Numbers and Topologies:

Two Aspects of Ramsey Theory

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Abstract

In graph Ramsey theory, Burr and Erdős in 1970s posed two conjectures which may be considered as initial steps toward the problem of characterizing the set of graphs for which Ramsey numbers grow linearly in their orders. One conjecture is that Ramsey numbers grow linearly for all degenerate graphs and the other is that Ramsey numbers grow linearly for cubes. Though unable to settle these two conjectures, we have contributed many weaker versions that support the likely truth of the first conjecture and obtained a polynomial upper bound for the Ramsey numbers of cubes that considerably improves all previous bounds and comes close to the linear bound in the second conjecture.

In topological Ramsey theory, Kojman recently observed a topological converse of Hindman's theorem and then introduced the so-called Hindman space and van der Waerden space (both of which are stronger than sequentially compact spaces) corresponding respectively to Hindman's theorem and van der Waerden's theorem. In this thesis, we will strengthen the topological converse of Hindman's theorem by using canonical Ramsey theorem, and introduce differential compactness that arises naturally in this context and study its relations to other spaces as well. Also by using compact dynamical systems, we will extend a classical Ramsey type theorem of Brown and Hindman et al on piecewise syndetic sets from natural numbers and discrete semigroups to locally connected semigroups.

Keywords:
Ramsey theory, graphs, topology, probabilistic methods

Zusammenfassung

In der Ramsey Theorie für Graphen haben Burr und Erdős vor nunmehr fast dreißig Jahren zwei Vermutungen formuliert, die sich als richtungsweisend erwiesen haben. Es geht darum diejenigen Graphen zu charakterisieren, deren Ramsey Zahlen linear in der Anzahl der Knoten wachsen. Diese Vermutungen besagen, daß Ramsey Zahlen linear für alle degenerierten Graphen wachsen und dass die Ramsey Zahlen von Würfeln linear wachsen. Ein Ziel dieser Dissertation ist es, abgeschwächte Varianten dieser Vermutungen zu beweisen.

In der topologischen Ramseytheorie bewies Kojman vor kurzem eine topologische Umkehrung des Satzes von Hindman und führte gleichzeitig sogenannte Hindman-Räume und van der Waerden-Räume ein (beide sind eine Teilmenge der folgenkompakten Räume), die jeweils zum Satz von Hindman beziehungsweise zum Satz von van der Waerden korrespondieren. In der Dissertation wird zum einen eine Verstärkung der Umkehrung des Satzes von van der Waerden bewiesen. Weiterhin wird der Begriff der Differentialkompaktheit eingeführt, der sich in diesem Zusammenhang ergibt und der eng mit Hindman-Räumen verknüpft ist. Dabei wird auch die Beziehung zwischen Differentialkompaktheit und anderen topologischen Räumen untersucht. Im letzten Abschnitt des zweiten Teils werden kompakte dynamische Systeme verwendet, um ein klassisches Ramsey-Ergebnis von Brown und Hindman et al. über stückweise syndetische Mengen über natürlichen Zahlen und diskreten Halbgruppen auf lokal zusammenhängende Halbgruppen zu verallgemeinern.

Schlagwörter:
Ramseytheorie, Graphen, Topologie, Probabilistische Methode

Preface

Ramsey Theory is concerned with the inevitable occurrence of certain substructures in any finite partition of a large structure. For example, van der Waerden's theorem: If the natural numbers are finitely partitioned then one part contains arithmetic progressions of arbitrary length. Ramsey's theorem: If a graph contains sufficiently many vertices then it must contain either a complete set or a stable set of vertices of a certain size. Hindman's theorem: If the natural numbers are finitely partitioned then one part contains an infinite set so that all finite sums of their elements are also in this part.

Though recognized for a cohesive subdiscipline of Discrete Mathematics, Ramsey Theory spans not only computer science but also many and diverse areas of mathematics: Algebra, Analysis, Dynamical System, Ergodic Theory, Geometry, Logic, Number Theory and Topology etc. In this thesis, we mainly study two aspects of them. One is in Graph Ramsey theory and the other in Topological Ramsey theory.

In graph Ramsey theory, the first question might be the problem of determining Ramsey numbers, the exact sizes of the graphs which are guaranteed by Ramsey's theorem. However, it is well-known that Ramsey numbers are very difficult to determine and even good asymptotic estimates are not easy to obtain. One of the currently pursuing problems is to characterize the set of graphs for which Ramsey numbers grow linearly in their orders. This problem turns out to be quite difficult and is still far from being solved. Almost thirty years ago, Burr and Erdős posed two conjectures which may be considered as initial steps toward the problem. One conjecture is that Ramsey numbers grow linearly for all degenerate graphs (defined as those graphs whose subgraphs all have bounded minimum degree) and the other is that Ramsey numbers grow linearly for cubes. Though unable to settle these two conjectures, we have contributed many weaker versions that support the likely truth of the first conjecture and obtained a polynomial upper bound for the Ramsey numbers of cubes that considerably improves all previous bounds and comes close to the linear bound in the second conjecture.

In topological Ramsey theory, compactness plays a central role and always comes from the finiteness of partitions in Ramsey theory. Actually, compact spaces have a character of possessing the Ramsey property because of the less obvious connection between Ramsey theory and ultrafilters noted by Hindman, and the relation between ultrafilters and compactness. Kojman recently observed a topological converse of Hindman's theorem and then introduced the so-called Hindman space and van der Waerden space (both of which are stronger than sequentially compact spaces) corresponding respectively to Hindman's theorem and van der Waerden's theorem. In this thesis, we will strengthen the topological converse of Hindman's theorem by using canonical Ramsey theorem, and introduce differential compactness that arises naturally in this context and is closely related to Hindman spaces, and study its relations to other spaces as well. Also by using compact dynamical systems, we will extend a classical Ramsey type theorem of Brown and Hindman et al on piecewise syndetic sets from natural numbers and discrete semigroups to locally connected semigroups.

This thesis is organized as follows. Chapter 1 contains a brief sketch of the basic concepts, notions and facts in Graph Theory and Topology which will be used later on. In Chapter 2, we give a short introduction to Graph Ramsey theory and Topological Ramsey theory, and summarize the results of this thesis that have been obtained in these two areas and describe their relationship to previous research. We only confine ourselves to stating some special cases of the results or describing them informally, postponing the detailed statements to the proper chapters. Chapter 3 deals with Ramsey numbers of various sparse graphs and Chapter 4 is concerned with various Ramsey topologies and spaces. The contents in Chapter 3 are taken from [67, 65, 66].

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Contents

Chapter 1

Preliminaries

In this chapter, we only sketch the basic concepts, notions and facts in Graph Theory and Topology, which will be used in this thesis. We refer the readers to the books [18, 54] for details.

1.1 Graphs

A *graph* is a pair $G = (V, E)$ of sets satisfying $E \subset \binom{V}{2}$, where $\binom{V}{2}$ stands for all two-element subsets of V. V is the set of *vertices* and E is the set of *edges*. Sometimes we write $V(G) := V$ and $E(G) := E$. The *order* of G is $|V|$ and the *size* of G is $|E|$. The *density* of G is $|E|/\binom{|V|}{2}$. The graph G is *complete* or called a *clique* if $E = \binom{V}{2}$. A subset U of V is *stable* or *independent* if there is no edge in U. $H = (V', E')$ is a *subgraph* of G (and denoted by $H \subset G$) if $V' \subset V$ and $E' \subset E$. Two vertices u and v of G are *adjacent* or *neighbors* if $\{u, v\} \in E$ (or briefly $uv \in E$). The set of neighbors of a vertex v in G is denoted by $N_G(v)$, or briefly by $N(v)$. The *degree* $d_G(v) = d(v)$ of a vertex v is $|N(v)|$. The number $\delta(G) := \min\{d(v)|v \in V\}$ is the *minimum degree* of G, the number $\Delta(G) := \max\{d(v)|v \in V\}$ its *maximum degree*. The number $d(G) := \sum_{v\in V} d(v)/|V|$ is the *average degree* of G. A graph G is *d-regular* if $d(v) = d$ for all vertices of G. The graph $G^c = (V, \binom{V}{2} - E)$ is called the *complement* of the graph $G = (V, E)$.

A *path* is a non-empty graph $P = (V, E)$ of the form $V = \{v_0, v_1, ..., v_k\}$ and $E = \{v_0v_1, v_1v_2, ..., v_{k-1}v_k\}$, where the vertices v_i are all distinct. The *length* of a path is defined to be its size. If $k > 1$ and (V, E) is a path, then the graph $C := (V, E \cup \{v_0v_k\})$ is called a *cycle*. A graph is *acyclic* if it contains no cycles. A non-empty graph G is called *connected* if any two of its vertices are linked by a path in G. An acyclic connected graph is called a *tree*. A *star* is a tree with at most one vertex of degree at least two.

An *automorphism* of a graph $G = (V, E)$ is a bijective map $f : V \to V$ so that $f(u)f(v) \in E$ if and only if $uv \in E$. It is clear that all automorphisms of a graph form a group.

A graph is *planar* if it can be embedded in the plane.

A graph G is a *subdivided* graph from a graph H if it is obtained by replacing some edges of H with some independent paths. In this case, H is called a *topological minor* of a graph that contains G as a subgraph.

A graph $G = (V, E)$ is called *bipartite* if V admits a partition into two classes U and W such that every edge has its ends in different classes: vertices in the same partition class must not be adjacent. In this case, we sometimes write $G = (U, W; E)$. G is called *complete bipartite* if $|E| = |U||W|$.

Assume n is a natural number. Let $[n] := \{1, 2, ..., n\}$. The *chromatic number* $\chi(G)$ of a graph $G = (V, E)$ is the least integer k such that there exists a map $c : V \to [k]$ with $c(u) \neq c(v)$ whenever u and v are adjacent.

The following property may be found in [18].

Proposition 1.1.1 *Every graph G satisfies* $\chi(G) \leq 1 + \max\{\delta(H)|H \subset G\}$.

1.2 Topology

The concept of a topological space grew out of the study of the real line and Euclidean space and the study of continuous functions on these spaces.

Definition 1.2.1 *A* topology *on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:*

(1) $\emptyset$ and X are in $\mathcal{T}$.

(2) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

(3) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A set X for which a topology $\mathcal{T}$ has been specified is called a topological space.

$\mathcal{T}$ is called the *discrete topology* if it is the collection of all subsets of X. If X is a topological space with topology $\mathcal{T}$, we say that a subset U of X is an *open set* of X if U belongs to the collection $\mathcal{T}$. A subset A of X is said to be *closed* if the set $X - A$ is open. Given a subset B of X, the *closure* of B, denoted by $\mathrm{cl}(B)$, is defined as the intersection of all closed sets containing B. A subset A of X is said to be *dense* in X if $\mathrm{cl}(A) = X$. A space having a countable dense subset is often said to be *separable*. We shorten the statement "U is an open set containing x"to the phrase "U is a *neighborhood* of x". A sequence (x_n) of points of a space X is said to *converge* to the point x of X if for every neighborhood U of x there is an $N \in \mathbb{N}$ such that $x_n \in U$ for $n > N$.

Definition 1.2.2 *A space X is said to be* compact *if every open covering of X contains a finite subcollection that also covers X.*

A space X is said to be sequentially compact *if for every sequence (x_n) in X there exists a convergent subsequence (x_{n_i}).*

Definition 1.2.3 *A space X is said to have a* countable basis at the point *$x \in X$ if there is a countable collection $\{U_n\}_{n \in \mathbb{N}}$ of neighborhoods of x such that any neighborhood U of x contains at least one of the sets U_n. A space X that has a countable basis at each of its points is said to satisfy the* first countability axiom.

Definition 1.2.4 *A space X is called a* Hausdorff space *if for each pair x and y of distinct points of X, there exist neighborhood U and V of x and y, respectively, that are disjoint.*

A space X is called a T_1 space *if for each pair x and y of distinct points of X, each has a neighborhood not containing the other.*

Connectedness for a topological space is a useful property and the definition is quite natural. One says that a space can be "separated"if it can be broken up into two parts (disjoint open sets). Otherwise, one says that it is connected.

Definition 1.2.5 *Let X be a topological space. A* separation *of X is a pair U and V of disjoint nonempty open subsets of X whose union is X. The space X is said to be* connected *if there does not exist a separation of X.*

Another way of formulating the definition of connectedness is as follows. *A space X is connected if and only if the only subsets of X that are both open and closed in X are the empty set and X itself.* For if A is a nonempty proper sebset of X which is both open and closed in X, then the set $U = A$ and $V = X - A$ form a separation of X, for they are open, disjoint, and nonempty, and their union is X. Conversely, if U and V form a separation of X, then U is nonempty and different from X, and it is both open and closed in X.

The following theorem shows how to form new connected spaces from given ones.

Theorem 1.2.1 *The union of a collection of connected sets that have a point in common is connected.*

Given an arbitrary space X, there is a natural way to break it up into pieces that are connected. This leads us to the following.

Definition 1.2.6 *Given X, define an equivalence relation on X by setting $x \sim y$ if there is a connected subset of X containing both x and y. The equivalence classes are called the* components *(or "connected components") of X.*

Symmetry and reflexivity of the relation are obvious. Transitivity follows by noting that if A is a connected set containing x and y, and if B is a connected set containing y and z, then $A \cup B$ is a set containing x and z, which is connected because A and B have the point y in common.

The components of X can also be described as follows:

Theorem 1.2.2 *The components of X are connected disjoint subsets of X whose union is X.*

For some purposes, it is more important that a space satisfy a connectivity condition *locally* rather than that it be connected. Roughly speaking, local connectivity means that each point has "arbitrarily small" neighborhoods that are connected. More precisely, one has the following definition:

Definition 1.2.7 *A space X is said to be* locally connected *if for every x in X and every neighborhood U of x, there is a connected neighborhood V of x contained in U.*

It is clear that X is locally connected if there is a basis for X consisting of connected sets. Local connectedness and connectedness of a space are not related to each other; a space may have one or both of these properties, or neither.

The following theorem shows an important fact about locally connected spaces.

Theorem 1.2.3 *A space X is locally connected if and only if for every open set U of X, each component of U is open in X.*

The following is a easy consequence of Theorems 1.2.2 and 1.2.3.

Corollary 1.2.1 *All components of a locally connected space are both open and closed.*

In the following, we will introduce compact-open topology and equicontinuity that are important for us to state a general version of Ascoli's thoerem.

Definition 1.2.8 *Let* X *and* Y *be topological spaces. Denote by* $\mathcal{C}(X, Y)$*, the set of all continuous maps from* X *into* Y*. If* K *is a compact subset of* X *and* U *is an open subset of* Y*, define*

$$S(K, U) = \{f | f \in \mathcal{C}(X, Y) \text{ and } f(K) \subset U\}.$$

The sets $S(K, U)$ *form a subbasis for a topology on* $\mathcal{C}(X, Y)$*, called the* compact-open topology*.*

Definition 1.2.9 *Let* (Y, d) *be a metric space and* $\mathcal{F}$ *a subset of the space* $\mathcal{C}(X, Y)$*. The set* $\mathcal{F}$ *is* equicontinuous *if for each* $x_0 \in X$ *and* $\epsilon > 0$*, there is a neighborhood* U *of* x_0 *such that for all* $x \in U$ *and all* $f \in \mathcal{F}$*,* $d(f(x), f(x_0)) < \epsilon$*.*

Ascoli's Theorem. *Let* Y *be a metric space. If a subset* $\mathcal{F}$ *of* $\mathcal{C}(X, Y)$ *is equicontinuous and the subset* $\mathcal{F}_x = \{f(x) | f \in \mathcal{F}\}$ *of* Y *has compact closure for each* x*, then* $\mathcal{F}$ *has compact closure in the compact-open topology of* $\mathcal{C}(X, Y)$*.*

Chapter 2

Introduction

Though the origins are diffuse, Ramsey theory, roughly speaking, asserts the philosophy that "complete disorder is impossible" by Motzkin and is concerned with the inevitable occurrence of certain substructures in any finite partition of a large structure. For example, Schur [63] proved in 1916 that if the natural numbers are finitely partitioned then one part contains l, m and n with $l + m = n$; van der Waerden [73] proved in 1927 that if the natural numbers are finitely partitioned then one part contains arithmetic progressions of arbitrary length; Ramsey [59] proved in 1930 that if a graph contains sufficiently many vertices (dependent on k) then it must contain either a clique or a stable set of vertices of size k. Computing these functions that are associated with precisely how large such structures need be has turned out to be extremely difficult. The best results on Ramsey numbers (for Ramsey's theorem itself), are still much unsatisfied (see the survey paper of Radziszowski [58]). The books [35, 64] present an exciting development of Ramsey theory. In this thesis, we mainly discuss two aspects of Ramsey theory. One is in Graph Ramsey theory and the other in Topological Ramsey theory.

