Direct and Inverse Spectral Problems for Hybrid Manifolds

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Chapter 1

Introduction

1.1 Motivation

Spectral theory on compact Riemannian manifolds has been studied for a long time and takes its roots in physical problems. A great number of important results has been obtained and this subject has a lot of ramifications. One of the main objects of investigation in spectral theory are Laplace type operators on a compact manifold, constructed from a Riemannian metric. These operators are generalizations of the usual Laplace operator on $\mathbb{R}^n$.

To such an operator we can associate a sequence of numbers, called the spectrum of this operator, each element of this sequence is an eigenvalue of the operator. Spectral theory aims at understanding the structure of the spectrum and its relations to the geometry and the topology of the manifold we begin with. For example, from the spectrum of the Laplace operator we can recover the dimension of the manifold, its volume and its Euler characteristic.

Moreover, the spectrum determines an infinite number of local geometric invariants, so that it was asked if it determines the manifold up to isometry. This is the famous question "Can one hear the shape of a drum?" raised by Kac [1966]. The answer to this question is negative and besides the original counter-examples of Milnor [1964], there exist by now large families of non-isometric isospectral manifolds (see for example Sunada [1985]). A related subject which is developing very actively is spectral theory on manifolds which are possibly singular (see for example Cheeger [1983]). In this case it is not clear a priori how to define some analogue of the Laplace operator, but once this is done, the spectral properties can be investigated.

Generalizing in another related direction, it is also possible to do spectral theory on graphs. This is the study of what are now called "quantum graphs".

Geometrically, a quantum graph is a set of one-dimensional segments with some end points identified. Each segment can be regarded as a segment in $\mathbb{R}$ with the standard metric. We then define a Laplace operator on the graph as follows. On each edge, it is the usual Laplace operator $-d^2/dx^2$, and we have to specify some boundary conditions at the vertices in order to obtain a self-adjoint operator. It is known that for generic finite quantum graphs, the spectrum determines completely the graph (i.e the lengths of the edges and the structure of the graph) Gutkin and Smilansky [2001], Kurasov and Nowaczyk [2005].

One of the reasons why quantum graphs are important is that they are supposed to model so-called "nano-structures". These are mathematical models for physical systems in which several dimensions are too small for classical physics and too large for quantum physics (typically the characteristic dimensions are a few nanometers). One hopes that the spectrum of the "nano-structure", which is very difficult to compute in general, is related to the spectrum of the corresponding quantum graph. Of course, this latter is easier to get. This is an important open question and there are many articles devoted to this problem (see the survey Kuchment [2002]). Some results concerning the behavior of the spectrum of a compact manifold which is "shrinking to a graph" can be found in Exner and Post [2005].

In this work, we are interested in more general objects than quantum graphs, the so-called "hybrid manifolds". Roughly speaking, a hybrid manifold is a union of manifolds connected by segments. If the manifolds are zero-dimensional, then we have a quantum graph. Such an object may be a good model for molecular-type nano-structures consisting of manifolds connected by nano-tubes. Of course, lots of questions arise when we use this model: besides the typical spectral problems it is interesting to understand how the spectral properties of a hybrid manifold are related to the properties of the corresponding nano-structure.

1.2 Plan and principal results

In the second chapter we define a hybrid manifold as a topological space, and find its Euler characteristic. Our next task will be to construct a Laplace operator on a hybrid manifold. To do this, we first consider the operator given by the direct sum of Laplace operators on the different parts of the hybrid space. We restrict this operator by letting it act on functions which vanish at the gluing points and finally take a self-adjoint extension of this restriction.

It can be shown that any such self-adjoint extension is defined by some boundary conditions, which describe how the different parts of our hybrid manifold "interact" at a gluing point. A priori all these boundary conditions are on the same footing, but it is possible that some of them will be preferred if we consider our hybrid space as the limit of a sequence of nano-structures (see Exner and Post [2005]). Nevertheless, we take all boundary conditions into consideration and parametrize any self-adjoint extension by a certain matrix describing the boundary conditions.

The spectral properties of the operators obtained in this way can be studied using their resolvents or some function of it. In our approach we consider the trace of the squared resolvent (taking the trace is a standard procedure in spectral theory, but the resolvent itself is not trace class in general, so we take into consideration the second power of the resolvent, which is trace class) and construct its expansion as the spectral parameter tends to ∞. In fact, due to the singular structure of the hybrid space, this expansion contains also powers of the logarithm of the spectral parameter.

In the third chapter we give a short review of the theory of self-adjoint extensions of symmetric operators. In particular, we describe Krein's theory of self-adjoint extensions. This formalism is well suited to the description of the resolvent of Laplace operators on a hybrid manifold. Indeed, it allows us to express the resolvent of the Laplacian defined by some boundary conditions through the resolvent of a fixed self-adjoint extension. In other words, all self-adjoint extensions are parametrized be the matrix of boundary conditions and one fixed self-adjoint extension. In our situation it is natural to choose this fixed operator as the direct sum of the Neumann Laplacians on the segments and the ordinary Laplacians on the manifolds constituting our hybrid space. Moreover, it is relatively convenient to perform the necessary computations for this operator.

In the Chapter 4 we find the expression for the trace of the second power of the resolvent for any Laplace operator on a hybrid space:

Theorem 1. *Consider the hybrid manifold H, consisting in manifolds M_i and N segments L_j. Let S be a Laplace operator on it, corresponding to the matrix Λ of boundary conditions. For $z \in \mathbb{C} \setminus [0, \infty)$, denote by $R(z) = (S + z^2)^{-1}$ the resolvent of S. Then for large z and all $q > 0$ there holds*

$$\operatorname{Tr} R^2(z) = \sum_{m=1}^{M}\sum_{k=0}^{q}\frac{a_{km}\,\Gamma(k+1)}{4\pi z^{2k+2}}) + \sum_j \left(\frac{l_j}{4z^3}+\frac{1}{2z^4}\right)$$

$$-\sum_{i=1}^{N}\frac{(F_i)''_{zz}(\frac{1}{z}-\lambda_{i+N,i+N})+\frac{2}{z^3}(F_i-\lambda_{i,i})}{4z^2(F_i-\lambda_{i,i})(\frac{1}{z}-\lambda_{i+N,i+N})-|\lambda_{i,i+N}|^2}$$

$$+\sum_{i=1}^{N}\frac{(F_i)'_z(\frac{1}{z}-\lambda_{i+N,i+N})-\frac{1}{z^2}(F_i-\lambda_{i,i})}{4z^3(F_i-\lambda_{i,i})(\frac{1}{z}-\lambda_{i+N,i+N})-|\lambda_{i,i+N}|^2}$$

$$+\sum_{i=1}^{N}\frac{((F_i)'_z)^2(\frac{1}{z}-\lambda_{i+N,i+N})^2-\frac{2}{z^2}(F_i)'_z|\lambda_{i,i+N}|^2+\frac{1}{z^4}(F_i-\lambda_{i,i})^2}{4z^2((F_i-\lambda_{i,i})(\frac{1}{z}-\lambda_{i+N,i+N})-|\lambda_{i,i+N}|^2)^2}$$

$$+O(z^{-2(q+2)}),$$

where a_{km} is the global k-th heat kernel coefficient on the m-th manifold M_m, l_j is the length of the segment L_j, λ_{ij} are elements of Λ and $F_i = F(q_i, q_i, z)$, F is the regular part of the Green function of the Laplacian on the manifold to which q_i belongs. Moreover, for all $p \geqslant 1$,

$$F(x,x,z) = \frac{1}{4\pi}\left(-2\gamma - \ln z^2 + \sum_{n=1}^{p}\frac{\Gamma(n)a_n(x,x)}{z^{2n}}\right) + O(z^{-2(p+1)}),$$

where $a_n(x,x)$ is the local n-th heat kernel coefficient on the manifold M to which the point x belongs.

In Section 4.4, we will give the definition of a z-pseudoasymptotic expansion for a function, depending on z and $\ln z^2$. Using the formula for the regular part of the Green function on the diagonal we will find the z-pseudoasymptotic expansion of $\operatorname{Tr} R^2(z)$ for large z.

Theorem 2. *Consider the hybrid manifold H, consisting in manifolds M_i and N segments L_j, and consider a Laplace operator on H (corresponding to boundary conditions determined by a matrix Λ, and disjoint with D_0). Suppose also that for all i the coefficients $\lambda_{i+N,i+N}$ do not vanish. Then the square of the resolvent $R(z)$, obtained in Theorem 4.4.1 has a z-pseudoasymptotic expansion which has the form:*

$$\operatorname{Tr} R^2(z) = \frac{\sum_i Vol(M_i)}{4\pi z^2} + \frac{\sum_j l_j}{4z^3}$$

$$+\frac{c_4(\ln z^2)}{z^4}+\frac{c_5(\ln z^2)}{z^5}+\frac{c_6(\ln z^2)}{z^6}+\frac{c_7(\ln z^2)}{z^7}+O\left(\frac{1}{z^8}\right)$$

The coefficients c_n are rational functions and have the following expansions:

$$c_4 = \frac{\sum_i \chi(M_i)}{6} + \frac{N}{4} + \frac{N}{\ln z^2}$$
$$+ \sum_{i=1}^{N} \frac{1 - 2\gamma - 4\pi\lambda_{i,i} + 4\pi\frac{|\lambda_{i,i+N}|^2}{\lambda_{i+N,i+N}}}{\ln^2 z^2} + O\left(\frac{1}{\ln^3 z^2}\right),$$
$$c_5 = \sum_{i=1}^{N} \frac{3}{4\lambda_{i+N,i+N}} + \sum_{i=1}^{N} \frac{3\pi|\lambda_{i,i+N}|^2}{\lambda_{i+N,i+N}^2 \ln z^2} + O\left(\frac{1}{\ln^2 z^2}\right),$$
$$c_6 = \sum_{M_i} \frac{a_{2i}}{2\pi} + \sum_{i=1}^{N} \frac{1}{\lambda_{i+N,i+N}^2} + \sum_{i=1}^{N} \frac{2a_{1i}\lambda_{i+N,i+N}^3 + 8\pi|\lambda_{i,i+N}|^2}{\lambda_{i+N,i+N}^3 \ln z^2} + O\left(\frac{1}{\ln^2 z^2}\right),$$
$$c_7 = \sum_{i=1}^{N} \frac{5}{4\lambda_{i+N,i+N}^3} + \sum_{i=1}^{N} \frac{15\pi|\lambda_{i,i+N}|^2}{\lambda_{i+N,i+N}^4 \ln z^2} + O\left(\frac{1}{\ln^2 z^2}\right),$$

where a_{ki} is the k-th heat kernel coefficient for the manifold M_i, $Vol(M_i)$ and $\chi(M_i)$ are the volume and Euler characteristic of M_i, l_j is the length of the segment L_j, γ is Euler's constant and λ's are elements of the boundary condition matrix Λ.

The coefficients in this expansion depend on topological and spectral properties of the hybrid space and its components. The expansion is recursive, but, unfortunately, we are not able to solve the recursion. Nevertheless, with the help of the obtained formulas one can compute an arbitrarily large number of the coefficients and find a general form of terms having some special structure.

The inverse spectral theory is presented in Chapter 5. By inverse spectral theory we mean the following problem: assume that we have an asymptotic expansion of the squared resolvent of a Laplace operator on some hybrid manifold. What kind of geometric and topological information about the manifold can we extract from this expansion? The answer to this question is given by the following theorems:

Theorem 3. *Consider the expansion of the trace of the square of the resolvent of a Laplace operator on a hybrid manifold. The knowledge of TrR^2 determines:*

- *whether this manifold is hybrid or "normal";*
- *the sum of the volumes of all manifolds taking part in the construction;*
- *the sum of the Euler characteristics of all manifolds;*

- *the number of segments used in this hybrid manifold;*
- *the sum of the lengths of these segments;*
- *the Euler characteristic of the hybrid manifold.*

One can also obtain information about the matrix of boundary conditions if we have some additional information about the initial system:

Theorem 4. *Consider the z-pseudoasymptotic expansion of the trace of the square of the resolvent expansion. If we assume that we know the heat kernel coefficients for all manifolds composing the hybrid manifold, and that the coefficients $\lambda_{i+N,i+N}$ are mutually distinct and nonzero, we can find the diagonal elements of the matrix of boundary conditions Λ and the absolute values of its non-diagonal elements up to permutation.*

In Chapter 6, we study two degenerate cases of hybrid manifolds: on the one hand a quantum graph and on the other hand a system of manifolds glued together at some points (that is to say, we glue our manifolds with segments of length zero). The direct and inverse spectral theory are considered in these cases. We show that in these degenerate cases we obtain additional information for the inverse spectral problem. In fact, in the case of a quantum graph one has

Theorem 5. *From the expansion of the trace of the square of the resolvent of a Laplace operator on a quantum graph it is possible to find the number of edges of the quantum graph, the sum of the lengths of all segments and the matrix Λ of boundary conditions up to unitary transformation.*

And in the case of a hybrid manifold without segments we have

Theorem 6. *From the expansion of $\operatorname{Tr} R^2$ for a Laplace operator on a system of N glued manifolds it is possible to find the number of manifolds, the sum of the volumes of all manifolds and the matrix Λ of boundary conditions up to a unitary transformation.*

Chapter 2

Hybrid manifolds

2.1 Definition of a hybrid manifold

The main object to be investigated is a so-called hybrid manifold. Let us describe it from the topological point of view. Consider a set of M 2-dimensional compact Riemannian manifolds $M_1, ..., M_M$ and a set of N segments $L_1, ..., L_N$. On each manifold M_i we fix some points q_{is}, $s = 1, \dots, \mu_i$, $\mu_i > 0$, $i = 1, \dots, M$.

First of all we consider the disjoint union of all initial elements: $M_1 \sqcup \dots \sqcup M_m \sqcup L_1 \sqcup \dots \sqcup L_N$. Then we construct a one-to-one correspondence between the set of end points of all segments and the set of points $q_{is}, s = 1, \dots, \mu_i, i = 1, \dots, M$. The following natural condition on the number of elements must be satisfied

$$\sum_{i=1}^{M} \mu_i = 2N.$$

Finally, according to this correspondence, we glue each end of each segment to the corresponding point on one of the manifolds. The resulting object is a topological space. We assume it to be path connected which immediately implies that $N \geqslant M - 1$. One can also define a metric structure on this space, but a metric tensor cannot be defined. It reflects the fact that this object is not a manifold in the standard sense, but consists of parts of different dimensions. Nevertheless we can give a

Definition 2.1.1. The topological space obtained by gluing the initial manifolds and segments as described is called a *hybrid manifold*.

2.2 The Euler characteristic of hybrid manifolds

As for any topological space, we can define the Euler characteristic for a hybrid manifold. Let us recall some facts and definitions from algebraic topology (see, for example Spanier [1981]).

Definition 2.2.1. Let X be a topological space. Denote by β_n the $n-$th Betti number of X, i.e. the dimension of the n-th real homology group, $H_n(X)$, of X. Assume that β_n is finite for all n. Then the **Euler characteristic** $\chi(X)$ is the alternating sum of the Betti numbers $\chi(X) = \sum_{n=0} (-1)^n \beta_n$.

Proposition 2.2.2. *If two topological spaces X and Y have the same homotopy type then their homology groups are isomorphic, $H_n(X) \cong H_n(Y)$, for all $n \geqslant 0$.*

Proposition 2.2.3. *Let X be a topological space and $A, B \subset X$ be such that $X = \mathrm{int}A \cup \mathrm{int}B$ then there is an exact sequence (the* ***Mayer - Vietoris sequence****)*

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to \cdots .$$

Proposition 2.2.4. *For an oriented surface M_g of genus g one has*

$$H_k(M_g, \mathbb{R}) = \begin{cases} \mathbb{R}, & \text{if } k = 0, 2, \\ \underbrace{\mathbb{R} \oplus \cdots \oplus \mathbb{R}}_{2g}, & \text{if } k = 1, \\ 0, & k > 2. \end{cases} \tag{2.1}$$

It is interesting to find a relation between the Euler characteristic of a hybrid manifold and that of the surface which one obtains by replacing all segments in the hybrid manifold by thin tubes. The answer is provided by the following theorems.

Theorem 2.2.5. *The Euler characteristic of a hybrid manifold obtained from M manifolds M_i and N segments L_j is equal to $\sum_{i=1}^{M} \chi(M_i) - N$.*

The proof of the theorem requires some additional lemmas. In what follows, we will say that M surfaces M_i, $i = 1, \ldots, M$, connected with the help of $M - 1$ segments, form an open simple chain if for $i = 1, \ldots, M - 1$, M_i is connected to M_{i+1} by exactly one segment.

Lemma 2.2.6. *M surfaces M_i of genus g_i connected with the help of $M-1$ segments in an open simple chain form a hybrid manifold whose Euler characteristic is equal to $1+M-\sum 2g_i$.*

Proof. We start with the case of a hybrid manifold X consisting of two surfaces and one segment. This topological space is the union of two parts A and B, where A is the union of the first surface and the segment, and B is the union of the second surface and the segment. Both A and B are homotopy equivalent to a surface without segment, and the intersection $D = A \cap B$ is homotopy equivalent to a point. So the Mayer-Vietoris sequence can be written as

$$0 \to H_2(D) \to H_2(A) \oplus H_2(B) \to H_2(X) \to H_1(D) \to H_1(A) \oplus H_1(B)$$
$$\to H_1(X) \to H_0(D) \to H_0(A) \oplus H_0(B) \to H_0(X) \to 0.$$

Using the well-known facts that $H_0(Y, \mathbb{R}) = \mathbb{R}$ if Y is path connected, $H_k(D) = 0,\ k > 0$, and the Propositions above, we can rewrite it, denoting by g_a and g_b the genus of the surface A and B respectively, as

$$0 \to 0 \to \mathbb{R} \oplus \mathbb{R} \to H_2(X) \to 0 \to \mathbb{R}^{2g_a} \oplus \mathbb{R}^{2g_b} \to$$
$$H_1(X) \to \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \to 0.$$

The exactness of this sequence implies that

$$H_k(X, \mathbb{R}) = \begin{cases} \mathbb{R}, & \text{if } k = 0, \\ \mathbb{R}^{2g_a+2g_b}, & \text{if } k = 1, \\ \mathbb{R}^2, & \text{if } k = 2, \\ 0, & \text{if } k > 2. \end{cases} \tag{2.2}$$

In the same way one can show that the procedure of "gluing" one surface of genus g with the help of one segment to an open simple chain of surfaces, denoted by Y, in such a way that the resulting object is also an open simple chain, denoted by X, gives us the following:

$$H_0(X, \mathbb{R}) = H_0(Y, \mathbb{R}) = \mathbb{R}; \quad H_1(X, \mathbb{R}) = H_1(Y, \mathbb{R}) \oplus \mathbb{R}^{2g},$$
$$H_2(X, \mathbb{R}) = H_2(Y, \mathbb{R}) \oplus \mathbb{R}.$$

By induction, we find that for an open simple chain of M manifolds the Euler characteristic is equal to

$$\chi = 1 - \sum 2g_i + M \tag{2.3}$$

□

Lemma 2.2.7. *Gluing a segment to a hybrid manifold reduces the Euler characteristic by 1.*

Proof. We will use again the Mayer-Vietoris exact sequence. We denote the original hybrid manifold by Y, the segment by D, and the result of gluing by X. So, topologically $X = Y \cup D$, and the intersection $Y \cap D$ consists of two gluing points. The exact sequence is

$$0 \to H_2(Y \cup D) \to H_2(Y) \oplus H_2(D) \to H_2(X) \to H_1(Y \cup D) \to H_1(Y) \oplus H_1(D)$$

$$\to H_1(X) \to H_0(Y \cup D) \to H_0(Y) \oplus H_0(D) \to H_0(X) \to 0,$$

or

$$0 \to 0 \oplus 0 \to H_2(Y) \oplus 0 \to H_2(X) \to 0 \to H_1 \oplus 0$$
$$\to H_1(X) \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \to 0.$$

Now we easily find

$$H_0(X, \mathbb{R}) = H_0(Y, \mathbb{R}); \quad H_1(X, \mathbb{R}) = H_1(Y, \mathbb{R}) \oplus \mathbb{R},$$
$$H_2(X, \mathbb{R}) = H_2(Y, \mathbb{R}).$$

and direct calculation of the Euler characteristic finishes the proof of the lemma. □

Proof. This is now straightforward: the hybrid manifold composed of M manifolds and N segments can be treated as a chain of M manifolds connected by $M - 1$ segments and $N - M + 1$ additional segments. Then by Lemmas 1 and 2,

$$\chi = 1 + M - 2 \sum g_i - (N - M + 1) = 2M - 2 \sum g_i - N = \sum \chi_i - N.$$

□

The following result allows us to compare the Euler characteristic of a hybrid manifold H with the Euler characteristic of the surface S obtained by gluing the manifolds constituting H with thin tubes instead of segments.

Theorem 2.2.8. *The Euler characteristic of the surface S composed from M manifolds M_i with Euler characteristics χ_i and N tubes is equal to $\sum_i \chi_i - 2N$. In other words, $\chi(S) = \chi(H) - N$.*

Proof. Let us compute the genus of this surface. As before we treat this object as a chain of M manifolds connected with $M-1$ tubes and then "add" the remaining $N-M+1$ tubes. The chain of manifolds has genus $\sum g_i$ and each additional tube increases the genus by 1, because adding a tube is just gluing a handelbody to our surface. Finally the genus of the system is $\sum g_i + N - M + 1$. Using the relation between genus and Euler characteristic, we find

$$\chi = 2 - 2\sum g_i - 2(N - M + 1) = 2M - 2\sum g_i - 2N = \sum \chi_i - 2N.$$

□

2.3 Laplace operator on the hybrid manifold

Since the constructed object is not really a manifold, we cannot define Laplace operators in the standard way. Nevertheless, we can define a self-adjoint analogue of Laplace operators for hybrid manifolds. The idea (Pavlov [1987]) is to take first a direct sum of Laplace operators on the initial parts without taking any interaction into account (it means that we restrict somehow the domain of the operators); this gives us a symmetric operator. Then we can extend it to a self-adjoint one, depending on the choice of "boundary conditions" at the gluing points.

