American Options in incomplete Markets: Upper and lower Snell Envelopes and robust partial Hedging

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Abstract

This thesis studies American options in an incomplete financial market and in continuous time. It is composed of two parts.

In the first part we study a stochastic optimization problem in which a robust convex loss functional is minimized in a space of stochastic integrals. This problem arises when the seller of an American option aims to control the shortfall risk by using a partial hedge. We quantify the shortfall risk through a robust loss functional motivated by an extension of classical expected utility theory due to Gilboa and Schmeidler. In a general semimartingale model we prove the existence of an optimal strategy. Under additional compactness assumptions we show how the robust problem can be reduced to a non-robust optimization problem with respect to a worst-case probability measure.

In the second part, we study the notions of the upper and the lower Snell envelope associated to an American option. We construct the envelopes for stable families of equivalent probability measures, the family of local martingale measures being an important special case. We then formulate two robust optimal stopping problems. The stopping problem related to the upper Snell envelope is motivated by the problem of monitoring the risk associated to the buyer's choice of an exercise time, where the risk is specified by a coherent risk measure. The stopping problem related to the lower Snell envelope is motivated by a robust extension of classical expected utility theory due to Gilboa and Schmeidler. Using martingale methods we show how to construct optimal solutions in continuous time and for a finite horizon.

Keywords:
American options, Optimal exercise, Robust optimization, Shortfall risk

Zusammenfassung

In dieser Dissertation werden Amerikanischen Optionen in einem unvollständigen Markt und in stetiger Zeit untersucht. Die Dissertation besteht aus zwei Teilen.

Im ersten Teil untersuchen wir ein stochastisches Optimierungsproblem, in dem ein konvexes robustes Verlustfunktional über einer Menge von stochastichen Integralen minimiert wird. Dies Problem tritt auf, wenn der Verkäufer einer Amerikanischen Option sein Ausfallsrisiko kontrollieren will, indem er eine Strategie der partiellen Absicherung benutzt. Hier quantifizieren wir das Ausfallsrisiko durch ein robustes Verlustfunktional, welches durch die Erweiterung der klassischen Theorie des erwarteten Nutzens durch Gilboa und Schmeidler motiviert ist. In einem allgemeinen Semimartingal-Modell beweisen wir die Existenz einer optimalen Strategie. Unter zusätzlichen Kompaktheitsannahmen zeigen wir, wie das robuste Problem auf ein nicht-robustes Optimierungsproblem bezüglich einer ungünstigsten Wahrscheinlichkeitsverteilung reduziert werden kann.

Im zweiten Teil untersuchen wir die obere und die untere Snellsche Einhüllende zu einer Amerikanischen Option. Wir konstruieren diese Einhüllenden für eine stabile Familie von äquivalenten Wahrscheinlichkeitsmassen; die Familie der äquivalenten Martingalmassen ist dabei der zentrale Spezialfall. Wir formulieren dann zwei Probleme des robusten optimalen Stoppens. Das Stopp-Problem für die obere Snellsche Einhüllende ist durch die Kontrolle des Risikos motiviert, welches sich aus der Wahl einer Ausübungszeit durch den Käufer bezieht, wobei das Risiko durch ein kohärentes Risikomass bemessen wird. Das Stopp-Problem für die untere Snellsche Einhüllende wird durch eine auf Gilboa und Schmeidler zurückgehende robuste Erweiterung der klassischen Nutzentheorie motiviert. Mithilfe von Martingalmethoden zeigen wir, wie sich optimale Lösungen in stetiger Zeit und für einen endlichen Horizont konstruieren lassen.

Schlagwörter:
Amerikanische Optionen, optimale Ausübung, robuste Optimierung, Ausfallsrisiko

Contents

Chapter 0

Introduction

The dynamic analysis of financial contracts is an important topic in the modern theory of finance. Derivative contracts such as call options have been playing a significant role both in the theory and in real financial markets. A call option is the right but not the obligation to buy a certain asset at a specified price until or at a predetermined maturity date. If the option specifies that the option holder may exercise the option only at the maturity date, the contract is termed European. If the option can be exercised at any time prior to the given expiration date, then the option is called American. Early exercise makes American options more interesting and more complex to analyze.
In a complete financial market the arbitrage free price of the American call option with strike price K coincides with the value function of an optimal stopping problem with payoff function $(x-K)^+$ which is formulated in terms of the unique equivalent martingale measure. This allows one to solve both the problem of optimal exercise for the buyer and the problem of hedging for the seller. In the more realistic case of an incomplete market, valuation, exercise and hedging of an American option become more involved. In this case, the no arbitrage principle admits a whole set of prices, and additional criteria are needed in order to specify a price.

From the point of view of the seller, who wants to protect himself against his contractual obligation, a possible approach consists in *superhedging* the American option by using a strategy which generates enough capital to cover the payoff at any stopping time used by the buyer to exercise the option.
This superhedging cost is finite under mild conditions and the existence of a superhedging strategy is a consequence of the optional decomposition theorem. A first version was proved by El Karoui and Quenez(1995) for a diffusion model. Generalizations of the optional decomposition theorem for a gen-

eral semimartingale model were obtained by Kramkov(1996), Föllmer and Kabanov(1998) and Föllmer and Kramkov(1997), first for locally bounded processes and then for the general unbounded case. From a practical point of view, however, the cost of superhedging is usually too high. This suggests to use strategies of partial hedging which are in some sense optimal under a given capital constraint. The problem of partial hedging has been investigated primarily in the case of European options, where criteria such as Efficient hedging, or Mean variance hedging have been proposed and are by now well understood. For American options, however, the problem of partial hedging is more complex.

In this thesis we are interested in the problems of partial hedging and of optimal exercise of an American option in an incomplete market in continuous time. These problems are studied independently of the problem of valuation.

In the first part of the thesis we adopt the perspective of the *seller* of an American option whose initial wealth is less than the cost of superhedging. Clearly, the value process of any self-financed strategy constructed with this initial capital produces a nontrivial shortfall, and so the seller will try to control the shortfall risk. With this motivation we propose an optimization problem which involves minimization over a family of stochastic integrals and maximization over the family of stopping times. In our formulation two streams of ideas are involved. In the first, decision criteria are based on subjective preferences which take model uncertainty into account. The numerical representation of such robust preferences leads to robust utility functionals defined on a set of random variables. In selecting a partial hedge for an American option the seller faces the uncertainty of which stopping time will be used by the buyer. Here we adopt a worst-case approach which reflects a pessimistic attitude against this source of uncertainty. In this way we obtain a utility-based criterion which incorporates both the exercise and the model uncertainty.
In the first part we construct hedging strategies which are optimal for such criteria. We also discuss the existence of worst-case measures which allow us to reduce the robust partial hedging problem to the partial hedging problem for a single probability measure.

In the second part of the thesis we study two robust stopping problems. In the first, we adopt the point of view of the *seller* of an American option in an incomplete market and in continuous time. The seller is uncertain about the stopping time used by the buyer to exercise the option. In addi-

tion to partially hedging the American option, the seller may try to identify stopping times with maximal risk quantified by some coherent risk measure which is represented by a stable family $\mathcal{Q}$. This monitoring problem leads to a robust optimal stopping problem. Our analysis will cover the special case of coherent risk measures which are time-consistent. We will show that stopping times with maximal risk exist if the American option satisfies a mild continuity property with respect to the risk measure. The construction of such stopping times will involve an upper Snell envelope associated to the American option with respect to the time-consistent coherent risk measure and to the family $\mathcal{Q}$.
In the second problem, we adopt the point of view of the *buyer* of an American option in an incomplete market and in continuous time. In a complete market the problem of optimal exercise of the American option is solved by an optimal stopping time with respect to the unique martingale measure of the market. In an incomplete market, one possible recipe for exercising the option consists in specifying a particular martingale measure and in solving the corresponding stopping problem. Here we follow a different approach. Instead of specifying a particular martingale measure, we assume that the buyer uses a robust functional to quantify the utility of the American option if exercised at some stopping time. This leads to a robust stopping problem whose solution we may interpret as a stopping time of maximal utility. Our analysis will consider the case where this robust utility functional is represented by a stable family of equivalent probability probability measures, a property related to time consistency of the underlying preferences as explained by Epstein and Schneider. We show the existence of such stopping times of maximal utility for American options which are sufficiently regular and integrable.

Summary of results

Part one. In chapter 2 we consider the problem of selecting a partial hedge. Our criterion asks for a strategy $\xi^* \in Ad_c$ attaining the value of the "robust partial hedging problem"

$$PH(c) := \inf_{\xi \in Ad_c} \sup_{\theta \in \mathcal{T}} \sup_{Q \in \mathcal{Q}} E_Q[f(H_\theta, V_\theta^{c,\xi})]$$

where Ad_c is a space of admissible strategies satisfying a budget constraint given by the initial capital c, $\mathcal{T}$ is the family of stopping times with values in the finite time interval $[0, T]$, $\mathcal{Q}$ is a convex family of absolutely continuous probability measures, and f denotes a generalized loss function. The most interesting cases of this function are of the form $f(h,v) = l((h-v)^+)$ for some convex loss function l, and $f(h,v) = (1-\frac{v}{h})^+$. The first case corresponds to robust efficient hedging for American options. Our analysis thus extends the efficient hedging approach of Föllmer and Leukert[24] from European to American options. In addition we take model uncertainty into account by considering a whole class $\mathcal{Q}$ of possible probabilistic models. The second case of the function f corresponds to robust quantile hedging for American options; here we extend the quantile hedging approach of Föllmer and Leukert[25] from European to American options.

For the robust partial hedging problem $PH(\cdot)$ we obtain two results. First we prove the existence of optimal strategies. Our second result shows that model uncertainty and uncertainty on the stopping times can be reduced into a non-robust problem with respect to a worst-case probability measure $Q^* \in \mathcal{Q}$ and with respect to a quasi-randomized stopping time $\gamma \in \overline{\mathcal{A}}$. This reduction takes the form

$$PH(c, Q^*, \gamma^*) := \inf_{\xi \in Ad_c} E_{Q^*}\left[\int_0^{T-} f(V_s^{c,\xi}, H_s)dA_s + \int_{0+}^{T} f(V_{s-}^{c,\xi}, H_{s-})dB_s\right],$$

where (A, B) is a pair of increasing processes representing the quasi-randomized stopping time $\gamma \in \overline{\mathcal{A}}$. This reduction is sharp in the sense that it does not change the value $PH(c)$. Moreover, solutions to the original robust partial hedging problem are solutions to the reduced partial hedging problem. From this reduction we also conclude that Q^* is a worst-case probability measure for our original problem in the sense that

$$PH(c) = \inf_{\xi \in Ad_c} \sup_{\theta \in \mathcal{T}} E_{Q^*}[f(H_\theta, V_\theta^{c,\xi})].$$

In Chapter 3 we specialize to a non-robust setting for the partial hedging problem of chapter 2, where the family of priors reduces to $\mathcal{Q} = \{R\}$ and the function f has the form $f(v,h) = (1-\frac{v}{h})^+$. This specification corresponds to quantile hedging for American options. We then consider an upper bound for the value of the resulting optimization problem and obtain a dual representation. This can be considered a first step towards a dual convex approach to the problem of quantile hedging for American options.

Part two. Chapters 4 and 5 are devoted to the problem of robust monitoring and robust exercise of American options.

In chapter 4 we consider a coherent risk measure of the form

$$\rho(X) = \sup_{P \in \mathcal{P}} E_P[-X],$$

where $\mathcal{P}$ is a stable convex family of probability measures equivalent to a reference probability measure. The concept of stability is crucial for our analysis. For a given American option $H := \{H_t\}_{0\leq t\leq T}$ we are led to the robust stopping problem

$$\sup_{\theta\in\mathcal{T}} \rho(-H_\theta) = \sup_{\theta\in\mathcal{T}} \sup_{P\in\mathcal{P}} E_P[H_\theta].$$

Under appropriate conditions on H we prove the existence of a stopping time $\tau^* \in \mathcal{T}$ which is optimal in following sense

$$\sup_{P\in\mathcal{P}} E_P[H_{\tau^*}] = \sup_{\theta\in\mathcal{T}} \sup_{P\in\mathcal{P}} E_P[H_\theta].$$

The construction will involve the upper Snell envelope associated to the American option H with respect to the family of probability measures $\mathcal{P}$.

In chapter 5 we study the lower Snell envelope associated to the American option H with respect to a stable family $\mathcal{P}$ of probability measures. This process is defined as

$$U_t^\downarrow = U_t^\downarrow(\mathcal{P}, H) = \operatorname{ess\,inf}_{P\in\mathcal{P}} \operatorname{ess\,sup}_{\theta\in\mathcal{T}[t,T]} E_P[H_\theta \mid \mathcal{F}_t].$$

Our first question is, whether the lower Snell envelope satisfies a minimax identity of the form

$$U_0^\downarrow(\mathcal{P}, H) = \sup_{\theta\in\mathcal{T}} \inf_{P\in\mathcal{P}} E_P[H_\theta].$$

This question is motivated by the analysis of arbitrage free prices of chapter 1. Another motivation is the identification of stopping times with maximal robust utility. To establish the minimax identity, a robust optimal stopping time problem arises asking for a stopping time τ^* with

$$\inf_{P\in\mathcal{P}} E_P[H_{\tau^*}] = \inf_{P\in\mathcal{P}} \sup_{\theta\in\mathcal{T}} E_P[H_\theta].$$

Under suitable regularity conditions on H we prove the existence of such a stopping time τ^*.

Our second question is whether the lower Snell envelope admits a version with good properties. In contrast to the upper Snell envelope, the lower Snell envelope is neither a submartingale nor a supermartingale. This creates a major difficulty since we can no longer apply the standard methods of constructing a nice version. Instead, we will use a result of Dellacherie[12] on the essential infimum of a class of stochastic processes to show that an optional version of the lower Snell envelope exists.

Part I

Robust partial hedging of American options

Chapter 1

Superhedging and arbitrage free prices

In complete financial markets the problems of Exercising and Hedging of American options are well-understood, due to the uniqueness of the equivalent martingale measure and the corresponding representation of martingales as stochastic integrals of the underlying price process; see Bensoussan[3] and Karatzas[34]. In the context of incomplete markets, these problems become more complex and require new techniques. Our goal in this chapter is to review the solution of the hedging problem in a general semimartingale model. Our exposition is based on the discussion of chapter six in Föllmer and Schied[27]. But here we discuss the case of continuous time, and we incorporate the optional decomposition theorem developed in Föllmer and Kramkov[23].

This exposition will be important for the first part of the thesis, because here we introduce all the necessary concepts and results related to the hedging of an American option in an incomplete market. This exposition will also be important for the second part, since here we introduce and motivate the basic objects to be studied, namely the upper and lower Snell envelopes associated to a stochastic process with respect to a stable family of equivalent probability measures.

Our exposition will begin by introducing a standard semimartingale model for a financial market and a class of processes modeling American options. We explain the basic concept of superhedging and summarize the main results:
The cost of superhedging is finite, and a superhedging strategy constructed at this cost does exist. The set of arbitrage free prices is a positive finite

interval, and the cost of superhedging is the supremum of this interval.

We first consider complete markets. In this setting some important concepts appear that will be needed hereafter, in particular, the problem of optimal stopping in continuous time with respect to the unique martingale measure P, whose solution is provided by the Snell envelope associated to the American option.

In incomplete markets, there is a whole class of equivalent martingale measures, and the analysis relies on the properties of two processes associated to an American option: The upper Snell envelope $\{U_t^{\uparrow}\}_{0\leq t\leq T}$, and the lower Snell envelope $\{U_t^{\downarrow}\}_{0\leq t\leq T}$. The fact that the supremum of the set of arbitrage free prices coincides with the superhedging cost, and the existence of a superhedging strategy constructed at minimal cost, depend on the structure of the upper Snell envelope as clarified by the optional decomposition theorem 1.15. The identification of the infimum of the set of arbitrage free prices depends on a minimax identity involving the lower Snell envelope.

The chapter is organized as follows. In section 1.1 we fix some notation and introduce the standard semimartingale model for a financial market. In section 1.2, we define the cost of superhedging of an American option. In section 1.3 we characterize the cost of superhedging in a complete market, and in section 1.4 in the case of an incomplete market. In section 1.5 we introduce the concept of an arbitrage free price and describe in terms of the upper and lower Snell envelopes the set of these prices.

1.1 Notation

We assume given a stochastic base

$$(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}, R),$$

with finite time horizon $T < \infty$. The filtration $\mathbb{F}$ satisfies the usual conditions of right continuity and completeness, and we assume that R is $0-1$ on $\mathcal{F}_0$.

Expectation with respect to R is denoted by $E_R[\cdot]$. Equality of random variables always means R-a.s. equality. The conditional expectation $E_R[\cdot \mid \mathcal{F}_0]$ is a random variable that is constant R-a.s.; we identify this random variable with the corresponding constant and write $E_R[\cdot \mid \mathcal{F}_0] = E_R[\cdot]$.

We next define in which sense a right continuous process $\{Y_t\}_{0\leq t\leq T}$ is said to dominate another right-continuous process $\{Z_t\}_{0\leq t\leq T}$.

Definition 1.1 *For two processes* $\{Y_t\}_{0\leq t\leq T}$ *and* $\{Z_t\}_{0\leq t\leq T}$ *with right continuous paths, we write* $Y \geq Z$ *if*

$$R[\{\omega \in \Omega \mid Y_t(\omega) \geq Z_t(\omega), \text{ for all } t \in [0,T]\}] = 1.$$

Note that due to right continuity, it is enough to have $Y_t \geq Z_t$ *R-a.s. for all t in a dense countable subset of* $[0,T]$.□

We model the discounted price of an asset in a **financial market** by an $\mathbb{F}$-adapted semimartingale $X := \{X_t\}_{0\leq t\leq T}$, defined in the domain $\Omega\times[0,T]$, whose trajectories are right continuous and have finite left limits (càdlàg). We assume the market is free of arbitrage opportunities in the sense that the set of equivalent local martingale measures

$$\mathcal{M} := \{P \sim R \mid X \text{ is a local martingale under } P\} \tag{1.1}$$

is nonempty. For the precise formulation of the relationship between the notion of an arbitrage free market and the family of martingale measures we refer to Delbaen and Schachermayer[9] and references therein. For any martingale measure $P \in \mathcal{M}$, we denote by $E_P[\cdot]$ the corresponding P-expectation.

The family of $\mathbb{F}$-stopping times with values in $[0,T]$ is denoted by $\mathcal{T}$. Recall that $\tau : \Omega \to [0,T]$ is a stopping time if for any $t \in [0,T]$

$$\{\tau \leq t\} \in \mathcal{F}_t.$$

If $\tau \in \mathcal{T}$ is a stopping time we define the class of stopping times

$$\mathcal{T}[\tau,T] := \{\theta \in \mathcal{T} \mid \tau \leq \theta \leq T\}. \tag{1.2}$$

In the next definition we introduce a class of processes which will be used to model American options, respectively a class of random variables modeling European options.

Definition 1.2 *We say that a process* $H := \{H_t\}_{0\leq t\leq T}$ *is an American option if it is a positive* $\mathbb{F}$*-adapted process, if the trajectories are right-continuous and have finite left limits (càdlàg), and if it satisfies the following integrability condition*

$$\sup_{P\in\mathcal{M}} \sup_{\theta\in\mathcal{T}} E_P\left[H_\theta\right] < \infty. \tag{1.3}$$

A European option H_T *is a positive* $\mathcal{F}_T$*-measurable random variable with*

$$\sup_{P\in\mathcal{M}} E_P[H_T] < \infty.\square$$

Throughout this chapter we fix a process H satisfying the definition 1.2 and which will represent an American option.

1.2 The superhedging cost of an American option

The problem we are considering in this chapter is the superhedging of American options. To this end, we introduce admissible strategies and their corresponding value processes.

Definition 1.3 *An admissible strategy is a pair* (c, ξ) *where* $c \in \mathbb{R}_+$ *is a positive constant, and* $\xi := \{\xi_t\}_{0 \le t \le T}$ *is a* $\mathbb{F}$*-predictable process* $\xi : \Omega \times [0, T] \to \mathbb{R}$ *such that the stochastic integral*

$$V_t^\xi := \int_0^t \xi_s dX_s,$$

is well defined for all $t \in [0, T]$ *and the corresponding value process* $V_t^{c,\xi} := c + V_t^\xi$ *is nonnegative.*

In this case we say that ξ *is a c-admissible strategy and the family of c-admissible strategies is denoted by* Ad_c.□

It is natural to ask whether it is possible to hedge the risk in an American option completely. This leads us to the concept of a superhedging strategy.

Definition 1.4 *A superhedging strategy for* H *is a pair* $(c, \xi) \in \mathbb{R}_+ \times Ad_c$ *such that*

$$V^{c,\xi} \ge H.\square$$

In order to formulate a concept of replicability we first introduce a uniform martingale property, resp. supermartingale property, with respect to a whole class of probability measures.

Definition 1.5 *Let* $\mathcal{P}$ *be a family of probability measures equivalent to* R*, and let* $U := \{U_t\}_{0 \le t \le T}$ *be a positive càdlàg* $\mathbb{F}$*-adapted process such that for any* $P \in \mathcal{P}$ *the following integrability condition is satisfied*

$$\sup_{\theta \in \mathcal{T}} E_P[U_\theta] < \infty.$$

Then, we say that U *is a* $\mathcal{P}$*-supermartingale (resp.* $\mathcal{P}$*-submartingale,* $\mathcal{P}$*-martingale) if it is a supermartingale (resp. submartingale, martingale) with respect to any* $P \in \mathcal{P}$.□

Definition 1.6 *The American option H is called replicable if there exists a superhedging strategy (c, ξ) for some $c \in \mathbb{R}_+$, and a stopping time $\tau \in \mathcal{T}$, such that $V^{c,\xi}_{t\wedge\tau}$ is a $\mathcal{M}$-martingale and*

$$V^{c,\xi}_{\tau} = H_\tau.$$

In this case, we say that the strategy (c, ξ) replicates the American option H.□

We will see that superhedging strategies exist, and we are interested in the minimal capital that allows to construct such strategies.

Definition 1.7 *The superhedging cost of the American option H is defined by*

$$\inf\{c \geq 0 \mid \exists \xi \in Ad_c, (c, \xi) \text{ is a superhedging strategy}\}.$$

A superhedging strategy (c_0, ξ) with c_0 being equal to the superhedging cost is called minimal.□

An important result is that the superhedging cost is finite, and that a minimal superhedging strategy exists. To construct such a strategy the **upper Snell envelope** associated to H with respect to $\mathcal{M}$ will be crucial. This is a càdlàg process, denoted $U^\uparrow = U^\uparrow(H, \mathcal{M})$ (we borrow the notation $U^\uparrow$ from [27]), such that the equality

$$U^\uparrow_t = \text{ess sup}_{P\in\mathcal{M}} \text{ess sup}_{\theta\in\mathcal{T}[t,T]} E_P[H_\theta \mid \mathcal{F}_t], \tag{1.4}$$

holds for all $t \in [0, T]$. In terms of the upper Snell envelope, condition (1.3) reads

$$U^\uparrow_0 < \infty. \tag{1.5}$$

In chapter 4 we show how to construct this process.

