIA string. To get the type IIB string from M theory is a bit less straight forward. Therefore, an alternative version of M theory, namely F theory, has been proposed [7]. In compactifications of F theory, one starts in 12 dimensions and compactifies on elliptically fibered manifolds [7, 8, 9]. The elliptic structure geometrizes the $SL(2, \mathbb{Z})$ strong weak coupling duality of the IIB string. However, representation theory of superalgebras gives a fundamental meaning to eleven dimensions: It is the highest dimension, in which it can be expected to find a supersymmetric field theory. This was shown using representation theory of supersymmetry algebras in [10] and further established by constructing a 11-dimensional supergravity theory in [11]. A way to come to 12 dimensions is to consider two time directions. But up to now F theory was mainly successful when considering compactifications. F theory compactified on elliptically fibered Calabi-Yau fourfolds is dual to the heterotic string on a Calabi-Yau threefold. These compactifications are important, because they lead to the phenomenologically interesting case of $N = 1$ supersymmetry in 4 dimensions. These compactifications are an active area of research.

Hand in hand with the above mentioned progress goes another important development: It became more and more clear that higher dimensional extended objects play an important role in string theory. These so called p-branes were obtained as solutions to the low energy effective actions of string theory. The work of Polchinski [12] (see also the review [13]) made clear that certain p-branes are loci in space time where open strings can end. These branes are called "D-branes", where the D stands for Dirichlet. Open strings have Dirichlet boundary conditions in the directions transversal to the D brane world volume directions and Neumann boundary conditions parallel to the D branes. The end of the string is stuck to the brane, but free to move parallel to it. It became furthermore clear [12] that D branes carry the charge of the gauge potentials in the Ramond-Ramond sector of type IIA and type IIB strings. Fundamental strings are neutral under these fields and the discovery of charged objects was a success. The development of the understanding of the role of the extended objects is very closely related to the study of string dualities, sometimes fundamental strings in one theory are mapped to extended objects in the other theory.

In this thesis, the worldvolume theories of branes will play a major role. First, we will look at theories on branes to learn about quantum field theory. This is one application of brane physics, which has become an industry. Among other things, it was possible, to rediscover the famous results by Seiberg and Witten [14] for supersymmetric gauge theories in four dimensions using branes. Our main focus will be on six-dimensional field theories from branes. Super Yang-Mills theories in six dimensions are non-renormalizable and therefore become free theories in the infrared. There has been the conjecture that there exist interacting fixed points of the renormalization group, giving rise to non-trivial field theories in six-dimensions. Using branes, we can embed these theories in string theory and therefore show that they exist.

But we can not only learn from string theory about field theory: Vice versa, there has been the conjecture that we can learn something from field theory about eleven dimensional M theory or compactified M theory. The field theory under consideration is the field theory of D0 branes, a so called Matrix model. Matrix models are the first attempt to give a non-perturbative fundamental formulation of M-theory. We will discuss matrix models in chapter 3. In the last chapter, we will study the compactification of M- and F-theory on Calabi-Yau fourfolds. This leads to the phenomenologically interesting case of $N = 1$ supersymmetry in four dimensions. In these vacua, branes play again an important role. In this thesis, we will concentrate on one particular aspect of branes in M- and F-theory vacua, namely, the generation of a non-perturbative superpotential by branes wrapping certain divisors of the compactification manifold.

In the remaining part of this introductory chapter we review some properties of branes, which will be of importance in the main part of this thesis.

1.1 Branes in string theory

In type IIA and IIB string theory, the massless fields can be divided into two distinct sectors: The NS-NS sector and the RR sector. In the NS-NS sector, both string theories have the same field content, namely a metric $g_{\mu\nu}$, an antisymmetric tensor field $B_{\mu\nu}$ and the dilaton ϕ, which determines the string coupling via

$$g_s = e^{\phi}.$$

The coupling is therefore a dynamical field in string theory. The fundamental strings are charged under the NS antisymmetric two form. The tension of the string is

$$T = \frac{1}{l_s^2}$$

where l_s is the string length, the fundamental length in string theory. The two-form potential can be integrated over the two-dimensional string world sheet. The coupling of the fundamental string to the two-form is similar to the coupling of a point like particle to a one form potential, like in ordinary Maxwell theory. In ten dimensions, the two form field is dual to a six-form. A six form potential can be integrated over a 5+1-dimensional world volume. In fact, there exists such an object, the NS 5 brane, both in IIA and in IIB theory. They are the magnetic duals to the fundamental strings. Their tension is

$$T = \frac{1}{g_s^2 l_s^6},$$

such that they are heavy at weak string coupling.

In addition to the NS 5 branes, we have also branes in the theory, which are charged under the RR fields. The field content in the RR sector of type IIA and IIB string theory is different. In type IIA theory we have a one form and a three form potential and in type IIB we have a scalar, a two form and a self dual four form. The charged objects under these potentials are the D branes. In type IIA we have only even-dimensional D branes, which couple to the odd potentials, whereas in type IIB we have only odd dimensional D branes. We will call a D brane which stretches in p space like directions and the time direction a Dp brane.

D branes have a description as loci in space time, where open strings can end. It was the observation of Polchinski [12, 13] to realize that these loci provide the RR charged objects in string theory. If we do not have any D brane, then we have Neumann boundary conditions in all directions of space time and open string endpoints are free to move everywhere in space, but no momentum is allowed to flow off the string end point. In the presence of the Dp brane, we have $(9-p)$ directions where we have Dirichlet conditions and the string endpoints are confined to the D brane. The tension of a Dp brane is

$$T = \frac{1}{g_s l_s^{p+1}}.$$

Type IIA and type IIB string theory are related by T-duality, if we compactify to nine dimensions on a circle. T-duality inverts the radius of the circle on which we compactified, exchanges winding and momentum modes of the closed string and exchanges Dirichlet with Neumann boundary conditions. It is valid order by order in perturbation theory. As a consequence of the exchange of Dirichlet and Neumann boundary condition, a Dp brane is transformed into a D(p+1) brane if T duality is performed in a direction transversal to a Dp brane and into a D(p-1) brane if the T duality is performed along a world volume direction of the D branes. In this way, the D branes in IIA and IIB transform into each other.

1.1.1 Branes from M-branes

We have heard in the introduction, that the conjecture exists that all string theories can be obtained from M theory. Then of course also all branes have to be obtained from M theory branes [15]. As type IIA is the dimensional reduction of M theory on a circle of radius R, it is more straight forward to obtain the IIA branes from M branes. The IIB branes can then be obtained by performing a T-duality tranformation. The radius R on which we reduce determines together with the 11 dimensional Planck length l_p the IIA parameters, which are the string coupling g_s and string length l_s [4],

$$g_s^2 = \frac{R^3}{l_p^3} \qquad l_s^2 = \frac{l_p^3}{R} \tag{1.1}$$

We already mentioned in the introduction that in M theory we have membranes and five branes. They carry the electric (magnetic) charges of a three form potential in

M theory. Further solutions are the Brinkmann-wave, which describes momentum in a single direction and the Kaluza-Klein (KK) monopole, which requires the existence of a compact isometry direction (NUT-direction) and stretches in six flat world-volume directions. The KK monopole is magnetically charged under the KK gauge field coming from the compactification. The type IIA string is obtained by wrapping the membrane on R. The tension of the membrane is $T_{membrane} = \frac{1}{l_p^3}$. Wrapping on R gives

$$T_{string} = \frac{R}{l_p^3} = \frac{1}{l_s^2},$$

which is the string tension. The D2 brane is obtained from the M2 brane by a simple dimensional reduction without wrapping. The fourbrane is obtained from the M5 by double dimensional reduction. The M theory counterpart of the NS 5 brane is the M5 brane. Finally, the M theory origin of the D0 brane is a momentum mode along R, and the origin of the D6 brane is the KK monopole with NUT direction R.

1.1.2 Branes and superalgebras

It was pointed out by Townsend and others (see [16] for a review) that a good deal of information about M theory can be extracted from its superalgebra. In particular, it can be seen, which kinds of branes may exist in a certain theory. So let us write down the general superalgebra in 11 dimensions:

$$\{Q_\alpha, Q_\beta\} = (,\ ^{M}C)_{\alpha\beta}P_M + \frac{1}{2}(,\ ^{MN}C)_{\alpha\beta}Z^{MN} + \frac{1}{5!}(,\ _{MNPQR}C)_{\alpha\beta}Y^{MNPQR} \tag{1.2}$$

We see that in addition to the momentum we have two central terms on the right hand side. The first one is related to the existence of the membrane and the second one of the five brane. The presence of an extended object in space time breaks translational invariance. Therefore, we can expect that the extended objects only preserve a fraction of the supersymmetry. Indeed, a single brane preserves half of the supersymmetry. This can be shown by looking at explicit solutions. The conservation of a fraction of supersymmetry is related to the existence of covariantly constant Killing spinors. We can see from the algebra which fraction of supersymmetry is preserved if we turn on a single brane. Let us do this for the membrane, following the lectures of [16]. We work in a representation, where the charge conjugation matrix C is given by $,\ ^{0}$. Let us consider the case where a static membrane stretches in the $1,2$ direction. On the level of the algebra, this means:

$$\{Q, Q\} = P^0 + ,\ ^{012}Z_{12} \tag{1.3}$$

The left hand side of the equation is non-negative. If the membrane preserves some fraction of the supersymmetry, there have to be some zero eigenvalues of the matrix given by the left hand side. Equivalently, the membrane has to saturate the bound

$$P^0 \geq |Z_{12}| \tag{1.4}$$

For arbitrary states "$\geq$" holds, but for the membrane we have equality, it is BPS-saturated. Plugging this into equation (1.3) gives

$$\{Q, Q\} = P^0 \left(1 \pm ,\ ^{012}\right) \tag{1.5}$$

The eigenspinors of the anticommutator with zero eigenvalue are given by

$$,\ ^{012}\epsilon = \pm\epsilon \tag{1.6}$$

The sign decides whether we have a membrane or antimembrane. The square of , ^{012} is the identity and the trace is zero, therefore it has sixteen eigenspinors with eigenvalue 1 and sixteen with eigenvalue -1. So we have sixteen eigenspinors of eigenvalue zero of the anticommutator of the supercharges. The membrane therefore preserves one half of the supersymmetry. An antimembrane preserves the opposite part of SUSY. Similar arguments can also be given for the M5 brane or for branes in string theory. Quite generally, the objects preserving 1/2 of supersymmetry give rise to some constraint of the form , $\epsilon = \epsilon$ for a traceless product of , -matrices, which squares to 1.

1.1.3 Bound states

In this thesis, we will be particularly interested in brane configurations involving more than one brane. Branes can form bound states in two ways: Either they form non-threshold bound states with a non-zero binding energy or threshold bound states. In the case of a non-threshold bound state, the energy of the bound state is smaller than the energy of the two single branes. The "compound" object is a new object which preserves 1/2 of the supersymmetry. These bound states have been studied for example in [17]. Threshold boundstates do not lead to a difference in the energy. They can be interpreted as "intersecting branes". The supergravity solutions corresponding to such configurations were studied in e.g. [18, 19]. A brane intersection of two branes preserves 1/4 of the supersymmetry (or none). Let us try to understand the different behaviour of the two types of bound states from the superalgebra, again following [16]. Townsend looks at the algebra with two charges turned on:

$$\{Q, Q\} = P^0 \left(1 + q,\ + q',\ '\right) \tag{1.7}$$

(where the q's are the ratios Z/P^0.) If , and , $'$ anticommute, the bound is saturated if

$$q^2 + q'^2 = 1 \tag{1.8}$$

Therefore, in this case we have a non-zero binding energy. The state preserves 1/2 of the supersymmetry. We can find a constraint of the form $,''\epsilon = \epsilon$, where $,''$ is a linear combination of the form $,'' = ,\, cos\vartheta + ,' sin\vartheta$. An example for this type of bound state is a bound state of a momentum mode and an extended brane, where the momentum is in a direction transversal to a brane. This means, that we “kick” the brane in some direction. The result is a “moving” brane. Maybe the example shows the quality of the bound state: It is a single objects, having some properties of both of its ingredients. Another example is the membrane lying inside the five brane [20] .

The other case we have to consider is the case that , and $,'$ commute. In this case the bound is

$$|q| + |q'| = 1. \tag{1.9}$$

There is no binding energy. We can simultaneously diagonalise the , 's because they commute. Each of the diagonalized , 's has 16 eigenvalues +1 and 16 −1. If the product $,\,,'$ is traceless, the diagonalization of $,'$ on an eigenspace of , gives eight positive and eight negative eigenvalues of $,'$. Note, that if we take a brane and its antibrane stretching in the same directions, the , 's commute but the tracelessness condition is not met. But for two different usual branes the condition is met. An example for a threshold bound state is the following configuration:

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
M5	0	1	2	3	4	5	-	-	-	-
M2	0	1	-	-	-	-	6	-	-	-

This is the intersection of an M5 and M2 brane. The intersection area is one-dimensional. From the point of view of the M5 brane, it looks like a string moving in its world volume. Remember that we have extracted the brane content of M-theory from its superalgebra. In the same way, we can write down the most general (0,2) supersymmetric algebra in six dimensions [21]. In this algebra it can be seen that we should have threebranes and strings living in the five brane world volume. In addition, we have non-dynamical five-branes. All these world volume objects have an 11 dimensional interpretation in terms of brane intersections. So it is possible to conclude in both directions: From the 11-dimensional space time point of view, we can predict the brane content of the 5 brane world volume theory. Vice versa, the supersymmetry algebra of the M5 brane world volume theory contains already some information about the 11-dimensional theory. The transversal rotation group appears as an R-symmetry of the six dimensional SUSY algebra.

On the other hand, the non-threshold bound states will have an interpretation as excitations of the brane, like fluxes of a world volume gauge field. The low energy theory on the world volume of the branes will be the subject of the next section.

1.2 Low energy field theory of the branes

1.2.1 D branes

The world volume theory of D branes is much better understood than the world volume theory of other branes, because we have a microscopic description of D branes as loci in space time, where open strings can end. This enables us to deduce properties of D branes from properties of strings. The low energy field theory on the world volume of a single Dp brane is a $U(1)$ gauge theory with sixteen supercharges in $p+1$ dimensions. These theories can be obtained by dimensional reduction of 10 dimensional $N=1$ Maxwell theory to p dimensions. The bosonic part of the massless spectrum of such a theory contains a $U(1)$ vector field A_μ, and $9-p$ scalar fields X^a. The scalar fields have their origin in the reduction of the 10-d vector. They describe the fluctuations of the brane in the transversal directions. Furthermore, we have fermions required by SUSY. The action for the theory is

$$S = \frac{1}{4g_{YM}^2}\int d^{p+1}\left(-F_{\mu\nu}F^{\mu\nu} - \frac{1}{l_s^4}\partial X^a \partial X_a\right) \quad + fermions. \tag{1.10}$$

The coupling on the brane is given by

$$g_{YM}^2 = g_s l_s^{p-3} \tag{1.11}$$

How does this theory arise from string theory? We know that Dp branes are defined to be the places where open strings can end. These open strings have a vector fields coupling to their ends. This vector field is the origin of the vector living on the D brane world volume. The presence of the 16 supercharges is explained by the arguments in the previous section: the brane preserves one half of the 32 supercharges which we have in type IIA/B. It should also be mentioned that of course normally the open strings which give rise to these modes can interact with closed strings from the bulk and there can also be higher excitations of open strings. If we want to talk about the worldvolume theory of the brane, we suppose that there is an energy gap between these additional excitations and the excitations giving rise to the gauge theory on the brane. To decouple the massive modes, we take the string length to zero. If the theory on the brane is supposed to be interacting, we need to take a limit of the string coupling in such a way that the coupling on the brane (1.11) is non-vanishing.

What happens if we have several parallel Dp-branes stretching in the same directions? This means, that there are several branes available on which our open strings can end. As before, there will be strings having both ends on the same brane, leading to a gauge theory with gauge group $U(1)^N$ for N branes. Furthermore, we can now also have strings stretching from one D brane to another. In general, there is a distance between the two D branes and the stretched string gives rise to massive degrees of freedom. The mass is proportional to the distance. But we can also imagine, that the N D branes lie on top of each other. The strings then give new massless degrees of freedom. This leads to an enhancement of the gauge symmetry to $U(N)$. The theory on the worldvolume is the reduction of 10 dimensional $N = 1$ super Yang-Mills, instead of super Maxwell theory. The scalars on the worldvolume transform in the adjoint representation of the gauge group. They do not necessarily commute anymore. If we reduce 10-dimensional Yang-Mills, we obtain a potential for the scalars

$$V \sim \mathrm{Tr}[X^a, X^b]^2. \tag{1.12}$$

This potential is minimal, if all scalars commute. In this case the matrices X^a can be simultaneously diagonalized and the eigenvalues $x_1^a \dots x_N^a$ of the matrix X^a can be interpreted as the positions of the N D branes in the transversal direction a.

1.2.2 Orientifolds

So far, we have considered $U(N)$ gauge groups on the brane. We can also obtain orthogonal and symplectic gauge groups by using orientifold planes parallel to the Dp branes. An orientifold plane is the fixed plane of a $\mathbb{Z}_2$ symmetry. This $\mathbb{Z}_2$ operation involves both a geometrical $\mathbb{Z}_2$ symmetry in space-time and a world sheet parity inversion. The orientifold plane carries the same type of RR-charge as a Dp brane of the same dimension. The sign of the orientifold charge is determined by the type of projection which is performed. The amount of charge measured in D-brane units depends on the dimension. It is given by

$$q_O = \pm 2^{p-5} q_D \tag{1.13}$$

This result can be obtained from a world sheet computation [13]. Here, we counted *physical* D-branes. Usually, when considering orientifolds, we find it convenient to look at the whole space (instead of the space modded out by the $\mathbb{Z}_2$) and take into account the orientifold by adding a mirror object to any object in the setup. It is then often convenient to count the brane and its mirror separately, and this will be our convention in this thesis. Then, the relation between the D brane and orientifold charge is modified to

$$q_O = \pm 2 \times 2^{p-5} q_D = \pm 2^{p-4} q_D \tag{1.14}$$

The sign of the charge determines the modification of the gauge theory on a set of D branes lying on top of an orientifold. If we have N D branes lying on top of each other, we have usually a $U(N)$ gauge symmetry enhancement. The presence of the orientifold leads to a projection of the $N \times N$ matrices describing the gauge fields A_μ. Positively charged orientifolds lead to a symmetric projection and therefore symplectic gauge groups, whereas negatively charged orientifolds lead to an antisymmetric projection and SO groups. Our convention is that N branes (=physical brane + mirror!) lead to an $Sp(N)$ (or $SO(N)$) gauge group, $Sp(2) \sim SU(2)$. The branes can be moved away in pairs from the orientifold plane. If all the branes are separated and away from the orientifold, we get a $U(1)^{\frac{N}{2}}$ gauge group. We can have some branes at the orientifold and some lying on top of each other away from the orientifold, leading to a gauge group $SO \times U \times \ldots \times U$. The highest symmetry enhancement is obtained if all branes lie on top of the orientifold.

The supersymmetry breaking due to an orientifold is the same as due to the D brane of equal dimension.

1.2.3 NS branes

D branes are not the only branes which appear in string theory. In addition we have the NS branes, which are the magnetically charged objects under the antisymmetric two form field in the NS sector. We have an NS brane both in type IIA and IIB theory (all closed string theories have an NS 5 brane), but the theory on the world volume of the NS branes is very different. On the type IIB fivebrane, we have a (1,1) supersymmetric Yang Mills theory in $5+1$ dimensions. The gauge coupling in this theory is given by

$$g_{YM}^2 = l_s^2 \tag{1.15}$$

The theory on the NS 5 in IIA, which is of course related to the theory of the M5 brane, is slightly more exotic. It is the (0,2) supersymmetric theory of a tensor multiplet in six dimensions. The tensor multiplet contains five scalars, which correspond to fluctuations of the five brane in the transversal directions (from the M-theory point of view). The tensor field is self dual (the antiselfdual tensor in 6d is contained in the gravity multiplet). Therefore, the coupling constant is ~ 1. Again, we can look at a stack of NS branes lying on top of each other. For the IIA NS branes, this leads to an interesting interacting theory. We can imagine that virtual membranes stretch between the 5 branes. This is similar to the situation with D branes, where we considered strings stretching between different D branes, leading to new massless degrees of freedom and gauge symmetry enhancement. In the case of the theory of a tensor field, the stretched membranes lead to tensionless strings on the worldvolume of the brane.

It is interesting to note that the NS 5 brane of the chiral IIB theory leads to a non-chiral theory, whereas the 5 brane of the non-chiral IIA theory leads to a chiral theory. Theories with tensionless strings can also be obtained in other string theory setups. For example, in type IIB theory we can obtain a six-dimensional theory with (0,2) supersymmetry by compactification on a K3 manifold. Here, we get tensionless strings from threebranes which are wrapped around shrinking two cycles of K3, as studied in [22]. Note that a K3 locally looks like a space with an ADE type singularity. Performing a T-duality orthogonal to k IIA five branes leads to a type IIB theory on an A_{k-1} singular space [23]. T-duality along a world volume direction relates the two different NS 5 branes.

1.2.4 Bound states and curvature terms

In section 1.1.3 we have already mentioned that bound states of branes can have an interpretation in terms of the world volume theory of one of the branes. Let us come back to this interpretation in this section. We have seen in section 1.2.1 that the world volume theory of a Dp brane is super Yang-Mills theory in $p+1$ dimensions. Bound states of a Dp brane with a lower dimensional D brane can be interpreted as excitations of the Yang-Mills theory. This was first observed in [24] and generalized in [25]. The Yang-Mills theory on the brane under consideration can have a non-trivial gauge bundle. The possible curvature terms correspond to bound states with a D brane of lower dimension. (An excellent review is [26].) More concretely, a non-trivial first Chern class (magnetic flux) corresponds to a bound state with a D(p-2) brane, a non-vanishing second Chern class (instanton number) to a bound state with a D(p-4) brane, the third Chern class to a D(p-6) brane. The reason for this is that we have a Chern Simons coupling between the RR potentials under which the lower dimensional D branes are charged and the field strength of the other D brane. The Chern Simons couplings in the full D brane action are schematically of the form [27]

$$\int A^{(k)} \wedge e^F, \tag{1.16}$$

where $A^{(k)}$ stands for all possible RR k – form potentials. The integration is over the world volume of the Dp brane. Taking the exponential of the field strength F means that we pick an appropriate power of the curvature, such that the integral leads to a non-vanishing contribution. We see that for example for a D2 brane we have a contribution involving the 1-form potential under which a D0 brane is charged. This term contains the field strength to the first power. This means that a D2 brane with non-trivial magnetic flux will also carry D0 brane charge. The property of the bundle gets reinterpreted as a bound state of the D2 brane with a D0 brane. More generally, we get a non-vanishing contribution in (1.16) if we take the RR $p-1$ – form potential and wedge it with the field strength. The charged object under the $p-1$ – form potential is a $p-2$ brane. Therefore, we see that a magnetic flux is related to a bound state of the Dp brane with a D(p-2) brane. The same argument can be given for higher powers of the field strength and D(p-4) and D(p-6) branes.

These observations will be crucial in our discussion of Matrix theory. Here, the attempt is made to understand M theory from the point of view of the field theory on branes.

1.3 Branes suspended between branes

So far, we have only studied theories with 16 supercharges, corresponding to $N=4$ in four dimensions, now we will break some more supersymmetry. There are two basic approaches to study theories with lower supersymmetry. One is, to put the branes into a non-trivial geometric backgrounds. A part of the supersymmetry is then broken by the background, and if the background geometry is suitably chosen, there is some supersymmetry left. The other idea is to put more branes into the setup. In this thesis, we will mainly consider the latter approach (and use the other idea only for comparisons with our results). We will specialize to a type of brane setup,which was first studied by Hanany and Witten in [28]. In this type of setup, branes are suspended between other branes. This means, they end on the other brane, and as a consequence they are finite in one direction. We have already seen in section 1.1.3 that threshold bound states lead to theories where one quarter of the supersymmetry is broken. There, all branes were infinite and we interpreted the resulting configuration as a brane intersection rather than the ending of one brane on another. Let us recall the configuration discussed in section 1.1.3. A membrane was intersecting with a five brane. Two world volume directions of the membrane were inside the M5 brane and one, the x_6 direction, was orthogonal. We can now imagine that we have two parallel M5 branes intersecting the membrane and having a finite distance in x_6 and sharing all world volume directions. This clearly does not break any further supersymmetry. We can now think of "breaking" the M2 branes at the M5 branes and moving the finite piece between the two branes independently from the semi-infinite M2 brane pieces. This is already almost a type of setup, which we will use throughout the paper. The other even more popular Hanany-Witten type possibility is to forget about the semi-infinite pieces and consider only membranes stretched between five branes. The membranes have a finite direction. We are not totally unfamiliar with this situation: It already occured when we discussed the non-trivial theories living on several M5 – or IIA NS 5 – branes. We will mainly use brane setups in a string theory context instead of M theory. Of course, there are many ways to relate the M5-M2 intersection to string theory. One way is to do a double dimensional reduction in the x_1 direction (the coordinates refer to the setup in section 1.1.3), which is a direction shared by the M5 and M2 brane, and obtain a string ending on a D4 brane. Here, the "ending" is not so strange anymore, as D branes have per definition strings ending on them. T duality along a world volume direction of the D4 leads us to a IIB situation, where a fundamental IIB string ends on a threebrane. S-duality takes us to a IIB D string ending on a threebrane. This is already one of the situations, which will frequently occur in the following: A Dp brane stretching between two Dp+2 branes. By performing T-dualities transversal to the 3 brane, we can obtain all values for p. Another possibility to push down the M5-M2 system to string theory is to use a simple dimensional reduction to get a IIA D2 brane stretched between two NS 5 branes. By performing T-dualities along the world volume of the NS 5 we get D3,4,5,6 branes stretched between NS 5 branes. These are the favourite setups of many papers following the original Hanany Witten paper. Before we go on, let us write down the particular configuration which was used in their paper. They considered D5, D3 and NS 5 branes in type IIB string theory, stretching in the following directions:

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
NS 5	0	1	2	3	4	5	-	-	-	-
D 5	0	1	2	-	-	-	-	7	8	9
D 3	0	1	2	-	-	-	6	-	-	-

It can be checked that this configuration preserves 1/4 of the supersymmetries, that is 8 supercharges. We want to consider the situation that the D3 branes are suspended between the D5 or NS 5 branes in the x_6 direction. That means, we are precisely in one of the situations we obtained above from the M5-M2 system: The Dp stretched between two D(p+2)'s or the Dp stretched between two NS branes. The field theory, which Hanany and Witten studied, is realized on the world volume of the D3 branes. This is a 3 dimensional theory because x_6 is a finite direction. This theory has $N = 4$ supersymmetry because 8 supercharges are left unbroken in the configuration. The point of view we take is that the 5 branes are much heavier than the 3 branes because they have two extra dimensions. The low energy dynamics is determined by the lowest dimensional brane in the setup. The 3 brane observer sees the degrees of freedom of the 5 brane as a classical non-dynamical background.