We start with the definition of the Laplace operator on manifolds. Consider on each M_i the symmetric operator in $L^2(M_i)$ with domain $C_0^\infty(M_i)$ defined in local coordinates by

$$-(\sqrt{g_i(x)})^{-1}\partial_\mu(\sqrt{g_i(x)}g_i^{\mu\nu}(x))\partial_\nu,$$

where $g_i^{\mu\nu}(x)$ is the inverse of the Riemannian metric $g_{i,\mu\nu}(x)$ on M_i and $g_i(x)$ is the determinant of $g_{i,\mu\nu}(x)$. The closure of each of these operators in $L^2(M_i)$ is a self-adjoint Laplace operator Δ_{M_i} with domain $\mathcal{D}(\Delta_{M_i})$, the second Sobolev space of M_i (a function belongs to this space, if in each local chart, the function as well as its first and second distributional derivatives are square integrable). Note that we assume $\dim M_i = 2$, hence the Sobolev Embedding Theorem implies that

$$\mathcal{D}(\Delta_{M_i}) \subset C^0(M_i).$$

Denote by D_i restriction of Δ_{M_i} to the domain

$$\mathcal{D}(D_i) = \{f \in \mathcal{D}(\Delta_{M_i}) : f(q_{is}) = 0, \quad q_{is} \in M_i, \quad s = 1, \dots, \mu_i\}. \qquad (2.4)$$

D_i is a symmetric operator in $L^2(M_i)$ with deficiency indices (μ_i, μ_i) (recall that μ_i is the number of marked points on M_i). This fact follows from Lemma 4.2.3 below, see also [Geĭler et al., 1995, Theorem 3], [Brüning and Geyler, 2003, Lemma 4].

We parametrize the segments L_j by $\{x_j \in \mathbb{R} : x_j \in [0, l_j]\}$, where $j = 1, \dots, N$ and denote by D_j^s the closure in $L^2(L_j)$ of the operator $-\frac{d^2}{dx_j^2}$ defined on $C_0^\infty(L_j)$. Each operator D_j^s is a symmetric operator with deficiency indices $(2, 2)$.

The operator D defined by

$$D = D_1 \oplus \cdots \oplus D_M \oplus D_1^s \oplus \cdots \oplus D_N^s \tag{2.5}$$

is a symmetric operator in $L^2(M_i) \oplus \cdots \oplus L^2(L_j) \oplus \cdots \oplus L^2(L_N)$ with deficiency indices $(4N, 4N)$.

By considering different self-adjoint extensions of this symmetric operator we will obtain a description of different types of interactions between the manifolds and the segments. There exist different ways to define self-adjoint operators which will describe some non-trivial interaction on the hybrid space, i.e. will satisfy some boundary condition at the gluing points.

Definition 2.3.1. A Laplace operator H on a hybrid manifold is a self-adjoint extension of the operator D.

Our aim is to construct suitable self-adjoint extensions of D to perform spectral theory on the hybrid manifold. We will use Krein's extension theory which will allow us to describe all self-adjoint extensions of a symmetric operator. This is reasonable, because we cannot say which of those extensions (i.e. which boundary condition) is preferable. In the somewhat similar case of the manifold obtained by replacing the edges by tubes and vertices by balls in a quantum graph, it is known that the relative velocity of shrinking tubes to segments can influence the boundary condition at the vertices for the original graph (see [Exner and Post, 2005, Theorems 5.2, 6.2, 7.1, 8.1]).

Chapter 3

Self-adjoint extensions of symmetric operators

In this section we will give a short introduction to the theory of self-adjoint extension of symmetric operators. Of course, the subject is too rich to be covered in all detail, so we restrict ourselves to give the basic definitions and facts.

3.1 Definitions and preliminaries

We state some basics fact which can be found for example in [Reed and Simon, 1980, Chapter X]. Let $\mathcal{H}$ be a Hilbert space with scalar product $\langle\cdot,\cdot\rangle \equiv \langle\cdot,\cdot\rangle_{\mathcal{H}}$ and let S be a linear operator with domain $\mathcal{D}(S)$. The set $\mathrm{gr}\,(S) := \{(x, Sx),\ x \in \mathcal{D}(S)\} \subset \mathcal{H}\times\mathcal{H}$ is called the *graph* of S. The operator S is called *closed* if its graph is a closed set. An operator S_1 is an *extension* of S if $\mathrm{gr}\,(S) \subset \mathrm{gr}\,(S_1)$, i.e. if $\mathcal{D}(S_1) \supset \mathcal{D}(S)$ and $S_1 x = Sx,\ \forall x \in \mathcal{D}(S)$. We denote by $\sigma(S)$ and $\rho(S)$ the spectrum and the resolvent set of S, respectively.

Assume that the domain of S is dense in $\mathcal{H}$. Set

$$\mathcal{D}(S^*) := \{x \in \mathcal{H} : \exists y \in \mathcal{H}\ \langle x, Sv\rangle = \langle y, v\rangle\ \forall v \in \mathcal{D}(S)\}.$$

In this notation, for each $x \in \mathcal{D}(S^*)$, set $S^*x = y$. The operator S^* defined in this way on the domain $\mathcal{D}(S^*)$ is called the *adjoint* of S. S is called *symmetric* if S^* is an extension of S and is called *self-adjoint* if $S = S^*$.

For a symmetric operator S and $z \in \mathbb{C} \setminus \mathbb{R}$ set $N_z = \mathrm{Ker}(S^* - z)$; these sets are called the *deficiency subspaces*. It is known [Reed and Simon, 1980, Theorem X.1] that the dimension of N_z does not vary as z lies in the upper half-plane or in the lower half-plane. Put $N_\pm = N_{\pm i}$. The numbers $n_\pm = \dim N_\pm$ are called the *deficiency indices* of S (they can also be infinite).

3.2 von Neumann theory of self-adjoint extensions

The classical theory of self-adjoint extensions goes back to von Neumann and is presented in a number of textbooks on functional analysis. We give here only the main facts, the details can be found, for example, in [Akhiezer and Glazman, 1993, Chapter 8] or in [Reed and Simon, 1980, Chapter X].

Lemma 3.2.1 (Theorem on page 98 in Akhiezer and Glazman [1993])**.** *For a densely defined closed symmetric operator S, the domain of S^* admits a decomposition $\mathcal{D}(S^*) = \mathcal{D}(S) \oplus N_z \oplus N_{\bar{z}}$, where z is any non-real number.*

Proof. Clearly, we have the inclusion

$$\mathcal{D}(S) \oplus N_z \oplus N_{\bar{z}} \subset \mathcal{D}(S^*).$$

We will show that, conversely, each $x \in \mathcal{D}(S^*)$ can be represented in the form

$$x = x_0 + x_z + x_{\bar{z}},$$

where $x_0 \in \mathcal{D}(S)$, $x_z \in N_z$ and $x_{\bar{z}} \in N_{\bar{z}}$. This representation would imply

$$S^*x = Sx_0 + zx_z + \bar{z}x_{\bar{z}}.$$

Let $x \in \mathcal{D}(S^*)$. We decompose $S^*x - zx$ into its components in the orthogonal subspaces $\operatorname{Ran}(S - z)$ and $N_{\bar{z}}$:

$$S^*x - zx = (Sx_0 - zx_0) + (\bar{z} - z)x_{\bar{z}}.$$

By definition $S^*x_{\bar{z}} = \bar{z}x_{\bar{z}}$ and

$$S^*(x - x_0 - x_{\bar{z}}) = z(x - x_0 - x_{\bar{z}}).$$

It means that $x - x_0 - x_{\bar{z}} = x_z \in N_z$ and

$$x = x_0 + x_z + x_{\bar{z}}.$$

To prove the uniqueness of this representation we suppose that

$$x_0 + x_z + x_{\bar{z}} = 0,$$

apply S^* to both sides of this equation, multiply by z and substract one from the other:

$$Sx_0 - zx_0 + (\bar{z} - z)x_{\bar{z}} = 0.$$

From the orthogonality of the summands we have $(\bar{z} - z)x_{\bar{z}} = 0$. In the same way we find $(\bar{z} - z)x_z = 0$ and

$$x_0 = x_z = x_{\bar{z}} = 0.$$

□

Lemma 3.2.2. *Let S be a closed densely defined symmetric operator. For any self-adjoint extension $\tilde{S}$ of S and any $z \in \rho(\tilde{S})$ there holds*

$$\mathcal{D}(S^*) = \mathcal{D}(\tilde{S}) \oplus N_z.$$

Proof. Let $x \in \mathcal{D}(S^*)$, and set $x_0 := (\tilde{S}-z)^{-1}(S^*-z)x$. Clearly, $x_0 \in \mathcal{D}(\tilde{S})$. For $y = x - x_0$ one has

$$\begin{aligned}(S^*-z)y &= (S^*-z)x - (S^*-z)(\tilde{S}-z)^{-1}(S^*-z)x \\ &= (S^*-z)x - (\tilde{S}-z)(\tilde{S}-z)^{-1}(S^*-z)x = 0,\end{aligned}$$

therefore, $y \in N_z$.

Now assume that for some $z \in \rho(\tilde{S})$ one has $x_0 + y_0 = x_1 + y_1$ for some $x_0, x_1 \in \mathcal{D}(\tilde{S})$ and $y_0, y_1 \in N_z$. Then $x_0 - x_1 = y_1 - y_0 \in N_z$ and we have

$$(\tilde{S}-z)(x_0-x_1) = (S^*-z)(x_0-x_1) = 0.$$

As $(\tilde{S}-z)$ is invertible, one has $x_0 = x_1$ and $y_0 = y_1$. □

Proposition 3.2.3 (Theorem X.2 in Reed and Simon [1980])**.** *Let S be a closed symmetric operator. The closed symmetric extensions of S are in one-to-one correspondence with the set of partial isometries of N_+ into N_-. If U is such an isometry with initial space $I(U) \subset N_+$, then the corresponding closed symmetric extension $\tilde{S}$ has domain*

$$\mathcal{D}(\tilde{S}) = \{x + x_+ + Ux_+ : x \in \mathcal{D}(S), x_+ \in I(U)\},$$

and

$$\tilde{S}(x + x_+ + Ux_+) = Sx + ix_+ - iUx_+.$$

The Cayley transform of S is the unique partially defined linear operator C_S acting from $\operatorname{Ran}(S+i)$ to $\operatorname{Ran}(S-i)$ defined by the equality

$$C_S(S+i)(x) = (S-i)(x), \quad x \in \mathcal{D}(S).$$

The operator C_S is isometric on its domain. Clearly, S is self-adjoint iff its Cayley transform C_S is unitary (global isometry of N_+ into N_-), and S has self-adjoint extensions iff C_S has unitary extensions.

Proposition 3.2.4. *The self-adjoint extensions of a closed symmetric operator S are in one-to-one correspondence with the unitary operators from N_+ to N_-. The domain of a self-adjoint extension $\tilde{S}$ corresponding to a unitary operator U is $\mathcal{D}(\tilde{S}) = \{x + x_+ + Ux_+ : x \in \mathcal{D}(S), x_+ \in N_+\}$. Moreover, $U = \big(C_{\tilde{S}}|_{N_-}\big)^{-1}$. For $x = x_0 + x_+ + Ux_+$ with $x_0 \in \mathcal{D}(S)$ and $x_+ \in N_+$ there holds $\tilde{S}x = Sx_0 + ix_+ - iUx_+$.*

The von Neumann theory gives a complete description of all self-adjoint extensions but the objects used are difficult to construct. We present below some alternative approaches to self-adjoint extensions.

3.3 Basic facts on linear relations

In many situations it is necessary to generalize the definition of a linear operator in order to admit multivalued maps. Such generalizations are usually called linear relations. Let us recall some basic facts in this context.

Any linear subspace of $\mathcal{H} \oplus \mathcal{H}$ will be called a *linear relation* on $\mathcal{H}$. For a linear relation Λ on $\mathcal{H}$ the sets

$$\begin{aligned} \mathcal{D}(\Lambda) &= \{x \in \mathcal{H} : \exists y \in \mathcal{H} \text{ with } (x,y) \in \Lambda)\}, \\ \operatorname{Ran}(\Lambda) &= \{y \in \mathcal{H} : \exists x \in \mathcal{H} \text{ with } (x,y) \in \Lambda)\}, \\ \operatorname{Ker}(\Lambda) &= \{x \in \mathcal{H} : (x,0) \in \Lambda\} \end{aligned}$$

will be called the *domain*, the *range* and the *kernel* of Λ, respectively. The linear relations

$$\begin{aligned} \Lambda^{-1} &= \{(x,y) :\ (y,x) \in \Lambda\}, \\ \Lambda^* &= \{(x_1,x_2) : \langle x_1, y_2\rangle = \langle x_2, y_1\rangle \quad \forall (y_1,y_2) \in \Lambda\} \end{aligned}$$

are called *inverse* and *adjoint* to Λ, respectively. For $\alpha \in \mathbb{C}$ we put

$$\alpha\Lambda = \{(x, \alpha y) :\ (x,y) \in \Lambda\}.$$

For two linear relations $\Lambda', \Lambda'' \subset \mathcal{H} \oplus \mathcal{H}$ one can define their *sum*

$$\Lambda' + \Lambda'' = \{(x, y' + y''),\ (x,y') \in \Lambda',\ (x,y'') \in \Lambda''\};$$

clearly, one has $\mathcal{D}(\Lambda' + \Lambda'') = \mathcal{D}(\Lambda') \cap \mathcal{D}(\Lambda'')$. The graph of any linear operator L on $\mathcal{H}$ is a linear relation, which we denote by gr L. Clearly, if L is invertible, then gr $L^{-1} = (\text{gr } L)^{-1}$. For arbitrary linear operators L', L'' one has gr L' + gr L'' = gr $(L' + L'')$. Therefore, the set of linear operators is naturally embedded into the set of linear relations. In analogy with the notion of closed operators, which is important in spectral theory, we can also define closed linear relations, i.e. relations which are closed linear subspaces in $\mathcal{H} \oplus \mathcal{H}$. In what follows we consider mostly closed linear relations.

A linear relation Λ on $\mathcal{H}$ is called *symmetric* if $\Lambda \subset \Lambda^*$ and is called *self-adjoint* if $\Lambda = \Lambda^*$. A linear operator L in $\mathcal{H}$ is symmetric (respectively, self-adjoint), iff its graph is a symmetric (respectively, self-adjoint) linear relation.

Proposition 3.3.1 (Theorem 3.1.4 in Gorbachuk and Gorbachuk [1984]). *There exists a one-to-one correspondence between self-adjoint linear relations in $\mathcal{H}$ and unitary operators acting on $\mathcal{H}$. For a given linear relation Λ in $\mathcal{H}$*

there is a unique unitary operator C_Λ in $\mathcal{H}$ (called the Cayley transform *of Λ) such that the condition $(x_1, x_2) \in \Lambda$ is equivalent to $(C_\Lambda - I)x_2 + i(C_\Lambda + I)x_1 = 0$. Conversely, this condition defines a self-adjoint linear relation for any unitary operator C_Λ.*

Clearly, the Cayley transform of a linear relation generalizes the notion of the Cayley transform for linear operators. Indeed, assume that the linear relation Λ is given by the graph of a closed symmetric operator S. Thus the condition $(x_1, x_2) \in \Lambda$ means that x_1 is in the domain of S and that $x_2 = Sx_1$. Moreover, the Cayley transform of Λ defined in the previous proposition satisfies $(C_\Lambda - I)x_2 + i(C_\Lambda + I)x_1 = 0$, which can be written as

$$(C_\Lambda - I)Sx_1 = -i(C_\Lambda + I)x_1,$$

or

$$C_\Lambda(S + i)x_1 = (S - i)x_1, \quad x_1 \in \mathcal{D}(S).$$

This is precisely the same relation which defines the Cayley transform C_S of S, hence $C_\Lambda = C_S$.

3.4 Abstract boundary conditions

It is well known that, in a functional analytic sense, the definition of an elliptic operator in a domain with boundary involves boundary conditions. A similar approach can be used in more abstract situations, namely, for the description of self-adjoint extensions with the help of abstract boundary values and the symplectic language.

Definition 3.4.1. Let S be a densely defined closed symmetric linear operator acting on a Hilbert space $\mathcal{H}$. Let Γ_1, Γ_2 be two linear mappings from $\mathcal{D}(S^*)$ into a Hilbert space $\mathcal{G}$. The triple $(\mathcal{G}, \Gamma_1, \Gamma_2)$ is called a *boundary value space* for S if

- for all $x, y \in \mathcal{D}(S^*)$

$$\langle x, S^*y\rangle - \langle S^*x, y\rangle = \langle \Gamma_1 x, \Gamma_2 y\rangle - \langle \Gamma_2 x, \Gamma_1 y\rangle, \tag{3.1}$$

- for any $u, v \in \mathcal{G}$ there exists $x \in \mathcal{D}(S^*)$ such that

$$\Gamma_1 x = u, \quad \Gamma_2 x = v. \tag{3.2}$$

The construction of the boundary value space for a given operator S is not a trivial problem. There is a standard procedure (described in the following proposition) of such a construction, but the boundary value space which we get is neither unique nor of practical use. The "right" choice usually comes from the nature of the problem under consideration.

Proposition 3.4.2 (Theorem 3.1.5 in Gorbachuk and Gorbachuk [1984]). *If S has equal deficiency indices (n, n) then there exists a boundary value space for this operator with $\dim \mathcal{G} = n$.*

Proof. We give a proof for the sake of completeness. As we have already noticed, $\mathcal{D}(S^*) = \mathcal{D}(S) \oplus N_- \oplus N_+$, where the decomposition is orthogonal relative to the graph inner product of $\mathcal{D}(S^*)$:

$$\langle x, y \rangle_{graph} = \langle x, y \rangle + \langle S^* x, S^* y \rangle.$$

Denote by P_- and P_+ the orthogonal projectors of $\mathcal{D}(S^*)$ on N_- and N_+ respectively, with respect to the graph inner product. Since $\dim N_- = \dim N_+$ there exists an isometric mapping U from N_+ to N_-. Define $\mathcal{G} = N_-$ with metric induced from scalar product on $\mathcal{H}$, and $\Gamma_1 = -iP_- + iUP_+$, $\Gamma_2 = P_- + UP_+$. Let us check that the triple $(\mathcal{G}, \Gamma_1, \Gamma_2)$ is a boundary value space for S.

In fact, if $x, y \in \mathcal{D}(S^*)$ then $x = x_0 + P_- x + P_+ x$ and $y = y_0 + P_- y + P_+ y$, where $x_0, y_0 \in \mathcal{D}(S)$. Taking into account the fact that S is symmetric and noting the equalities $S^* P_+ = iP_+$ and $S^* P_- = -iP_-$, we obtain

$$\langle x, S^* y \rangle - \langle S^* x, y \rangle = 2i \left(\langle P_+ x, P_+ y \rangle - \langle P_- x, P_- y \rangle \right).$$

Due to isometry of U we have

$$\langle \Gamma_1 x, \Gamma_2 y \rangle - \langle \Gamma_2 x, \Gamma_1 y \rangle = 2i \left(\langle P_+ x, P_+ y \rangle - \langle P_- x, P_- y \rangle \right),$$

and we see that this triple satisfies the condition (3.4.1).

If $u, v \in \mathcal{G}$ we choose $x \in \mathcal{D}(S^*)$ such that $x = x_0 + x_- + x_+$, where x_0 is an arbitrary vector from $\mathcal{D}(S)$, $x_- = \frac{1}{2i}(iv - u) \in N_-$ and $x_+ = \frac{1}{2i} U^{-1}(iv + u) \in N_+$. One can easily see that $\Gamma_1 x = u$ and $\Gamma_2 x = v$, which finishes the proof. □

Proposition 3.4.3 (Theorem 3.1.6 in Gorbachuk and Gorbachuk [1984]). *Let S be a densely defined closed symmetric operator with equal deficiency indices and let $(\mathcal{G}, \Gamma_1, \Gamma_2)$ be a boundary value space. The self-adjoint extensions of S are in one-to-one correspondence with self-adjoint linear relations in $\mathcal{G}$. The self-adjoint extension S^Λ corresponding to a self-adjoint linear relation Λ is the restriction of the adjoint operator S^* to the domain*

$$\mathcal{D}(S^\Lambda) = \{x \in \mathcal{D}(S^*) : (\Gamma_1 x, \Gamma_2 x) \in \Lambda\}. \tag{3.3}$$

3.5 Krein formalism

A very powerful tool in the spectral theory of self-adjoint extensions is the famous Krein formula, which we present in this subsection.

Suppose that S is a densely defined closed symmetric linear operator with equal deficiency indices and that S_0 is a fixed self-adjoint extension of S. For $z \in \rho(S_0)$ denote the resolvent of S_0 by $R_0(z) = (S_0 - z)^{-1}$. We need some additional constructions to describe all self-adjoint extensions of S.

Definition 3.5.1. *A Krein γ-field* γ of the pair (S, S_0) is an operator-valued function from $\rho(S_0)$ into the Banach space of linear bounded operators from $\mathcal{G}$ to $\mathcal{H}$, γ: $\rho(S_0) \to L(\mathcal{G}, \mathcal{H})$, such that

- $\gamma(z)$ is a linear topological isomorphism from $\mathcal{G}$ to the deficiency space N_z of the operator S,
- for any $z_1, z_2 \in \rho(S_0)$ there holds

$$\frac{\gamma(z_1) - \gamma(z_2)}{z_1 - z_2} = R_0(z_1)\gamma(z_2). \tag{3.4}$$

A *Krein Q-function* corresponding to the pair (S, S_0) and a γ-field $\gamma(z)$ is a map from $\rho(S_0)$ into $L(\mathcal{G}, \mathcal{G})$ with the property

$$Q(z_1) - (Q(z_2))^* = (z_1 - \overline{z_2})(\gamma(z_2))^*\gamma(z_1), \quad z_1, z_2 \in \rho(S_0). \tag{3.5}$$

The γ-field and the Q-function are not defined uniquely. To see this, note that (3.4) can be rewritten in the following way:

$$\gamma(z_1) = \gamma(z_2) + (z_1 - z_2)R_0(z_1)\gamma(z_2). \tag{3.6}$$

Therefore, if we define $\gamma(z_2)$ as an arbitrary isomorphism between $\mathcal{G}$ and N_{z_2}, then $\gamma(z)$ extends uniquely to $\rho(S_0)$. Moreover, as the resolvent is an analytic function, formula (3.6) also shows us that $\gamma(z)$ is holomorphic in $\rho(S_0)$.