In the next two sections we show that the superhedging cost is equal to the value of the upper Snell envelope at time $t = 0$.

1.3 Hedging in complete markets

In this section we review the well-known solution of the superhedging problem in a complete market. In this case the price process X admits a unique equivalent martingale measure P so that $\mathcal{M} = \{P\}$. For the American option H, proposition 1.14 shows that the superhedging cost is

$$\sup_{\theta\in\mathcal{T}} E_P[H_\theta].$$

Moreover, under a mild regularity condition H is replicable, so that this cost is actually that of a replicating superstrategy. This important result is well known and was first established by Bensoussan[3] and Karatzas[34].

We will apply a general theorem for the Snell envelope and optimal stopping times. To this end, we need the concepts of *class*(D) and of upper semicontinuity in expectation from the left.

Definition 1.8 *A nonnegative process $\{K_t\}_{0\leq t\leq T}$ is said to be of class(D) with respect to the probability measure P, if the family of random variables*

$$\{K_\theta \mid \theta \in \mathcal{T}\}$$

is uniformly integrable with respect to P, that is

$$\lim_{i\to\infty} \sup_{\theta\in\mathcal{T}} E_P[K_\theta; K_\theta > i] = 0.$$

In particular, this implies that $\sup_{\theta\in\mathcal{T}} E_P[K_\theta] < \infty$.□

The next definition is motivated by definition 2.11 p. 110 and remark 2.42 p. 142 in El Karoui[18].

Definition 1.9 *A process $\{K_t\}_{0\leq t\leq T}$ is said to be upper semicontinuous in expectation from the left with respect to the probability measure P if for any increasing sequence of stopping times $\{\tau_i\}_{i=1}^{\infty}$ converging to τ, we have*

$$\limsup_{i\to\infty} E_P[K_{\tau_i}] \leq E_P[K_\tau].\square \tag{1.6}$$

Theorem 1.10 *Let $K := \{K_t\}_{0\leq t\leq T}$ be a positive càdlàg $\mathbb{F}$-adapted process with*

$$\sup_{\theta\in\mathcal{T}} E_P[K_\theta] < \infty.$$

Then

1. *There exists a càdlàg supermartingale denoted $U^P = U^P(K)$, such that*

$$U^P_\tau = \operatorname{ess\,sup}_{\theta\in\mathcal{T}[\tau,T]} E_P[K_\theta \mid \mathcal{F}_\tau], \quad P-a.s.,$$

   for any stopping time $\tau \in \mathcal{T}$. U^P is the minimal càdlàg supermartingale that dominates K. If K is of class(D), then U^P is also of class(D).

2. *A stopping time* $\tau_t^* \in \mathcal{T}[t,T]$ *is optimal in the sense that*

$$U_t^P = E_P[K_{\tau_t^*} \mid \mathcal{F}_t], \quad P - a.s.,$$

   if and only if

   (a) *The process* $\{U^P_{s\wedge\tau_t^*}\}_{t\le s\le T}$ *is a martingale*

   (b) $K_{\tau^*} = U^P_{\tau^*}, \quad P - a.s.$

3. *If* K *is upper semicontinuous in expectation from the left, then optimal stopping times exist, and the minimal one is given by*

$$\tau_t^* := \inf\{s \ge t \mid K_t \ge U_t^P\}. \tag{1.7}$$

Proof. See theorems 2.28 p. 126, 2.31 p. 129, 2.39 p. 138 and 2.41 p. 140 in El Karoui[18].□

Definition 1.11 *The stochastic process* $U^P = U^P(K)$ *constructed in theorem 1.10 is called the Snell envelope of* K *with respect to* P.□

In the remark below we recall some consequences of uniform integrability, related to the de la Vallée Poussin criterion.

De la Vallée Poussin criterion of uniform integrability 1.12 *Let* $\mathcal{C}$ *be an arbitrary family of random variables. Then* $\mathcal{C}$ *is uniformly integrable with respect to* R *if and only if there exists an increasing convex function* ϕ *with* $\lim_{x\to\infty}\frac{\phi(x)}{x} = \infty$, *and such that*

$$\sup_{X\in\mathcal{C}} E_R[\phi(X)] < \infty.$$

In particular, $\mathcal{C}$ *is uniformly integrable with respect to* R *if for* $p > 1$

$$\sup_{X\in\mathcal{C}} E_R[|X|^p] < \infty.$$

Proof. See for example theorem 11 section I.2 in Protter[48].□

Remark 1.13 *Now we relate the integrability assumptions of the previous theorem with our conditions on the process* H. *The following implications are well known.*

1. *If H is of class(D) it satisfies the apparently stronger condition that the family*
$$\Psi := \{E_P[H_\theta \mid \mathcal{F}_\tau] \mid \theta, \tau \in \mathcal{T}, \theta \geq \tau\}$$
is uniformly integrable.

2. *The weaker assumption (1.11) below implies that for any fixed $\tau \in \mathcal{T}$, the family*
$$\Psi_\tau := \{E_P[H_\theta \mid \mathcal{F}_\tau] \mid \theta \in \mathcal{T}, \theta \geq \tau\} \tag{1.8}$$
is uniformly integrable.

3. *Moreover, any of the hypotheses*
$$E_P\left[\sup_{t\in[0,T]} H_t\right] < \infty \tag{1.9}$$
or
$$\sup_{\theta\in\mathcal{T}} E_P[(H_\theta)^p] < \infty, \text{ with } p > 1, \tag{1.10}$$
imply that Ψ is uniformly integrable and the maximal expected reward is finite
$$\sup_{\theta\in\mathcal{T}} E_P[H_\theta] < \infty. \tag{1.11}$$

4. *A right continuous process of class(D) is upper-semicontinous in expectation for decreasing sequences of stopping times, and in fact, continuous.*

Proof.

1. If H is of *class*(D), according to lemma 1.12 there exists a convex increasing function such that
$$\sup_{\theta\in\mathcal{T}} E_P[f(H_\theta)] < \infty.$$
Jensen's inequality yields
$$E_P[f(E_P[H_\theta \mid \mathcal{F}_\tau])] \leq E_P[f(H_\theta)] \leq \sup_{\theta\in\mathcal{T}} E_P[f(H_\theta)] < \infty,$$
and thus, Ψ is uniformly integrable.

2. To prove that Ψ_τ is uniformly integrable, we observe that Ψ_τ is directed upwards (see lemma 4.15), and deduce the existence of an increasing sequence $\{E_P[H_{\theta_n} \mid \mathcal{F}_\tau]\}_{n=1}^{\infty}$ converging to $\operatorname{ess\,sup} \Psi_\tau$. An application of monotone convergence gives
$$E_P[\operatorname{ess\,sup} \Psi_\tau] = \lim_{n\to\infty} E_P[H_{\theta_n}] \leq \sup_{\theta \in \mathcal{T}[\tau,T]} E_P[H_\theta] < \infty.$$
This means that all the member in the family Ψ_τ are dominated by the P-integrable random variable $\operatorname{ess\,sup} \Psi_\tau$, and so this family is uniformly integrable.

3. The statement is trivial under condition (1.9). Under condition (1.10), we set $\mathcal{C} = \{H_\theta \mid \theta \in \mathcal{T}\}$ and the statement follows by De La Vallée criterion of uniform integrability.
4. For a given decreasing sequence of stopping times $\{\tau_i\}_{i=1}^{\infty}$, the inequality in (1.6) is in fact an identity due to Lebesgue's convergence theorem and the fact that the sequence $\{H_{\tau_i}\}_{i=1}^{\infty}$ is uniformly integrable.$\diamondsuit$

In the following proposition we determine the cost of superhedging.

Proposition 1.14 *Assume that the process H is of $class(D)$ and upper semicontinuous in expectation from the left. Then*

1. *The cost of superhedging is $U_0^P(H)$, and there exists a superhedging strategy constructed at this cost.*
2. *H is replicable.*

Proof(Sketch). Let U^P be the Snell envelope of H with respect to P. It is a nonnegative supermartingale, and according to theorem 1.10 it is of $class(D)$. Therefore, it admits the Doob-Meyer decomposition:
$$U_t^P = U_0^P + M_t^P - A_t^P, \tag{1.12}$$
where M^P is a uniformly integrable martingale of $class(D)$ and A^P is a predictable increasing process with $A_0^P = 0$. Now we use the assumption that the market is complete. This implies that the martingale M^P can be represented as an stochastic integral $M_t^P = \int_0^t \xi_s dX_s$.
Any value process of a strategy in definition 1.7 is a supermartingale dominating H, and we have just seen that the process $U_0^P(H) + M_t^P$ is itself included. This implies the first assertion.

Now we prove the second part. Due to theorem 1.10, there exists a stopping time τ^* such that $\{U^P_{t\wedge\tau^*}\}_{0\le t\le T}$ is a martingale and $U^P_{\tau^*} = H_\tau$. This implies that $U^P_0 + M^P_{\tau^*} = H_{\tau^*}$, and so H is replicable by definition.□

1.4 Superhedging in incomplete markets

Turning back to the general situation of incomplete markets, the set $\mathcal{M}$ of martingale measures is infinite, and the relationship of the previous section generalizes to the fact that the superhedging cost of H is equal to

$$\sup_{P\in\mathcal{M}} \sup_{\theta\in\mathcal{T}} E_P\left[H_\theta\right].$$

The proof will require a special uniform decomposition of the upper Snell envelope $U^\uparrow$, in a sense we explain below. Motivated by this financial problem, El Karoui and Quenez[19] obtained a uniform decomposition for European options in a model driven by Brownian motion. Kramkov[41] generalized to a locally bounded semimartingale model. Föllmer and Kabanov[22] and Föllmer and Kramkov[23] removed the restriction of local boundedness.

Let us recall the optional decomposition theorem in the following form.

Optional decomposition theorem 1.15 *Let $\{U_t\}_{0\le t\le T}$ be a positive càdlàg $\mathcal{M}$–supermartingale with*

$$U_0 = \sup_{P\in\mathcal{M}} \sup_{\theta\in\mathcal{T}} E_P[U_\theta] < \infty.$$

Then, there exists $\xi \in Ad_{U_0}$, and an increasing optional process $\{C_t\}_{0\le t\le T}$ with $C_0 = 0$, such that

$$U_t = U_0 + \int_0^t \xi_s dX_s - C_t, \qquad \textit{for all } t \in [0,T].\square$$

In theorem 4.3 we will see that the upper Snell envelope admits a version that satisfies the hypotheses of the optional decomposition theorem, and is furthermore, the minimal $\mathcal{M}$-supermartingale dominating H. From this fact we can determine the cost of superhedging and construct a minimal strategy as stated in the following theorem.

Theorem 1.16 *The superhedging cost of the American option H is equal to*

$$U_0^\uparrow = \sup_{P\in\mathcal{M}} \sup_{\theta\in\mathcal{T}} E_P[H_\theta], \tag{1.13}$$

and there exists a minimal superhedging strategy $(U_0^\uparrow, \xi)$.

Proof. The main steps of the proof are contained in that of corollary 7.9 in Föllmer and Schied[27]. First notice that for any $\xi \in Ad_c$ the value process $V^{c,\xi}$ is a $\mathcal{M}$-supermartingale. Then, $V^{c,\xi} \geq H$, and consequently $V^{c,\xi} \geq U^{\uparrow}$. This implies that the superhedging cost of H is an upper bound of (1.13).

To prove the opposite inequality, the optional decomposition theorem allows us to decompose the upper Snell envelope as

$$U_t^{\uparrow} = U_0^{\uparrow} + \int_0^t \xi_s dX_s - C_t,$$

where $\{C_t\}_{0 \leq t \leq T}$ is an increasing process with $C_0 = 0$, and ξ is a $U_0^{\uparrow}$-admissible strategy such that $V^{U_0^{\uparrow},\xi} \geq H$. This implies that the cost of superhedging is a lower bound of (1.13).□

In some cases, the seller of an American option will not implement a superhedging strategy. One possible reason could be that the cost is too expensive. This phenomenon is theoretically predicted, see e.g., Eberlein and Jacod[17]. Then, the seller of an American option H could be interested in controlling risk given an initial capital constraint.

Definition 1.17 *Let $c \geq 0$ and $\xi \in Ad_c$. The shortfall process associated to the admissible strategy (c, ξ) is the stochastic process defined by*

$$\{(H_t - V_t^{c,\xi})^+\}_{0 \leq t \leq T}.\square$$

The following proposition says that any admissible strategy in Ad_c, with $c < U_0^{\uparrow}$, generates a nontrivial shortfall risk.

Proposition 1.18 *Let $l : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous strictly increasing convex function. Then, for any constant c with $0 < c < U_0^{\uparrow}$ and for $\xi \in Ad_c$*

$$\sup_{\theta \in \mathcal{T}} E_R[l((H_\theta - V_\theta^{c,\xi})^+)] > 0.$$

Proof. By Jensen's inequality it is enough to consider the case $l(x) = x$. By way of contradiction assume that

$$\sup_{\theta \in \mathcal{T}} E_R[(H_\theta - V_\theta^{c,\xi})^+] = 0.$$

Then, from the fact that H and $V^{c,\xi}$ are càdlàg and by theorem 13 p.73 in Dellacherie[11], we conclude that $(H - V^{c,\xi})^+ \leq 0$. This implies that $V^{c,\xi} \geq H$, a clear contradiction to our assumption that $c < U_0^{\uparrow}$.□

We interpret this proposition as follows: *The expected shorfall risk of any partial hedge is nontrivial.* This suggests to control shortfall risk subject to a cost constraint, an approach that we take in chapter 2.

1.5 Arbitrage free prices

In the previous section we studied the superhedging cost of an American option. With this capital, a superhedge can be constructed. A natural question is: Can this cost be interpreted as a price? In order to give an answer, we analize the structure of arbitrage free prices, extending the discussion in [27] from discrete to continuous time.

Arbitrage free prices 1.19 *A real number c is called an arbitrage free price for H if the following two conditions are satisfied.*

- *There exists a stopping time $\tau \in \mathcal{T}$ and a martingale measure $P \in \mathcal{M}$ such that $c \leq E_P[H_\tau]$.*
- *For any stopping time $\tau' \in \mathcal{T}$ there exists $P' \in \mathcal{M}$ such that $c \geq E_{P'}[H_{\tau'}]$.*

The set of all arbitrage free prices for H is denoted $\Pi(H)$, and we set

$$\pi_{\inf}(H) := \inf \Pi(H) \text{ and } \pi_{\sup}(H) := \sup \Pi(H). \square$$

According to this definition, given $c \in \Pi(H)$, the following inequality holds

$$\sup_{\theta \in \mathcal{T}} \inf_{P \in \mathcal{M}} E_P\left[H_\theta\right] \leq c \leq \sup_{\theta \in \mathcal{T}} \sup_{P \in \mathcal{M}} E_P\left[H_\theta\right]. \tag{1.14}$$

The right-hand term equals $\pi_{\sup}(H)$, and is finite because we have assumed condition (1.3). We will see that $\pi_{\sup}(H) = U_0^{\uparrow}$, or in words: The supremum of the set of arbitrage free prices coincides with the superhedging cost.

In corollary 5.8 of chapter 5 we will prove a *minimax identity* in the sense that the exchange of infimum and supremum in the left-hand term of (1.14) holds:

$$\sup_{\theta \in \mathcal{T}} \inf_{P \in \mathcal{M}} E_P\left[H_\theta\right] = \inf_{P \in \mathcal{M}} \sup_{\theta \in \mathcal{T}} E_P\left[H_\theta\right]. \tag{1.15}$$

This identity will allow us to characterize the set $\Pi(H)$ as an interval with the bounds given in (1.14). This characterization involves the lower Snell envelope.

The **lower Snell envelope** associated to H with respect to $\mathcal{M}$, is an $\mathbb{F}$-adapted process which we denote by $U^{\downarrow} = U^{\downarrow}(H, \mathcal{M})$ (following the notation of [27]), such that the equality

$$U_t^{\downarrow} = \operatorname{ess\,inf}_{P \in \mathcal{M}} \operatorname{ess\,sup}_{\theta \in \mathcal{T}[t,T]} E_P[H_\theta \mid \mathcal{F}_t], \tag{1.16}$$

holds for all $t \in [0,T]$. In theorem 5.21 of chapter 5 we will show how to construct an optional version of this process.

In the next theorem we characterize the infimum and supremum of the set of arbitrage free prices. The condition on H of boundedness and upper semicontinuity in expectation from the left are imposed in order to guarantee the existence of optimal stopping times and to guarantee the minimax identity (1.15). See definition 1.9 for the concept of semicontinuity in expectation from the left. The condition of uniform boundedness for H can be relaxed for the first part of the theorem. However, for the second part we use boundedness in an essential way.

Theorem 1.20 *Assume that $H \leq K$ for some constant $K > 0$, and that H is upper semicontinuous in expectation from the left with respect to any probability measure $P \in \mathcal{M}$. Then, the set of arbitrage free prices $\Pi(H)$ is an interval with infimum*

$$\pi_{\inf}(H) = \inf_{P\in\mathcal{M}} \sup_{\theta\in\mathcal{T}} E_P\left[H_\theta\right] = \sup_{\theta\in\mathcal{T}} \inf_{P\in\mathcal{M}} E_P\left[H_\theta\right]$$

and supremum

$$\pi_{\sup}(H) = \sup_{P\in\mathcal{M}} \sup_{\theta\in\mathcal{T}} E_P\left[H_\theta\right] = \sup_{\theta\in\mathcal{T}} \sup_{P\in\mathcal{M}} E_P\left[H_\theta\right].$$

Moreover, the supremum $\pi_{\sup}(H)$ is not an arbitrage free price unless the interval $\Pi(H)$ consists of a single point.

Proof. Let $c \in \mathbb{R}_+$ be such that

$$\sup_{\theta\in\mathcal{T}} \inf_{P\in\mathcal{M}} E_P\left[H_\theta\right] < c < \sup_{\theta\in\mathcal{T}} \sup_{P\in\mathcal{M}} E_P\left[H_\theta\right].$$

If we prove that c is an arbitrage free price, then the first assertion of the theorem will follow. To verify the first condition of an arbitrage price, we observe that there exist a probability measure $P \in \mathcal{M}$ and a stopping time $\theta \in \mathcal{T}$ such that $c \leq E_P[H_\theta]$, since $\pi_{\sup}(H) < \infty$.

Let $\theta' \in \mathcal{T}$ be a fixed stopping time. We must find a probability measure $P' \in \mathcal{M}$ such that $E_{P'}[H_{\theta'}] \leq c$. According to the minimax identity (1.15), it must be the case that $\inf_{P\in\mathcal{M}} \sup_{\theta\in\mathcal{T}} E_P\left[H_\theta\right] < c$. We infer that there exists P' such that $\sup_{\theta\in\mathcal{T}} E_{P'}[H_\theta] < c$, and this is the desired probability measure.

Now we prove the last part. If $\pi_{\sup}(H) \in \Pi(H)$, then from the first part there exist $P^* \in \mathcal{M}$ and $\theta^* \in \mathcal{T}$ such that $\pi_{\sup}(H) = E_{P^*}[H_{\theta^*}]$. We deduce that

$$U_0^{\uparrow} = \pi_{\sup}(H) = U_0^{P^*}(H),$$

and then, according to proposition 1.21, H is replicable and $\Pi(H) = \{U_0^{P^*}\}$.□

Proposition 1.21 *Let us assume the conditions of theorem 1.20. Then, the following conditions are equivalent*

1. *H is replicable.*
2. *There exists $P_0 \in \mathcal{M}$ such that $U_0^{P_0} = U_0^{\uparrow}$.*

In this case, we have that $U_0^P = U_0^{\uparrow}$, for arbitrary $P \in \mathcal{M}$.

Proof. We first show the implication $1 \Rightarrow 2$. Let (c, ξ) be a replicating strategy for H. Let $\tau \in \mathcal{T}$ be a stopping time such that $V_\tau^{c,\xi} = H_\tau$ and $\{V_{t\wedge\tau}^{c,\xi}\}_{0\leq t\leq T}$ is a $\mathcal{M}$-martingale. Let P be any probability measure in $\mathcal{M}$. For $t \in [0, T]$, we have the inequalities

$$H_t \leq U_t^P \leq U_t^{\uparrow} \leq V_t^{c,\xi}, \quad P-a.s.,$$

where the last inequality follows from the fact that $V^{c,\xi}$ is a $\mathcal{M}$-supermartingale above H, and $U^{\uparrow}$ is the minimal càdlàg $\mathcal{M}$-supermartingale with this property. We evaluate in τ and take P-expectation to obtain

$$U_0^{\uparrow} \leq V_0^{c,\xi} = E_P[V_\tau^{c,\xi}] = E_P[H_\tau] \leq U_0^P \leq U_0^{\uparrow}.$$

We conclude that $U_0^{\uparrow} = U^P$. Since P was arbitrary, we conclude the last assertion in the proposition.

Now we show $2 \Rightarrow 1$. Let $P \in \mathcal{M}$ be such that $U_0^P = U_0^{\uparrow}$, and let $\tau \in \mathcal{T}$ be an optimal stopping time of H with respect to P:

$$U_0^P = E_P[H_\tau].$$

We have that

$$E_P[U_\tau^P] \geq E_P[U_\tau^{\uparrow}],$$

since $U^{\uparrow}$ is a P-supermartingale. But we know that $U^{\uparrow} \geq U^P$, and we conclude that

$$U_\tau^{\uparrow} = U_\tau^P. \tag{1.17}$$

Then, $U^{\uparrow}$ is a $\mathcal{M}$-supermartingale with the property that $U^{\uparrow} \geq H$ and

$$U^{\uparrow}_{\tau} = H_{\tau}. \tag{1.18}$$

We now apply the optional decomposition theorem 1.15 to the upper Snell envelope, to obtain

$$U^{\uparrow}_t = V_t^{U_0^{\uparrow},\xi} - C_t, \tag{1.19}$$

where $C := \{C_t\}_{0 \leq t \leq T}$ is an optional increasing process with $C_0 = 0$ and $\xi \in Ad_{U_0^{\uparrow}}$ is an admissible strategy.