2.1 Ramsey numbers

Graph Ramsey theory began about 30 years ago with work on Ramsey numbers of graphs done by Erdős et al. Now it becomes one of the presently most active areas in Ramsey theory. The following paragraph quoted from the book "Ramsey Theory" of Graham, Rothschild and Spencer [35] tells its story.

> A major impetus behind the early development of Graph Ramsey theory was the hope that it would eventually lead to meth-

> ods for determining large values of the classical Ramsey numbers $R(m,n)$. However, as so often happens in mathematics, this hope has not been realized; rather, the field has blossomed into a discipline of its own. In fact, it is probably safe to say that the results arising from Graph Ramsey theory will prove to be more valuable and interesting than knowing the exact value of $R(5,5)$ [or even $R(m,n)$].

The Ramsey number $R(m,n)$ is the least integer r so that any graph of order r contains either a clique of order m or a stable set of order n. It is easy to see that one may extend the definition of Ramsey numbers from integers to graphs. More precisely, for an arbitrary (fixed) graph G, the Ramsey number $R(G)$ is the least integer r so that, whenever the edges of the complete graph K_r are bicolored, there is always a monochromatic subgraph isomorphic to G. They are the classical Ramsey numbers when the graph G itself is taken to be a complete graph. When k colors are used instead of two, we will denote the corresponding Ramsey numbers by $R(G;k)$. One may also consider the more general "off-diagonal" situation as follows. For graphs $G_1, G_2, ..., G_k$, we let $R(G_1, G_2, ..., G_k)$ denote the least integer r so that, whenever the edges of K_r are k-colored, there is a monochromatic copy of G_i for some i. The existence of $R(G_1, G_2, ..., G_k)$ is of course guaranteed by Ramsey's theorem.

The determination or estimation of these numbers is usually a very difficult problem. For the classical Ramsey numbers of complete graphs, the only values that are known precisely are those of $R(3,n)$ for $n = 3,\ 4,\ 5$ [37], 6 [42], 7 [42, 36], 8 [38, 53], 9 [42, 38], $R(4,4)$ [37], $R(4,5)$ [41, 52] and $R(3,3,3)$ [37]. Some nontrivial bounds of small Ramsey numbers can be found in [58, 68, 69]. Apart from computing Ramsey numbers of single graphs, we would also like to know the rate of growth of Ramsey numbers for a set of graphs. But even the asymptotic behavior of Ramsey numbers for a set of graphs up to a constant factor is hard to determine, and despite a lot of efforts by various researchers (see, eg. [14, 35]), there are only a few infinite sets of graphs for which this behavior is known. A particular interesting such an example is the result of Kim [44] together with that of Ajtai, Komlós and Szemerédi [1]: $R(3,n) = \Theta(n^2/\ln n)$. We say that the Ramsey numbers *grow linearly* for a set of graphs $\mathcal{G}$ if there is a constant $c = c(\mathcal{G}, k) \geq 1$ so that $R(G;k) \leq cn$ for all $G \in \mathcal{G}$ of order n and *grow polynomially* if there is a constant $c = c(\mathcal{G}, k) > 1$ so that $R(G;k) \leq n^c$ for all $G \in \mathcal{G}$ of order n. We also say that the Ramsey numbers *grow linearly* for a set of graphs $\mathcal{G}$ versus a set of graphs $\mathcal{H}$ if there is a constant $c = c(\mathcal{G}, \mathcal{H}) \geq 1$ so that $R(G,H) \leq cn$ for all $G \in \mathcal{G}$ of order n and $H \in \mathcal{H}$ of order n. It has been known for a long time that Ramsey numbers grow exponentially for sets of certain dense graphs. For example, the classic papers of Erdős and Szekeres [23] as well as Erdős [20] give $2^{n/2} \leq R(K_n) < 2^{2n}$. Meanwhile, it has also been known that Ramsey numbers grow linearly for very sparse graphs. For example, they grow linearly for paths (see Geréncser and Gyárfás [29]), for cycles (see Faudree and Schelp [24] as well as Rosta [61]) and for stars and trees (see Burr [11] as well as Erdős and Graham [21]). A very general result in Graph Ramsey theory that is particularly useful for computing the magnitude of Ramsey numbers is the following proved by Chvátal and Harary [15] in 1972.

Theorem 2.1.1 (Chvátal-Harary [15]) *Let G be a graph of order n and size m, and let s be the order of the automorphism group of the graph G. Then*

$$R(G;k) \geq (sk^{m-1})^{1/n}.$$

So in order to force Ramsey numbers to grow linearly, the set must consist of relatively sparse graphs with average degree at most $2\log_k n$. Let d be a positive number. A graph is called *d-degenerate* if its subgraphs all have minimum degree at most d. In 1973, Burr and Erdős [12] offered a total of \$25 for settling the following conjecture.

Conjecture. *The Ramsey numbers grow linearly for d-degenerate graphs.*

But they also wrote in [12], "However, it seems to be quite difficult, and probably further work must continue to be in the direction of partial results". In fact, some weakened versions of this conjecture were obtained in the last two decades. In 1983, Chvátal, Rödl, Szemerédi, and Trotter [17] proved that the Ramsey numbers grow linearly for all graphs with bounded maximum degree. In 1993, Chen and Schelp [13] extended this to *p-arrangeable* graphs which are those whose vertices can be ordered as $v_1, v_2, ..., v_n$ so that for each integer i with $1 \leq i \leq n$, at most p vertices among $\{v_1, v_2, ..., v_i\}$ have a neighbor $v \in \{v_{i+1}, v_{i+2}, ..., v_n\}$ adjacent to v_i. They also showed that a planar graph is 761-arrangeable, which was later improved to 10-arrangeable by Kierstead and Trotter [43]. Thus their results imply that the Ramsey numbers grow linearly for all planar graphs. In 1997, Rödl and Thomas [60] extended this to graphs with bounded genus and proved that all graphs without any topological minor of a clique of order p are p^8-arrangeable and so that their Ramsey numbers grow linearly. As pointed out in [13], subdivided graphs need not be p-arrangeable. But in 1994, Alon [2] proved that the Ramsey numbers still grow linearly for them. In fact, he proved a stronger version showing that the Ramsey numbers grow linearly for graphs without any pair of adjacent vertices of degree at least three. To illustrate his result, let us first introduce some specific graphs. See Figure 1.

Denote by H_i, the graph obtained by adding a path of length i between the centers of two disjoint stars of order three for all natural numbers i,
A, the graph obtained by adding an edge between the second and fourth vertices of a path of order five,
and Θ, the complete graph of order four dropping an edge.

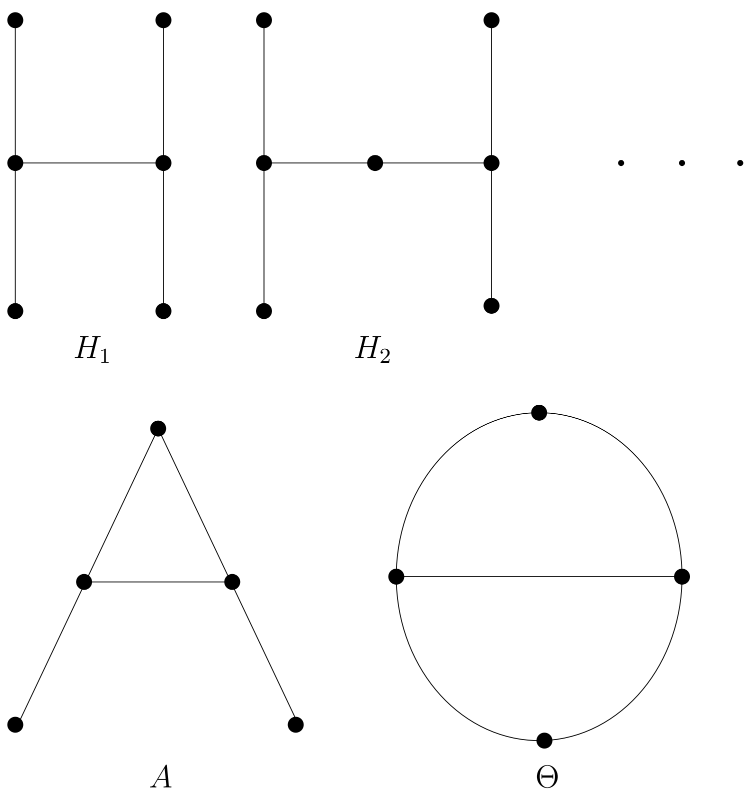

Figure 1

Note that a graph without any pair of adjacent vertices of degree at least three is exactly one containing a copy of neither H_1, A nor Θ. So Alon actually proved that the Ramsey numbers grow linearly for graphs containing a copy of neither H_1, A nor Θ. In Section 3.1.2, we will prove the following result.

Theorem 2.1.2 *The Ramsey numbers grow linearly for graphs containing a copy of neither H_1 nor H_2.*

This result is obtained when we study the Ramsey numbers of graphs without containing any copy of a fixed tree. The reason to study this set of graphs is as follows. If a graph is dense, then it contains a copy of all trees of some order. In other words, the graphs without containing any copy of a fixed tree are sparse and turn out to be degenerate. Thus the conjecture would imply that the Ramsey numbers grow linearly for such graphs. Kühn found that a result of Kostochka and Rödl [49] implies the almost linear bound $R(G) < n^{1+o(1)}$ for all such graphs G of order n, which is also a consequence of the very recent result of Kostochka and Sudakov [50] that $R(G) < n^{1+o(1)}$ for the larger set of all d-degenerate graphs G of order n. Actually, instead of showing Theorem 2.1.2, we will show a slightly stronger result. Given a tree T, denote by $\mathcal{T}(T)$ the set of all trees obtained by replacing some edges of T by internally disjoint paths of length two between their ends. In Section 3.1.1, we will show that for any tree T of order p (> 2), if a graph is not $4p^4$-arrangeable then it contains a copy of some tree in $\mathcal{T}(T)$. Then by the result of Chen and Schelp, we know that the Ramsey numbers grow linearly for graphs without a copy of any tree in $\mathcal{T}(T)$. However, we have not been able to solve the following special case of this problem.

Problem. *The Ramsey numbers grow linearly for graphs without containing any copy of* H_1.

In 2001, Kostochka and Rödl [49] extended Alon's result to crowns (a special kind of bipartite sparse graphs, see the definition in Section 3.3), which confirms a conjecture by Burr and Erdős [12] as well as by Trotter [49]. In Section 3.3, we will extend the result of Kostochka and Rödl to degenerate graphs versus crowns.

Recently, Kostochka and Rödl [48] also proved that the Ramsey number $R(G_1, G_2)$ of a d-degenerate graph G_1 of order n and a d-degenerate graph G_2 of order n with maximum degree Δ is bounded by $cn\Delta$. If Δ is bounded, this implies that the Ramsey numbers grow linearly for degenerate graphs versus graphs with bounded maximum degree. We will in Section 3.2 extend this and the result of Chen and Schelp to degenerate graphs versus p-arrangeable graphs. Now by combining the previous results, we know that the Ramsey numbers grow linearly for degenerate graphs versus planar graphs, graphs without any topological minor of a fixed clique or graphs without a copy of any tree in $\mathcal{T}(T)$ for each tree T.

Theorem 2.1.1 may be used to give a polynomial lower bound of the Ramsey numbers for $\mathrm{O}(\ln n)$-degenerate graphs of order n. This leads us to the following question.

Question 2.1.1 *Do the Ramsey numbers grow polynomially for* $O(\ln n)$*-degenerate graphs of order* n*?*

In Section 3.5, we will show that the answer is positive for bipartite $\mathrm{O}(\ln n)$-degenerate graphs of order n. As pointed out by Alon, a weaker polynomial bound may also be deduced from a Turán type result of Alon, Krivelevich and Sudakov [3].

One of the simplest and most general results in Graph Ramsey theory is the following proved by Chvátal and Harary [16] in 1972: For a graph G, let $c(G)$ denote the order of the largest connected component of G.

Theorem 2.1.3 (Chvátal and Harary [16])

$$R(G, H) \geq (\chi(G) - 1)(c(H) - 1) + 1.$$

This theorem shows that for linear growth of Ramsey numbers, it is necessary to bound the chromatic number of the graphs. In 1973, Burr and Erdős [12] also conjectured that the Ramsey numbers grow linearly for $2\log_k n$-degenerate graphs of order n with bounded chromatic numbers. An interesting test case is the set of cubes. They offered a total of $25 for deciding the following question.

Question 2.1.2 *Do the Ramsey numbers grow linearly for cubes?*

Though unable to answer this question, we will in Section 3.4 deduce the polynomial upper bound 2^{cn}, which improves the old bound 2^{cn^2} due to Beck [5] and the recent bound $2^{cn \ln n}$ due to Graham, Rödl and Ruciński [32, 33].

2.2 Ramsey topologies

Topological Ramsey theory started with Ellentuck's theorem [19] in 1974 and was anticipated by work of Nash-Williams [55], Galvin and Prikry [28] and Silver [70] by giving a fairly abstract treatment of infinitary Ramsey's theorem. The interplay between Ramsey theory and Topology was also demonstrated by ergodic proofs of a large amount of Ramsey type theorems by Fürstenberg, Katznelson and Weiss. The monograph [26] is a beautiful account of some applications of the modern theory of topological dynamics and ergodic theory to Ramsey theory. The book [51] is a good introduction to Ergodic Ramsey theory and the survey paper [6] shows the recent development.

Roughly speaking, the interplay between Ramsey theory and Topology is based on the connection between the *finiteness* of partitions in Ramsey theory and the *compactness* in Topology. To make it clear, let us first recall more classical theorems in Ramsey theory besides those mentioned already. In 1968, Brown [9, 10] proved a simpler result analogous to van der Waerden's theorem that if the natural numbers are finitely partitioned then one part is piecewise syndetic (see the definition in Section 4.3). In 1969, among others, Rado [57] and Sanders [62] independently proved Folkman's theorem (which extends Schur's theorem) that if the natural numbers are finitely partitioned then one part contains arbitrarily large finite sets so that all finite sums of their elements are still in this part. Graham and Rothschild [34] conjectured that under the condition, one part contains an infinite set with the same property. This was proved by Hindman [39] in 1974.

Note that all of these theorems are instances of the following general statement: For a set D and a collection of "good"subsets $\mathcal{G}$ of D, no matter how the set D is *finitely* partitioned then one part will contain a set of $\mathcal{G}$. By this point of view, Hindman and Strauss wrote in their book [40] that "It would not be entirely accurate, but certainly not far off the mark, to define *Ramsey Theory* as the classification of pairs $(D, \mathcal{G})$ for which the above statement is true. We see now that under this definition, any question in Ramsey Theory is a question about ultrafilters"in the sense of the following general connection between Ramsey theory and ultrafilters noted by Hindman (see [35, 40]). We omit the definition of an ultrafilter as we will not need it later on (it can be thought of as a certain {0,1}-valued measure).

Theorem 2.2.1 *The following are equivalent:*

(i) If D is finitely partitioned then one part contains a set of $\mathcal{G}$.

(ii) There exists an ultrafilter p on D such that all sets of p contain a set of $\mathcal{G}$.