Moving around the 3 brane corresponds to changing the moduli of our theory. If we move around 5 branes this corresponds to changing parameters like masses, coupling constants, FI-terms.

What is the field theory on a 3 brane suspended between two D5 or NS 5 branes? On an infinite 3 brane there is a theory of a vector multiplet with 16 supercharges. This corresponds to $N = 8$ in 3 dimensions. The multiplet contains a vector and scalars corresponding to the fluctuations of the 5 brane in the transversal directions, as explained in section 1.2.1. In our setup, the x_6 direction is finite, therefore we are left with a $2 + 1$ dimensional field theory. A further effect of the 5 branes is that SUSY is broken to $N = 4$. If the supersymmetry is broken, the supermultiplets of the larger supersymmetry decompose into multiplets of the lower supersymmetry. The $3 + 1$ dimensional vector turns to a $2 + 1$ dimensional vector and a scalar, which is the component of the vector in the compactified direction, $b = A_6$. Furthermore, we have the fluctuations of the brane in the 789 and 345 direction. The fluctuations in 345 form together with the vector A_μ the bosonic part of the $2 + 1$ dimensional $N = 4$ vector multiplet. The other 3 scalars corresponding to the fluctuations in 789 pair together with the scalar b coming from the dimensional reduction to a hypermultiplet transforming in the adjoint representation of the gauge group.

The five branes provide boundary conditions for the fields living on the D3 branes. As a consequence, some part of the bosonic spectrum is projected out. These boundary conditions are different for the D5 and NS 5 branes. Therefore, we will study these effects separately. Let us start with the D5 brane. Here, the three brane is free to fluctuate in the 789 directions along the D5 brane. However, it can not fluctuate in 345. In these directions the three brane position is fixed by the boundary conditions on the scalars. As a consequence, the hypermultiplet survives the projection, but the vector multiplet gets projected out. The theory on a D3 brane suspended between D5 branes is the theory of an $N = 4$ hypermultiplet.

Let us turn to the D3 brane suspended between NS branes. Here, the effect of the boundary conditions is that the hypermultiplet gets projected out. The fluctuation in the 789 direction is prevented by the boundary conditions. The three brane positions have to agree with the 5 brane positions in these directions in order to be able to connect the two NS branes. The D3 is free to fluctuate in the 345 direction. The vector survives and we are left with an $N = 4$ vector multiplet. To enhance the gauge group to $U(N)$, we can put N D3 branes on top of each other. The gauge coupling of the theory is related to the distance between the NS 5 branes:

$$\frac{1}{g_{YM}^2} = \frac{\Delta x_6}{g_s}$$

Here, g_s is the string coupling and $1/g_s$ would be the gauge coupling on an infinite D3 (1.11). So the above formula is obtained from the gauge coupling on an infinite D3 via Kaluza-Klein reduction on the finite interval in x_6.

Like in the case with infinite D3 branes, we can also study SO and Sp groups by putting orientifold planes parallel to the D3 branes.

In principle, it could also be possible to stretch a D3 brane between an NS and a D5 brane. We see, that none of the possible fluctuations survive: The D3 brane position in 345 is fixed by the D5 brane and the D3 brane position in 789 is given by the position of the NS5. For this type of configuration, the Hanany-Witten "s-rule" holds. This rule states that in configurations, where D branes stretch between D and NS branes in such a way that all moduli are frozen, we can suspend only one D-brane. Configurations with more than one D brane are not supersymmetric. [28] come to this conclusion mainly by consistency arguments and comparisons with field theory rather than by a pure D brane derivation.

Let us comment on the global symmetry of the brane setup. The presence of the branes breaks the full Lorentz symmetry to a subgroup. This subgroup contains the Lorentz group for the field theory on the D3 branes and further SO factors which act as a global symmetry for the field theory on the brane. If we look at the Hanany-Witten setup, we see that we have a symmetry corresponding to the Lorentz group in $2+1$ dimensions realized in directions 012 and that we have invariance under rotations in the 345 direction and 789 direction. Therefore, the symmetry splits in the following way:

$$SO(1,9) \to SO(1,2) \times SO(3)_{345} \times SO(3)_{789} \tag{1.17}$$

The two SO factors can be interpreted as the R-symmetries. From field theory, we know that the R-symmetry of the $N = 4$ algebra in $d = 3$ is $SO(4) \sim SU(2) \times SU(2)$. The two

SU-factors can be interpreted as the double covers of the SO factors extracted from the brane setup.

So far, we have considered theories with gauge fields and hyper multiplets in the adjoint. We can also include matter multiplets in the fundamental representation by putting N_f D5 branes in between the NS branes. Strings can then stretch between the D5 and D3 branes yielding matter in the fundamental representation. The mass of these multiplets corresponds to the distance (in the 345 direction) between the D5 and D3 branes. On the D5 branes, we have an $SU(N_f)$ gauge theory. From the point of view of the D3 branes, this is seen as a global symmetry. An alternative way to include matter is to add semi-infinite 3 branes to the left and right of the NS branes. Matter arises from strings stretching between a semi-infinite and finite piece of the 3 brane. Here, the masses correspond to the distance between the semi-infinite and finite D3 brane pieces. The two descriptions of matter multiplets are related by the Hanany Witten effect: We move the D5 branes off to infinity. When they cross an NS brane a new D3 brane is created, which ends on the NS brane.

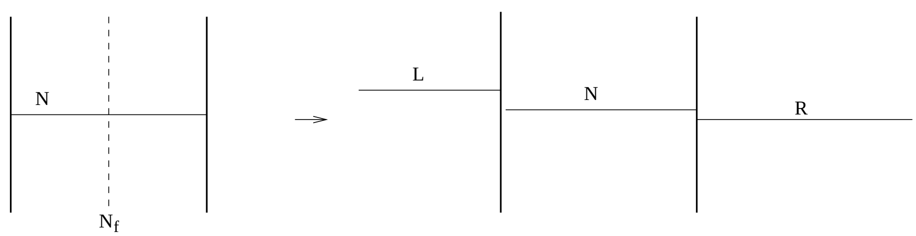

Figure 1: To ways to include matter in the fundamental representation: Either by using higher dimensional D branes or semi-infinite D branes of the same dimension as the color-branes.

The Hanany Witten setup can be T-dualized along the directions 3,4,5. The dimensions of the D brane stretched between the two NS branes increases in each step. Of course, we can also T-dualize in the 2,1 direction to study lower dimensional field theory or quantum mechanics. This enables us to study theories with 8 supercharges in various dimensions.

Another modification, which was first used in [29], is to rotate one of the NS branes to an NS' branes, which stretches in 012389. This rotation breaks one half of the 8 supersymmetries, this is therefore a setup to study theories with 4 supercharges.

It is also possible to include orientifolds in our setup in various ways [30, 31]. We will discuss this in more detail later on.

1.3.1 The classical moduli space

The discussion of the classical moduli space is very similar in all dimensions. Let us briefly describe the moduli space of the 3 dimensional gauge theory as seen in the brane picture: We expect to see a Coulomb and a Higgs branch. The Coulomb branch is parametrized by the expectation value of the scalars in the vector multiplet. The 3 brane ending on the 5 brane is free to fluctuate in the three, four and five directions. These fluctuations are described by the scalar in the vector multiplet. For lower dimensional gauge theories the vector contains more scalars, which of course corresponds to more transversal directions on the five brane in the brane picture. At a generic point on the Coulomb branch, all the scalars have non-vanishing and distinct expectation value and the gauge group is broken to $U(1)^N$. The case of three dimensions is somewhat special, because in addition to the three scalars describing the position of the D3 on the NS5, we have one more scalar. This is because the three dimensional vector is dual to a scalar. Therefore, the moduli space of the three dimensional gauge theory is $4N$ dimensional, for N color branes. It is equivalent to the Hyper Kähler manifold of N monopoles in a $SU(2)$ gauge theory. This is precisely the theory seen by a five-brane observer, as pointed out by Hanany and Witten. Here, the moduli space of monopoles is a classical moduli space, in agreement with the idea that we treat the 5 branes classically, because they are much heavier than the D3 branes.

Note, that if we look at theories with an NS' brane instead of the NS brane, (a theory with lower supersymmetry), then the motions of the D3 in the 45 directions are locked, in agreement with the fact that the $N = 2$ vector in three dimensions has only one scalar, corresponding to the motion in the three direction. If we perform a T duality along 3 to come to a 4 dimensional gauge theory, we do not have transversal directions left. This of course reflects the fact that the $N = 1$ vector in 4d does not contain any scalars.

The Higgs branch can be visualized in the brane picture, if we move the D5 branes in the 345 direction until they touch the 3 branes. In this situation the hypermultiplets become massless. The threebranes can now "break" at the D5 brane: If the threebrane is spanned between two D5 branes, the boundary conditions enforce that not the vector survives, but the hyper. The brane is free to move in the 789 direction. These degrees of freedom are the same in any dimension. For the D3 (or the lightest brane in T-dualized setups) stretching between one of the NS branes and a D brane we have to take care of the Hanany-Witten s-rule, which states that in a given configurations only one single D3 is allowed to connect a D5 with an NS5 brane. There are no degrees of freedom left for these brane pieces: Both the vector and the scalar are projected out by the boundary conditions and all directions of the brane are locked. If we are in an $N = 1$ situation with one rotated 5 brane, the only difference is that the color branes connecting a flavor giving brane with the NS' brane has two real degrees of freedom corresponding to motions of the color branes in the 89 direction, which is common to the flavor giving brane and the NS'. The following picture shows a point on the Higgs branch (in an $N = 1$ theory), where all color branes are broken, which means that the gauge group is completely higgsed.

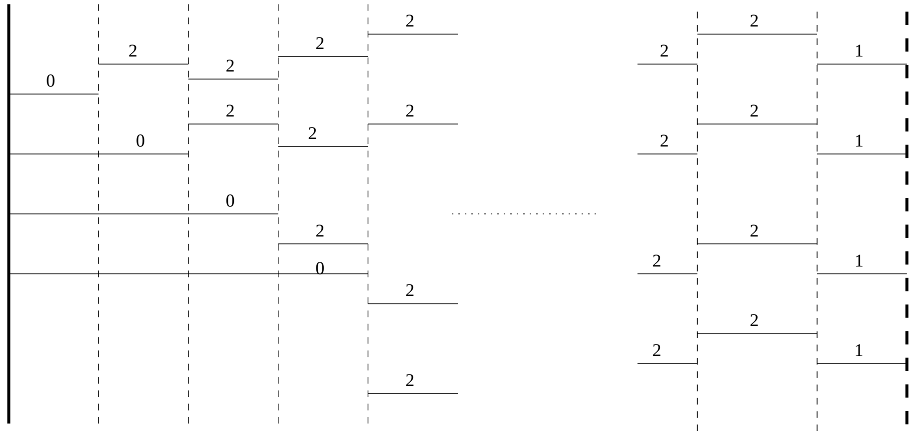

Figure 2: A maximally broken situation on the Higgs branch in a theory with 4 supercharges

In the picture, the fat line is an NS brane, the fat broken line is an NS' brane. For a theory with 8 supercharges, the NS' has to be replaced by an NS branes and both sides of the picture look the same. The horizontal lines denote the color giving branes and the thin broken lines the flavor giving D branes. The numbers on the color branes count the degrees of freedom, two for a D-brane piece between flavor giving D branes, one for a brane between a D-brane and a rotated NS-brane (NS'-brane), and zero for a D brane between an NS 5 brane and a D brane. The picture looks exactly the same for 2,3,4 dimensional gauge theories with 4 supercharges. Note also that it was essential to use the D5 branes to introduce matter to visualize the Higgs branch. The Higgs branch is not visible, when we use semi-infinite three branes.

For theories with eight supercharges, the distance between the NS branes in the 789 direction corresponds to a Fayet-Iliopoulos term in field theory. Note that in the Higgs phase we can have a non zero Fayet-Iliopoulos term without supersymmetry breaking: If a D brane breaks on the flavor giving brane, we can move the NS branes in the 789 direction. The two parts of the broken D brane separate in the brane picture. For theories with only four supercharges, there is only the 7 direction, in which neither NS or NS' nor the color giving D branes stretch. Therefore, we have only one real parameter corresponding to a Fayet-Iliopoulos term.

One of the main results in the Hanany Witten paper was to recover the mirror symmetry between the Coulomb branch of a gauge theory and the Higgs branch of a different gauge theory in the brane picture: In IIB we have an $SL(2,\mathbb{Z})$ symmetry which maps D and NS 5 branes upon each other. This exchange reproduces mirror symmetry.

1.3.2 Bending and RR charge conservation

In higher dimensions it becomes very important to take into account the disturbance caused by the D branes ending on the NS branes [32]. The end of a Dd brane looks like a magnetic monopole in the worldvolume of the NS 5 brane or as a charged particle on the $6-d$ dimensional subspace transverse to the Dd brane. Dd branes ending from different sides on the NS brane contribute with opposite charge. The consequence is that the NS branes do not have a definite x_6 position, but the x_6 coordinate obeys a Laplace equation:

$$\Delta x_6(y) = 0,$$

where y parametrizes the transversal space. The "true" x_6 coordinate of the NS brane is the x_6 value far away from the disturbance. We can analyze the behaviour in various dimensions by looking at the solutions to the Laplace equation in various dimensions. In three dimensions, the case analyzed in the previous paragraph, the solution behaves as

$$x_6 = \frac{1}{|y|} + constant,$$

such that for $|y| \to \infty$ we get a definite value, which we can call the x_6 position of the 5 brane.

In $d = 4$ we obtain a logarithmic behaviour. The distance in x_6 between two NS branes is proportional to the 4 dimensional gauge coupling, which is known to diverge logarithmically in 4 dimensions. This is reproduced in the brane picture. Indeed, it was even possible to reproduce the entire Seiberg-Witten curves from branes. This was done in [32] by lifting the IIA configuration to M-theory. Both the NS 5 branes and the D4 branes stretching between them get mapped to M5 branes. We can interpret the situation as one M5 brane wrapping a holomorphic curve Σ, which is embedded in the 456 10 plane. 10 denotes the direction which relates type IIA and M-theory. The four directions are the directions where either the NS 5 or the D4 brane stretches, but not both. The holomorphic curve Σ is the Seiberg-Witten curve. For SU-groups, the curves were obtained from branes in [32]. For SO and Sp groups the curves were found in [33, 34] using orientifolds.

In 5 dimensions, the transversal space is one dimensional and we obtain a linear bending of the NS brane. This can also be seen from the fact that RR charge has to be conserved at the vertices where different 5 branes come together. If we characterize a 5 brane by its charge under the NS and RR 2 forms, then a D brane has charge (0,1) and an NS brane (1,0). If they end on each other, a (1,1) brane emerges from the vertex. In this way, the IIB 5 branes can form webs, which can be used to study five dimensional superconformal field theories. This was done in [35, 36]. Again, 5 brane configurations can be lifted to M-theory, which was done in [37, 38, 36].

The following chapter is dedicated to six-dimensional fixed points from branes, so we do not mention them at this point.

Chapter 2

Six-dimensional Fixed Points from Branes

In this chapter, we will study the brane realization of six-dimensional quantum filed theories. Until very recently, it was generally believed that no interesting quantum field theories exist in six dimensions. The reason for this assumption is that in six dimensions the gauge coupling has a positive length dimension

$$[g^2_{YM}] = l^2.$$

As a consequence, the gauge kinetic term is an irrelevant operator and the theory flows to a free field theory in the infra red. It was argued in [39] that nevertheless there might be interacting quantum field theories at non-trivial fixed points of the renormalization group. The coupling at these non-trivial fixed points becomes infinite.

Recent results in string theory have indicated that these exotic theories can be realized as the theories on branes. This will be the subject of this chapter. Our interest is in six dimensional field theories with minimal supersymmetry, that is eight supercharges in $d = 6$. These theories are chiral (1,0) theories. The massless representations of low spin are a vector, a self dual tensor and a hyper multiplet. Because we are dealing with a chiral theory, the field content is restricted by anomaly considerations from the field theory point of view. We will reproduce the anomaly cancellation conditions in a Hanany-Witten like brane setup as the condition of RR-charge conservation.

2.1 The basic 6d brane setup

Let us study a Hanany-Witten like brane setup to investigate six dimensional gauge theories with a strong coupling fixed point [40, 41, 42, 43]. The ingredients are IIA D6 branes and NS 5 branes. They occupy the following directions:

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
NS 5	0	1	2	3	4	5	-	-	-	-
D 6	0	1	2	3	4	5	6	-	-	-

This setup can be obtained by applying T-duality to the configuration in section 1.3 in the 345 direction. However, T-dualizing the D5 of section 1.3 would lead to a D8 brane in the directions 012345789. We will postpone the discussion of D8 branes to the next section. We consider D6 branes stretching between NS branes and semi-infinite D6 branes.

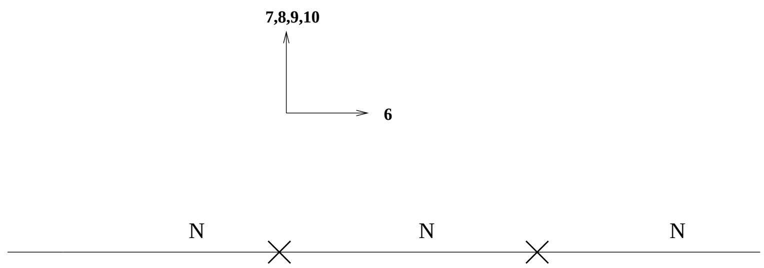

Figure 3: The brane configuration under consideration, giving rise to a 6 dimensional field theory. Horizontal lines represent D6 branes, the crosses represent NS 5 branes.

The configuration of NS5 and D6 branes is shown in figure 3 . The worldvolume of the NS brane lies completely inside the worldvolume of the D6 which ends on it. We include matter by semi-infinite D6 branes extending to both sides of the NS branes. Because there are no transversal directions of the NS 5 brane left, there is no room for bending. The RR charge has to cancel exactly at each vertex. The net charge is given by the number of D6 branes ending from one side minus the number of D6 branes ending from the other side. For an $SU(N)$ gauge group we supspend N finite D6 branes between the NS branes. Thus, we only get a consistent picture if:

$$N = L = R,$$

where L (R) denotes the number of D6 ending from the left (right). The total number of flavor giving semi infinite D6 is therefore

$$N_f = L + R = 2N \tag{2.1}$$

Later on, we will modify this basic building block in various ways. But let us discuss the field theory of this basic brane setup first.

2.2 The low energy field theory

What is the low energy field theory interpretation of this brane setup? According to the philosophy explained in section 1.3, we have to look for the lowest dimensional brane in the setup. In our setup, the NS branes are as light as the finite D6 brane pieces. Only the semi infinite D6 branes are heavy. This is different from brane configurations leading to lower dimensional field theories, where the NS branes could always be considered as heavy and their motions determined parameters in the theory. The theory on a IIA NS 5 brane is the theory of a (0,2)– tensor multiplet. This multiplet consists of a tensor and 5 scalars (and fermions). Because of the presence of the D6 branes, one half of the SUSY is broken and we are left with a (0,1) theory. The tensor multiplet decomposes into a (0,1) tensor, which only contains one scalar, and a hypermultiplet, which contains 4 scalars. The hypermultiplet is projected out from the massless spectrum because the position of the semi-infinite D6 branes fixes the position of the NS branes, so that fluctuations in the transversal directions are suppressed. The scalar in the tensor multiplet corresponds to motions of the 5 branes in the x_6 direction. We have two NS 5 branes and therefore two tensor multiplets, but effectively we keep only one of them because one of the scalars can be taken to describe the center of mass motion of the system. The vev of the other scalar gives us the distance between the NS branes. On the other hand, we know that the distance between the NS 5 branes is related to the Yang-Mills coupling of the six-dimensional gauge theory. This result was obtained in section 1.3 from the Kaluza-Klein reduction. If the two 5 branes come together we arrive at a strong coupling fixed points. This theory contains tensionless strings coming from virtual membranes stretching between the 5 branes.

Altogether, the branes describe an SU(N) theory with a tensor and N_f hypers. The brane analysis gives the result that for a consistent theory the number of fundamentals has to be $N_f = 2N$. It predicts a strong coupling fixed point with this matter content.

2.3 Inclusion of D8 branes

So far, we included fundamental matter multiplets by semi-infinite D6 branes. It should also be possible to describe the matter content by higher dimensional D branes between the NS branes [42]. In our case, these flavor branes are D8 branes. This causes some complication, because D8 branes are not solutions in standard IIA, but require massive IIA [44]. D8 branes are charged under a RR nine form potential. The dual field strength of it is a constant m. This constant is related to the cosmological constant appearing in massive IIA supergravity. The D8 branes divide space-time into different regions with different cosmological constant. Whenever we cross a D8 brane, the cosmological constant jumps by one unit. This is important for our brane configuration because there is a term in the action of massive IIA supergravity which is proportional to the IIA mass parameter and which couples the NS two form field B to the field strength of the 7 form potential under which the D6 is charged. The coupling reads:

$$-m \int B \wedge *F^{(2)}$$

($F^{(2)}$ is the field strength of the dual one form potential.) This modifies the equations of motion for the 7 form potential and therefore the RR charge cancellation condition. If a D6 brane ends on an NS brane, the equations of motion for the 7 form potential, or equivalently the Bianchi identity for the dual two form field strength is

$$dF^{(2)} = d * F^{(8)} = \theta(x_6)\delta^{(789)} - mH,$$

where $H = dB$. The δ term is the source term coming from the D6 brane ending on the NS brane. In the presence of m D8 branes the number of D6 branes ending from left and right on the NS branes should differ by m. In this way, a D8 placed in between the two NS branes has the same effect as a semi infinite D6 ending on the NS brane. To fix signs, we will use the convention that by passing an D8 brane from the left to the right, the massive IIA parameter m increases by one unit. In a background with given m, th number of D6 branes ending on an NS brane from the left is by m bigger than the number of D6 branes ending from the right.

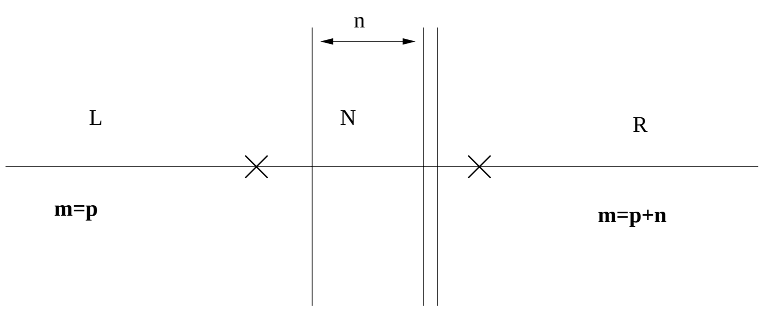

Figure 4: Basic Hanany Zaffaroni Setup

Figure 4 shows the basic brane configuration involving D8 branes. The n D8 branes in the middle give rise to 8 flavors for the $SU(N)$ gauge group. In addition they raise the cosmological constant m from p to $p+n$. In the background of these values of m the modified RR charge conservation tells us

$$\begin{aligned} N &= L - p \\ R &= N - (p+n) = L - 2p - n \end{aligned}$$

With the $R + L$ flavors from the semi-infinite D6 branes the total number of flavors is $L + R + n = 2N$.