The Q-function is defined up to a bounded self-adjoint summand. Taking $z_2 = z$ and $z_1 = \overline{z}$ in (3.5) we conclude that $Q(z) = (Q(\overline{z}))^*$ for any $z \in \rho(S_0)$. Rewriting (3.5) as

$$Q(z_1) = Q(z_2) + (z_1 - z_2)(\gamma(\bar{z}_2))^*\gamma(z_1),$$

we conclude that $Q(z)$ is holomorphic in $\rho(S_0)$.

Now we are ready to formulate the main tool of this work: the Krein formula. This formula was first obtained in [Kreĭn and Langer, 1971, Theorem 5.1] and further discussed with some variations in [Derkach and Malamud, 1991, Section 2]. Of course this is not an exhaustive list of works dealing with this formalism.

Theorem 3.5.2 (Krein resolvent formula)**.** *Let S be a symmetric operator and let S_0 be a fixed self-adjoint extension. There exists a one-to-one correspondence between self-adjoint linear relations Λ in $\mathcal{G}$ and resolvents of self-adjoint extensions S^Λ of S. More precisely, for any self-adjoint relation Λ and any $z \in \rho(S^\Lambda) \cap \rho(S_0)$ the linear relation $[Q(z) - \Lambda]^{-1}$ is the graph of a certain bounded linear operator, and*

$$R^\Lambda(z) = R_0(z) - \gamma(z)[Q(z) - \Lambda]^{-1}(\gamma(\bar{z}))^*, \tag{3.7}$$

where $R^\Lambda(z)$ is the resolvent of S^Λ. The operators S_0 and S^Λ are disjoint (i.e. $\mathcal{D}(S^\Lambda) \cap \mathcal{D}(S_0) = \mathcal{D}(S)$) iff Λ is a self-adjoint operator.

We will not give here the rather technical proof of this theorem. Instead, in the next section we will prove a slightly modified version of this result (see Theorem 3.6.1 below).

3.6 Boundary value space and Krein formula

The choice of γ-field and Q-function in the previous subsection contains a lot of arbitrariness. It is useful to relate this choice with a boundary value space for S, and we are going to describe now this relationship, see [Derkach and Malamud, 1991, Section 1] for details. Assume that we have already chosen a boundary value space $(\mathcal{G}, \Gamma_1, \Gamma_2)$ of S. Clearly, the restriction of S^* to the set of elements x satisfying $\Gamma_1 x = 0$ is a self-adjoint extension of S; denote it by S_0. Actually, for any self-adjoint extension of S there is a boundary value space such that the extension is defined by the above equality.

For $z \in \mathbb{C} \setminus \mathbb{R}$ we denote by $\Gamma_1(z)$ the restriction of Γ_1 to the deficiency subspace N_z. Then

$$\gamma(z) = (\Gamma_1(z))^{-1} \tag{3.8}$$

is a bijective bounded operator from $\mathcal{G}$ to the deficiency space N_z of the operator S, and satisfies condition (3.4). Moreover, this map has an analytic continuation to $\rho(S_0)$ and, therefore, is a Krein γ-field for the pair (S, S_0). The operator

$$Q(z) = \Gamma_2 \gamma(z) \tag{3.9}$$

is a Q-function of the pair (S, S_0) corresponding to γ.

Theorem 3.6.1 (Proposition 2 in Derkach and Malamud [1991])**.** *Let S be a symmetric operator with equal deficiency indices. Fix a boundary value space $(\mathcal{G}, \Gamma_1, \Gamma_2)$ of S and consider the γ-field and the Q-function given by* (3.8) *and* (3.9) *respectively. Denote by S_0 the self-adjoint extension of S defined by $\Gamma_1 = 0$. Let Λ be a self-adjoint linear relation in $\mathcal{G}$.Then the resolvent of the operator S^Λ defined in* (3.3) *is given by:*

$$R^\Lambda(z) = R_0(z) - \gamma(z)[Q(z) - \Lambda]^{-1}(\gamma(\bar{z}))^*, \quad z \in \rho(S_0) \cap \rho(S^\Lambda). \tag{3.10}$$

We will prove this theorem along the lines kindly communicated to us by K. Pankrashkin Pankrashkin [2005]. We begin with the following lemma:

Lemma 3.6.2. *In the same notation as in Theorem 3.6.1, for any $z \in \rho(S_0)$ we have*

1. *For any $x \in \mathcal{D}(S_0)$ there holds $(\gamma(\bar{z}))^*(S_0 - z)x = \Gamma_2 x$.*

2. $\operatorname{Ker}(S^\Lambda - z) = \gamma(z) \operatorname{Ker}(Q(z) - \Lambda)$.

Proof. For the first assertion, we notice that for any $y \in \mathcal{G}$ we have

$$\begin{aligned} \langle y, (\gamma(\bar{z}))^*(S_0 - z)x\rangle &= \langle \gamma(\bar{z})y, (S_0 - z)x\rangle = \langle \gamma(\bar{z})y, S^* x\rangle - z\langle \gamma(\bar{z})y, x\rangle \\ &= \langle S^*\gamma(\bar{z})y, x\rangle - z\langle \gamma(\bar{z})y, x\rangle + \langle \Gamma_1\gamma(\bar{z})y, \Gamma_2 x\rangle - \langle \Gamma_2\gamma(\bar{z})y, \Gamma_1 x\rangle \\ &= \langle (S^* - \bar{z})\gamma(\bar{z})y, x\rangle + \langle y, \Gamma_2 x\rangle = \langle y, \Gamma_2 x\rangle, \end{aligned}$$

which proves that $\Gamma_2 x = (\gamma(\bar{z}))^*(S_0 - z)x$.

To show the second assertion, we proceed as follows. Assume that x is an element of $\operatorname{Ker}\big(\Lambda - Q(z)\big)$, which means that there exists $y \in \mathcal{G}$ such that $(x, y) \in \Lambda$ and $y - Q(z)x = 0$. This means that $(x, Q(z)x) \in \Lambda$. Consider the element $h = \gamma(z)x$, and notice that $(S^* - z)h = 0$. Moreover we have $(\Gamma_1 h, \Gamma_2 h) = (x, Q(z)x)$ which is an element of Λ, so that $h \in \mathcal{D}(S^\Lambda)$ and $(S^\Lambda - z)h = 0$. This implies the inclusion

$$\gamma(z) \operatorname{Ker}\big(Q(z) - \Lambda\big) \subset \operatorname{Ker}(S^\Lambda - z).$$

Conversely, let $h \in \operatorname{Ker}(S^\Lambda - z)$, $z \in \rho(S_0)$. Then also $(S^* - z)h = 0$ and there exists $x \in \mathcal{G}$ with $h = \gamma(z)x$. Clearly,

$$(x, Q(x)) = (\Gamma_1 h, \Gamma_2 h) \in \Lambda,$$

so $x \in \operatorname{Ker}\big(Q(z) - \Lambda\big)$. This finishes the proof of the lemma. ∎

Proof of Theorem 3.6.1. Let $z \in \rho(S_0) \cap \rho(S^\Lambda)$. Take any $h \in \mathcal{H}$ and set $x = (S^\Lambda - z)^{-1}h$; clearly, $x \in \mathcal{D}(S^\Lambda)$, and by Lemma 3.2.2 there exist uniquely determined elements $x_z \in \mathcal{D}(S_0)$ and $y_z \in N_z$ with $x = x_z + y_z$. There holds

$$\begin{aligned} h &= (S^\Lambda - z)x = (S^* - z)x \\ &= (S^* - z)x_z + (S^* - z)y_z = (S^* - z)x_z \\ &= (S_0 - z)x_z \end{aligned}$$

and $x_z = (S_0 - z)^{-1}h$. Moreover, from $\Gamma_1 x_z = 0$ one has $\Gamma_1 x = \Gamma_1 y_z$, $y_z = \gamma(z)\Gamma_1 x$, and

$$x = (S^\Lambda - z)^{-1}h = (S_0 - z)^{-1}h + \gamma(z)\Gamma_1 x. \tag{3.11}$$

If we apply the operator Γ_2 to both sides of the equality $x = x_z + \gamma(z)\Gamma_1 x$ we get $\Gamma_2 x = \Gamma_2 x_z + Q(z)\Gamma_1 x$ and

$$\Gamma_2 x - Q(z)\Gamma_1 x = \Gamma_2 x_z. \tag{3.12}$$

When h runs through the whole space $\mathcal{H}$, then x_z runs through $\mathcal{D}(S)$ and the values $\Gamma_2 x_z$ cover the whole space $\mathcal{G}$. At the same time x runs through $\mathcal{D}(S^\Lambda)$ and the values $(\Gamma_1 x, \Gamma_2 x)$ cover the whole Λ. It follows then from (3.12) that $\operatorname{Ran}\big(\Lambda - Q(z)\big) = \mathcal{G}$. On the other hand, by the second assertion of Lemma 3.6.2 one has $\operatorname{Ker}\big(\Lambda - Q(z)\big) = 0$ and $0 \in \rho(\big(\Lambda - Q(z)\big))$. From (3.12) one obtains

$$\Gamma_1 x = \big(\Lambda - Q(z)\big)^{-1}\Gamma_2 x_z. \tag{3.13}$$

By the first assertion of Lemma 3.6.2 there holds $\Gamma_2 x_z = (\gamma(\bar{z}))^* h$. Substituting this equality into (3.13) and then into (3.11) one arrives at the conclusion. □

3.7 Examples

3.7.1 Krein's formula in terms of Green functions

In this section we discuss a realization of the Krein formula which will be useful for applications. It is not the general case, but it is similar to the situation concerned in this work. In [Geĭler et al., 1995, Theorem 4] it was shown that one can rewrite the Krein formula for the resolvents using Green functions. As we will use intensively this form, it is useful to recall briefly the corresponding machinery.

Let M be a compact Riemannian manifold of dim ≤ 3. We start from the Laplacian $S_0 = \Delta$ in $L^2(M)$. Fix a finite subset A of M and denote by S the restriction of Δ to the domain

$$f \in \mathcal{D}(\Delta) : f(a) = 0 \quad \forall a \in A.$$

This definition makes sense because the condition $d \leq 3$ and the Sobolev imbedding theorem imply the inclusion $\mathcal{D}(\Delta) \subset C^0(M)$, so we can speak about $f(a)$ for any element $f \in \mathcal{D}(\Delta)$. S is a symmetric operator whose self-adjoint extensions we are going to describe using the so-called Krein formalism. Clearly, S_0 is a self-adjoint extension of S.

Definition 3.7.1. Let T be a self-adjoint operator in $L^2(M)$. Assume that for complex z with $-z^2 \in \rho(T)$ the operator $(T+z^2)^{-1}$ has an intergal kernel, i.e. there exists a measurable function $T(x, y, z)$ such that

$$\text{for all } f \in L^2(M)(T + z^2)^{-1} f(x) = \int_M T(x, y, z) f(y)\, dy, \text{a.e.}, \tag{3.14}$$

then $T(x, y, z)$ is called the *Green function* of T.

Denote the Green function of S_0 by $G_0(x, y, z)$. For each fixed z, $-z^2 \in \rho(S_0)$, the function $G_0(x, y, z)$ is in $C^\infty(M \times M \setminus \{(x, x),\, x \in M\})$, and for each $y \in M$ the function $G_0(\cdot, y, z)$ belongs to $L^2(M)$. Moreover, for any fixed pair $(x, y) \in M \times M$, $x \neq y$ the function $G(x, y, z)$ is holomorphic for all z such that $-z^2 \in \rho(S_0)$, [Brüning and Geyler, 2005, Theorem 23].

Denote the number of elements in A by n and enumerate all points in A: $A = \{a_i,\, i = 1 \ldots n\}$.

Proposition 3.7.2 (Theorem 3 in Geĭler et al. [1995])**.** *The deficiency indices of S are (n, n) and the deficiency subspaces N_z of S are spanned by the functions $G_0(\cdot, a, z)$, $a \in A$.*

To define a Q-function we need an additional construction. Let us represent G_0 as the sum of two terms,

$$G_0(x, y, z) = F(x, y) + R(x, y, z),$$

where

$$F(x, y) = \begin{cases} 0, & \text{for } d = 1, \\ -\frac{1}{2\pi} \log \frac{1}{r(x,y)}, & \text{for } d = 2, \\ \frac{1}{4\pi r(x,y)}, & \text{for } d = 3, \end{cases}$$

is the standard singularity of the Green function, $r(x, y)$ is the geodesic distance between the points x and y. The function R is then continuous in the whole space $M \times M$.

The Krein formula (3.7) can be rewritten now in terms of the Green functions:

Theorem 3.7.3 (Theorem 4 in Geĭler et al. [1995]). *For z with $-z^2 \in \rho(S_0)$ define an $n \times n$ matrix by*

$$\begin{cases} Q_{ij}(z) = G_0(a_i, a_j, z), & i \neq j, \\ Q_{ii}(z) = R(a_i, a_i, z), & \textit{otherwise}. \end{cases} \tag{3.15}$$

The Green function of the operator S^Λ from Theorem 3.5.2 is given by the expression

$$G^\Lambda(x, y, z) = G_0(x, y, z) - \sum_{i,j=1}^{n} G_0(x, a_i, z)[Q(z) - \Lambda]^{-1}_{ij} G_0(a_j, y, z), \tag{3.16}$$

where Λ is a self-adjoint linear relation in $\mathbb{C}^n$.

As we will see later, this approach to the description of the self-adjoint extensions of a symmetric operator is relatively easy to use and, if combined with boundary value space techniques, provides a good description of these extensions in terms of "boundary conditions".

3.7.2 Laplacian on a half-line

Now we will illustrate these techniques with the simplest example. Let us consider the half-line $[0, \infty)$ parameterized by the coordinate x. Define the operator S on $[0, \infty)$ as the closure of the operator $-\frac{d^2}{dx^2}$ with initial domain $C_0^\infty(0, \infty)$ in the Hilbert space $L^2(0, \infty)$. This operator is obviously symmetric, but not self-adjoint. In fact, let us find its deficiency indices. First of all we should describe $\mathcal{D}(S^*)$. According to the definition, $f \in \mathcal{D}(S^*)$ if there exists $h \in L^2(0, \infty)$ such that $\langle f, Sg \rangle = \langle h, g \rangle$ for all $g \in \mathcal{D}(S)$ and then we define $S^* f = h$. In our case, for all $g \in C_0^\infty(0, \infty)$

$$\langle f, Sg \rangle = -\int_0^\infty f(x) g''(x)\, dx = \int_0^\infty h(x) g(x)\, dx = \langle h, g \rangle. \tag{3.17}$$

Then we have that $-f'' = h \in L^2$ in the distributional sense, which implies $\mathcal{D}(S^*) = W^{2,2}(0, \infty)$ and $S^* f = -f''$.

Now we can find $N_+ = \mathrm{Ker}(S^* - i)$:

$$-f'' - if = 0,\ f \in \mathcal{D}(S^*) = W^{2,2}(0, \infty) \Rightarrow \mathrm{Ker}(S^* - i) = \mathrm{Span}\{e^{-\sqrt{-i}x}\},$$

and similarly for the space N_-. So $n_\pm = \dim N_\pm = 1$ and the operator S has deficiency indices $(1, 1)$.

As the next step we define a boundary value space, namely operators $\Gamma_1, \Gamma_2 : \mathcal{D}(S^*) \to \mathbb{C}$ satisfying the conditions of Definition 3.4.1. They can be chosen as

$$\Gamma_1(f) = -f'(0), \quad \Gamma_2(f) = f(0). \tag{3.18}$$

In fact, for $f, g \in \mathcal{D}(S^*) = W^{2,2}(0,\infty)$ we have $f, g, f', g' \to 0$ as $x \to +\infty$, and then

$$\begin{aligned}\langle f, S^* g\rangle_{L^2} - \langle S^* f, g\rangle_{L^2} &= f'(x)g(x)|_0^\infty - f(x)g'(x)|_0^\infty \\ &= -f'(0)g(0) + f(0)g'(0) = \langle \Gamma_1(f), \Gamma_2(g)\rangle_{\mathbb{C}^2} - \langle \Gamma_1(g), \Gamma_2(f)\rangle_{\mathbb{C}^2}. \end{aligned} \tag{3.19}$$

As we have seen before we can rewrite the Krein formula using the Green function for one fixed self-adjoint extension, described by the condition $\Gamma_1 f = 0$. In our case with the chosen boundary value space this condition takes the form $f'(0) = 0$. It is the well-known Neumann boundary condition. The Green function for the Neumann Laplacian $(\Delta + z^2)$ on the half-line can be found directly or with the help of the following formula

$$G(x,y,z) = \begin{cases} \dfrac{\varphi_1(x,z)\varphi_2(y,z)}{w(z)}, & x \geq y, \\ \dfrac{\varphi_1(y,z)\varphi_2(x,z)}{w(z)}, & x \leq y. \end{cases}$$

where $\varphi_1(x,z), \varphi_2(x,z)$ are solutions of the equation $-\varphi'' + z^2\varphi = 0$ with the L^2 condition at infinity ($x = \infty$) and Neumann boundary condition at $x = 0$, respectively and where $w(z)$ is the Wronskian of these functions (it doesn't depend on x):

$$w(z) = \begin{vmatrix} \varphi_1(x,z) & \varphi_2(x,z) \\ \varphi_1'(x,z) & \varphi_2'(x,z) \end{vmatrix}.$$

In any case, after some computations one finds that

$$G_0(x,y,z) = -\frac{e^{-z(x+y)} + e^{-z|x-y|}}{2z}. \tag{3.20}$$

The γ-field and Q-matrix can be found explicitly using their expression in terms of the Green function:

$$\gamma(z)(\xi) = G_0(x,0,z)\,\xi = -\frac{e^{-zx}}{z}\,\xi, \quad Q(z) = G_0(0,0,z) = -\frac{1}{z}. \tag{3.21}$$

According to the Krein formula, the Green function (integral kernel of the resolvent) of each self-adjoint extension of the operator S can be expressed in terms of the Green function corresponding to the Neumann boundary condition (denoted by $G_0(x,y,z)$), the γ-field, the Q-matrix and some parameterization constant. Namely, each self-adjoint extension of S is defined by the boundary condition $\Gamma_2(f) = \lambda\Gamma_1(f)$ where $\lambda \in \mathbb{C}$, or, equivalently $f(0) + \lambda f'(0) = 0$. Using the explicit form of the γ-field and the Q-matrix, we find that the Green function for this self-adjoint operator has the form

$$G^\lambda(x,y,z) = G_0(x,y,z) - G_0(x,0,z)\left(G_0(0,0,z) - \lambda\right)^{-1} G_(0,y,z) =$$

$$-\frac{e^{-z(x+y)} + e^{-z|x-y|}}{2z} + \frac{e^{-z(x+y)}}{z(1+z\lambda)}.$$

3.7.3 Laplacian on a segment

As we will see, this case has no principal difference with the case of a half-line. Nevertheless, we will construct all necessary objects, because we will use them later. On the segment $[a,b]$ parameterized by the coordinate x we define the operator S as the closure of the operator $-\frac{d^2}{dx^2}$ with domain $C_0^\infty(a,b)$ in the Hilbert space $L^2(a,b)$.

This operator is symmetric but not self-adjoint. We describe $\mathcal{D}(S^*)$ with the same method as for the half-line. According to the definition, $f \in \mathcal{D}(S^*)$ if there exists $h \in L^2(a,b)$ such that $\langle f, Sg\rangle = \langle h, g\rangle$ for all $g \in \mathcal{D}(S)$ and then we define $S^* f = h$. In our case for all $g \in C_0^\infty$

$$\langle f, Sg\rangle = -\int_a^b f(x)g''(x)\,dx = \int_a^b h(x)g(x)\,dx = \langle h, g\rangle.$$

Then we have that $-f'' = h \in L^2$ in the distributional sense, which implies $\mathcal{D}(S^*) = W^{2,2}(a,b)$ and $S^* f = -f''$. The next step is to find $N_+ = \mathrm{Ker}(S^* - i)$:

$$-f'' - if = 0,\ f \in \mathcal{D}(S^*) = W^{2,2} \Rightarrow \mathrm{Ker}(S^* - i) = \mathrm{Span}\{e^{\sqrt{-i}x}, e^{-\sqrt{-i}x}\},$$

and the same for the space N_-. So $n_\pm = \dim N_\pm = 2$ and the operator S has deficiency indices $(2,2)$.

A boundary value space, namely operators $\Gamma_1, \Gamma_2 : \mathcal{D}(S^*) \to \mathbb{C}^2$, can be chosen as

$$\Gamma_1(f) = (-f'(a), f'(b)), \quad \Gamma_2(f) = (f(a), f(b)).$$

This can be proved by showing that for $f, g \in \mathcal{D}(S^*)$

$$\begin{aligned}\langle f, S^* g\rangle_{L^2} - \langle S^* f, g\rangle_{L^2} &= f'(x)g(x)|_a^b - f(x)g'(x)|_a^b \\ &= f'(b)g(b) - f'(a)g(a) + f(a)g'(a) - f(b)g'(b) \\ &= \langle \Gamma_1(f), \Gamma_2(g)\rangle_{\mathbb{C}^2} - \langle \Gamma_1(g), \Gamma_2(f)\rangle_{\mathbb{C}^2}.\end{aligned}$$

We need also the Green function for the self-adjoint extension, described by the condition $\Gamma_1 f = 0$. For the case of the operator on the segment with the chosen boundary value space this condition takes form $f'(a) = f'(b) = 0$. It is the well-known Neumann boundary condition. The Green function for the Neumann Laplacian on the segment can be found directly or with the help of the following formula

$$G(x, y, z) = \begin{cases} \dfrac{\varphi_1(x, z)\varphi_2(y, z)}{w(z)}, & x \geq y, \\ \dfrac{\varphi_1(y, z)\varphi_2(x, z)}{w(z)}, & x \leq y. \end{cases}$$

where $\varphi_1(x, z), \varphi_2(x, z)$ are solutions of the equation $-\varphi'' + z^2\varphi = 0$ with the Neumann boundary condition on the right $(x = b)$ and the left boundary $(x = a)$ respectively and $w(z)$ is the Wronskian of these functions.