We now introduce the notion of p-limit originally due to Frolík [25] where p is an ultrafilter on D. The definition of p-limit is very natural and similar to that of nets. For an indexed family $\{x_s\}_{s\in D}$ in a topological space X and y in X, we write $p\text{-lim}_{s\in D} x_s = y$ if and only if for every neighborhood V of y, $\{s \in D | x_s \in V\} \in p$. This means that x_s is *often close* to y. *Closeness* is of course determined by neighborhoods of y while *often* is determined by members of the ultrafilter p. The following theorem (see [40] for example) builds the connection between ultrafilters and compact topological spaces.

Theorem 2.2.2 *A topological space X is compact if and only if for any ultrafilter p on a set D and any indexed family $\{x_s\}_{s\in D}$, $p\text{-lim}_{s\in D} x_s$ exists.*

Now the relation between Ramsey theory and Topology becomes clear from Theorems 2.2.1 and 2.2.2. Namely, a topological space X is compact if and only if for any "good"pair $(D, \mathcal{G})$ and any indexed family $\{x_s\}_{s\in D}$ in X there exists y in X so that for any neighborhood V of y, $\{s \in D | x_s \in V\}$ contains a set of $\mathcal{G}$.

Another example of the relation between Ramsey theory and Topology is the topological converse of Hindman's theorem due to Kojman [45]. Define an IP set to be an infinite set of natural numbers with all finite sums of its elements (see Section 4.2 for formal definition). Then Hindman's theorem may topologically be restated as for every sequence $(x_n)_{n\in\mathbb{N}}$ in a finite space there exists a convergent subsequence $(x_n)_{n\in A}$ for some IP set A. Kojman [45] observed that the converse is also true: If for every sequence $(x_n)_{n\in\mathbb{N}}$ in a T_1 space X there exists a convergent subsequence $(x_n)_{n\in A}$ for some IP set A, then X is finite. If we have not the condition on the set of indices for the convergent subsequence, then X is sequentially compact. So the extra condition of the IP set, a kind of *good* set, forces X to be finite. In Section 4.2, we will strengthen this result by weakening the condition on the set of indices, namely by replacing the IP set with some DP set (see the definition in Section 4.2).

Along this direction, Kojman [45, 46] recently introduced two new spaces, Hindman spaces and van der Waerden spaces corresponding respectively to Hindman's theorem and van der Waerden's theorem. A topological space X is *Hindman* if for every sequence (x_n) in X there exists a subsequence (x_{n_i}) convergent to a point x so that for every neighborhood V of x there exists m and $x_n \in V$ for all $n = \sum_{i\in F} n_i,\ m < \min F \leq \max F < \infty$. A topological space X is *van der Waerden* if for every sequence (x_n) in X there exists a convergent subsequence (x_{n_i}) so that the set of indices $\{n_i\}$ contains arbitrarily long arithmetic progressions. Then it is easy to see that Hindman's theorem and van der Waerden's theorem may topologically be restated as finite spaces are Hindman and van der Waerden respectively. In 1978, Furstenburg and Weiss [27] extended Hindman's theorem to a statement implying that compact metric spaces are Hindman and used it to study the phenomenon of uniformly recurrence in compact dynamical systems. Kojman further extended this to be true for first countable compact spaces by proving that if the closure of every countable set of a space X is compact and first countable then X is Hindman and van der Waerden. By the definitions, it is also clear that Hindman spaces and van der Waerden spaces are both sequentially compact. But sequentially compact spaces need be neither Hindman nor van der Waerden. In fact, Kojman constructed some spaces that are compact Hausdorff, sequentially compact, separable and first countable at all points but one, but neither Hindman nor van der Waerden. He also proved that the product of two Hindman (resp. van der Waerden) spaces is Hindman (resp. van der Waerden). His proof for Hindman spaces is somewhat complex. In Section 4.2, we will simplify the proof of this result in the framework of set theory as introduced by Fürstenberg [26]. Also as for Hindman spaces, we will introduce differential compactness by using DP sets instead of IP sets and study its relations to other spaces in Section 4.2.

Shortly after van der Waerden proved his celebrated theorem on arithmetic progressions, Gallai [56] and Witt [74] independently extended this to higher dimensions. Their theorem may be stated as the following: Any map from $\mathbb{N}^m$ to a finite set is constant on a homothetic copy $bF+\mathbf{a},\ \mathbf{a}\in\mathbb{N}^m,\ b\in\mathbb{N}$ for any finite subset F of $\mathbb{N}^m$. Fürstenberg (Theorem 2.9 in [26]) extended this again to maps with values in a compact metric space i.e., if X is a compact metric space and $f:\mathbb{N}^m\to X$ is an arbitrary function with values in X then for any $\epsilon>0$ and finite set $F\subset\mathbb{N}^m$, we can find a homothetic copy $bF+\mathbf{a},\ \mathbf{a}\in\mathbb{N}^m,\ b\in\mathbb{N}$, for which the image under f, $f(bF+\mathbf{a})$, is a set of diameter at most ϵ in X. This result can be used as a tool in diophantine approximation (see examples in [26]). In Section 4.1, we will further extend this to maps with ranges in nonmetric spaces:

Theorem 2.2.3 *If any closed separable subspace of a space X is compact, then for any map $f:\mathbb{N}^m\to X$ and any finite set $F\subset\mathbb{N}^m$, there exists a point x so that for any neighborhood V of x one can find a homothetic copy $bF+\mathbf{a},\ \mathbf{a}\in\mathbb{N}^m,\ b\in\mathbb{N}$, for which $f(bF+\mathbf{a})\subset V$.*

Theorem 2.2.3 will enable us to deduce a shorter proof of Kojman's extension of van der Waerden's theorem.

It is also easy to see that Brown's result may be restated as any map from the natural numbers to a finite set is constant on a piecewise syndetic set. Hindman et al [40] extended this from natural numbers to discrete semigroups via algebraic methods. In Section 4.3, we will further extend this to locally connected abelian semigroups with continuous maps via topological dynamics following the method of Fürstenberg, Katznelson and Weiss.

Chapter 3

Ramsey numbers

In this chapter, we show that Ramsey numbers grow linearly for graphs without containing some trees, for degenerate graphs versus arrangeable graphs or crowns. We also show polynomial upper bounds of Ramsey numbers for cubes and bipartite O(ln$n$)-degenerate graphs of order n.

3.1 Trees

In this section, we first introduce the notion "admissibility"of a graph that originated in coloring theory due to Kierstead and Trotter [43], which is closely related to arrangeability. Then we show the relation between adimissibility and forbidden trees of a graph, which builds the bridge between arrangeability and forbidden trees. At last, we give the applications to linear Ramsey numbers and game chromatic numbers.

3.1.1 Admissibility and trees

Let G be a graph, $M \subset V(G)$, and $v \in M$. A set $A \subset V(G)$ is an M-*blade* with center v if either $A = \{a\}$ and $a \in M$ is adjacent to v, or $A = \{a, b\}$, $a \in M$, $b \in V(G) - M$, and b is adjacent to both v and a. An M-*fan* with center v is a set of pairwise disjoint M-blades with center v. Let p be an integer. A graph G is p-*admissible* if the vertices of G can be ordered as $v_1, v_2, ..., v_n$ so that for all $i = 1, 2, ..., n$, G has no $\{v_1, v_2, ..., v_i\}$-fan with center v_i of size $p + 1$. See Figure 2.

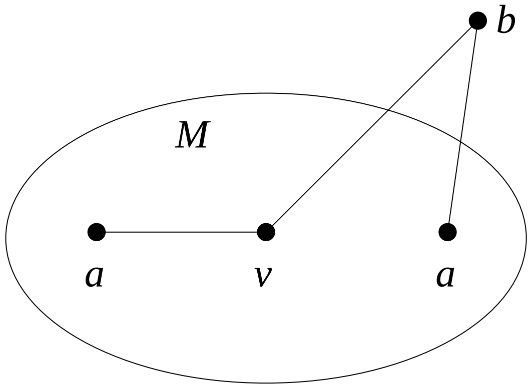

Figure 2

As pointed out in [43], the concepts of arrangeability and admissibility are asymptotically equivalent in the sense that if a graph is p-arrangeable then it is $2p$-admissible, and if it is p-admissible then it is $p^2 - p + 1$-arrangeable.

The following is our main result in this section. It exhibits the relation between admissibility and forbidden trees of a graph.

Theorem 3.1.1 *For each tree T of order p (> 2), if a graph is not $2p^2 - 7p + 5$-admissible then it contains a copy of some tree in $\mathcal{T}(T)$.*

We first need the following Turán type result that may be deduced from Proposition 1.2.2 and Corollary 1.5.4 in [18].

Lemma 3.1.1 *Let $p > 2$ be an integer. If a graph of order n has size at least $(p-2)n$ then it contains all trees of order p.*

Proof. Let G be a graph of order n and size at least $(p-2)n$ and let T be a tree of order p. First note that $n > 1$. Otherwise, if $n = 1$ then G has size at least $p - 2 > 0$, which is impossible for G is simple. Then G is never empty since $n > 1$ and $p > 2$.

Now we try to find a subgraph H of G with minimum degree $\delta(H) > p-2$ and then embed all trees of order p into H. We do this by deleting vertices of small degree one by one, until only vertices of large degree remain. It is clear that up to $d(v) = p - 2$, we may afford to delete a vertex v without the average degree of the graph becoming less than $2(p-2)$, since then the number of vertices decreases by one and the number of edges by at most $p - 2$, so the overall average degree $2(p-2)$ will not decrease.

Formally, we construct a sequence $G = G_0 \supset G_1 \supset ...$ of induced subgraphs of G as follows. If G_i has a vertex v_i of degree $d(v_i) < p - 1$, we let $G_{i+1} = G_i - v_i$; if not, we terminate our sequence and set $H := G_i$. By the choice of v_i, we have the average degree $d(G_{i+1}) \geq d(G_i)$ for all i and hence $d(H) \geq d(G)$.

Since $|E(K_1)| = 0 < |E(G)|$, none of the graphs in our sequence is trivial, so in particular $V(H) \neq \emptyset$. The fact that H has no vertex suitable for deletion thus implies that $\delta(H) > p - 2$ as claimed.

Now we try to find an embedding $f : T \to H$. It is clear that T is 1-degenerate, i.e., the vertices of T may be ordered as $v_1, v_2, ..., v_p$ and let $T_i = T[v_1, v_2, ..., v_i]$ be the induced tree so that $d_{T_i}(v_i) = 1$. We first choose any vertex $f(v_1)$ of H as the image of v_1 and at step j, we have embedded the vertices $v_1, v_2, ..., v_j$ and assume that v_{j+1} is adjacent to the vertex v_k where $k \leq j$. Since $d(f(v_k)) \geq \delta(H) > p - 2$, we can choose any neighbor of $f(v_k)$ that is not used before as the image of v_{j+1}. Then this process surely embeds T into H, which completes the proof. □

We then prove the following lemma using a method that is similar to that of Rödl and Thomas [60].

Lemma 3.1.2 *Let T be a tree of order p (> 2), let G be a graph and let M be a non-empty subset of $V(G)$. If for all $v \in M$ there is an M-fan in G with center v of size $2p^2 - 7p + 6$, then G contains a copy of some tree in $\mathcal{T}(T)$.*

Proof. For $v \in M$ let F_v be a fan in G with center v of size $2p^2 - 7p + 6$. We may assume that G is minimal subject to $M \subset V(G)$ and the existence of all F_v $(v \in M)$. Let $|M| = m$, let e_1 be the number of edges of G with both ends in M, and let e_2 be the number of edges of G with one end in M and the other in $V(G) - M$. Then from the existence of the F_v for $v \in M$ we have

$$2e_1 + e_2 \geq (2p^2 - 7p + 6)m.$$

We claim that if $|V(G) - M| \geq 2(p-2)m - e_1$ then G contains a copy of some tree in $\mathcal{T}(T)$. Indeed, by our minimality assumption for all $w \in V(G) - M$ there exist vertices $u, v \in M$ such that $\{u, w\} \in F_v$. For $w \in V(G) - M$ let us denote by $e(w)$ some such pair of vertices. Let H be the multigraph obtained from G by deleting $V(G) - M$ and for all $w \in V(G) - M$ adding an edge between the vertices in $e(w)$. Then

$$|E(H)| \geq 2(p-2)m,$$

and since each pair of vertices is joined by at most two (parallel) edges, H has a simple subgraph H' on the same vertex set with at least $(p-2)m$ edges. By Lemma 3.1.1, H' has a copy of tree T. Any edge of this copy that is not in G joins two vertices u and v with $\{u, v\} = e(w)$ for some $w \in V(G) - M$. By replacing each such edge by the edges uw and vw we obtain a copy of some tree in $\mathcal{T}(T)$. This proves our claim, and so we may assume that

$$|V(G) - M| < 2(p-2)m - e_1.$$

Now $|V(G)| < (2p-3)m - e_1$, and

$$\begin{aligned}|E(G)| \geq & e_1 + e_2 = 2e_1 + e_2 - e_1 \geq (2p^2 - 7p + 6)m - e_1 \\ \geq & (p-2)[(2p-3)m - e_1] > (p-2)|V(G)|,\end{aligned}$$

hence G contains a copy of tree T. □

Proof of Theorem 3.1.1. Let G be a graph without a copy of any tree in $\mathcal{T}(T)$. We show that G is $2p^2-7p+5$-admissible by giving a proper ordering of $V(G)$. Let $i \in \{0, 1, ..., n\}$ be the least number such that there exist vertices $v_{i+1}, v_{i+2}, ..., v_n$ with the property that for all $j = i, i+1, ..., n$, G has no $(V(G) - \{v_{j+1}, v_{j+2}, ..., v_n\})$-fan with center v_j of size $2p^2 - 7p + 6$. We claim that $i = 0$. Otherwise by Lemma 3.1.2 applied to $M = V(G) - \{v_{i+1}, v_{i+2}, ..., v_n\}$, there exists a vertex v_i without M-fan with center v_i of size $2p^2 - 7p + 6$, and so the sequence $v_i, v_{i+1}, ..., v_n$ contradicts the choice of i. Hence $i = 0$ and $v_1, v_2, ..., v_n$ is the desired ordering of the vertices of G. □

3.1.2 Applications

The following result on linear Ramsey numbers is a direct consequence of Theorem 2.1.1 and the result of Chen and Schelp.

Corollary 3.1.1 *For each tree T, the Ramsey numbers grow linearly for all graphs without containing a copy of any tree in $\mathcal{T}(T)$.*

Proof. By Theorem 3.1.1, the graphs without a copy of any tree in $\mathcal{T}(T)$ are $2p^2-7p+5$-admissible. Then they are $4p^4$-arrangeable and their Ramsey numbers grow linearly by the result of Chen and Schelp. □

Recall that H_i is defined to be the graph obtained by adding a path of length i between the centers of two disjoint stars of order three for all natural numbers i. The following corollary implies Theorem 2.1.2 by letting $i = 1$.

Corollary 3.1.2 *For all natural numbers i, the Ramsey numbers grow linearly for all graphs without any copy of H_i, H_{i+1},..., H_{2i}.*

Proof. Note that for the tree $T = H_i$ for some i, a graph without a copy of any tree in $\mathcal{T}(T)$ is exactly one without any copy of H_i, H_{i+1},..., H_{2i}. Then the conclusion follows from Corollary 3.1.1 with the tree $T = H_i$. □

For another application of Theorem 3.1.1, we now introduce the following two-person game, first considered by Bodlaender [8]. Let G be a graph of order n and let t be an integer. The game is played by two players A and B. A is trying to properly color the graph and B is trying to prevent that from happening. They alternate turns with A having the first move. A move consists of selecting a previously uncolored vertex v and assigning it a color from $\{1, 2, ..., t\}$ distinct from the colors assigned previously (by either player) to neighbors of v. If after n moves the graph is (properly) colored, A wins, otherwise B wins. More precisely, B wins if after less than n steps either player cannot make his next move. The *game chromatic number* of a graph is the least integer t such that A has a winning strategy in this game. Kierstead and Trotter [43] proved the following.