2.4 Anomaly cancellation

We have seen that the brane picture indicates that only certain theories are consistent: They have to fulfill the relation (2.1). This restriction has to be reproduced from a pure field theory point of view. In six dimensions, the matter content of field theories is restricted by anomaly considerations. The anomaly in six dimensions can be characterized by an anomaly eight form. For a theory with a product gauge group $G = \prod_\alpha G_\alpha$ with matter transforming in the representation R, the anomaly polynomial reads

$$I = \sum_\alpha \left(\mathrm{Tr} F_\alpha^4 - \sum_R n_R \mathrm{tr} F^4 \right) - 6 \sum_{\alpha\alpha' \, RR'} n_{R,R'} \mathrm{tr}_R F_\alpha^2 \mathrm{tr}_{R'} F_{\alpha'}^2 . \tag{2.2}$$

Here, Tr denotes the trace in the adjoint and tr_R is the trace in Representation R. The symbol tr is reserved for the trace in the fundamental representation. n_R denotes the number of matter multiplets transforming in a particular representation R and $n_{RR'}$ is the number of multiplets transforming in the representation $R \times R'$ of a product group $G \times G'$. For a consistent theory, the anomaly should be cancelled in some way: Either the anomaly polynomial vanishes or we cancel the anomaly by a Green–Schwarz mechanism [45]. So let us look at the anomaly in more detail. First, we want to rewrite the anomaly polynomial using only traces in the fundamental representation. Formally, the polynomial then looks like

$$I = \sum_\alpha a_\alpha \mathrm{tr} F_\alpha^4 + \sum_{\alpha\alpha'} c_{\alpha\,\alpha'} \left(\mathrm{tr} F_\alpha^2 \right) \left(\mathrm{tr} F_{\alpha'}^2 \right) \tag{2.3}$$

To write this more explicitely we have to use some group theoretical identities for the particular gauge groups we want to consider. In the previous section we have discussed brane configurations leading to a single SU gauge group and matter in the fundamental representation. In the following, we will also consider gauge SO and Sp gauge groups and products thereof and matter in the antisymmetric/symmetric tensor representation. Let us collect some formulas from [46], which we want to apply to convert all traces in the anomaly polynomial to traces in the fundamental representation.

$$\begin{aligned} \mathrm{Tr} F_{SU(N)}^4 &= 2N \mathrm{tr} F_{SU(N)}^4 + 6 \left(\mathrm{tr} F_{SU(N)}^2 \right)^2 \\ \mathrm{Tr} F_{SO(N)}^4 &= (N-8) \mathrm{tr} F_{SO(N)}^4 + 3 \left(\mathrm{tr} F_{SO(N)}^2 \right)^2 \\ \mathrm{Tr} F_{Sp(N)}^4 &= (N+8) \mathrm{tr} F_{Sp(N)}^4 + 3 \left(\mathrm{tr} F_{Sp(N)}^2 \right)^2 \end{aligned} \tag{2.4}$$

The second and third relation can also be used to convert traces in the antisymmetric or symmetric tensor representation of an $SU-$ group to traces in the fundamental representation. For theories without fourth Casimir element, we can rewrite the F^4 terms as squares. For $SU(2)$ and $SU(3)$ we have

$$\mathrm{tr} F_{SU(2)}^4 = \frac{1}{2} \left(\mathrm{tr} F_{SU(2)}^2 \right)^2 \tag{2.5}$$

$$\mathrm{tr}F^4_{SU(3)} \;=\; \frac{1}{2}\left(\mathrm{tr}F^2_{SU(3)}\right)^2$$

First, let us consider theories with only one group factor, the case studied from the brane perspective in the previous section. Here, the anomaly (2.3) simplifies to

$$I = a\mathrm{tr}F^4 + c\left(\mathrm{tr}F^2\right)^2 \tag{2.6}$$

To have a consistent theory, it is necessary that the anomaly can be cancelled in some way. As we will undedstand, this requires only that the F^4 term vanishes – the quadratic terms can be cancelled by a Greem-Schwarz mechanism [45]. Vanishing of the F^4 term is guaranteed for gauge groups without fourth order Casimir operator. For an $SU(2)$ gauge group with N_f matter multiplets in the fundamental representation the anomaly can be easily calculated to be

$$I = \frac{1}{2}\left(16 - N_f\right). \tag{2.7}$$

This means that for 16 flavors the anomaly vanishes and we have a consistent theory. Similarly, the anomaly for $SU(3)$ gauge group with N_f fundamentals is

$$I = \frac{1}{2}\left(18 - N_f\right), \tag{2.8}$$

such that the anomaly vanishes if we have 18 fundamentals. But what happens if the anomaly does not cancel? In the case that the prefactor of the F^4 term does not vanish, we have an anomalous theory which is inconsistent. In the case that it is zero, there can still be the possibility to cancel the anomaly by the Green–Schwarz mechanism. This requires that the anomaly eight form factorizes into a product of two anomaly four forms. (In general, the requirement is that the anomaly $d+2$ form factorizes into a $d-2$ form and a 4 form.) Also, it is necessary that an antisymmetric tensor field is available.

$$I = X_4 \wedge \tilde{X}_4 \tag{2.9}$$

In this case the anomaly can be cancelled by adding a tree level counterterm involving the tensor field to the action. The tree-level counterterm is of the form

$$\int B \wedge \tilde{X}_4 \tag{2.10}$$

The Bianchi identity and equation of motion for the 3 form field strength H of the antisymmetric tensor field are modified by the two factors of the anomaly polynomial:

$$\begin{aligned} dH &= X_4 \\ d\star H &= \tilde{X}_4 \end{aligned} \tag{2.11}$$

In six dimensions we are in a special situation: Here the tensor splits into an antiselfdual and a selfdual part. The antiselfdual part is contained in the gravity multiplet, whereas the selfdual part is contained in a matter multiplet. In ordinary compactification of the heterotic string, we have one tensor multiplet in a matter multiplet, which forms together with the tensor in the gravity multiplet one tensor, which is not subject to any selfduality constraint.

Of course, if we want to cancel the anomaly in a six dimensional *field theory* – the field theory on the brane– we do not want to get gravity into the game in the anomaly cancellation mechanism. We want to cancel the anomaly using only the self dual part of the tensor multiplet. Therefore, the corrections to the equations of motion and the Bianchi identity induced by the Green-Schwarz mechanism should be the same. It is not enough that the anomaly factorizes in an arbitrary way like in ordinary string compactification, but it should be a square.

$$I = (X_4)^2\,, \tag{2.12}$$

Or, in other words, the prefactor c in equation (2.6) has to be positive. If c is negative, the anomaly can be cancelled if we use gravity. We have derived the anomaly for $SU(2)$ and for $SU(3)$. The number of flavors for which the anomaly vanishes now becomes an upper bound. For $SU(2)$ we can cancel the anomaly with the self dual part of a tensor for $N_f < 16$ and for $SU(3)$ it can be cancelled for $N_f < 18$ [47]. We have not considered global anomalies here, but let us remark that global anomalies restrict the possible matter content further: For $SU(2)$ the allowed matter content is $N_f = 4, 10, 16$ and for $SU(3)$ it is $N_f = 0, 6, 12, 18$ [48]. For the SU groups of higher rank, the allowed number of fundamentals is fixed by anomaly cancellation.

The mechanism can be generalized to product gauge groups. In this case, we have to write the anomaly (2.3) as a sum of squares. The number of squares has to be less or equal to the number of tensor multiplets.

Let us now compare field theory and brane considerations. We have seen in section 2.2 that the brane configuration is restricted by the requirement of RR conservation. We derived the restriction that $N_f = 2N$ for gauge group $SU(N)$. This should be reproduced by anomaly considerations. For $SU(N)$ with N_f we compute the anomaly by using the formulas (2.4) in (2.6):

$$I = (2N - N_f)\mathrm{tr}F^4 + 6\left(\mathrm{tr}F^2\right)^2. \tag{2.13}$$

Therefore, the deadly F^4 term cancels precisely for $N_f = 2N$, as predicted from the brane picture. The prefactor c of the $(F^2)^2$ term is bigger than zero, so that all requirements for the application of the Green–Schwarz mechanism are met. The additional possibilities for $SU(2)$ and $SU(3)$ cannot be seen in our brane configurations.

From the brane picture we have predicted a strong coupling fixed point, when the expectation value of the scalar in the tensor multiplet vanishes. To verify this from a field theory point of view, we look at the following part of the action [39]:

$$\frac{1}{g^2}\mathrm{tr}F_{\mu\nu}^2 + \sqrt{c}\phi\mathrm{tr}F_{\mu\nu}^2$$

Here, ϕ denotes the scalar in the tensor multiplet. We see, that one can absorb the bare gauge coupling into the expectation value of the tensor to get an effective coupling

$$\frac{1}{g_{eff}^2} = \sqrt{c}\phi$$

This is the effective coupling we see in the brane picture. At $\phi = 0$ there is a strong coupling fixed point [39].

2.5 Generalizations – Product gauge groups and orientifolds

In this section we want to extend the discussion of RR charge conservation and anomaly cancellation for more general brane setups and field theories. This has been done in the Hanany Witten framework in [41, 43]. A discussion using branes at orbifolds can be found in [49, 50].

2.5.1 Product gauge groups

Let us first consider a very simple generalization: We want to discuss product gauge groups. From the brane picture, we simply have to take k NS 5 branes and suspend D6 branes between them. RR charge conservation requires the number of D6 branes to be the same everywhere. This is shown in the following picture:

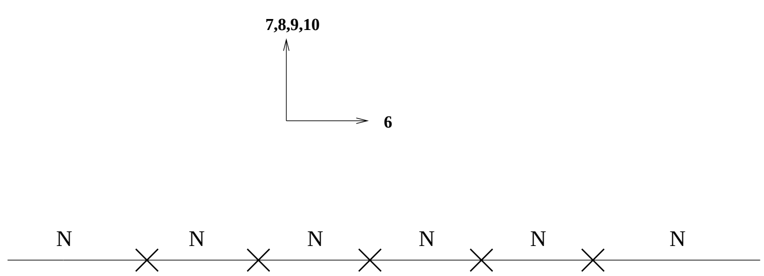

Figure 5: Brane configuration yielding a $SU(N)^k$ gauge group with bifundamentals

This leads to a product of SU gauge groups with fundamental matter in the first and last factor and bifundamentals for neighbouring gauge groups.

From the field theory point of view, we have to consider the anomaly eight form for products of gauge groups (2.3). We have to show that the prefactor of each F^4 term vanishes and that the rest of the anomaly eight form can be written as a sum of squares.

Each of these squares can be cancelled by one tensor multiplet, so the number of squares should not exceed the number of tensor multiplets available. If we again separate off the center off mass motion, we see that the number of tensors is the number of NS branes minus one. Therefore, the number of tensors equals the number of factors in the gauge group. Let us write down the anomaly (2.3) for a product of gauge groups $G = \prod_{\alpha=1}^{k-1} SU(N_\alpha)$ with additional fundamental matter N_0 in the first and N_k in the $k-1$'th factor and bifundamentals:

$$I = \sum_{\alpha=1}^{k-1} (2N_\alpha - N_{\alpha-1} - N_{\alpha+1})\ \mathrm{tr}F_\alpha^4 + 6\sum_{\alpha=1}^{k-1}\left(\mathrm{tr}F_\alpha^2\right)^2 - 6\sum_{\alpha=1}^{k-2}\left(\mathrm{tr}F_\alpha\right)^2\left(\mathrm{tr}F_{\alpha+1}\right)^2 \tag{2.14}$$

We can easily see that the F^4 term is zero in the case that all N are equal. This would be the case corresponding to the brane picture in figure 5 . However, the anomaly polynomial would allow for other possibilities, namely that there is a difference between $N_{\alpha-1}$ and $N_{\alpha+1}$ but the sum still gives $2N_\alpha$ in each factor. In the brane picture such a situation can be realized using D8 brane as flavors in the individual factors which do not transform under the other gauge groups. So we have a different field content, but this does not affect the anomaly polynomial.

We now have to show that we can write the anomaly in the form

$$I = \sum_{\alpha=1}^{k-1}\left(\sum_{\alpha'} c_{\alpha\alpha'}\mathrm{tr}F_{\alpha'}^2\right)^2 \tag{2.15}$$

For the case of two $SU(N)$ group factors with N matter multiplets in the fundamental representation in each factor and a bifundamental, we can explicitely write down the polynomial and factorize it:

$$\begin{aligned} I &= \mathrm{Tr}F_1^4 + \mathrm{Tr}F_2^4 - 2N\mathrm{tr}F_1^4 - 2N\mathrm{tr}F_2^4 - 6\mathrm{tr}F_1^2\mathrm{tr}F_2^2 \\ &= 6\left(\mathrm{tr}F_1^2\right)^2 + 6\left(\mathrm{tr}F_2\right)^2 - 6\mathrm{tr}F_1^2\mathrm{tr}F_2^2 \\ &= \frac{3}{2}\left(\mathrm{tr}F_1^2 + \mathrm{tr}F_2^2\right)^2 + \frac{9}{2}\left(\mathrm{tr}F_1^2 - \mathrm{tr}F_2^2\right)^2 \end{aligned} \tag{2.16}$$

We see that the polynomial can be written as a sum of two squares with positive coefficients. Therefore, the anomaly can be cancelled by two self dual tensor multiplets. Let us also write down an explicit factorization for the case of three group factors $SU(N)$ and bifundamentals of the first and second group and the second and third group. In this case the polynomial is

$$\begin{aligned} I &= 6\left(\mathrm{tr}F_1^2\right)^2 + 6\left(\mathrm{tr}F_2\right)^2 + 6\left(\mathrm{tr}F_3^2\right)^2 - 6\mathrm{tr}F_1^2\mathrm{tr}F_2^2 - 6\mathrm{tr}F_2^2 - 6\mathrm{tr}F_3^2 \\ &= \frac{3\sqrt{2}+6}{4}\left(\mathrm{tr}F_1^2 - \sqrt{2}\mathrm{tr}F_2^2 + \mathrm{tr}F_3^2\right)^2 + \frac{-3\sqrt{2}+6}{4}\left(\mathrm{tr}F_1^2 + \sqrt{2}\mathrm{tr}F_2^2 + \mathrm{tr}F_3^2\right)^2 \\ &\quad +3\left(F_1^2 - F_3^2\right)^2 \end{aligned} \tag{2.17}$$

This time, we have a sum of three squares and can cancel the anomaly with three tensors, as predicted from the brane picture. The polynomial for $k-1$ factors (k is the number of NS 5 branes) reads:

$$I = \sum_{\alpha=1}^{k-1} 6\left(\mathrm{tr}F_\alpha^2\right)^2 - \sum_{\alpha=1}^{k-2} 6\mathrm{tr}F_\alpha^2 \mathrm{tr}F_{\alpha+1}^2 \tag{2.18}$$

The formulas for an explicit factorization become more complicated with increasing number of group factors, which can already be seen in the above example for two and three group factors. But let us give an argument why the factorization is possible in principle. Let us interpret the square of the field strengths of the various group factors as components of a vector in $\mathbf{C}^n$, $F_\alpha^2 = x_\alpha$. The anomaly eight form can then be interpreted as a bilinear form in this space. It is given by the following $(k-1)\times(k-1)$ matrix:

$$\begin{pmatrix} 6 & -3 & & & & \\ -3 & 6 & -3 & & & \\ & -3 & 6 & -3 & & \\ & & & \ddots & & \\ & & & & 6 & -3 \\ & & & & -3 & 6 \end{pmatrix} \tag{2.19}$$

This matrix is symmetric and therefore can be diagonalized. It furthermore has full rank. It is also quite obvious that the bilinear form does not have negative eigenvalues. To see this we only need to write down the bilinear form evaluated on two equal vectors and see that it is bigger than zero:

$$I = 3\sum_{\alpha=1}^{k-2} (x_\alpha - x_{\alpha+1})^2 + 3x_1^2 + 3x_{k-1}^2 \tag{2.20}$$

Therefore, the required factorization property is easily seen and we can cancel the anomaly by a Green Schwarz mechanism. Let us remark that the matrix is proportional to the Cartan matrix of A_{k-1}, (which of course implies the properties discussed above). The appearance of the Cartan matrix might be related to the theory on the NS5 branes: Upon compactification we obtain an ordinary 5d gauge theory with gauge group A_{k-1}.

2.5.2 SU(N) product gauge groups with compact x_6

A slight modification of the above setup is to consider the compactification of the x_6 direction on a circle. This is shown in the following picture:

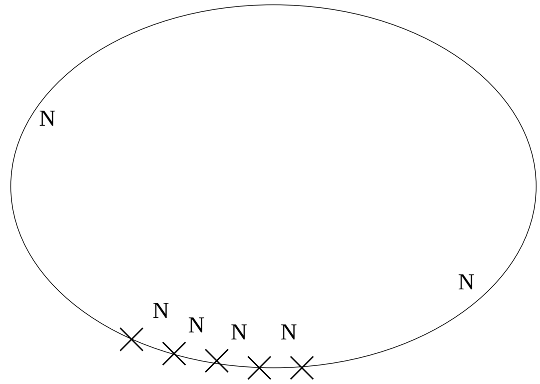

Figure 6: Brane configuration with x_6 compact

Let us check the anomaly cancellation: As a consequence of the compactification, we also have bifundamentals for the first and last factor in the gauge group. We turned a global symmetry in a local symmetry group. The quadratic terms in the anomaly polynomial read:

$$\begin{aligned} I &= 6\sum_{\alpha=1} k\left(\mathrm{tr}F_\alpha^2\right)^2 - 6\sum_{\alpha=1} k\mathrm{tr}F_\alpha^2\mathrm{tr}F_{\alpha+1}^2 \\ &= 3\sum_\alpha \left(\mathrm{tr}F_\alpha - \mathrm{tr}F_{\alpha+1}\right)^2, \end{aligned} \tag{2.21}$$

where $F_k \equiv F_1$. If we want to see an infrared fixed point in the picture with compactified x_6, we again have to move all the NS 5 branes on top of each other. The coupling constant is related to the distance between two neighbouring NS branes. This means that we can never adjust the moduli in such a way that all factors are strongly coupled. One factor is always weakly coupled and becomes free in the infrared. Therefore, at the fixed point always one of the group factors becomes a global symmetry. In the picture with a compact x_6 direction we can easily relate our brane picture of six dimensional fixed points to the analysis of [49, 50]. In these papers, the authors use branes at orbifolds to analyze the fixed points. Starting from our picture, we can T-dualize along the x_6 direction. Performing a T duality transverse to k NS 5 branes leads to a singular space with an A_{k-1} singularity. The space looks locally like $\mathbb{R}^3 \times S^1/\mathbb{Z}_k$. The appearance of the A_{k-1} might also clarify the appearance of the Cartan matrix in the anomaly polynomial. Furthermore, we have D5 branes coming from the D6 branes at the singularity. Performing a further S-duality, we come to a picture of IIB NS 5 branes at an orbifold singularity. This is precisely the situation studied by [49, 50]. Let us discuss the relation of our approach to this approach. They analyze IIB 5 branes at orbifold singularities $\mathbb{C}^2/, _G$. Here, $, _G$ is a discrete group of $SU(2)$ of rank k. These discrete groups have an ADE classification. To obtain the A_{k-1} singularities one divides by the cyclic groups $\mathbb{Z}_k$. The other possible discrete groups are the dihedral, tetrahedral, octahedral and icosahedral group where the first type leads to the D-type singularities and the other types to E-type singularities. In the spirit of [51] it should not be expected that the NS 5 branes can distinguish between $(\mathbb{R}^3 \times S^1)/,_G$ and $\mathbb{C}^2/,_G$. The information about compactness is lost, but the singularity type matters. We can analyze the theories using IIB NS or D5 branes. The gauge group and matter content for the branes at orbifold singularities can be determined using the quiver diagrams of [52]. The case we can reproduce in the Hanany-Witten setup corresponds to an A-type singularity, since it is not known what the T-dual picture of 5 branes at an D or E type singularity is. Let us analyze the field theory in the branes from the T-dual Intriligator-Blum point of view. On N 5 branes we have a $U(N)$ gauge symmetry. For the A-type singularity, the gauge group on the worldvolume of the 5 branes is

$$\prod_{\alpha=1}^{k} U(N) \tag{2.22}$$

More generally, the gauge group is

$$\prod_{\alpha=1}^{k} U(Nn_\alpha), \tag{2.23}$$

where n_α denotes the Dynkin indices. Each node in the extended Dynkin diagram of the ADE group G corresponds to the irreducible representation R_α of the discrete group $,_G$, with $|R_\alpha| = n_\alpha$. The matter multiplets in the theory transform as $\frac{1}{2}\oplus_{\alpha\beta} a_{\alpha\beta}(\Box,\bar{\Box})$. $a_{\alpha\beta}$ is one if the node in the extended Dynkin diagram are connected and zero otherwise. In our case, $a_{\alpha\beta} = \delta_{\alpha+1,\beta}$ (and the first and last factor are connected). The ALE space of type A_{k-1} has $k-1$ non-trivial two-cycles,. Reducing the type IIB two form and four form on these cycles leads to $k-1$ tensors and $k-1$ hypers of the 6d (1,0) theory. The diagonal $U(1)$ in the gauge group has no charged matter and decouples. The other k $U(1)$'s do have charged matter and are anomalous. These anomalies can be cancelled using the K hypers. They can pair up with the $U(1)$ gauge fields to give them a mass and effectively become Fayet Iliopoulos terms in the gauge group. Turning on a Fayet Iliopoulos term in the orbifold picture corresponds to a blow-up mode, reducing the A_{k-1} singularity to A_{k-2}. In the dual HW picture it corresponds to moving one NS brane up in the 7,8,9 direction and reconnecting the D6 branes, thereby reducing the number of gauge groups. Thus we see that we can obtain the basic features of the theories in both pictures.

2.5.3 Product gauge groups and D8 branes

Let us briefly discuss the inclusion of D8 branes into the product gauge group setups in some more detail. It is only possible to put in D8 branes in the uncompactified situation, otherwise there would be obstructions because the total D8 brane charge in the compact direction would have to add to zero. This is only possible if we include negatively charged orientifolds, as we discuss later on. For now, consider the case that we put n_α branes between the αth and $\alpha+1$ th NS brane in the uncompactified situation. This means that we have n_α fundamentals in the group factor α. The brane configuration is shown in the following figure.

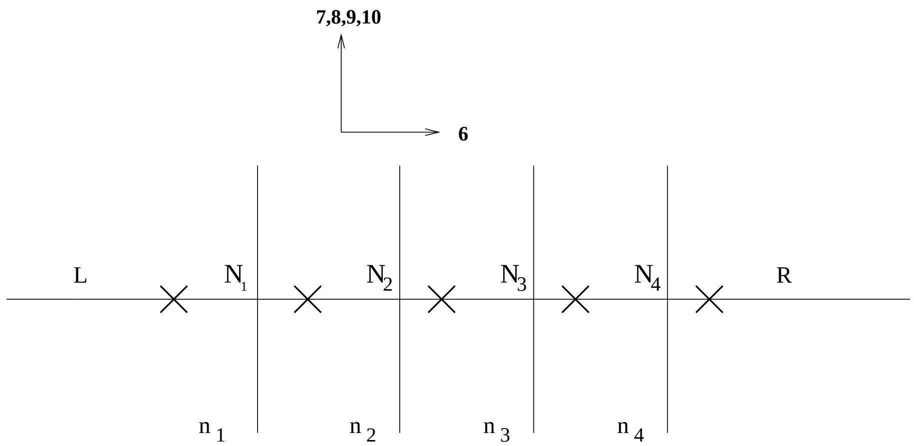

Figure 7: Brane configuration for product gauge groups with fundamental matter included by D8 branes

Each D8 brane raises the cosmological constant by one unit, as explained in section 2.3. Charge cancellation in the background with non zero cosmological constant due to the D8 branes requires that

$$\begin{aligned} N_1 &= L \\ N_2 &= N_1 - n_1 \\ &\vdots \\ N_\alpha &= N_{\alpha-1} - \sum_{\beta=1}^{\alpha-1} n_\beta, \end{aligned} \tag{2.24}$$

where we have taken into account that the cosmological constant jumps by n_α units when crossing n_α D8 branes. The quartic anomaly cancels:

$$\begin{aligned} &\mathrm{Tr}F_\alpha^4 - (N_{\alpha-1} + N_{\alpha+1} + n_\alpha)\,\mathrm{tr}F_\alpha^4 \\ &= \left[2N_\alpha - \left(N_{\alpha-1} + N_\alpha - \sum_{\beta=1}^{\alpha} n_\alpha + n_\alpha\right)\right] \mathrm{tr}F_\alpha^4 = 0 \end{aligned} \tag{2.25}$$

The quadratic anomaly is unaffected compared to the case without fundamentals included by D8 branes but only bifundamentals. Therefore the anomaly cancels, as expected.

2.5.4 Orientifold Six-Planes

So far, we have only analyzed theories with SU gauge groups. We can also study theories with orthogonal or symplectic gauge groups by including orientifold planes parallel to the D6 branes.

An orientifold six plane carries (up to a sign) twice the charge of a physical D6 brane, or four times the charge of a brane-mirror pair, see section 1.2.2. In the following, we will always count D branes and their mirrors separately and therefore use the factor of four. The charge of the orientifold determines whether the resulting gauge group is an SO or Sp gauge group. For a positive charge we have a symmetric projection and a resulting Sp group and for negative charge an SO group. Our conventions for the Sp is that $Sp(N)$ has rank $\frac{N}{2}$ or in other words $Sp(2) = SU(2)$ and only even N can appear. Let us now consider the configuration shown in figure 3 and put an orientifold six plane parallel to the D6 branes. We know from [30] that an orientifold plane, which passes an NS 5 brane, changes its sign. This means that if we have gauge group SO (Sp) the global symmetry is Sp (SO). Our charge cancellation condition gets modified in the presence of the orientifold: Because of the sign flip at the NS 5 brane the orientifold "pulls" from the one side and "pushes" from the other side. Let us first consider an Sp gauge group and SO global symmetry. The orientifold in the left part of the figure has charge $+4$, then changes to -4 between the two NS branes and then is again $+4$ right of the rightmost NS brane. The charge cancellation condition is therefore:

$$\begin{aligned} -4 + L &= N + 4 \\ -4 + R &= N + 4 \end{aligned} \tag{2.26}$$

We can fulfill the RR charge cancellation condition, if we put $N + 8$ D6 branes on the left and right hand side. Each of the flavor giving D6 branes corresponds to half a hyper multiplet in the low energy field theory on the brane. The consistency condition read off from the brane picture is that for an $Sp(N)$gauge group we need $N + 8$ fundamental hypers to cancel the anomaly. In complete analogy we can derive that for $SO(N)$ we need $N - 8$ flavors. Let us compare this to the anomaly cancellation in field theory: Here, we have the following anomaly polynomial

$$I = (N \pm 8)\mathrm{tr}F^4 - \frac{1}{2}\left(2N \mp 16\right)\mathrm{tr}F^4 + 3\left(\mathrm{tr}F^2\right)^2, \tag{2.27}$$

where we have used (2.4). The quartic anomaly cancels and the prefactor of the quadratic anomaly is positive. Therefore, we can use the Green-Schwarz mechanism to obtain a consistent theory. Generalizing to product gauge groups, we see that we can have products of groups

$$\prod_\alpha SO(N + 8) \times Sp(N) \tag{2.28}$$

Of course, there is again the possibility to use D8 branes. Similar to the cause without orientifolds, the consequence is that we have different gauge groups instead of a product of the same groups. If we put n D8 branes between NS brane α and NS brane $\alpha + 1$ and the group factor is Sp, the RR charge conservation condition is

$$N_\alpha = N_{\alpha-1} - 8 - \sum_{\beta=1}^{\alpha-1} n_\beta. \tag{2.29}$$

For the SO group factors we have to put a $+8$ instead of -8. The anomaly analysis is completely analogous to the case without orientifold.