We evaluate (1.19) in the stopping time τ and take expectation with respect to P to obtain that

$$U_0^{\uparrow} = E_P[V_{\tau}^{U_0^{\uparrow},\xi} - C_{\tau}],$$

where we have applied (1.17) and the optimality of τ with respect to P. It follows that $E_P[V_{\tau}^{\xi} - C_{\tau}] = 0$, and we conclude that

$$C_{\tau} = 0 \quad P - a.s.,$$

since C is an increasing process and V^{ξ} is a P-local martingale bounded from below by $-U_0^{\uparrow}$, and hence a P-supermartingale. But then, from the optional decomposition (1.19)

$$U^{\uparrow}_{\tau \wedge t} = V_{\tau \wedge t}^{U_0^{\uparrow},\xi} \quad P - a.s. \quad t \in [0, T].$$

We conclude that the process $\{V_{\tau \wedge t}^{U_0^{\uparrow},\xi}\}_{0 \leq t \leq T}$ is a $\mathcal{M}$-local martingale, which in fact is a $\mathcal{M}$-martingale, *since it is upper bounded by* K and positive. From (1.18) we see that $V_{\tau}^{U_0^{\uparrow},\xi} = H_{\tau}$. We have proved that the strategy $(U_0^{\uparrow}, \xi)$ replicates H.□

Remark 1.22 *Proposition 1.21 is an important result in mathematical finance. For European options it goes back to Ansel and Stricker[1] and Jacka[33]. Here we have extended the result to the case of bounded American options and given a different proof based on the optional decomposition theorem.*♢

Chapter 2

Robust partial hedging of American options

In this chapter we take the point of view of the *seller* of an American option who aims to control the shortfall risk by trading in the financial market. It is reasonable to assume that he trades in a self-financing way and is limited by an initial budget constraint c. If c is not less than the cost of superhedging, then we know that the seller can reduce the shortfall risk to zero by constructing a superhedge. Both from a theoretical and practical point of view it is more interesting to consider the case where c is insufficient for this purpose. In this case, any admissible strategy yields a nontrivial shortfall risk. This suggests to search for a strategy minimizing shortfall risk, specified by the convex loss functional

$$\xi \mapsto \sup_{\theta \in \mathcal{T}} E_R[l((H_\theta - V_\theta^{c,\xi})^+)] \tag{2.1}$$

in terms of some convex loss function l. We will take this point of view, and our first goal is to show that such minimizing strategies exist. For European options, the problem of minimizing the shortfall risk was introduced and solved in Föllmer and Leukert[24]. In the American case, we have to take the supremum over the family of stopping times $\mathcal{T}$, and thus the optimization problem becomes more complex.

We actually will consider a more general class of loss functionals where model uncertainty is explicitly taken into account. Our motivation comes from robust numerical representations of preference orders on asset profiles due to Gilboa and Schmeidler[30]. This leads us to a convex loss functional of the form

$$\xi \mapsto \sup_{\theta \in \mathcal{T}} \sup_{Q \in \mathcal{Q}} E_Q[l((H_\theta - V_\theta^{c,\xi})^+)], \tag{2.2}$$

where the second supremum is taken over a whole class $\mathcal{Q}$ of probability measures Q. The resulting convex optimization problem combines aspects of optimal control and optimal stopping. The convexity of the problem will allow us to attack directly the primal problem in order to show the existence of optimal strategies. Our solution will thus extend the analysis of Föllmer and Leukert[24] from European to American options.
Let us emphasize that although we are going to study a utility-based optimization problem, here we focus on the primal problem. A first step towards a convex duality approach to the problem of partial hedging of American options will be developed in chapter 3.

The second goal we pursue in this chapter is to reduce the optimization of the robust functional (2.2) to the optimization of a non-robust functional of the form (2.1) with respect to a worst-case probability measure $Q^* \in \mathcal{Q}$, based on the assumption that $\mathcal{Q}$ is weakly compact.

This chapter is organized as follows. In section 2.1 we introduce an optimization problem asking for an optimal partial hedging strategy. In section 2.1.1 we motivate our criterion by showing how it is related to efficient hedging of European options. In section 2.2.1 we prove the existence of an optimal strategy. In section 2.2.2 we prove the existence of a worst-case probability measure.

2.1 Problem formulation

Throughout this chapter we fix a stochastic base $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}, R)$ satisfying the usual conditions of right continuity and completeness. We furthermore fix a semimartingale $\{X_t\}_{0\leq t\leq T}$ representing a discounted price process as presented in section 1.1. The corresponding family of equivalent local martingale measures is denoted by $\mathcal{M}$ and we assume it is nonempty.
For a positive constant $c \in \mathbb{R}_+$, Ad_c is the family of c-admissible strategies of definition 1.3. We recall that the value process of a c-admissible strategy $\xi \in Ad_c$ is defined by

$$V_t^{c,\xi} := c + \int_0^t \xi_s dX_s, \text{ for all } t \in [0, T].$$

We fix a process $H := \{H_t\}_{0\leq t\leq T}$ representing an American option and satisfying the conditions of definition 1.2.

Let us now introduce the following definition.

Definition 2.1 *We say that* $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ *is a generalized loss function if*

1. $f(\cdot, v)$ *is a continuous increasing function for any* $v \in \mathbb{R}_+$*,*
2. $f(h, \cdot)$ *is a convex continuous decreasing function for any* $h \in \mathbb{R}_+$*.*□

The most important examples of a generalized loss function which we have in mind are of the form

$$f(h, v) = l((h - v)^+)$$

for a loss function l and

$$f(h, v) = \left(1 - \frac{v}{h}\right)^+ .$$

Motivated by definition 1.17 and proposition 1.18 of chapter 1, we now define a robust partial hedging problem for American options which takes model uncertainty into account explicitly.

Definition 2.2 *Let* f *be a generalized loss function. Let* $\mathcal{Q}$ *be a convex family of probability measures which are absolutely continuous with respect to* R*. Let* c *be a positive constant satisfying*

$$0 \leq c < \pi_{\sup}(H).$$

The value function $PH(\cdot)$ *of the robust partial hedging problem is defined by*

$$PH(c) := \inf_{\xi \in Ad_c} \sup_{\theta \in \mathcal{T}} \sup_{Q \in \mathcal{Q}} E_Q[f(H_\theta, V_\theta^{c,\xi})].$$

We say that a c-admissible strategy $\xi^* \in Ad_c$ *is optimal if the corresponding value process attains the value* $PH(c)$*:*

$$PH(c) = \sup_{\theta \in \mathcal{T}} \sup_{Q \in \mathcal{Q}} E_Q[f(H_\theta, V_\theta^{c,\xi^*})].\square$$

In section 2.1.1 we discuss how the robust optimization problem $PH(\cdot)$ corresponds to a *robust* efficient hedging approach for American options in the special case

$$f(h, v) = l((h - v)^+),$$

for a loss function l. This discussion will also motivate the robust functional

$$\sup_{\theta \in \mathcal{T}} \sup_{Q \in \mathcal{Q}} E_Q[\cdot].$$

In chapter 3 we will see that the robust problem $PH(\cdot)$ corresponds to a *robust* quantile hedging approach for American options in the special case

$$f(h,v) = \left(1 - \frac{v}{h}\right)^+ .$$

Let us note that the initial wealth c was required to be positive and strictly smaller than the cost of superhedging. A natural restriction would be to require $\pi_{\inf}(H) \leq c$. However, our discussion covers the case $c < \pi_{\inf}(H)$.

2.1.1 Robust efficient hedging

Let us explain how the efficient hedging problem solved by Föllmer and Leukert[24] and the problem of definition 2.2 are related. For this purpose, let H_T be a European option with superhedging cost $\pi_{\sup}(H_T) = \sup_{P\in\mathcal{M}} E_P[H_T]$, and let c be an initial wealth with $0 \leq c < \pi_{\sup}(H_T)$. For a loss function l, the efficient hedging asks for a c-admissible strategy $\xi^* \in Ad_c$ with

$$E_R[l((H_T - V_T^{c,\xi^*})^+)] = \inf_{\xi\in Ad_c} E_R[l((H_T - V_T^{c,\xi})^+)].$$

Loosely speaking, the value process of any c-admissible strategy $\xi \in Ad_c$ yields a nontrivial shortfall $(H_T - V_T^{c,\xi})^+$, and the strategy is selected through a loss-based criterion specified by the loss function l.

But in the utility or loss representation of a preference order, it has been assumed that the probabilistic structure specified by the probability measure R is well determined. A more realistic formulation should allow for model uncertainty where some probabilistic aspects are unclear. This is captured by the robust formulation of preferences due to Gilboa and Schmeidler[30]. Accordingly, we assume that the agent has a convex set $\mathcal{Q}$ of probability measures or priors Q, and valuates a payoff-profile W through the utility functional

$$\inf_{Q\in\mathcal{Q}} E_Q[u(W)] \tag{2.3}$$

where u is a utility function. Alternatively, a loss-profile S is valuated according to the loss functional

$$\sup_{Q\in\mathcal{Q}} E_Q[l(S)]$$

where l is a loss function. Thus, we are led to quantify the robust shortfall risk by

$$\xi \in Ad_c \to \sup_{Q\in\mathcal{Q}} E_Q[l((H_T - V_T^{c,\xi})^+)].$$

This can be seen as a robust version of efficient hedging for European options, a problem which was introduced and discussed by Kirch[39].

Let us now move on to the American case.

We are taking the point of view of the seller, and so the liquidation date is uncertain. If the option is exercised in a stopping time $\theta \in \mathcal{T}$, then the correspondence

$$\xi \in Ad_c \to \sup_{Q\in\mathcal{Q}} E_Q[l((H_\theta - V_\theta^{c,\xi})^+)]$$

gives a robust quantification of the shortfall risk at time θ. But the seller has no control over the time of exercise. If he takes a worst-case attitude regarding stopping times, then this is quantified by the functional

$$\xi \in Ad_c \to \sup_{\theta\in\mathcal{T}} \sup_{Q\in\mathcal{Q}} E_Q[l((H_\theta - V_\theta^{c,\xi})^+)].$$

In this robust framework, efficient hedging for American options asks for a c-admissible strategy $\xi^* \in Ad_c$ with

$$\sup_{\theta\in\mathcal{T}} \sup_{Q\in\mathcal{Q}} E_Q[l((H_\theta - V_\theta^{c,\xi^*})^+)] = \inf_{\xi\in Ad_c} \sup_{\theta\in\mathcal{T}} \sup_{Q\in\mathcal{Q}} E_Q[l((H_\theta - V_\theta^{c,\xi})^+)].$$

This is the robust partial hedging problem 2.2 in the special case $f(h,v) = l((h-v)^+)$.

Stochastic optimization of utility with discretionary stopping. In the previous paragraph we explained that the robust partial hedging problem $PH(\cdot)$ is motivated by a robust version of efficient hedging of American options, where model uncertainty is explicitly taken into account. Our formulation combined two lines of ideas. In the first, preferences are represented by robust utility or loss functionals. In the second, in order to incorporate the dynamic nature of American options, we assumed a worst-case attitude whereby the seller is pessimistic regarding the buyer's selection of a stopping time. In this way, we obtained a robust stochastic optimization problem of expected shortfall with discretionary stopping. The class of problems where expected utility optimization is combined with discretionary stopping is quite recent, and it has been previously studied in the financial literature with purposes other than partial hedging. Let us cite a few papers.
Davis and Zariphopoulou[6] and Oberman and Zariphopoulou[46] studied two stochastic problems of maximizing expected utility with discretionary stopping. They adopted an indifference-price approach in order to valuate

early exercise contingent claims. Their analysis was based on variational inequalities.
Karatzas and Wang[37] studied an optimal portfolio management problem combined with discretionary stopping. Their analysis was based on the martingale-method and they established a criterion to apply convex-duality which in the cases of logarithmic and moment utilities led to explicit results.
Letting aside the different motivations, a common feature in the afore mentioned papers is that stopping and portfolio selection are decision variables under our control. This is the main conceptual difference with our problem here.
In the indifference-pricing approach studied in [6, 46], a price is given to an early exercise contingent claim from the perspective of an investor having a long position on the claim. The investor simultaneously searches an optimal exercise and an optimal portfolio allocation, and hence a utility functional is maximized over portfolio strategies and over stopping times.
In [37] the problem is of utility maximization from consumption and terminal wealth, stopping times are introduced to search for the best liquidation date. Here again, a utility functional is maximized over portfolio strategies and over stopping times.
In the robust partial hedging problem 2.2 we have taken the point of view of the seller of an American option, portfolios are not investment opportunities but hedging strategies, and stopping times are adverse variables. Loosely speaking, the criteria in the afore mentioned papers are of "maxmax" type, while here we are considering a "minimax" criterion.

Robust utility maximization. We conclude this section with some remarks about numerical representations of preference orders and about robust utility maximization. The axiomatic treatment on preference orders and its numerical representations began with Von Neumann and Morgenstern[53] and Savage[50]. They formulated a set of axioms to be satisfied by a preference order, and constructed a numerical representations of the form

$$E_Q[u(\cdot)].$$

The interpretation is that, given two payoffs X_1 and X_2, the first is "more preferred" than the second if and only if $E_Q[u(X_1)] > E_Q[u(X_2)]$, see e.g., section 2.5 in Föllmer and Schied[27]. However, Ellsberg's paradox (see example 2.75 in [27]) illustrates that this numerical representation does not account for model-uncertainty aversion. An *uncertainty aversion axiom* led Gilboa and Schmeidler[30] to obtain a robust numerical representation of the form (2.3):

$$\inf_{Q \in \mathcal{Q}} E_Q[u(\cdot)].$$

Maximization of robust utility in the context of financial markets is recent. Let us give a partial list of the related literature.

Schied[51] studies the problem of robust utility maximization in a complete market. For the special case of priors

$$\mathcal{Q}_\lambda := \left\{ Q \ll P \mid \frac{dQ}{dP} \leq \frac{1}{\lambda} \right\},$$

corresponding to the risk measure AVaR, explicit solutions are obtained, using the robust version of the Neyman-Pearson lemma due to Huber and Strassen[32].

Kirch[39] studies a robust version of efficient hedging for European options. His solution reduces the problem into a Neyman-Pearson type problem for composite hypotheses against composite alternatives and for non linear power functions.

Föllmer and Gundel[21] consider the robust utility maximization problem in incomplete markets. They extend the method of Goll and Rüschendorf[31] and obtain a least favorable pair of probability measures (Q^*, P^*) where Q^* is an element of the set of priors and P^* is an extended martingale measure. This pair reduces the robust problem to a classical problem of utility maximization with respect to Q^* and for P^*-affordable claims. Their approach allows to obtain the least favorable pair as the solution of a dual optimization problem.

Schied and Wu[52] consider the problem of robust utility in an incomplete market. Their approach extends the duality results of Kramkov and Schachermayer[42] to the robust setting.

2.2 Solution

2.2.1 Existence of an optimal strategy

In this subsection we show that the robust partial hedging problem $PH(\cdot)$ introduced in definition 2.2 has a solution. We will apply a convergence theorem for a sequence of supermartingales as obtained in lemma 5.2 by Föllmer and Kramkov[23]. The following definition of Fatou convergence for processes is taken from [23].

Definition 2.3 *Let $\mathcal{D}$ be a countable dense subset of $[0,T]$. A sequence of processes $\{Y^n\}_{n=1}^{\infty}$ is Fatou convergent on $\mathcal{D}$ to a process Y if Y^n is uniformly bounded from below and if for any $t \in [0,T]$ the following equalities hold R-a.s.*

$$\begin{aligned} X_t &= \limsup_{s \downarrow t, s \in \mathcal{D}} \limsup_{n \to \infty} X_s^n \\ &= \liminf_{s \downarrow t, s \in \mathcal{D}} \liminf_{n \to \infty} X_s^n . \square \end{aligned} \tag{2.4}$$

The next result is lemma 5.2 in [23]. We give a formulation which is convenient for our application.

Lemma 2.4 *Let $\{X^i\}_{i=1}^{\infty}$ be a sequence of positive càdlàg supermartingales indexed by $[0,T]$ with $X_0^i = c$. Let $\mathcal{D} \subset [0,T]$ be a dense countable subset. Then, there exists a càdlàg supermartingale $\{Y_t\}_{0 \le t \le T}$ with $Y_0 \le c$, and a sequence of convex combinations*

$$Y^i \in \operatorname{conv}\left\{X^i, X^{i+1}, \ldots\right\},$$

such that the sequence $\{Y^i\}_{i=1}^{\infty}$ is Fatou convergent to Y on $\mathcal{D}$.$\square$

The next theorem is formulated in the setting of definition 2.2. Recall that c is a constant with $0 \le c < \pi_{\sup}(H)$.

Theorem 2.5 *Let us assume that the American option H satisfies the following integrability condition*

$$\sup_{\theta \in \mathcal{T}} \sup_{Q \in \mathcal{Q}} E_Q[f(H_\theta, 0)] < \infty. \tag{2.5}$$

Then, there exists an optimal strategy $\xi^* \in Ad_c$ for the robust partial hedging problem of definition 2.2.

Proof.

1. Let $\{\xi^i\}_{i=1}^{\infty} \subset Ad_c$ be a minimizing sequence in the following sense

$$PH(c) = \lim_{i \to \infty} \sup_{\theta \in \mathcal{T}} \sup_{Q \in \mathcal{Q}} E_Q[f(H_\theta, V_\theta^{c,\xi^i})].$$

   The value process V^{c,ξ^i} of the strategy ξ^i is a nonnegative local martingale and thus is a $\mathcal{M}$-supermartingale. Let $\mathcal{D}$ be a countable dense subset of $[0,T]$. We apply lemma 2.4 to obtain a sequence of convex combinations

$$\widetilde{V}^i \in \text{conv}\left\{V^{c,\xi^i}, V^{c,\xi^{i+1}}, \dots\right\},$$

which is Fatou convergent to a positive $\mathcal{M}$–supermartingale V^*. We prove that the sequence $\{\widetilde{V}^i\}_{i=1}^{\infty}$ is also minimizing. It will be convenient to write $\widetilde{V}^i$ explicitly as a convex combination:

$$\widetilde{V}^i = \sum_{k=i}^{\infty} \lambda_k^i V^{\xi^k}.$$

Let $Q^0 \in \mathcal{Q}$ be a fixed probability measure, and let $\tau \in \mathcal{T}$ be a fixed stopping time. We use the fact that f is a generalized loss function to obtain the following inequalities

$$\begin{aligned} E_{Q^0}[f(H_\tau, \widetilde{V}^i_\tau)] &\le \sum_{k=i}^{\infty} \lambda_k^i E_{Q^0}[f(H_\tau, V_\tau^{\xi^k})] \\ &\le \sum_{k=i}^{\infty} \lambda_k^i \sup_{\theta\in\mathcal{T}} \sup_{Q\in\mathcal{Q}} E_Q[f(H_\theta, V_\theta^{\xi^k})] \\ &\le \sup_{k\ge i} \left\{ \sup_{\theta\in\mathcal{T}} \sup_{Q\in\mathcal{Q}} E_Q[f(H_\theta, V_\theta^{\xi^k})] \right\}. \end{aligned}$$

It follows that

$$\begin{aligned} \lim_{i\to\infty} \sup_{\theta\in\mathcal{T}} \sup_{Q\in\mathcal{Q}} E_Q[f(H_\theta, \widetilde{V}^i_\theta)] &\le \limsup_{k\to\infty} \left\{ \sup_{\theta\in\mathcal{T}} \sup_{Q\in\mathcal{Q}} E_Q[f(H_\theta, V_\theta^{\xi^k})] \right\} \\ &= PH(c). \end{aligned} \tag{2.6}$$

This means that the sequence $\{\widetilde{V}^i\}_{i=1}^{\infty}$ is also minimizing.

2. There exists a dense subset $\mathcal{D}' \subset [0,T]$ such that for any $t \in \mathcal{D}'$

$$V_t^* = \liminf_{i\to\infty} \widetilde{V}_t^i, R - a.s, \tag{2.7}$$

due to the right continuity of the supermartingale V^*. Now we prove that

$$\sup_{Q\in\mathcal{Q}} \sup_{\theta\in\mathcal{T}} E_Q[f(H_\theta, V_\theta^*)] \le PH(c). \tag{2.8}$$

Let $Q \in \mathcal{Q}$ and $\tau \in \mathcal{T}$ be arbitrary but fixed. By a usual discretization procedure, there exists a sequence of stopping times $\{\tau^i\}_{i=1}^{\infty} \subset \mathcal{T}$ such that $\tau^i \searrow \tau$, and τ^i takes a finite number of values in $\mathcal{D}'$. Continuity of $f(h,\cdot)$ and Fatou's lemma implies that

$$E_Q[f(H_\tau, V^*_\tau)] \le \liminf_{i\to\infty} E_Q[f(H_{\tau^i}, V^*_{\tau^i})]. \tag{2.9}$$

For $i \in \mathbb{N}$ fixed, we represent the stopping time τ^i explicitly by

$$\tau^i = \sum_{j=1}^{n^i} d^i_j 1_{\{\tau^i = d^i_j\}},$$

where $d^i_j \in \mathcal{D}'$. Then

$$\begin{aligned} V^*_{\tau^i} &= \sum_{j=1}^{n^i} 1_{\{\tau^i=d^i_j\}} V^*_{d^i_j} \\ &= \sum_{j=1}^{n^i} 1_{\{\tau^i=d^i_j\}} \liminf_{k\to\infty} \widetilde{V}^k_{d^i_j} &(2.10) \\ &= \liminf_{k\to\infty} \sum_{j=1}^{n^i} 1_{\{\tau^i=d^i_j\}} \widetilde{V}^k_{d^i_j} \\ &= \liminf_{k\to\infty} \widetilde{V}^k_{\tau^i}, &(2.11) \end{aligned}$$

where we have used (2.7) in the equality (2.10). We now can conclude that

$$\begin{aligned} E_Q[f(H_{\tau^i}, V^*_{\tau^i})] &= E_Q[\liminf_{k\to\infty} f(H_{\tau^i}, \widetilde{V}^k_{\tau^i})] &(2.12) \\ &\le \liminf_{k\to\infty} E_Q[f(H_{\tau^i}, \widetilde{V}^k_{\tau^i})] \le PH(c), &(2.13) \end{aligned}$$

(2.12) holds by continuity of $f(h,\cdot)$ and (2.11). The first inequality in (2.13) follows from Fatou's Lemma, and the second from the previous step.

The inequalities (2.9) and (2.13) imply that

$$E_Q[f(H_\tau, V^*_\tau)] \le PH(c).$$

Since Q and τ were arbitrary, (2.8) holds true.