Theorem 3.1.2 *Let p and t be positive integers. If a p-admissible graph has chromatic number t then its game chromatic number is at most $pt + 1$.*

By Lemma 3.1.1, Theorems 3.1.1 and 3.1.2 we have the following.

Corollary 3.1.3 *Any graph without a copy of any tree in $\mathcal{T}(T)$ for any tree T of order p (> 2) has game chromatic number at most $4p^3 - 22p^2 + 38p - 19$.*

Proof. By Theorem 3.1.1 with a tree T of order p, a graph G without a copy of any tree in $\mathcal{T}(T)$ is $2p^2 - 7p + 5$-admissible. By Lemma 3.1.1, any subgraph H of G has size less than $(p-2)|V(H)|$. Then the minimum degree of H satisfies

$$\delta(H) \leq d(H) < 2(p-2).$$

So G is $2p - 5$-degenerate. By Proposition 1.1.1, The chromatic number $\chi(G) \leq 2p - 4$. Then by Theorem 3.1.2, G has game chromatic number at most

$$(2p^2 - 7p + 5)(2p - 4) + 1 = 4p^3 - 22p^2 + 38p - 19.$$

□

We remark that Erdős and Sós [23] conjectured in 1963 that a graph of average degree at least p contains every tree of size p. This conjecture which is better than Lemma 3.1.1 by a factor of 2, would improve all results in this section.

3.2 Arrangeable graphs

In this section, we show that Ramsey numbers grow linearly for degenerate graphs versus arrangeable graphs. From now on, we will omit ceilings and floors throughout this chapter, as these will not affect the proofs. Also when neccessary, we will always assume that n is sufficiently large for our estimates to hold.

We start with some definitions. A graph H is called (d, s)-*thick*, if for every $s \leq k \leq |V(H)|$ and every subgraph G of H of order k, $|E(G)| \geq \frac{1}{2d}\binom{k}{2}$.

By Lemmas 2 and 3 in [48], we have the following lemma, which will later allow us to consider only the relations between thick graphs and arrangeable graphs or crowns.

Lemma 3.2.1 *If $|V(H)| > 4n$ and for some s, $4n \le s \le |V(H)|$, H is not (d,s)-thick, then its complement H^c contains every d-degenerate graph of order n.*

Since the proof is short and shows the basic idea of how to deal with Ramsey numbers of degenerate graphs, we include it here for completeness.
Proof. Suppose that H is not (d,s)-thick and D is a d-degenerate graph of order n. By definition, there exists a subgraph G of H of order k so that

$$s \le k \le |V(H)| \text{ and } |E(G)| < \frac{1}{2d}\binom{k}{2}.$$

Then

$$|E(G^c)| \ge [1 - 1/(2d)]\binom{k}{2}.$$

Claim. There exists a subgraph G' of G such that

$$d_{G'^c}(v) \ge \frac{d-1}{d}(|V(G')| - 1) + k/(4d), \ \forall v \in V(G').$$

Otherwise, we can order the vertices of G as $v_1, v_2, ..., v_k$ and let $G_i = G^c - \{v_1, v_2, ..., v_{i-1}\}$ for $i = 1, 2, ..., n-1$ so that

$$d_{G_i}(v_i) < \frac{d-1}{d}(k-i) + k/(4d).$$

Then

$$\begin{aligned} |E(G^c)| = &\sum_{i=1}^{k-1} d_{G^i}(v_i) < \sum_{i=1}^{k-1}[\frac{d-1}{d}(k-i) + k/(4d)] \\ =&\frac{d-1}{d}\sum_{i=1}^{k-1}(k-i) + \binom{k}{2}/(2d) \\ =&[1-1/(2d)]\binom{k}{2} \le |E(G^c)|. \end{aligned}$$

This contradiction proves the claim. Then it follows that for all $v_1, v_2, ..., v_d \in V(G')$,

$$\begin{aligned} |\cap_{i=1}^{d} N_{G'^c}(v_i)| \ge &|V(G')| - d - (|V(G') - 1) + k/4 \\ =&1 - d + k/4 > n - d. \end{aligned}$$

We now construct an embedding $f : D \to G'^c$, and then we are done since G'^c is a subgraph of H^c. Since D is d-degenerate, we can order the vertices of D as $u_1, u_2, ..., u_n$ so that for every $i = 1, 2, ..., n$ at most d neighbors of u_i have indices less than i. We first let $f(u_1)$ be an arbitrary vertex of G'. Suppose we have embedded vertices $u_1, u_2, ..., u_{i-1}$ and u_i is adjacent only to $u_{i_1}, u_{i_2}, ..., u_{i_h}$ among embedded vertices. Then $h \le d$. Since there are at least $n - d$ vertices in $\cap_{j=1}^{h} N_{G'^c}(f(u_{i_j}))$, we choose as $f(u_i)$ any of them that is different from $f(u_1), f(u_2), ..., f(u_{i-1})$. This completes the proof. □

Let G be a graph of order n. An a-tuple $A \subset V(G)$ is called (G, m)-*good* if $|\cap_{v \in A} N(v)| \ge n/m^a$, and is called (G, m)-*bad* otherwise. We will need two lemmas from [48] which later let us reduce the proofs of the theorems to the cases when in *big* subgraphs of a graph, every good a-tuple is contained in *few* bad $(a + 1)$-tuples.

For a graph H of order n, an (H, r, m)-*reducing pair* is a pair of disjoint subsets R and S of $V(H)$ such that

$$|R| = r,\ |S| \ge 3n/(4m^{d-1}) \text{ and } |N_H(v) \cap S| \le 4|S|/(3m)\ \forall v \in R.$$

Lemma 3.2.2 [48] *Let $m \ge 2$. Let $H_1 \subset H$, where $|V(H)| = M$, $|V(H_1)| = M_1$, and $M_1 \ge Mm^{-4d^2}$. Let $r \le M_1/(2m^d)$. If for some $0 \le a < d$, an (H_1, m)-good a-tuple A is contained in at least r (H_1, m)-bad $(a + 1)$-tuples, then H_1 contains an (H_1, r, m)-reducing pair.*

Lemma 3.2.3 [48] *Let $|V(H)| = M$, $d \ge 2$, $r \ge 2$, and $m \ge 8d$. If every subgraph H_1 of H with $|V(H_1) \ge Mm^{-4d^2}$ contains an (H_1, r, m)-reducing pair, then H contains a subgraph H' on $4dr$ vertices with $|E(H')| < \frac{1}{2d}\binom{4dr}{2}$. In particular, H is not $(d, 4dr)$-thick.*

Corollary 3.2.1 *Let $0 \le a < d$, $m \ge 8d$ and $r \le n/(2m^{4d^2+d})$. If a graph H of order n is $(d, 4dr)$-thick, then there is a subgraph G of H of order at least n/m^{4d^2} so that every (G, m)-good a-tuple is contained in at most r (G, m)-bad $(a + 1)$-tuples.*

This is a direct consequence of Lemmas 3.2.2 and 3.2.3 which enables us to deduce the following Turán type result from which our main result on linear Ramsey numbers will follow.

Theorem 3.2.1 *Any $(d, 4dn)$-thick graph of order at least $(1+2^{d-1})(8d)^{4d^2+d}n$ contains all d-arrangeable graphs of order n.*

Proof. Let a graph F of order $(1+2^{d-1})(8d)^{4d^2+d}n$ be $(d,4dn)$-thick and a graph G of order n be d-arrangeable. By Corollary 3.2.1, there is a subgraph H of F of order at least $N=(1+2^{d-1})(8d)^d n$ so that every $(G,8d)$-good a-tuple is contained in at most n $(G,8d)$-bad $(a+1)$-tuples. And by definition, the vertices of G can be ordered as $v_1,v_2,...,v_n$ so that for each integer i with $1\le i\le n$, at most d vertices among $\{v_1,v_2,...,v_i\}$ have a neighbor $v\in\{v_{i+1},v_{i+2},...,v_n\}$ adjacent to v_i. Denote by $L(i,j)$ the set of neighbors of v_i with indices less than j. We now construct an embedding f from $V(G)$ into $V(H)$ by maintaining after step $k-1$ the property that $f(L(i,k))$ is $(H,8d)$-good for all $i>k$.

Step 1. Since there are fewer than n $(H,8d)$-bad vertices, choose a good one for $f(v_1)$.

Step k. Now we have the property $f(L(i,k))$ is $(H,8d)$-good for all $i>k$. Assume that taking $v=f(v_k)\in\cap_{v\in f(L(k,k))}N_H(v)-\{f(v_1),f(v_2),...,f(v_{k-1})\}$ makes an $(H,8d)$-bad a-tuple $A=\{f(v_{i_1}),...,f(v_{i_a}),v\}$ for some $0<a\le d$ and the $(a-1)$-tuple $A-v$ is some good $f(L(i,k))$. Since in H, $A-v$ sits in at most n $(H,8d)$-bad a-tuples, at most n vertices v can make an $(H,8d)$-bad a-tuple of this $A-v$. The total number of such $A-v$ is at most 2^{d-1}. Then

$$|\cap_{v\in f(L(k,k))}N_H(v)|-(k-1)-n2^{d-1}>N/(8d)^d-n(1+2^{d-1})\ge 0.$$

Now we can choose $f(v_k)$ still with the property that we want, which completes the proof. □

By Lemma 3.2.1 and Theorem 3.2.1, we have the following consequence that shows that Ramsey numbers grow linearly for degenerate graphs versus arrangeable graphs.

Corollary 3.2.2 *Let $c=(1+2^{d-1})(8d)^{4d^2+d}$. Then for every d-arrangeable graph G_1 and d-degenerate graph G_2 of order n, $R(G_1,G_2)\le cn$.*

Similarly we can prove the following extension of Theorem 3 in [48] and Corollary 3.2.2 with a smaller constant c for d-degenerate graphs with chromatic number less than d.

Theorem 3.2.2 *Let a graph G_1 of order n be d-arrangeable and a graph G_2 of order n be d-degenerate with chromatic number χ. Let $m=4(d+1)(\chi-1)$ and $c=(1+2^{d-1})m^{d+1}(4m^{d-1})^{\chi-2}$. Then $R(G_1,G_2)\le cn$. In particular, if G_2 is bipartite, then $R(G_1,G_2)\le(1+2^{d-1})[4(d+1)]^{d+1}n$.*

The proof is similar to that of Kostochka and Rödl [48].

Proof. Let H be an arbitrary graph of order cn. If some subgraph $H_1\subset H$ of order at least $2m^d(1+2^{d-1})(d+1)n$ has no $(H_1,2(d+1)n,m)$-reducing pair, then, by Lemmas 3.2.2 and 3.2.3 and Theorem 3.2.1, H_1 contains G_1.

Thus, we may assume that every subgraph $H_1 \subset H$ of order at least $2m^d(1+2^{d-1})(d+1)n$ has an $(H_1, 2(d+1), m)$-reducing pair.

Let $H_0 = H$ and for $k = 1, 2, ..., \chi - 1$ we do the following:

(a) Choose an $(H_{k-1}, 2(d+1)n, m)$-reducing pair (R_k, S_k);

(b) Since $|N_H(v) \cap S_k| \leq 4|S_k|/(3m)$ $\forall v \in R_k$, there exists $S'_k \subset S_k$ such that $|S'_k| \geq |S_k|/3$ and

$$|N_H(v) \cap R_k| \leq 2|R_k|/m \ \forall v \in S'_k.$$

(c) Take $H_k = H(S'_k)$ and note that by the definitions of S'_k and reducing pairs,

$$|V(H_k)| \geq |S_k|/3 \geq |V(H_{k-1})|/(4m^{d-1}) \geq ... \geq |V(H_0)|/(4m^{d-1})^k.$$

Observe that since $|V(H)| = cn = (1+2^{d-1})m^{d+1}(4m^{d-1})^{\chi-2}n$, for $k < \chi - 1$ we have $|V(H_k)| \geq 2m^d(1+2^{d-1})(d+1)n$ and we can make Step $k+1$.

Denote by R_χ any subset of $S'_{\chi-1}$ of cardinality $2(d+1)n$.

Observe that

(1) $|R_1| = |R_2| = ... = |R_\chi| = 2(d+1)n$;

(2) for every $i > k$ and every $v \in R_i$,

$$|N_H(v) \cap R_k| \leq 2|R_k|/m = 4(d+1)n/m = n/(\chi - 1).$$

Now, we construct $T_1, T_2, ..., T_\chi$ as follows. Let T_χ be any subset of R_χ of size $(d+1)n$. Suppose that sets $T_\chi \subset R_\chi$, $T_{\chi-1} \subset R_{\chi-1}, ..., T_{k+1} \subset R_{k+1}$ of size $(d+1)n$ are chosen. By (2), $|E_H(R_k, T_i)| \leq (d+1)n^2/(\chi - 1)$ for every $i > k$. Hence the number of vertices in R_k having more than n neighbors in T_i is at most $(d+1)n/(\chi - 1)$. It follows that there are at least

$$|R_k| - (\chi - k)(d+1)n/(\chi - 1) \geq |R_k| - (d+1)n = (d+1)n$$

vertices in R_k with at most n neighbors in each of T_χ, $T_{\chi-1}, ..., T_{k+1}$. Take as T_k any set of $(d+1)n$ such vertices.

Now, we have

(i) $|T_1| = |T_2| = ... = |T_\chi| = (d+1)n$;

(ii) for every $i \neq k$ and every $v \in R_i$, $|N_H(v) \cap R_k| \leq n$.

Denote by F the complement of the subgraph of H induced by $\cup_{k=1}^{\chi} T_k$. By (i) and (ii), the graph F has the property that $\forall i \in \{1, 2, ..., \chi\}$, $\forall v_1, v_2, ..., v_d \in V(F) - T_i$,

$$| \cap_{j=1}^{d} N_F(v_j) \cap T_i| \geq n.$$

Then we can embed G_2 into F simply by repeating the second part of the proof of Lemma 3.2.1 with the only change that the image $f(u_i)$ of u_i must belong to $T_{f(u_i)}$. $\square$

We remark that by careful optimization, the constant $(1+2^{d-1})(8d)^{4d^2+d}$ may be improved to $(1+2^{d-1})[(4+\epsilon)d]^{4d^2+d}$, and the constant m from $4(d+1)(\chi-1)$ to $(1+\epsilon)(d+1)(\chi-1)$ for any sufficiently small $\epsilon > 0$.

3.3 Crowns

In this section, we show that Ramsey numbers grow linearly for degenerate graphs versus crowns.

Let $d > 0$ and $r \in (0,1)$. A (d,r)-*crown* is a bipartite graph $C = (U,V;E)$ with $|V| \le |U|^r$ and $d(v) \le d$ for every $v \in U$.

Lemma 3.3.1 *If a bipartite graph $G = (U,V;E)$ with $|V| = n^r,\ r \in (0,1)$ has the property that for every d-tuple of V there are at least $n-d$ common neighbors of the d-tuple in U, then G contains a copy of any (d,r)-crown of order n.*

Proof. Let a (d,r)-crown $C = (A,B;F)$ be given. We need to construct an embedding $f : A\cup B \to U\cup V$ from C into G. Let f be an arbitrary bijection from B to V. We extend this map to $A = \{v_1, v_2, ..., v_m\},\ m < n$ as follows: for all i, since the degree of v_i is at most d, there are at least $n-d$ vertices in U adjacent to all vertices in $f(N_G(v_i))$ by assumption. We choose for $f(v_i)$ any vertex that has not been used before. This completes the proof. $\square$

We will use the following Chernoff-Hoeffding type inequality (cf. [4], Appendix A).

Lemma 3.3.2 *Let Y be the sum of mutually independent indicator random variables, $\mu = \mathbf{E}(Y)$. For all $\epsilon > 0$,*

$$\mathbf{P}[Y < \mu(1-\epsilon)] < e^{-\epsilon^2\mu/2};$$
$$\mathbf{P}[Y \ge \mu(1+\epsilon)] \le [e^{\epsilon}/(1+\epsilon)^{1+\epsilon}]^{\mu}.$$

The following extends the result of Erdős [20] that each graph contains a bipartite subgraph of size at least half of its size in the sence that we require the two parts are almost equal.