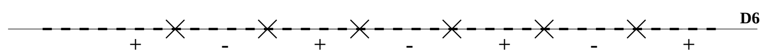

Figure 8: The product gauge group configuration of SO and Sp groups. The broken line is the orientifold plane on top of the D6 branes.

2.5.5 Orientifold 8 planes

Let us consider the inclusion of O8 planes [41, 43] These O8 planes are parallel to the flavor giving D8 planes. They carry D8 brane charge: If we again count D8 branes and their mirrors separately, we have to assign a 16 or -16 times the charge of a D8 brane to any orientifold in the setup. The sign of the charge again depends on the type of projection. We will consider the uncompactified setup, therefore, we are free to choose the sign. If we compactify x_6 on a circle like we did before, we can either pick a negative sign and put additional D8 branes or we can pick a negatively and a positively charged plane.

If we want to put an O8 plane (e.g. at $x_6 = 0$) we can only consider configurations which are symmetric under the reflection $x_6 \to -x_6$. There are several ways to achieve a symmetric configuration. The field theory interpretation of the various brane configurations were discussed in [31] in the T dual setup involving color D4 branes and flavor giving D6 branes. First, we can consider an O8 and an NS brane and its mirror at opposite sides of the orientifold. We span N D6 branes between these NS branes. The orientifold affects the cosmological constant like the equivalent amount of D8 branes would. For symmetry reasons, a negatively charged O8 causes a jump of the cosmological constant from 8 to -8 and a positively charged orientifold from -8 to $+8$. The brane configuration is shown in the following figure:

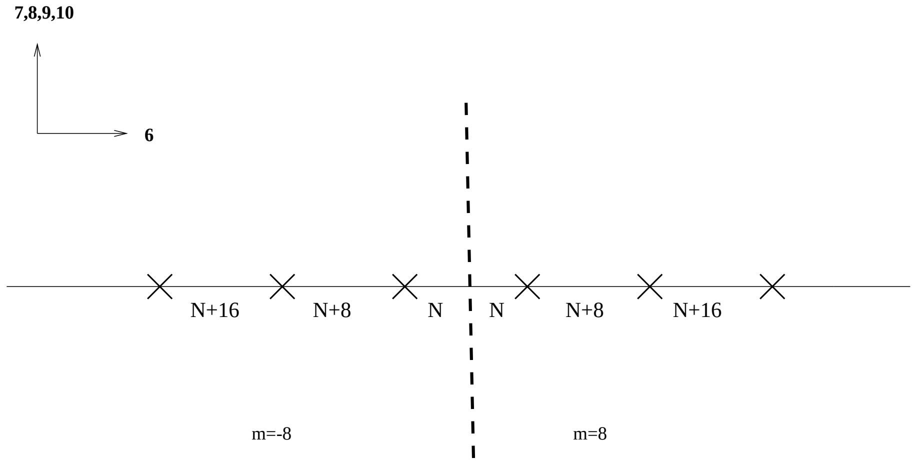

Figure 9: Brane configuration with positively charged O8 plane, leading to an Sp gauge group and a product of SU groups.

The O8 with negative charge projects to an SO group on the D8 branes but to an Sp group on the D6 branes. Similarly, we get a global Sp and local SO from the orientifold with positive charge. Charge conservation requires to include $N+8$ flavor giving branes (+mirrors). This is the same condition we already discovered before for orthogonal and symplectic gauge groups. We can also include further group factors. This is done by putting more NS5 branes along the x_6 factors in a mirror symmetric way with respect to the orientifold. These NS branes are (symmetrically) connected by D6 branes. The new group factors are simply SU factors. The orientifold only identifies the D6 branes stretching between the additional NS branes with their mirror images, but does not project further on Chan Paton factors. Therefore, we obtain the gauge groups

$$Sp(N) \times \prod_{\alpha=1}^{\frac{k-2}{2}} SU(N+8\alpha) \tag{2.30}$$

and

$$SO(N) \times \prod_{\alpha=1}^{\frac{k-2}{2}} SU(N-8\alpha) \tag{2.31}$$

The other possibility [31] considered is to take an odd number of NS branes. One of the NS branes has to be self mirror and therefore be stuck to the orientifold plane. The brane configuration is shown in the following figure:

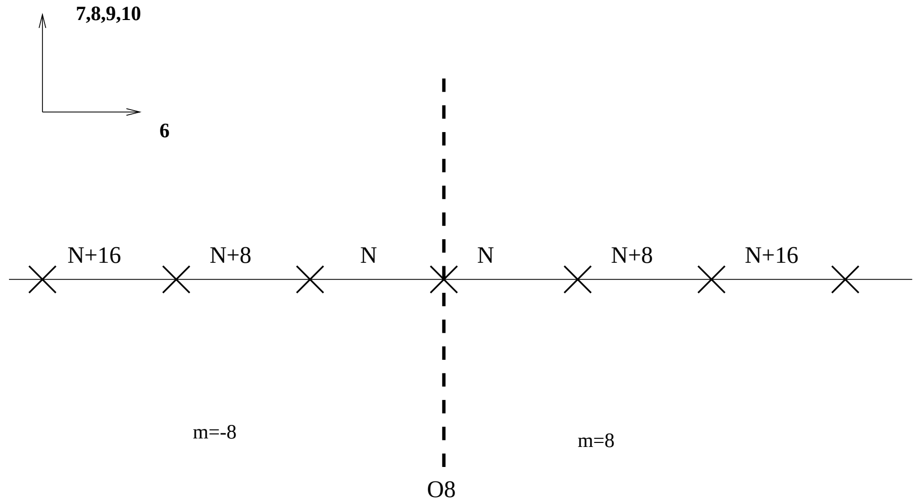

Figure 10: Brane configuration using a positively charged O8 plane leading to a product of SU-groups with a symmetric tensor in the central group factor.

The gauge groups on the left and right of the orientifold are projected upon each other, so that we have effectively only half the number of gauge groups. All the groups stay SU groups. For the middle gauge group we have open strings connecting the D6 branes with the mirror D6 branes "across" the orientifold plane. These strings lead to matter in the symmetric tensor representation for the positively charged orientifold and to an antisymmetric tensor for an orientifold with negative charge. Furthermore, we get bifundamentals for neighbouring gauge groups. The orientifold causes a jump from -8 to 8 (8 to -8) in the cosmological constant for positive (negative) orientifold charge. Taking this into account, the gauge group is

$$\prod_{\alpha=0}^{\frac{k-3}{2}} SU(N \pm 8\alpha) \tag{2.32}$$

The factor with the $\alpha = 0$ is the factor with the (anti-) symmetric tensor. The theories obtained in this way are anomaly free [47].

2.5.6 O8 planes and a compactified x_6 direction

If we have a compact direction x_6 the total charge has to sum up to zero on the circle. Let us consider k NS 5 branes on the circle connected by D6 branes. In addition, let us place two orientifold eight planes. To cancel the total charge, we pick orientifolds of negative charge and cancel their charge by also putting 32 D8 branes. The orientifolds are the T dual of the orientifold 9 planes, which project type IIB string theory to type I. The group $SO(32)$ appears as the highest possible global symmetry of the 6 dimensional theory in our setup. In general, it is broken to a subgroup which is a product of two SO groups and several SU groups. This corresponds to the possibility of placing D8 branes between different NS branes. However, there is the restriction that the configuration has to be symmetric with respect to the orientifold planes.

$$n_\alpha = n_{k-\alpha} \tag{2.33}$$

Here, n_α denotes the number of D8 in the group factor α. See the figure for notation. (The labelling of the group factors is always chosen in such a way that we start with the group $\alpha = 0$ at one orientifold and end up at the second orientifold, where we find the highest α.)

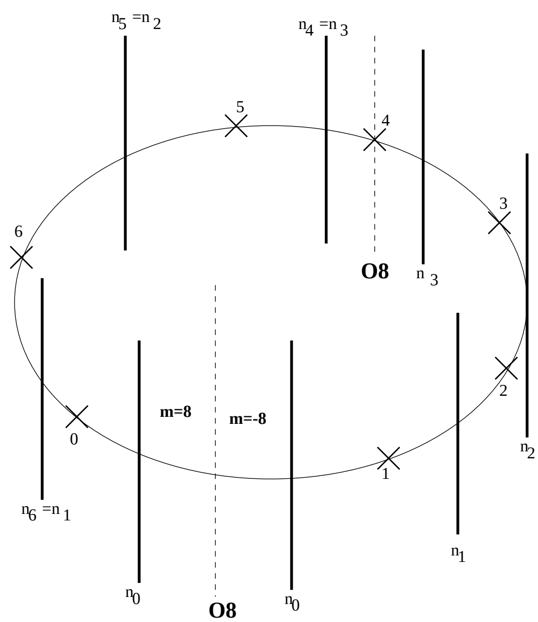

Figure 11: Brane configuration with compact x_6, O8 planes and D8 branes

There are two different cases to consider: k odd and k even. For k odd, one of the NS branes has to be stuck to an orientifold for the configuration to be symmetric. A configuration with $k = 7$ is drawn in the figure. The 32 D8 branes can be distributed in a symmetric way obeying (2.33) on the circle. The D6 branes in factor 0 are crossed by the orientifold. Therefore, half of the D8 branes for this factor have to be placed on either side of the orientifold. Our convention is that n_0 refers to only half of the D0 branes in factor 0, see figure. (An analogous convention holds in the case that the orientifold crosses also the gauge group with highest α, i.e. that we have two Sp groups in the case of even k.)

The gauge group of such a configuration is given by

$$Sp(N_0) \times \prod_{\alpha=1}^{\frac{k-1}{2}} SU(N_\alpha) \tag{2.34}$$

We have bifundamentals for each pair of neighbouring gauge groups and an antisymmetric tensor for group $\frac{k-1}{2}$. To determine the N_α one has to take into account the jump in the cosmological constant caused by the D8 and O8 planes.

$$\begin{aligned} N_\alpha &= N_{\alpha-1} + 8 - \sum_{\beta=0}^{\alpha-1} n_\beta \\ &= N_0 + 8\alpha - \sum_{\beta=0}^{\alpha-1} (\alpha-\beta) n_\beta \\ &= N_0 - 8\alpha + \sum_{\beta=0}^{(k-1)/2} min(\alpha,\beta) n_\beta, \end{aligned} \tag{2.35}$$

where we have used that the total number of D8 counted up to the factor in the middle is 16.

For k even, we have two possibilities: We can either have k free NS branes, or we can stick one NS brane to each orientifold plane. In the first case, the gauge group is of the form

$$Sp(N_0) \times \prod_{\alpha=1}^{\frac{k}{2}-1} SU(N_\alpha) \times Sp(N_{\frac{k}{2}}) \tag{2.36}$$

again with bifundamentals between neighbouring gauge groups and fundamental matter coming from 6-8 strings. The counting of the D6 branes which are necessary to cancel the anomaly (given a distribution of D8 branes and O8) is completely analogous to the previous case;

$$N_\alpha = N_0 - 8\alpha + \sum_{\beta=0}^{k/2} min(\alpha,\beta) n_\beta \tag{2.37}$$

The second case is the case where we have one NS brane stuck at each orientifold planes. As already discussed for the noncompact theories, this leads to a product of SU factors. We have matter in the antisymmetric tensor representation in the first and last group factor.

The same theories were already discussed in [49, 50]. Here, they used D5 branes in type I theories placed at an orbifold singularity. This is related to our approach by applying a T-duality in the 6 direction. The k NS 5 branes turn into an A_{k-1} singularity and the O8 planes become the O9 branes which project type IIB string theory onto type one. The D6 branes become D5 branes. D5 branes in type I theories are small $SO(32)$ instantons. N D5 branes inside 32 D9 branes can be interpreted as N instantons in $SO(32)$ (taking also into account the orientifold). Including the singularity, we consider $SO(32)$ instantons at an orbifold singularity. We can compare the data required to determine an instanton configuration with the Hanany-Witten data, which consists of brane positions. We have already mentioned implicitly that the instanton number corresponds to the number of D6 branes. An instanton gauge connection is flat, but we have a non-trivial fundamental group of our orbifold space. Therefore, we can have Wilson lines of the group $SO(32)$ at infinity: These are specified by non-trivial group elements ρ_∞. The non-trivial ρ_∞ breaks $SO(32)$ to a subgroup. In the dual HW picture, $SO(32)$ was broken to a subgroup by the distribution of the D8 branes along the circle. If we compare the gauge groups and matter content of [50] to our analysis, we find indeed that we should map the positions of the D8 branes to the Wilson line data.

In our approach we had a discrete choice in the case of an even number of NS 5 branes. Namely, we could either have only free NS 5 branes or one fivebrane stuck at each orientifold. This corresponds to the distinction of “with or without” vector structure [53] in the orbifold approach of [49, 50]. The gauge group of the type I or heterotic string is actually $Spin(32)/\mathbb{Z}_2$. The $\mathbb{Z}_2$ is generated by an element w, which acts as -1 on the vector, -1 on the spinor of negative chirality and $+1$ on the spinor of positive chirality. If we project w to $SO(32)$, it becomes the generator -1 in the center of $SO(32)$. For a $\mathbb{Z}_k$ orbifold, the element ρ_∞^k can be either mapped to 1 or w . This leads to the case of with or without vector structure. Comparing the field content of the theories obtained in [49, 50] with our results, we see that for even k the case with only free NS branes corresponds to the case with vector structure. If we have two stuck NS branes, we get the same results as for theories without vector structure in the orbifold approach.

Chapter 3

Matrix Theory

It is conjectured [54] that there exists a formulation of M theory in terms of supersymmetric quantum mechanics. This formulation is called Matrix theory. The hope is that Matrix theory provides us with a non-perturbative approach to string theory and quantum gravity. Matrix theory is formulated in light cone quantization or even discrete light cone quantization. In the following, we will review the Matrix conjecture and some basic features of discrete light cone quantization. After that we discuss toroidal compactifications in somewhat more detail and explain the problems with compactifications on higher dimensional tori. For extensive reviews, see [55, 56].

3.1 The conjecture

The matrix model conjecture of Banks, Fischler, Shenker and Susskind (BFSS) states that the uncompactified 11 dimensional M-theory can be formulated as the large N limit of the quantum mechanics describing N D0 branes. That means that M theory is described by the Lagrangian

$$\mathcal{L} = \frac{1}{2R}\mathrm{Tr}\left(\dot{X}^a\dot{X}_a + \sum_{a<b}[X^a, X^b]^2 + \theta^T\left(i\dot{\theta} - ,\ _a[X^a, \theta]\right)\right) \tag{3.1}$$

This Lagrangian is obtained by reducing the Lagrangian for $N = 1$ super Yang-Mills theory in ten dimensions to $0 + 1$ dimensions and choosing a gauge, where $A_0 = 0$. The X^a, where $a = 1, \ldots, 9$, are $U(N)$ matrices. The Lagrangian (3.1) is written down in 11-dimensional units. The argument of [54] makes heavily use of the fact that in special reference frames many degrees of freedom decouple. The reference frame they picked in the original conjecture is the infinite momentum frame (IMF). Here, one picks one special direction $x_{||}$, the longitudinal direction. The system is then infinitely boosted along this direction, so that $p_{||}$ of all constituents is much larger than any other scale in the problem. BFSS considered a compact longitudinal direction, such that the momenta

$p_{||}$ are quantized

$$p_{||} = \frac{N}{R} \tag{3.2}$$

The energy of a particle in the IMF is

$$E = \sqrt{p_{||}^2 + p_\perp^2 + m^2} \sim |p_{||}| + \frac{p_\perp^2 + m^2}{2|p_{||}|}. \tag{3.3}$$

The approximation holds for large $|p_{||}|$. In the limit of large $p_{||}$, states with negative or zero longitudinal momentum become infinitely heavy. These states may be integrated out. However, they play a crucial role, because the process of integrating out can influence the Hamiltonian of the remaining modes crucially. The effective Hamiltonian for the motion of the dynamical degrees of freedom, which have $p_{||} > 0$ in the transversal directions is

$$H = E - p_{||} = \frac{p_\perp^2 + m^2}{2p_{||}}. \tag{3.4}$$

BFSS suggested that to obtain M-theory in uncompactified space one has to take the limit

$$N,\, R,\, p_{||} \to \infty \tag{3.5}$$

of the supersymmetric quantum mechanics (3.1). There are two sources in the literature, from which the proposal [54] could draw. The quantum mechanics of D0 branes was studied extensively in [57, 58]. A matrix model of membranes was developed in [59].

Later, Susskind modified the original proposal and suggested that the conjecture might even be true at finite N [60]. Instead of the IMF, where one compactifies on a space-like circle, he proposed to consider discrete light cone quantization (DLCQ). Here, the theory is compactified on a "light-like circle". We will describe light cone quantization and discrete light cone quantization in the next section. For reviews on quantization on the light front and discrete light cone quantization see [61, 62].

3.2 Quantization on the light front and DLCQ

In light cone quantization we label the coordinates of our D-dimensional space-time in the following way:

$$x^\mu = \left(t,\, z,\, x^1, \ldots, x^{D-2}\right) \tag{3.6}$$

Light cone coordinates are then defined by

$$x^\pm = \frac{t \pm z}{\sqrt{2}} \tag{3.7}$$

The coordinates $x^1, \ldots, x^{D-2}$ parametrize the transversal space and will also be collectively denoted by $x_\perp$ We have conjugate momenta for the light cone coordinates

$$p_\pm = \frac{1}{\sqrt{2}} \left(p_t \pm p_z \right). \tag{3.8}$$

The coordinate x^+ is interpreted as time in light front quantization. The conjugate momentum p_+ generates translations in x^+ and therefore is the light cone Hamiltonian. For free fields, we have the following condition:

$$p_\mu p^\mu = M^2 \Rightarrow 2p_+p_- - p_\perp^2 = M^2. \tag{3.9}$$

We can solve this equation for the Hamiltonian

$$H \equiv p_+ = \frac{p_\perp^2}{2p_-} + \frac{M^2}{2p_-}. \tag{3.10}$$

The first part of the Hamiltonian looks like the kinetic energy of a non-relativistic particle, the second part like an internal energy.

3.2.1 Canonical Quantization in the Light Front

Light cone quantization is a Hamiltonian formulation and in some sense similar to ordinary canonical quantization at equal time. In both frameworks, one starts with an initial field configuration on an initial surface. In the usual equal time quantization the initial surface is the surface

$$t = x^0 = 0. \tag{3.11}$$

For light cone quantization, the role of time is played by x^+. Therefore, we study the development of the field theory starting from an initial surface

$$x^+ = 0. \tag{3.12}$$

In both cases, the fields away from the surface are expanded in terms of the fields on the initial surface. These are different surfaces, so we expect the physics on these surfaces to be different. Therefore, in this way discrete light cone physics differs from standard methods. It should also be mentioned that the operator P_- leaves the initial surface in light cone quantization invariant. However, the zero modes of P_- may still influence the dynamics of the other modes.

If we want to canonically quantize the theory, we impose canonical commutation relations between the fields and the conjugate momenta. The zero mode does not have a conjugate momentum. Therefore, it imposes a constraint on the dynamics, which can be very difficult to solve.

3.2.2 The vacuum

The structure of the vacuum on the light front is extremely simplified. That is one of the reasons, which makes light cone quantization attractive. The simplification occurs because for positive light cone energies P_+ (3.10), P_- must also be non-negative. This is in contrast to ordinary quantization, where for positive energies the transversal momenta can be negative or positive. As a consequence, generally in interacting theories we have a very complicated physical vacuum. On the light front, only zero-mode excitations are degenerate with the vacuum. This means that we have a trivial vacuum! Of course, the light front theory must somehow contain the non-trivial physics, which is ordinarily described by the non-trivial vacuum structure. The obvious guess is that this physics has to be encoded somehow in the zero modes. If we look at the relation (3.10), we see that the zero modes are heavy – therefore one can hope to be able to integrate them out. This is possible in principle, but can be hard in practice. It should be rather clear now that the zero mode has far reaching consequences on the dynamics of the other fields. Therefore, the expectation is that integrating them out is complicated.

3.2.3 Discrete Light Cone Quantization (DLCQ)

Discrete light cone quantization has been used as a technique to handle the zero modes in [63, 64]. In the continuum, the zero modes can accumulate in the limit $P_- \to 0$. The advantage of going to DLCQ is then that the zero mode is separated out and has a gap from the other modes. It has also been discussed in the literature as a method to control infrared singularities.

In discrete light cone quantization, one compactifies on a light like circle. This means, that we identify (for a field ϕ):

$$\phi(x^- + R, x^+) \sim \phi(x^-, x^+). \tag{3.13}$$

It is useful to think of such a light like compactification as a limit of a spatial compactification. One reason is, that the points which we wish to identify, namely x^- and $x^- + R$, are separated by a light like distance. Relating these points by imposing boundary conditions might violate causality. Therefore, we perform a slight shift in x^+ and consider the coordinates:

$$\begin{aligned} x^-_\epsilon &= x^- \\ x^+_\epsilon &= x^+ + \frac{\epsilon^2}{2} x^- \end{aligned} \tag{3.14}$$

The metric on the light cone before performing the shift

$$ds^2 = dx_i^2 - 2dx_+ dx_- \tag{3.15}$$

and is now modified to

$$ds^2 = dx_i^2 - 2dx_+dx_- - \epsilon^2 dx_-dx_-. \tag{3.16}$$

The points which we want to identify therefore have a space like distance. Finally, the limit $\epsilon \to 0$ has to be taken.

3.3 M theory in DLCQ

Seiberg and Sen [65, 66] have given a prescription for the description of M theory in DLCQ. They think of DLCQ as a limit of a compactification on a small spatial circle. This small circle is related to a light like circle by a large boost. The dynamical degrees of freedom should of course survive this boost. The large boost causes Lorentz contraction in the direction in which the boost is performed. That is the reason why the circle is small. So we want to describe M theory compactified on a light like circle L with the help of an auxiliary theory $\tilde{M}$, which is compactified on a small spatial circle R. The Hamiltonian for the auxiliary theory is given by (3.4). Of course the limit we take should be chosen in such a way that the relevant degrees of freedom survive. In other words:

$$\lim_{R\to 0} \frac{p_\perp^2 + m^2}{2p_{||}} = constant = \frac{\mathcal{P}_\perp^2 + \mathcal{M}^2}{2p_-} \tag{3.17}$$

The quantities on the rhs are the physical quantities in M-theory and the quantities on the lhs are in the auxiliary theory. The equation expresses that the energies of particles in the auxiliary theory equal the energies of M-theory. The parameters and quantities in the auxiliary theory are subject to a limiting procedure, whereas the parameters in M-theory can of course be freely chosen and have fixed values. Let us derive in which way the parameters of the auxiliary $\tilde{M}$ theory are given by the M theory parameters and the properties of the limit. The only parameter of the uncompactified M theory is the Planck length L_p. For DLCQ we have in addition the size of the light like circle L. For $\tilde{M}$ theory we have the Planck length l_p and in addition the size of the radius of the space like circle R. We can express the quantities in M theory and the auxiliary theory in terms of these parameters:

$$\begin{aligned} \mathcal{M} &= \frac{\mathcal{M}'}{L_p} \qquad & m &= \frac{m'}{l_p} \\ p_- &= N/L \qquad & p_{||} &= N/R, \end{aligned} \tag{3.18}$$

where $\mathcal{M}'$ and m' are dimensionless numbers. Our condition for the Hamiltonian reads

$$\lim_{R\to 0} \frac{\frac{1}{l_p^2}}{1/R} = \frac{\frac{1}{L_p^2}}{1/L}. \tag{3.19}$$

From this we can deduce that the limit should be taken in such a way that

$$\frac{R}{l_p^2} = \frac{L}{L_p^2}. \tag{3.20}$$

Let us now consider the case that we have additional compact directions. We will restrict ourselves to toroidal compactifications, but the arguments given in this paragraph are in principle valid for arbitrary backgrounds. In the auxiliary theory the length of a transversal compact circle is called L_i and in M theory it is called R_i. Our condition takes the form

$$\lim_{R\to 0} \frac{\frac{1}{L_i^2} + \frac{1}{l_p^2}}{1/R} = \frac{\frac{1}{R_i^2} + \frac{1}{L_p^2}}{1/L} \tag{3.21}$$

This leads to the condition that

$$\frac{L_i}{l_p} = \frac{R_i}{L_p}. \tag{3.22}$$

Putting everything together, we conclude that M theory in DLCQ is given by an auxiliary theory compactified on a spatial circle R in the limit that

$$R \to 0 \quad \text{such that} \quad \frac{R}{l_p^2} = \frac{L}{L_p^2}, \quad \frac{L_i}{l_p} = \frac{R_i}{L_p}. \tag{3.23}$$

From now on, we will work in the auxiliary theory. The physical M-theory quantities can be re-obtained using the formulas (3.23). What did we gain by this procedure? We do not really know what M theory is, but we know very well what M theory on a small circle is: It is simply weakly coupled IIA string theory. Furthermore, we have infinitely boosted in one direction. Therefore from the previous sections we expect that only states with positive momentum in this direction are dynamical degrees of freedom. We also expect problems due to the zero modes, to which we come back in the discussion at the end of this chapter. From the IIA point of view, the states with positive momentum are the D0 branes. They are electrically charged under the KK gauge field coming from the reduction of the 11 dimensional metric to 10 dimensions. The IIA parameters, the string coupling constant g_s and the string length l_s can be related to the $\tilde{M}$ theory parameters in the following way [4]:

$$\begin{aligned} g_s^2 &= R^3/l_p^3 \\ l_s^2 &= l_p^3/R \\ L_i &= L_i. \end{aligned} \tag{3.24}$$

We see that the infinite boost limit (3.23) corresponds to

$$\begin{aligned} g_s &\to 0 \\ M_s = 1/l_s &\to \infty \\ L_i &\to 0, \\ \text{keeping} \qquad g_{YM_0}^2 &= \frac{g_s}{l_s^3} \qquad \text{fixed.} \end{aligned} \tag{3.25}$$

This means that our IIA theory is weakly coupled. Furthermore, massive string modes are decoupled because of the behaviour of the string scale. The fixed quantity is the coupling constant of the D0 brane physics, as written down in (1.11). If we have no compact transverse dimensions we recover the original DLCQ version of the matrix description. The full quantum theory of the sector with N units of momentum $p_{||}$ is described by the dynamics of N D0 branes in the limit that the string coupling is zero, the Planck and the string scale go to infinity while the gauge coupling of the quantum mechanics on the D0 worldvolume is fixed.