In any case, after some computations one finds that

$$G_0(x, y, z) = \begin{cases} \dfrac{(e^{z(x-b)} + e^{z(b-x)})(e^{z(y-a)} + e^{z(a-y)})}{2z(e^{z(b-a)} - e^{z(a-b)})}, & x \geq y, \\ \dfrac{(e^{z(y-b)} + e^{z(b-y)})(e^{z(x-a)} + e^{z(a-x)})}{2z(e^{z(b-a)} - e^{z(a-b)})}, & x \leq y. \end{cases} \tag{3.22}$$

This expression can be rewritten in another form to make more obvious the fact that some terms decay as z tends to infinity (we will use this property later):

$$G_0(x, y, z) = \frac{e^{z(x+y-2b)} + e^{-z|x-y|} + e^{z(|x-y|-2(b-a))} + e^{z(2a-x-y)}}{2z(1 - e^{2z(a-b)})}. \tag{3.23}$$

The γ-field and Q-matrix can be founexplicitlyly using this expression for the Green function:

$$\begin{aligned}\gamma(z)(\xi_1, \xi_2) &= G_0(x, a, z)\xi_1 + G_0(x, b, z)\xi_2, \\ Q(z) &= \begin{pmatrix} G_0(a, a, z) & G_0(a, b, z) \\ G_0(b, a, z) & G_0(b, b, z) \end{pmatrix}.\end{aligned}$$

And as we already see for this simple example, the explicit calculation of $G(x, y, z)$ with the help of formulas (3.7) or (3.16) gives a rather complicated answer. But if one is interested in some specific question, for example in spectral properties or in the asymptotic behavior of the resolvent, this formula can give a desired answer.

Chapter 4

Spectral theory on hybrid manifolds

4.1 History of the question

We have already seen how we can define a Laplace operator on a manifold with the help of the Krein formula. This formula actually describes the resolvent of these operators. In order to do some spectral theory, it is therefore natural to use the resolvents directly, rather than the operators themselves. For a usual compact manifold there are two powerful instruments of spectral theory (in fact they are equivalent): the heat kernel expansion and the expansion of the trace of a suitable power of the resolvent. The simplest construction i.e. the expansion of the trace of the resolvent does not exist if the manifold has dimension at least two, because in this case the resolvent is not in trace class. These two expansions are equivalent, more precisely, $\operatorname{Tr} R^2 = \operatorname{Tr}(\Delta + z^2)^{-2}$ can be obtained from $e^{-\Delta t}$ with the Mellin transform. It seems that in our situation the resolvent approach is more natural, as we know the expression for the resolvent.

First of all let us recall the well-known result concerning the heat kernel and resolvent expansion for compact smooth manifolds.

Definition 4.1.1. Let (M, g) be a Riemannian manifold with associated Laplace operator Δ and heat semigroup $e^{-t\Delta}$, $t > 0$. The heat kernel $e(t, \cdot, \cdot)$ of M is the integral kernel of $e^{-t\Delta}$. Existence and construction are described for example in Rosenberg [1997], chapter 3.

Theorem 4.1.2 (Rosenberg [1997], Propositions 3.23, 3.26, 3.29)**.** *Let (M^n, g) be a compact Riemannian manifold of dimension n. Then $e(t, x, x)$ has the* *asymptotic expansion*

$$e(t,x,x) \sim (4\pi t)^{-n/2} \sum_{k=0}^{\infty} a_k(x,x)t^k,\ t \to 0+,$$

where $a_0(x,x) = 1, a_1(x,x) = \frac{1}{6}S(x)$, *where* $S(x)$ *is the scalar curvature, and moreover, all coefficients* $a_i(x,x)$ *are universal polynomial expressions in the Riemannian curvature tensor at* x *and its covariant derivatives.*

As an easy corollary, we get

Theorem 4.1.3. *With the notation of the previous theorem, the trace of the heat kernel has the following expansion as* $t \to 0$*:*

$$\operatorname{Tr} e^{-\Delta t} \sim (4\pi t)^{-n/2} \sum_{k=0}^{\infty} a_k t^k,$$

where $a_k = \int\limits_M a_k(x,x)\, dvol(x)$.

Proposition 4.1.4 (Seeley [1967]). *If* $\dim M = 2$, *then* $R^2(z) = (\Delta + z^2)^{-2}$, *has the asymptotic expansion as* $z \to \infty$ *: for all* $N \geqslant 0$,

$$\operatorname{Tr} R^2(z) = \sum_{k=0}^{N} \frac{a_k\, \Gamma(k+1)}{4\pi z^{2k+2}} + O(z^{-2(N+2)}),$$

where the coefficients a_k *are as in Theorem 4.1.3.*

The first terms of this expansion reveal important geometric and topological characteristics of the surface:

$$\operatorname{Tr} R^2 \sim \frac{Vol(M)}{4\pi z^2} + \frac{\chi(M)}{6z^4} + \cdots . \tag{4.1}$$

Our aim is to obtain some similar results for the resolvent of a Laplace operator on a hybrid manifold.

4.2 Krein's formula for a hybrid manifold

Let us recall the formula for the resolvent of a self-adjoint extension of the symmetric operator as it was introduced in (3.7). This Krein formula says

$$R^{\Lambda} = R_0 - \gamma(z)[Q(z) - \Lambda]^{-1}(\gamma(\bar{z}))^*,$$

where R_0 is the resolvent of one fixed self-adjoint extension, γ is a Krein γ-field and Q is a Krein Q-matrix, Λ is the matrix representing the boundary conditions in the points of gluing.

We want to apply this formula to the Laplace operator H on the hybrid manifold defined in Section 2.3. In fact, the method of Section 3.7.1 will be used with some modifications. First of all we need to define the Green function for the differential operator on the hybrid manifold, then we will use it to define the objects needed for the Krein formalism.

4.2.1 Boundary value space

It is clear that due to the special structure of the symmetric operator D on the hybrid manifold (see 2.5) it is enough to choose boundary value spaces for each component of the hybrid manifold and take their direct sum. We start from the case of a segment and then consider a manifold.

In the case of the operator D_j^s defined on the segment $[0, l_j]$ we already have proved the following result (see 3.7.3)

Proposition 4.2.1. *The boundary value space for each of the operators D_j^s defined on the segment $[0, l_j]$ can be chosen as the operators*

$$\Gamma_1(f) = (-f'(0), f'(l_j)), \quad \Gamma_2(f) = (f(0), f(l_j)). \tag{4.2}$$

with values in $\mathcal{G} = \mathbb{C}^2$.

The case of a surface is more complicated. In fact, we need some additional facts before we can construct an appropriate boundary value space for the operator D_i defined in (2.4). Let us fix the manifold $M_i =: M$. As before, denote by $R = R_i$ and $G(x, y, z) = G_i(x, y, z)$ the resolvent and the Green function of a Laplace operator on M. We denote by R_0 and G_0 the resolvent and the Green function for the standard Laplacian on M.

If one studies the behavior of the Green function on a two-dimensional manifold in a neighborhood of a fixed point $q \in M_i$ in case of fixed $z \in \rho(D_i)$, one can find (see for example [Avramidi, 1998, section II.2]) that the following Lemma holds:

Lemma 4.2.2. *The Green function for the operator Δ on the manifold M_i has the following expansion near the point q_j:*

$$G_0(x, q_j, z) = -\frac{c(x, q_j)}{2\pi} \ln r(x, q_j) + F(x, q_j, z) + P(x, q_j, z), \tag{4.3}$$

where $c(x, q_j)$ does not depend on z and is continuous with respect to x, $c(q_j, q_j) = 1$, $r(x, q_j)$ is the geodesic distance between the points x and q_j,

$F(x, q_j, z)$ is continuous with respect to x, and the remainder $P(x, q_j, z)$ is $o(1)$ as $x \to q_j$. The functions F and P are analytic functions of z in $\rho(\Delta)$.

As we know the domain of D^* is

$$\mathcal{D}(D^*) = \mathcal{D}(D) \oplus N_+ \oplus N_-, \tag{4.4}$$

and the result of the article [Geĭler et al., 1995, Theorem 3], analogous to Proposition 3.7.2 describes the structure of the deficiency subspaces for the operator D:

Lemma 4.2.3. *The functions $G_0(\cdot, q_j, z)$ form a basis for the deficiency subspace N_z, for $z \in \mathbb{C}\setminus\mathbb{R}$. The deficiency indices for the operator D are (μ, μ), where μ is the number of gluing points on this manifold M.*

So we can decompose a function $f(x)$ from $\mathcal{D}(D^*)$ into the sum of a function $g(x)$ from $\mathcal{D}(D)$ and linear combination of the Green functions $G(x, q_j, z)$, for any $z \notin \mathbb{R}$. We are interested in the behavior of this function for x near q_j. We know that $g(x) \in \mathcal{D}(D_i)$ is continuous at q_j, and Lemma 4.2.2 describes the behavior of the Green functions. So we find that

$$f(x) = a_j(f)(-c(x, q_j)\ln r(x, q_j)) + b_j(f) + P(x), \tag{4.5}$$

where $a_j(f), b_j(f) \in \mathbb{C}$, $P(x, q_j, z) = o(1)$ as $x \to q$, and the function c is the same as in Lemma 4.2.2. Now we have enough information to formulate

Proposition 4.2.4. *The triple $(\mathbb{C}^\mu, \Gamma_1, \Gamma_2)$, where the linear operators Γ_1, Γ_2 act from $\mathcal{D}(D^*)$ into $\mathcal{G} = \mathbb{C}^\mu$ as follows*

$$\begin{aligned} \Gamma_1(f) &= (a_1(f), \ldots, a_\mu(f)), \\ \Gamma_2(f) &= (b_1(f), \ldots, b_\mu(f)), \end{aligned} \tag{4.6}$$

is a boundary value space for the operator D on a manifold M with μ marked points. The proof is straightforward but technical and can be found in [Brüning and Geyler, 2003, Lemma 5].

Taking into account the fact that the operator D has a direct sum structure (2.5) and using the two previous propositions we obtain

Theorem 4.2.5 (Theorem 1 in Brüning and Geyler [2003])**.** *For the operatordefined as*

$$D = D_1 \oplus \cdots \oplus D_i \oplus D_1^s \oplus \cdots \oplus D_j^s$$

on the hybrid manifold consisting of M 2-dimensional compact Riemannian manifolds $M_1, ..., M_M$ and a set of N segments $L_1, ..., L_N$, The triple

$(\mathbb{C}^{\mu}, \Gamma_1, \Gamma_2)$ *with*

$$\begin{aligned} \mathcal{G} &= \mathbb{C}^{\mu_1} \oplus \cdots \mathbb{C}^{\mu_M} \oplus \mathbb{C}^{2N} = \mathbb{C}^{4N}, \\ \Gamma_1 &= (\Gamma_1)_{M_1} \oplus \cdots (\Gamma_1)_{M_M} \oplus (\Gamma_1)_{L_1} \oplus \cdots (\Gamma_1)_{L_N}, \\ \Gamma_2 &= (\Gamma_2)_{M_1} \oplus \cdots (\Gamma_2)_{M_M} \oplus (\Gamma_2)_{L_1} \oplus \cdots (\Gamma_2)_{L_N}, \end{aligned} \tag{4.7}$$

where the boundary value spaces $(\Gamma)_M$ *for manifolds and* $(\Gamma)_L$ *for segments were described in* (4.2), (4.6)*, is a boundary value space.*

4.2.2 Krein's formula for the hybrid manifold

Here we formulate some results about Laplace operators and the Krein formula on a hybrid manifold in terms of given boundary value space.

First of all, for application of the Krein theory we need to fix some self-adjoint extension of the operator D (defined in 2.5). Due to Proposition 3.6.1, if for the boundary value space fixed in Theorem 4.2.5 we define the self-adjoint operator D_0 as the restriction of D^* to $\operatorname{Ker}\Gamma_1$ then the resolvents of self-adjoint extensions of D are described by the Krein formula (3.7).

In fact, this operator is the direct sum of the smooth Laplacians on each manifold and the Neumann Laplacians D_j^N on each segment of the hybrid space. For a segment, the condition $\Gamma_1 f = 0$ takes the form $f'(0) = f'(l) = 0$, indeed the Neumann boundary condition. The condition $\Gamma_1 f = 0$ is equivalent to $a_i(f) = 0$ for $f \in \mathcal{D}(D_0)$ on a manifold, and this means that we deal with the class of functions f from $\mathcal{D}$ which have no singularity at the marked points of the manifold, i.e. we take the smooth Laplacian on this manifold as one fixed self-adjoint extension.

This choice of D_0 gives us the possibility to use many known results for the Laplacian on a smooth manifold. For example, we will use the heat expansion for D_0. We denote by R_0 and G_0 the resolvent and the Green function of D_0, i.e. the direct sum of the resolvents and Green functions for each component.

Once we have chosen a boundary value space and fixed a self-adjoint extension of D, we can construct the γ-field and the Q-matrix, necessary for the Krein formalism.

By [Geĭler et al., 1995, Lemma 5] and Proposition 3.7.3 we can use the Green function for the construction of the γ-field and Q-matrix. We always suppose that the boundary value space is chosen as in (4.7). We start from the construction for each component of a hybrid manifold.

Proposition 4.2.6. *For the pair of operators* D_j^s, D_j^N *on the segment* $L_j = [0, l_j]$

the operator-valued function γ_j^s: $\rho(D_j^N) \to L(\mathcal{G}, \mathcal{H})$, $\mathcal{G} = \mathbb{C}^2$ defined by

$$\gamma_j^s(z)(\xi) = G^N(x, 0, z)\xi_1 + G^N(x, l_j, z)\xi_2, \quad \xi = (\xi_1, \xi_2) \in \mathbb{C}^2, \tag{4.8}$$

where $G^N(x, y, z)$ is the Green function for the Laplacian on this segment with Neumann boundary conditions, is a Krein γ-field;

the operator-valued function from $\rho(D_j^N)$ to $L(\mathcal{G}, \mathcal{G})$ given by

$$Q_j^s(z) = \begin{pmatrix} G^N(0,0,z) & G^N(0,l_j,z) \\ G^N(l_j,0,z) & G^N(l_j,l_j,z) \end{pmatrix}, \tag{4.9}$$

is a Krein Q-matrix for D_j^s.

Proof. Using the exact form of the Green function obtained in (3.23) it is easy to see that $\Gamma_1(\gamma(z))$ is the identity on $\mathcal{G}$ because of

$$-(G^N)'_x(x, 0, z)|_{x=0} = (G^N)'_x(x, l_j, z)|_{x=l_j} = 1,$$

and, consequently, condition (3.8) is satisfied. The condition (3.9) $Q = \Gamma_2(\gamma(z))$ is obvious in this case. So $\gamma_j^s(z)$ is the γ-field and Q_j^s is the Q-matrix. □

The case of a manifold is similar to the considered one. We use once more the chosen boundary value space and state

Proposition 4.2.7. *For the pair of operators D_i, D_{0i} on the manifold M_i with μ_i marked points*

the operator-valued function γ: $\rho(D_j) \to L(\mathcal{G}, \mathcal{H})$, $\mathcal{G} = \mathbb{C}^{\mu_i}$ defined by

$$\gamma_i(z)(\xi) = \sum_{k=1}^{\mu_i} G(x, q_k, z)\xi_k, \quad \xi = (\xi_1, \ldots, \xi_{\mu_i}) \in \mathbb{C}^{\mu_i}, \tag{4.10}$$

where $G(x, y, z)$ is the Green function for the Laplacian on M_i and q_k for $k = 1, \ldots, \mu_i$ are the marked points, is a Krein γ-field;

the $\mu_i \times \mu_i$ matrix defining an operator-valued function from $\rho(D_i)$ into $L(\mathcal{G}, \mathcal{G})$

$$Q(z)_{ij} = \begin{cases} F(q_i, q_i, z), & i = j, \\ G(q_i, q_j, z), & i \neq j, \end{cases}$$

where $G(x, y, z)$ is the Green function for the Laplacian on M_i and F is the regular part of this Green function near q_j, as defined in Lemma 4.2.2,is a Krein Q-matrix.

Proof. We should verify the conditions of Definition 3.5.1. But as we know for the fixed boundary value space it is sufficient to verify the properties (3.8), (3.9).

Condition (3.8) is equivalent to $\Gamma_1(\gamma(z)(\xi)) = \xi$. From the definition of a_i we easily conclude that $a_l(G(x, q_k, z)) = \delta_{lk}$ and

$$\begin{aligned}\Gamma_1(\gamma(z)(\xi)) &= \{a_l(\sum_{k=1}^{\mu_i} G(x, q_k, z)\xi_k),\ l = 1, \dots, \mu_i\} \\ &= \{\sum_{k=1}^{\mu_i} \delta_{lk}\xi_k,\ l = 1, \dots, \mu_i\} = \xi.\end{aligned}$$

The condition (3.9) $Q = \Gamma_2(\gamma(z))$ is proved in the same way because of $b_i(G(x, q_j, z)) = Q_{ij}$. So, $\gamma(z)$ is the γ-field and Q is the Q-matrix. □

Applying these Propositions to each component of the hybrid manifold and taking the direct sum, we obtain

Theorem 4.2.8. *For a hybrid manifold consisting of 2-dimensional compact Riemannian manifolds $M_1, ..., M_M$ and a set of segments $L_1, ..., L_N$ and its boundary value space chosen as in Theorem 4.2.5, the operator-valued functions, defines as:*

$$\begin{aligned}\gamma(z) &= \gamma_{M_1}(z) \oplus \cdots \gamma_{M_M}(z) \oplus \gamma^s_{L_1}(z) \oplus \cdots \gamma^s_{L_N}(z), \\ Q(z) &= Q_{M_1}(z) \oplus \cdots Q_{M_M}(z) \oplus Q^s_{L_1}(z) \oplus \cdots Q^s_{L_N}(z),\end{aligned} \tag{4.11}$$

where the γ-field and the Q-matrix for each component of the hybrid manifold are defined in Propositions 4.2.6, 4.2.7, are Krein γ-field and Q-matrix for the operator D.

Theorem 4.2.9. *The Laplace operators on the hybrid space, disjoint with the operator D_0, are in one-to-one correspondence with the Hermitian $4N \times 4N$-matrices Λ, i.e. for each such matrix there exists a self-adjoint extension of D defined by the boundary condition*

$$\Gamma_2 x = \Lambda \Gamma_1 x, \quad x \in \mathcal{D}(D^*),$$

for the boundary value space fixed in Theorem 4.2.5.

Proof. This is a direct consequence of Theorem 3.4.3 while any Hermitian Λ defines a self-adjoint linear relation. □

From this theorem and Krein's formula (3.7) we now derive the main tool in our spectral analysis of hybrid Laplace operators:

Theorem 4.2.10. *The resolvent R^Λ of any Laplace operator disjoint with D_0 and corresponding to the the Hermitian matrix Λ, can be written as*

$$R^\Lambda(z) = R_0(z) - \gamma(z)[Q(z) - \Lambda]^{-1}(\gamma(\bar{z}))^*,$$

where R_0 is the resolvent of D_0 and $\gamma(z)$ and $Q(z)$ were defined in Theorem 4.2.8.

Remark. Let us notice that we consider Hermitian matrices Λ (which define the self-adjoint operators Λ in Theorem 3.5.2) and, therefore, all obtained self-adjoint extensions of D are disjoint with D_0. In fact, it is natural to restrict ourselves to more special classes of matrices. As we know, this matrix defines the relation between some characteristics (such as $f, f', a_i(f), \dots$) of a function from $\mathcal{D}(D^*)$ in a neighborhood of the gluing points. It is natural to suppose that such characteristics are "local", i.e. that their values in one point cannot influence the values in another point. For example, if we consider the first row of the matrix Λ we see that these elements are coefficients relating $b_1(f)$ with $a_i(f)$ and $f'(q_i)$ for $i = 1, \dots, N$:

$$b_1(f) = \sum_{i=1}^{N} \lambda_{1,i} a_i(f) + \sum_{i=1}^{N} \lambda_{1,N+i} f'(q_i).$$

The described "locality condition" means that in fact $b_1(f)$ depends only on $a_1(f)$ and $f'(q_1)$, i.e. only from local characteristics of the function at the point q_1. We see that under this assumption only $\lambda_{1,1}$ and $\lambda_{1,N+1}$ are non-zero in the first row. A a consequence, the matrix Λ has the following block structure:

$$\Lambda = \begin{pmatrix} \lambda_{1,1} & 0 & \dots & 0 & \lambda_{1,N+1} & 0 & \dots & 0 \\ 0 & \lambda_{2,2} & \dots & 0 & 0 & \lambda_{2,N+2} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_{N,N} & 0 & 0 & \dots & \lambda_{N,2N} \\ \lambda_{N+1,1} & 0 & \dots & 0 & \lambda_{N+1,N+1} & 0 & \dots & 0 \\ 0 & \lambda_{N+2,2} & \dots & 0 & 0 & \lambda_{N+2,N+2} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_{2N,N} & 0 & 0 & \dots & \lambda_{2N,2N} \end{pmatrix}$$

Note also that $\lambda_{N+l,l} = \overline{\lambda_{l,N+l}}$ and $\lambda_{l,l} \in \mathbb{R}$ due to the Hermitian structure of Λ.

4.3 The resolvent expansion

As mentioned before, we try to compute asymptotically the trace of the square of the resolvent of a Laplace operator on a hybrid space. This is now possible with the help of Theorem 4.2.10. The matrix Λ is fixed during the rest of this section so we do not specify the dependence on Λ. First of all we prove the following technical lemma:

Lemma 4.3.1. *For the Green function, i.e. the integral kernel of the operator* $(\Delta + z^2)^{-1}$ *on a compact manifold* M*, one has*

$$\int_M G(x,u,z)G(u,y,z)\,du = -\frac{1}{2z}G'_z(x,y,z).$$

Proof. For the resolvent of Δ we have

$$\frac{\partial}{\partial z}(\Delta + z^2)^{-1} = -2z(\Delta + z^2)^{-2},$$

and it is enough to rewrite this identity in terms of the operator kernels, noting that G is analytic in z [Brüning and Geyler, 2005, Theorem 23]. The identity holds for all $x, y \in M$ because of the continuity of the Green function outside the diagonal [Brüning and Geyler, 2005, Theorem 23], and [Brüning and Geyler, 2005, Proposition 6]. □

Now we have all the necessary tools for the computation of $\operatorname{Tr} R^2$.