Splitting Theorem. *Let $\epsilon > 0$, m and n sufficiently large. Then a graph G of order n and size m contains a bipartite subgraph $(V_1, V_2; E)$ of size $|E| \ge m(1-\epsilon)/2$ and $||V_i| - n/2| \le \epsilon n/2$ for $i = 1$ and 2.*

Proof. Let $V_1 \subset V(G)$ be a random subset given by $\mathbf{P}(v \in V_1) = 1/2$, these choices being mutually independent. Then $\mathbf{E}(|V_1|) = n/2$. Let $V_2 = V(G) - V_1$, X the number of edges between V_1 and V_2 and X_e the indicator random variable for the edge e being between V_1 and V_2. Then $X = \sum_{e \in E(G)} X_e$. $\mathbf{E}(X_e) = 1/2$ as two fair coin flips have probability $1/2$ of being different. Then $\mathbf{E}(X) = \sum_{e \in E(G)} \mathbf{E}(X_e) = m/2$. It is easy to check that all X_e are mutually independent. Then by Lemma 3.3.2, we have

$$\begin{aligned} \mathbf{P}[|V_1| < n(1-\epsilon)/2] &< e^{-\epsilon^2 n/4} \to 0, \\ \mathbf{P}[|V_1| \geq n(1+\epsilon)/2] &\leq [e^{\epsilon}/(1+\epsilon)^{1+\epsilon}]^{n/2} \to 0, \\ \mathbf{P}[X < m(1-\epsilon)/2] &< e^{-\epsilon^2 m/4} \to 0. \end{aligned}$$

Thus there is some choice of V_1 so that the bipartite subgraph induced by the two parts V_1 and V_2 has the property that we want. □

We will often use the following standard bounds.

Lemma 3.3.3

$$\binom{n}{a} / \binom{m}{a} \leq \left(\frac{n}{m}\right)^a \textit{ for } m > n,$$

$$\binom{n}{a} / \binom{m}{a} \leq \left(\frac{n-a}{m-a}\right)^a \textit{ for } m < n.$$

The following Turán type result plays a key role in this section from which our main result on linear Ramsey numbers will follow. In the proof, we first apply Splitting Theorem to a dense graph G to obtain a relatively dense bipartite subgraph with almost equal parts and then show that this bipartite graph contains the crowns by combining an idea of a simple lemma of Kostochka and Rödl [49] which will be refined and also applied in the following sections.

Theorem 3.3.1 *Let $d > 0$ and $r, \rho \in (0,1)$. Then a graph of order $2\rho^{(d-r+1)/(r-1)}n + o(n)$ and density at least ρ contains a copy of any (d,r)-crown of order n.*

Proof. For $\epsilon > 0$, let $a = \rho(1-\epsilon)/(1+\epsilon)^2$, $b = a^{(d-r+1)/(r-1)}/(1+\epsilon)$ and let G a graph of order $2(b+\epsilon)n$ with density at least ρ. Since $(b+\epsilon)(1-\epsilon) \to \rho^{(d-r+1)/(r-1)} > 1$ as $\epsilon \to 0$, We can choose $\epsilon > 0$ small enough so that $(b+\epsilon)(1-\epsilon) > 1$.

By the Splitting Theorem, G contains a bipartite subgraph $(U, V; E)$ of size

$$|E| \geq |E(G)|(1-\epsilon)/2 \geq \rho \binom{2(b+\epsilon)n}{2} (1-\epsilon)/2,$$

such that

$$||U| - (b+\epsilon)n| \leq \epsilon(b+\epsilon)n$$

and

$$||V| - (b+\epsilon)n| \leq \epsilon(b+\epsilon)n.$$

An $(r-1)\log_a n$-tuple $\{v_1, v_2, ..., v_{(r-1)\log_a n}\}$ of vertices in U is called *bad* if it is contained in the intersection of neighborhoods $\cap_{i=1}^d N(u_i) \cap U$ for some d-tuple $\{u_1, u_2, ..., u_d\}$ in V with $|\cap_{i=1}^d N(u_i) \cap U| < n$. Other $(r-1)\log_a n$-tuples in U are called *good*. See Figure 3.

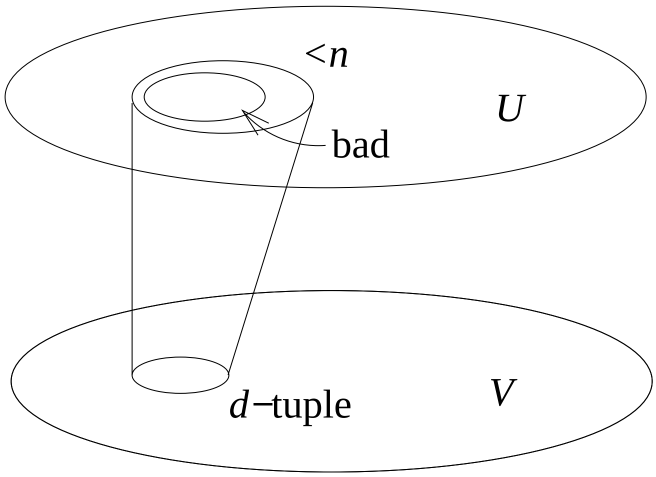

Figure 3

Thus the number of pairs (v, L) such that $v \in V$ and $L \subset N(v) \cap U$ is a bad $(r-1)\log_a n$-tuple is at most $(b+\epsilon)n(1+\epsilon)\binom{(b+\epsilon)n(1+\epsilon)}{d}\binom{n-1}{(r-1)\log_a n}$. On the other hand, the total number of pairs (v, L) such that $v \in V$ and $L \subset N(v) \cap U$ is an $(r-1)\log_a n$-tuple is at least

$$\begin{aligned}\sum_{v\in V}\binom{d_U(v)}{(r-1)\log_a n} \geq& |V|\binom{\sum_{v\in V} d_U(v)/|V|}{(r-1)\log_a n} \\ \geq& (b+\epsilon)n(1-\epsilon)\binom{\rho[(b+\epsilon)n - 1/2]\frac{1-\epsilon}{1+\epsilon}}{(r-1)\log_a n}\end{aligned}$$

by Jensen's inequality. Now by using Lemma 3.3.3, we have

$$\begin{aligned}&\frac{(b+\epsilon)n(1+\epsilon)\binom{(b+\epsilon)n(1+\epsilon)}{d}\binom{n-1}{(r-1)\log_a n}}{(b+\epsilon)n(1-\epsilon)\binom{\rho[(b+\epsilon)n-1/2]\frac{1-\epsilon}{1+\epsilon}}{(r-1)\log_a n}} \\ \leq& \frac{1+\epsilon}{1-\epsilon}[(b+\epsilon)n(1+\epsilon)]^d[\frac{\rho(b+\epsilon)(1-\epsilon)}{1+\epsilon}]^{(1-r)\log_a n} \\ =& \frac{1+\epsilon}{1-\epsilon}[(b+\epsilon)(1+\epsilon)]^d[1+\epsilon/b]^{(1-r)\log_a n} \to 0\end{aligned}$$

and

$$\frac{n^r\binom{(b+\epsilon)n(1+\epsilon)}{(r-1)\log_a n}}{(b+\epsilon)n(1-\epsilon)\binom{\rho[(b+\epsilon)n-\frac{1}{2}]\frac{1-\epsilon}{1+\epsilon}}{(r-1)\log_a n}}$$

$$\leq \frac{n^{r-1}}{(b+\epsilon)(1-\epsilon)}\{\frac{(b+\epsilon)n(1+\epsilon)-(r-1)\log_a n}{\rho[(b+\epsilon)n-\frac{1}{2}]\frac{1-\epsilon}{1+\epsilon}-(r-1)\log_a n}\}^{(r-1)\log_a n}$$

$$= \frac{\{1+\frac{[\rho(1+\epsilon)^2+\epsilon-1](r-1)\log_a n+(1-\epsilon^2)/2}{[(b+\epsilon)n-1/2](1-\epsilon^2)-\rho(1+\epsilon)^2(r-1)\log_a n}\}^{(r-1)\log_a n}}{(b+\epsilon)(1-\epsilon)}$$

$$\leq \frac{\exp\{\frac{[\rho(1+\epsilon)^2+\epsilon-1](r-1)^2\log_a^2 n+(1-\epsilon^2)(r-1)\log_a n/2}{[(b+\epsilon)n-1/2](1-\epsilon^2)-\rho(1+\epsilon)^2(r-1)\log_a n}\}}{(b+\epsilon)(1-\epsilon)}$$

$$\to 1/[(b+\epsilon)(1-\epsilon)] < 1.$$

It follows that the number of pairs (v, L) such that $v \in V$ and $L \subset N(v) \cap U$ is a good $(r-1)\log_a n$-tuple is at least $(b+\epsilon)n(1-\epsilon)\binom{\rho[(b+\epsilon)n-1/2]\frac{1-\epsilon}{1+\epsilon}}{(r-1)\log_a n} - (b+\epsilon)n(1+\epsilon)\binom{(b+\epsilon)n(1+\epsilon)}{d}\binom{n-1}{(r-1)\log_a n} \geq n^r\binom{(b+\epsilon)n(1+\epsilon)}{(r-1)\log_a n}$. So there exists a good $(r-1)\log_a n$-tuple $L \subset N(v) \cap U$ which is contained in at least n^r such pairs. Hence there is a subset N of V with $|N| = n^r$ such that $L \subset N(v) \cap U$ for all $v \in N$. As L is good, $|\cap_{i=1}^d N(u_i) \cap U| \geq n$ for all d-tuples $\{u_1, u_2, ..., u_d\}$ in N. Now by Lemma 3.3.1, G contains a copy of any (d, r)-crown of order n. $\square$

Lemma 3.2.1 and Theorem 3.3.1 yield the following consequence that shows that Ramsey numbers grow linearly for degenerate graphs versus crowns.

Corollary 3.3.1 *For any (d, r)-crown G_1 and d-degenerate graph G_2 of order n,*

$$R(G_1, G_2) < 4d(2d)^{d/(1-r)}n + o(n).$$

3.4 Cubes

In this section, we prove polynomial upper bounds for the Ramsey numbers of cubes. The proof is based on the following refined version of a simple lemma of Kostochka and Rödl [49], which was also proved and applied by other researchers including Gowers [31] and Sudakov [72]. The lemma asserts, roughly, that every bipartite graph with sufficiently many edges contains a large subset M of one part in which every d vertices have many common neighbors. The proof uses a control set called l-tuple. M is simply the set of all common neighbors of an appropriately chosen l-tuple L. Intuitively, it is clear that if some d vertices have only a few common neighbors, it is unlikely all the members of L will be chosen among these neighbors. Hence, we do not expect M to contain any such subset of d vertices. See Figure 4.

Lemma 3.4.1 *Let $\rho \in (0,1]$. Suppose that positive numbers N, d, l, m and n satisfy the inequality*

$$N\binom{\rho N}{l} - N\binom{N}{d}\binom{n-1}{l} \geq m\binom{N}{l}.$$

Then for every bipartite graph $(U, V; E)$ of size ρN^2 with equal parts $|U| = |V| = N$, there exists a subset $M \subset V$ with $|M| \geq m$ and the property that for every d-tuple D of M, there are at least n vertices of U adjacent to all vertices in D.

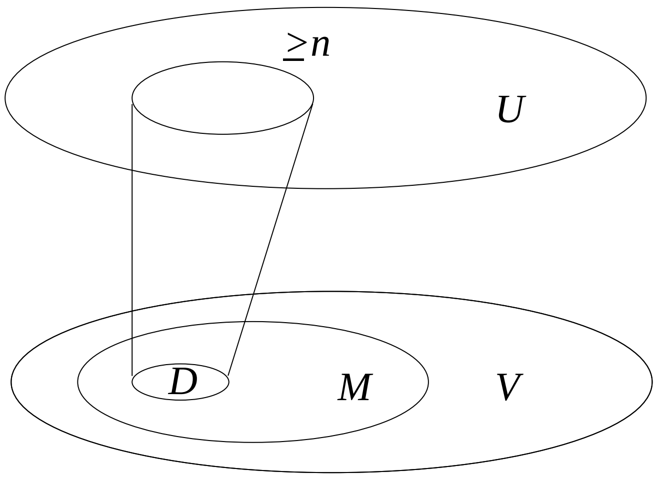

Figure 4

Proof. An l-tuple $\{v_1, v_2, ..., v_l\}$ of vertices in U is called *bad* if it is contained in the intersection of neighborhoods $\cap_{i=1}^{d} N(u_i)$ for some d-tuple $\{u_1, u_2, ..., u_d\}$ in V with $|\cap_{i=1}^{d} N(u_i)| < n$. Other l-tuples in U are called *good*. Thus the number of pairs (v, L) such that $v \in V$ and $L \subset N(v)$ is a bad l-tuple is at most $N\binom{N}{d}\binom{n-1}{l}$. On the other hand, the total number of pairs (v, L) such that $v \in V$ and $L \subset N(v)$ is an l-tuple is at least

$$\sum_{v\in V}\binom{d(v)}{l} \geq N\binom{\sum_{v\in V} d(v)/N}{l} \geq N\binom{\rho N}{l}$$

by Jensen's inequality. Since

$$N\binom{\rho N}{l} - N\binom{N}{d}\binom{n-1}{l} \geq m\binom{N}{l},$$

the number of pairs (v, L) such that $v \in V$ and $L \subset N(v)$ is a good l-tuple is at least $m\binom{N}{l}$. So there exists a good l-tuple $L \subset N(v)$ which is contained in at least m such pairs. Hence there is a subset M of V with $|M| = m$ such that $L \subset N(v)$ for all $v \in M$. As L is good, $|\cap_{i=1}^{d} N(u_i)| \geq n$ for all d-tuples $\{u_1, u_2, ..., u_d\}$ in M. This proves the lemma. □

In the following, the logarithm is to base 2.

Theorem 3.4.1 *For any positive constant* c*, let* $l = 1 + (c + \sqrt{c^2 + 4c})/2$ *and* $G = (U, V; E)$ *be a bipartite graph of order* n *where the maximum degree of vertices in* U *is at most* $c \log n$*. Then for any bicoloring of the edges of bipartite complete graph* $K_{n^{l+o(1)}, n^{l+o(1)}}$*, the subgraph induced by the more frequent color contains a copy of* G*. In particular,* $R(G) < n^{l+o(1)}$.

Since the cube Q_n of dimension n is bipartite and n-regular, the following polynomial upper bound for the Ramsey numbers of cubes is a special case of Theorem 3.4.1 with $c = 1$.

Corollary 3.4.1 $R(Q_n) < 2^{(3+\sqrt{5})n/2+o(n)}$.

Proof of Theorem 3.4.1. For any $\epsilon > 0$ and any bicoloring of the edges of the bipartite complete graph $K_{n^{l+\epsilon}, n^{l+\epsilon}} = (A, B; E_K)$ there is a monochromatic subgraph S with at least $n^{2(l+\epsilon)}/2$ edges. By using Lemma 3.3.3, we have

$$\begin{aligned}\frac{n^{l+\epsilon}\binom{n^{l+\epsilon}}{c \log n}\binom{n-1}{(l-1)\log n}}{n^{l+\epsilon}\binom{n^{l+\epsilon}/2}{(l-1)\log n}} \leq & n^{(l+\epsilon)c\log n}(2/n^{l-1+\epsilon})^{(l-1)\log n} \\ = & n^{\epsilon(c-\sqrt{c^2+4c})\log n/2} 2^{(c+\sqrt{c^2+4c})\log n/2} \longrightarrow 0\end{aligned}$$

and

$$\begin{aligned}\frac{n\binom{n^{l+\epsilon}}{(l-1)\log n}}{n^{l+\epsilon}\binom{n^{l+\epsilon}/2}{(l-1)\log n}} \leq & n^{1-l-\epsilon}[\frac{n^{l+\epsilon} - (l-1)\log n}{n^{l+\epsilon}/2 - (l-1)\log n}]^{(l-1)\log n} \\ = & n^{1-l-\epsilon} 2^{(l-1)\log n}[1 + \frac{(l-1)\log n}{n^{l+\epsilon} - 2(l-1)\log n}]^{(l-1)\log n} \\ \leq & n^{-\epsilon}\exp\{\frac{(l-1)^2\log^2 n}{n^{l+\epsilon} - 2(l-1)\log n}\} \to 0.\end{aligned}$$

It follows that

$$n^{l+\epsilon}\binom{n^{l+\epsilon}/2}{(l-1)\log n} - n^{l+\epsilon}\binom{n^{l+\epsilon}}{c\log n}\binom{n-1}{(l-1)\log n} \geq n\binom{n^{l+\epsilon}}{(l-1)\log n}.$$

By Lemma 3.4.1, there exists subset N of B with $|N| = n$ such that $|\cap_{i=1}^{c\log n} N_S(u_i)| \geq n$ for all $c\log n$-tuples $\{u_1, u_2, ..., u_{c\log n}\}$ in N.