3. It remains to construct an optimal strategy $\xi^* \in Ad_c$. V^* is a non-negative $\mathcal{M}$-supermartingale with $V^*_0 \le c$, the optional decomposition theorem 1.15 allows us to represent V^* as

$$V^*_t = V^*_0 + \int_0^t \xi^*_s dX_s - C_t,$$

where $\xi^* \in Ad_{V_0^*}$ and C is an optional increasing process with $C_0 = 0$. We certainly have that $\xi^* \in Ad_c$, and ξ^* is optimal since $f(h, \cdot)$ is decreasing and

$$V_t^* \leq V_0^* + \int_0^t \xi_x^* dX_s. \square$$

Remark 2.6 *Efficient hedging for American options has been studied by other authors. Pérez [47] obtained an existence result similar to our theorem 2.5 in a discrete time model. Mulinacci[45] obtained an existence theorem in continuous time. More recently, Dolinsky and Kifer[14] formulated and studied efficient hedging for game options. None of these papers, however, consider a robust setting.*$\diamondsuit$

A natural question is the uniqueness of an optimal strategy. But the fact that the problem depends on the family of stopping times suggests that unicity holds only in a restricted way. However, the convexity of the function f implies that the value processes of optimal strategies share an optimal stopping time in the sense of the following remark.

Remark 2.7 *Let us assume that the family of probability measures $\mathcal{Q}$ is a singleton $\{Q\}$. Moreover, let us assume that the price process X, the American option H, and the filtration $\mathbb{F}$ are continuous. Assume furthermore that the process $f(H, 0)$ is of class(D) with respect to Q. Let $(c, \xi^1), (c, \xi^1)$ be two optimal strategies for the problem of partial hedging $PH(c)$. Then, the corresponding value processes V^{c,ξ^1} and V^{c,ξ^2} have a common optimal stopping time $\tau^* \in \mathcal{T}$ in the following sense:*

$$E_Q[f(H_{\tau^*}, V_{\tau^*}^{c,\xi^1})] = \sup_{\theta \in \mathcal{T}} E_Q[f(H_\theta, V_\theta^{c,\xi^1})]$$

and

$$E_Q[f(H_{\tau^*}, V_{\tau^*}^{c,\xi^2})] = \sup_{\theta \in \mathcal{T}} E_Q[f(H_\theta, V_\theta^{c,\xi^2})].$$

Proof. We take a convex combination of the optimal strategies $(c, \xi^1), (c, \xi^1)$ of the form

$$\xi^3 := \frac{1}{2}\xi^1 + \frac{1}{2}\xi^2.$$

The corresponding value process satisfies

$$V^{c,\xi^3} = \frac{1}{2}V^{c,\xi^1} + \frac{1}{2}V^{c,\xi^2},$$

and convexity of f implies that it is also optimal for $PH(c)$. The hypotheses imply the existence of an optimal stopping time τ^* for V^{c,ξ^3} in the following sense:

$$E_Q[f(H_{\tau^*}, V_{\tau^*}^{c,\xi^3})] = \sup_{\theta \in \mathcal{T}} E_Q[f(H_\theta, V_\theta^{c,\xi^3})].$$

The inequality

$$E_Q[f(H_{\tau^*}, V_{\tau^*}^{c,\xi^3})] \le \frac{1}{2}(E_Q[f(H_{\tau^*}, V_{\tau^*}^{c,\xi^1})] + E_Q[f(H_{\tau^*}, V_{\tau^*}^{c,\xi^2})])$$

implies that the stopping time τ^* is optimal for V^{c,ξ^1} and V^{c,ξ^2}.$\diamondsuit$

2.2.2 Existence of a worst-case measure

The partial hedging problem of definition 2.2 is of a robust nature. In this subsection we show that it can be reduced to a non-robust hedging problem with respect to a worst-case probability measure $Q^* \in \mathcal{Q}$, if we assume further regularity conditions.

Definition 2.8 *A probability measure $Q^* \in \mathcal{Q}$ is said to be a worst-case probability measure for the problem of robust partial hedging at cost c if*

$$PH(c) = \inf_{\xi \in Ad_c} \sup_{\theta \in \mathcal{T}} E_{Q^*}[f(H_\theta, V_\theta^{c,\xi})].\square$$

We will prove the existence of such a worst-case probability measure under the

Assumption 2.9 *$\mathcal{Q}$ is a convex family of probability measures whose elements are absolutely continuous with respect to R with the following two properties:*

1. *The family of densities*

$$dens(\mathcal{Q}) := \left\{ \frac{dQ}{dR} \mid Q \in \mathcal{Q} \right\}$$

   is $\sigma(L^1(R), L^\infty(R))$-compact.

2. *Let $\{Z_t^Q\}_{0 \le t \le T}$ denote a càdlàg version of the density process of a probability measure $Q \in \mathcal{Q}$ with respect to R. Then*

$$E_R \left[\sup_{0 \le t \le T} Z_t^Q \right] < \infty.\square \qquad (2.14)$$

The property of weak compactness will be applied in proposition 2.22. The property (2.14) will be applied in lemma 2.23.

It will be convenient to introduce the space

$$\mathbb{V}(c) := \{V \text{ a càdlàg process} \mid 0 \leq V \leq H \text{ and } \pi_{\sup}(V) \leq c\}, \tag{2.15}$$

and to reformulate the robust partial hedging problem 2.2 in terms of $\mathbb{V}(c)$.

Proposition 2.10 *Consider the setting of the optimization problem 2.2 and assume that $f(h, v) = 0$ for $v \geq h$. Then, the value $PH(c)$ can be equivalently computed as*

$$PH(c) = \inf_{V\in\mathbb{V}(c)} \sup_{\theta\in\mathcal{T}} \sup_{Q\in\mathcal{Q}} E_Q[f(H_\theta, V_\theta)].$$

Proof. Let us set $\widetilde{PH}(c) = \inf_{V\in\mathbb{V}(c)} \sup_{\theta\in\mathcal{T}} \sup_{Q\in\mathcal{Q}} E_Q[f(H_\theta, V_\theta)]$. For $\xi \in Ad_c$ we have that $V^{c,\xi} \wedge H \in \mathbb{V}(c)$. Then, according to the hypothesis on f

$$f(H_t, V_t^{c,\xi} \wedge H_t) = f(H_t, V_t^{c,\xi}).$$

We get immediately the inequality $\widetilde{PH}(c) \leq PH(c)$.

Let V be an element in $\mathbb{V}(c)$ and let U denote its upper Snell envelope as introduced in (1.4). According to the optional decomposition theorem 1.15, we can represent U as

$$U_t = U_0 + \int_0^t \xi_s dX_s - C_t,$$

where $0 \leq U_0 \leq c$, $\xi \in Ad_{U_0}$ and C is an increasing optional process with $C_0 = 0$. It is clear that

$$c + \int_0^t \xi_s dX_s \geq V_t,$$

and we then get that

$$f(H_t, c + \int_0^t \xi_s dX_s) \leq f(H_t, V_t),$$

since $f(h, \cdot)$ is a decreasing function for a fixed $h \geq 0$. This implies the opposite inequality $\widetilde{PH}(c) \geq PH(c)$.□

We now state the main theorem of this subsection.

Theorem 2.11 *Let the generalized loss function f be such that $f(h, v) = 0$ for $v \geq h$. Let $\mathcal{Q}$ be a convex family of probability measures satisfying the assumption 2.9. Moreover, assume that $H \leq K$ for some constant $K > 0$. Then, there exists a worst-case probability measure $Q^* \in \mathcal{Q}$.*

Proof. See section 2.2.5.□

The proof of theorem 2.11 will require some preparation. In a first step, we introduce an enlarged class of stopping times $\overline{\mathcal{A}}$. The value function $PH(c)$ will stay invariant under this enlargement. In a second step we topologize the spaces $\mathcal{Q} \times \overline{\mathcal{A}}$ and $\mathbb{V}(c)$ in such a way that the topology associated to the product space $\mathcal{Q} \times \overline{\mathcal{A}}$ is compact and $\mathbb{V}(c)$ is a convex subset of a Banach space. In the third step, we apply a topological minimax theorem based on connectedness of level sets.

2.2.3 Randomized stopping times

Let us recall that we have fixed a stochastic base $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}, R)$. We now introduce a Banach space of processes. Let us denote by $\mathcal{V}$ the space of càdlàg $\mathbb{F}$-adapted processes $\{Y_t\}_{0\leq t\leq T}$ with finite norm:

$$\|Y\|_{\mathcal{V}} := E_R \left[\sup_{0\leq t\leq T} |Y_t| \right] < \infty. \tag{2.16}$$

The following theorem characterizes the dual space $\mathcal{V}^*$ of $\mathcal{V}$.

Theorem 2.12 *Let $\gamma \in \mathcal{V}^*$ be a continuous linear functional on $\mathcal{V}$. Then γ admits the representation*

$$\gamma(Y) = E_R \left[\int_{[0,T[} Y_s dA_s + \int_{]0,T]} Y_{s-} dB_s \right], \quad \text{for } Y \in \mathcal{V}$$

where $A := \{A_t\}_{0\leq t\leq T}$ and $B := \{B_t\}_{0\leq t\leq T}$ are adapted processes whose trajectories are right continuous and of integrable variation. The process B is predictable with $B_0 = 0$, and can be chosen so that it charges only a sequence of predictable stopping times. In this case, the pair (A, B) is unique. The functional γ is positive if and only if A and B are increasing processes.

Proof. c.f., theorem 1.2 of Bismut[4].□

The next definition is taken from [4].

Definition 2.13 *A positive continuous functional $\gamma \in \mathcal{V}^*$ is a quasi-randomized stopping time if the representation (A, B) given by theorem 2.12 satisfies*

$$A_T + B_T = 1.$$

We denote the family of quasi-randomized stopping times by $\overline{\mathcal{A}}$.

Moreover, if in the representation (A, B) the process B vanishes, then we say that γ is a randomized stopping time. We denote this family by $\mathcal{A}$.□

Let us note that the specification of the probability measure R determines the duality pairing in theorem 2.12.

Notation 2.14 *Let $K := \{K_t\}_{0 \le t \le T}$ be a càdlàg $\mathbb{F}$-adapted process. The process K sampled in a stopping time θ is denoted by K_θ. We extend this notation to quasi-randomized stopping times as follows.*
If $\gamma \in \overline{\mathcal{A}}$ is a quasi-randomized stopping time represented by a pair (A, B) as in theorem 2.12, we set

$$K_\gamma := \int_{[0,T[} K_s dA_s + \int_{]0,T]} K_{s-} dB_s.$$

For a randomized stopping time $\kappa \in \mathcal{A}$, this notation simplifies to

$$K_\kappa := \int_0^T K_s dA_s. \square$$

The reason to consider the families $\mathcal{A}$ and $\overline{\mathcal{A}}$ for the problem of optimal stopping is justified by the following theorems.

Theorem 2.15 *A continuous linear functional $\gamma \in \mathcal{V}^*$ is an element of $\overline{\mathcal{A}}$ if and only if $\gamma(1) = 1$ and the following inequality holds*

$$\gamma(Z) \le Z_0$$

for any bounded right-continuous R-supermartingale $\{Z_t\}_{0 \le t \le T}$.

Proof. c.f., proposition 1.4 in [4].$\square$

Theorem 2.16 *The family of quasi-randomized stopping times $\overline{\mathcal{A}}$ is $\sigma(\mathcal{V}^*, \mathcal{V})$-compact, and $\mathcal{A}$ is a dense subset of $\overline{\mathcal{A}}$.*

Proof. c.f., theorem 1.1 in [4].$\square$

2.2.4 A compact weak-topology associated to the product space $\mathcal{Q} \times \overline{\mathcal{A}}$

In this subsection we associate a weakly compact set of continuous linear functionals $\mathfrak{L}(\mathcal{Q} \times \overline{\mathcal{A}})$ to the product space $\mathcal{Q} \times \overline{\mathcal{A}}$. To this end, let us introduce a linear space of processes.

Definition 2.17 *By $\mathbb{L}^\infty$ we denote the space of càdlàg $\mathbb{F}$-adapted processes $V : \Omega \times [0,T] \to \mathbb{R}$ such that $|V| \leq K$ for some constant $K > 0$. We introduce a norm in $\mathbb{L}^\infty$ by*

$$\|V\|_{\mathbb{L}^\infty} := \|V^*\|_{L^\infty(R)} \quad for \quad V \in \mathbb{L}^\infty, \tag{2.17}$$

where $V^* := \sup_{0\leq t\leq T} |V_t|$.□

Remark 2.18 *The linear space $\mathbb{L}^\infty$ is complete with the norm $\|\cdot\|_{\mathbb{L}^\infty}$; see Dellacherie and Meyer[13].*◇

In the next definition we associate a set of continuous linear functionals $\mathcal{L}(\mathcal{Q} \times \overline{\mathcal{A}})$ to the product space $\mathcal{Q} \times \overline{\mathcal{A}}$. Recall that $\mathcal{Q}$ is a convex family of probability measures which are absolutely continuous with respect to R, and that the family of densities is $\sigma(L^1, L^\infty)$-compact.

Definition 2.19 *Let $(Q,\gamma) \in \mathcal{Q} \times \overline{\mathcal{A}}$. To the pair (Q,γ) we associate a continuous linear functional $q^{(Q,\gamma)} \in (\mathbb{L}^\infty)^*$ by*

$$q^{(Q,\gamma)}(V) := E_Q[V_\gamma], \quad for \quad V \in \mathbb{L}^\infty.$$

We say that $q^{(Q,\gamma)}$ is represented by the pair (Q,γ), and denote by $\mathcal{L}(\mathcal{Q} \times \overline{\mathcal{A}})$ the class of continuous linear functionals $q^{(Q,\gamma)}$.□

Remark 2.20 *Let us note that $\mathcal{L}(\mathcal{Q} \times \overline{\mathcal{A}})$ is not necessarily convex. Moreover, the correspondence*

$$(Q,\gamma) \to q^{(Q,\gamma)}$$

is not necessarily injective. Indeed, let $t \in [0,T]$ be a constant time and let $Q^1, Q^2 \in \mathcal{Q}$ be two probability measures such that $Q^1 = Q^2$ in $\mathcal{F}_t$. Then we have that $q^{(Q^1,t)} = q^{(Q^2,t)}$. However, we do not need injectivity in what follows.◇

Let us recall the Eberlein-Šmulian theorem in the following form.

Theorem 2.21 *Let A be a subset of a Banach space X. The following are equivalent*

1. *A is weakly compact.*
2. *A is weakly sequentially compact.*

Proof. See theorem V.6.1 [16].□

In the proof of proposition 2.22 below we will apply the Eberlein-Šmulian theorem to the family $\mathcal{L}(\mathcal{Q} \times \overline{\mathcal{A}})$.

Proposition 2.22 *Let $\mathcal{Q}$ be a family of probability measures satisfying the assumption 2.9. Then, the class $\mathfrak{L}(\mathcal{Q} \times \overline{\mathcal{A}})$ is $\sigma((\mathbb{L}^\infty)^*, \mathbb{L}^\infty)$-compact.*

Proof.

1. Let $\{q_i\}_{i=1}^\infty \subset \mathfrak{L}(\mathcal{Q} \times \overline{\mathcal{A}})$ be an arbitrary sequence. We are going to construct a subsequence converging with respect to the $\sigma((\mathbb{L}^\infty)^*, \mathbb{L}^\infty)$-topology to some $q^0 \in \mathfrak{L}(\mathcal{Q} \times \overline{\mathcal{A}})$. By the Eberlein-Šmulian theorem 2.21 we conclude that $\mathfrak{L}(\mathcal{Q} \times \overline{\mathcal{A}})$ is $\sigma((\mathbb{L}^\infty)^*, \mathbb{L}^\infty)$-compact.

   But we observe that $\mathfrak{L}(\mathcal{Q} \times \overline{\mathcal{A}}) \subset B_{(\mathbb{L}^\infty)^*}$, where $B_{(\mathbb{L}^\infty)^*}$ is the unitary ball of the dual space $(\mathbb{L}^\infty)^*$. According to the Banach-Alaoglu theorem, $B_{(\mathbb{L}^\infty)^*}$ is $\sigma((\mathbb{L}^\infty)^*, \mathbb{L}^\infty)$-compact. It follows that $\mathfrak{L}(\mathcal{Q} \times \overline{\mathcal{A}})$ is relatively compact, and by the Eberlein-Šmulian theorem, it is sequentially relatively compact. Passing to a subsequence if necessary, we thus can assume that $\{q_i\}_{i=1}^\infty$ converges to some $q^0 \in (\mathbb{L}^\infty)^*$. It remains to show that $q^0 \in \mathfrak{L}(\mathcal{Q} \times \overline{\mathcal{A}})$. To this end, we must find a pair $(Q^0, \gamma^0) \in \mathcal{Q} \times \overline{\mathcal{A}}$ such that for any process $V \in \mathbb{L}^\infty$

$$q^0(V) = E_{Q^0}[V_{\gamma^0}]. \tag{2.18}$$

2. Assume that q_i is represented by the pair $(Q_i, \gamma_i) \in \mathcal{Q} \times \overline{\mathcal{A}}$. Let us recall the Banach space $\mathcal{V}$ introduced in (2.16). According to theorem 2.16, $\overline{\mathcal{A}}$ is $\sigma(\mathcal{V}^*, \mathcal{V})$-compact. Thus we obtain a subsequence $\{\gamma_{n_i}\}_{i=1}^\infty$ converging weakly to $\gamma^0 \in \overline{\mathcal{A}}$. The $\sigma(\mathcal{V}^*, \mathcal{V})$-convergence of the subsequence $\{\gamma_{n_i}\}_{i=1}^\infty$ means that for any $V \in \mathcal{V}$

$$\lim_{i \to \infty} E_R[V_{\gamma_{n_i}}] = E_R[V_{\gamma^0}].$$

3. Lemma 2.23 allows us to conclude that for $Q \in \mathcal{Q}$ fixed and any $V \in \mathbb{L}^\infty$

$$\lim_{i \to \infty} E_Q[V_{\gamma_{n_i}}] = E_Q[V_{\gamma^0}]. \tag{2.19}$$

4. In this step, we do not distinguish between a probability measure $Q \in \mathcal{Q}$ and the corresponding density with respect to R. Now we recall the subsequence $\{\gamma_{n_i}\}_{i=1}^\infty$ constructed in the second step and extract a subsequence as follows. Since $\mathcal{Q}$ is $\sigma(L^1(R), L^\infty(R))$-compact, we obtain a subsequence $\{Q_{n_{i_j}}\}_{j=1}^\infty$ converging to a probability measure $Q^0 \in \mathcal{Q}$ in the $\sigma(L^1(R), L^\infty(R))$-topology. This means that for any $Y \in L^\infty(R)$

$$\lim_{j \to \infty} E_{Q_{n_{i_j}}}[Y] = E_{Q^0}[Y]. \tag{2.20}$$

We let $Z^{n_{i_j}}$ denote the density of $Q_{n_{i_j}}$ with respect to R and Z^0 the density of Q^0.

5. To conclude the proof, we show that the pair (Q^0, γ^0) satisfies (2.18). To this end, we only have to verify that for any $V \in \mathbb{L}^\infty$:

$$\lim_{j\to\infty} E_{Q_{n_{i_j}}}[V_{\gamma_{n_{i_j}}}] = E_{Q^0}[V_{\gamma^0}]. \tag{2.21}$$

6. According to corollary V.3.14 [16], there exist a sequence of convex combinations

$$\widetilde{Z}^{n_{i_j}} \in \operatorname{conv}\{Z^{n_{i_j}}, Z^{n_{i_{j+1}}}, \ldots\}$$

such that

$$\lim_{j\to\infty} \widetilde{Z}^{n_{i_j}} = Z^0, \quad \text{in} \quad L^1(R). \tag{2.22}$$

Clearly $\widetilde{Z}^{n_{i_j}}$ is the density of a probability measure $\widetilde{Q}_{n_{i_j}} \in \mathcal{Q}$ since the family $\mathcal{Q}$ is convex. Moreover

$$\lim_{j\to\infty} E_{Q_{n_{i_j}}}[V_{\gamma_{n_{i_j}}}] = \lim_{j\to\infty} E_{\widetilde{Q}_{n_{i_j}}}[V_{\gamma_{n_{i_j}}}]$$

since we have taken convex combinations of a convergent sequence. But now, the identity

$$\begin{aligned} E_{\widetilde{Q}_{n_{i_j}}}[V_{\gamma_{n_{i_j}}}] - E_{Q^0}[V_{\gamma^0}] = {} & E_{\widetilde{Q}_{n_{i_j}}}[V_{\gamma^0}] - E_{Q^0}[V_{\gamma_{n_{i_j}}}] \\ & + E_R[(\widetilde{Z}^{n_{i_j}} - Z^0)(V_{\gamma_{n_{i_j}}} - V_{\gamma^0})] \end{aligned}$$

together with (2.19), (2.20), (2.22) and the boundedness of V, imply that

$$\lim_{j\to\infty} E_{\widetilde{Q}_{n_{i_j}}}[V_{\gamma_{n_{i_j}}}] = E_{Q^0}[V_{\gamma^0}].$$

We have proved the equality (2.21), and this concludes the proof of the proposition.□

We have applied the following lemma in proposition 2.22. In the proof we are going to apply condition (2.14) of the assumption 2.9.