4.3.1 Computation of $\operatorname{Tr} R^2$

Let us start with the computation of $\operatorname{Tr} R^2$ using Theorem 4.2.10. We denote the linear operator $\gamma(z)[Q(z) - \Lambda]^{-1}(\gamma(\bar{z}))^*$ by $A(z)$. Then we obtain

$$\operatorname{Tr} R^2(z) = \operatorname{Tr} R_0^2(z) - \operatorname{Tr}(R_0(z)A(z)) - \operatorname{Tr}(A(z)R_0(z)) + \operatorname{Tr} A^2(z).$$

We recall that the resolvent R_0 is the direct sum of the resolvents of the ordinary Laplacians on the manifolds M_i (denoted by R_{0,M_i}) and the resolvents for the Neumann Laplacians on the segments L_j (denoted by R_{0,L_j}) forming the hybrid manifold. According to the definition of the Green function for the self-adjoint operator Δ on a compact manifold M we can represent the resolvent as an integral operator (under the notation dy here and in what follows we understand the volume form $dvol(y)$ on the manifold):

$$(\Delta + z^2)^{-1}f(x) = R(z)f(x) = \int_M G(x,y,z)f(y)\,dy.$$

This representation holds for all $f \in L^2(M)$, and since $\mathcal{D}(\Delta) \subset C(M)$ and the continuity of the map $x \to \int_M G(x,y,z)f(y)\,dy$ for all $f \in L^2(M)$ hold also for all $x \in M$, see [Brüning and Geyler, 2005, Theorem 23]. This representation allows us to find the trace of R^2_{0,M_i} on any of the manifolds M_i, using Lemma 4.3.1:

$$R^2_{0,M_i}(z)f(x) = \int\limits_{M_i}\int\limits_{M_i} G_{M_i}(x,u,z)G_{M_i}(u,y,z)f(y)\,dy\,du$$

$$\operatorname{Tr} R^2_{0,M_i}(z) = \int\limits_{M_i}\int\limits_{M_i} G_{M_i}(x,u,z)G_{M_i}(u,x,z)\,dx\,du = -\int\limits_{M_i}\frac{1}{2z}(G_{M_i})'_z(x,x,z)\,dx.$$

For the Laplacian on a segment $L_j = [0,l_j]$ with Neumann boundary conditions we have the exact formula (3.22) for the Green function and we find

$$\begin{aligned}\operatorname{Tr} R^2_{0,L_j}(z) &= -\int\limits_0^{l_j}\frac{1}{2z}\frac{\partial}{\partial z}\left(\frac{1+e^{2(x-l_j)z}+e^{-2l_jz}+e^{-2xz}}{2z(1-e^{-2l_jz})}\right)dx\\ &= \frac{e^{-4l_jz}(2-l_jz)+4e^{-2l_jz}(-1+l_j^2z^2)+2+l_jz}{4(1-e^{-2l_jz})^2z^4}\end{aligned}$$

The linear operator $A(z)$ can be also rewritten as an integral operator as follows. Denote the entries of the $2N\times 2N$ matrix $[Q(z)-\Lambda]^{-1}$ by $c_{ij}(z)$. Using the expression for $\gamma(z)$ obtained in (4.8), (4.10) and the fact that $G(x,y,z)=\overline{G(y,x,\bar z)}$, we get:

$$\begin{aligned}&A(z)f(y) = \gamma(z)[Q(z)-\Lambda]^{-1}(\gamma(\bar z))^*f(y) =\\ &\sum_{i,j}\int c_{ij}(z)G(y,q_i,z)f(u)G(q_j,u,z)\,du.\end{aligned}$$

The integral here is in fact the sum of the integrals over manifolds and segments. Using these integral representations of the operators A and R_0, we calculate the remaining terms of $\operatorname{Tr} R^2$ as follows:

$$R_0(z)A(z)f(x) = \sum_{i,j}c_{ij}(z)\int\int G(x,y,z)G(y,q_i,z)f(u)G(q_j,u,z)\,dy\,du.$$

This operator has the integral kernel

$$K(x,t,z) = \sum_{i,j}c_{ij}(z)\int G(x,y,z)G(y,q_i,z)G(q_j,t,z)\,dy,$$

hence

$$\operatorname{Tr} R_0(z)A(z) = \sum_{i,j} c_{ij}(z) \int\int G(x,y,z)G(y,q_i,z)G(q_j,x,z)\,dx\,dy,$$

or, using Lemma 4.3.1,

$$\operatorname{Tr} R_0(z)A(z) = \frac{1}{2}\sum_{i,j} c_{ij}(z) \int \left(\int G(x,y,z)G(y,q_i,z)\,dy\right) G(q_j,x,z)\,dx$$

$$+\frac{1}{2}\sum_{i,j} c_{ij}(z) \int \left(\int G(x,y,z)G(q_j,x,z)\,dx\right) G(y,q_i,z)\,dy$$

$$= \frac{1}{2}\sum_{i,j} c_{ij}(z)\Big(\int -\frac{1}{2z}G'_z(x,q_i,z)G(q_j,x,z)\,dx +$$

$$\int -\frac{1}{2z}G'_z(q_j,y,z)G(y,q_i,z)\,dy\Big)$$

$$= -\frac{1}{4z}\sum_{i,j} c_{ij}(z) \int \left(G(q_j,x,z)G'_z(x,q_i,z) + G'_z(q_j,x,z)G(x,q_i,z)\right)\,dx$$

$$= -\frac{1}{4z}\sum_{i,j} c_{ij}(z) \int \left(G(q_j,x,z)G(x,q_i,z)\right)'_z\,dx$$

$$= \frac{1}{8z}\sum_{i,j} c_{ij}(z) \left(\frac{1}{z}G'_z(q_j,q_i,z)\right)'_z$$

$$= -\sum_{i,j} c_{ij}(z) \left(\frac{G'_z(q_j,q_i,z)}{8z^3} - \frac{G''_{zz}(q_j,q_i,z)}{8z^2}\right).$$

Let us now find $\operatorname{Tr} A^2(z)$. We will use the same method and represent the operator $A^2(z)$ as an integral operator:

$$A^2(z)f = \sum_{i,j} c_{ij}(z) \int G(x,q_i,z)G(q_j,y,z)A(z)f(y)\,du$$

$$= \sum_{i,j,k,l} c_{ij}(z)\,c_{kl}(z) \int G(x,q_i,z)G(q_j,y,z)\Big(\int G(y,q_k,z)G(q_l,u,z)f(u)\,du\Big)\,dy$$

$$= \sum_{i,j,k,l} c_{ij}(z)\,c_{kl}(z) \int G(x,q_i,z)G(y,q_k,z)G(q_j,y,z)G(q_l,u,z)f(u)\,du\,dy.$$

Now we find:

$$\operatorname{Tr} A^2(z) = \sum_{i,j,k,l} c_{ij}(z)\, c_{kl}(z) \int G(x,q_i,z)G(q_l,x,z)G(y,q_k,z)G(q_j,y,z)\,dy\,dx$$
$$= \frac{1}{4z^2} \sum_{i,j,k,l} c_{ij}(z)\, c_{kl}(z) G'_z(q_l,q_i,z)G'_z(q_j,q_k,z).$$

Summarizing the results obtained, we state

Theorem 4.3.2. *Consider a hybrid manifold H, consisting of the manifolds M_i and the segments L_j, and a Laplace operator (corresponding to the boundary conditions determined by a matrix Λ, and disjoint with D_0) on H. Then the following formula for the trace of the square of the resolvent of this operator holds:*

$$\operatorname{Tr} R^2(z) = -\int\limits_H \frac{1}{2z} G'_z(x,x,z)\,dx + 2\sum_{i,j=1}^{2N} c_{ij}(z)\left(\frac{G'_z(q_j,q_i,z)}{8z^3} - \frac{G''_{zz}(q_j,q_i,z)}{8z^2}\right)$$
$$+\frac{1}{4z^2}\sum_{i,j,k,l=1}^{2N} c_{ij}(z)\, c_{kl}(z) G'_z(q_l,q_i,z)G'_z(q_j,q_k,z).$$

Here $G(x,y,z)$ is the Green function of S_0 on the hybrid manifold and the entries of the matrix $[Q(z)-\Lambda]^{-1}$ are denoted by $c_{ij}(z)_{1\leqslant i,j\leqslant 2N}$.

4.3.2 The asymptotic expansion

The result of Theorem 4.3.2 is given in terms of the Green functions for the Laplacians on the smooth parts of the hybrid manifold. But using it in applications is practically impossible for two reasons: first of all, inverting the $2N \times 2N$ matrix $Q(z) - \Lambda$ can be difficult, and secondly, the explicit computation of the Green function is almost never possible. Nevertheless, there are ways to get some simplifications if we restrict attention to large z. As we will see below, the special structure of the matrix $(Q(z) - \Lambda)$ allows us to find its inverse asymptotically as $z \to \infty$. Likewise, one can use the representation of R_0 as an asymptotic series in powers of z (analog of heat kernel expansion). Thus we may attempt to generalize (4.1) to the case of hybrid manifolds.

For further calculations the following lemma is crucial.

Lemma 4.3.3. *Let M be a compact two-dimensional manifold and fix some distinct points q_i and q_j on it. For the Green function of the Laplace operator* *on M and for $|z|$ large enough, $z \notin Z_\varepsilon = \{z \in \mathbb{C} : |\arg z| < \varepsilon\}$, one has the estimates*

$$
\begin{aligned}
|G(q_i, q_j, z)| &= O(e^{-C|z|}),\\
|G'_z(q_i, q_j, z)| &= O(e^{-C|z|}),\\
|G''_{zz}(q_i, q_j, z)| &= O(e^{-C|z|}),
\end{aligned}
$$

where C is some positive constant.

Proof. The first estimate is a direct consequence of [Hörmander, 1969, Proposition 4.8], which asserts that for two distinct points x and y we have

$$
|G(x, y, z)| \leqslant C \frac{e^{-Cr(x,y)|z|}}{r(x, y)|z|}, \tag{4.12}
$$

where $r(x, y)$ is the geodesic distance between x and y and C is some positive constant.

Now we turn to the proof of the second estimate. For this fix some small $0 < \varepsilon < r(q_i, q_j)/2$ so that the geodesic balls $B(q_i, \varepsilon)$ and $B(q_j, \varepsilon)$ are disjoint. Using Lemma 4.3.1 we can write

$$
\begin{aligned}
|G'_z(q_i, q_j, z)| &\leqslant 2|z| \int_M |G(q_i, u, z)||G(u, q_j, z)|\, du\\
&= 2|z| \Big[\int_{M \setminus (B(q_i,\varepsilon) \cup B(q_j,\varepsilon))} |G(q_i, u, z)||G(u, q_j, z)|\, du\\
&+ \int_{B(q_i,\varepsilon)} |G(q_i, u, z)||G(u, q_j, z)|\, du + \int_{B(q_j,\varepsilon)} |G(q_i, u, z)||G(u, q_j, z)|\, du \Big]
\end{aligned}
$$

Due to 4.12, the first integral in the last inequality is easily seen to decay exponentially as z goes to infinity. It remains to estimate the second one (as the reasoning is similar for the third one). By 4.12 we have

$$
\int_{B(q_i,\varepsilon)} |G(q_i, u, z)||G(u, q_j, z)|\, du \leqslant C \frac{e^{-C\varepsilon|z|}}{\varepsilon|z|} \int_{B(q_i,\varepsilon)} |G(u, q_j, z)|\, du.
$$

Using the $z-$asymptotic expansion of G near the diagonal [Avramidi, 1998, Formula (38)], we see that $\int_{B(q_i,\varepsilon)} |G(u, q_j, z)|\, du = O(1)$. This proves the desired estimate for G'_z. For G''_{zz}, the proof is similar. □

Thus, if the points x and y do not coincide, the Green function $G(x, y, z)$, as well as its derivatives $G'_z(x, y, z)$ and $G''_{zz}(x, y, z)$ decay exponentially as z tends to infinity.

In the statement of Theorem 4.3.2 the terms $G(q_i, q_j, z)$ appear, where the distance between the points q_j and q_j is fixed by the configuration of our hybrid space. Hence all terms of such type for non-coinciding points $q_j \neq q_j$ are exponentially small as $z \to \infty$, so we need to consider only the terms $G(q_i, q_i, z)$. Using this observation and performing calculations similar to the proof of Theorem 4.3.2, but neglecting terms of exponential small order of z we prove

Theorem 4.3.4. *With R as in Theorem 4.3.2 we have the following asymptotic relation for some positive constant c, as $z \to \infty$*

$$\operatorname{Tr} R^2(z) = -\int\limits_H \frac{1}{2z} G'_z(x,x,z)\,dx + 2\sum_{i=1}^{2N} c_{ii}(z) \left(\frac{G'_z(q_i,q_i,z)}{8z^3} - \frac{G''_{zz}(q_i,q_i,z)}{8z^2} \right)$$
$$+\frac{1}{4z^2} \sum_{i,j=1}^{2N} c_{ij}(z)\, c_{ji}(z) G'_z(q_i,q_i,z) G'_z(q_j,q_j,z) + O(e^{-cz}).$$

4.3.3 Matrix formula

The result of Theorem 4.3.2 can be rewritten in a compact matrix-form. Let us denote by G'_z the diagonal matrix with entries $G'_z(q_i, q_i, z)$. Denote also by G''_{zz} the diagonal matrix with entries $G''_{zz}(q_i, q_i, z)$. This is a slight abuse of notation, since G'_z and G''_{zz} are also well defined operators; it will be clear from the context what is meant by this notation. Then it is easy to see that Theorem 4.3.4 can be formulated as follows.

Theorem 4.3.5. *We have the following asymptotic relation as $z \to \infty$:*

$$\operatorname{Tr} R^2(z) = -\int\limits_H \frac{G'_z(x,x,z)}{2z}\,dx - \frac{1}{4z^2} \operatorname{Tr}\left(G''_{zz}\,[Q-\Lambda]^{-1} \right)$$
$$+\frac{1}{4z^3} \operatorname{Tr}\left(G'_z\,[Q-\Lambda]^{-1} \right) + \frac{1}{4z^2} \operatorname{Tr}\left(G'_z\,[Q-\Lambda]^{-1} \right)^2 + O(e^{-cz}).$$

Proof. The proof is trivial - we need just to notice that in the matrix sense

$$\sum_{i,j=1}^{2N} c_{ii}\,(z) G'(q_i,q_i,z) = \operatorname{Tr}\left(G'_z\,[Q-\Lambda]^{-1} \right),$$
$$\sum_{i,j,k,l=1}^{2N} c_{ij}(z)\, c_{ji}(z)\, G'_z(q_i,q_i,z)\, G'_z(q_j,q_j,z) =$$
$$\operatorname{Tr}\left(G'_z\,[Q-\Lambda]^{-1}\, G'_z\,[Q-\Lambda]^{-1} \right).$$

□

Now we need only to compute the inverse matrix $Q-\Lambda$, and fortunately, this is possible in this approximation.

4.3.4 Asymptotic representation of $[Q(z)-\Lambda]^{-1}$

Let us recall that under the natural "locality" conditions discussed in the Remark after the Statement of Theorem 4.2.10, the matrix Λ is a Hermitian matrix consisting of four diagonal blocs

$$\Lambda = \begin{pmatrix} (\lambda_{i,i}) & (\lambda_{i,i+N}) \\ \overline{(\lambda_{i,i+N})} & (\lambda_{i+N,i+N}) \end{pmatrix}$$

As we have already shown in Theorem 4.2.8,the matrix $Q(z)$ has the form

$$Q(z) = \begin{pmatrix} Q_1 & 0 & ... & 0 & 0 \\ 0 & Q_2 & ... & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & ... & 0 & Q_M & 0 \\ 0 & ... & 0 & 0 & G \end{pmatrix}$$

where Q_i is the Q-matrix for the i-th manifold M_i in the hybrid manifold, for example

$$Q_1 = \begin{pmatrix} F(q_1, q_1, z) & ... & G(q_1, q_{\mu_1}, z) \\ ... & ... & ... \\ G(q_{\mu_1}, q_1, z) & ... & F(q_{\mu_1}, q_{\mu_1}, z) \end{pmatrix}$$

and $G = (G_{kl})$ is an $N \times N$-matrix consisting of the Green functions (denoted by G_s) for the Neumann Laplacian on the segments:

$$G_{kl} = \begin{cases} G_s(q_k, q_l, z) & \text{if } q_k \text{ and } q_l \text{ belong to the same segment,} \\ 0 & \text{otherwise.} \end{cases}$$

Nevertheless, the matrix $Q(z)-\Lambda$ is too complicated to find its inverse explicitly. But letting $z \to \infty$ and applying Lemma 4.3.3 the situation improves radically! More precisely, all non-diagonal terms of $Q(z)$ are either zero or of type $G(q_i, q_j, z)$, $i \neq j$, and decay exponentially as z tends to infinity. Therefore, we can write $Q(z) = Q_d(z) + \tilde{Q}(z)$, where $\tilde{Q}(z)$ is exponentially small as z goes to infinity, and the leading order approximation

$Q_d(z)$ to $Q(z)$ is given by

$$Q_d(z) = \begin{pmatrix} F(q_1,q_1,z) & ... & 0 & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & ... & F(q_N,q_N,z) & 0 & ... & 0 \\ 0 & ... & 0 & G_s(q_1,q_1,z) & ... & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & ... & 0 & 0 & ... & G_s(q_N,q_N,z) \end{pmatrix}.$$

Then $Q(z)-\Lambda$ to first order, i.e. $Q_d(z)-\Lambda$, consists of four diagonal $N\times N$ blocs:

$$Q_d(z)-\Lambda = \begin{pmatrix} [F(q_i,q_i,z)-\lambda_{i,i}] & [-\lambda_{i,i+N}] \\ \overline{[-\lambda_{i,i+N}]} & [G_s(q_i,q_i,z)-\lambda_{i+N,i+N}] \end{pmatrix},\ i=1\dots N.$$

The inverse matrix can be found using the Frobenius formula for a block matrix consisting of matrices A,B,C,D:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A-BD^{-1}C)^{-1} & A^{-1}B(CA^{-1}B-D)^{-1} \\ (CA^{-1}B-D)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix}$$

Applying this to the matrix $Q_\Lambda(z)-\Lambda$, we obtain its inverse as four $N\times N$ diagonal blocks

$$[Q_d(z)-\Lambda]^{-1} = \begin{pmatrix} U & W \\ \overline{W} & V \end{pmatrix},$$

where the $N\times N$-matrices U,W,V are defined as

$$U_{ii} = \frac{G_i-\lambda_{i+N,i+N}}{(F_i-\lambda_{i,i})(G_i-\lambda_{i+N,i+N})-|\lambda_{i,i+N}|^2},\quad U_{ij}=0,\ i\neq j,$$

$$W_{ii} = \frac{\lambda_{i,i+N}}{(F_i-\lambda_{i,i})(G_i-\lambda_{i+N,i+N})-|\lambda_{i,i+N}|^2},\quad W_{ij}=0,\ i\neq j,$$

$$V_{ii} = \frac{F_i-\lambda_{i,i}}{(F_i-\lambda_{i,i})(G_i-\lambda_{i+N,i+N})-|\lambda_{i,i+N}|^2},\quad V_{ij}=0,\ i\neq j.$$

Here again F_i denotes $F(q_i,q_i,z)$ and G_i denotes $G_s(q_i,q_i,z)$. From this, we deduce:

Lemma 4.3.6. *With the notation above, we have*

$$[Q(z)-\Lambda]^{-1} = [Q_d(z)-\Lambda]^{-1} + O(e^{-cz}),$$

for some positive constant c.

Proof. First, we write

$$\begin{aligned}[Q(z)-\Lambda]^{-1} &= [Q_d(z)+\tilde{Q}(z)-\Lambda]^{-1}\\ &= [Q_d(z)-\Lambda]^{-1}[1+[Q_d(z)-\Lambda]^{-1}\tilde{Q}(z)]^{-1}.\end{aligned}$$

Assume for the moment that we are able to prove that $[Q_d(z)-\Lambda]^{-1} = O(z^2)$. As $\tilde{Q}(z)$ decays exponentially fast for large z, the product $[Q_d(z)-\Lambda]^{-1}\tilde{Q}(z)$ will then also be exponentially small. By using the Neuman series to compute $[1+[Q_d(z)-\Lambda]^{-1}\tilde{Q}(z)]^{-1}$, we see that this term is of the form $1+O(e^{-cz})$, and this immediately implies the conclusion of the Lemma. Now it remains to prove that $[Q_d(z)-\Lambda]^{-1} = O(z^2)$. By the computations done before the statement of the Lemma, we have to show that the terms U_{ii}, W_{ii} and V_{ii} have the order $O(z^2)$. This follows easily from the following two facts. First, using the explicit form of $G_i = G_s(q_i, q_i, z)$ obtained in equation 3.23, we get $G_i \sim 1/z$ as z goes to infinity. Next, by [Avramidi, 1998, Formula (38)] (see also the appendix) we have $F_i = c' \ln z + O(1)$ for some nonvanishing constant c'. This finishes the proof of the Lemma. □

This form of the matrix $[Q(z)-\Lambda]^{-1}$ allows us to get the following result.