Now we construct an embedding $f : U \cup V \to A \cup N$ from $G = (U, V; E)$ into the induced subgraph $S[A \cup N]$. Let f be an arbitrary bijection from V to N. We extend this map to $U = \{v_1, v_2, ..., v_n\}$ as follows: for all i, since the degree of v_i is at most $c\log n$, there are at least n vertices in A adjacent to all vertices in $f(N_G(v_i))$. We choose for $f(v_i)$ any vertex that has not been used before. This completes the proof. □

We remark that by careful optimization, one may get

$$R(Q_n) < 2^{(3+\sqrt{5})n/2-(1+\sqrt{5})/2+[1+o(1)]/n}.$$

3.5 O(ln$n$)-degenerate graphs of order n

In this section, we show the following polynomial upper bound of the Ramsey numbers for bipartite O(ln$n$)-degenerate graphs of order n.

Theorem 3.5.1 *For any positive constant c and natural number k, let* $l = l(k, c) = \inf\{h/(h-d) | h(d-2c) = (h-d)(1+h\ln k)(d-c),\ 2c < d < h\}$. *Then for any k-coloring of the edges of the bipartite complete graph* $K_{(kn)^{l+o(1)},(kn)^{l+o(1)}}$*, the subgraph induced by the most frequent color contains bipartite* $c\ln n$*-degenerate graphs G of order n. In particular,* $R(G; k) < 2(kn)^{l+o(1)}$.

We first prove the following Turán-type result.

Theorem 3.5.2 *For* $0 < a \leq 1$ *and* $c > 0$*, let* $l = l(a, c) = \inf\{h/(h-d) | h(d-2c) = (h-d)(1-h\ln a)(d-c),\ 2c < d < h\}$. *Any bipartite graph* $G = (U, V; E)$ *with* $|U| = |V| = (n/a)^{l+o(1)}$ *and* $|E| = a(n/a)^{2l+o(1)}$ *contains every bipartite* $c\ln n$*-degenerate graphs of order n.*

The proof is a combination of Lemma 3.4.1 and a probabilistic method of Kostochka and Sudakov [50].

Proof of Theorem 3.5.2. Let $h > d > 2c$ and satisfying $h(d-2c) = (h-d)(1-h\ln a)(d-c)$, and let $l = h/(h-d)$. For any $\epsilon > 0$, let $G = (U, V; E)$ be a bipartite graph with $|U| = |V| = (n/a)^l + \epsilon n^l$ and $|E| = a[(n/a)^l + \epsilon n^l]^2$. By using Lemma 3.3.3, we have

$$\begin{aligned}\frac{[(n/a)^l + \epsilon n^l]\binom{(n/a)^l+\epsilon n^l}{d\ln n}\binom{n-1}{h\ln n}}{[(n/a)^l + \epsilon n^l]\binom{a[(n/a)^l+\epsilon n^l]}{h\ln n}} \leq& [(n/a)^l + \epsilon n^l]^{d\ln n}\{n/\{a[(n/a)^l + \epsilon n^l]\}\}^{h\ln n} \\ =& 1/(1+\epsilon a^l)^{h\ln n/l} \to 0\end{aligned}$$

and

$$\begin{aligned}\frac{n^{l+h\ln a}\binom{(n/a)^l+\epsilon n^l}{h\ln n}}{[(n/a)^l+\epsilon n^l]\binom{a[(n/a)^l+\epsilon n^l]}{h\ln n}} \leq & \frac{n^{h\ln a}}{1/a^l+\epsilon}\{\frac{[(n/a)^l+\epsilon n^l]-h\ln n}{a[(n/a)^l+\epsilon n^l]-h\ln n}\}^{h\ln n} \\ =&\{1+\frac{(1-a)h\ln n}{a[(n/a)^l+\epsilon n^l]-h\ln n}\}^{h\ln n}/(1/a^l+\epsilon) \\ \leq & \exp\{\frac{(1-a)h^2\ln^2 n}{a[(n/a)^l+\epsilon n^l]-h\ln n}\}/(1/a^l+\epsilon) \\ \to & 1/(1/a^l+\epsilon) < a^l \leq 1.\end{aligned}$$

It follows that $[(n/a)^l+\epsilon n^l]\binom{a[(n/a)^l+\epsilon n^l]}{h\ln n} - [(n/a)^l+\epsilon n^l]\binom{(n/a)^l+\epsilon n^l}{d\ln n}\binom{n-1}{h\ln n} \geq n^{l+h\ln a}\binom{(n/a)^l+\epsilon n^l}{h\ln n}$. Now by Lemma 3.4.1, there exists a subset V_2 of V with $|V_2| = n^{l+h\ln a}$ such that $|\cap_{i=1}^{d\ln n} N(u_i)| \geq n$ for all $d\ln n$-tuples $\{u_1, u_2, ..., u_{d\ln n}\}$ in V_2.

Now let $m = (d-c)\ln n$. Take a sequence $v_1, v_2, ..., v_m$ of not necessarily distinct vertices of V_2, which we choose uniformly and independently at random and let $A = \{v_1, v_2, ..., v_m\}$, $V_1 = \cap_{i=1}^m N(v_i)$. Note that a set of vertices $W \in U$ is contained in V_1 if and only if $A \in \cap_{v\in W} N(v)$ for all $i = 1, 2, ..., m$ and the probability that this happens equals $(|\cap_{v\in W} N(v)|/|V_2|)^m$. Denote by Z the number of subsets W of V_1 of size $c\ln n$ with $|\cap_{v\in W} N(v)| < n$. Then

$$\begin{aligned}\mathbf{E}(Z) \leq & \binom{(n/a)^l+\epsilon n^l}{c\ln n}[(n-1)/|V_2|]^m \\ \leq & (1/a^l+\epsilon)^{c\ln n} n^{cl\ln n} n^{(1-l-h\ln a)m}/(c\ln n)! \\ = & n^{c\ln(1/a^l+\epsilon)}/(c\ln n)! \to 0.\end{aligned}$$

Since Z is an integer, by the definition of expectation, there exists a particular choice of $v_1, v_2, ..., v_m$ for which $Z = 0$. Fix such $v_1, v_2, ..., v_m$ and the corresponding set V_1. By construction, all $c\ln n$ vertices in V_1 have at least n common neighbors in V_2. And, vice versa, all $c\ln n$ vertices in V_2 have at least n common neighbors in V_1. Indeed let B be a subset of V_2 of order $c\ln n$. Then the set $A\cup B$ is a subset of V_2 of order at most $m+c\ln n = d\ln n$. By the choice of V_2 there are at least n vertices in U adjacent to all vertices in $A\cup B$. And all these vertices are in V_1, since V_1 contains all common neighbors of A. Let $D = (U_1, U_2; E_D)$ be any bipartite $c\ln n$-degenerate graph of order n. By the definition of d-degenerate graphs, there exists a labeling $v_1, v_2, ..., v_n$ of vertices of D such that for all i, the number of neighbors v_j of v_i with $j < i$ is at most $c\ln n$. Now we construct an embedding $f : D \to G$ greedily so that the vertices in U_l will be embedded into set V_l, $l = 1, 2$. Without loss of generality we assume that $v_1 \in U_1$ and let $f(v_1)$ be an arbitrary vertex in V_1. Suppose that we have already embedded vertices $v_1, v_2, ..., v_{i-1}$ and suppose that $v_i \in U_l$. Let $C = \{f(v_j)|(v_i, v_j) \in E_D,\ j < i\}$. Then C is a subset of V_{3-l} of order at most $c \ln n$. Since there are at least n vertices in V_l adjacent to all vertices in C, we choose for $f(v_i)$ any vertex that has not been used before. This process surely embeds D into G. □

We remark that one could only use the probabilistic method to prove this theorem, but with some weaker polynomial upper bound, which in fact was done by Alon et al [3] when they studied a Turán type problem mentioned in the introduction.

Proof of Theorem 3.5.1. This follows from Theorem 3.5.2 by setting $a = 1/k$ and noting that for any k-coloring of the edges of a bipartite complete graph, the size of the subgraph induced by the most frequent color is at least as large as a $1/k$ fraction of the size of the bipartite complete graph. □

Chapter 4

Ramsey topologies

In this chapter, we study various Ramsey spaces including Hindman spaces, van der Waerden spaces and differentially compact spaces. We will also extend the results of Brown [9, 10] and Hindman et al [40] on piecewise syndetic sets from natural numbers and discrete semigroups to locally connected abelian semigroups.

4.1 Van der Waerden spaces

In this section, we prove Theorem 2.2.3 that extends Fürstenberg's result to maps with ranges in nonmetric spaces and apply it to give the shorter proof of Kojman's extension of van der Waerden's theorem. This short proof avoids using ultrafilters which Kojman did in his proof.

Proof of Theorem 2.2.3. Let a map $f : \mathbb{N}^m \to X$ and a finite set $F \subset \mathbb{N}^m$ be given. Assume that for any point $x \in \mathrm{cl}(f(\mathbb{N}^m))$ there exists a neighborhood V_x of x so that $f^{-1}(V_x)$ fails to contain a homothetic copy $bF + \mathbf{a}$, $\mathbf{a} \in \mathbb{N}^m$, $b \in \mathbb{N}$. Since $\mathrm{cl}(f(\mathbb{N}^m))$ is compact, a finite collection of V_x covers $\mathrm{cl}(f(\mathbb{N}^m))$ and the corresponding collection of $f^{-1}(V_x)$ covers $\mathbb{N}^m$, a contradition to Gallai's theorem. □

Theorem 2.2.2 also has the following consequence.

Corollary 4.1.1 *If X is compact, then for any map $f : \mathbb{N}^m \to X$ and any finite set $F \subset \mathbb{N}^m$, there exists a point x so that for any neighborhood V of x one can find a homothetic copy $bF + \mathbf{a}$, $\mathbf{a} \in \mathbb{N}^m$, $b \in \mathbb{N}$, for which $f(bF + \mathbf{a}) \subset V$.*

Definition 4.1.1 *A set A of natural numbers is called an* AP *set if it contains arbitrarily long arithmetic progressions.*

A topological space X is van der Waerden *if for every sequence (x_n) in X there exists a convergent subsequence (x_{n_i}) so that the index set $\{n_i | i \in \mathbb{N}\}$ is an AP set.*

Corollary 4.1.2 (Kojman [46]) *If the closure of every countable set of a space X is compact and first countable then X is van der Waerden.*

Proof. Let a sequence (x_n) in X be given. By Theorem 2.2.3, there exists a point x in $\mathrm{cl}\{x_n\}$ so that for any neighborhood V of x, $f^{-1}(V)$ contains arbitrarily long arithmetic progressions. Since $\mathrm{cl}\{x_n\}$ is first countable, we can refine a decreasing base (V_m) of x. For any m, choose an arithmetic progression P_m of length m so that $\{x_n | n \in P_m\} \subset V_m$. Then $(x_n)_{n \in \cup_m P_m}$ converges to x. □

4.2 Hindman spaces and differential compactness

In this section, we strengthen the topological converse of Hindman's theorem observed by Kojman [45], and then introduce and study differential compactness that has close relation to Hindman spaces and in the end give the short proof of the fact that Hindman spaces are closed under products in the framework of set theory.

Definition 4.2.1 *A set A of natural numbers is called an* IP *set if there exists an infinite set D of natural numbers so that $FS(D) \subset A$, where $FS(D) = \{\sum_{n \in F} n | F$ is a finite subset of natural numbers$\}$.*

Definition 4.2.2 *A set A of natural numbers is called a* DP *set if there exists an infinite set S of natural numbers so that the* difference set *$D(S) \subset A$, where $D(S) = \{m - n | m > n,\ m, n \in S\}$.*

It is known that both IP sets and DP sets belong to the class of so called *Poincaré sequences* that plays important role in the study of recurrences in topological dynamics (see [26] for example). The relation of themselves is shown in the following.

Proposition 4.2.1 *The class of IP sets is a proper subclass of DP sets.*

Proof. Let S be a IP set with a set $D = \{d_1 < d_2 < ...\}$. Set $n_i = \sum_{j=1}^{i} d_j$. Clearly $n_i - n_j \in S$ for $i > j$. So S contains a DP set.

Now we construct a DP set which is not an IP set. Let $n_1 < n_2$ be two natural numbers and $S_i = \{n_j | j \le i\}$. Define

$$n_{i+1} = 1 + n_i + \sum_{n \in D(S_i)} n,$$

and let

$$S = \bigcup_{i \in \mathbb{N}} S_i = \{n_i | i \in \mathbb{N}\}.$$

Claim. The DP set $D(S)$ is not an IP set.

Assume that there is an infinite subset D of $D(S)$ so that $FS(D) \subset D(S)$. Then at least one of the following two cases occurs.

Case 1. There are $i < j < k < l$ so that $n_j - n_i$ and $n_l - n_k$ are both in D. Then $n_j - n_i + n_l - n_k \notin D(S)$ since by the construction of S,

$$\begin{aligned} \sum_{n \in D(S_k)} n <& n_{k+1} - n_k \le n_l - n_k < n_j - n_i + n_l - n_k \\ <& n_k - n_{k-1} + n_l - n_k = n_l - n_{k-1}, \end{aligned}$$

a contradiction.

Case 2. There are $i \le j < k \le l$ so that $n_k - n_j$ and $n_l - n_i$ are both in D. Then $n_k - n_j + n_l - n_i \notin D(S)$ since by the construction of S,

$$n_l - n_1 = n_i - n_1 + n_l - n_i < n_k - n_j + n_l - n_i \le \sum_{n \in D(S_l)} n < n_{l+1} - n_l,$$

a contradiction. □

The following properties show that IP (and DP) sets need not be AP sets and AP sets need not be DP (or IP) sets.

Proposition 4.2.2 *There exists an IP set, which is not an AP set.*

Proof. Let n_1 be an arbitrary natural number and for $i \in \mathbb{N}$, let

$$n_{i+1} = 1 + 2 \sum_{j=1}^{i} n_j.$$

Then let

$$D = \{n_i | i \in \mathbb{N}\}.$$

Now we show that the IP set $FS(D)$ has no arithmetic progressions of length three by using induction on the subsets $D_i = \{n_j | j \le i\}$ of D. First for $i = 1$, $FS(D_1) = \{n_1\}$ has only one element and no arithmetic progressions of length three. Now suppose that for $i = k$, $FS(D_k)$ has no arithmetic progressions of length three. Then we show that $FS(D_{k+1})$ also has no arithmetic progressions of length three.