If we do have more compact directions, we expect that for example D2 branes wrapped on a two dimensional torus give also contributions. Compactifications of Matrix theory on tori have been considered e.g. in [67, 68, 69, 70, 71, 72] before the Seiberg-Sen proposal. It was conjectured that the Matrix description for M theory on a torus is given by super Yang-Mills on the dual torus. However, for higher dimensional tori this is not the full story. The compactification on T^4 was considered in [70] and on T^5 in [71, 72]. The T^5 compactifications are given by new $5+1$ dimensional string theories living on the worldvolume of 5 branes. (See [73] for a review on "microstring" theories.) Matrix theory on T^6 was discussed in [74, 75, 76]. We have seen that Matrix theory can be obtained as a Seiberg-Sen limit for an arbitrary number of compact directions. Thus, Seiberg-Sen provides us with a unified proposal of how to treat Matrix theory compactified to lower dimensions. This will be the topic in the following sections. We recover the original proposals from the Seiberg-Sen procedure.

3.4 Bound states and Matrix theory

We have seen in the previous section that in the DLCQ we are left with all states which have positive momentum in the 11 direction. We motivated that for Matrix theory it is useful to look at these states from a IIA perspective in a particular limit in an auxiliary $\tilde{M}$ theory. In the following, we will discuss all possibilities. First of all, we have the D0 branes, whose light cone energy vanishes. These branes are simply purely gravitational waves travelling in the 11 direction from the 11 dimensional point of view. Of course we can get more states by looking at waves which also have a component in an additional transversal direction. Furthermore, we can give momentum in the 11 direction to other branes. The light-cone energy of an M brane wrapping transversal directions is given by:

$$\lim E = \sqrt{\left(\frac{N}{R}\right)^2 + (TV)^2} - \frac{N}{R}, \tag{3.26}$$

where lim denotes the Seiberg-Sen limit. T is the tension of the brane and V the volume which it wraps. This translates in the 10 dimensional IIA picture to bound states of N D0 branes with other branes. (The number of "other" branes will be denoted by n

throughout the text). It has been indicated how to compute the energies for true bound states and bound states at threshold in section 1.1.3. For a non-threshold bound state of a Dp brane with a D0 brane we get using the results in section 1.1.3

$$\lim \sqrt{\left(\frac{N}{l_s g_s}\right)^2 + (T_{Dp}V)^2} - \frac{N}{l_s g_s}, \tag{3.27}$$

where lim is the 10-dimensional version of the Seiberg-Sen limit. The tension of the Dp brane was denoted by T_{Dp} and the D0 brane tension was explicitely written down, see the formulas in (1.1). The (IIA version of the) Seiberg-Sen limit is precisely chosen in such a way that these bound states remain of finite energy and the D0 brane contribution is dominant. Subtracting the D0 brane energy corresponds to subtracting a ground state energy. The computation of the light cone energy of a BPS state obtained from a wrapped M brane (3.26) and the bound state energy of a D0 and wrapped D brane in IIA (3.27) is the same.

Let us first look at two gravitational waves, one travelling in direction 11 and one travelling in some transversal direction L_i. If L_i is compact, the momentum in this direction is quantized, $p_i = \frac{n}{L_i}$. Let us compute the energy of the resulting wave in the light cone frame:

$$\lim \sqrt{\left(\frac{N}{R}\right)^2 + \left(\frac{n}{L_i}\right)^2} - \frac{N}{R} = \frac{1}{2}\frac{R}{N}\left(\frac{n}{L_i}\right)^2 = \frac{p_\perp^2}{2p_{||}} \tag{3.28}$$

As expected, this is the light cone energy of some particle travelling in a perpendicular direction. Now let us look at bound states of higher dimensional branes with D0 branes. We will consider branes which are totally wrapped along compact transversal directions. We can give some momentum in 11 direction to n M2 branes wrapping only transversal directions. From the IIA perspective, this corresponds to building a bound state of the D2 and a D0 brane. We can compute the energy of such a bound state in our limit:

$$\lim \sqrt{\left(\frac{N}{l_s g_s}\right)^2 + \left(\frac{nL_iL_j}{g_s l_s^3}\right)^2} - \frac{N}{l_s g_s} = \frac{n^2 L_i^2 L_j^2}{2 g_{YM_0}^2 l_s^8 N} \tag{3.29}$$

Note that the result is a finite quantity. That the bound state has a finite energy in our limit is simply a reformulation of the fact that the limit is chosen in such a way that states with positive momentum $p_{||}$ survive. The resulting energy can be rewritten in 11 dimensional units as

$$\frac{n^2 L_i^2 L_j^2}{2 l_p^6 p_{||}} \tag{3.30}$$

If we compute the mass of this state using

$$m^2 = 2p_{||}E \tag{3.31}$$

we get back that this is simply the energy of an M2 brane wrapped on a transversal two torus. Before we do this, let us consider the other bound states we can have, namely the bound state of N D0 brane and n NS 5 branes. Its energy is

$$\lim \sqrt{\left(\frac{N}{l_s g_s}\right)^2 + \left(\frac{nL_iL_jL_kL_lL_m}{g_s^2 l_s^6}\right)^2} - \frac{N}{l_s g_s} = \frac{n^2 L_i^2 L_j^2 L_k^2 L_l^2 L_m^2}{2N g_{YM_0}^6 l_s^{20}} \tag{3.32}$$

In 11 dimensional units this reads

$$\frac{n^2 R L_i^2 L_j^2 L_k^2 L_l^2 L_m^2}{2N l_p^{12}}. \tag{3.33}$$

Using (3.31) we find the energy of a wrapped M5 brane. There are also bound states of D0 and D4 branes. From the eleven dimensional point of view, we see that this time the wave travels in a direction along which the brane stretches. The consequence of this is that the corresponding bound states are bound states of a somewhat different quality: we get bound states at threshold, see section 1.1.3. The energy of n wrapped D4 branes in IIA (or longitudinal M5 branes in M theory) is

$$E = \frac{nL_iL_jL_kL_m}{g_s l_s^5} = \frac{nL_iL_jL_kL_mR}{l_p^6} \tag{3.34}$$

This energy is finite in the limit we take and therefore the threshold bound state with the D0 brane also survives the limit. Another threshold bound state is the bound state of a D0 brane and a fundamental string. From the M theory perspective the string is a membrane wrapping the R direction. The energy of such a string or membrane is:

$$E = \frac{L_i}{l_s^2} = \frac{L_i R}{l_p^3} \tag{3.35}$$

This survives the limit and therefore also the threshold bound state has this finite energy. Let us now consider the D6 brane or KK6 monopole in M theory. There is no bound state of a D0 with a D6 because we have a repulsive force between these two branes. The energy of the wrapped D6 is

$$E = \frac{L_iL_jL_kL_mL_nL_o}{g_s l_s^7} = \frac{L_iL_jL_kL_mL_nL_oR^2}{l_p^9} \tag{3.36}$$

The energy is going to zero in the limit we take. The D6 brane is becoming light. This causes some problems in compactifications of M(atrix) theory on six (or higher) dimensional manifolds, as we will explain in more detail below.

3.5 Matrix Theory in Lower Dimensions

The philosophy in Matrix theory is to describe M theory in 11 dimensions or compactified M theory in terms of a lower dimensional field theory. This lower dimensional field theory should be easier to treat than full fledged 11 dimensional M theory. In this section we will review the Matrix theory description for M theory compactified on tori. We explain the difficulties with compactification on six dimensional tori.

In the previous section we have already discussed that there are several bound states of D0 branes and higher dimensional branes wrapping transversal compact directions, which lead to states of finite energy. In this section we will explain how the bound states discussed above find an interpretation in terms of lower dimensional field theory quantities. Here, we will use the interpretation of bound states as excitations of the world volume theory, which we introduced in section 1.2.4. We will refer to the field theory which gives the Matrix description of M theory as a "base space" theory. This is to make a distinction between these auxiliary theories and the true "space-time" M theory they are supposed to describe.

In the case without compactified transversal dimensions we have seen that Matrix theory is the quantum mechanics of D0 branes. If we have compact (toroidal) transversal dimensions then our limit dictates that the length of these directions goes to zero. Therefore, the original IIA description is not a good picture anymore. We should perform T dualities in all compact transversal dimensions, thereby mapping the D0 branes to Dd branes wrapping the dual torus. Here, d denotes the number of compact dimensions. The field theory we are looking for is the field theory on the wrapped Dd brane. We will see that in our limit the bulk physics decouples from the brane and we are left with the theory on the brane (in certain dimensions). The parameters in our theory transform under T-duality. The following formulas can be obtained by applying the usual T duality.

$$\begin{aligned} l_s^2 &= \frac{l_p^3}{R} \quad \text{(unchanged)} \\ \Sigma_i &= \frac{l_p^3}{RL_i} \\ g_s^2 &= \frac{l_p^{3d-3}}{R^{d-3}V^2} \\ M_P^8 &= g_s^{-2}M_s^8 = \frac{V^2R^{d+1}}{l_p^{3d+9}} \\ g_{YM}^2 &= g_s l_s^{d-3} = \frac{l_p^{3d-6}}{R^{d-3}V}, \end{aligned} \tag{3.37}$$

where $V = \prod_{i=1}^d L_i$, M_P is the Planck mass and $M_s = 1/l_s$. The last two formulas can be obtained from the others using the relationship between Einstein and String frame and the fact that the gauge coupling on a Dd brane is proportional to g_s and has a length dimension $d-3$. The string length l_s remains unchanged under T-duality, so we are still in the limit that l_s goes to zero. The transversal lengths Σ_i are finite after T duality, so we found a reasonable string theory description. We can read off from these formulas how the various other parameters behave in the infinite boost limit:

$$\begin{aligned}
g_s^2 &\sim \begin{cases} 0 & \text{for } d<3 \\ \text{finite} & \text{for } d=3 \\ \infty & \text{for } d>3 \end{cases} \\
M_s^2 &\sim \infty \text{for all } d \\
M_P^8 &\sim \begin{cases} \infty & \text{for } d<7 \\ \text{finite} & \text{for } d=7 \\ 0 & \text{for } d>7 \end{cases} \\
\Sigma_i &\sim \text{finite for all } d \\
g_{YM}^2 &\sim \text{finite for all } d
\end{aligned} \tag{3.38}$$

The quantity which is kept fixed in all dimensions is the gauge coupling on the brane. This means that the physics on the wrapped brane is an interacting theory. In the limit that g_s is finite or vanishes and the Planck and string scale become infinite, the physics on a Dd brane is well described by super Yang Mills theory. Therefore, for $d \leq 3$ the Matrix description of M theory on T^d is super Yang Mills theory on the dual torus. For higher dimensions, the story becomes more complicated. For $d \geq 7$ there is no hope to describe physics in terms of a lower dimensional decoupled theory. The Planck scale stays finite or even becomes zero in the limit. This means that we will not be able to decouple bulk gravity. The conclusion is that the Matrix description of M theory (with maximal supersymmetry) compactified to four or less dimensions is a ten dimensional theory. For dimensions $4 \leq d \leq 6$ the Planck scale still goes to infinity, but the string coupling becomes also infinite. Theories at strong coupling can be described by using S-duality in the type IIB context or the relationship with M theory in the type IIA situation. Using this we will demonstrate that we can still come to an appropriate description in a weakly coupled theory for $d=4,5$. For $d=6$ we have to use that IIA at strong coupling is better thought of as 11 dimensional M theory. Unfortunately, the 11 dimensional Planck scale is finite, whereas the 10 dimensional Planck scale still goes to infinity. Therefore, as for the higher dimensions d, bulk gravity does not decouple.

3.5.1 Bound states after T duality

In section 3.4 we have discussed bound states of D0 branes with other branes. We now want to study what happens to these bound states after T duality. The D0 branes turn into Dd branes wrapping the torus, so we are looking for bound states of D0 with some p branes, where the p brane can be a fundamental string, a Dp brane, or NS brane. The energy of such a bound state is:

$$\lim \sqrt{(T_{Dd}V_d)^2 + (T_pV_p)^2} - T_{Dd}V_d, \tag{3.39}$$

where the limit to be taken was discussed before. T_{Dd} and T_p denote the tensions of the branes. The tension of the Dd brane is

$$T_{Dd} = \frac{1}{g_s l_s^{d+1}} = \frac{1}{g_{YM}^2 l_s^4}. \tag{3.40}$$

Equation (3.39) is the T-dual version of (3.27). First of all, we had transversal momentum modes around some direction L_i. These momentum modes become after T-duality in direction L_i strings winding on L_i. Applying formula (3.39) we obtain for the energy:

$$\lim_{l_s\to 0} \sqrt{\left(\frac{NV_\Sigma}{g_{YM}^2 l_s^4}\right)^2 + \left(\frac{n\Sigma_i}{l_s^2}\right)^2} - \frac{NV_\Sigma}{g_{YM} l_s^4} = \frac{\Sigma_i^2 n^2 g_{YM}^2}{2NV_\Sigma}, \tag{3.41}$$

where $V_\Sigma = \Pi_i \Sigma_i$. This energy corresponds to the energy of a Wilson line in the Yang-Mills theory on the dual torus. Using the T duality relations we can of course verify that the left hand side of the formula is simply (3.28). The result is that transversal momentum modes in the compact space time directions are described by Wilson lines in the dual direction in a Yang Mills theory on the dual torus. Similarly, we can look at the T-dual of the bound state between the N D0 and n D2 brane, which comes from a transversal M2 with momentum in the 11 direction in M theory. After T duality, this turns into a bound state of N Dd brane with n D(d-2) brane. The $d-2$ brane tension is

$$T_{D(d-2)} = \frac{1}{g_s l_s^{d-1}} = \frac{1}{g_{YM}^2 l_s^2}$$

The energy of the bound state is:

$$\lim_{l_s\to 0} \sqrt{\left(\frac{NV_\Sigma}{g_{YM}^2 l_s^4}\right)^2 + \left(\frac{nV_{d-2}}{g_{YM}^2 l_s^2}\right)^2} - \frac{NV_\Sigma}{g_{YM}^2 l_s^4} = \frac{n^2 V_\Sigma}{2N\Sigma_i^2\Sigma_j^2 g_{YM}^2} \tag{3.42}$$

This time, this corresponds to the energy of a magnetic flux in the super Yang Mills theory. The correspondence between D brane bound states and curvature terms was discussed in section 1.2.4. The result is that magnetic fluxes in the SUSY YM describe wrapped transversal membranes in space time. As we have seen, we should also consider M2 branes wrapping the 11-direction. This corresponds to a string and D0 brane threshold bound state in the IIA theory. After T duality, the string winding in the compact direction turns into a momentum mode along this direction. The Matrix description of the longitudinal M2 brane is therefore a momentum mode:

$$\frac{n}{\Sigma_i} = \frac{nL_i}{l_s^2} = \frac{L_i R}{l_p^3} \tag{3.43}$$

So far, we have seen that we have a Matrix description of space time momentum modes in terms of electric flux, transversal wrapped membranes in terms of magnetic flux and longitudinal membranes in terms of base space momentum. In the case that we have three or less transversal compact dimensions, these are the only BPS states we can expect.

The BPS states transform under the so-called U-duality group (see below). Here, we have two types of states: First there are fluxes in the Yang-Mills theory, which correspond to non-threshold bound states in terms of branes and to transversally wrapped branes in space-time. They form the so-called flux multiplet of the U-duality group. Then, there are the momentum modes, or, more generally, we also have non-threshold bound states. These modes correspond to longitudinally wrapped branes in space-time and form the momentum multiplet.

In [77] it was conjectured that all string dualities can be unified by a so called U-duality, which includes the S- and T-dualities discussed up to now. In fact, indications of the presence of such a symmetry group were already found in the context of toroidally compactified 11-dimensional supergravity: Here, the equations of motion of the theories are invariant under certain symmetry groups. The U-duality groups are a discrete version of these groups, which survives the quantization procedure. U-duality acts on the BPS spectrum of the theory. A nice feature of Matrix theory is that it makes the U-duality groups more visible. For example, the U-duality group of M-theory on a 3-torus is $SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$, which is seen in the Matrix picture as the geometrical symmetry of the dual 3-torus times the strong-weak coupling symmetry of the Yang-Mills theory. We will see what Matrix theory has to say about the U-dualities for compactifications on higher dimensional tori in the following sections. U-dualities in Matrix theory have been discussed in detail in [78, 79].

The four-torus

For a compactification on the four torus we expect one more state from M-theory considerations: For the first time we should be able to recover a wrapped M5 brane, namely the M5 wrapping R and all transversal directions. Also from the field theory point of view it is clear that the YM theory cannot be the full story: In dimensions higher than 3+1 these theories are non-renormalizable. So they can only be effective descriptions in a certain energy range and we have to add more degrees of freedom to obtain the full theory. At the beginning of this section we have learnt that for T^4 compactification the string coupling blows up. Therefore, YM on the dual torus is no longer the appropriate description. We have to consider a wrapped D4 brane at strong coupling. The strong coupling limit of this is best thought of as an M-theory fivebrane wrapped in an additional direction, which we call Σ_5. Σ_5 has to fullfill:

$$g_s^2 = \frac{\Sigma_5^3}{\lambda_p^3} \quad \text{and} \quad l_s^2 = \frac{\lambda_p^3}{\Sigma_5}, \tag{3.44}$$

where λ_p is the Planck length of the 11 dimensional theory. We see immediately that this implies

$$\Sigma_5 = l_s g_s = g_{YM}^2 = \frac{l_p^6}{RL_1L_2L_3L_4} \tag{3.45}$$

In our limit the Planck mass $1/\lambda_p$ goes to infinity. It is related to space time quantities by

$$\lambda_p^3 = \frac{l_p^9}{L_1L_2L_3L_4R^2} \tag{3.46}$$

This formula can be obtained using the relation

$$M_P^{(11)} = g_s^{-\frac{1}{12}} M_P^{(10)} \tag{3.47}$$

between 10 and 11 dimensional Planck masses. To understand relation (3.47), note that the gravitational coupling in 11 dimensions is proportional to $(1/M_P^{(11)})^9$. The ten-dimensional gravitational coupling is proportional $(1/M_P^{(10)})^8$ and can be obtained from the eleven dimensional coupling and Σ_5 by the usual KK-procedure. Inserting the M-theory-IIA relation between the compactification radius and the string coupling constant [4] (see the formulas (3.24 and replace R by Σ_5) leads to (3.47).

We are left with the theory of N M5 branes in the limit that the Planck mass goes to infinity. The bulk gravity decouples. The theory on the M5 is an interacting theory at a non-trivial fixed point of the renormalization group. It is the (0,2) superconformal theory of a self dual tensor multiplet. In the limit that the Planck scale becomes infinite, the theory on the M5 decouples from the bulk and leaves a consistent six-dimensional theory. The coupling of the theory of a self dual tensor is one, so the theory on the brane remains interacting even in this limit. Upon compactification on a circle, the theory flows at long distances to 5d super Yang Mills theory. The self dual tensor gives one vector in 5 dimensions. The coupling of the 5 dimensional theory is

$$\frac{1}{g_{YM}^2} = \frac{1}{\Sigma_5}, \tag{3.48}$$

in agreement with the M theory consideration in (3.45). The gauge coupling has negative mass dimensions, which means that the gauge kinetic term is an irrelevant operator in a non-renormalizable field theory. The coupling determines the scale at which the description in terms of a 5d gauge theory breaks down.

What are the excitations in the Matrix picture? We know now that our Matrix description is the 6 dimensional (0,2) fixed point. The excitations should be the fluxes of the two form tensor field. These fluxes can also be interpreted as bound states of the M2 brane with the M5 brane. This interpretation is similar to the argument given for D-branes in section 1.2.4. For D-branes, the existence of certain Chern Simons term showed the equivalence of particular bound states involving lower dimensional D branes with non-trivial curvature for the gauge field living on the brane. For the M5 we have a term of the form

$$C_3 \wedge H$$

in the world volume action for the M5 brane [80]. Here, C_3 denotes the M-theory three-form, under which the membrane is charged and H is the field strength of the tensor field living on the M5 brane. This coupling is completely analogous to the Chern-Simons term considered in the D brane context. Let us write down the energy of a bound state of an M5 brane with an M2 brane in its worldvolume:

$$\lim \sqrt{\left(\frac{N\,\Sigma_1\Sigma_2\Sigma_3\Sigma_4\Sigma_5}{\lambda_p^6}\right)^2 + \left(\frac{n\Sigma_i\Sigma_j}{\lambda_p^3}\right)^2} - \frac{N\,\Sigma_1\Sigma_2\Sigma_3\Sigma_4\Sigma_5}{\lambda_p^6} = \frac{n^2\Sigma_i^2\Sigma_j^2}{2N\,\Sigma_1\Sigma_2\Sigma_3\Sigma_4\Sigma_5} \qquad (3.49)$$

For $i, j \neq 5$ the corresponding BPS states in space time are transversally wrapped membranes. If i (or j) equals 5, we obtain a momentum mode in the L_j (L_i) direction. To obtain the longitudinal branes we have to look at the momenta on the world volume. The momentum modes along $\Sigma_1, \ldots, \Sigma_4$ give the wrapped space time membranes and the momentum in the 5 direction gives the wrapped longitudinal 5 brane.