Theorem 4.3.7. *As z tends to infinity we have for some positive constant c*

$$\begin{aligned}\operatorname{Tr} R^2(z) = &-\int_H \frac{G'_z(x,x,z)}{2z}\,dx\\ &-\frac{1}{4z^2}\sum_{i=1}^{N}\frac{(F_i)''_{zz}(G_i-\lambda_{i+N,i+N})+(G_i)''_{zz}(F_i-\lambda_{i,i})}{(F_i-\lambda_{i,i})(G_i-\lambda_{i+N,i+N})-|\lambda_{i,i+N}|^2}\\ &+\frac{1}{4z^3}\sum_{i=1}^{N}\frac{(F_i)'_z(G_i-\lambda_{i+N,i+N})+(G_i)'_z(F_i-\lambda_{i,i})}{(F_i-\lambda_{i,i})(G_i-\lambda_{i+N,i+N})-|\lambda_{i,i+N}|^2}\\ &+\frac{1}{4z^2}\sum_{i=1}^{N}\frac{((F_i)'_z(G_i-\lambda_{i+N,i+N}))^2+2(F_i)'_z(G_i)'_z|\lambda_{i,i+N}|^2+((G_i)'_z(F_i-\lambda_{i,i}))^2}{((F_i-\lambda_{i,i})(G_i-\lambda_{i+N,i+N})-|\lambda_{i,i+N}|^2)^2}\\ &+O(e^{-cz})\end{aligned}$$

Proof. Due to the special structure of the matrices G' and G'' it is easy to find the traces $\operatorname{Tr}(G''_{zz}[Q-\Lambda]^{-1})$, $\operatorname{Tr}(G'_z[Q-\Lambda]^{-1})$ and $\operatorname{Tr}(G'_z[Q-\Lambda]^{-1})^2$. We recall that these matrices are diagonal:

$$G' = \begin{pmatrix} F'_z & 0\\ 0 & G'_z\end{pmatrix}, \quad G'' = \begin{pmatrix} F''_{zz} & 0\\ 0 & G''_{zz}\end{pmatrix}.$$

Moreover, both matrices G' and G'' decay exponentially as z goes to infinity. To see this, we use the explicit expression of the Green function of a segment found in equation 3.23 to estimate G'_z and G''_{zz}, and [Avramidi, 1998, Formula (38)] to estimate F'_z and F''_{zz}. By Lemma 4.3.6, it follows that

$$G''[Q-\Lambda]^{-1} = \begin{pmatrix} F''U & F''W \\ G''\overline{W} & G''V \end{pmatrix} + O(e^{-cz}),$$

$$G'[Q-\Lambda]^{-1} = \begin{pmatrix} F'U & F'W \\ G'\overline{W} & G'V \end{pmatrix} + O(e^{-cz}),$$

and

$$(G'[Q-\Lambda]^{-1})^2 = \begin{pmatrix} (F'U)^2 + F'G'|W|^2 & (F')^2UW + F'G'WV \\ F'G'\overline{W}U + (G')^2\overline{W}V & F'G'|W|^2 + (G'V)^2 \end{pmatrix} + O(e^{-cz}),$$

where U, V, W are diagonal matrices forming the approximating matrix for $[Q-\Lambda]^{-1}$. So, using their diagonal form, we have

$$\begin{aligned}
\operatorname{Tr}\left(G''_{zz}\,[Q-\Lambda]^{-1}\right) &= \operatorname{Tr}(F''U + G''V) + O(e^{-cz}) \\
= \sum_{i=1}^{N} &\frac{(F_i)''_{zz}(G_i - \lambda_{i+N,i+N}) + (G_i)''_{zz}(F_i - \lambda_{i,i})}{(F_i - \lambda_{i,i})(G_i - \lambda_{i+N,i+N}) - |\lambda_{i,i+N}|^2} + O(e^{-cz}), \\
\operatorname{Tr}\left(G'_z\,[Q-\Lambda]^{-1}\right) &= \operatorname{Tr}(F'U + G'V) + O(e^{-cz}) \\
= \sum_{i=1}^{N} &\frac{(F_i)'_z(G_i - \lambda_{i+N,i+N}) + (G_i)'_z(F_i - \lambda_{i,i})}{(F_i - \lambda_{i,i})(G_i - \lambda_{i+N,i+N}) - |\lambda_{i,i+N}|^2} + O(e^{-cz}),
\end{aligned}$$

$$\operatorname{Tr}\left(G'_z\,[Q-\Lambda]^{-1}\right)^2 = \operatorname{Tr}((F'U)^2 + F'G'|W|^2 + F'G'|W|^2 + (G'V)^2) + O(e^{-cz})$$

$$= \sum_{i=1}^{N} \frac{((F_i)'_z(G_i - \lambda_{i+N,i+N}))^2 + 2(F_i)'_z(G_i)'_z|\lambda_{i,i+N}|^2 + ((G_i)'_z(F_i - \lambda_{i,i}))^2}{((F_i - \lambda_{i,i})(G_i - \lambda_{i+N,i+N}) - |\lambda_{i,i+N}|^2)^2} + O(e^{-cz}).$$

Substituting this in Theorem 4.3.5 finishes the proof. □

This theorem describes the large spectral parameter asymptotic of the trace of the second power of the resolvent for a Laplace operator through the Green functions of the manifolds and segments forming the hybrid manifold. In the next section, we will write this expansion in terms of heat kernel coefficients for the manifolds forming the hybrid manifold.

4.4 Resolvent expansion in terms of heat kernel coefficients

In this section we will use the results obtained by Avramidi [Avramidi, 1998, Section II.2] (see Appendix). In this work he has shown that the Green function for a compact Riemannian manifold has the following form near the diagonal:

$$G = G^{sing} + G^{non-anal} + G^{reg}$$

where each term (singular, non-analytic or regular with respect to to the geodesic distance σ between two arguments of the Green function) can be expanded in powers of the geodesic distance. If the dimension of the manifold is equal to 2, for the operator $\Delta + z^2$ on the manifold M_i, the regularized Green function on the diagonal asymptotically is (we denoted it by $F(x, x, z)$) [Avramidi, 1998, Formula (38)]: for all $N \geqslant 1$,

$$\begin{aligned} F(x, x, z) &:= G^{reg}(x, x, z) \\ &= \frac{1}{4\pi}\left(-\ln z^2 - 2\gamma + \sum_{n=1}^{N} \frac{\Gamma(n) a_{ni}(x, x)}{z^{2n}}\right) + O(z^{-2(N+1)}). \end{aligned}$$

Here $a_{ni}(x, x)$ is the n-th heat kernel coefficient on the i-th manifold, $\psi(z) = \frac{d}{dz}\Gamma(z)$, $\gamma = -\psi(1) = 0.577...$ is Euler's constant.

We need also the Green function for segments as a series in powers of z. Now, if we look at the exact form of this function for the segment $[a, b]$ (3.23)

$$G(x, y, z) = \frac{e^{z(x+y-2b)} + e^{-z|x-y|} + e^{z(|x-y|-2(b-a))} + e^{z(2a-x-y)}}{2z(1 - e^{2z(a-b)})}.$$

we can see that as $z \to \infty$ if $x, y \in [a, b]$ this expression can be simplified as

$$G(x, y, z) = \frac{e^{-z|x-y|} + e^{z(x+y-2b)} + e^{z(2a-x-y)}}{2z} + O(e^{-cz})$$

for some constant c.

In our previous notation $G_i = G(q_i, q_i, z)$ where q_i is one of the end points of the segment. And as we can see, for all q_i these expressions are the same and do not depend on the segment length:

$$G_i = G(a, a, z) = G(b, b, z) = \frac{1}{z} + O(e^{-cz}).$$

To complete the expression of the trace in terms of heat kernel coefficients, we need to consider the term $\int\limits_H \frac{G'_z(x,x,z)}{2z}\,dx$. In fact, this term consists of two types of integrals: over the manifolds and over the segments. As we already know ((4.1.4)), for manifolds the following expansion holds as $z \to \infty$: for all $q \geqslant 0$

$$-\int\limits_M \frac{G'_z(x,x,z)}{2z}\,dx = \operatorname{Tr} R_0^2(z) = \sum_{k=0}^{q} \frac{a_k\,\Gamma(k+1)}{4\pi z^{2k+2}} + O(z^{-2(q+2)}),$$

where R_0 is the resolvent of the Laplace operator on this manifold and a_k's are heat kernel coefficients for it. The integral over segments can be easily computed in this approximation, using the exact form of the Green function:

$$-\int\limits_{[a,b]} \frac{G'_z(x,x,z)}{2z}\,dx$$

$$= -\int\limits_{[a,b]} \frac{1}{2z}\left(\frac{1+e^{2z(x-b)}+e^{2z(a-x)}}{2z}\right)'_z dx = \frac{l}{4z^3} + \frac{1}{2z^4},$$

where l is the length of this segment.

Now, taking all these facts together, we can write

Theorem 4.4.1. *For large z and all $q > 0$ there holds*

$$\operatorname{Tr} R^2(z) = \sum_{m=1}^{M}\sum_{k=0}^{q} \frac{a_{km}\,\Gamma(k+1)}{4\pi z^{2k+2}}) + \sum_j \left(\frac{l_j}{4z^3} + \frac{1}{2z^4}\right)$$

$$-\sum_{i=1}^{N} \frac{(F_i)''_{zz}(\frac{1}{z} - \lambda_{i+N,i+N}) + \frac{2}{z^3}(F_i - \lambda_{i,i})}{4z^2(F_i - \lambda_{i,i})(\frac{1}{z} - \lambda_{i+N,i+N}) - |\lambda_{i,i+N}|^2}$$

$$+\sum_{i=1}^{N} \frac{(F_i)'_z(\frac{1}{z} - \lambda_{i+N,i+N}) - \frac{1}{z^2}(F_i - \lambda_{i,i})}{4z^3(F_i - \lambda_{i,i})(\frac{1}{z} - \lambda_{i+N,i+N}) - |\lambda_{i,i+N}|^2}$$

$$+\sum_{i=1}^{N} \frac{((F_i)'_z)^2(\frac{1}{z} - \lambda_{i+N,i+N})^2 - \frac{2}{z^2}(F_i)'_z|\lambda_{i,i+N}|^2 + \frac{1}{z^4}(F_i - \lambda_{i,i})^2}{4z^2((F_i - \lambda_{i,i})(\frac{1}{z} - \lambda_{i+N,i+N}) - |\lambda_{i,i+N}|^2)^2}$$

$$+O(z^{-2(q+2)}),$$

where a_{km} is the global k-th heat kernel coefficient on the m-th manifold M_m, l_j is the length of the segment L_j, λ_{ij} are elements of Λ and $F_i = F(q_i, q_i, z)$. Moreover, for all $p \geqslant 1$,

$$F(x,x,z) = \frac{1}{4\pi}\left(-2\gamma - \ln z^2 + \sum_{n=1}^{p} \frac{\Gamma(n)a_n(x,x)}{z^{2n}}\right) + O(z^{-2(p+1)}),$$

where $a_n(x,x)$ *is the local* n*-th heat kernel coefficient on the manifold* M *to which the point* x *belongs.*

To proceed we need the following

Lemma 4.4.2. *Suppose that the function* $f = f(z, \ln z^2)$ *has the following expansion for large* z

$$f(z, \ln z^2) = \sum_{n=0}^{N} \frac{c_n(\ln z^2)}{z^n} + O\left(\frac{1}{z^{N+1}}\right),$$

where $c_n(\ln z^2) = \frac{P_n(\ln z^2)}{Q_n(\ln z^2)}$ *are rational functions in* $\ln z^2$ *and* $\deg P_n \leqslant \deg Q_n$*. Then this expansion is unique and satisfies the property* $\frac{c_{n+1}(\ln z^2)}{z^{n+1}} = O\left(\frac{1}{z^{n+1}}\right)$ *(we will call such an expansion* z*-pseudoasymptotic).*

Proof. First, we notice that if P and Q are two polynoms with $\deg P \leqslant \deg Q$, then $\frac{P(\ln z^2)}{Q(\ln z^2)} = O(1)$ for large z. Therefore, we obtain that $\frac{c_{n+1}(\ln z^2)}{z^{n+1}} = O(\frac{1}{z^{n+1}})$. Now, suppose that there exists another expansion of f with the same properties but with other coefficients d_n:

$$f(z, \ln z^2) = \sum_{n=0}^{N} \frac{d_n(\ln z^2)}{z^n} + O\left(\frac{1}{z^{N+1}}\right).$$

Subtracting it from first one we obtain

$$c_0(\ln z^2) - d_0(\ln z^2) = O\left(\frac{1}{z}\right) \Leftrightarrow z(c_0(\ln z^2) - d_0(\ln z^2)) = O(1),$$

But $c_0(\ln z^2) - d_0(\ln z^2)$ is a rational function in $\ln z^2$, so that its product with z has to go to infinity as z goes to infinity, unless it is identically zero. Hence we get $c_0 = d_0$. By an immediate induction on n, we show similarly that we will always obtain $z(c_n(\ln z^2) - d_n(\ln z^2)) = O(1)$ and consequently the uniqueness of the expansion. □

Let us remark now that if a function f possesses a z-pseudoasymptotic expansion, then each coefficient c_n can be uniquely expanded as a $\ln z^2$-asymptotic series: $c_n(\ln z^2) = \sum_{k=0}^{K} \frac{c_{nk}}{(\ln z^2)^k} + O\left(\frac{1}{(\ln z^2)^{K+1}}\right)$, for some constants c_{nk}.

Theorem 4.4.3. *Consider the hybrid manifold* H*, consisting in manifolds* M_i *and* N *segments* L_j*, and consider a Laplace operator on* H *(corresponding to boundary conditions determined by a matrix* Λ*, and disjoint with* D_0*). Suppose also that for all* i *the coefficients* $\lambda_{i+N,i+N}$ *do not vanish. Then the square* *of the resolvent $R(z)$, obtained in Theorem 4.4.1 has a z-pseudoasymptotic expansion which has the form:*

$$\operatorname{Tr} R^2(z) = \frac{\sum_i Vol(M_i)}{4\pi z^2} + \frac{\sum_j l_j}{4z^3}$$
$$+\frac{c_4(\ln z^2)}{z^4} + \frac{c_5(\ln z^2)}{z^5} + \frac{c_6(\ln z^2)}{z^6} + \frac{c_7(\ln z^2)}{z^7} + O\left(\frac{1}{z^8}\right)$$

The coefficients c_n are rational functions and have the following $\ln z^2$-expansions:

$$c_4 = \frac{\sum_i \chi(M_i)}{6} + \frac{N}{4} + \frac{N}{\ln z^2}$$
$$+\sum_{i=1}^{N} \frac{1 - 2\gamma - 4\pi\lambda_{i,i} + 4\pi\frac{|\lambda_{i,i+N}|^2}{\lambda_{i+N,i+N}}}{\ln^2 z^2} + O\left(\frac{1}{\ln^3 z^2}\right),$$
$$c_5 = \sum_{i=1}^{N} \frac{3}{4\lambda_{i+N,i+N}} + \sum_{i=1}^{N} \frac{3\pi|\lambda_{i,i+N}|^2}{\lambda^2_{i+N,i+N}\ln z^2} + O\left(\frac{1}{\ln^2 z^2}\right),$$
$$c_6 = \sum_{M_i} \frac{a_{2i}}{2\pi} + \sum_{i=1}^{N} \frac{1}{\lambda^2_{i+N,i+N}} + \sum_{i=1}^{N} \frac{2a_{1i}\lambda^3_{i+N,i+N} + 8\pi|\lambda_{i,i+N}|^2}{\lambda^3_{i+N,i+N}\ln z^2} + O\left(\frac{1}{\ln^2 z^2}\right),$$
$$c_7 = \sum_{i=1}^{N} \frac{5}{4\lambda^3_{i+N,i+N}} + \sum_{i=1}^{N} \frac{15\pi|\lambda_{i,i+N}|^2}{\lambda^4_{i+N,i+N}\ln z^2} + O\left(\frac{1}{\ln^2 z^2}\right),$$

where a_{ki} is the k-th heat kernel coefficient for the manifold M_i, $Vol(M_i)$ and $\chi(M_i)$ are the volume and Euler characteristic of M_i, l_j is the length of the segment L_j, γ is Euler's constant and λ's are elements of the boundary condition matrix Λ.

Proof. Substituting $F(x, x, z)$ into the expression in Theorem 4.4.1 we see that the $\ln z^2$-part of $\operatorname{Tr} R^2$ depends on z and $\ln z^2$ in the following way:

$$\operatorname{Tr} R^2 = \sum_{i=1}^{N} D_1 \frac{\frac{\ln z^2}{z^5} + \sum_{k=0}^{K} \frac{d'_k}{z^k} + O(\frac{1}{z^{K+1}})}{d''_0 + \ln z^2 + \frac{\ln z^2}{z} + \sum_{k=1}^{K} \frac{d''_k}{z^k} + O(\frac{1}{z^{K+1}})}$$
$$+\sum_{i=1}^{N} D_2 \frac{\frac{(\ln z^2)^2}{z^6} + \frac{e'_0 \ln z^2}{z^6} + \sum_{k=0}^{K} \frac{e'_k}{z^k} + O(\frac{1}{z^{K+1}})}{\left(e''_0 + \ln z^2 + \frac{\ln z^2}{z} + \sum_{k=1}^{K} \frac{e''_k}{z^k} + O(\frac{1}{z^{K+1}})\right)^2} + O(e^{-cz}),$$

for some constants $D_1, D_2, d', d'', e', e''$.

Consider the first fraction:

$$\frac{\frac{\ln z^2}{z^5}+\sum_{k=0}^K \frac{d'_k}{z^k}+O(\frac{1}{z^{K+1}})}{d''_0+\ln z^2+\frac{\ln z^2}{z}+\sum_{k=1}^K \frac{d''_k}{z^k}+O(\frac{1}{z^{K+1}})}$$
$$=\frac{\frac{\ln z^2}{z^5}+\sum_{k=0}^K \frac{d'_k}{z^k}+O(\frac{1}{z^{K+1}})}{\ln(z^2e^{d''_0})\left(1+\frac{\ln z^2}{z\ln(z^2e^{d''_0})}+\sum_{k=1}^K \frac{d''_k}{z^k\ln(z^2e^{d''_0})}+O(\frac{1}{z^{K+1}})\right)}.$$

For large z the last term in the denominator can be expanded as

$$\left(1+\frac{\ln z^2}{z\ln(z^2e^{d''_0})}+\sum_{k=1}^K \frac{d''_k}{z^k\ln(z^2e^{d''_0})}+O(\frac{1}{z^{K+1}})\right)^{-1}$$
$$=1+\sum_{l=1}^L \frac{g_l\ln^{s_1}(z^2)}{\ln^{s_2}(z^2e^{d''_0})z^l}+O\left(\frac{1}{z^{L+1}}\right),$$

for some constant g_l and non-negative powers s_1, s_2. On can see also that $s_1 \leqslant s_2$. Multiplying both series in powers of z in the expression for the first fraction we obtain an expansion of type

$$\sum_{l=0}^L \frac{g'_l\ln^{t_1}(z^2)}{\ln^{t_2}(z^2e^{d''_0})z^l}+O\left(\frac{1}{z^{L+1}}\right)=\sum_{l=0}^L \frac{c_l(\ln z^2)}{z^l}+O\left(\frac{1}{z^{L+1}}\right),$$

with coefficients $c_l(\ln z^2) = \frac{P_l(\ln z^2)}{Q_l(\ln z^2)}$ which are rational in $\ln z^2$. There still holds $\deg P_l \leqslant \deg Q_l$.

The second fraction can be considered in the same way and we obtain the same structure of the expansion. We have proved that $\operatorname{Tr} R^2$ has a z-pseudoasymptotic expansion. In order to find the coefficients c_l we should perform the procedure described above more thoroughly and summarize the coefficients over all gluing points. We find that the first non-zero coefficient is c_4 and

$$c_4=\sum_{i=1}^N \frac{\lambda_{i+N,i+N}(\lambda_{i+N,i+N}(1+2\gamma+\ln z^2+4\pi\lambda_{i,i})-4\pi|\lambda_{i,i+N}|^2)}{(\lambda_{i+N,i+N}(2\gamma+\ln z^2+4\pi\lambda_{i,i})-4\pi|\lambda_{i,i+N}|^2)^2}.$$

It can be expanded in powers of $\ln z^2$, which gives us

$$c_4=\frac{N}{\ln z^2}+\sum_{i=1}^N \frac{1-2\gamma-4\pi\lambda_{i,i}+4\pi\frac{|\lambda_{i,i+N}|^2}{\lambda_{i+N,i+N}}}{\ln^2 z^2}+O\left(\frac{1}{\ln^3 z^2}\right).$$

One can compute in the same way further coefficients and their expansion in powers of $\ln z^2$, but it requires complicated computations and we do not write them here. □

Chapter 5

Inverse spectral theory of hybrid manifolds

Before we state the results concerning the inverse spectral problem on hybrid manifolds we should state some additional results. Let us first note that the formula for the $\operatorname{Tr} R^2$ obtained in Theorem 4.4.1 depends on the heat kernel coefficients for the smooth parts of the hybrid manifold, and it is well known that heat kernel coefficients are recursive. The recursivity of the coefficients in the expansion of $\operatorname{Tr} R^2$ follows as well. But as series inversion is required, we cannot obtain this recursive formula in an explicit way. Nevertheless, it is possible to find the formula for some terms in this expansion.

To simplify notation in this section, we will use the following convention. If the function $f = f(z, \ln z^2)$ has a z-pseudoasymptotic expansion with coefficients $c_n(\ln z^2)$, we noticed before the statement of Theorem 4.4.3 that each $c_n(\ln z^2)$ can be expanded as $c_n(\ln z^2) = \sum_{k=0}^{K} \frac{c_{nk}}{(\ln z^2)^k} + O\left(\frac{1}{(\ln z^2)^{K+1}}\right)$, for some constants c_{nk}. We will then write

$$f \sim \sum_{n,k=0}^{\infty} \frac{c_{nk}}{z^n (\ln z^2)^k}.$$

5.1 First terms in the resolvent expansion.

Lemma 5.1.1. *Fix some integer $n \geq 4$ and consider the coefficient c_n which appears in Theorem 4.4.3. The coefficient c_n has an expansion in $\ln(z^2)$.*

Moreover, the first term of this expansion has the following form:

$$\frac{\sum_i \chi(M_i)}{6} + \frac{N}{4}, \quad n = 4;$$

$$\sum_{i=1}^{N} \frac{2k-1}{4\lambda_{i+N,i+N}^{2k+1}}, \quad n = 2k+1,\ k > 1;$$

$$\sum_i \frac{a_{ki}\Gamma(k+1)}{4\pi} + \sum_{i=1}^{N} \frac{2k}{4\lambda_{i+N,i+N}^{2k-2}}, \quad n = 2k+2,\ k > 1.$$

Proof. The fact, that the coefficients c_n has an expansion in $\ln(z^2)$ was observed before the statement of Theorem 4.4.3. The terms in c_4 and the first term in c_n for even n arise from the expansions of the second power of the resolvents for all manifolds and segments. The other terms require some calculation. Let us look once more at the expression in Theorem 4.4.1 and analyze it more carefully. As we can see the expansion of $\frac{1}{(F-\lambda_1)(\frac{1}{z}-\lambda_2)-|\lambda_3|^2)}$ always has $\ln z^2$ in the denominator, i.e. for some non-zero constants k''_{nm} one has the z-pseudoasymptotic expansion

$$\frac{1}{D_i} \sim \frac{4\pi}{\lambda_{i+N,i+N}\ln z^2} + \sum_{n,m=0}^{\infty} \frac{k''_{nm}}{z^n(\ln z^2)^{m+1}},$$

where we denote $(F_i - \lambda_{i,i})(\frac{1}{z} - \lambda_{i+N,i+N}) - |\lambda_{i,i+N}|^2)$ by D_i. In the corresponding numerators we have terms of type F_i'', F_i' and F_i. Their expansions as $z \to \infty$ are: for all $p \geqslant 1$

$$F_i(x,x,z) = \frac{1}{4\pi}\left(-2\gamma - \ln z^2 + \sum_{n=1}^{p} \frac{\Gamma(n)a_{ni}(x,x)}{z^{2n}}\right) + O(z^{-2p-2})),$$

$$F_i'(x,x,z) = \frac{-1}{2\pi z} - \sum_{n=1}^{p} \frac{2n\Gamma(n)a_{ni}(x,x)}{4\pi z^{2n+1}} + O(z^{-2p-3})),$$

$$F_i''(x,x,z) = \frac{1}{2\pi z^2} + \sum_{n=1}^{p} \frac{2n(2n+1)\Gamma(n)a_{ni}(x,x)}{4\pi z^{2n+2}} + O(z^{-2p-4}).$$

It is clear now that the first terms in the expansion of c_n (i.e. terms without $\ln z^2$ can appear only in terms containing F_i. All other terms will contain this logarithm in denominator. Moreover, only the part $\frac{-\ln z^2}{4\pi}$ in F_i plays a role.