Lemma 2.23 *Let $\{\gamma^i\}_{i=1}^\infty \subset \overline{\mathcal{A}}$ be a sequence converging to $\gamma^0 \in \overline{\mathcal{A}}$ with respect to the weak topology $\sigma(\mathcal{V}^*, \mathcal{V})$. Then, for any $Q \in \mathcal{Q}$ and $V \in \mathbb{L}^\infty$ we have that*

$$\lim_{i\to\infty} E_Q[V_{\gamma^i}] = E_Q[V_{\gamma^0}].$$

Proof. Let us denote by (A^i, B^i) the associated pair of increasing processes to γ^i constructed in theorem 2.12. Let $Z^Q = \{Z_t^Q\}_{0\le t\le T}$ be a càdlàg version of the density process of Q with respect to R. The formula of integration by parts allow us to compute

$$Z_T^Q \int_0^T V_s dA_s^i = \int_0^T \left(Z_s^Q V_s\right) dA_s^i + \int_0^T \left(\int_0^{s-} V_z dA_z^i\right) dZ_s^Q.$$

If we take R-expectation in this formula we get that

$$E_R\left[Z_T^Q \int_0^T V_s dA_s^i\right] = E_R\left[\int_0^T \left(Z_s^Q V_s\right) dA_s^i\right]. \tag{2.23}$$

In a similar way

$$Z_T^Q \int_0^T V_{s-} dB_s^i = \int_0^T \left(Z_s^Q V_{s-}\right) dB_s^i + \int_0^T \left(\int_0^{s-} V_{z-} dB_z^i\right) dZ_s^Q.$$

If we take R-expectation in this formula we get that

$$E_R\left[Z_T^Q \int_0^T V_{s-} dB_s^i\right] = E_R\left[\int_0^T \left(Z_{s-}^Q V_{s-}\right) dB_s^i\right], \tag{2.24}$$

where we have used the fact that the process B^i is predictable. Now from (2.23) and (2.24) we conclude that

$$E_Q[V_{\gamma^i}] = E_R[(Z^Q V)_{\gamma^i}].$$

The process $\{(Z^Q V)_t\}_{0\le t\le T}$ is an element of $\mathcal{V}$ since the density process Z^Q satisfies (2.14) and $V \in \mathbb{L}^\infty$. Then

$$\lim_{i\to\infty} E_Q[V_{\gamma_i}] = \lim_{i\to\infty} E_R[(Z^Q V)_{\gamma_i}] = E_R[(Z^Q V)_{\gamma^0}] = E_Q[V_{\gamma^0}].\square$$

2.2.5 Proof of theorem 2.11

Proof. Recall that the set of processes $\mathbb{V}(c)$ was defined in (2.15). We start with the following equalities

$$\begin{aligned} PH(c) &= \inf_{V\in\mathbb{V}(c)} \sup_{\theta\in\mathcal{T}} \sup_{Q\in\mathcal{Q}} E_Q[f(H_\theta, V_\theta)] \\ &= \inf_{V\in\mathbb{V}(c)} \sup_{Q\in\mathcal{Q}} \sup_{\theta\in\mathcal{T}} E_Q[f(H_\theta, V_\theta)] \\ &= \inf_{V\in\mathbb{V}(c)} \sup_{Q\in\mathcal{Q}} \sup_{\gamma\in\overline{\mathcal{A}}} E_Q[f(H, V)_\gamma]. \end{aligned}$$

The first equality was proved in proposition 2.10. The second equality is trivial. In the last equality we have applied theorem 2.15.

In proposition 2.24 below we prove the existence of a pair $(Q^*, \gamma^*) \in \mathcal{Q}\times\overline{\mathcal{A}}$ such that

$$\inf_{V\in\mathbb{V}(c)} E_{Q^*}[f(H,V)_{\gamma^*}] = \inf_{V\in\mathbb{V}(c)} \sup_{Q\in\mathcal{Q}} \sup_{\gamma\in\overline{\mathcal{A}}} E_Q[f(H,V)_\gamma].$$

This identity implies that

$$PH(c) = \inf_{V\in\mathbb{V}(c)} E_{Q^*}[f(H,V)_{\gamma^*}] \leq \inf_{V\in\mathbb{V}(c)} \sup_{\theta\in\mathcal{T}} E_{Q^*}[f(H_\theta, V_\theta)] \leq PH(c),$$

which proves that Q^* is a worst-case probability measure.□

Proposition 2.24 *Assume the conditions of theorem 2.11. Then, there exist a pair* $(Q^*, \gamma^*) \in \mathcal{Q} \times \overline{\mathcal{A}}$ *such that*

$$\inf_{V\in\mathbb{V}(c)} E_{Q^*}[f(H,V)_{\gamma^*}] = \inf_{V\in\mathbb{V}(c)} \sup_{Q\in\mathcal{Q}} \sup_{\gamma\in\overline{\mathcal{A}}} E_Q[f(H,V)_\gamma]. \tag{2.25}$$

Proof. Let us note that the equality (2.25) can be written as

$$\inf_{V\in\mathbb{V}(c)} q^*[f(H,V)] = \inf_{V\in\mathbb{V}(c)} \sup_{q\in\mathfrak{L}(\mathcal{Q}\times\overline{\mathcal{A}})} q[f(H,V)], \tag{2.26}$$

where $\mathfrak{L}(\mathcal{Q}\times\overline{\mathcal{A}})$ is the set of functionals of definition 2.19 and $q^* \in \mathfrak{L}(\mathcal{Q}\times\overline{\mathcal{A}})$.

We are going to verify the hypotheses of theorem 2.25 below. To this end, let us specify the elements in that theorem. The compact Hausdorff topological space X corresponds to $\mathfrak{L}(\mathcal{Q} \times \overline{\mathcal{A}})$. The topological vector space F corresponds to $\mathbb{L}^\infty$ and the convex subset Y to $\mathbb{V}(c)$. We define a function

$$G : \mathfrak{L}(\mathcal{Q} \times \overline{\mathcal{A}}) \times \mathbb{L}^\infty \to \mathbb{R}$$

by

$$G(q, V) := q[V].$$

Note that if $q \in \mathfrak{L}(\mathcal{Q} \times \overline{\mathcal{A}})$ is represented by a pair $(Q, \gamma) \in \mathcal{Q} \times \overline{\mathcal{A}}$ then

$$G((Q,\gamma), V) = E_Q\left[\int_0^{T-} f(H_s, V_s)dA_s + \int_{0+}^{T} f(H_{-s}, V_{-s})dB_s\right].$$

Now we check the conditions in theorem 2.25. For arbitrary $q \in \mathfrak{L}(\mathcal{Q}\times\overline{\mathcal{A}})$ it will be convenient to work with a representing pair $(Q, \gamma) \in \mathcal{Q} \times \overline{\mathcal{A}}$.

1. The functional G is convex in the variable $V \in \mathbb{V}(c)$ since $f(h,\cdot)$ is convex. Let $(Q^0,\gamma^0) \in \mathcal{Q} \times \overline{\mathcal{A}}$ be a fixed pair, we verify continuity of $G((Q^0,\gamma^0),\cdot)$ with respect to the norm of $\mathbb{L}^\infty$.
   Let $\{V_i\}_{i=1}^{\infty} \subset \mathbb{V}(c)$ be a sequence converging to $V \in \mathbb{V}(c)$ in $\mathbb{L}^\infty$. The random variable $f(H,V_i)_{\gamma^0}$ converges to the random variable $f(H,V)_{\gamma^0}$ R-a.s. since $f(h,\cdot)$ is continuous. We recall that $V \in \mathbb{V}(c)$ satisfies $0 \le V \le H$, and that $H \le K$ for some constant $K > 0$. Thus, we are aloud to apply Lebesgue dominated convergence theorem to conclude that
$$\lim_{i\to\infty} E_{Q^0}[f(H,V_i)_{\gamma^0}] = E_{Q^0}[f(H,V)_{\gamma^0}].$$

2. Now we verify continuity of G in the first argument with respect to the weak topology $\sigma((\mathbb{L}^\infty)^*,\mathbb{L}^\infty)$. Let $\{q^\lambda\}_{\lambda\in\Lambda} \subset \mathfrak{L}(\mathcal{Q} \times \overline{\mathcal{A}})$ be a net converging weakly to $q^0 \in \mathfrak{L}(\mathcal{Q} \times \overline{\mathcal{A}})$. For $V \in \mathbb{V}(c)$ the convergence $G(q^\lambda,V) \to G(q^0,V)$ in the weak topology is immediate because H is uniformly bounded and hence $f(V,H)$ does as well.

3. Now for $l \in \mathbb{R}$ and $V \in \mathbb{V}(c)$ we define
$$L(V) := \{(Q,\gamma) \in \mathcal{Q} \times \overline{\mathcal{A}} \mid G((Q,\gamma),V) \ge l\}.$$
   For $V^1,\cdots,V^n \in \mathbb{V}(c)$ we prove that
$$L := \bigcap_{i=1}^{n} L(V^i)$$
   is either connected or empty. Assume it is nonempty and let $(Q^1,\gamma^1),(Q^2,\gamma^2)$ be two elements in the intersection L. Since G is *linear separately* in Q and γ, we see that for any $\lambda \in (0,1)$, the pair
$$(Q^1, \lambda\gamma^1 + (1-\lambda)\gamma^2)$$
   is an element of L, and so does the pair
$$(\lambda Q^1 + (1-\lambda)Q^2, \gamma^2).$$
   We define a function $q : [0,2] \to \mathcal{Q} \times \overline{\mathcal{A}}$ by
$$q(\lambda) := \begin{cases} (Q^1, (1-\lambda)\gamma^1 + \lambda\gamma^2) & \text{if } \lambda \in [0,1], \\ ((2-\lambda)Q^1 + (\lambda-1)Q^2, \gamma^2) & \text{if } \lambda \in [1,2]. \end{cases}$$
   Note that $q(1)$ is well defined, and that $q(0) = (Q^1,\gamma^1)$ and $q(2) = (Q^2,\gamma^2)$.

In order to conclude that L is connected it is enough to show that q is continuous. That is, we have to show that for any $V \in \mathbb{L}^\infty$, $r > 0$ and $t_0 \in [0, 2]$ then

$$B(t_0, V, r) := \{t \in [0, 2] \mid |q(t)(V) - q(t_0)(V)| < r\}$$

is an open subset of the interval $[0, 2]$. We only verify the case $t_0 = 1$, the other cases being similar. First take $t \geq 1$, then

$$q(t)(V) = (2 - t)E_{Q^1}[V_{\gamma^2}] - (t - 1)E_{Q^2}[V_{\gamma^2}],$$

so that

$$|q(t)(V) - q(t_0)(V)| = (t - 1)\left|E_{Q^1}[V_{\gamma^2}] - E_{Q^2}[V_{\gamma^2}]\right|.$$

Then we see that any $t \in [1, 2]$ satisfying the inequality

$$t < 1 + r\left|E_{Q^1}[V_{\gamma^2}] - E_{Q^2}[V_{\gamma^2}]\right|^{-1}$$

is in $B(t_0, V, r)$.

Now let us take $t \leq 1$, then

$$q(t)(V) = (1 - t)E_{Q^1}[V_{\gamma^1}] + tE_{Q^1}[V_{\gamma^2}],$$

so that

$$|q(t)(V) - q(t_0)(V)| = (1 - t)\left|E_{Q^1}[V_{\gamma^1}] - E_{Q^2}[V_{\gamma^1}]\right|.$$

Then we see that any $t \in [0, 1]$ satisfying the inequality

$$1 - t < r\left|E_{Q^1}[V_{\gamma^1}] - E_{Q^2}[V_{\gamma^1}]\right|^{-1}$$

is in $B(t_0, V, r)$. This shows that $B(t_0, V, r)$ is in fact an open subset of $[0, 2]$.

We have verified all the hypotheses of the topological minimax theorem 2.25. This theorem implies (2.25).

The proof of the proposition is now complete.□

In the proof of theorem 2.24 we have applied the following topological minimax theorem.

Theorem 2.25 *Let X be a compact Hausdorff topological space, and let Y be a nonempty convex subset of a Hausdorff topological vector space F. Suppose that $G : X \times Y \to \mathbb{R}$ is a function satisfying the following conditions*

1. *$G(x, \cdot)$ is lower semicontinuous and convex.*
2. *$G(\cdot, y)$ is upper semicontinuous.*
3. *for $l \in \mathbb{R}$, $m \in \mathbb{N}$ and $y_i \in Y$, the set*

$$\bigcap_{i=1}^{m} \{x \in X \mid G(x, y_i) \geq c\}$$

is either connected or empty.

Then we have

$$\max_{x \in X} \inf_{y \in Y} G(x, y) = \inf_{y \in Y} \max_{x \in X} G(x, y).$$

Proof. See theorem 3.2 in [54].□

2.2.6 Reduction of $PH(c)$

Now we combine theorems 2.5 and 2.11 to reduce the problem $PH(c)$. Let V^{c,ξ^*} be the optimal value process constructed in theorem 2.5. Moreover, let (Q^*, γ^*) be the pair constructed in theorem 2.11. From (2.25) and the optimality of V^{c,ξ^*} we get the following identities

$$PH(c) = \sup_{\gamma \in \overline{\mathcal{A}}} E_{Q^*}[f(H, V^{c,\xi^*})_\gamma] = \inf_{V \in \mathbb{V}} E_{Q^*}[f(H, V)_{\gamma^*}].$$

This means that the pair (V^{c,ξ^*}, γ^*) is a saddle point. That is, for any other pair $(V, \gamma) \in \mathbb{V} \times \overline{\mathcal{A}}$ the following inequality holds

$$E_{Q^*}[f(H, V^{c,\xi^*})_\gamma] \leq E_{Q^*}[f(H, V^{c,\xi^*})_{\gamma^*}] \leq E_{Q^*}[f(H, V)_{\gamma^*}]. \tag{2.27}$$

Moreover, the quasi-randomized time γ^* simplifies our original problem $PH(c)$ to the problem

$$PH(c, Q^*, \gamma^*) := \inf_{V \in \mathbb{V}} E_{Q^*}[f(V, H)_{\gamma^*}], \tag{2.28}$$

in the sense that in order to find a solution to $PH(c)$ we can search among the solutions of $PH(c, Q^*, \gamma^*)$ since V^{c,ξ^*} is itself a solution for $PH(c, Q^*, \gamma^*)$.

Now we show that the optimal value process V^{c,ξ^*} and the worst-case pair (Q^*, γ^*) are interconnected. We will assume that the processes X and H are continuous. Continuity of these processes implies that we can identify the quasi-randomized stopping time γ^* with an element $\kappa^* \in \mathcal{A}$, that is, we can simplify from a quasi-randomized to a randomized time. Let $\{c_s\}_{0\leq s\leq 1}$ be the inverse process of κ^* defined by

$$c_s := \inf\{0 \leq t \leq T \mid \kappa_t^* \geq s\}.$$

Then $R(c_s \leq T) = 1$ since $\kappa_T^* = 1$.

Proposition 2.26 *Assume that the processes X and H are continuous. Let λ denote the Lebesgue measure in $[0,1]$. Then, for λ-almost all $s \in [0,1]$, c_s is an optimal stopping time with respect to Q^* for the process $\{f(H_t, V_t^{c,\xi^*})\}_{0\leq t\leq T}$.*

Proof. In the left inequality of (2.27) we specialize γ to be a stopping time $\tau \in \mathcal{T}$, we then get

$$\begin{aligned} E_{Q^*}[f(H_\tau, V_\tau^{c,\xi^*})] &\leq E_{Q^*}\left[\int_0^T f(H, V^{c,\xi^*})d\kappa^*\right] \\ &= E_{Q^*}\left[\int_0^1 f(H_{c_s}, V_{c_s}^{c,\xi^*})ds\right] \\ &= \int_0^1 E_{Q^*}\left[f(H_{c_s}, V_{c_s}^{c,\xi^*})\right] ds, \end{aligned}$$

where in the second identity we have applied a change of variable and on the third identity we have applied Fubini's theorem. This inequality implies that

$$\sup_{\theta\in\mathcal{T}} E_{Q^*}[f(H_\theta, V_\theta^{c,\xi^*})] \leq \int_0^1 E_{Q^*}\left[f(H_{c_s}, V_{c_s}^{c,\xi^*})\right] ds,$$

and then, for λ-almost all $s \in [0,1]$

$$\sup_{\theta\in\mathcal{T}} E_{Q^*}[f(H_\theta, V_\theta^{c,\xi^*})] = E_{Q^*}\left[f(H_{c_s}, V_{c_s}^{c,\xi^*})\right].\square$$

Chapter 3

An upper bound for Quantile Hedging

In this chapter we specialize the problem of partial hedging 2.2 to the non-robust case $\mathcal{Q} = \{R\}$ and to the function

$$f(h, v) := \left(1 - \frac{v}{h}\right)^+. \tag{3.1}$$

First we explain why this specification of the function f corresponds to a quantile hedging problem for American options. We then consider an upper bound for the resulting value function and obtain a dual representation formula.

3.1 Problem formulation

Throughout this chapter we fix a stochastic base $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}, R)$ satisfying the usual conditions of right continuity and completeness. Moreover, we fix a positive càdlàg $\mathbb{F}$-adapted stochastic process $H := \{H_t\}_{0\le t\le T}$ which represents an American option.

In definition 2.2 of chapter 2 we introduced a general robust partial hedging problem for American options and we then explained how the special case with a function of the form $l((h - v)^+)$ corresponds to robust efficient hedging for American options, extending efficient hedging from European to American options. In this chapter we specialize to the function (3.1) and explain how it corresponds to quantile hedging for American options. We will obtain an optimization problem with value function given by

$$QH(c) := \sup_{\xi \in Ad_c} \inf_{\theta \in \mathcal{T}} E_R \left[1_{\{V_\theta^{c,\xi} \geq H_\theta\}} + \frac{V_\theta^{c,\xi}}{H_\theta} 1_{\{V_\theta^{c,\xi} < H_\theta\}} \right].$$

The goal in this chapter is to show that the upper bound

$$QH^+(c) := \inf_{\theta \in \mathcal{T}} \sup_{\xi \in Ad_c} E_R \left[1_{\{V_\theta^{c,\xi} \geq H_\theta\}} + \frac{V_\theta^{c,\xi}}{H_\theta} 1_{\{V_\theta^{c,\xi} < H_\theta\}} \right],$$

admits the dual representation of the next theorem.

Theorem 3.1 *The upper bound $QH^+(c)$ admits the dual representation*

$$QH^+(c) = \inf_{\lambda > 0} \inf_{P \in \mathcal{M}} \inf_{\theta \in \mathcal{T}} \left\{ E_R[(1 - \lambda Z_\theta^P H_\theta)^+] + \lambda c \right\},$$

where Z^P denotes a càdlàg version of the density process of the probability measure $P \in \mathcal{M}$ with respect to R.

Proof. See corollary 3.14.□

Let us explain the approach we are going to take in the proof of theorem 3.1. In a first step we reformulate the problem of quantile hedging in terms of randomized test processes as in definition 3.6. In a second step, in lemma 3.9 the optimization problem is reduced from processes to random variables. And in the last step we apply a criterion of optimality from convex analysis.

3.2 Solution

3.2.1 Quantile Hedging

In this subsection we explain how the partial hedging problem of definition 2.2 when specialized to the above setup corresponds to quantile hedging for American options extending the analysis of Föllmer and Leukert[25] from European to American options. We start with two definitions.

Definition 3.2 *A randomized test process ϕ is a càdlàg $\mathbb{F}$-adapted process taking values in* $[0, 1]$. *We denote by $\mathcal{R}$ the family of randomized test processes.*□

This definition is a process-version of the randomized tests used in [25]. Note that we require regularity of the trajectories.

Definition 3.3 *For a c-admissible strategy $\xi \in Ad_c$ the success ratio process associated to ξ is defined by*

$$\phi^{\xi} := \left\{ 1_{\{V^{c,\xi} \geq H\}} + \frac{V^{c,\xi}}{H} 1_{\{V^{c,\xi} < H\}} \right\}.$$

The value function of the quantile hedging problem is defined by

$$QH(c) = \sup_{\xi \in Ad_c} \inf_{\theta \in \mathcal{T}} E_R[\phi^{\xi}_{\theta}].$$

We say that $\xi^* \in Ad_c$ has maximal success ratio process if it attains the value $QH(c)$, that is, for any $\xi \in Ad_c$ the following inequality holds

$$\inf_{\theta \in \mathcal{T}} E_R[\phi^{\xi^*}_{\theta}] \geq \inf_{\theta \in \mathcal{T}} E_R[\phi^{\xi}_{\theta}].\Box \tag{3.2}$$

Remark 3.4 *The success ratio process of an admissible strategy $\xi \in Ad_c$ is the process version of the success ratio introduced in [25] with the form*

$$\left\{ 1_{\{V^{c,\xi}_T \geq H_T\}} + \frac{V^{c,\xi}_T}{H_T} 1_{\{V^{c,\xi}_T < H_T\}} \right\}.$$

Note that the success ratio process can equivalently be written as

$$\phi^{\xi} = 1 - \left(1 - \frac{V^{c,\xi}}{H}\right)^{+},$$

and the value function $QH(c)$ is equal to

$$QH(c) = 1 - \inf_{\xi \in Ad_c} \sup_{\theta \in \mathcal{T}} E_R\left[\left(1 - \frac{V^{c,\xi}_{\theta}}{H_{\theta}}\right)^{+}\right]. \tag{3.3}$$

Moreover, a strategy $\xi^* \in Ad_c$ has maximal success ratio process if

$$\sup_{\theta \in \mathcal{T}} E_R\left[\left(1 - \frac{V^{c,\xi^*}_{\theta}}{H_{\theta}}\right)^{+}\right] \leq \sup_{\theta \in \mathcal{T}} E_R\left[\left(1 - \frac{V^{c,\xi}_{\theta}}{H_{\theta}}\right)^{+}\right].\diamondsuit \tag{3.4}$$

The value function $QH(c)$ in the form (3.3) will allow us to apply theorem 2.5 to obtain a strategy ξ^* with maximal success ratio.

Theorem 3.5 *There exists a strategy $\xi^* \in Ad_c$ with maximal success ratio process.*

Proof. The optimization problem

$$\inf_{\xi\in Ad_c}\sup_{\theta\in\mathcal{T}} E_R\left[\left(1-\frac{V_\theta^{c,\xi}}{H_\theta}\right)^+\right]$$

is a special case of the robust partial hedging problem 2.2 with $\mathcal{Q}=\{R\}$ and generalized loss function $f(h,v)=(1-\frac{v}{h})^+$. The integrability condition of theorem 2.5 is trivially satisfied. Then, there exists $\xi^*\in Ad_c$ such that

$$\sup_{\theta\in\mathcal{T}} E_R\left[\left(1-\frac{V_\theta^{c,\xi^*}}{H_\theta}\right)^+\right]\le \sup_{\theta\in\mathcal{T}} E_R\left[\left(1-\frac{V_\theta^{c,\xi}}{H_\theta}\right)^+\right],$$

for any $\xi\in Ad_c$, which implies that ξ^* is a strategy with maximal success ratio process.□

We conclude this section with an equivalent formulation of problem 3.3 which will be applied in the subsection 3.2.2 below.

Definition 3.6 *Let $\mathcal{R}_c$ be the family of elements $\phi\in\mathcal{R}$ satisfying the budget constraint*

$$\sup_{P\in\mathcal{M}}\sup_{\theta\in\mathcal{T}} E_P[\phi_\theta H_\theta]\le c. \tag{3.5}$$

The value function of the optimal testing problem is defined by

$$T(c):=\sup_{\phi\in\mathcal{R}_c}\inf_{\theta\in\mathcal{T}} E_R[\phi_\theta]. \tag{3.6}$$

We say that $\phi^*\in\mathcal{R}_c$ is an optimal randomized test process if it attains the value $T(c)$, that is

$$\inf_{\theta\in\mathcal{T}} E_R[\phi^*_\theta]=T(c).\square \tag{3.7}$$

Proposition 3.7 *The value function of the quantile hedging problem $QH(c)$ and the value function of the testing problem $T(c)$ are equal: $T(c)=QH(c)$.*

Proof. First note that for any $\xi\in Ad_c$, the success ratio process ϕ^ξ is an element in $\mathcal{R}_c$. In fact:

$$H\phi^\xi = H1_{\{V^{c,\xi}\ge H\}}+H\frac{V^{c,\xi}}{H}1_{\{V^{c,\xi}<H\}}\le V^{c,\xi}.$$

Then

$$\inf_{\theta\in\mathcal{T}} E_R[\phi^\xi_\theta]\le T(c).$$

Since ξ was arbitrary we conclude the inequality $QH(c) \leq T(c)$.