The five-torus

Let us proceed to compactification on a five dimensional torus. One way to find a Matrix picture is simply to think of the M5 which described the T^4 compactifaction with an additional compact direction U transversal to the world volume [81]. To be a suitable Matrix description for an M theory compactification on a higher dimensional torus it is clear that this theory has to have additional degrees of freedom compared to the M5 theory without compact transversal directions. We now have to find a base space interpretation of those states, which were not present in the compactification on the four torus. These are the longitudinal five branes wrapping L_5, the longitudinal membrane wrapping L_5 and the transversal M5 brane wrapping $L_1, \ldots, L_5$ and four transversal membranes wrapping L_5 and one further direction. Our base space theory is the theory of a M5 brane with one compact scalar

$$U = L_5. \qquad (3.50)$$

This theory is a string theory on the M5 brane: the strings come from membranes wrapping the compact direction and extending in one of the directions on the M5 brane. The tension of these strings is:

$$T_f = \frac{U}{\lambda_p^3} \tag{3.51}$$

This is finite in our limit. We therefore have strings of finite energy living on our world volume. The energy is for $i = 1, \ldots, 4$

$$E = \frac{U\Sigma_i}{\lambda_p^3} = \frac{nRL_jL_mL_kL_5}{l_p^6}, \tag{3.52}$$

where $j, m, k \in \{1, \ldots, 4\}$ and $j, m, k \neq i$. Therefore, their space time interpretation are M5 branes wrapping R and L_5 and three more directions. For $i = 5$ we get

$$E = \frac{U\Sigma_5}{\lambda_p^3} = \frac{nL_5R}{l_p^3} \tag{3.53}$$

giving the missing longitudinal membrane. The momentum mode on L_5 simply remains the momentum mode on U in the base space theory. Now let us look for the missing transversal objects. Indeed, we now have the possibility to also form a non threshold bound state in the base space theory between our basic M5 brane and M5 branes which wrap U and four further directions. The energy of such a bound state is

$$\begin{aligned} \lim \sqrt{\left(\frac{N\,\Sigma_1\Sigma_2\Sigma_3\Sigma_4\Sigma_5}{\lambda_p^6}\right)^2 + \left(\frac{nU\Sigma_i\Sigma_j\Sigma_k\Sigma_l}{\lambda_p^6}\right)^2} &- \frac{N\,\Sigma_1\Sigma_2\Sigma_3\Sigma_4\Sigma_5}{\lambda_p^6} \\ &= \frac{n^2}{2N}\left(\frac{U}{\lambda_p^3}\right)^2 \frac{\Sigma_1\Sigma_2\Sigma_3\Sigma_4\Sigma_5}{\Sigma_m^2} \end{aligned} \tag{3.54}$$

For $m = 1, 2, 3, 4$ this leads to the four membranes wrapping L_5 and L_m and for $m = 5$ we obtain the M5 wrapping $L_1, \ldots L_5$. This means that the base space theory has reproduced all space time BPS states. It is possible to think of this theory in various ways. We described it as the theory on the M5 brane with one additional compact scalar. Of course, this situation can also be interpreted in a IIA context. The additional compact direction is small, therefore the IIA point of view is appropriate. We are then left with a IIA NS5 brane. The parameters of the IIA theory are

$$\begin{aligned} \tilde{g}_s^2 &= \frac{U^3}{\lambda_p^3} = \frac{L_1L_2L_3L_4L_5^3R^2}{l_p^9} \\ \tilde{l}_s^2 &= \frac{\lambda_p^3}{U} = \frac{l_p^9}{L_1L_2L_3L_4L_5R^2} \\ \lambda_p^{(10)8} &= \frac{\lambda_p^9}{U} = \frac{l_p^{27}}{L_5L_1^3L_2^3L_3^3L_4^3R^6} \end{aligned} \tag{3.55}$$

In this picture the base space theory is obtained by taking an NS5 brane in the limit that the Planck scale goes to infinity and the string coupling goes to zero. This means that the bulk theory has vanishing coupling constant and the bulk modes decouple from the brane physics. Fundamental IIA strings are captured on the brane. They cannot leave the brane because of the vanishing string coupling. We are again left with a six dimensional string theory. The threshold bound states of membranes and fivebranes in our M theory picture became fundamental strings on the NS5. A further T duality along Σ_5, which is contained in the world volume of the NS5 brane, leads us to a IIB picture. Performing this T duality the string coupling becomes

$$g_{s_{IIB}}^2 = \frac{R^2 L_1^2 L_2^2 L_3^2 L_4^2 L_5^2}{l_p^{12}} \tag{3.56}$$

Note that this is simply the inverse of the string coupling computed by applying T duality along all directions of a 5 torus starting from the D0 brane picture, given in (3.37). We could have come more directly to this IIB description by performing 5 T dualities (3.37) and an additional S duality. Again, there are arguments that we have an interacting theory on the brane, which is decoupled from the bulk physics. The string coupling goes to zero, so we have captured strings on the brane. Note that the coupling of the theory on the IIB NS5 is $\sim M_s^2$ (this can be seen by applying S-duality to the formula for g_{YM} for $d = 5$ in (3.37)), which remains constant in the limit. Therefore, the theory on the brane can remain interacting. In this picture, the excitations are given by non-threshold bound states with D1, D3 and D5 branes. The threshold bound states leading to the transversal membranes and fivebranes are given by the captured strings and momentum modes in the base space theory.

The six-torus

We will discuss some candidates for the Matrix description of M theory on a six torus [74, 75]. All of these candidates are again obtained by applying the Seiberg-Sen limit. They are theories on branes related by a sequence of S- and T-dualities (or by going to M-theory). The first candidate is to consider the M theory fivebrane, which described the four and five torus, with two compact directions. The second possibility is to perform the T-dualities (3.37), which lead to a D6 brane at strong coupling [74, 75]. This D6 is better thought of as the M theory KK monopole.

Let us start with the description in terms of the M5. In complete analogy to the 5-torus description in terms of the M5 brane with one compact scalar U, we now look at an M5 brane with two compact scalars, U and V,

$$V = L_6 \tag{3.57}$$

The limit we take is in terms of the quantities of the base space theory:

$$\lambda_p, U, V \to 0 \qquad \text{but} \qquad \frac{U}{\lambda_p^3}\,,\ \frac{V}{\lambda_p^3} \quad \text{fix.} \tag{3.58}$$

This means that we now have two types of strings which come from M theory membranes wrapping the different cycles of the torus. In addition, we have threebranes living on the worldvolume coming from five branes wrapping both compact directions. Their tension is

$$T = \frac{UV}{\lambda_p^6} \tag{3.59}$$

and therefore finite. The theory looks very similar to type IIB theory, where we have fundamental and D strings. This is the reason why it was baptised *iib* theory in [82]. In addition to the 3 branes and the two types of strings living on the world volume we have also world volume 5 branes, which come from KK monopoles with NUT direction U or V. We have to consider KK monopoles for the first time, because they need 7 compact directions to be completely wrapped: the NUT direction and 6 further directions to wrap the world volume. The energy of a wrapped KK6 with NUT=U is

$$E = \frac{U^2 \Sigma_1 \Sigma_2 \Sigma_3 \Sigma_4 \Sigma_5 V}{\lambda_p^9}. \tag{3.60}$$

For NUT direction V we have of course a similar formula. We see that this energy remains finite in our limit. Of course all these threshold bound states have a space time interpretation, which was worked out for *iib*-like setups in [83]. The strings coming from membranes wrapping U or V give rise to 4+4 = 8 longitudinal 5 branes and 1+1 = 2 longitudinal membranes. The reason for the different space time interpretation is due to the special role of Σ_5. The 3 branes give rise to 4 space time KK monopoles (if Σ_5 is not in the 3 brane worldvolume) and 6 longitudinal 5 branes (if Σ_5 is in the 3 brane world volume). The KK6 monopoles of (3.60) lead to two further wrapped KK6. The 5 world volume momenta give rise to 4 longitudinal membranes and a longitudinal 5 brane, as already discussed in the context of compactification in four and five dimensional tori. Altogether, we have found 6 membranes, 15 5 branes and 6 KK monopoles, precisely as expected from space time considerations. The 27 states transform as the **27** of $E_{6(6)}$, the U-duality group for a compactification to 5 dimensions. We have found all threshold bound states or equivalently, we have found a base space description of all space time wrapped branes.

Let us now turn to the non-threshold bound states. Here we have as before 10 stuck membranes, corresponding to 6 transversal space time membranes and 4 momenta. Then there are 10 bound states of the basic M5 with another M5 wrapping U or V. This gives 8 transversal membranes and 2 transversal 5 branes. Furthermore, we have to consider the world volume KK6 with NUT direction $\Sigma_1, \ldots, \Sigma_5$. They lead to 4 transversal 5 branes and 1 transversal membrane. The momentum modes along U and V remain momentum modes in space time. Altogether we found 6 space time momenta, 15 membranes and 6 5 branes. Together, these states transform as the **27** of the U duality group.

So far, everything looks fine. But we have left out one important state: The M2 brane can wrap both compact transversal directions U and V. These states have energy

$$E = \frac{UV}{\lambda_p^3} \tag{3.61}$$

They become light in our limit! This indicates that the theory on the M5 is not the full story, but that we have to consider the bulk physics, too.

We have compactified on a T^6, and furthermore, we have a compact direction R. So we should expect to be able to see the full U-duality group E_7 realized on the charges. The **56** of E_7 decomposes under the E_6 as

$$56 \rightarrow 27 + \bar{27} + 1 + 1 \tag{3.62}$$

The two times 27 states are the states we have discussed already. One of the singlets is the momentum on R, which is represented by the M5 brane itself in the base space theory. The other missing singlet is the KK monopole with NUT direction R. It is represented by the light wrapped M2 branes in the base space theory, as we could also have expected from (3.36). Therefore, these states are essential also from U-duality considerations [84].

Let us look at the theory from the point of view of type IIB theory. M theory compactified on a two-torus is equivalent to type IIB on a circle. The radius of the circle is

$$\Sigma_B = \frac{\lambda_p^3}{UV}. \tag{3.63}$$

(This can be obtained by first going to type IIA on U and then applying a T duality on V.) In the limit we take, this circle decompactifies. This means that we have a ten dimensional IIB theory. The string coupling in the type IIB is given by

$$g_s = \frac{V}{U} \tag{3.64}$$

The M5 branes translate into a IIB KK monopole with NUT direction Σ_B. In the limit we take $\Sigma_B \rightarrow \infty$. The resulting space becomes $R^4/\mathbb{Z}_N$. The light wrapped M2 branes become momentum modes in Σ_B-direction. We are therefore left with a singular space with momentum travelling in one of the directions. The string coupling is finite. Therefore, we do not expect a decoupling of the brane physics.

Let us again have a look from the IIA perspective [74, 75, 84]. This time, we imagine that we have applied the formulas (3.37) and have obtained a D6 brane at strong coupling. We describe this D6 best as a KK6 monopole in M theory. However, if we compute the 11-dimensional Planck scale using (3.47), we obtain:

$$M_{P_{11}}^3 = \frac{V_6 R^3}{l_p^1 2}, \tag{3.65}$$

which is finite in our limit. So we do not expect gravity to decouple. The NUT direction has length

$$\Sigma_7 = g_s l_s \tag{3.66}$$

and blows up in our limit. From the type IIA perspective this means that the D0 branes become light. The D0 branes are the remnants of 11-dimensional gravity from the 10-dimensional point of view. Their space time interpretation is the KK6 monopole with NUT direction R, which can be checked using the T-duality formulas (3.37). The conclusion of all these considerations is that the brane physics does not decouple from the bulk.

In our limit, we have taken the IIB string coupling to be a constant. For decoupling, one would take the string coupling to zero. Seiberg and Sethi [51] have shown that if one takes such a limit, one of the compact direction automatically decompactifies. So the resulting theory is again the theory obtained in [72].

Compactification on Calabi-Yau manifolds

In the paper [85] it is argued that Matrix theory on Calabi-Yau does not suffer from these problems. The argument is that the gravity multiplet of the higher supersymmetry decomposes into a gravity multiplet of the lower supersymmetry plus a hypermultiplet. It is argued that the massless state coming from the D6 wrapping a shrinking CY (in the Seiberg-Sen limit) gives rise to a hypermultiplet. This means that the theory decouples from bulk *gravity*. Still, there is a light hypermultiplet and as a second step there might be the hope that this hyper multiplet also decouples. The D6 brane under consideration is the analog of the D6 in (3.36), where the background is not toroidal but a Calabi-Yau manifold. The reason for the possible decoupling is roughly speaking that the D6 which wraps the whole Calabi Yau is mapped by mirror symmetry to a 3 brane wrapping a vanishing three-cycle of the mirror Calabi-Yau. This can be understood if one thinks of the mirror-operation as T duality on a T^3 fiber of a Calabi Yau. The physical interpretation of 3 branes wrapping shrinking three cycles has been given in [86]: A hypermultiplet becomes massless. On the other hand, the D0 brane would also transform into a D3 brane under the T duality along the T^3 fiber. This D3 brane wraps the T^3 fiber. The hope is that the hypermultiplet coming from the other D3 wrapping the other directions does not interact with the physics on this brane. For a general Calabi Yau manifold, the Matrix description would be given by the physics of branes near a particular conifold point. This conifold point would have to have the property that it is mirror to the point in the moduli space, where the volume of the mirror Calabi-Yau collapses.

An optimistic interpretation of this might be, that Matrix theory only wants to be compactified on manifolds which break some fraction of the supersymmetry.

3.6 Discussion

We have reviewed in some detail the problems connected to Matrix descriptions of M theory compactified to lower dimensions. Here, we concluded that the Matrix description of compactifications to 5 and lower dimensions with full supersymmetry is again an eleven dimensional theory. It is not fully captured by the lower dimensional physics on the world volume of a brane, like it was the case for higher dimensions. A possible way out of this dilemma might be the compactification on manifolds breaking some supersymmetry, if one is very optimistic.

There are further problems with Matrix theory: In compactifications on K3 manifolds or on ALE spaces some discrepancies occured [87]. Here, the physics of the D0 branes at weak coupling does not reproduce the supergravity interactions. If Matrix theory is simply given by applying the Seiberg-sen procedure in a naive way, this is a contradiction.

Further problems arise, when we try to reproduce the scattering of supergravitons in the Yang Mills theory. Here, good agreement has been found in the papers [88], where two graviton scattering amplitudes are discussed. On the other hand, if three graviton amplitudes are considered, a disagreement was found in [89]. Up to now, it is not fully understood, why in the one case we get agreement, but in other cases no agreement is found. An argument which has been given by many people is that the amplitudes computed in [88] is protected by supersymmetry and that we should expect disagreement in all other cases.

Let us recall the field theory considerations we made at the beginning of this section. Here, we pointed out that in field theory the consideration of the D0 branes are particularly important. Any background can modify the light cone dynamics of the dynamical degrees of freedom. It was pointed out in [87], that the boost to the infinite momentum frame is complicated. Considering the zero modes correctly and renormalizing the Matrix theory Lagrangian correctly might lead to a reasonable description of the eleven dimensional physics. In other words: The correct Hamiltonian for Matrix theory has not yet been found in all cases.

So far, the attempt was to describe the eleven dimensional physics using D0 branes in the limit of weak string coupling. The physics of D0 branes probing non-trivial geometrical backgrounds was studied in detail in [57]. It was pointed out in that paper that D0 branes probe the substringy regime. Therefore, we are exploring another energy regime if we use the D0 brane physics. It might be used in addition to other methods probing long distances, but not instead of them.

Chapter 4

Calabi-Yau Fourfold Compactifications

This chapter will be somewhat different than the other chapters of this thesis. In the other chapters we considered brane configurations in flat space time and branes wrapped on tori. In the context of Matrix theory, we discussed compactifications on tori. Toroidal compactifications have the property that they preserve the full amount of supersymmetry of the uncompactified theory. More interesting are compactifications on K3 surfaces, which break one half of the supersymmetry, or Calabi-Yau threefolds, which break one quarter of the supersymmetry. For phenomenological reasons, we would preferably like to compactify to four dimensions on manifolds which lead to $N = 1$ supersymmetry. This will be the subject of this chapter. In this context, branes will again play a significant role. They can wrap cycles of the manifolds on which we compactify and in this way modify the physics. For M-theory compactifications on Calabi-Yau fourfolds, it was shown in [90] that branes wrapping certain divisors of the Calabi-Yau fourfold lead to the generation of a non-perturbative superpotential. In some cases, it can be necessary to include branes to obtain consistent vacua, as pointed out in [91].

To get non-perturbative information about the string vacua obtained by compactification, we can use duality symmetries. Dualities relate perturbatively different vacua. An important step was the discovery of the duality between the heterotic string on T^4 and the IIA string on a K3 surface [4, 77, 92, 93]. The non-abelian gauge structure of the heterotic string is related to the ADE singularities of the K3 surface. Here, we have a six-dimensional duality with $N = 2$ supersymmetry. It was possible to extend this duality to a four dimensional duality, breaking a further half of the supersymmetry [94, 95]. More precisely, the heterotic string on K3$\times T^2$ is dual to a type IIA string on a Calabi-Yau threefold. It was understood that the structure of the heterotic string couplings requires the Calabi-Yau manifold on the type II side to be a K3 fibration [96]. We will see other examples, where the fibration structure plays an important role below. To come to $N = 1$ in four dimensions, we need to break a further half of the supersymmetry. This can be reached if we compactify the heterotic string on a Calabi-Yau manifold. A dual description can be obtained from F-theory on Calabi-Yau fourfolds.

Before we go on, let us briefly recall some more facts about F-theory.

The strong-weak coupling symmetry of type IIB theory was a motivation to consider not only M-theory but also F-theory as a candidate for a unifying theory. The SL(2,$\mathbb{Z}$) symmetry of the type IIB string transforms the combination $\lambda = a + i\, e^{-\phi}$ of the RR-scalar a and the dilaton ϕ as the modular parameter of a torus. In the context of M-theory this symmetry appears when M-theory is compactified on a torus and the modular parameter of this torus is compared with the coupling λ of a type IIB string on a circle [97, 98]. F-theory provides us with a more geometric interpretation of this symmetry. The compactifications of F-theory can be interpreted as special kinds of type IIB compactifications where the coupling λ is allowed to vary over the internal manifold. Let us consider IIB theory compactified on a base space $\mathcal{B}$. We can imagine that at each point of the base space of the compactification we erect a torus. The base space $\mathcal{B}$ together with the torus give rise to a space X which is the compactification space of F-theory. The complexified dilaton is identified with the modular parameter of this torus. It is allowed to undergo undergo SL(2,$\mathbb{Z}$) monodromies, if we move on a closed circle of the base space. More precisely, the internal manifold X on which F-theory is compactified, has to admit an elliptic fibration

$$\mathrm{T}^2 \longrightarrow X \longrightarrow \mathcal{B}. \tag{4.1}$$

The modular parameter of the fiber is identified with the coupling of the IIB string, so that F-theory on X is type IIB on $\mathcal{B}$ with varying coupling constant. The Kähler paramter of the torus does not correspond to a physical modulus and is frozen. Note the difference to ordinary perturbative vacua, where the dilaton is always a constant. The dilaton determines the coupling constant of type IIB string theory. The fact that λ undergoes monodromies in F-theory compactifications means that there can be points where the theory is strongly coupled. Therefore, F-theory compactifications are intrinsically non-perturbative. F-theory describes the non-perturbative region of the type IIB string, in a similar way that M-theory describes the strongly coupled IIA string.

The relation of F-theory to M-theory can be understood, if we compactify on a further circle. F-theory on $X \times S^1$ is type IIB on $\mathcal{B} \times S^1$. But IIB on S^1 is M-theory on T^2. Applying this relation fiber wise, we see that F-theory on $X \times S^1$ is M-theory on X. Compactifying on a further circle leads to IIA on X, giving a relation between F-theory and IIA strings. Because of this, we are interested in the case that the manifolds are Calabi-Yau manifolds.

F-theory is related by duality to the heterotic string. The basic duality relation is that in eight dimensions F theory on K3 is dual to the heterotic string on T^2 [7]. This duality was derived in [99] by going to a special point in the moduli space, where λ becomes a constant, and applying known duality relations. Duality in lower dimensions can then be obtained by applying the basic eight dimensional duality fiber-wise. F-theory on X is dual to a heterotic string theory, if the base $\mathcal{B}$ is a $\mathbb{P}_1$ fibration over a base S. Altogether, we have the following fibration structure:

$$\begin{array}{ccc} \mathrm{T}^2 & \longrightarrow & \mathrm{CY}_4 \\ & & \downarrow \\ \mathbb{P}_1 & \longrightarrow & \mathcal{B} \\ & & \downarrow \\ & & S, \end{array}$$

The dual heterotic string is compactified on a Calabi-Yau manifold, which is elliptically fibered over S. In particular, F-theory on an elliptically fibered CY-fourfold, whose base is a $\mathbb{P}_1$-fibration, is dual to the heterotic string on a Calabi-Yau threefold, which is elliptically fibered over the base of the $\mathbb{P}_1$-fibration. These vacua lead to $N = 1$ in four dimensions.

In this chapter, we are interested in some aspects of Calabi-Yau fourfolds and the compactification of F- and M-theory on them. We introduce a class of examples, namely complete intersections in products of projective spaces, in section 4.1. Using these manifolds, many new vacua can be constructed. However, as we will see in section 4.2.1, the moduli spaces of these manifolds are connected by transitions involving singular fourfolds. This discussion is similar to the discussion for threefolds in [100]. After that, we will discuss the generation of non-perturbative superpotentials in M-theory compactifications on fourfolds. Not all fourfolds lead to superpotentials, it is necessary that the fourfold contains a certain type of divisors. We show, that the transitions discussed before can introduce the necessary divisors in some cases.

4.1 A class of examples

In this section, we will explain a class of examples of Calabi-Yau manifolds. These are hypersurfaces in weighted projective spaces. Many other examples can be found as hypersurfaces in toric varieties. A (complex) projective space of (complex) dimension n can be parametrized by $n+1$ coordinates $z_1 \dots z_n$, which are subject to the identification

$$z_i \sim \lambda z_i \tag{4.2}$$

for any $\lambda \in \mathbb{C}^*$. The point $z_1 = \ldots = z_n = 0$ is not contained in the projective space. We call an n-dimensional projective space $\mathbb{P}_n$. A hypersurface in the projective space is given by a homogeneous polynomial p in the z_i. A homogeneous polynomial of degree d has the property that

$$p(\lambda z_1, \ldots, \lambda z_{n+1}) = \lambda^d p(z_1, \ldots z_{n+1}). \tag{4.3}$$

The hypersurface is given by the zero locus of the homogenous polynomial. We are interested in transversal polynomials, which means that the only solution to the equations $p = 0$ and $dp = 0$ is the origin $z_1 = \ldots = z_n = 0$, which is not in the manifold. In this case the hypersurface is smooth. Furthermore, it is compact and Kähler, because the embedding space is.

The concept of projective spaces can be easily generalized to weighted projective spaces. Here, the z_i scale with different weights k_i:

$$z_i \sim \lambda^{k_i} z_i \tag{4.4}$$

We will denote a weighted projective space by $\mathbb{P}_{(k_1,\ldots,k_{n+1})}$. In general, these spaces can be singular, because the projective equivalence leads to orbifold singularities. Again, we can consider submanifolds which are given by the zero locus of a homogeneous polynomial

$$p(\lambda^{k_1} z_1, \ldots, \lambda^{k_{n+1}} z_{n+1}) = \lambda^d p(z_1, \ldots, z_{n+1}). \tag{4.5}$$

These submanifolds can hit the singularities of the embedding space. Therefore, to obtain a smooth manifold, we need to resolve the singularities. This is done by replacing the singular sets by smooth codimension one sets, which are called exceptional divisors. The process of resolution induces new cycles and therefore also new cohomology. An additional requirement for the manifolds to be smooth is that they are given by transversal polynomials as in the non-weighted case.

A further straight forward generalization is to consider hypersurfaces in products of projective spaces. More interesting examples are obtained if we consider the transverse intersection of several polynomials in products of projective or weighted projective spaces. Such a configuration will be denoted in the following way:

$$\begin{matrix} \mathbb{P}_{(k_1^1,\ldots,k_{n_1+1}^1)} \\ \mathbb{P}_{(k_1^2,\ldots,k_{n_2+1}^2)} \\ \vdots \\ \mathbb{P}_{(k_1^F,\ldots,k_{n_F+1}^F)} \end{matrix} \begin{bmatrix} d_1^1 & d_2^1 & \ldots & d_N^1 \\ d_1^2 & d_2^2 & \ldots & d_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ d_1^F & d_2^F & \ldots & d_N^F \end{bmatrix} \quad = \quad X. \tag{4.6}$$

This configuration describes a complete intersection manifold in the product of weighted projective spaces which are listed on the left hand side of the matrix. Each column of the matrix describes one polynomial: The entry d is the degree of the polynomial in the variables of the respective projective space. The configurations describes the intersection of the zero locus of N polynomials embedded in a product of weighted projective spaces, where $N = \left(\sum_{i=1}^{F} n_i - D\right)$ is the number of polynomials p_a of F-degree $(d_a^1, \ldots, d_a^F)$ and D is the dimension of the manifold described by this degree matrix. So far, our considerations were completely general. However, we will mainly be interested in manifolds of vanishing first Chern class, which can be used to compactify string theories consistently.

For string compactifications, the "right" dimension of the internal space is 6, whereas for F (or also M theory) we are more interested in Calabi-Yau fourfolds. The first Chern class can be computed directly from the degree matrix:

$$c_1(X) = \sum_{i=1}^{F} \left[\sum_{l=1}^{n_i+1} k_l^i - \sum_{a=1}^{N} d_a^i \right] h_i \tag{4.7}$$

Here we denote by $h_i, i = 1, ..., F$ the pullback of the Kähler form of $\mathbb{P}_{(k_1^i,...,k_{n_1+1}^i)}$. We read off that the complete intersection is Calabi-Yau, if the sum of the weights of each projective space equals the sum of the degrees of the polynomials in the variables of the particular projective space.

For later use, let us also write down the formulas for the higher Chern classes:

$$\begin{aligned} c_2(X) &= \frac{1}{2}\left[\sum_{a=1}^{N}\left(\sum_{i=1}^{F} d_a^i h_i\right)^2 - \sum_{i=1}^{F}\sum_{r=1}^{n_i+1}(k_r^i h_i)^2\right] \\ c_3(X) &= -\frac{1}{3}\left[\sum_{a=1}^{N}\left(\sum_{i=1}^{F} d_a^i h_i\right)^3 - \sum_{i=1}^{F}\sum_{r=1}^{n_i+1}(k_r^i h_i)^3\right] \\ c_4(X) &= \frac{1}{4}\left[\sum_{a=1}^{N}\left(\sum_{i=1}^{F} d_a^i h_i\right)^4 - \sum_{i=1}^{F}\sum_{r=1}^{n_i+1}(k_r^i h_i)^4 + 2c_2^2\right]. \end{aligned} \tag{4.8}$$

We will mainly consider weighted projective spaces or products of ordinary projective spaces. For concreteness and illustration let us look at an example for a Calabi-Yau fourfold:

This first example is the most simple example: Let us look at the manifold given by the degree six polynomial in $\mathbb{P}_5$, the sextic $\mathbb{P}_5[6]$. We can compute the Hodge numbers of the manifold: First of all, we compute the Euler number using the formulas (4.8). For the top Chern class we obtain:

$$c_4 = 435h^4.$$

We have to integrate this over the hypersurface. To do this, we use the theorem (taken from Hübsch's book [101], or see [100]) that the integral on a complex submanifold X of $\prod \mathbb{P}$ can be computed by doing the integral in the embedding space :

$$\int_X \omega = \int_{\prod \mathbb{P}} \mu \wedge \omega \tag{4.9}$$

Here, ω is any closed (D, D) form (for a D-dimensional hypersurface) and μ restricts integration from the embedding space $\prod \mathbb{P}$ to the hypersurface X like a delta function. μ is given by the top Chern class of the normal bundle. In our case, $\mu = 6h$, and therefore we compute that the Euler number is 2610. Since our manifold is Calabi-Yau,

$$h^{(0,0)} = h^{(D,D)} = h^{(D,0)} = h^{(0,D)} = 1$$

and all other $h^{(i,0)}$ vanish. The non-trivial cohomologies, which need to be determined are $h^{(1,1)}, h^{(1,2)}, h^{(2,2)}$ and $h^{(3,1)}$. To obtain the Hodge numbers, we can apply Lefschetz' theorem. Doing this, we get all Hodge numbers except the ones in the middle cohomology. This means that we get $h^{(1,2)} = 0$ and $h^{(1,1)} = 1$. So we are left with two more numbers and therefore need one more relation in addition to the Euler number. This relation is provided by an index-calculation in [91]. The result is

$$-h^{(1,1)} + h^{(1,2)} - h^{(1,3)} = 8 - \frac{\chi}{6} \tag{4.10}$$

This can be rewritten as a constraint on the Hodge numbers

$$44 + 4h^{(1,1)} + 4h^{(3,1)} - 2h^{(2,1)} - h^{(2,2)} = 0 \tag{4.11}$$

Using this relation and the result for the Euler number, we get for the Hodge diamond:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 1 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 1 & & 426 & & 1752 & & 426 & & 1. \end{array} \tag{4.12}$$

More examples can be found later in the text. Let us mention at this point that in the case of weighted projective spaces the exceptional divisors introduced when resolving the singularity have to be taken into account when we determine the Hodge numbers.