The same arguments are valid for the last summand whose denominator contains $(\ln z^2)^2$, and consequently we should take only the term $(F_i - \lambda_{i,i})^2$

into account. Finally, we try to find the terms containing only powers of z in

$$\frac{-2F_i - F_i}{4z^5 D_i} + \frac{(F_i - \lambda_{i,i})^2}{4z^6 (D_i)^2}.$$

First of all let us treat the denominator more carefully

$$\frac{1}{D_i} \sim \frac{4\pi}{\lambda_{i+N,i+N} \ln z^2} \left(1 - \frac{1}{z\lambda_{i+N,i+N}} + \sum_{n=0}^{\infty} \frac{k'_n}{z^n \ln z^2}\right)^{-1}$$
$$\sim \frac{4\pi}{\lambda_{i+N,i+N} \ln z^2} \left(1 + \sum_{n=1}^{\infty} \frac{1}{(z\lambda_{i+N,i+N})^n} + \sum_{n,m=0}^{\infty} \frac{k''_n}{z^n (\ln z^2)^{m+1}}\right),$$

with some coefficients k'_n, k''_n, whose explicit expressions are not important now.

Then we see that in $\frac{-3F_i}{4z^5 D_i}$ the terms containing only powers of z (without logarithm) are

$$\frac{-3}{4z^5} \cdot \frac{4\pi}{\lambda_{i+N,i+N} \ln z^2} \left(1 + \sum_{n=1}^{\infty} \frac{1}{(z\lambda_{i+N,i+N})^n}\right) \cdot \frac{-\ln z^2}{4\pi}$$
$$= \frac{3}{4} \sum_{n=0}^{\infty} \frac{1}{z^{n+5} \lambda_{i+N,i+N}^{n+1}}.$$

Performing the same reasoning we find that

$$\frac{1}{D_i^2} \sim \frac{(4\pi)^2}{(\lambda_{i+N,i+N} \ln z^2)^2} \left(\sum_{n=0}^{\infty} \frac{n+1}{z^n \lambda_{i+N,i+N}^n} + \sum_{n,m=0}^{\infty} \frac{k''_n}{z^n (\ln z^2)^{m+1}}\right).$$

The only term which cancels this logarithm in $(F_i - \lambda_{i,i})^2$ is $\frac{(\ln z^2)^2}{(4\pi)^2}$. And the contribution to the "pure" polynomial part in z is

$$\frac{1}{4z^6} \cdot \frac{(4\pi)^2}{(\lambda_{i+N,i+N} \ln z^2)^2} \sum_{n=0}^{\infty} \frac{n+1}{z^n \lambda_{i+N,i+N}^n} \cdot \frac{(\ln z^2)^2}{(4\pi)^2} = \frac{1}{4} \sum_{n=0}^{\infty} \frac{n+1}{z^{n+6} \lambda_{i+N,i+N}^{n+2}}.$$

Finally, we find (not taking into account those which arise from the expansion of R_0)

$$\frac{3}{4} \sum_{n=0}^{\infty} \frac{1}{z^{n+5} \lambda_{i+N,i+N}^{n+1}} + \frac{1}{4} \sum_{n=0}^{\infty} \frac{n+1}{z^{n+6} \lambda_{i+N,i+N}^{n+2}} = \frac{1}{4} \sum_{n=0}^{\infty} \frac{n+3}{z^{n+5} \lambda_{i+N,i+N}^{n+1}}.$$

Collecting the terms with the same power of z we prove the lemma. □

The next result requires more complicated calculations of the same nature, so we just state the main steps of its proof.

Lemma 5.1.2. *Fix some integer* $n \geq 4$ *and consider the coefficient* c_n *which appears in Theorem 4.4.3. The second term of the expansion of* c_n *has the following form:*

$$\sum_{i=1}^{N} \frac{\pi(n-4)(n-2)|\lambda_{i,i+N}|^2}{\lambda_{i+N,i+N}^{n-3} \ln z^2}, \quad n = 2k+1,\ k > 1;$$

$$\sum_{i=1}^{N} \frac{\pi(n-4)(n-2)|\lambda_{i,i+N}|^2}{\lambda_{i+N,i+N}^{n-3} \ln z^2} + \sum_{i=1}^{N} \frac{a_{li}(l+1)!}{\ln z^2}, \quad n = 2l+4,\ l \geqslant 0.$$

Proof. The idea is the same as before: one can obtain the terms of this form only from specific terms in the expansion. Due to the special structure of the denominator we have terms of type $\frac{f(z)}{\ln z^2}$ and terms $\frac{g(z)\ln z^2}{\ln^2 z^2}$ and the fact that in the numerator only the terms with F_i contain a logarithm, allows us to restrict our calculation to some specific terms.

We need as before the expansion of $\frac{1}{D_i}$ but up to the second power of $\ln z^2$ and the expansion of $\frac{1}{D_i^2}$ up to the third power. To simplify the expressions we denote the following quantity by W_i

$$W_i = -2\gamma - 4\pi\left(\lambda_{i,i} - \frac{|\lambda_{i,i+N}|^2}{\lambda_{i+N,i+N}}\right) + \frac{2\gamma + 4\pi\lambda_{i,i}}{z\lambda_{i+N,i+N}} + \left(1 - \frac{1}{z\lambda_{i+N,i+N}}\right)\sum_{n=1}^{\infty} \frac{(n-1)!a_{ni}}{z^{2n}}$$

and state that up to the second power of $\ln z^2$

$$\frac{1}{D_i} \sim \frac{4\pi}{\lambda_{i+N,i+N} \ln z^2} \sum_{n=0}^{\infty} \left(\frac{1}{z\lambda_{i+N,i+N}}\right)^n + \frac{4\pi W_i}{\lambda_{i+N,i+N} \ln^2 z^2} \sum_{n=1}^{\infty} n \left(\frac{1}{z\lambda_{i+N,i+N}}\right)^{n-1}$$

and up to the third power

$$\frac{1}{D_i^2} \sim \frac{16\pi^2}{\lambda_{i+N,i+N}^2 \ln^2 z^2} \sum_{n=0}^{\infty} \frac{n}{(z\lambda_{i+N,i+N})^{n-1}} + \frac{4\pi^2 W_i}{\lambda_{i+N,i+N}^2 \ln^3 z^2} \sum_{n=0}^{\infty} \frac{n(n-1)}{(z\lambda_{i+N,i+N})^{n-2}}.$$

Then the terms we are looking for appear in

$$\frac{1}{D_i}\left(\frac{-F_i''(\frac{1}{z}-\lambda_{i+N,i+N})}{4z^2}-\frac{3(F_i-\lambda_{i,i})}{4z^5}+\frac{F_i'(\frac{1}{z}-\lambda_{i+N,i+N})}{4z^3}\right),$$

and in

$$\frac{1}{D_i^2}\left(\frac{(F_i-\lambda_{i,i})^2}{4z^6}\right).$$

Expanding and summing these expressions we find that the terms containing the first power of logarithm in the denominator can be arranged in two sums as in the statement of the Lemma. □

In the same way but using much more complicated calculations we can get the following result whose proof we omit.

Lemma 5.1.3. *In Theorem 4.4.3 the terms with* $\ln^2 z^2$ *in the denominator of the* c_n *are (we do not separate the different powers of* z *in order to not complicate the expression):*

$$\frac{1}{\ln^2 z^2}\sum_{i=1}^{N}\left(\frac{1}{z^4}-\frac{2\gamma}{z^4}-\frac{4\pi\lambda_{2i}}{z^4}+\sum_{n=1}^{\infty}\frac{(2n+1)(n-1)!a_{ni}}{z^{2n+4}}+(2\pi\lambda_{2i}\right.$$
$$-\gamma)\frac{2\pi\lambda_{3i}^2}{z^4\lambda_{1i}}\sum_{n=1}^{\infty}\frac{n(n+2)}{(z\lambda_{1i})^n}-\gamma\sum_{n=1}^{\infty}\frac{2(n+1)n!a_{ni}}{z^{2n+4}}$$
$$+\frac{2\pi^2\lambda_{3i}^4}{\lambda_{1i}^2z^4}\sum_{n=1}^{\infty}\frac{n(n^2-1)}{(z\lambda_{1i})^{n-1}}-4\pi\lambda_{2i}\sum_{n=1}^{\infty}\frac{(n+1)n!a_{ni}}{z^{2n+4}}+\frac{4\pi\lambda_{3i}^2}{z^4\lambda_{1i}}\sum_{n=0}^{\infty}\frac{n+1}{(z\lambda_{1i})^n}$$
$$+\sum_{n=1}^{\infty}\frac{(n-1)!a_{ni}}{z^{2n}}\sum_{k=1}^{\infty}\frac{(k+1)k!a_{ki}}{z^{2k+4}}+\frac{\pi\lambda_{3i}^2}{z^4\lambda_{1i}}\left(\sum_{n=1}^{\infty}\frac{(n-1)!a_{ni}}{z^{2n}}\sum_{k=1}^{\infty}\frac{k(k+2)}{(z\lambda_{1i})^k}\right.$$
$$\left.\left.+\sum_{n=1}^{\infty}\frac{2(2n+1)n!a_{ni}}{z^{2n}}\sum_{k=0}^{\infty}\frac{1}{(z\lambda_{1i})^k}+\sum_{n=1}^{\infty}\frac{2n!a_{ni}}{z^{2n}}\sum_{k=0}^{\infty}\frac{1+2k}{(z\lambda_{1i})^k}\right)\right),$$

where $\lambda_{1i}=\lambda_{i+N,i+N}$, $\lambda_{2i}=\lambda_{i,i}$ *and* $\lambda_{3i}=|\lambda_{i,i+N}|$.

5.2 Inverse spectral data

We consider now the inverse spectral problem, i.e. the question "Which information about the initial system can one obtain using the expansion of the second power of the resolvent?". As we will see, it is possible to find some geometric characteristics of the system and some information about the operator on it.

Theorem 5.2.1. *Consider the expansion of the trace of the square of the resolvent as in the Theorem 4.4.3. The knowledge of* TrR^2 *determines:*

- *whether this manifold is hybrid or "normal";*
- *the sum of the volumes of all manifolds taking part in the construction;*
- *the sum of the Euler characteristics of all manifolds;*
- *the number of segments used in this hybrid manifold;*
- *the sum of the lengths of these segments;*
- *the Euler characteristic of the hybrid manifold.*

Proof. The presence of $\log z^2$-type terms is a criteria of singularity. If in the expansion of TrR^2 there are no such terms, this means that the considered manifold is a "normal" manifold without any singular points. Indeed, the log-terms appear only from the singularities of the Green function (and not its derivatives) at the points of gluing.

The coefficient of z^{-2} is equal to $\frac{\sum VolM_i}{4\pi}$ and provides us with the sum of the volumes of all manifolds.

The coefficient of z^{-3} is equal to $\frac{\sum_{L_j} l_j}{4}$ and provides us with the sum of the lengths of all segments.

Considering the term of type $\frac{1}{z^4 \log z^2}$ we find the number N of all segments.

The coefficient of z^{-4} is equal to $\frac{\sum \chi(M_i)}{6} + \frac{N}{4}$ and gives as the sum of the Euler characteristics of all manifolds since we already know the number N.

As it was shown in Theorem 2.2.5 the Euler characteristic of the hybrid manifold is equal to $\sum \chi(M_i) - N$ and is easy to find now. □

The results obtained in this work do not let us find the volume of each manifold and the length of each segment separately. But it can be possible to find these quantities with the use of scattering theory Kurasov and Nowaczyk [2005]. The resolvent expansion provides us also with some information about how we glue the segments to the manifold in the hybrid space. Namely, we can obtain some information about the matrix of boundary conditions which defines the Laplace operator on the hybrid space.

Theorem 5.2.2. *Consider the z-pseudoasymptotic expansion of the trace of the square of the resolvent expansion of type 4.4.3. If we assume that we know the heat kernel coefficients for all manifolds composing the hybrid manifold, and that the coefficients* $\lambda_{i+N,i+N}$ *are mutually distinct and nonzero, we can find the diagonal elements of the matrix of boundary conditions* Λ *and the absolute values of its non-diagonal elements up to permutation.*

Proof. As it was shown in Lemma 5.1.1 the first terms in asymptotic expansion of c_n are of the following form:

$$\frac{\sum_{M_i} \chi(M_i)}{6} + \frac{N}{4}, \quad n = 4;$$
$$\sum_{i=1}^{N} \frac{2k-1}{4\lambda_{i+N,i+N}^{2k+1}}, \quad n = 2k+1,\ k > 1;$$
$$\sum_{M_i} \frac{a_{ki}\Gamma(k+1)}{4\pi} + \sum_{i=1}^{N} \frac{2k}{4\lambda_{i+N,i+N}^{2k-2}}, \quad n = 2k+2,\ k > 1.$$

Consider $x_i = \frac{1}{\lambda_{i+N,i+N}}$, $i = 1, \cdots, N$. Taking the first N coefficients of powers of z starting from z^5 and denoting them by d_{n+4}, $n = 1, \cdots, N$ we obtain N equations which have the following structure

$$d_{n+4} = \sum_i x_i^n$$

To find the solution of such a system we find the following symmetric polynomials in terms of c_n using a well-known recursive procedure:

$$\begin{gathered} S_1 = x_1 + \cdots + x_N, \\ S_2 = x_1x_2 + x_1x_3 + \cdots + x_2x_3 + \ldots, \\ S_3 = x_1x_2x_3 + x_1x_2x_4 + \cdots + x_2x_3x_4 + \ldots, \\ \cdots \\ S_N = x_1x_2 \ldots x_N. \end{gathered}$$

Due to the Vieta Theorem we state now that $x_1, \ldots x_N$ are the roots of the equation

$$x^N - S_1x^{N-1} + S_2x^{N-2} + \cdots + S_N = 0$$

and can be found up to permutation because we assume that they are mutually distinct.

If we substitute the obtained values of $\lambda_{i+N,i+N}$ in the expression obtained in Lemma 5.1.2 we obtain a system of linear equations for $|\lambda_{i,i+N}|$ and can find them.

As soon as we have found these elements, the result of Lemma 5.1.3 give us a system of linear equations for $\lambda_{i,i}$ and we can also find them. The Theorem is proved. □

Chapter 6

Degenerated cases

6.1 Quantum graph

Our hybrid manifold in this case is a system of N segments L_j parameterized by $\{x_j \in \mathbb{R} : x_j \in [0, l_j]\}$, where $j = 1, \ldots, N$. We consider the disjoint union $L_1 \sqcup \cdots \sqcup L_N$ and construct the topological space from this space by gluing the end points 0 and l_j of each segments L_j to one of the other end points. There are no independency conditions in this case, i.e. no restriction on the structure of the graph. Two or more segments can be glued in the same vertex.

Let us enumerate all ends of the segments by $q_1, \ldots, q_{2N}$. Taking the direct sum of the Laplace operator on those segments and appling the extension theory we obtain a symmetric operator with deficiency indices $(2N, 2N)$ as for a general hybrid manifold. As we have already shown, a boundary value space for one segment $[0, l]$ can be chosen as

$$\Gamma_1(f) = (-f'(0), f'(l)); \quad \Gamma_2(f) = (f(0), f(l)).$$

And as z tend to ∞ the Q-matrix is asymptotically

$$Q(z) = \begin{pmatrix} \frac{1}{z} & 0 \\ 0 & \frac{1}{z} \end{pmatrix} + O(e^{-cz}).$$

We construct the direct sum of all such boundary value spaces, and obtain a diagonal Q-matrix for the whole system ($Q = \frac{1}{z}I$, where I is the identity matrix). As before, Λ denotes the matrix of boundary conditions $\Gamma_2(f) = \Lambda\Gamma_1(f)$. The resolvent of each self-adjoint operator on the obtained space defined by Λ can be expressed with the help of the Krein formula in terms of Green functions $G(x, y, z)$ for the segments with Neumann boundary condition ($\Gamma_1 = 0$) and the matrices Q, Λ.

We note that

$$R^2(z) = -\frac{1}{2z}\frac{\partial}{\partial z}(R_0(z) - A(z)) = R_0^2(z) + \frac{1}{2z}\frac{\partial}{\partial z}A(z),$$

where $A(z) = \gamma(z)[Q(z) - \Lambda]^{-1}(\gamma(\bar{z}))^*$. The first term was already discussed before and we try to simplify the second one in order to have some additional information for the inverse problem in this case.

As it was already shown, $\operatorname{Tr} A(z) = \frac{-1}{2z}\operatorname{Tr} Q'[Q - \Lambda]^{-1}$. Note that if we conjugate Λ with a unitary transformation U, this does not change $\operatorname{Tr} A$ due to the diagonal structure of Q:

$$\begin{aligned}\operatorname{Tr} Q'[Q - U\Lambda U^{-1}]^{-1} &= -\operatorname{Tr}\frac{I}{z^2}\left[\frac{I}{z} - U\Lambda U^{-1}\right]^{-1} \\ &= \operatorname{Tr} U(Q'[Q-\Lambda]^{-1})U^{-1} = \operatorname{Tr} Q'[Q-\Lambda]^{-1}.\end{aligned}$$

We suppose then that the matrix Λ is a diagonal $2N \times 2N$ matrix with k non-zero eigenvalues $\lambda_1, \ldots, \lambda_k$. It is easy to find $[Q - \Lambda]^{-1}$ in this case:

$$[Q-\Lambda]^{-1} = \begin{pmatrix} \frac{1}{z} - \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{z} - \lambda_k & 0 & 0 & 0 \\ 0 & 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & z \end{pmatrix}^{-1} + O(e^{-cz}).$$

And finally we have

$$\begin{aligned}\operatorname{Tr} A(z) &= \frac{-1}{2z}\operatorname{Tr} Q'[Q-\Lambda]^{-1} \\ &= \frac{1}{2z^3}\left(\sum_{i=1}^{k}\frac{-1}{\lambda_i(1 - \frac{1}{z\lambda_i})} + z(2N-k)\right) + O(e^{-cz}) \\ &= \frac{2N-k}{2z^2} - \sum_{n=0}^{\infty}\frac{1}{2z^{n+3}}\sum_{i=1}^{k}\lambda_i^{-n-1} + O(e^{-cz}).\end{aligned}$$

and

$$\begin{aligned}\operatorname{Tr} R^2 &= \operatorname{Tr} R_0^2 + \frac{1}{2z}\operatorname{Tr} A(z) \\ &= \frac{\sum l_j}{4z^3} + \frac{N}{2z^4} - \frac{2N-k}{2z^4} + \sum_{n=0}^{\infty}\frac{(n+3)\sum_{i=1}^{k}\lambda_i^{-n-1}}{4z^{n+5}} + O(e^{-cz}) \\ &= \frac{\sum l_j}{4z^3} + \frac{k-N}{2z^4} + \sum_{n=0}^{\infty}\frac{(n+3)\sum_{i=1}^{k}\lambda_i^{-n-1}}{4z^{n+5}} + O(e^{-cz}).\end{aligned}$$

We have proved the following

Theorem 6.1.1. *Consider the hybrid manifold consisted only of N segments and a Laplace S operator on it, defined by a matrix Λ of boundary conditions (possibly degenerate, with k non-zero eigenvalues). Then the expansion of the square of the resolvent of S in terms of z as z tends to infinity is*

$$\operatorname{Tr} R^2 = \frac{\sum l_j}{4z^3} + \frac{k-N}{2z^4} + \sum_{n=0}^{\infty} \frac{(n+3)\sum_{i=1}^{k} \lambda_i^{-n-1}}{4z^{n+5}} + O(e^{-cz}).$$

Considering the inverse spectral problem for this case we state

Theorem 6.1.2. *From the expansion above it is possible to find the number, the sum of the lengthes of all segments, and the matrix Λ of boundary conditions up to a unitary transformation.*

Proof. It is easy to see that in the case when the matrix Λ is degenerate we cannot find its size directly from this expansion. Nevertheless this is possible if we use the following proposition

Proposition 6.1.3. *Consider a hermitian matrix Λ (possibly degenerate) of an unknown even size $2N \times 2N$. Suppose that we know the following functions of its non-zero eigenvalues: $\sum_{i=1}^{k} \lambda_i^{-n-1}$ for all natural numbers n and the difference between the half-size of the matrix and the number of non-zero eigenvalues (i.e $N-k$). Then we can find the size of Λ and the matrix itself up to a unitary transformations.*

Proof. Let us first of all consider the diagonal matrix A with entries λ_i^{-1} for non-zero eigenvalues of Λ. Then we know $\operatorname{Tr} A^n$ for all natural numbers n. We use the following formula to find the eigenvalue of A with the greatest absolute value:

$$\ln |\lambda_{max}| = \lim_{n\to\infty} \frac{\ln \operatorname{Tr} \Lambda^{2n}}{2n}$$

We use only the trace of even powers of A to assure the existence of the logarithm. As soon as we have found the greatest eigenvalue, we can subtract it in the corresponding power from the trace: $\operatorname{Tr} A^{2n} - |\lambda_{max}|^{2n}$ and repeat the procedure. In this way, we obtain all the eigenvalues of A up to sign.