Now we prove the converse inequality. Let $\phi \in \mathcal{R}_c$ be an admissible randomized test process, and let $U^{\uparrow}$ be the upper Snell envelope of the modified process $\widetilde{H} := \phi H$. The optional decomposition theorem 1.15 allow us to represent $U^{\uparrow}$ as

$$U_t^{\uparrow} = U_0^{\uparrow} + \int_0^t \xi_s dX_s - C_t,$$

where $\{C_t\}_{0\leq t\leq T}$ is an optional increasing process with $C_0 = 0$ and $\xi \in Ad_{U_0^{\uparrow}}$. It is clear that $U_0^{\uparrow} \leq c$ and $\xi \in Ad_c$.

The success ratio process of the strategy ξ satisfies $\phi^{\xi} \in \mathcal{R}_c$. It is clear that on the set $\{V^{c,\xi} \geq H\}$ we have the inequality $\phi^{\xi} \geq \phi$, since $\phi^{\xi} = 0$. Moreover, we have the inclusion $\{\phi = 1\} \cup \{\phi = 0\} \subset \{V^{c,\xi} \geq H\}$. Now, on the set $\{\phi \in (0,1)\} \cap \{H > V^{c,\xi}\}$ we have the equality $\phi^{\xi} = \frac{V^{c,\xi}}{H}$ and it follows that $H\phi^{\xi} \geq H\phi$. We conclude that $\phi^{\xi} \geq \phi$. Thus

$$QH(c) \geq \inf_{\theta\in\mathcal{T}} E_R[\phi_\theta^{\xi}] \geq \inf_{\theta\in\mathcal{T}} E_R[\phi_\theta]. \tag{3.8}$$

Since ϕ was arbitrary we conclude the converse inequality $QH(c) \geq T(c)$.□

3.2.2 The upper values $QH^+(c)$ and $T^+(c)$

Let us introduce the upper values

$$T^+(c) := \inf_{\theta\in\mathcal{T}} \sup_{\phi\in\mathcal{R}_c} E_R[\phi_\theta] \tag{3.9}$$

$$QH^+(c) := \inf_{\theta\in\mathcal{T}} \sup_{\xi\in Ad_c} E_R\left[1_{\{V_\theta^{c,\xi}\geq H_\theta\}} + \frac{V_\theta^{c,\xi}}{H_\theta} 1_{\{V_\theta^{c,\xi}<H_\theta\}}\right]. \tag{3.10}$$

We clearly have that $T^+(c) \geq T(c)$ and $QH^+(c) \geq QH(c)$. We proved in proposition 3.7 that $T(c) = QH(c)$. The upper values $T^+(c)$ and $QH^+(c)$ are related in the same way:

Proposition 3.8 *The value function $T^+(c)$ and the value function $QH^+(c)$ are equal: $T^+(c) = QH^+(c)$.*

Proof. The proof is similar to the one in proposition 3.7.□

The goal in this section is to show that the upper bound $T^+(c)$ admits the following representation

$$T^+(c) = \inf_{\lambda>0} \inf_{P\in\mathcal{M}} \inf_{\theta\in\mathcal{T}} \left\{ E_R[(1-\lambda Z_\theta^P H_\theta)^+] + \lambda c \right\}, \tag{3.11}$$

where Z^P denotes a càdlàg version of the density process of the probability measure $P \in \mathcal{M}$ with respect to R.

Let $\tau \in \mathcal{T}$ be a fixed stopping time. We introduce the following set of random variables

$$\mathcal{R}_c^\tau := \{\psi : \Omega \to [0,1] \mid \psi \text{ is } \mathcal{F}_\tau - \text{measurable and } \sup_{P\in\mathcal{M}} E_P[\psi H_\tau] \le c\}. \tag{3.12}$$

Moreover, we define the values

$$\widetilde{T}^+(\tau, c) := \sup_{\psi\in\mathcal{R}_c^\tau} E_R[\psi],$$
$$T^+(\tau, c) := \sup_{\phi\in\mathcal{R}_c} E_R[\phi_\tau].$$

Note that $T^+(c) = \inf_{\theta\in\mathcal{T}} T^+(\theta, c)$.

Lemma 3.9 *The value $T^+(\tau, c)$ can be computed as*

$$T^+(\tau, c) = \widetilde{T}^+(\tau, c).$$

Proof. Let us prove the inequality $T^+(\tau, c) \le \widetilde{T}^+(\tau, c)$. For $\phi \in \mathcal{R}_c$ we define $\psi := \phi_\tau$. It is clear that $\psi \in \mathcal{R}_c^\tau$ and $E_R[\psi] = E_R[\phi_\tau]$. This proves the desired inequality.

In order to prove the opposite inequality, let $\psi \in \mathcal{R}_c^\tau$. Without loss of generality we can assume that

$$\sup_{P\in\mathcal{M}} E_P[\psi H_\tau] \le c' < c, \tag{3.13}$$

otherwise, for $0 < \delta < 1$ we can consider the modified randomized test $\delta\psi$ and then let $\delta \nearrow 1$. For $\epsilon > 0$ let us define the following objects

$$\tau^\epsilon := \inf\{s > \tau \mid H_s \ge H_\tau + \epsilon\} \wedge T,$$
$$B^\epsilon := \frac{1}{1+\epsilon} 1_{\{H_\tau=0\}} + \frac{1}{1+\epsilon(H_\tau)^{-1}} 1_{\{H_\tau>0\}},$$
$$\phi_t^\epsilon := \begin{cases} \psi B^\epsilon, & \text{if } \tau \le t < \tau^\epsilon, \text{ or } t = \tau = T, \\ 0, & \text{otherwise.} \end{cases}$$

The random variable τ^ϵ is a stopping time, B^ϵ is a $\mathcal{F}_\tau$-measurable random variable, and ϕ^ϵ is a càdlàg $\mathbb{F}$-adapted process taking values in $[0,1]$.

From the definition it follows that $\lim_{\epsilon\to 0}\phi^\epsilon_\tau = \psi$, and Lebesgue's dominated convergence implies that $E_R[\phi^\epsilon_\tau] \to E_R[\psi]$. It remains to show that $\phi^\epsilon \in \mathcal{R}_c$.

Let $\theta \in \mathcal{T}$ be a stopping time, and let us set

$$A := \{\tau \le \theta < \tau^\epsilon\} \cup \{\theta = \tau = T\}.$$

Notice that $\phi^\epsilon_\theta H_\theta 1_{A^c} = 0$. The following relationships hold

$$H_\theta \phi^\epsilon_\theta = H_\theta \psi B^\epsilon 1_A = \psi \frac{H_\theta}{1+\epsilon} 1_{\{H_\tau = 0\}} 1_A + \psi H_\tau \frac{H_\theta}{H_\tau + \epsilon} 1_{\{H_\tau > 0\}} 1_A,$$

and we conclude that for any $P \in \mathcal{M}$

$$E_P[\phi^\epsilon_\theta H_\theta] \le \epsilon + E_P[\psi H_\tau],$$

so that for sufficiently small ϵ, the equation (3.13) implies that $\phi^\epsilon \in \mathcal{R}_c$. The inequality $\ge$ is now established.□

This lemma reduces the problem of computing $T^+(c)$ from processes to random variables since now we have

$$T^+(c) = \inf_{\theta\in\mathcal{T}} \widetilde{T}^+(\theta, c).$$

This reduction will be crucial in the next proposition. We use the notation

$$V(P,\tau,c) := \inf_{\lambda>0} \left\{ E_R[(1-\lambda Z^P_\tau H_\tau)^+] + \lambda c \right\},$$

for $\tau \in \mathcal{T}$ and $P \in \mathcal{M}$.

Proposition 3.10 *The inequality $\le$ in (3.11) holds. That is:*

$$T^+(c) \le \inf_{\lambda>0} \inf_{P\in\mathcal{M}} \inf_{\theta\in\mathcal{T}} \left\{ E_R[(1-\lambda Z^P_\theta H_\theta)^+] + \lambda c \right\}.$$

Proof. According to lemma 3.9 we have

$$T^+(c) = \inf_{\theta\in\mathcal{T}} \widetilde{T}^+(\theta, c).$$

To prove the proposition it suffices to show that

$$\widetilde{T}^+(\theta, c) \le \inf_{\lambda>0} \inf_{P\in\mathcal{M}} V(P,\theta,c), \tag{3.14}$$

for $\theta \in \mathcal{T}$ fixed.

Let $\psi \in \mathcal{R}_c^\theta$ be arbitrary. Then the following holds

$$E_R[\psi] = E_R[\psi - \lambda Z_\theta^P H_\theta \psi] + \lambda E_P[H_\theta \psi] \leq E_R[(1 - \lambda Z_\theta^P H_\theta)^+] + \lambda c, \quad (3.15)$$

where Z^P is a càdlàg version of the density process with respect to R of the equivalent martingale measure $P \in \mathcal{M}$. If we take supremum over $\psi \in \mathcal{R}_c^\theta$ in (3.15), we conclude that

$$\widetilde{T}^+(\theta, c) \leq E_R[(1 - \lambda Z_\theta^P H_\theta)^+] + \lambda c.$$

If we take infimum over $P \in \mathcal{M}$ and $\lambda > 0$, then we conclude the inequality (3.14). The proof of the proposition is complete. □

Now we prove the converse inequality. We need a result from convex analysis.

Theorem 3.11 *Let $\mathbb{X}$ be a Banach space and let $f : \mathbb{X} \to \mathbb{R}$ be a convex function. Let $C \subset \mathbb{X}$ be a closed convex set. Let $x^* \in C$. The normal cone of C in x^* and the subdifferential of f in x^* are defined by*

$$N_C(x^*) := \{l \in \mathbb{X}^* \mid l(y - x^*) \leq 0, \forall y \in C\},$$
$$\partial f(x^*) := \{l \in \mathbb{X}^* \mid f(y) - f(x^*) \geq l(y - x^*), \forall y \in \mathbb{X}\}.$$

Then, $x^* \in C$ is a minimum of f in C if and only if

$$0 \in \partial f(x^*) + N_C(x^*). \quad (3.16)$$

Proof. See e.g., chapter 4 in Aubin and Ekeland[2].□

Theorem 3.12 *For any stopping time $\tau \in \mathcal{T}$ we have the equality*

$$\widetilde{T}^+(\tau, c) = \inf_{P \in \mathcal{M}} V(P, \tau, c). \quad (3.17)$$

Proof. The inequality $\leq$ in (3.17) follows from (3.15).

Now we prove the converse inequality. Let us introduce the space

$$\mathbb{X} := \mathbb{R} \times L^1(R, \mathcal{F}_\tau).$$

We will use the notation

$$x := (\lambda, Z) \in \mathbb{X}.$$

This is a Banach space if endowed with the norm

$$\|x\| := |\lambda| + E_R[|Z|].$$

On this space we define a function $f : \mathbb{X} \to \mathbb{R}$ by

$$f(x) = f(x,\tau) := E_R[(1 - ZH_\tau)^+] + \lambda c.$$

Through Lebesgue dominated convergence theorem we can see that f is continuous with respect to the norm of $\mathbb{X}$ and is a convex function in the following sense. For any $\alpha \in (0,1)$

$$f(\alpha\lambda^1 + (1-\alpha)\lambda^2, \alpha Z^1 + (1-\alpha)Z^2)) \leq \alpha f(\lambda^1, Z^1) + (1-\alpha) f(\lambda^2, Z^2).$$

Let us define the sets

$$C^0 := \{(\lambda, \lambda Z_\tau^P) \in \mathbb{X} \mid \lambda \geq 0, P \in \mathcal{M}\}, \tag{3.18}$$
$$C := \{(\lambda, \lambda Z) \in \mathbb{X} \mid \lambda \geq 0, Z \in \mathcal{M}_\tau\}, \tag{3.19}$$

where

$$\mathcal{M}_\tau := \{Z \in L^1(\mathcal{F}_\tau) \mid \exists \{P^i\}_{i=1}^\infty \subset \mathcal{M}, Z_\tau^{P^i} \to Z, R - a.s.\}. \tag{3.20}$$

Note that $\mathcal{M}_\tau$ is well defined since

$$E_R[Z] = E_R[\liminf_{i\to\infty} Z_\tau^{P^i}] \leq \liminf_{i\to\infty} E_R[Z_\tau^{P^i}] \leq 1,$$

due to Fatou's lemma. Let us show that $\mathcal{M}_\tau$ is closed with respect to pointwise convergence. Let $\{Z^i\}_{i=1}^\infty \subset \mathcal{M}_\tau$ be a sequence converging to $Z \in L^1$. We want to show that there exists a sequence of probability measures $\{P^i\}_{i=1}^\infty \subset \mathcal{M}$ such that $Z_\tau^{P^i} \to Z$ pointwise. Let $\{P^{i,j}\}_{j=1}^\infty \subset \mathcal{M}$ be a sequence of probability measures such that $Z_\tau^{P^{i,j}} \to Z^i$. According to Egoroff's theorem on almost uniform convergence, there exists a measurable set Ω^i with $R(\Omega^i) \geq 1 - \frac{1}{2^i}$ and

$$\left| Z_\tau^{P^{i,j}}(\omega) - Z^i(\omega) \right| \leq \frac{1}{2^i},$$

for $\omega \in \Omega^i$ and $j \geq j(i) \in \mathbb{N}$. We only have to show that

$$\lim_{i\to\infty} \left| Z_\tau^{P^{i,j(i)}} - Z^i \right| = 0 \quad R - a.s. \tag{3.21}$$

to conclude that

$$\lim_{i\to\infty} \left| Z_\tau^{P^{i,j(i)}} - Z \right| = 0 \quad R - a.s.$$

since

$$\left|Z_\tau^{P^{i,j(i)}} - Z\right| \leq \left|Z_\tau^{P^{i,j(i)}} - Z^i\right| + \left|Z^i - Z\right|.$$

But

$$\left|Z_\tau^{P^{i,j(i)}}(\omega) - Z^i(\omega)\right| \leq \frac{1}{2^i}$$

for $\omega \in \bigcap_{k=i}^\infty \Omega^k$. Note that

$$R\left(\bigcup_{k=i}^\infty (\Omega^k)^c\right) \leq \sum_{k=i}^\infty R((\Omega^k)^c) \leq \sum_{k=i}^\infty \frac{1}{2^k} = \frac{1}{2^{i-1}}$$

which allow us to conclude (3.21) by an application of the Borel-Cantelli lemma.

Note that

$$\inf_{P\in\mathcal{M}} V(P,\tau,c) = \inf_{x\in C^0} f(x) = \inf_{x\in C} f(x),$$

where the first equality follows from the definitions of $V(P,\tau,c)$ and f. The last equality follows due to Lebesgue dominated convergence theorem. We are going to show that the problem

$$\inf_{x\in C} f(x)$$

has a minimum $x^* \in C$, and $f(x^*) \leq \widetilde{T}^+(\tau,c)$. This will establish the equality (3.17).

1. We show that f has a minimum in C. Since we know that $\inf_{x\in C} f(x) \geq 0$ there exists a minimizing sequence $x^i = (\lambda^i, \lambda^i Z^i) \in C^0$ so that

   $$f(x^i) \searrow \inf_{x\in C} f(x).$$

   The sequence $\{\lambda^i\}_{i\in\mathbb{N}}$ must be bounded. By passing to a subsequence if necessary, we can assume that the sequence converges to some $\lambda^* \geq 0$. Moreover, we can select this sequence in such a way that

   $$\sum_{i=1}^\infty \left|\lambda^{i+1} - \lambda^i\right| < \infty. \tag{3.22}$$

   Due to Komlós' principle of convergence [40], there exists a sequence of convex combinations

   $$\widetilde{Z}^i \in \operatorname{conv}\left\{Z^i, Z^{i+1}, \ldots\right\},$$

and $Z^* \in \mathcal{M}_\tau$ such that $Z^n \to Z^*$ R-a.s.

We get that

$$f(\lambda^*, \lambda^* Z^*) = \lim_{i\to\infty} f(\lambda^i, \lambda^i \widetilde{Z}^i),$$

due to Lebesgue dominated convergence theorem.

The convexity of f together with (3.22) imply that the sequence $(\lambda^i, \lambda^i \widetilde{Z^i})$ is also minimizing. It follows that $x^* = (\lambda^*, \lambda^* Z^*) \in C$ is a minimum of the function f in C.

2. Now we apply the optimality criterion theorem 3.11 to $x^* = (\lambda^*, \lambda^* Z^*)$. According to (3.16) in theorem 3.11 we have that

$$0 \in \partial f(x^*) + N_C(x^*).$$

Thus, there exists $l \in \mathbb{X}^*$ such that

$$\begin{aligned} f(y) - f(x^*) - l(y - x^*) &\geq 0, \forall y \in \mathbb{X}, \\ l(y - x^*) &\geq 0, \forall y \in C. \end{aligned}$$

The continuous linear functional l acts in the following form

$$l(y) = a\lambda + E_R[bZ], \quad for \quad y = (\lambda, Z) \in \mathbb{X},$$

where $a \in \mathbb{R}$ and $b \in L^\infty(R, \mathcal{F}_\tau)$. We now write with more detail the optimality conditions

$$E_R[(1 - ZH_\tau)^+ - (1 - \lambda^* Z^* H_\tau)^+ - b(Z - \lambda^* Z^*)] \\ + (\lambda - \lambda^*)(c - a) \geq 0, \tag{3.23}$$

$$a(\lambda - \lambda^*) + E_R[b(\lambda Z - \lambda^* Z^*)] \geq 0. \tag{3.24}$$

The inequality (3.23) holds for any $y \in \mathbb{X}$, and (3.24) holds for any $y \in C$. In (3.23) the expectation is finite and this implies that $a = c$. In (3.24) $\lambda = \lambda^*$ yields

$$E_R[bZ] \geq E_R[bZ^*].$$

On the other hand, setting $Z = Z^*$ results in

$$(c + E_R[bZ^*])(\lambda - \lambda^*) \geq 0.$$

If $\lambda^* = 0$ we are in a trivial case, so we can assume $\lambda^* > 0$. We can take $\lambda = \lambda^* \pm \frac{1}{2}\lambda^*$ to conclude that

$$c = -E_R[bZ^*].$$

We have proved that b is a random variable, $\mathcal{F}_\tau$-measurable, and with

$$E_R[bZ] \geq E_R[bZ^*] = -c. \tag{3.25}$$

3. We show that $-H_\tau \leq b \leq 0$. If we set $Z^m := \lambda^* Z^* + m1_{\{b>0\}}$ for $m \in \mathbb{N}$, then (3.23) implies

$$0 \leq E_R[(1 - Z^m H_\tau)^+ - (1 - \lambda^* Z^* H_\tau)^+ - bm1_{\{b>0\}}] \leq 2 - mE_R[b1_{\{b>0\}}],$$

if we let $m \nearrow \infty$, then we conclude that $R(b > 0) = 0$.

Now we show that $b \geq -H_\tau$. In fact, let us define

$$Z^{k,N} := \lambda^* Z^* + k\frac{b}{H_\tau}1_{\{-H_\tau - N \leq b < -H_\tau\}},$$

where $\frac{b}{0} := -1$. Then (3.23) implies

$$\begin{aligned} 0 &\leq E_R[(1 - Z^{k,N} H_\tau)^+ - (1 - \lambda^* Z^* H_\tau)^+ - kb\frac{b}{H_\tau}1_{\{-H_\tau - N \leq b < -H_\tau\}}] \\ &\leq 2 - kE_R[b\frac{b}{H_\tau}1_{\{-H_\tau - N \leq b < -H_\tau\}}]. \end{aligned}$$

If we let $k \nearrow \infty$, then we conclude that $R(\{-H_\tau - N \leq b < -H_\tau\}) = 0$. The statement is proved. Note that $H_\tau = 0 \Rightarrow b = 0$.

4. We show that $b = -H_\tau$ if $1 - \lambda^* Z^* H_\tau > 0$. If we set $Z^m := \lambda^* Z^* - m1_{\{1-\lambda^* Z^* H_\tau > 0\}}$ for $m \in \mathbb{N}$, then (3.23) implies that

$$\begin{aligned} 0 &\leq E_R[(1 - Z^m H_\tau)^+ - (1 - \lambda^* Z^* H_\tau)^+ + mb1_{\{1-\lambda^* Z^* H_\tau > 0\}}] \\ &= E_R[m1_{\{1-\lambda^* Z^* H_\tau > 0\}}(b + H_\tau)]. \end{aligned}$$

If we let $m \searrow -\infty$, then we conclude that $b = -H_\tau$ in the event $\{1 - \lambda^* Z^* H_\tau > 0\}$ as claimed before.

5. $b = 0$ if $1 - \lambda^* Z^* H_\tau < 0$. In fact, from the previous steps we can write

$$b = -H_\tau 1_{\{1-\lambda^* Z^* H_\tau > 0\}} - H_\tau \widetilde{b} 1_{\{1-\lambda^* Z^* H_\tau \leq 0\}},$$

and we want to show that $\widetilde{b} = 0$ in $\{1 - \lambda^* Z^* H_\tau < 0\}$. Let $\delta > 0$ and let us define

$$Z^\delta := \lambda^* Z^* 1_{\{1-\lambda^* Z^* H_\tau > 0\}} + \frac{1}{H_\tau + \delta} 1_{\{1-\lambda^* Z^* H_\tau \leq 0\}},$$

then (3.23) reads

$$\begin{aligned} 0 &\leq E_R[(1 - Z^\delta H_\tau)^+ - (1 - \lambda^* Z^* H_\tau)^+ - b(Z^\delta - \lambda^* Z^*)] \\ &= E_R[(1 - Z^\delta H_\tau) 1_{\{1-Z^\delta H_\tau > 0\}} - (1 - \lambda^* Z^* H_\tau) 1_{\{1-\lambda^* Z^* H_\tau > 0\}} \\ &+ H_\tau (Z^\delta - \lambda^* Z^*) 1_{\{1-\lambda^* Z^* H_\tau > 0\}} + \widetilde{b} H_\tau (Z^\delta - \lambda^* Z^*) 1_{\{1-\lambda^* Z^* H_\tau \leq 0\}}] \\ &= E_R[(1 - Z^\delta H_\tau)(1_{\{1-Z^\delta H_\tau > 0\}} - 1_{\{1-\lambda^* Z^* H_\tau > 0\}}) \\ &+ \widetilde{b} H_\tau (Z^\delta - \lambda^* Z^*) 1_{\{1-\lambda^* Z^* H_\tau \leq 0\}}]. \end{aligned}$$

The first term on the last equality reduces to zero because $\{1 - Z^\delta H_\tau > 0\} = \{1 - \lambda^* Z^* H_\tau > 0\}$ and we arrive to the following inequality

$$0 \leq E_R[\widetilde{b} H_\tau (Z^\delta - \lambda^* Z^*) 1_{\{1-\lambda^* Z^* H_\tau \leq 0\}}] = E_R[\widetilde{b}(\frac{H_\tau}{H_\tau + \delta} - \lambda^* Z^* H_\tau) 1_{\{1-\lambda^* Z^* H_\tau \leq 0\}}].$$

We can let $\delta \searrow 0$ and apply monotone convergence to conclude that

$$0 \leq E_R[\widetilde{b}(1 - \lambda^* Z^* H_\tau) 1_{\{1-\lambda^* Z^* H_\tau \leq 0\}}],$$

this last inequality allow us to conclude that $\widetilde{b} = 0$ in the event $\{1 - \lambda^* Z^* H_\tau < 0\}$ as desired.