In [91] it was shown that there are restrictions on the possible compactification manifolds for F-theory compactifications to four dimensions or equivalently M-theory compactifications to three dimensions or IIA compactifications to two dimensions. These restrictions arise from the requirement that tadpoles for the F-theory four form gauge field (or M-theory three form potential or IIA antisymmetric tensor) have to be cancelled. [91] argue that these tadpoles can be cancelled, if the Euler number of the compactification manifold is divisible by 24. We see that our simple example does not fulfill this requirement. We will discuss manifolds whose Euler number is a multiple of 24 below.

4.1.1 Fibrations

In addition to the requirement that tadpoles need to be cancelled, we need for F-theory compactifications that the manifold is an elliptic fibration. We have seen that fibered manifolds are also of particular interest in the context of dualities. Some easy examples of fibered manifolds can be obtained using hypersurfaces in weighted projective space. For F-theory, we are interested in elliptically fibered fourfolds. For $N = 2$ string dualities in four dimensions, K3 fibered Calabi-Yau threefolds are important. As a special case, we can study nested fibrations, which means we can study Calabi-Yau fourfolds which are fibrations of Calabi-Yau threefolds which are themselves fibrations of elliptically fibered K3-surfaces. More clearly, this can be expressed in the following diagram:

$$\begin{array}{ccccccc} T^2 & \longrightarrow & K3 & \longrightarrow & CY_3 & \longrightarrow & CY_4 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \mathbb{P}_1 & & \mathbb{P}_1 & & \mathbb{P}_1 . \end{array} \tag{4.13}$$

As an example, let us start with $\mathbb{P}_2[3]$, the cubic in $\mathbb{P}_2$. This is a curve of genus 1, as can easily be checked using the formula (4.7). We can build a K3 surface having this curve as its generic fiber. Such a surface is given by $\mathbb{P}_{(1,1,2,2)}[6]$. To see this, let us write down a typical transversal configuration:

$$z_1^6 + z_2^6 + z_3^3 + z_4^3 = 0 \tag{4.14}$$

We can now consider the locus, where

$$z_1 = \lambda z_2. \tag{4.15}$$

Substituting this into equation (4.14) gives:

$$(\lambda^6 + 1) z_2^6 + z_3^3 + z_4^3 = 0 \tag{4.16}$$

Here, we should make the substitution

$$z_2 \to \sqrt{z_2}, \tag{4.17}$$

which is single valued because the projective equivalence gives

$$(z_2, z_3, z_4) \sim (-z_2, z_3, z_4). \tag{4.18}$$

This leads to the equation

$$(1 + \lambda^6) z_2^3 + z_3^3 + z_4^3, \tag{4.19}$$

which is clearly a configuration in $\mathbb{P}_2[3]$. λ can be considered as a parameter parametrizing the base of the elliptic fibration. For most values of λ (4.19) gives a torus. However, at the points where

$$1 + \lambda^6 = 0 \tag{4.20}$$

z_2 becomes undetermined. The equation $z_3^3 + z_4^3 = 0$ describes three points. The fiber degenerates. We can go on and consider the threefold $\mathbb{P}_{(1,1,2,4,4)}[12]$, which is a K3 fibration with generic fiber $\mathbb{P}_{(1,1,2,2)}$. This can be seen using similar arguments as above. Finally, we can construct the fourfold $\mathbb{P}_{(1,1,2,4,8,8)}[24]$, which is a threefold-fibration with fiber $\mathbb{P}_{(1,1,2,4,4)}[12]$. It is also K3 fibered, because the threefold is and it is elliptically fibered because the K3 is. The Euler number of this manifold is 4896, which is divisible by 24. Therefore, the tadpoles of [91] can be cancelled. In a similar way, we can build elliptically fibered fourfolds from any elliptically fibered K3 surface. Some examples where considered in [102].

4.2 Vacuum degeneracy

A longstanding problem in string theory is the issue of the vacuum degeneracy. Soon after the discovery of anomaly free low energy field theories of various types of strings the existence of a plethora of consistent ground states was shown to exist among different compactification schemes and 4D string constructions. This appears to create a difficulty which in the early days of the first string revival led to some disenchantment. Namely, if all these consistent vacua are disjoint then this raises the question whether even in principle the string could ever determine its own ground state and eventually make some detailed predictions. If on the other hand vacua with different spectra are connected then this implies that a singularity must occur somewhere. It is then not a priori clear that the string can consistently propagate in the background of the singular configuration. Thus one has to face the issues of connecting different vacua and providing a physical interpretation of the resulting singularities. Both of these problems admit a solution in the context of Calabi-Yau compactifications.

The first of these problems was solved in [100] by showing that complete intersection Calabi-Yau manifolds with different moduli spaces can indeed be connected. In the simplest instance the Calabi-Yau spaces degenerate at the transition locus at a number of nodes, leading to conifold configurations. The resulting conifold transitions have been shown to connect all complete intersection Calabi-Yau manifolds [103] and have also been shown to generalize to the class of weighted complete intersection spaces [104].

Many more vacua can be obtained by using the framework of toric geometry. Also here, it should be possible to proof that the Calabi-Yau manifolds which can be obtained as hypersurfaces in toric varieties are connected by singular transitions [105].

At the time when conifold transitions [100] were introduced it was unclear what the correct physical interpretation is of the divergences associated to the conifold transition [103]. Only recently has it been understood through the work of Greene, Morrison and Strominger [86, 106] in type II compactifications, following similar ideas in Seiberg-Witten theory, that the divergences in the low energy effective action arise because of new massless states which are generated at the conifold configuration when the volume of vanishing cycles degenerates. This then shows that at least in type II string theory the conifold transitions of [100] admit a physically reasonable interpretation.

The same problem arises in maybe even more pronounced form in F-theory and M-theory. One consistent compactification type of both of these theories involves Calabi-Yau fourfolds, of which there are many more than there are threefolds. The class of Fermat type hypersurface threefolds embedded in weighted $\mathbb{P}_4$, for instance, consists of only 147 configurations, whereas the number of Fermat hypersurface fourfolds is 3462, more than an order of magnitude larger. Preliminary results obtained by Lynker, Schimmrigk and Wißkirchen show that the number of all weighted hypersurfaces consists of at least several hundred thousand configurations. This suggests that there are millions of complete intersection Calabi-Yau fourfolds, providing possible vacua for 4D F-theory. However, out of all these fourfolds we still have to extract the elliptically fibered manifolds (for F-theory), which obey the tadpole cancellation condition.

4.2.1 The splitting transition between complete intersections

In this section, we will look at a particular kind of transition between two topologically distinct Calabi-Yau manifolds, the so-called splitting transitions. For simplicity, let us look at intersections in non-weighted projective spaces. Our considerations can be generalized to the weighted case. The degree matrix (4.6) simplifies in the following way:

$$\begin{array}{c} \mathbb{P}_{n_1} \\ \mathbb{P}_{n_2} \\ \vdots \\ \mathbb{P}_{n_F} \end{array} \begin{bmatrix} d_1^1 & d_2^1 & \dots & d_N^1 \\ d_1^2 & d_2^2 & \dots & d_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ d_1^F & d_2^F & \dots & d_N^F \end{bmatrix} \quad = \quad X. \tag{4.21}$$

We are interested in Calabi-Yau manifolds, that is we want to impose the requirement that the first Chern class of the manifold vanishes. A formula for the first Chern class was given in (4.7). In the non-weighted case, the vanishing of the first Chern class translates to the following condition:

$$\sum_{a=1}^{N} d_a^i = n_i + 1 \qquad \text{for} \quad i = 1, \dots F. \tag{4.22}$$

A splitting type transition for two manifolds of type (4.21) can be written down in the following way:

Introducing two vectors u, v such that

$$(u^i + v^i) = d_1^i$$

and denoting the remaining $(F \times (N-1))$-matrix by M, we write the manifolds (4.6) as $Y[(u+v) \quad M]$. The simplest kind of transition is the $\mathbb{P}_1$-split which is defined by

$$X \;=\; Y[(u+v) \quad M] \;\longleftrightarrow\; \begin{array}{c} \mathbb{P}_1 \\ Y \end{array} \begin{bmatrix} 1 & 1 & 0 \\ u & v & M \end{bmatrix} \;=\; X_{\text{split}}. \tag{4.23}$$

The split variety of the rhs is described by the polynomials of the original manifold and two additional polynomials, which we can write as

$$\begin{aligned} p_1 &= x_1 Q(y_i) + x_2 R(y_i) \\ p_2 &= x_1 S(y_i) + x_2 T(y_i), \end{aligned} \tag{4.24}$$

where $Q(y_i), R(y_i)$ are of multi-degree u and $S(y_i), T(y_i)$ are of degree v. In (4.24) we collectively denote the coordinates of the space Y by y_i whereas the x_i are the coordinates of the projective line $\mathbb{P}_1$. The degree matrix on the rhs of (4.23) furthermore determines that the polynomials p_1 and p_2 are linear in the coordinates of the $\mathbb{P}_1$. The polynomials encoded in the $(F \times (N-1))$-matrix M were not written down explicitely here. To understand the relation between the two manifolds in (4.23) we consider (4.24) in more detail. We can regard the vanishing locus of (4.24) as a linear equation system in x_1 and x_2. These equations only have nontrivial solutions if the determinant

$$p_{det} = QT - RS$$

vanishes. Together with the original polynomials the determinant defines a determinantal variety $X^\sharp$. The polynomial p_{det} is of multidegree $u+v$ in the variables of Y and therefore $X^\sharp \in Y[(u+v) \quad M]$ where

$$X^\sharp = \{\; p_{\mathrm{det}} = QT - RS = 0,\; p_a = 0,\; a = 2, ..., N\} \tag{4.25}$$

The space $X^\sharp$ however is not a smooth manifold. It is singular on the locus where the determinant has a double zero because the polynomials Q, R, S, T vanish simultaneously. The singular set is described by

$$\Sigma = Y[u \quad u \quad v \quad v \quad M]. \tag{4.26}$$

The manifold on the lhs in (4.23) is D-dimensional, therefore Σ has dimension $(D-3)$. The polynomial p_{det} is not transversal. There are two ways to smooth out the singularity. This leads to a relation between the two manifolds in (4.23). They are different resolutions of the same singular space. The first way to smooth out the singularity, is to add a transversal piece to p_{det}

$$p_{\mathrm{def}} = p_{\mathrm{det}} + t \cdot p_{\mathrm{trans}}. \tag{4.27}$$

In this way, we get a smooth configuration in $Y[(u+v) \quad M]$. This explains how to get from the singular configuration (4.25) to the configuration on the lhs in (4.23). The important point is that the singularity in (4.25) also admits a small resolution. This takes us from the singular variety to the smooth configuration on the rhs of (4.23). The moduli spaces of the manifolds in (4.23) are connected by a singular configuration. We can get from one moduli space to the other by passing a singular variety. Put differently, we can start from a determinantal variety and smooth out the singularities in two distinct ways

$$X \longleftarrow X^\sharp \longrightarrow X_{\mathrm{split}}.$$

Equivalently, we arrive at the same singular set starting from two different smooth manifolds and degenerating them in distinct ways:

$$X \longrightarrow X^\sharp \longleftarrow X_{\mathrm{split}}.$$

The process of going from the lhs to the rhs via the singular variety is refered to as splitting and the process to go from the rhs to the lhs is called contraction.

The transitions can in principle be studied for manifolds of any dimension (equal or bigger than three). For string-theory compactifications threefolds are of particular importance and have been studied in [100] and for F-theory applications fourfolds are relevant [107].

4.2.2 Splitting and contracting Calabi-Yau threefolds and fourfolds

In the following we will focus on varieties of complex dimension three and four. For a threefold split (4.26) describes a number of points, i.e. a conifold configuration, whereas for a fourfold split the singular set is an algebraic curve, i.e. a real two-dimensional surface, with in general several components. Performing a small resolution means that the singular set is smoothed out using an exceptional set of codimension two. For threefolds this involves the projective line $\mathbb{P}_1$ and for fourfolds the projective plane $\mathbb{P}_2$. Performing such a small resolution leads to the higher codimension split manifold. To support this picture a relation between the Euler numbers of X and X_{split} was proven in [100] for threefolds and in [107] for fourfolds. For threefolds the result is

$$\chi(X_{split}) = \chi(X) + 2n, \tag{4.28}$$

where n denotes the number of singular points and the factor 2 originates from the Euler number of the $\mathbb{P}_1$ used to smooth out the manifold. The analogous formula for fourfolds

$$\chi(X_{\text{split}}) = \chi(X) + 3\chi(\Sigma) \tag{4.29}$$

describes the resolution involving a $\mathbb{P}_2$ whose Euler number is three.

A simple example for a split between two threefolds is

$$\mathbb{P}_4[5]_{-200} \quad \longleftrightarrow \quad \begin{matrix} \mathbb{P}_1 \\ \mathbb{P}_4 \end{matrix} \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}_{-168}. \tag{4.30}$$

This conifold transition connects the quintic in $\mathbb{P}_4$ with Hodge numbers $(h^{(1,1)}, h^{(2,1)}) = (1, 101)$ to the codimension two configuration on the rhs with Hodges $(h^{(1,1)}, h^{(2,1)}) = (2, 86)$. The physical interpretation in the context of type II string theory of this transition has been discussed in [106].

The perhaps simplest example of a splitting transition involving fourfolds is the split of the sextic

$$\mathbb{P}_5[6]_{2610} \quad \longleftrightarrow \quad \begin{matrix} \mathbb{P}_1 \\ \mathbb{P}_5 \end{matrix} \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix}_{2160}, \tag{4.31}$$

where the subscripts denote the Euler numbers. The smooth hypersurface can be defined by the Fermat polynomial $p = \sum_i z_i^6$ and a transverse choice of the split configuration is provided by

$$\begin{aligned} p_1 &= x_1 y_1 + x_2 y_2 \\ p_2 &= x_1 \left(y_2^6 + y_4^6 + y_6^6\right) + x_2 \left(y_1^6 + y_3^6 + y_5^6\right). \end{aligned}$$

The singular set of the determinantal variety is given by the genus $g = 76$ curve $\Sigma = \mathbb{P}_3[5\ \ 5]$. Thus we can verify the relation (4.29). More precisely the split (4.31) connects the Hodge diamond of the sextic hypersurface
with the Hodge diamond

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 2 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 1 & & 350 & & 1452 & & 350 & & 1. \end{array} \qquad (4.32)$$

of the codimension two complete intersection manifold of (4.31). The Hodge numbers can be computed using the formulas (4.8) to get the Euler number. The Lefschetz-theorem gives $h^{(1,1)} = 2$ and $h^{(1,2)} = 0$ because the embedding space consists of a product of two $\mathbb{P}_n$'s. Using also the relation (4.11) gives us all Hodge numbers.

4.3 Superpotentials

In the previous section we have seen that some regions of M- and F-theory vacua are connected by transitions. In the present section we will see that these vacua can be distinguished in an intrinsic manner by the property that some of them lead to a non-perturbatively generated superpotential. In [90] it was shown that a nonperturbative superpotential is generated for compactifications of M-theory on certain Calabi-Yau four-folds. An example which shows modular behaviour for the superpotential was described in [108]. The model (and some related models) was discussed in the F-theory context and compared to a superpotential on the heterotic side generated by world sheet instantons in [109]. In [107] the variety of [108] was connected to a manifold which does not generate a superpotential by a splitting transition. Before we describe this transition we will briefly review the generation of superpotentials in three-dimensional field theory and M-theory compactifications to three dimensions.

4.3.1 The superpotential in 3D field theory

The generation of a superpotential by instantons in $d = 3$, $N = 2$ field theory was considered in [110] via dimensional reduction of an $N = 1$ supersymmetric gauge theory with gauge group $SU(2)$ from four dimensions. It was shown that due to instanton effects there are potential terms for the scalar φ arising in the three-dimensional theory as a mode of the 4-dimensional gauge field.

Instanton corrections lead to a factor e^{-I} in the effective action, where I is the one-instanton action. In the BPS-case it is given by

$$I = \frac{4\pi\varphi}{g}$$

In 3D we are in the situation that the gauge field is dual to a scalar ϕ. If one computes the effective action in 3D it can be seen that the scalar field φ combines with the field ϕ to a complex field $Z = \varphi + i\phi$. The instanton correction becomes

$$e^{-(I+i\phi)} \tag{4.33}$$

Actually, φ and ϕ have combined to give the scalar component of a chiral superfield. In the presence of the instanton there are two fermion zero modes. Integrating the function (4.33) over chiral superspace leads to a superpotential

$$\int d^2\theta \;\; e^{-(I+i\phi)}. \tag{4.34}$$

4.3.2 Superpotentials in M-theory compactifications

Consider now the compactification of M-theory to three dimensions on a Calabi-Yau fourfold. It was shown in [90] that a superpotential can be generated by wrapping 5-branes over certain divisors in the fourfold, resulting in the M-theory analogs of the field theory quantities reviewed in the previous section.

Gauge fields in three dimensions are obtained as modes of the 3-form potential C

$$C_{\mu i \bar{j}} \; (x,y) = \sum_{\Lambda=1}^{h^{(1,1)}} A_\mu^\Lambda(x) \; \omega_{i\bar{j}}^{(\Lambda)}(y). \tag{4.35}$$

Here, μ is the three-dimensional spacetime index and i, j are internal indices. $\omega_{i\bar{j}}^{(\Lambda)}$ are a basis of (1,1)-forms of the fourfold. x and y are the coordinates of three-dimensional space time and the internal manifold, respectively. Again, the A_μ^Λ's are dual to scalars. The next thing we have to look for is the analog of the BPS-monopoles of the field theory. The magnetic source for C is the M-theory M5-brane. Thus, we can produce something that looks like an instanton in 3D by wrapping the world-volume of the M5-brane around 6-cycles in the Calabi-Yau. As we have seen in the previous section we need the property that the instanton must be invariant under two of the four supersymmetries, so the cycle must be a complex divisor. The field theory analysis of the previous section leads us to expect a superpotential of the form

$$e^{-(V_D+i\phi_D)} \; P, \tag{4.36}$$

where V_D is the instanton-action determined by the volume of the divisor D around which the M5-brane wraps, and ϕ_D is a linear combination of dual scalars. P is a one-loop determinant of world-volume fields, whose zeroes determine the vacua of the theory. In [90] the fermion zero modes were discussed. Using an anomaly-cancellation argument it was derived that the divisor has to fulfill additional properties to give a nonvanishing contribution to the superpotential. A particularly simple case is the situation where the only fermionic zero modes are the two fermion zero modes coming from the supercharges. Here a superpotential is generated (see section 4.3.1). One way to guarantee that there are no further fermion zero modes is to require that the divisor has to satisfy the conditions

$$\dim \Omega^{(0,1)} = \dim \Omega^{(0,2)} = \dim \Omega^{(0,3)} = 0. \tag{4.37}$$

The reason for this is that on a Kähler manifold the zero modes of the Dirac operator are given by the Dolbeault- cohomology. We then obtain a superpotential similar to (4.34).

The condition (4.37) implies that the arithmetic genus of the divisor equals one.

$$\chi(D, \mathcal{O}_D) = \sum_{n=0}^{3} (-1)^n \ \dim \Omega^{(0,n)} = 1. \tag{4.38}$$

The precise anomaly cancellation argument shows that (4.38) has to hold also in more general situations where a superpotential is generated, i.e. (4.38) is a necessary (but not sufficient) condition for superpotential generation, whereas (4.37) is sufficient but not necessary.

4.3.3 Generating a superpotential via splitting

We will now consider the behaviour of the superpotential under splitting transitions. Consider the manifold

$$X = \begin{matrix} \mathbb{P}_1 \\ \mathbb{P}_2 \\ \mathbb{P}_2 \end{matrix} \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}. \tag{4.39}$$

From Lefschetz' hyperplane theorem we know that $h^{(1,1)} = 3$ and $h^{(2,1)} = 0$. The other Hodge numbers can be determined using the relation (4.11) and the Euler number computed using (4.8). This leads to the Hodge half-diamond:

$$\begin{matrix} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 3 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 1 & & 280 & & 1176 & & 280 & & 1. \end{matrix} \tag{4.40}$$

It has been shown in [90] that manifolds of this type, i.e. hypersurfaces embedded in products of ordinary projective spaces, do not contain divisors fulfilling (4.38). However, the manifold above can be split into one that does contain divisors which generate a superpotential, namely the manifold studied in [108].

$$\begin{matrix}\mathbb{P}_1\\ \mathbb{P}_2\\ \mathbb{P}_2\end{matrix}\begin{bmatrix}2\\3\\3\end{bmatrix} \longleftrightarrow \begin{matrix}\mathbb{P}_1\\ \mathbb{P}_1\\ \mathbb{P}_2\\ \mathbb{P}_2\end{matrix}\begin{bmatrix}1 & 1\\ 2 & 0\\ 3 & 0\\ 0 & 3\end{bmatrix} = X_{\text{split}}. \tag{4.41}$$

Both of these spaces are elliptic fibrations and the split manifold is also a K3-fibration with generic elliptic K3 fibers.

The determinantal hypersurface

$$X^\sharp = \{p_{\text{det}} = QT - RS = 0\} \in \begin{matrix}\mathbb{P}_1\\ \mathbb{P}_2\\ \mathbb{P}_2\end{matrix}\begin{bmatrix}2\\3\\3\end{bmatrix} \tag{4.42}$$

is singular at the locus

$$\begin{matrix}\mathbb{P}_1\\ \mathbb{P}_2\\ \mathbb{P}_2\end{matrix}\begin{bmatrix}2 & 2 & 0 & 0\\ 3 & 3 & 0 & 0\\ 0 & 0 & 3 & 3\end{bmatrix} = 9 \times \Sigma, \tag{4.43}$$

where $\Sigma = \begin{matrix}\mathbb{P}_1\\ \mathbb{P}_2\end{matrix}\begin{bmatrix}2 & 2\\ 3 & 3\end{bmatrix}$ and $\mathbb{P}_2[3 \;\; 3] = 9pts$. The curve Σ has Euler number $\chi(\Sigma) = -54$ and hence genus $g(\Sigma) = 28$. Thus the singular set has 9 different components and the splitting formula (4.29) becomes

$$\chi(X_{\text{split}}) = \chi(X) + 3 \cdot 9 \; \chi(\Sigma) = 288. \tag{4.44}$$

We see from this that it is precisely the small resolution of the curve Σ which introduces the divisors in X_{split} which are responsible for the superpotential [107]. But this does not necessarily have to be so. A counterexample is the sextic split (4.31). Here, both sides do not contain divisors fulfilling (4.38).

Summary

In this thesis, we have discussed various aspects of branes in string theory and M-theory. In chapter 2 we were able to construct six-dimensional chiral interacting field theories from Hanany-Witten like brane setups. The field theory requirement that the anomalies cancel was reproduced by RR-charge conservation in the brane setup. We compared our results to the construction of [49, 50], who use branes at orbifolds to construct the same type of fixed points. This method is T-dual to our approach. The data of the Hanany-Witten setup, which consists of brane positions, was mapped to the instanton data of [49, 50]. Our brane configurations could reproduce and extend the results of [49, 50] for A_k-type singularities. The orbifold construction can be extended to D and E type singularities. This is still unsolved in the HW approach because it is unknown what the T-dual of a D or E type singularity is.

In chapter 3 we discussed the Matrix conjecture of [54]. The claim of [54] is that M-theory in the light cone gauge is described by the quantum mechanics of D0 branes. Toroidal compactifications of M-theory have a description in terms of super Yang-Mills theory an the dual torus. For more than three compactified dimensions, more degrees of freedom have to be added. In some sense, the philosophy in this chapter is orthogonal to the previous chapter: Here, we want to get M-theory results from field theory considerations, whereas in the previous chapter we obtained field theory results by embedding the theories in string theory. Our main focus was on the compactification on T^6, which leads to complications. Here, the Matrix model is again given by an eleven dimensional theory, not by a lower dimensional field theory. Other problems and possible resolutions of Matrix theory are discussed at the end of chapter 3.

In the last chapter we considered M- and F-theory compactifications on Calabi-Yau fourfolds. After explaining some basics of fourfolds, we showed that the web of fourfolds is connected by singular transitions. The two manifolds which are connected by the transition are different resolutions of the same singular manifold. The resolution of the singularities can lead to a certain type of divisors, which lead to non-perturbative superpotentials, when branes wrap them. The vacua connected by the transitions can be physically very different. It would be nice to get a better understanding of the physics of the transition , similar to the case of threefold compactifications in string theory [86, 106].