We will now find the sign of each eigenvalue using the trace of the odd powers of A. Let us rewrite it as

$$\operatorname{Tr} A^{2n+1} = \sum_i c_i |\lambda_i|^{2n+1},$$

where $c_i = \pm 1$. We can find the number

$$\lim_{n\to\infty} \frac{\operatorname{Tr} A^{2n+1}}{|\lambda_m ax|} = \sum_{|\lambda_i|=|\lambda_{max}|} c_i = d_{max}.$$

In fact, we know also the number of eigenvalues with the absolute value $|\lambda_{max}|$ (denote it by k_{max}). This information is sufficient to find the number of positive (k_+) and negative (k_-) eigenvalues:

$$\begin{cases} k_+ + k_- = k_{max} \\ k_+ - k_- = d_{max} \end{cases} \rightarrow k_+ = (k_{max} + d_{max})/2, \quad k_- = (k_{max} - d_{max})/2.$$

We know now that A has k_+ eigenvalues λ_{max} and k_- eigenvalues $-\lambda_{max}$. Subtracting the corresponding powers of $\pm\lambda_{max}$ from the traces of the odd powers of A and iterating the procedure, we find all eigenvalues of A and the number of them.

The non-zero eigenvalues of the matrix Λ are inverse to the eigenvalues of A. Due to the additional information about the difference between half-size of the matrix and the number of non-zero eigenvalues we can find also the size $2N$ of the matrix Λ. Using that all other eigenvalues are zero we prove the proposition. □

Coming back to the proof of the theorem we see that due to the proposition 6.1.3 we find the size of Λ, what gives us the number of segments, Λ itself up to a unitary transformation, and the summary length of the segments from the term of order z^{-3}. □

It is also obvious that using only the information obtained from the expansion of $\operatorname{Tr} R^2$ for large spectral parameter we cannot find the separate length of each segment. There are some other methods allowing to do this (see Gutkin and Smilansky [2001],Kostrykin and Schrader [2000]).

6.2 Manifolds without segments

In this case we consider a system of M compact Riemannian manifolds $M_1 ... M_M$ of dimension 2. On each manifold M_i we fix μ_i points $q_{is}, s = 1, \ldots, \mu_i$ so that $\sum_i \mu_i$ is an even number $2N$. First of all we construct the disjoint union of these manifolds. Then we glue these manifolds together by identifying two points q' and q'' from two different manifolds. More precisely we identify N pairs of points $(q'_j, q''_j), j = 1, \ldots, N$, with the following conditions: $q'_j \in M_{c_j}, q''_j \in M_{d_j}; c_j, d_j \in \{1, \ldots, K\}$ and $c_j \neq d_j$.

In order to construct Laplace operators we perform the same steps as for a general hybrid manifold and obtain a symmetric operator with deficiency indices $(2N, 2N)$. Now we should choose a boundary value space. To simplify the further calculation we choose it in the following particular way: as before for each point q we take $\Gamma^1(f) = a(f, q), \Gamma^2(f) = b(f, q)$; for the complete system

$$\Gamma^1(f) = (a(f, q'_1), \ldots, a(f, q'_K), a(f, q''_1), \ldots, a(f, q''_N)),$$

$$\Gamma^2(f) = (b(f, q'_1), \ldots, b(f, q'_K), b(f, q''_1), \ldots, b(f, q''_N)).$$

As $z \to \infty$ the Q-matrix asymptotically is a diagonal matrix

$$Q(z) \begin{pmatrix} F(q'_1, q'_1, z) & 0 & \ldots & 0 \\ \ldots & F(q'_N, q'_N, z) & \ldots & \ldots \\ \ldots & \ldots & F(q''_1, q''_1, z) & \ldots \\ 0 & \ldots & 0 & F(q''_N, q''_N, z) \end{pmatrix} + O(e^{-cz})$$

and denoting $F(q'_i, q'_i, z)$ by $F(q_i)$ and $F(q''_i, q''_i, z)$ by $F(q''_i)$ we can write this in the block-matrix form

$$Q(z) = \begin{pmatrix} [F(q'_i)] & [0] \\ [0] & [F(q''_i)] \end{pmatrix} + O(e^{-cz}), \quad i = 1, \ldots, N.$$

Performing calculation similar to the general case we state

Theorem 6.2.1. *Consider the hybrid manifold obtained by the gluing of M two-dimensional manifolds in N points and a Laplace operator on it defined* *by a matrix of the boundary conditions. The following expansion holds:*

$$\operatorname{Tr} R^2 = \sum_{M_i}\sum_{k} \frac{a_{ki}\Gamma(k+1)}{4\pi z^{2k+2}}$$
$$-\sum_{i=1}^{N} \frac{F(q_i')''_{zz}(F(q_i'')-\lambda_{i+N,i+N}) + F(q_i'')''_{zz}(F(q_i')-\lambda_{i,i})}{4z^2(F(q_i')-\lambda_{i,i})(F(q_i'')-\lambda_{i+N,i+N}) - |\lambda_{i,i+N}|^2}$$
$$+\frac{F(q_i')'_z(F(q_i'')-\lambda_{i+N,i+N}) + F(q_i'')'_z(F_i-\lambda_{i,i})}{4z^3(F(q_i')-\lambda_{i,i})(F(q_i'')-\lambda_{i+N,i+N}) - |\lambda_{i,i+N}|^2}$$
$$+\frac{(F(q_i')'_z)^2(F(q_i'')-\lambda_{i+N,i+N})^2 + 2F(q_i')'_zF(q_i'')'_z|\lambda_{i,i+N}|^2}{4z^2\Big((F(q_i')-\lambda_{i,i})(F(q_i'')-\lambda_{i+N,i+N}) - |\lambda_{i,i+N}|^2\Big)^2}$$
$$+\frac{(F(q_i'')'_z)^2(F(q_i')-\lambda_{i,i})^2}{4z^2\Big((F(q_i')-\lambda_{i,i})(F(q_i'')-\lambda_{i+N,i+N}) - |\lambda_{i,i+N}|^2\Big)^2} + O(e^{-cz}),$$

where for all $p \geqslant 1$

$$F(q_i') = F(q_i', q_i', z) = \frac{1}{4\pi}\left(-2\gamma - \ln z^2 + \sum_{n=1}^{p} \frac{\Gamma(n)a_n(q_i', q_i')}{z^{2n}}\right) + O(z^{-2p-2}),$$
$$F(q_i'') = F(q_i'', q_i'', z) = \frac{1}{4\pi}\left(-2\gamma - \ln z^2 + \sum_{n=1}^{p} \frac{\Gamma(n)a_n(q_i'', q_i'')}{z^{2n}}\right) + O(z^{-2p-2}).$$

One can see that this expression is rather difficult to analyze. It is why we recalculate $\operatorname{Tr} R^2$ in some other form to proceed with the inverse spectral problem. We write

$$R^2(z) = -\frac{1}{2z}\frac{\partial}{\partial z}(R_0(z) - A(z)) = R_0^2(z) + \frac{1}{2z}\frac{\partial}{\partial z}A(z),$$

where $A(z) = \gamma(z)[Q(z)-\Lambda]^{-1}(\gamma(\bar{z}))^*$, and try to simplify the second term in order to have some additional information about the inverse problem.

As $z \to \infty$, the matrix $(Q-\Lambda)^{-1} = (I - Q^{-1}\Lambda)^{-1}Q^{-1}$ can be expanded as a series $\sum_{n=0}^{\infty}(Q^{-1}\Lambda)^n Q^{-1}$ because all entries of Q^{-1} are of order $O(\frac{1}{\ln z^2})$. And if we look now for the trace $\operatorname{Tr} A$ as an integral operator, we see that $\operatorname{Tr} A(z) = -\frac{1}{2z}\operatorname{Tr} Q_z' \sum_{n=0}^{\infty} Q(z)^{-n-1}\Lambda^n$. First of all we perform a change of variables which allows us to simplify the calculations.

As we know, for all $p \geqslant 1$

$$F(q_i, q_i, z) = F_i = \frac{1}{4\pi}\left(-2\gamma - \ln z^2 + \sum_{n=1}^{p} \frac{\Gamma(n)a_n(q_i, q_i)}{z^{2n}}\right) + O(z^{-2p-2}),$$

we change the variable $u = ze^{\gamma}$ and rewrite F_i in this new variable:

$$F(q_i, q_i, u) = \frac{1}{4\pi}\left(-\ln u^2 + \sum_{n=1}^{p} \frac{b_{ni}}{u^{2n}}\right) + O(u^{-2p-2}),$$

where $b_{ni} = \Gamma(n)a_n(q_i, q_i)e^{2n\gamma}$. Now we will find the inverse matrix $Q^{-1}(u)$. As in our case Q is asymptotically a diagonal matrix, we can use a similar reasoning to the one in the proof of Lemma 4.3.6 to write the u-pseudoasymptotic expansion of the diagonal entries of Q^{-1}:

$$Q_{ii}^{-1} = -\frac{4\pi}{\ln u^2}\left(1 - \sum_{n=1}^{p} \frac{b_{ni}}{u^{2n}\ln u^2} + O(u^{-2p-2}(\ln u^2)^{-1})\right)^{-1}$$
$$\sim -\frac{4\pi}{\ln u^2} - \sum_{n,k=1}^{\infty} \frac{c_{nik}}{u^{2nk}(\ln u^2)^{k+1}},$$

for some coefficients c_{nik} independent of u. We also rewrite Q' in this new coordinate

$$(Q'_z)_{ii} = -\frac{1}{2\pi z} - \sum_{n=1}^{p} \frac{2n\Gamma(n)a_n(q_i, q_i)}{4\pi z^{2n+1}} + O(z^{-2p-3}) =$$
$$-\frac{e^{\gamma}}{2\pi u} - \sum_{n=1}^{p} \frac{d_{ni}}{u^{2n+1}} + O(u^{-2p-3}),$$

where $d_{ni} = 2n\Gamma(n)a_n(q_i, q_i)/(4\pi)$.

And now

$$\operatorname{Tr} A(u) = -\frac{e^{\gamma}}{2u}\operatorname{Tr}(Q'_u \sum_{n=0}^{\infty} Q(u)^{-n-1}\Lambda^n)$$
$$= \sum_{n=0}^{\infty} \frac{\operatorname{Tr}\Lambda^n}{2z}\left(\frac{e^{\gamma}}{2\pi u} + \sum_{n=1}^{\infty} \frac{d_{ni}}{u^{2n+1}}\right)\left(-\frac{4\pi}{\ln u^2} - \sum_{l,k=1}^{\infty} \frac{c_{lik}}{u^{2lk}(\ln u^2)^{k+1}}\right)^{n+1}.$$

We are looking for a term containing $\ln u^2$ with the lowest power of u. Such a term is of the form

$$\frac{(-1)^{n+1}(4\pi)^n e^{2\gamma}\operatorname{Tr}\Lambda^n}{(\ln u^2)^{n+1}u^2},$$

it is obvious that all the other terms containing $\ln u^2$ have a higher power of u.

After differentiation (using $\frac{1}{2z}\frac{\partial}{\partial z} = \frac{e^\gamma}{2u}\frac{\partial}{\partial u}\frac{\partial u}{\partial z}$) such a term gives us

$$\frac{e^\gamma}{2u}\frac{\partial}{\partial u}\left(\frac{(-1)^{n+1}(4\pi)^n e^{2\gamma}\operatorname{Tr}\Lambda^n}{(\ln u^2)^{n+1}u^2}\right)e^\gamma$$
$$= \frac{(-1)^n(4\pi)^n e^{4\gamma}\operatorname{Tr}\Lambda^n(n+1)}{(\ln u^2)^{n+2}u^4} + \frac{(-1)^n(4\pi)^n e^{4\gamma}\operatorname{Tr}\Lambda^n}{(\ln u^2)^{n+1}u^4}.$$

And now, if we gather together the terms with the same power of $\ln u^2$ for different n's we obtain

$$\frac{(-1)^k(4\pi)^{k-2}e^{4\gamma}}{(\ln u^2)^k u^4}\left((k-1)\operatorname{Tr}\Lambda^{k-2} - 4\pi\operatorname{Tr}\Lambda^{k-1}\right), \quad k>1,$$
$$\frac{Ne^{4\gamma}}{\ln u^2 u^4}, \quad k=1.$$

We see now that if we look for the terms containing $\ln u^2$ and u^4 in the asymptotic expansion of $\operatorname{Tr} R^2$, we can extract the following values:

$$N, \quad N-4\pi\operatorname{Tr}\Lambda, \quad \ldots(k-1)\operatorname{Tr}\Lambda^{k-2} - 4\pi\operatorname{Tr}\Lambda^{k-1}, \ldots$$

and recursively find also $\operatorname{Tr}\Lambda^n$ for all natural number n.

Theorem 6.2.2. *From the expansion of* $\operatorname{Tr} R^2$ *for the Laplace operator on a system of* N *glued manifolds it is possible to find the number of manifolds, the sum of the volumes of all manifolds and the matrix* Λ *of boundary conditions up to a unitary transformation.*

Proof. The number of the manifolds and the summary volumes can be found from the terms of order z^{-2} and z^{-4}, and the information about $\operatorname{Tr}\Lambda^n$ provides us with the eigenvalues of Λ (see proposition 6.1.3). □

Chapter 7

Appendix

Here we will give some details from the results of Avramidi Avramidi [1998] concerning the expansion of the Green function for the Laplace operator near the diagonal for large spectral parameter. The technique is similar to the one used for the expansion of the heat kernel for small time values. However the result is an expansion, using not only the coefficients a_k of the heat kernel expansion, but also some generalization of them, being in some sense "the analytical continuation" of a_k to the whole complex plane. Note also that there seems to be a sign error in the main formula 7.5 of Avramidi [1998], which we correct here.

Let us take the Laplacian Δ on a d-dimensional manifold M. We consider the operator $F = \Delta + z^2$ and the heat kernel $U(t) = e^{-tF}$ defined by the heat equation

$$\left(\frac{\partial}{\partial t} + F\right) U(x, x', t) = 0,$$

with initial condition:

$$\lim_{t \to 0} \int_M U(x, x', t) f(x')\, dx = f(x),\ a.e.$$

for all $f \in L^2(M)$. For any $t > 0$ the heat kernel is a smooth analytic bounded function on M.

For further computations we suppose that x and x' are close together. As a first step we factorise out the semi-classical factor

$$U(x, x', t) = \frac{1}{(4\pi t)^{d/2}} \Delta^{1/2}(x, x') e^{-\frac{r^2(x,x')}{4t}} \Omega(x, x', t),$$

where $r(x, x')$ is the distance between x and x', and $\Delta(x, x')$ is the so-called Van Vleck-Morette determinant. As we are interested finally in the Green function on the diagonal it is worth noting that $\Delta(x, x) = 1$ and therefore it does not play any important role for us. We can define also the so-called world function

$$\sigma = \sigma(x, x') = r^2(x, x')/2$$

and rewrite the exponential therm as $\sigma = e^{-\frac{\sigma(x,x')}{2t}}$.

The heat equation can be rewritten as an equation on $\Omega(x, x', t)$, as well as boundary conditions. See the article Avramidi [1998] for more details. Then it can be shown that the function Ω satisfies the following asymptotic conditions: for any $\alpha > 0$ and $N \geqslant 0$ when t goes to 0 or ∞ we have

$$\lim_{t} t^{\alpha} \left(\frac{\partial}{\partial t}\right)^{N} \Omega(t) = 0. \tag{7.1}$$

We consider now the Mellin transform of $\Omega(t)$

$$b_q = \frac{1}{\Gamma(-q)} \int_0^{\infty} t^{-q-1} \Omega(t)\, dt,$$

where the integral converges in $\{\Re q < 0\}$ due to the properties (7.1). Integrating by parts allows us to obtain for $\Re q < N$ where $N > 0$

$$b_q = \frac{1}{\Gamma(-q+N)} \int_0^{\infty} t^{-q-1+N} \left(-\frac{\partial}{\partial t}\right)^{N} \Omega(t)\, dt,$$

and conclude that b_q can be defined in the whole complex plane by analytic continuation.

It can be also found that the functions b_q have the following values at the positive integer points $q = k, k = 0, 1, 2, \ldots$

$$b_k = \left(-\frac{\partial}{\partial t}\right)^{N} \Omega(t)|_{t=0}$$

and the following asymptotic property: for any $N > 0$ and $\Re q < N$

$$\lim_{|q| \to \infty} \Gamma(-q+N) b_q = 0. \tag{7.2}$$

Inverting the Mellin transform we write

$$\Omega(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^q \Gamma(-q) b_q \, dq, \tag{7.3}$$

for some constant $c < 0$. Deforming the contour of integration and using the properties of b_q we obtain

$$\Omega(t) = \sum_{k=0}^{N-1} \frac{(-t)^k}{k!} b_k + \frac{1}{2\pi i} \int_{c_N - i\infty}^{c_N + i\infty} t^q \Gamma(-q) b_q \, dq,$$

where $c_N \in (N-1, N)$. The fact that the integral here is of order $O(t^N)$ as $t \to 0$ gives us the asymptotic expansion of $\Omega(t)$ in this limit:

$$\Omega(t) \sim \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} b_k.$$

Comparing this expansion with the standard one for the heat kernel

$$U(x, x, t) \sim \frac{1}{4\pi t} \sum_{k=0}^{\infty} a_k(x, x) t^k,$$

we find that the the coefficients b_k (i.e. b_q in the integer points) are related to the heat kernel coefficients $a_k(\Delta + z^2)$ and $a_k = a_k(\Delta)$ by

$$b_n = (-1)^k k! a_k(\Delta + z^2) = \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!} z^{2(n-k)} a_k.$$

The definition of Ω provides us now with a partial differential equation on b_q together with initial conditions. The solution has the form

$$b_q = \frac{1}{2\pi i \Gamma(-q)} \int_{c-i\infty}^{c+i\infty} \Gamma(-p) \Gamma(p-q) z^{2(q-p)} \tilde{b}_p \, dp,$$

for $\Re q < c < 0$ where $\tilde{b}_p = b_p|_{z=0}$. Deforming the contour of integration we get for some integer $N \geq 1$ and for $\Re q \leq N - 1$

$$b_q = \sum_{k=0}^{N-1} \frac{\Gamma(q+1) z^{2(q-k)}}{k! \Gamma(q-k+1)} \tilde{b}_k + \frac{1}{2\pi i \Gamma(-q)} \int_{c_N - i\infty}^{c_N + i\infty} z^{2(q-p)} \Gamma(-p) \Gamma(p-q) \tilde{b}_p \, dp,$$

where $c_N \in (N-1, N)$.

Using the view 7.3 of Ω we can write the Green function of the operator $\Delta + z^2$ as

$$
\begin{aligned}
G(x,x',z) &= \int_0^\infty \frac{1}{4\pi t} \Delta^{1/2}(x,x') e^{-\frac{\sigma(x,x')}{2t}} \Omega(x,x',t)\, dt \\
&= \int_0^\infty \frac{1}{4\pi t} \Delta^{1/2}(x,x') e^{-\frac{\sigma(x,x')}{2t}} \left(\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^q \Gamma(-q) b_q \, dq \right) dt \\
&= \frac{1}{4\pi} \Delta^{1/2}(x,x') \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(-q) b_q \left(\int_0^\infty e^{-\frac{\sigma(x,x')}{2t}} t^{q-1}\, dt \right) dq \\
&= \frac{1}{4\pi} \Delta^{1/2}(x,x') \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(-q) b_q \left(\frac{\sigma}{2}\right)^q \Gamma(-q)\, dq \\
&= \frac{1}{4\pi} \Delta^{1/2}(x,x') \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(-q)^2 b_q \left(\frac{\sigma}{2}\right)^q dq
\end{aligned}
$$

where $c < -1/2$. We recall here that we consider the two-dimensional situation and in this case the integrand has double poles at the points $q = 0, 1, 2, \dots$. Moving the contour of integration to the right we obtain an expansion of the Green function in powers of σ.

$$G = G^{non-anal} + G^{reg}$$

In general we have also some singular part which is polynomial in powers of $(\sigma)^{-1/2}$, but for dimension 2 it is equal to zero.

These two terms can be calculated using the Cauchy theorem:

$$
\begin{aligned}
G = &- \frac{\Delta^{1/2}(x,x')}{4\pi} \log\left(\frac{\sigma}{2}\right) \sum_{k=0}^{n-1} \frac{1}{(k!)^2} \left(\frac{\sigma}{2}\right)^k b_k \\
&- \frac{\Delta^{1/2}(x,x')}{4\pi} \sum_{k=0}^{n-1} \frac{1}{(k!)^2} \left(\frac{\sigma}{2}\right)^k (b_k' - 2\psi(k+1) b_k) \\
&+ \frac{\Delta^{1/2}(x,x')}{4\pi} \frac{1}{2\pi i} \int_{c_n - i\infty}^{c_n + i\infty} \left(\frac{\sigma}{2}\right)^q \Gamma(-q)^2 b_q \, dq,
\end{aligned}
$$

where $n-1<c_n<n$ and $\psi(z)=\frac{d}{dz}\Gamma(z)$. If we let now $n\to\infty$ we obtain

$$G^{non-anal}\sim-\frac{\Delta^{1/2}(x,x')}{4\pi}\log\left(\frac{\sigma}{2}\right)\sum_{k=0}^{\infty}\frac{1}{(k!)^2}\left(\frac{\sigma}{2}\right)^k b_k, \tag{7.4}$$

$$G^{reg}\sim-\frac{\Delta^{1/2}(x,x')}{4\pi}\sum_{k=0}^{\infty}\frac{1}{(k!)^2}\left(\frac{\sigma}{2}\right)^k\left(b'_k-2\psi(k+1)b_k\right). \tag{7.5}$$

As we are interested only in the expansion of the Green function on the diagonal $x=x'$ we can rewrite the regular part of the Green function as

$$G^{reg}\sim-\frac{1}{4\pi}\left(b'_0-2\psi(1)b_0\right).$$

Using the form of b_q, obtained before, one find that

$$b'_0=\ln z^2+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\tilde{b}_n(x,x)}{nz^{2n}},$$

We want to note that from the definition of $\tilde{b}_n$ on can see that

$$\tilde{b}_n=b_n|_{z=0}=(-1)^n n!a_n.$$

where a_n's are the usual heat kernel expansion coefficients for the Laplacian. That allows us to rewrite

$$b'_0=\ln z^2-\sum_{n=1}^{\infty}\frac{\Gamma(n)a_n(x,x)}{z^{2n}},$$

and the regular part of Green function on the diagonal as

$$G^{reg}\sim\frac{1}{4\pi}\left(-\ln z^2-2\gamma+\sum_{n=1}^{\infty}\frac{\Gamma(n)a_n(x,x)}{z^{2n}}\right).$$

We use this computation in Section 4.3.3.

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Selbständigkeitserklärung

Hiermit versichere ich, dass die vorliegende Dissertation selbständig und ohne unerlaubte Hilfe angefertigt wurde.