6. Now let us define the randomized test

$$\psi^* = \begin{cases} 0 & \text{if } 1 - \lambda^* Z^* H_\tau < 0 \\ 1 & \text{if } 1 - \lambda^* Z^* H_\tau > 0 \\ \frac{-b}{H_\tau} & \text{if } 1 - \lambda^* Z^* H_\tau = 0. \end{cases}$$

It is clear that ψ^* is $\mathcal{F}_\tau$-measurable, and step number four implies that $\psi^* H_\tau = -b$. In particular for $P \in \mathcal{M}$, (3.25) reads

$$E_P[H_\tau \psi^*] \leq E_R[Z^* H_\tau \psi^*] = c. \tag{3.26}$$

This means that $\psi^* \in \mathcal{R}_c^\tau$. On the other hand, we get

$$\begin{aligned} \widetilde{T}^+(\tau, c) \geq E_R[\psi^*] &= E_R[\psi^*(1 - \lambda^* Z^* H_\tau)] + \lambda^* E_R[Z^* \psi^* H_\tau] \\ &= E_R[(1 - \lambda^* Z^* H_\tau)^+] + \lambda^* c \\ &= \inf_{x \in C} f(x) \\ &= \inf_{x \in C^0} f(x) \\ &= \inf_{P \in \mathcal{M}} V(P, \tau, c). \end{aligned} \tag{3.27}$$

We have proved (3.17).□

Remark 3.13 *The optimality conditions of convex analysis applied in the proof of theorem 3.12 is motivated by theorem 4.1 in Cvitaníc and Karatzas[5].*◇

A corollary of this theorem is formula (3.17).

Corollary 3.14 *With the notation of theorem 3.12, formula (3.17) holds:*

$$T^{+}(c) = \inf_{\lambda \geq 0} \inf_{P \in \mathcal{M}} \inf_{\theta \in \mathcal{T}} \left\{ E_R[(1 - \lambda Z^P_\theta H_\theta)^+] + \lambda c \right\}.$$

Equivalently, in terms of $QH^+(c)$ we have

$$QH^{+}(c) = \inf_{\lambda \geq 0} \inf_{P \in \mathcal{M}} \inf_{\theta \in \mathcal{T}} \left\{ E_R[(1 - \lambda Z^P_\theta H_\theta)^+] + \lambda c \right\}.$$

Proof. The first part follows from lemma 3.9, proposition 3.10 and theorem 3.12. The second part follows from proposition 3.8 and the first part.□

Part II

The upper and lower Snell envelopes

Chapter 4

The upper Snell envelope and stopping times of maximal risk

In this chapter we study the upper Snell envelope $U^{\uparrow}(H, \mathcal{M})$ introduced in equation (1.4) of chapter 1. This envelope was associated to a process H, representing an American option, and to the family of equivalent martingale measures $\mathcal{M}$. In chapter 1 we explained how this process is involved in the solution to the problem of superhedging; see theorem 1.16.
This upper envelope was introduced in continuous time by El Karoui and Quenez[19] for European options in an incomplete market model based on Brownian motion. Kramkov[41] and Föllmer and Kramkov[23] generalized the construction to American options in a general semimartingale model, incorporating portfolio constraints. Karatzas and Kou[35] constructed upper envelopes for American options in a model driven by a multidimensional Brownian motion. These papers were motivated by the problem of superhedging under incompleteness. Föllmer and Schied[27] generalize the notion of the upper Snell envelope to a general *stable* family of probability measures, but in discrete time.

An important step in [19, 23, 35] was to construct a càdlàg version of the upper Snell envelope. We are going to show that, more generally, for a given *stable* family of equivalent probability measures $\mathcal{P}$ and a process H satisfying mild conditions, we can construct a process $U^{\uparrow}(H, \mathcal{P})$ which enjoys the same properties as the envelope $U^{\uparrow}(H, \mathcal{M})$: $U^{\uparrow}(H, \mathcal{P})$ is a $\mathcal{P}$-supermartingale as in definition 1.5, it admits a càdlàg version, and $U^{\uparrow}(H, \mathcal{P}) \geq H$. Moreover, $U^{\uparrow}(H, \mathcal{P})$ is characterized as the minimal process with these properties. This is proved in theorem 4.3.

The concept of a *stable* family of equivalent probability measures is introduced in definition 4.2 and further studied in section 4.2.1. Our main reference here are sections 6.4 and 6.5 of Föllmer and Schied[27]. There, they developed the concept of stability and constructed an envelope associated to a process H with respect to a stable family of equivalent probability measures in discrete time. We extend their analysis to continuous time.

As we mentioned above, the solution of the superhedging problem involves the upper Snell envelope $U^{\uparrow}(H, \mathcal{M})$. The second goal of this chapter is to clarify the role of $U^{\uparrow}(H, \mathcal{P})$ in the analysis of a *robust optimal stopping problem* formulated in definition 4.5 in terms of a class $\mathcal{P}$. In theorem 4.27 we will construct a t-optimal stopping time for discrete time, and then in theorem 4.20 for continuous time. In theorems 4.27 and 4.20, the envelope $U^{\uparrow}(H, \mathcal{P})$ will play a key role. As explained in remark 4.6, Zamfirescu[55] contains a similar discussion.
In section 4.3.4 we motivate the robust optimal stopping problem 4.5 from the point of view of convex risk measures and interpret a 0-optimal stopping time for the upper Snell envelope as a *time of maximal risk*.

4.1 Problem formulation

Throughout this chapter we fix a stochastic base $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t \in [0,T]}, R)$. The probability measure R is a reference measure, and we assume that the filtration $\mathbb{F}$ satisfies the usual assumptions of right continuity and completeness. We assume furthermore that $\mathcal{F} = \mathcal{F}_T$. We start with the pasting operation and the concept of stability for a family of equivalent probability measures, c.f., for example, Föllmer and Schied[27] section 6.5.

Definition 4.1 *Let $\tau \in \mathcal{T}$ be a stopping time and P_1 and P_2 be probability measures equivalent to R. The probability measure defined through*

$$P_3(A) := E_{P_1}[P_2[A \mid \mathcal{F}_\tau]], A \in \mathcal{F}_T$$

is called the pasting of P_1 and P_2 in τ.

As pointed out in the discussion of definition 6.41 in Föllmer and Schied[27], P_3 is indeed a probability measure.

Definition 4.2 *A family of probability measures $\mathcal{P}$ defined in the probability space $(\Omega, \mathcal{F}, R)$ is called stable under pasting or simply stable if every $P \in \mathcal{P}$ is equivalent to R, and if for any P_1 and P_2 in $\mathcal{P}$ and any stopping time $\tau \in \mathcal{T}$, the pasting of P_1 and P_2 in τ is an element of $\mathcal{P}$.*□

We will study further the concept of stability in section 4.2.1. Delbaen[7] studies the concept of m-stability which is closely related to stability.

We now fix some notation. Throughout this chapter, $\mathcal{P}$ will denote a fixed stable family of probability measures and $H := \{H_t\}_{0\le t\le T}$ will be a fixed positive càdlàg $\mathbb{F}$-adapted stochastic process satisfying the integrability condition

$$\sup_{P\in\mathcal{P}}\sup_{\theta\in\mathcal{T}} E_P[H_\theta] < \infty. \tag{4.1}$$

Our first goal in this chapter is to prove the next theorem.

Theorem 4.3 *There exists a càdlàg $\mathcal{P}$-supermartingale*

$$U^{\uparrow}(H,\mathcal{P}) := \{U^{\uparrow}_t(H,\mathcal{P})\}_{0\le t\le T}$$

such that

$$U^{\uparrow}_\tau(H,\mathcal{P}) = \text{ess sup}_{P\in\mathcal{P}}\text{ess sup}_{\theta\in\mathcal{T}[\tau,T]}E_P[H_\theta \mid \mathcal{F}_\tau], \quad R-a.s., \tag{4.2}$$

for any stopping time $\tau\in\mathcal{T}$. Moreover, $U^{\uparrow}(H,\mathcal{P})$ is the smallest $\mathcal{P}$-supermartingale above H in the sense that $S\ge U^{\uparrow}(H,\mathcal{P})$ whenever S is a càdlàg $\mathcal{P}$-supermartingale such that $S\ge H$ as in definition 1.1.

Proof. See section 4.2.3.□

Definition 4.4 *We say that the stochastic process $\{U^{\uparrow}_t(H,\mathcal{P})\}_{0\le t\le T}$ constructed in theorem 4.3 is the upper Snell envelope of H with respect to $\mathcal{P}$.*□

As we noticed before, the upper Snell envelope with respect to the family of martingale measures $\mathcal{M}$ goes back to El Karoui and Quenez[19], Kramkov[41], Föllmer and Kramkov[23], and Karatzas and Kou[35]. Here we extend the construction to the case of a general stable family of probability measures and obtain a slightly stronger result in the sense that the equality (4.2) was proved only for constant stopping times, while in theorem 4.3 we obtain the equality (4.2) for any stopping time.

Our second goal in this chapter is to construct stopping times which are optimal in the following sense.

Definition 4.5 *For a fixed time $t\in[0,T]$, we say that a stopping time $\tau^*_t\in\mathcal{T}[t,T]$ is a t-optimal stopping time for the upper Snell envelope of H if*

$$\text{ess sup}_{P\in\mathcal{P}}E_P[H_{\tau^*_t}\mid\mathcal{F}_t] = \text{ess sup}_{\theta\in\mathcal{T}[t,T]}\text{ess sup}_{P\in\mathcal{P}}E_P[H_\theta\mid\mathcal{F}_t].\square$$

In subsection 4.3.4 we motivate this robust stopping problem for the case $t = 0$ from the point of view of convex risk measures. In theorem 4.27 we construct t-optimal stopping times for discrete time, and then, subject to appropriate conditions, in theorem 4.20 for continuous time.

Remark 4.6 *Zamfirescu[55] studies a robust stopping problem similar to 4.5. She considers a stochastic base in continuous time and infinite horizon, a class of stopping times whose elements can be infinite with positive probability, and a convex family of probability measures equivalent to a reference probability measure. However, the problem in [55] is formulated without the property of stability under pasting, which is crucial for our approach.*$\diamondsuit$

Remark 4.7 *Let $\{X_t\}_{0\le t\le T}$ be a price process as in section 1.1. In proposition 4.12 below, we show that the family of martingale measures $\mathcal{M}$ is a stable family. Now, let us specialize definition 4.5 and theorem 4.20 to the case where $\mathcal{P} = \mathcal{M}$ and $t = 0$. Then we obtain a stopping time τ_0^* such that*

$$\sup_{P\in\mathcal{M}} E_P[H_{\tau_0^*}] = \sup_{\theta\in\mathcal{T}} \sup_{P\in\mathcal{M}} E_P[H_\theta],$$

which means that the implied European option $H_{\tau_0^*}$ has the same cost of superhedging as the American option H.$\diamondsuit$

4.2 Solution

4.2.1 Stability under pasting

Let us recall definitions 4.1 and 4.2 on the pasting operation and stability.

Let $\tau \in \mathcal{T}$ be a stopping time and P_1 and P_2 be probability measures equivalent to R. The probability measure defined through

$$P_3(A) := E_{P_1}[P_2[A \mid \mathcal{F}_\tau]], A \in \mathcal{F}_T$$

is called the pasting of P_1 and P_2 in τ.$\square$

A family of probability measures $\mathcal{P}$ defined in the probability space $(\Omega, \mathcal{F}, R)$ is called stable under pasting or simply stable if every $P \in \mathcal{P}$ is equivalent to R, and if for any P_1 and P_2 in $\mathcal{P}$ and any stopping time $\tau \in \mathcal{T}$, the pasting of P_1 and P_2 in τ is an element of $\mathcal{P}$.$\square$

Trivial examples of stable families of probability measures are $\{R\}$ and $M^e(R) := \{P \text{ a probability measure } \mid P \sim R\}$. In example 4.14 we are going to see a stable family defined in terms of the Girsanov transformation. In proposition 4.12 we show that the family of equivalent local martingale measures is stable.
The definition of stability under pasting for families of probability measures deserves some comments. First notice that the definition of stability is only formulated for families whose elements are **equivalent** to the reference probability measure R. Thus, whenever a stable family of probability measures is given, we implicitely assume that its elements are equivalent to R. In section 6.5 in Föllmer and Schied[27], stability of the family of equivalent martingale measures plays a key role for the analysis of the upper and lower prices $\pi_{\sup}(\cdot)$ and $\pi_{\inf}(\cdot)$ of an American option H in discrete time. Another important application of the stability concept appears in the problem of representing dynamically consistent risk measures, see Föllmer and Penner[26] for details and references.

Let us now collect some simple properties of the pasting operation.

Lemma 4.8 *Let P_1 and P_2 be two equivalent probability measures and let $\{Z_t\}_{0\leq t\leq T}$ denote a càdlàg version of the density process of P_2 with respect to P_1. Let P_3 be the pasting of P_1 and P_2 in $\sigma \in \mathcal{T}$. Then P_3 is equivalent to P_1 and its density is given by*

$$\frac{dP_3}{dP_1} = \frac{Z_T}{Z_\sigma}.$$

Moreover, $P_3 = P_1$ in $\mathcal{F}_\sigma$.

Proof. The proof is similar to lemma 6.42 in [27].□

In lemma 4.10 we make use of the following result.

Lemma 4.9 *Let P_1 and P_2 be two equivalent probability measures and let $\{Z_t\}_{0\leq t\leq T}$ denote a càdlàg version of the density process of P_2 with respect to P_1.*

Let $\tau, \sigma \in \mathcal{T}$ be two stopping times with $\tau \geq \sigma$. Let $Y \geq 0$ be a $\mathcal{F}_\tau$-measurable random variable, integrable with respect to P_2. Then

$$E_{P_2}[Y \mid \mathcal{F}_\sigma] = \frac{1}{Z_\sigma} E_{P_1}[Y Z_\tau \mid \mathcal{F}_\sigma].$$

Proof. See e.g., lemma 3.5.3 in Karatzas and Shreve[36].□

Lemma 4.10 *Let P_1 and P_2 be two equivalent probability measures and let P_3 be the pasting of P_1 and P_2 in $\sigma \in \mathcal{T}$. Then the density process of P_3 with respect to R is given by*

$$Z_t^3 = \begin{cases} Z_t^1 & \text{if } t \leq \sigma \\ Z_\sigma^1 \frac{Z_t^2}{Z_\sigma^2} & \text{if } t > \sigma \end{cases},$$

where Z_t^i denotes a càdlàg version of the density process of P_i with respect to R, for $i = 1, 2$.

Proof. Due to lemmas 4.9 and 4.8, the following identity results

$$\frac{dP^3}{dR} = Z_T^2 \frac{Z_\sigma^1}{Z_\sigma^2}.$$

Now we consider separately the events $\{t \leq \sigma\}$ and $\{t > \sigma\}$. In the event $\{t \leq \sigma\}$ we get

$$Z_t^3 = E_R\left[\frac{Z_T^2}{Z_\sigma^2} Z_\sigma^1 \mid \mathcal{F}_t\right] = E_R[Z_\sigma^1 \mid \mathcal{F}_t] = Z_t^1.$$

In the event $\{t > \sigma\}$

$$Z_t^3 = \frac{Z_\sigma^1}{Z_\sigma^2} E_R[Z_T^2 \mid \mathcal{F}_t] = \frac{Z_\sigma^1}{Z_\sigma^2} Z_t^2. \square$$

The next lemma is a key result to compute conditional expectations with respect to the pasting of two probability measures.

Lemma 4.11 *Let P_3 be the pasting of P_1 and P_2 in σ. Let Y be a positive random variable $\mathcal{F}_T$-measurable and P_i-integrable for $i = 1, 2, 3$. Then, for any stopping time $\tau \in \mathcal{T}$ we have*

$$E_{P_3}[Y \mid \mathcal{F}_\tau] = E_{P_1}[E_{P_2}[Y \mid \mathcal{F}_{\sigma\vee\tau}] \mid \mathcal{F}_\tau].$$

Proof. This is the continuous-time version of lemma 6.43 in [27], and we follow their proof. Let P_3 be the pasting of P_1 and P_2 in σ. The lemma is proved if we show that for any $A \in \mathcal{F}_\tau$, the following formula holds

$$E_{P_3}[1_A Y] = E_{P_3}[1_A E_{P_1}[E_{P_2}[Y \mid \mathcal{F}_{\sigma\vee\tau}] \mid \mathcal{F}_\tau]].$$

Lemma 4.8 allows us to write

$$E_{P_3}[1_A Y] = E_{P_1}\left[\frac{Z_T}{Z_\sigma} 1_A Y\right],$$

and from lemma 4.9 we deduce that

$$E_{P_1}[E_{P_2}[Y \mid \mathcal{F}_{\sigma\vee\tau}] \mid \mathcal{F}_\tau] = E_{P_1}\left[\frac{Z_T}{Z_{\sigma\vee\tau}} Y \mid \mathcal{F}_\tau\right].$$

If we put together the right-hand terms, then we see that it is enough to verify

$$E_{P_1}\left[\frac{Z_T}{Z_\sigma} 1_A Y\right] = E_{P_1}\left[\frac{Z_T}{Z_\sigma} 1_A E_{P_1}\left[\frac{Z_T}{Z_{\sigma\vee\tau}} Y \mid \mathcal{F}_\tau\right]\right]. \tag{4.3}$$

Let $B = \{\tau \leq \sigma\} \in \mathcal{F}_{\sigma\wedge\tau}$. Note that $1_{A\cap B} E_{P_1}[\frac{Z_T}{Z_{\sigma\vee\tau}} Y \mid \mathcal{F}_\tau]$ is $\mathcal{F}_\sigma$-measurable, hence

$$E_{P_1}\left[\frac{Z_T}{Z_\sigma} 1_{A\cap B} E_{P_1}\left[\frac{Z_T}{Z_{\sigma\vee\tau}} Y \mid \mathcal{F}_\tau\right]\right] = E_{P_1}\left[1_{A\cap B} E_{P_1}\left[\frac{Z_T}{Z_\sigma} Y \mid \mathcal{F}_\tau\right]\right] = E_{P_1}\left[1_{A\cap B} \frac{Z_T}{Z_\sigma} Y\right].$$

In a similar way we get the equality

$$E_{P_1}\left[\frac{Z_T}{Z_\sigma} 1_{A\cap B^c} E_{P_1}\left[\frac{Z_T}{Z_{\sigma\vee\tau}} Y \mid \mathcal{F}_\tau\right]\right] = E_{P_1}\left[1_{A\cap B^c} \frac{Z_T}{Z_\sigma} Y\right].$$

Combining these two equalities for B and B^c we get (4.3).□

In the next proposition we illustrate how to use lemma 4.11 to show the well-known fact that the family of local martingale measures $\mathcal{M}$ is stable under pasting.

Proposition 4.12 *The family of equivalent local martingale measures $\mathcal{M}$ for a price process X, specified as in section 1.1, is stable under pasting.*

Proof. Delbaen[7] proved that $\mathcal{M}$ is m-stable, a concept equivalent to stability under pasting. His setup is in continuous time and infinite horizon for a locally bounded price process; see his proposition 9.1. Föllmer and Schied[27] proved the proposition in discrete time and finite horizon; see their proposition 6.45. Notice that in [27] they denoted the family of martingale measures by $\mathcal{P}$.

We now adapt the argument in the proof of proposition 6.45 of [27] to continuous time and finite horizon. Let $P_1, P_2 \in \mathcal{M}$ and $\sigma \in \mathcal{T}$. Let P_3 be the pasting of P_1 and P_2 in σ. Let $\{\theta_i^k\}_{i=1}^\infty$ be a localizing sequence of X with respect to P_k, for $k = 1, 2$. This means that (in our finite horizon setting)

1. θ_i^k is a stopping time in $\mathcal{T}$.

2. The sequence converges R-a.s. to the horizon T: $\lim_{i\to\infty}\theta_i^k = T$.

3. The stopped process $\{X_{t\wedge\theta_i^k}\}_{0\le t\le T}$ is a P_k-martingale, for $k=1,2$.

We define $\theta_i^3 := \theta_i^1 \wedge \theta_i^2$ and show that $\{\theta_i^3\}_{i=1}^\infty$ is a localizing sequence of X with respect to P_3. The first two properties of a localizing sequence are obvious for $\{\theta_i^3\}_{0\le t\le T}$. To prove the last property, take $s,t\in[0,T]$ with $s\ge t$. Then, for $i\in\mathbb{N}$

$$E_{P_3}[X_{s\wedge\theta_i^3} \mid \mathcal{F}_t] = E_{P_1}[E_{P_2}[X_{s\wedge\theta_i^3} \mid \mathcal{F}_{\sigma\vee t}] \mid \mathcal{F}_t] = E_{P_1}[X_{(s\wedge\theta_i^3)\wedge(\sigma\vee t)} \mid \mathcal{F}_t],$$

where in the first equality we have applied lemma 4.11. In the second equality we have applied Doob's optional sampling theorem for martingales. We are allowed to do so, since the stopping time $s\wedge\theta_i^3$ is bounded and $\theta_i^3\le\theta_i^2$. We apply once more Doob's optional sampling theorem to conclude that

$$E_{P_1}[X_{(s\wedge\theta_i^3)\wedge(\sigma\vee t)} \mid \mathcal{F}_t] = X_{(s\wedge\theta_i^3)\wedge(\sigma\vee t)\wedge t},$$

where we have used the fact that $\theta_i^3\le\theta_i^1$. It is easy to show that $(s\wedge\theta_i^3)\wedge(\sigma\vee t)\wedge t = t\wedge\theta_i^3$ and we conclude that

$$E_{P_3}[X_{s\wedge\theta_i^3} \mid \mathcal{F}_t] = X_{t\wedge\theta_i^3}.\square$$

In example 4.14 below, we construct a stable family of probability measures. It is a special case of theorem 1.3 in Delbaen[7]. We will need the stochastic exponential of a continuous martingale.