Assume that there is an arithmetic progression of length three $s_1 < s_2 < s_3$ in $FS(D_{k+1})$. Then the three cannot all be in $FS(D_k)$ since $FS(D_k)$ has no arithmetic progressions of length three by assumption. And also the three cannot all be the sum of n_{k+1} and others. So we must have $s_1 \in FS(D_k)$ and s_3 be the sum of n_{k+1} and others. Now suppose that $s_2 \in FS(D_k)$. Then

$$s_2 - s_1 < \sum_{i=1}^{k} n_i < n_{k+1} - \sum_{i=1}^{k} n_i < s_3 - s_2.$$

By the same argument, we have $s_2 - s_1 > s_3 - s_2$ if $s_2 \notin FS(D_k)$. So $FS(D_{k+1})$ has no arithmetic progressions of length three anyway. This completes the induction. □

Proposition 4.2.3 *There exists an AP set, which is not a DP set.*

Proof. Take a AP set $A = \{n_1 < n_2 < ...\}$ so that

$$\lim_{i\to\infty} n_{i+1} - n_i = \infty.$$

Then A cannot be a DP set. □

As mentioned in the introduction, Kojman [45] proved the topological converse of Hindman's theorem that if a T_1 space X satisfies that for every sequence $(x_n)_{n\in\mathbb{N}}$ there exists a convergent subsequence $(x_n)_{n\in A}$ for some IP set A, then X is finite. We strengthen this result by replacing the IP set by a DP set in the following.

Theorem 4.2.1 *Let X be a T_1 space. The following are equivalent.*

(1) X is finite;

(2) For every sequence $(x_n)_{n\in\mathbb{N}}$ in X there exists a convergent subsequence $(x_n)_{n\in A}$ for some IP set A;

(3) For every sequence $(x_n)_{n\in\mathbb{N}}$ in X there exists a convergent subsequence $(x_n)_{n\in A}$ for some DP set A.

In the proof, we will use the famous canonical Ramsey theorem due to Erdős and Rado [22].

Canonical Ramsey Theorem. *For every coloration c of all pairs of natural numbers there exists an infinite $T \subset \mathbb{N}$ on which c has one of the following four properties: For $i,\ j,\ k,\ l \in T$*

(i) distinct: $c(ij) = c(kl)$ *iff* $\{ij\} = \{kl\}$,
(ii) min: $c(ij) = c(kl)$ *iff* $\min\{i, j\} = \min\{k, l\}$,
(iii) max: $c(ij) = c(kl)$ *iff* $\max\{i, j\} = \max\{k, l\}$,
(iv) monochromatic: c *is constant.*

Proof of Theorem 4.2.1. (1)⇒(2). By Hindman's theorem, for every sequence $(x_n)_{n\in\mathbb{N}}$ in a finite space X there exists a constant subsequence $(x_n)_{n\in A}$ for some IP set A.

(2)⇒(3). This follows from Proposition 4.2.1.

(3)⇒(1). For every $n \in \mathbb{N}$ let

$$i(n) = \max\{i : 2^i|n\}.$$

Note that

$$i(n_1 + n_2) > i(n_1) \text{ if } i(n_1) = i(n_2)$$

and

$$i(n_1 + n_2) = i(n_1) \text{ if } i(n_1) < i(n_2).$$

Assume that X is infinite and $(x_n)_{n\in\mathbb{N}}$ is an injective sequence in X. Define a new sequence (y_n) in X by

$$y_n := x_{i(n)}.$$

Suppose now that $S = \{s_1 < s_2 < ...\}$ is an infinite subset of natural numbers so that $(y_n)_{n\in D(S)}$ is convergent. Define a coloring

$$c : \binom{\mathbb{N}}{2} \to \mathbb{N} \text{ by } c(jk) = i(|s_j - s_k|).$$

By Canonical Ramsey Theorem, there exists an infinite $T \subset \mathbb{N}$ on which one of the following occurs.

(1) $c(ij) = c(kl)$ iff $\{ij\} = \{kl\}$. In particular, there are in T, three numbers $j < k < l$ with

$$c(jk) < c(kl)$$

i.e.,

$$i(s_k - s_j) < i(s_l - s_k),$$

Then we have

$$i(s_l - s_j) = i(s_l - s_k + s_k - s_j) = i(s_k - s_j).$$

Thus $c(jk) = c(jl)$, a contradiction.

(2) $c(ij) = c(kl)$ iff $\min\{i, j\} = \min\{k, l\}$. Now order the numbers in T as $t_1 < t_2 < \ldots.$ Then

$$i(s_{t_k} - s_{t_j}) = i(s_{t_{j+1}} - s_{t_j}) \text{ for } j < k.$$

i.e.,

$$y_{s_{t_k} - s_{t_j}} = y_{s_{t_{j+1}} - s_{t_j}} \text{ for } j < k.$$

which contradicts the fact that $(y_n)_{n \in D(S)}$ will be eventually constant since X is T_1.

(3) $c(ij) = c(kl)$ iff $\max\{i, j\} = \max\{k, l\}$. In particular, there are three numbers in T, $j < k < l$ with

$$c(jk) < c(jl) = c(kl)$$

i.e.,

$$i(s_k - s_j) < i(s_l - s_j) = i(s_l - s_j)$$

which contradicts the following

$$i(s_l - s_j) = i(s_l - s_k + s_k - s_j) = i(s_k - s_j).$$

(4) c is constant. In particular, there are in T, three numbers $j < k < l$ with

$$c(jk) = c(jl) = c(kl)$$

i.e.,

$$i(s_k - s_j) = i(s_l - s_j) = i(s_l - s_k)$$

which contradicts the following

$$i(s_l - s_j) = i(s_l - s_k + s_k - s_j) > i(s_k - s_j).$$

□

In view of the equivalence of (1) and (2) in Theorem 3.2.1, Kojman [45] gave the following definition of a Hindman space by using the notion of IP-*convergence*, introduced by Fürstenberg and Weiss [27].

Definition 4.2.3 *Suppose that $D \subset \mathbb{N}$ is infinite. A sequence $(x_n)_{n \in FS(D)}$ in a topological space X* IP-converges *to a point $x \in X$ if for every neighborhood V of x there exists $m \in \mathbb{N}$ so that $\{x_n | n \in FS(D - [m])\} \subset V$.*

A topological space X is Hindman *if for every sequence (x_n) in X there exists an infinite set $D \subset \mathbb{N}$ so that $(x_n)_{n \in FS(D)}$ IP-converges to some $x \in X$.*

Similarly in the following, we define differential compactness by using the notion of differential convergence dual to Δ^*-*convergence*, also introduced by Fürstenberg et al [27].

Definition 4.2.4 *Suppose that $S \subset \mathbb{N}$ is infinite. A sequence $(x_n)_{n\in D(S)}$ in a topological space X* differentially converges *to a point $x \in X$ if for every neighborhood V of x there exists $m \in \mathbb{N}$ so that $\{x_n | n \in D(S - [m])\} \subset V$.*

A topological space X is differentially compact *if for every sequence (x_n) in X there exists an infinite set $S \subset \mathbb{N}$ so that $(x_n)_{n\in D(S)}$ is differentially convergent in X.*

By Proposition 4.2.1, it is clear that Hindman spaces are differentially compact and differentially compact spaces are sequentially compact by definition. But sequential compactness need not imply differential compactness (see Theorem 4.2.2). As mentioned in the introduction, Kojman [45] proved that if the closure of every countable set of a space X is compact and first countable then X is Hindman. Hence such space X is also differentially compact.

Recently, Kojman and Shelah [47] showed that van der Waerden spaces and Hindman spaces are not the same by constructing a compact, separable van der Waerden space which is not Hindman under the assumption of continuum hypothesis. Using the similar technique, we strengthen their result by constructing a van der Waerden space, which is not differentially compact. More precisely, we show the following.

Theorem 4.2.2 *Suppose the continuum hypothesis holds. Then there exists a compact Hausdorff, separable, van der Waerden space which is first countable at all points but one, and not differentially compact.*

Before proving this theorem, we need some notations and lemmas. Let $\mathcal{I}_{AP}$ denote the collection of all subsets of natural numbers which are not AP sets. $\mathcal{I}_{AP}$ is a proper ideal over ω and a set $A \subset \omega$ is an AP set if and only if $A \notin \mathcal{I}_{AP}$. Similarly, let $\mathcal{I}_{DP}$ denote the collection of all subsets of natural numbers which are not DP sets. $\mathcal{I}_{DP}$ is a proper ideal over ω and a set $A \subset \omega$ is a DP set if and only if $A \notin \mathcal{I}_{DP}$.

The following lemma relates $\mathcal{I}_{AP}$ to $\mathcal{I}_{DP}$.

Lemma 4.2.1 *Let A be an AP set and let $f : \omega \to \omega$. There exists an AP set $C \subset A$ such that either*

(1) $f(C)$ is constant or

(2) f is finite-to-one on C and if (x_n) enumerates $f(C)$ in ascending order, then

$$\lim_{n\to\infty} (x_{n+1} - x_n) = \infty.$$

In particular, $f(C) \in \mathcal{I}_{DP}$.

Proof. Suppose that for every AP set $C \subset A$, $f(C)$ is not constant. We construct an AP set $C \subset A$ for which the conclusion (2) holds.

For each $n \in \omega$, $A \cap f^{-1}([n])$ is not an AP set since it is the finite union of sets on which f is constant, and thus $A - f^{-1}([n])$ is an AP set by van der Waerden's theorem.

We construct inductively sets C_n for each $n \in \mathbb{N}$ such that

(a) for each $n \in \mathbb{N}$, C_n is an arithmetic progression of length n and

(b) for all m, $n \in \mathbb{N}$, all $x \in C_m$, and all $y \in C_n$, if $m < n$, then $f(y) \geq f(x) + n$ and if $m = n$, then either $f(x) = f(y)$ or $|f(x) - f(y)| \geq n$. Let C_1 be any singleton subset of A. Let $n \in \mathbb{N}$ and assume that we have chosen $C_1, C_2, ..., C_n$. Let $k = \max \cup_{i=1}^{n} f(C_i)$ and choose $i \in [n]$ such that $[A - f^{-1}([k+n])] \cap f^{-1}((n+1)\omega + i)$ is an AP set. Let C_{n+1} be an arithmetic progression of length $n+1$ contained in $[A - f^{-1}([k+n])] \cap f^{-1}((n+1)\omega + i)$. Given $m \leq n+1$, $x \in C_m$, and $y \in C_{n+1}$, if $m \leq n$, then $f(x) \leq k$ and $f(y) > k + n$, while if $m = n+1$, then either $f(x) = f(y)$ or $|f(x) - f(y)| \geq n + 1$.

Now let $C = \cup_{n \in \mathbb{N}} C_n$ as required. $\square$

Lemma 4.2.2 *Assume the continuum hypothesis holds. Then there exists a maximal almost disjoint family* $\mathcal{A} \subset \mathcal{I}_{DP}$ *so that for every AP set* $B \subset \omega$ *and every finite-to-one map* $f : B \to \omega$ *there exists an AP set* $C \subset B$ *and* $A \in \mathcal{A}$ *so that* $f(C) \subset A$.

Proof. We construct from continuum hypothesis, an almost disjoint family

$$\mathcal{A} = \{A_a | a < \omega_1\} \subset \mathcal{I}_{DP}$$

by induction on a. The enumeration $\{A_a | a < \omega_1\}$ may contain repetitions. Let $\{A_n | n < \omega\} \subset \mathcal{I}_{DP}$ be a collection of infinite and pairwise disjoint sets.

Fix a list $(f_a, B_a) : \omega \leq a < \omega_1$ of all pairs (f, B) in which $B \subset \omega$ is an AP set and $f : B \to \omega$ is a finite-to-one map.

Suppose $\omega \leq a < \omega_1$ and that A_b has been chosen for all $b < a$. Consider the pair (f_a, B_a). If there exists a finite set $\{b_0, b_1, ..., b_l\} \subset a$ so that $B_a \cap f^{-1}(\cup_{i \leq l} A_{b_i})$ is an AP set, then there exists $j \leq l$ so that $B_a \cap f^{-1}(A_{b_j})$ is an AP set by van der Waerden's theorem and let $A_a = A_{b_j}$.

Otherwise, order a as $b_i : i < \omega$ and now for all $n < \omega$ the set $B_a \cap f^{-1}(\cup_{i<n} A_{b_i})$ is not an AP set, hence $B_a - f^{-1}(\cup_{i<n} A_{b_i})$ is an AP set. Let an arithmetic progression $D_n \subset B_a - f^{-1}(\cup_{i<n} A_{b_i})$ of length n be chosen for all n. Then $D := \cup_{n \in \omega} D_n$ is an AP subset of B_a, $f_a(D)$ is infinite (since f_a is finite-to-one) and $|f_a(D) \cap A_b| < \infty$ for all $b < a$. Apply Lemma 3.2.1 to find an AP set $C \subset D$, so that $f_a(C) \in \mathcal{I}_{DP}$, and let $A_a = f_a(C)$.

The family $\mathcal{A} = \{A_a | a < \omega_1\}$ is surely an almost disjoint family of (infinite) sets, and $\mathcal{A} \subset \mathcal{I}_{DP}$.

Suppose now that $B \subset \omega$ is an AP set and $f : B \to \omega$ is finite-to-one. There is an index $\omega \leq a\omega_1$ for which $(f, B) = (f_a, B_a)$. At stage a of the construction of $\mathcal{A}$, either $B_a \cap f^{-1}(\cup_{i \leq l} A_{b_i})$ was an AP set for some finite set $\{b_0, b_1, ..., b_l\} \subset a$, hence $B_a \cap f^{-1}(A_b)$ was an AP set for some single $b < a$, or else $f^{-1}(A_a)$ was an AP set. In either case, there is an AP set $C \subset B$ and $A \in \mathcal{A}$ so that $f(C) \subset A$.

Finally, to see that $\mathcal{A}$ is maximal let an infinite set $D \subset \omega$ be given and let $f:\ \omega \to D$ be the ascending order of D. Since there is an AP set $C \subset \omega$ and $A \in \mathcal{A}$ so that $f(C) \subset A$ it is clear that $D \cap A = D$ is infinite. $\square$

Now we are ready to prove Theorem 4.2.1.

Proof of Theorem 4.2.1. Let $\mathcal{A}$ be as stated in Lemma 4.2.2. For each $A \in \mathcal{A}$ let $p_A \notin \omega$ be a distinct point. Define a topology $\mathcal{T}$ on $Y = \omega \cup \{p_A | A \in \mathcal{A}\}$ by requiring that $V \in \mathcal{T}$ if and only if for all $p_A \in V$ the set $A - V$ is finite. Then for each $A \in \mathcal{A}$, $A \cup \{p_A\}$ is a compact neighborhood of p_A, so $\mathcal{T}$ is a locally compact Hausdorff topology in which ω is a dense and discrete subspace.

Let $X = Y \cup \{p\}$ be the one-point compactification of $\mathcal{T}(T)$. Let us check that X is van der Waerden. Suppose $f : \omega \to X$ is given. Let $g : f(\omega) \to \omega$ be bijective. By Lemma 4.2.2 we can find an AP set $B \subset \omega$ so that gf is constant or finite-to-one on B, and hence f is constant or finite-to-one on B. In the former case, the sequence $(f(n))_{n \in B}$ is constant, and therefore converges. So assume that f is finite-to-one on B. Since either $B \cap f^{-1}(\omega)$ or $B - f^{-1}(\omega)$ is an AP set, we may assume, by shrinking B to some AP subset, that either $f(B) \subset \omega$ or $f(B) \subset X - \omega - \{p\}$.

In the former case, there is some $A \in \mathcal{A}$ and AP set $C \subset B$ so that $f(C) \subset A$. Since f is finite-to-one on B, $(f(n))_{n \in C}$ converges to p_A. In the latter case, we claim that the sequence $(f(n))_{n \in B}$ converges to p. To see this, let K be a compact subset of Y, so that $X - K$ is a basic neighborhood of p. Then $K - \omega$ is finite so, since f is finite-to-one on B, $(f(n))_{n \in B}$ is eventually in $X - K$.