Bibliography

[1] M. B. Green, J. H. Schwarz, and E. Witten, "Superstring theory. vol. 1: Introduction,".

[2] M. B. Green, J. H. Schwarz, and E. Witten, "Superstring theory. vol. 2: Loop amplitudes, anomalies and phenomenology,".

[3] D. Lust and S. Theisen, "Lectures on string theory,".

[4] E. Witten, "String theory dynamics in various dimensions," *Nucl. Phys.* **B443** (1995) 85–126, `hep-th/9503124`.

[5] P. Horava and E. Witten, "Eleven-dimensional supergravity on a manifold with boundary," *Nucl. Phys.* **B475** (1996) 94–114, `hep-th/9603142`.

[6] P. Horava and E. Witten, "Heterotic and type I string dynamics from eleven-dimensions," *Nucl. Phys.* **B460** (1996) 506–524, `hep-th/9510209`.

[7] C. Vafa, "Evidence for F theory," *Nucl. Phys.* **B469** (1996) 403–418, `hep-th/9602022`.

[8] D. R. Morrison and C. Vafa, "Compactifications of F theory on Calabi-Yau threefolds. 1," *Nucl. Phys.* **B473** (1996) 74–92, `hep-th/9602114`.

[9] D. R. Morrison and C. Vafa, "Compactifications of F theory on Calabi-Yau threefolds. 2," *Nucl. Phys.* **B476** (1996) 437–469, `hep-th/9603161`.

[10] W. Nahm, "Supersymmetries and their representations," *Nucl. Phys.* **B135** (1978) 149.

[11] E. Cremmer, B. Julia, and J. Scherk, "Supergravity theory in eleven-dimensions," *Phys. Lett.* **76B** (1978) 409.

[12] J. Polchinski, "Dirichlet branes and Ramond-Ramond charges," *Phys. Rev. Lett.* **75** (1995) 4724–4727, `hep-th/9510017`.

[13] J. Polchinski, "Tasi lectures on D-branes," hep-th/9611050.

[14] N. Seiberg and E. Witten, "Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory," *Nucl. Phys.* **B426** (1994) 19–52, hep-th/9407087.

[15] P. K. Townsend, "D-branes from M-branes," *Phys. Lett.* **B373** (1996) 68–75, hep-th/9512062.

[16] P. K. Townsend, "M theory from its superalgebra," hep-th/9712004.

[17] J. G. Russo and A. A. Tseytlin, "Waves, boosted branes and BPS states in m theory," *Nucl. Phys.* **B490** (1997) 121–144, hep-th/9611047.

[18] E. Bergshoeff, M. de Roo, E. Eyras, B. Janssen, and J. P. van der Schaar, "Intersections involving monopoles and waves in eleven- dimensions," *Class. Quant. Grav.* **14** (1997) 2757, hep-th/9704120.

[19] E. Bergshoeff, M. de Roo, E. Eyras, B. Janssen, and J. P. van der Schaar, "Multiple intersections of D-branes and M-branes," *Nucl. Phys.* **B494** (1997) 119–143, hep-th/9612095.

[20] J. M. Izquierdo, N. D. Lambert, G. Papadopoulos, and P. K. Townsend, "Dyonic membranes," *Nucl. Phys.* **B460** (1996) 560–578, hep-th/9508177.

[21] P. S. Howe, N. D. Lambert, and P. C. West, "The three-brane soliton of the M-five-brane," hep-th/9710033.

[22] J. Distler and A. Hanany, "(0,2) Noncritical strings in six-dimensions," *Nucl. Phys.* **B490** (1997) 75–90, hep-th/9611104.

[23] H. Ooguri and C. Vafa, "Two-dimensional black hole and singularities of CY manifolds," *Nucl. Phys.* **B463** (1996) 55–72, hep-th/9511164.

[24] E. Witten, "Small instantons in string theory," *Nucl. Phys.* **B460** (1996) 541–559, hep-th/9511030.

[25] M. R. Douglas, "Branes within branes," hep-th/9512077.

[26] I. Washington Taylor, "Lectures on D-branes, gauge theory and M(atrices)," hep-th/9801182.

[27] M. Li, "Boundary states of D-branes and Dy strings," *Nucl. Phys.* **B460** (1996) 351–361, hep-th/9510161.

[28] A. Hanany and E. Witten, "Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics," *Nucl. Phys.* **B492** (1997) 152–190, `hep-th/9611230`.

[29] S. Elitzur, A. Giveon, and D. Kutasov, "Branes and N=1 duality in string theory," *Phys. Lett.* **B400** (1997) 269–274, `hep-th/9702014`.

[30] N. Evans, C. V. Johnson, and A. D. Shapere, "Orientifolds, branes, and duality of 4-d gauge theories," *Nucl. Phys.* **B505** (1997) 251, `hep-th/9703210`.

[31] K. Landsteiner and E. Lopez, "New curves from branes," `hep-th/9708118`.

[32] E. Witten, "Solutions of four-dimensional field theories via M theory," *Nucl. Phys.* **B500** (1997) 3, `hep-th/9703166`.

[33] A. Brandhuber, J. Sonnenschein, S. Theisen, and S. Yankielowicz, "M theory and Seiberg-Witten curves: Orthogonal and symplectic groups," *Nucl. Phys.* **B504** (1997) 175, `hep-th/9705232`.

[34] K. Landsteiner, E. Lopez, and D. A. Lowe, "N=2 supersymmetric gauge theories, branes and orientifolds," *Nucl. Phys.* **B507** (1997) 197, `hep-th/9705199`.

[35] O. Aharony and A. Hanany, "Branes, superpotentials and superconformal fixed points," *Nucl. Phys.* **B504** (1997) 239, `hep-th/9704170`.

[36] O. Aharony, A. Hanany, and B. Kol, "Webs of (p,q) five-branes, five-dimensional field theories and grid diagrams," `hep-th/9710116`.

[37] B. Kol, "5-d field theories and M theory," `hep-th/9705031`.

[38] A. Brandhuber, N. Itzhaki, J. Sonnenschein, S. Theisen, and S. Yankielowicz, "On the M theory approach to (compactified) 5-d field theories," *Phys. Lett.* **B415** (1997) 127, `hep-th/9709010`.

[39] N. Seiberg, "Nontrivial fixed points of the renormalization group in six-dimensions," *Phys. Lett.* **B390** (1997) 169–171, `hep-th/9609161`.

[40] I. Brunner and A. Karch, "Branes and six-dimensional fixed points," *Phys. Lett.* **B409** (1997) 109–116, `hep-th/9705022`.

[41] I. Brunner and A. Karch, "Branes at orbifolds versus Hanany Witten in six-dimensions," `hep-th/9712143`.

[42] A. Hanany and A. Zaffaroni, "Chiral symmetry from type IIA branes," *Nucl. Phys.* **B509** (1998) 145, `hep-th/9706047`.

[43] A. Hanany and A. Zaffaroni, "Branes and six-dimensional supersymmetric theories," hep-th/9712145.

[44] L. J. Romans, "Massive N=2A supergravity in ten-dimensions," *Phys. Lett.* **169B** (1986) 374.

[45] M. B. Green and J. H. Schwarz, "Anomaly cancellations in supersymmetric d = 10 gauge theory require so(32)," *Phys. Lett.* **149B** (1984) 117.

[46] J. Erler, "Anomaly cancellation in six-dimensions," *J. Math. Phys.* **35** (1994) 1819–1833, hep-th/9304104.

[47] U. H. Danielsson, G. Ferretti, J. Kalkkinen, and P. Stjernberg, "Notes on supersymmetric gauge theories in five-dimensions and six-dimensions," *Phys. Lett.* **B405** (1997) 265–270, hep-th/9703098.

[48] M. Bershadsky and C. Vafa, "Global anomalies and geometric engineering of critical theories in six-dimensions," hep-th/9703167.

[49] K. Intriligator, "RG fixed points in six-dimensions via branes at orbifold singularities," *Nucl. Phys.* **B496** (1997) 177, hep-th/9702038.

[50] J. D. Blum and K. Intriligator, "New phases of string theory and 6-d RG fixed points via branes at orbifold singularities," *Nucl. Phys.* **B506** (1997) 199, hep-th/9705044.

[51] N. Seiberg and S. Sethi, "Comments on Neveu-Schwarz five-branes," hep-th/9708085.

[52] M. R. Douglas and G. Moore, "D-branes, quivers, and ALE instantons," hep-th/9603167.

[53] M. Berkooz *et. al.*, "Anomalies, dualities, and topology of d = 6 N=1 superstring vacua," *Nucl. Phys.* **B475** (1996) 115–148, hep-th/9605184.

[54] T. Banks, W. Fischler, S. H. Shenker, and L. Susskind, "M theory as a Matrix model: A Conjecture," *Phys. Rev.* **D55** (1997) 5112–5128, hep-th/9610043.

[55] T. Banks, "Matrix theory," hep-th/9710231.

[56] D. Bigatti and L. Susskind, "Review of Matrix theory," hep-th/9712072.

[57] M. R. Douglas, D. Kabat, P. Pouliot, and S. H. Shenker, "D-branes and short distances in string theory," *Nucl. Phys.* **B485** (1997) 85–127, hep-th/9608024.

[58] U. H. Danielsson, G. Ferretti, and B. Sundborg, "D particle dynamics and bound states," *Int. J. Mod. Phys.* **A11** (1996) 5463–5478, hep-th/9603081.

[59] B. de Wit, J. Hoppe, and H. Nicolai, "On the quantum mechanics of supermembranes," *Nucl. Phys.* **B305** (1988) 545.

[60] L. Susskind, "Another conjecture about M(atrix) theory," hep-th/9704080.

[61] M. Burkardt, "Light front quantization," *Adv. Nucl. Phys.* **23** (1996) 1–74, hep-ph/9505259.

[62] S. J. Brodsky, H.-C. Pauli, and S. S. Pinsky, "Quantum chromodynamics and other field theories on the light cone," hep-ph/9705477.

[63] K. Yamawaki, "Zero mode and symmetry breaking on the light front," hep-th/9707141.

[64] S. Tsujimaru and K. Yamawaki, "Zero mode and symmetry breaking on the light front," hep-th/9704171.

[65] N. Seiberg, "Why is the Matrix model correct?," *Phys. Rev. Lett.* **79** (1997) 3577–3580, hep-th/9710009.

[66] A. Sen, "D0-branes on T^n and Matrix theory," hep-th/9709220.

[67] I. Washington Taylor, "D-brane field theory on compact spaces," *Phys. Lett.* **B394** (1997) 283–287, hep-th/9611042.

[68] O. J. Ganor, S. Ramgoolam, and I. Washington Taylor, "Branes, fluxes and duality in M(atrix) theory," *Nucl. Phys.* **B492** (1997) 191–204, hep-th/9611202.

[69] W. Fischler, E. Halyo, A. Rajaraman, and L. Susskind, "The incredible shrinking torus," *Nucl. Phys.* **B501** (1997) 409, hep-th/9703102.

[70] M. Rozali, "Matrix theory and U duality in seven-dimensions," *Phys. Lett.* **B400** (1997) 260–264, hep-th/9702136.

[71] R. Dijkgraaf, E. Verlinde, and H. Verlinde, "Matrix string theory," *Nucl. Phys.* **B500** (1997) 43, hep-th/9703030.

[72] N. Seiberg, "New theories in six-dimensions and Matrix description of m theory on T^5 and $T^5/\mathbb{Z}_2$," *Phys. Lett.* **B408** (1997) 98–104, hep-th/9705221.

[73] R. Dijkgraaf, E. Verlinde, and H. Verlinde, "Notes on Matrix and micro strings," *Nucl. Phys. Proc. Suppl.* **62** (1998) 348, hep-th/9709107.

[74] I. Brunner and A. Karch, "Matrix description of M theory on T^6," *Phys. Lett.* **B416** (1998) 67, hep-th/9707259.

[75] A. Hanany and G. Lifschytz, "M(atrix) theory on T^6 and a m(atrix) theory description of KK monopoles," hep-th/9708037.

[76] O. J. Ganor, "On the M(atrix) model for M theory on T^6," hep-th/9709139.

[77] C. M. Hull and P. K. Townsend, "Unity of superstring dualities," *Nucl. Phys.* **B438** (1995) 109–137, hep-th/9410167.

[78] S. Elitzur, A. Giveon, D. Kutasov, and E. Rabinovici, "Algebraic aspects of Matrix theory on T^d," *Nucl. Phys.* **B509** (1998) 122, hep-th/9707217.

[79] M. Blau and M. O'Loughlin, "Aspects of U duality in Matrix theory," hep-th/9712047.

[80] A. Strominger, "Open p-branes," *Phys. Lett.* **B383** (1996) 44–47, hep-th/9512059.

[81] M. Berkooz, M. Rozali, and N. Seiberg, "Matrix description of M theory on T^4 and T^5," *Phys. Lett.* **B408** (1997) 105–110, hep-th/9704089.

[82] A. Losev, G. Moore, and S. L. Shatashvili, "M & m's," hep-th/9707250.

[83] K. Behrndt and I. Brunner, "Entropy of bound states in the Matrix description of M five-branes," hep-th/9712142.

[84] I. Brunner and A. Karch, "Matrix theory and the six torus," hep-th/9801154.

[85] S. Kachru, A. Lawrence, and E. Silverstein, "On the Matrix description of Calabi-Yau compactifications," hep-th/9712223.

[86] A. Strominger, "Massless black holes and conifolds in string theory," *Nucl. Phys.* **B451** (1995) 96–108, hep-th/9504090.

[87] M. R. Douglas and H. Ooguri, "Why Matrix theory is hard," hep-th/9710178.

[88] K. Becker and M. Becker, "A two loop test of M(atrix) theory," *Nucl. Phys.* **B506** (1997) 48, hep-th/9705091.

[89] M. Dine and A. Rajaraman, "Multigraviton scattering in the Matrix model," hep-th/9710174.

[90] E. Witten, "Nonperturbative superpotentials in string theory," *Nucl. Phys.* **B474** (1996) 343–360, hep-th/9604030.

[91] S. Sethi, C. Vafa, and E. Witten, “Constraints on low dimensional string compactifications,” *Nucl. Phys.* **B480** (1996) 213–224, `hep-th/9606122`.

[92] A. Sen, “String string duality conjecture in six-dimensions and charged solitonic strings,” *Nucl. Phys.* **B450** (1995) 103–114, `hep-th/9504027`.

[93] J. A. Harvey and A. Strominger, “The heterotic string is a soliton,” *Nucl. Phys.* **B449** (1995) 535–552, `hep-th/9504047`.

[94] S. Kachru and C. Vafa, “Exact results for N=2 compactifications of heterotic strings,” *Nucl. Phys.* **B450** (1995) 69–89, `hep-th/9505105`.

[95] A. Klemm, W. Lerche, and P. Mayr, “K3 Fibrations and heterotic type II string duality,” *Phys. Lett.* **B357** (1995) 313–322, `hep-th/9506112`.

[96] P. S. Aspinwall and J. Louis, “On the ubiquity of K3 fibrations in string duality,” *Phys. Lett.* **B369** (1996) 233–242, `hep-th/9510234`.

[97] P. S. Aspinwall, “Some relationships between dualities in string theory,” *Nucl. Phys. Proc. Suppl.* **46** (1996) 30, `hep-th/9508154`.

[98] J. H. Schwarz, “An $SL(2,\mathbb{Z})$ multiplet of type IIB superstrings,” *Phys. Lett.* **B360** (1995) 13–18, `hep-th/9508143`.

[99] A. Sen, “F theory and orientifolds,” *Nucl. Phys.* **B475** (1996) 562–578, `hep-th/9605150`.

[100] P. Candelas, A. M. Dale, C. A. Lutken, and R. Schimmrigk, “Complete intersection Calabi-Yau manifolds,” *Nucl. Phys.* **B298** (1988) 493.

[101] T. Hübsch, “Calabi-Yau manifolds: A Bestiary for physicists,”.

[102] I. Brunner and R. Schimmrigk, “F theory on Calabi-Yau fourfolds,” *Phys. Lett.* **B387** (1996) 750–758, `hep-th/9606148`.

[103] P. Candelas, P. S. Green, and T. Hubsch, “Rolling among Calabi-Yau vacua,” *Nucl. Phys.* **B330** (1990) 49.

[104] M. Lynker and R. Schimmrigk, “Conifold transitions and mirror symmetries,” *Nucl. Phys.* **B484** (1997) 562, `hep-th/9511058`.

[105] A. C. Avram, M. Kreuzer, M. Mandelberg, and H. Skarke, “The web of Calabi-Yau hypersurfaces in toric varieties,” *Nucl. Phys.* **B505** (1997) 625–640, `hep-th/9703003`.

[106] B. R. Greene, D. R. Morrison, and A. Strominger, "Black hole condensation and the unification of string vacua," *Nucl. Phys.* **B451** (1995) 109–120, hep-th/9504145.

[107] I. Brunner, M. Lynker, and R. Schimmrigk, "Unification of M theory and F theory Calabi-Yau fourfold vacua," *Nucl. Phys.* **B498** (1997) 156, hep-th/9610195.

[108] R. Donagi, A. Grassi, and E. Witten, "A nonperturbative superpotential with E(8) symmetry," *Mod. Phys. Lett.* **A11** (1996) 2199–2212, hep-th/9607091.

[109] G. Curio and D. Lüst, "A class of N=1 dual string pairs and its modular superpotential," *Int. J. Mod. Phys.* **A12** (1997) 5847, hep-th/9703007.

[110] I. Affleck, J. A. Harvey, and E. Witten, "Instantons and (super)symmetry breaking in (2+1)- dimensions," *Nucl. Phys.* **B206** (1982) 413.

Zusammenfassung

Die physikalischen Kräfte in der Natur werden durch zwei fundamentale Theorien beschrieben: Das Standard Modell der Elementarteilchenphysik kann die Wechselwirkungen zwischen Elementarteilchen, also die Physik auf mikroskopischer Skala, erklären. Das Gegenstück ist die allgemeine Relativitätstheorie, die die Physik auf makrokopischen, kosmologischen Skalen widergibt. Eine fundamentale Frage an die theoretische Physik ist es, diese beiden Theorien zu vereinigen. String Theorie ist ein Kandidat für eine solche vereinheitlichende Theorie.

In den letzten Jahren wurden große Fortschritte auf diesem Gebiet erzielt. Zum ersten Mal wurde es möglich, in den starken Kopplungsbereich vorzudringen. Das Mittel, das hier benutzt wird, sind die sogenannten Dualitätssymmetrien, die den starken Kopplungsbereich einer Theorie in den schwachen Kopplungsbereich einer anderen Theorie abbilden. Man glaubt, daß letztendlich die bekannten konsistenten Stringtheorien lediglich unterschiedliche perturbative Limites einer fundamentalen Theorie, der M- oder F-Theorie, sind. Außerdem wurde klar, daß Stringtheorien nicht nur eindimensional ausgedehnte Objekte, die Strings, enthalten, sondern auch höherdimensionale "Branen". Die Masse dieser Objekte ist bei schwacher Stringkopplung sehr groß, bei starker Kopplung geht ihre Masse hingegen gegen 0. Dualitätssymmetrien können Strings und höherdimensionale Branen in Beziehung setzen.

Diese Arbeit ist einigen Anwendungen der Branen gewidmet. Im Niederenergielimes ist die Weltvolumentheorie solcher Branen durch eine Feldtheorie gegeben. Dies wird benutzt, um Eigenschaften von Feldtheorien im Stringtheorie-Kontext zu reproduzieren. In Kapitel 2 verwenden wir bestimmte Konfigurationen von sich im Raum schneidenden Branen, um Theorien mit $(0,1)$ Supersymmetrie in sechs Dimensionen zu studieren. Von der Feldtheorieseite her ergibt sich die Anomaliefreiheit als Konsistenzbedingung an chirale Theorien. Diese Bedingung kann in der String-Theorie als Ladungserhaltung von dort auftretenden Eichpotentialen reproduziert werden, d.h. als eine Konsistenzbedingung an die möglichen Branenkonfigurationen.

Das Kapitel 3 beschäftigt sich mit Matrix-Modellen. Hierbei handelt es sich um einen Vorschlag, wie die fundamentale M-Theorie formuliert werden könnte. M-Theorie in der Lichtkegel-Eichung soll durch ein quantenmechanisches Modell erfaßt werden. Die Kompaktifizierungen von M-Theorie werden durch niederdimensionale Feldtheorien beschrieben, die wiederum als Feldtheorien auf Branen interpretiert werden können. Wir befassen uns mit der Kompaktifizierung von Matrix-Theorien auf Tori und diskutieren hierbei insbesondere Probleme, die bei Kompaktifizierungen auf sechs dimensionalen Tori auftreten. Hier scheint es – anders als für niederdimensionalere Tori und genau wie für höherdimensionale Tori – keine Beschreibung als niederdimensionale Feldtheorie zu geben.

Das letzte Kapitel behandelt die Kompaktifizierung von M- bzw F-Theorie auf Calabi-Yau 4-Mannigfaltigkeiten. Dies führt auf den phänomenologisch relevanten Fall von $N = 1$ Supersymmetrie in vier Dimensionen. Es wird gezeigt, daß topologisch verschiedene Calabi-Yau Mannigfaltigkeiten durch Übergänge über singuläre Konfigurationen verbunden werden können. Die Vakua, die durch Kompaktifizierung auf den durch die Singularität verbundenen Mannigfaltigkeiten erhalten werden, haben physikalisch unterschiedliche Eigenschaften. So können gewisse Divisoren, die durch Ausglätten der Singularität entstehen, zur Folge haben, daß ein nicht-perturbatives Superpotential durch Branen erzeugt wird, die sich um die betreffenden Divisoren herumwickeln.

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Selbständigkeitserklärung

Hiermit versichere ich, die vorliegende Arbeit selbständig angefertigt zu haben und keine weiteren als die angegebenen Hilfsmittel verwendet zu haben.

Ilka Brunner

Lebenslauf

Name: Ilka Marie-Louise Brunner

Nationalität: Deutsche

Geburtsort und -datum: Pittsburgh, Pa /USA, 25.01.1971

Bildungsweg:

seit 1996	Doktorandin an der Humboldt-Universität, Berlin Mitglied des Graduiertenkollegs "Strukturuntersuchungen, Präzisionstests und Erweiterungen des Standard Modells der Elementarteilchenphysik" Betreuer: Prof. Dr. D. Lüst
1995	Diplom in Physik an der Universität Bonn
1994 – 1995	Gaststudentin an der Universität Amsterdam, Niederlande Anfertigen der Diplomarbeit am mathematischen Institut der Universität Amsterdam Titel: "Area-Preserving Symmetry in Two-Dimensional Quantum Field Theory " Betreuer: Prof. Dr. R. Dijkgraaf, Universität Amsterdam und Prof. Dr. W. Nahm, Universität Bonn Stipendium der Carl-Duisberg-Stiftung
1990 – 1995	Studium der Physik, Universität Bonn
1981 – 1990	Nicolaus-Cusanus-Gymnasium, Bergisch Gladbach

Nebentätigkeiten im Studium :

1993 – 1994	Übungsleiterin in Linearer Algebra, Universität Bonn
1992 – 1993	Mitarbeit in der Fachschaft Physik, Studentische Vertreterin im Fachausschuß
1990 & 1991	Werksstudentin in der Abteilung für angewandte Physik, Bayer, Leverkusen

Publikationsliste

I. Brunner. A. Hanany, A. Karch und D. Lüst, **Brane Dynamics and Chiral non-Chiral Transitions**, hep-th/9801017, zur Veröffentlichung an *Nucl. Phys.* **B** geschickt.

I. Brunner und A. Karch, **Branes at Orbifolds versus Hanany-Witten in Six Dimensions** , JHEP Electronic Journal, hep-th/9712143.

K. Behrndt und I. Brunner, **Entropy of Bound States in the Matrix Description of M Five-branes**, hep-th/9712142.

I. Brunner und A. Karch, **Compactifications of Matrix Theory on $\mathbf{T}^6$** *Phys. Lett.* **B416**(1998) 67, hep-th/9707259.

I. Brunner und A. Karch, **Branes and Six-Dimensional Fixed Points.**, *Phys. Lett.* **B409** (1997) 109, hep-th/9705022.

I. Brunner, M. Lynker und R. Schimmrigk, **Unification of M Theory and F Theory Calabi-Yau Fourfold Vacua**, *Nucl. Phys.* **B498** (1997) 156, hep-th/9710195.

I.Brunner und R. Schimmrigk, **F-Theory on Calabi-Yau Fourfolds**, *Phys. Lett.* **B387** (1996) 750, hep-th/9606148.

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I. Brunner und A. Karch, **Field Theory from Branes**, wird veröffentlicht in den Proceedings of the 31st International Symposium Ahrenshoop on the Theory of Elementary Particles, Buckow, Germany, September 1997.

I. Brunner und A. Karch, **Matrix Theory and the Six Torus**, wird veröffentlicht in den Proceedings of the 31st International Symposium Ahrenshoop on the Theory of Elementary Particles, Buckow, Germany, September 1997.

I. Brunner, M. Lynker und R. Schimmrigk, **Dualities and Phase Transitions for Calabi-Yau Threefolds and Fourfolds**, in D. Lüst, H.-J. Otto, G. Weigt (eds.), Proceedings of the 30st International Symposium Ahrenshoop on the Theory of Elementary Particles, Buckow, Germany, September 1996. *Nucl.Phys.Proc.Suppl.* **56B** (1997) 120.