To see that X is not differentially compact, let $(x_n) = n$ for each $n \in \omega$ and suppose we have some infinite $B = \{n_1 < n_2 < ...\} \subset \omega$ such that $(x_n)_{n \in D(B)}$ differentially converges to $q \in X$. Then $q \notin \omega$. If $q = p_A$ for some $A \in \mathcal{A}$, then A is a DP set. So $q = p$. Now construct an infinite subset $S = \{s_1 < s_2 < ...\}$ of $D(B)$ by taking $s_i \in D(B - [n_i])$. Then by the maximality of $\mathcal{A}$, pick $A \in \mathcal{A}$ such that $A \cap S$ is infinite. But then $X - A - \{p_A\}$ is a neighborhood of p and for no $n \in \mathbb{N}$ does one have $D(B - [n]) \subset X - A - \{p_A\}$. $\square$

We now turn to give the short proof of the closure of Hindman spaces under products in the framework of set theory, due to Fürstenberg [26]. We start with the introduction of the notion of $\mathcal{F}$-compactness and show its equavalence to Hindman spaces.

Let $\mathcal{F}$ denote the set of all finite nonempty subsets of the natural numbers. The family $\mathcal{F}$ is closed under unions and nonempty intersections, and we consider lattice isomorphisms from $\mathcal{F}$ to $\mathcal{F}$, which are called homomorphisms. Explicitly, a *homomorphism* $\phi : \mathcal{F} \to \mathcal{F}$ is a map such that for any A and B of $\mathcal{F}$, $A \cap B = \emptyset \Rightarrow \phi(A) \cap \phi(B) = \emptyset$ and $\phi(A \cup B) = \phi(A) \cup \phi(B)$. An $\mathcal{F}$-*sequence* of elements in an arbitrary space X is a sequence (x_A) indexed by elements $A \in \mathcal{F}$. An $\mathcal{F}$-*subsequence* is defined by a homomorphism $\phi : \mathcal{F} \to \mathcal{F}$ and forming $\{x_{\phi(A)}\} \subset \{x_A\}$. Let (x_A) be an $\mathcal{F}$-sequence in a space X and $x \in X$. We say that $x_A \to x$ as an $\mathcal{F}$-sequence, if for any neighborhood V of x there is some $A_V \in \mathcal{F}$ such that $x_A \in V$ for all A with $\min A > \max A_V$. A space X is $\mathcal{F}$-*compact* if for any $\mathcal{F}$-sequence in X there exists an $\mathcal{F}$-subsequence converging as an $\mathcal{F}$-sequence.

Theorem 4.2.3 *A space is Hindman if and only if it is $\mathcal{F}$-compact.*

Proof. We first show that every Hindman space X is $\mathcal{F}$-compact. Suppose that an $\mathcal{F}$-sequence (x_A) is given in X. Map $\mathcal{F} \to \mathbb{N}$ by $A \to \sum_{n \in A} 2^n$. This is a bijective map and the $\mathcal{F}$-sequence (x_A) induces a sequence (x_n) in X. Since X is Hindman, there exists an infinite set $D \subset \mathbb{N}$ so that $(x_n)_{n \in FS(D)}$ IP-converges to some $x \in X$. Write this IP set $FS(D) = \{p_A | A \in \mathcal{F}\}$ where $p_A = \sum_{n \in A} p_n$. To each $A \in \mathcal{F}$ attach the set of exponents in the binary expansion of p_A:

$$p_A = \sum_{n \in \phi(A)} 2^n.$$

We then find that if $A \cap B = \emptyset$ would imply $\phi(A) \cap \phi(B) = \emptyset$, then $A \to \phi(A)$ would define a homomorphism. This is not necessarily the case, but we will find a homomorphism $\varphi : \mathcal{F} \to \mathcal{F}$ such that $A \cap B = \emptyset$ implies $\phi\varphi(A) \cap \phi\varphi(B) = \emptyset$. Then $\phi\varphi$ is the desired homomorphism and moreover we construst φ carefully so that the $\mathcal{F}$-subsequence $(x_A)_{A \in \phi\varphi(\mathcal{F})}$ converges to x as an $\mathcal{F}$-sequence.

We will define $\varphi(\{1\}) = A_1, \varphi(\{2\}) = A_2, ...$ inductively and extend φ to all of $\mathcal{F}$. We use the following property of IP sets in $\mathbb{N}$. If S is an IP set in $\mathbb{N}$ and n is an arbitrary positive integer, then there exists $s \in S$ with s divisible by 2^n (or any other number, for that matter). This amounts to showing that in any finite group an IP set must contain the identity. This is easily seen, and we omit the detail. Now let A_1 be arbitrary in $\mathcal{F}$, and let $2^{n_1} > p_{A_1}$. By the above remark there exists A_2 with $\max A_1 < \min A_2$ and p_{A_2} divisible by 2^{n_1}. Then $\varphi(A_1) \cap \varphi(A_2) = \emptyset$. Let $2^{n_2} > p_{A_1} + p_{A_2}$ and choose A_3 with $\max A_2 < \min A_3$ and p_{A_3} divisible by 2^{n_2}. Continuing in this manner, we obtain a sequence of monotone A_i with $\varphi(A_i)$ disjoint. Let

$$\varphi(A) = \cup_{n \in A} \varphi(\{n\}) = \cup_{n \in A} A_n,$$

then φ has the desired property:

$$A \cap B = \emptyset \Rightarrow \phi\varphi(A) \cap \phi\varphi(B) = \emptyset.$$

Now we check that the $\mathcal{F}$-subsequence $(x_A)_{A \in \phi\varphi(\mathcal{F})}$ converges to x as an $\mathcal{F}$-sequence. Since $(x_n)_{n \in FS(D)}$ IP-converges to x, for any neighborhood V of x there exists $m \in \mathbb{N}$ so that $\{x_n | n \in FS(D - [m])\} \subset V$. By the definition of φ, there exists k with $\min \varphi(\{k\}) = \min A_k > m$. Then for any set $B \in \mathcal{F}$ with $\min B > k$, we have

$$\min \varphi(B) = \min \varphi(\min B) = \min A_{\min B} > \max A_k \geq \min A_k > m$$

and hence

$$x_{\phi\varphi(B)} = x_{p_{\varphi(B)}} \in V.$$

Next we show that every $\mathcal{F}$-compact space X is Hindman. Suppose that (x_n) is a sequence in X. Using the bijective map $f : \mathcal{F} \to \mathbb{N}$ by $A \to \sum_{n \in A} 2^n$, we obtain an $\mathcal{F}$-sequence $(x_A)_{A \in \mathcal{F}}$ in X. Since X is $\mathcal{F}$-compact, there exists an $\mathcal{F}$-subsequence $(x_A)_{\phi(\mathcal{F})}$ converges to some $x \in X$ as an $\mathcal{F}$-sequence. Now let $D = \{f\phi(\{n\}) | n \in \mathbb{N}\}$, then $(x_n)_{n \in FS(D)}$ IP-converges to x. In fact, for any neighborhood V of x there exists a set $A_V \in \mathcal{F}$ so that $x_A \in V$ with $\min A > \max A_V$. Let $m = \max\{f\phi(\{n\}) | n \leq \max A_V\}$. Then it is easy to see that $(x_n)_{n \in FS(D-[m])} \subset V$. □

We will need the following lemma to prove the closure of Hindman spaces under products.

Lemma 4.2.3 *For any homomorphism $\phi : \mathcal{F} \to \mathcal{F}$ there exists a monotone homomorphism $\varphi : \mathcal{F} \to \phi(\mathcal{F})$ so that $\max \varphi(A) < \min \varphi(B)$ if and only if $\max A < \min B$.*

Proof. Since ϕ is a homomorphism, we can choose an infinite sequence $n_1 < n_2 < \ldots$ so that $\max \phi(n_i) < \min \phi(n_{i+1})$, $i \in \mathbb{N}$. Define $\varphi(A) = \cup_{i \in A} \phi(n_i)$. Then it is clear that $\varphi : \mathcal{F} \to \phi(\mathcal{F})$ is a homomorphism and $\max \varphi(A) < \min \varphi(B)$ if and only if $\max A < \min B$. □

Theorem 4.2.4 *The product of two Hindman spaces is Hindman.*

Proof. Let $Z = X \times Y$ where X and Y are Hindman and an $\mathcal{F}$-sequence (x_A, y_A) be given in Z. Since X is Hindman and hence $\mathcal{F}$-compact, there is an $\mathcal{F}$-subsequence $x_{\phi(A)} \to x$ as an $\mathcal{F}$-sequence. Since Y is also $\mathcal{F}$-compact, there is an $\mathcal{F}$-subsequence $y_{\phi\varphi(A)} \to y$ as an $\mathcal{F}$-sequence. Since φ is a homomorphism, there is by Lemma 4.2.3, a monotone homomorphism $\psi : \mathcal{F} \to \varphi(\mathcal{F})$ so that $\max \psi(A) < \min \psi(B)$ if and only if $\max A < \min B$. Then $(x_{\phi\psi(A)}, y_{\phi\psi(A)}) \to (x, y)$ as an $\mathcal{F}$-sequence. To see that, let a neighborhood $U \times V$ of (x, y) be given, there is a set A so that $x_{\phi(E)} \in U$ for all E with $\min E > \max A$ and a set B so that $y_{\phi\varphi(E)} \in V$ for all E with $\min E > \max B$. Choose a set C so that $\max \psi(C) > \max A$ and let $D = B \cup C$. Take a set E with $\min E > \max D$. Then $\min E > \max C,\ \min \psi(E) > \max \psi(C) > \max A$ and so $x_{\phi\psi(E)} \in U$. Note that $\min E > \max B,\ \psi(E) = \cup_{i \in E} \varphi(n_i)$ and $n_i \geq i,\ i \in \mathbb{N}$. We have $y_{\phi\psi(E)} \in V$ as desired. □

The following questions arise naturally.

Question 4.2.1 *Are Hindman spaces van der Waerden?*

Question 4.2.2 *Are differentially compact spaces Hindman or van der Waerden?*

Question 4.2.3 *Are differentially compact spaces closed under products?*

We suspect that the answers to the first two questions are negative, like Proposions 4.2.1 and 4.2.2, simply because Hindman's theorem, Ramsey's theorem and van der Waerden's theorem do not imply one another and the answer to the last one might be positive as usual in Topology.

4.3 Piecewise syndetic sets

In this section, we extend the results of Brown and Hindman et al on piecewise syndetic sets in Theorem 4.3.1. We start with a brief introduction of topological dynamics.

4.3.1 A dynamical prelude

We only sketch the basic concepts and facts in Topological Dynamics that we need and refer the readers to the book [26] for details.

Throughout this section, a *dynamical system* (X, G) will consist of a compact space X together with a (semi-)group G acting on X by continuous transformations. Let G be a (semi-)group and F a compact metric space.

Form $\Omega = F^G$, the compact metrizable space of all maps from G to F. An action of G on Ω (the *regular* action) is defined by letting

$$g_1\omega(g_2) = \omega(g_2 g_1),\ \omega \in \Omega,\ g_1, g_2 \in G.$$

This is indeed an action as $g_1(g_2\omega) = (g_1 g_2)\omega$. So (Ω, G) forms a dynamical system.

Definition 4.3.1 *A* Bebutov system *is a subsystem of* (Ω, G)*, i.e., it is a system* (X, G) *where* $X \subset \Omega$ *is a closed subset invariant under the regular action of* G.

If $\omega \in \Omega$, then the smallest G-invariant closed subset containing ω is the orbit closure of ω in Ω. This subsystem is denoted by $\mathrm{cl}(G\omega)$ and is refered to as the *Bebutov system generated by* ω.

Simple recurrence possessed by a point in a dynamical system means that the set of group elements that applied to the given point bring it close to itself (the "return times") does not merely reduce to the identity. Uniform recurrence means that the set is large in the sense of the following definition.

Definition 4.3.2 *A subset* S *of a topological semigroup* G *is* syndetic *if there exists a compact set* $K \subset G$ *so that for any* $g \in G$*, there exists* $k \in K$ *with* $gk \in S$.

If G is discrete, then K is finite, and so for a discrete group, a set S is syndetic if finitely many translates of it fill G. A subset of $\mathbb{R}^n$ is syndetic if there exists a positive number r such that all ball of radius r meet the set. A subset of natural numbers (or integers) is syndetic if it can be arranged as an ascending sequence $n_1 < n_2 < ...$ with bounded gaps $n_{i+1} - n_i$. Such sets have sometimes been called *relatively dense* sets. Syndetic sets play a fundamental role in dynamical systems.

Definition 4.3.3 *Let* (X, G) *be a dynamical system. A point* $x \in X$ *is* uniformly recurrent *for* (X, G) *if for any neighborhood* $V \ni x$*, the set* $\{g \in G | gx \in V\}$ *is syndetic.*

The following famous Birkhoff's recurrence theorem [7] will be a main ingredient in the proof of Theorem 4.3.1.

Birkhoff's Theorem. *For any dynamical system* (X, G) *with compact* X*, the set of uniformly recurrent points is nonempty.*

The following definition extends that of thickness for subsets of the natural numbers or integers to a general semigroup.

Definition 4.3.4 *A subset* T *of a topological semigroup* G *is* thick *if for any compact set* $K \subset G$ *there exists* $g \in G$ *with* $gK \subset T$.

So a subset T of the natural numbers or integers is thick if it contains arbitrarily long intervals, which is called *replete* in Gottschalk and Hedlund [30]. It is clear that a syndetic set nontrivially intersects each thick set and vice versa. This leads us to the following definition.

Definition 4.3.5 *A subset of a topological semigroup is* piecewise syndetic *if it is the intersection of a syndetic set and a thick set.*

4.3.2 The extension

As mentioned in the introduction, Brown [9, 10] in 1968 proved that any map from the natural numbers to a finite set is constant on a piecewise syndetic set. Straubing [71] used this result to give new and almost entirely combinatorial proofs of all of the key theorems dealing with the local finiteness of semigroups of matrices over an arbitrary field, and with the local finiteness of subsemigroup of rings satisfying a polynomial identity. Recently, Hindman et al [40] extended this result from natural numbers to discrete semigroups via algebra. Brown's proof is purely combinatorial. While Hindman et al observed that one can extend the operator of a discrete semigroup to its Stone-Čech compactification and a set is piecewise syndetic if and only if its closure intersects the smallest ideal of the Stone-Čech compactification (see Theorem 4.40 in [40]). So their extension follows from the fact that the smallest ideal is never empty. We now extend this further to the following via topology.

Theorem 4.3.1 *Any continuous map from a locally connected abelian semigroup to a discrete finite space is constant on a piecewise syndetic set.*

Proof. Let F be a discrete finite space and an abelian semigroup G locally connected. Let $f : G \to F$ be any continuous map. Let the dynamical system (F^G, G) equiped with compact-open topology on F^G. Consider the Bebutov system $(\mathrm{cl}(Gf), G)$ generated by f. For each $g_0 \in G$, take U to be the connected component containing g_0. Since G is locally connected, U is both open and closed by Corollary 1.2.1. So $gf(U)$ is constant for all $g \in G$. Then Gf is equicontinuous. Since the finite space F is surely compact, the subset $\mathcal{F}_x = \{gf(x)|g \in G\}$ of F has compact closure for each $x \in G$. Then by Ascoli's Theorem, $\mathrm{cl}(Gf)$ is compact in the compact-open topology of $\mathcal{C}(G, F)$ and so contains a uniformly recurrent point, say h, by Birkhoff's Theorem. Suppose that c is a value taken on by some $h(x)$. The value c occurs syndetically in h. Since $h \in \mathrm{cl}(Gf)$, there are translates gf arbitrarily close to h. This means that for each compact subset K of G there exists $g \in G$ so that $gf = h$ on K. So $f^{-1}(c)$ is piecewise syndetic. □

It seems reasonable to require the continuity of the map in Theorem 4.3.1 as usual in topological Ramsey theory though it might somehow be weakened. But we feel that the local connectivity and commutativity of the semigroup are really strong to force the ‘monochromatic’ piecewise syndetic set.

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Selbständigkeitserklärung

Hiermit erkläre ich, die vorliegende Arbeit selbständig ohne fremde Hilfe verfaßt und nur die angegebene Literatur und Hilfsmittel verwendet zu haben.

Lingsheng Shi
12. Mai 2003