Definition 4.13 *Let $M := \{M_t\}_{0\le t\le T}$ be a continuous local martingale with $M_0 = 0$. The stochastic exponential of M denoted $\{\mathcal{E}_t(M)\}_{0\le t\le T}$ is defined by*

$$\mathcal{E}_t(M) := \exp\left\{M_t - \frac{1}{2}\langle M\rangle_t\right\},$$

where $\{\langle M\rangle_t\}_{0\le t\le T}$ is the quadratic variation process of the local martingale M.$\square$

Example 4.14 *Let $\{W_t\}_{0\le t\le T}$ be a standard Brownian motion defined in the probability space $(\Omega,\mathcal{F},\mathbb{F}=\{\mathcal{F}_t\}_{t\in[0,T]},R)$ where $\mathbb{F}$ is the augmented Brownian filtration. Let furthermore $\{\xi_t^0\}_{0\le t\le T}$ be a predictable process with*

$$E_R\left[\int_0^T (\xi_s^0)^2 ds\right] < \infty. \tag{4.4}$$

Notice that this inequality implies that the stochastic integral

$$\xi^0 \cdot dW_t := \int_0^t \xi_s^0 dW_s,$$

is well defined and is a square integrable martingale. We furthermore require that

$$E_R\left[\exp\left\{\frac{1}{2}\int_0^T (\xi_s^0)^2 ds\right\}\right] < \infty. \tag{4.5}$$

Then

1. *If* $\xi := \{\xi_t\}_{0\le t\le T}$ *is a predictable process such that*

   $$R(\{\omega \in \Omega \mid |\xi_t(\omega)| \le \left|\xi_t^0(\omega)\right| \text{ for almost all } t \in [0,T]\}) = 1, \tag{4.6}$$

   then the stochastic integral $\xi \cdot N_t := \int_0^t \xi_s dN_s$ *is well defined and is a uniformly integrable martingale*

2. *Let* $\mathcal{P}$ *be the family of probability measures obtained from the family of densities with respect to* R *given by:*

   $$dens(\mathcal{P}) := \{\mathcal{E}_T(\xi \cdot W) \mid \{\xi_t\}_{0\le t\le T} \text{ is a predictable process satisfying } (4.6)\}.$$

   Then the probability measures in $\mathcal{P}$ *are equivalent to* R *and* $\mathcal{P}$ *is a convex stable family.*

Proof. Note that (4.5) is the Novikov criterion for Girsanov transformation theorem; see for example corollary 3.5.13 and theorem 3.5.1 in Karatzas and Shreve[36]. The first assertion in the proposition follows from (4.4) and (4.6). Now we verify the second assertion.

1. Let ξ be a predictable process satisfying (4.6). We show that

   $$E_R[\mathcal{E}_T(\xi \cdot W)] = 1 \tag{4.7}$$

   $$R(\mathcal{E}_T(\xi \cdot W) > 0) = 1. \tag{4.8}$$

   From the first step we know that $\xi \cdot W$ is a uniformly integrable martingale. From (4.6) and (4.5) we deduce that

   $$E_R\left[\exp\left\{\frac{1}{2}\int_0^T (\xi_s)^2 ds\right\}\right] < \infty.$$

   Hence, by Novikov's criterion for Girsanov transformation theorem, the process $\{\mathcal{E}_t(\xi \cdot W)\}_{0\le t\le T}$ is a uniformly integrable R-martingale. In particular this implies (4.7).

In order to prove (4.8), we first apply Itô isometry:

$$E_R\left[\left(\int_0^T \xi_s dW_s\right)^2\right] = E_R\left[\int_0^T (\xi_s)^2 ds\right] < \infty.$$

Then we see that $|\xi \cdot W_T| < \infty$, R-a.s. This implies (4.8) since $\langle \xi \cdot W\rangle_T = \int_0^T (\xi_s)^2 ds$ and then, by definition

$$\mathcal{E}_T(\xi \cdot W) = \exp\left\{\int_0^T \xi_s \cdot W_s - \frac{1}{2}\int_0^T (\xi_s)^2 ds\right\}.$$

2. In order to prove that $\mathcal{P}$ is convex, it is enough to show that $dens(\mathcal{P})$ is convex. Take two elements in $dens(\mathcal{P})$, $Z_T^1 = \mathcal{E}_T(\xi^1 \cdot W)$ and $Z_T^2 = \mathcal{E}_T(\xi^2 \cdot W)$. For $0 < \lambda < 1$ we must prove that $\lambda Z_T^1 + (1-\lambda)Z_T^2 \in dens(\mathcal{P})$. Let us define the process $\{Z_t^3\}_{0\le t\le T}$ by $Z_t^3 := \lambda Z_t^1 + (1-\lambda)Z_t^2$. Then Z^3 satisfies

$$dZ_t^3 = Z_{t-}^3 \xi_t^3 dW_t,$$

where

$$\xi_t^3 := \lambda \frac{Z_{t-}^1}{Z_{t-}^3}\xi_t^1 + (1-\lambda)\frac{Z_{t-}^2}{Z_{t-}^3}\xi_t^2$$

is a predictable process satisfying (4.6). We conclude that $Z_T^3 = \mathcal{E}_T(\xi^3 \cdot W)$, and thus $Z_T^3 \in dens(\mathcal{P})$. This proves the required convexity.

3. We now show that $\mathcal{P}$ is stable under pasting. Let $P_i \in \mathcal{P}$ for $i = 1, 2$, and let $Z_t^i = \mathcal{E}_t(\xi^i \cdot W)$ be the density process of P_i with respect to R. Let $\sigma \in \mathcal{T}$ be a stopping time and P_3 be the pasting of P_1 and P_2 in σ. We must show that $P_3 \in \mathcal{P}$. The process

$$\xi_t^3 := \xi_t^1 1_{\{t\le\sigma\}} + \xi_t^2 1_{\{t>\sigma\}},$$

is predictable (since $\mathbb{F}$ is the augmented Brownian filtration) and satisfies (4.6). Let $\{Z_t^3\}_{0\le t\le T}$ be a càdlàg version of the density of P_3 with respect to R, lemma 4.10 implies that $Z_T^3 = \mathcal{E}_T(\xi^3 \cdot W)$.□

4.2.2 Lattice properties

This section depends on the concept of stability under pasting as developed in section 4.2.1. Let us recall that we have fixed a stable family $\mathcal{P}$ of equivalent probability measures. For $P \in \mathcal{P}$ and for a stopping time $\tau \in \mathcal{T}$, let us define the *random variables* Z_τ^P and $Z_\tau^\uparrow$ by

$$Z_\tau^P := \text{ess sup}_{\theta\in\mathcal{T}[\tau,T]} E_P\left[H_\theta \mid \mathcal{F}_\tau\right], \tag{4.9}$$

$$Z_\tau^\uparrow := \text{ess sup}_{P\in\mathcal{P}} Z_\tau^P = \text{ess sup}_{P\in\mathcal{P}} \text{ess sup}_{\theta\in\mathcal{T}[\tau,T]} E_P\left[H_\theta \mid \mathcal{F}_\tau\right]. \tag{4.10}$$

Notice that

$$Z_\tau^\uparrow = \operatorname{ess\,sup}_{\theta \in \mathcal{T}[\tau,T]} \operatorname{ess\,sup}_{P \in \mathcal{P}} E_P \left[H_\theta \mid \mathcal{F}_\tau \right].$$

In this subsection we prepare the construction of the upper Snell envelope. Here we show that the family of random variables

$$Z^\uparrow := \{Z_\theta^\uparrow\}_{\theta \in \mathcal{T}} \tag{4.11}$$

has a $\mathcal{P}$-**supermartingale** property of the form (4.16) below. In lemma 4.15 we verify a lattice property of the conditional expectations appearing in (4.13), extending the method in section 6.5 of [27] from discrete time to continuous time. We then obtain two corollaries which will serve as lemmas for the proofs of theorems 4.3 and 4.20.

We fix the notation

$$\mathcal{P}(P_0, \tau) := \{P \in \mathcal{P} \mid P = P_0 \text{ in } \mathcal{F}_\tau\}, \tag{4.12}$$

for $P_0 \in \mathcal{P}$ and $\tau \in \mathcal{T}$.

Lemma 4.15 *Let $\tau \in \mathcal{T}$ be a stopping time and $P_0 \in \mathcal{P}$ be a fixed probability measure.*

1. *The family*

$$\Phi(\tau) := \{E_P[H_\theta \mid \mathcal{F}_\tau] \mid P \in \mathcal{P}, \theta \in \mathcal{T}[\tau, T]\} \tag{4.13}$$

   is upwards directed, that is, for any pair $P_1, P_2 \in \mathcal{P}$ of probability measures and for any pair of stopping times $\theta_1, \theta_2 \in \mathcal{T}[\tau, T]$, there exists $P_3 \in \mathcal{P}$ and $\theta_3 \in \mathcal{T}[\tau, T]$ such that

$$E_{P_3}[H_{\theta_3} \mid \mathcal{F}_\tau] = E_{P_1}[H_{\theta_1} \mid \mathcal{F}_\tau] \vee E_{P_2}[H_{\theta_2} \mid \mathcal{F}_\tau]. \tag{4.14}$$

2. *There exist a pair of sequences $\{P_i\}_{i=1}^\infty \subset \mathcal{P}(P_0, \tau)$ and $\{\theta_i\}_{i=1}^\infty \subset \mathcal{T}[\tau, T]$, such that*

$$E_{P_i}[H_{\theta_i} \mid \mathcal{F}_\tau] \nearrow Z_\tau^\uparrow. \tag{4.15}$$

3. *For any $\sigma \in \mathcal{T}[\tau, T]$*

$$E_{P_0}[Z_\sigma^\uparrow \mid \mathcal{F}_\tau] \le Z_\tau^\uparrow, \tag{4.16}$$

Proof.

1. Let $P_1, P_2 \in \mathcal{P}$ be two arbitrary probability measures and θ_1, θ_2 two stopping times in $\mathcal{T}[\tau, T]$. It is convenient to define the following objects

$$\begin{aligned} B &:= \{E_{P_1}[H_{\theta_1} \mid \mathcal{F}_\tau] \geq E_{P_2}[H_{\theta_2} \mid \mathcal{F}_\tau]\}, \\ \theta_3 &:= \theta_1 1_B + \theta_2 1_{B^c} \in \mathcal{T}[\tau, T], \\ \sigma &:= T 1_B + \tau 1_{B^c}, \\ P_3 &:= \text{The pasting of } P_1 \text{ and } P_2 \text{ in } \sigma. \end{aligned}$$

From lemma 4.11 we can write

$$E_{P_3}[H_{\theta_3} \mid \mathcal{F}_\tau] = E_{P_1}[E_{P_2}[H_{\theta_3} \mid \mathcal{F}_{\sigma \vee \tau}] \mid \mathcal{F}_\tau],$$

and we compute easily the equalities:

$$\begin{aligned} 1_B E_{P_3}[H_{\theta_3} \mid \mathcal{F}_\tau] &= 1_B E_{P_1}[E_{P_2}[H_{\theta_1} \mid \mathcal{F}_T] \mid \mathcal{F}_\tau] = 1_B E_{P_1}[H_{\theta_1} \mid \mathcal{F}_\tau], \\ 1_{B^c} E_{P_3}[H_{\theta_3} \mid \mathcal{F}_\tau] &= 1_{B^c} E_{P_1}[E_{P_2}[H_{\theta_2} \mid \mathcal{F}_\tau] \mid \mathcal{F}_\tau] = 1_{B^c} E_{P_2}[H_{\theta_2} \mid \mathcal{F}_\tau]. \end{aligned}$$

We conclude that

$$E_{P_3}[H_{\theta_3} \mid \mathcal{F}_\tau] = E_{P_1}[H_{\theta_1} \mid \mathcal{F}_\tau] \vee E_{P_2}[H_{\theta_2} \mid \mathcal{F}_\tau],$$

which is (4.14).

2. Since the family $\Phi(\tau)$ is upwards directed, there exist two sequences $\{\widetilde{P}_i\}_{i=1}^\infty \subset \mathcal{P}$ and $\{\theta_i\}_{i=1}^\infty \subset \mathcal{T}[\tau, T]$ such that the sequence of random variables $\left\{E_{\widetilde{P}_i}[H_{\theta_i} \mid \mathcal{F}_\tau]\right\}_{i \in \mathbb{N}}$ is increasing and converges to $Z_\tau^\uparrow$ R-a.s. Let $P_0 \in \mathcal{P}$ be fixed. For $i \in \mathbb{N}$ let us define inductively

$$\begin{aligned} B_i &:= \{E_{P_{i-1}}[H_{\theta_i} \mid \mathcal{F}_\tau] \geq E_{\widetilde{P}_i}[H_{\theta_i} \mid \mathcal{F}_\tau]\}, \\ \sigma_i &:= 1_{B_i} \tau + 1_{B_i^c} T, \\ P_i &:= \text{ The pasting of } P_{i-1} \text{ and } \widetilde{P}_i \text{ in } \sigma_i. \end{aligned} \tag{4.17}$$

Note that $P_i = P_{i-1}$ in $\mathcal{F}_{\sigma_i}$. This implies that $P_i = P_0$ in $\mathcal{F}_\tau$, so that $P_i \in \mathcal{P}(P_0, \tau)$. A computation as in the first part shows that

$$E_{P_i}[H_{\theta_i} \mid \mathcal{F}_\tau] = E_{P_{i-1}}[H_{\theta_i} \mid \mathcal{F}_\tau] \vee E_{\widetilde{P}_i}[H_{\theta_i} \mid \mathcal{F}_\tau],$$

hence the sequence $\{P_i\}_{i=1}^\infty$ has the desired property.

3. For the last statement take a stopping time $\sigma \in \mathcal{T}[\tau, T]$ and $A \in \mathcal{F}_\tau$. We must show that

$$E_{P_0}[1_A Z_\sigma^\uparrow] \leq E_{P_0}[1_A Z_\tau^\uparrow].$$

Let $\{P_i\}_{i=1}^{\infty} \subset \mathcal{P}$ and $\{\theta_i\}_{i=1}^{\infty} \subset \mathcal{T}[\sigma, T]$ be two sequences as constructed in the previous step with

$$E_{P_i}[H_{\theta_i} \mid \mathcal{F}_\sigma] \nearrow Z_\sigma^{\uparrow},$$

and $P_i = P_0$ in $\mathcal{F}_\sigma$. Then we have the following

$$\begin{aligned}
E_{P_0}[1_A Z_\sigma^{\uparrow}] &= E_{P_0}[1_A \lim_{i\to\infty} E_{P_i}[H_{\theta_i} \mid \mathcal{F}_\sigma]] \\
&= E_{P_0}[\lim_{i\to\infty} 1_A E_{P_i}[H_{\theta_i} \mid \mathcal{F}_\sigma]] \\
&= \lim_{i\to\infty} E_{P_0}[1_A E_{P_i}[H_{\theta_i} \mid \mathcal{F}_\sigma]] \\
&= \lim_{i\to\infty} E_{P_0}[1_A E_{P_i}[H_{\theta_i} \mid \mathcal{F}_\tau]] \\
&\le E_{P_0}[1_A Z_\tau^{\uparrow}].
\end{aligned}$$

The third equality is justified by monotone convergence, the fourth equality follows from the fact that $P_i = P_0$ in $\mathcal{F}_\sigma$, and the last inequality is obvious.□

Lemma 4.16 *Let $P_0 \in \mathcal{P}$ be a fixed probability measure and $\tau \in \mathcal{T}$ be a fixed stopping time. Then, for $\sigma \in \mathcal{T}[\tau, T]$, R-a.s.*

$$\begin{aligned}
E_{P_0}[Z_\sigma^{\uparrow} \mid \mathcal{F}_\tau] &= \operatorname{ess\,sup}_{P\in\mathcal{P}} E_{P_0}[Z_\sigma^P \mid \mathcal{F}_\tau] \\
&= \operatorname{ess\,sup}_{P\in\mathcal{P}} \operatorname{ess\,sup}_{\theta\in\mathcal{T}[\sigma,T]} E_{P_0}[E_P[H_\theta \mid \mathcal{F}_\sigma] \mid \mathcal{F}_\tau] \\
&= \operatorname{ess\,sup}_{P\in\mathcal{P}(P_0,\sigma)} \operatorname{ess\,sup}_{\theta\in\mathcal{T}[\sigma,T]} E_P[H_\theta \mid \mathcal{F}_\tau].
\end{aligned}$$

Proof. The following inequalities hold R-a.s.

$$\begin{aligned}
E_{P_0}[Z_\sigma^{\uparrow} \mid \mathcal{F}_\tau] &\ge \operatorname{ess\,sup}_{P\in\mathcal{P}} E_{P_0}[Z_\sigma^P \mid \mathcal{F}_\tau] \\
&\ge \operatorname{ess\,sup}_{P\in\mathcal{P}} \operatorname{ess\,sup}_{\theta\in\mathcal{T}[\sigma,T]} E_{P_0}[E_P[H_\theta \mid \mathcal{F}_\sigma] \mid \mathcal{F}_\tau] \\
&\ge \operatorname{ess\,sup}_{P\in\mathcal{P}(P_0,\sigma)} \operatorname{ess\,sup}_{\theta\in\mathcal{T}[\sigma,T]} E_P[H_\theta \mid \mathcal{F}_\tau].
\end{aligned}$$

And thus, it is enough to show that

$$E_{P_0}[Z_\sigma^{\uparrow}] \le E_{P_0}[\operatorname{ess\,sup}_{P\in\mathcal{P}(P_0,\sigma)} \operatorname{ess\,sup}_{\theta\in\mathcal{T}[\sigma,T]} E_P[H_\theta \mid \mathcal{F}_\tau]].$$

According to lemma 4.15 there exists a sequence of measures $\{P_i\}_{i=1}^{\infty} \subset \mathcal{P}(P_0, \sigma)$ and stopping times $\{\theta_i\}_{i=1}^{\infty} \subset \mathcal{T}[\sigma, T]$ such that

$$E_{P_i}[H_{\theta_i} \mid \mathcal{F}_\sigma] \nearrow Z_\sigma^{\uparrow}.$$

Now we have

$$\begin{aligned} E_{P_0}[Z^{\uparrow}_{\sigma}] &= E_{P_0}[\lim_{i\to\infty} E_{P_i}[H_{\theta_i} \mid \mathcal{F}_{\sigma}]] \\ &= \lim_{i\to\infty} E_{P_0}[E_{P_i}[H_{\theta_i} \mid \mathcal{F}_{\sigma}]] \\ &= \lim_{i\to\infty} E_{P_0}[E_{P_i}[H_{\theta_i} \mid \mathcal{F}_{\tau}]] \\ &\leq E_{P_0}[\operatorname{ess\,sup}_{P\in\mathcal{P}(P_0,\sigma)} \operatorname{ess\,sup}_{\theta\in\mathcal{T}[\sigma,T]} E_P[H_\theta \mid \mathcal{F}_{\tau}]], \end{aligned}$$

where the second equality is justified by monotone convergence, and the third by the fact that $P_i \in \mathcal{P}(P_0, \sigma)$. The last inequality is obvious.□

Lemma 4.17 *Let $\tau \in \mathcal{T}$ be a fixed stopping time and $\sigma \in \mathcal{T}[\tau, T]$. Then*

$$\operatorname{ess\,sup}_{P\in\mathcal{P}} E_P[Z^{\uparrow}_{\sigma} \mid \mathcal{F}_{\tau}] = \operatorname{ess\,sup}_{P\in\mathcal{P}} \operatorname{ess\,sup}_{\theta\in\mathcal{T}[\sigma,T]} E_P[H_\theta \mid \mathcal{F}_{\tau}]. \tag{4.18}$$

Proof. The inequality $\leq$ follows from lemma 4.16. We now show the converse inequality $\geq$. Let $P \in \mathcal{P}$ and $\theta \in \mathcal{T}[\sigma, T]$ be fixed. Notice that

$$E_P[Z^{\uparrow}_{\sigma} \mid \mathcal{F}_{\tau}] \geq E_P[E_P[H_\theta \mid \mathcal{F}_{\sigma}] \mid \mathcal{F}_{\tau}] = E_P[H_\theta \mid \mathcal{F}_{\tau}].$$

If we take the essential supremum over $\theta \in \mathcal{T}[\sigma, T]$ and $P \in \mathcal{P}$ in this inequality, we then obtain the inequality $\geq$.□

4.2.3 Proof of theorem 4.3

In this section we prove theorem 4.3. For convenience of the reader we recall the statement.

There exists a càdlàg $\mathcal{P}$-supermartingale

$$U^{\uparrow}(H, \mathcal{P}) := \{U^{\uparrow}_t(H, \mathcal{P})\}_{0\leq t\leq T}$$

such that

$$U^{\uparrow}_{\tau}(H, \mathcal{P}) = \operatorname{ess\,sup}_{P\in\mathcal{P}} \operatorname{ess\,sup}_{\theta\in\mathcal{T}[\tau,T]} E_P[H_\theta \mid \mathcal{F}_{\tau}], \quad R - a.s.,$$

for any stopping time $\tau \in \mathcal{T}$. Moreover, $U^{\uparrow}(H, \mathcal{P})$ is the smallest $\mathcal{P}$-supermartingale above H in the sense that $S \geq U^{\uparrow}(H, \mathcal{P})$ whenever S is a càdlàg $\mathcal{P}$-supermartingale such that $S \geq H$ as in definition 1.1.

Proof. We simplify notation and write $U^{\uparrow} = U^{\uparrow}(H, \mathcal{P})$. Let $P_1 \in \mathcal{P}$ be fixed but arbitrary. Let us recall that in (4.11) we defined the family of random variables $Z^{\uparrow} = \{Z^{\uparrow}_{\theta}\}_{\theta\in\mathcal{T}}$.

1. In this first step we show that the process $\{Z_t^{\uparrow}\}_{0\leq t\leq T}$ has a càdlàg modification. We use the fact that $\{Z_t^{\uparrow}\}_{0\leq t\leq T}$ has the P_1-supermartingale property as stated in lemma 4.15. The stopping time defined in (4.20) and the argument involved in (4.22) are important in this step and they were first considered by Föllmer and Kramkov[23]. The existence of a càdlàg modification will follow after proving that the correspondence $t \to E_{P_1}[Z_t^{\uparrow}]$ is right-continuous (see e.g., theorem 3.1 in Lipster and Shiryayev[43]).

   Let $\{t_i\}_{i=1}^{\infty} \subset [t, T]$ be a decreasing sequence converging to t. We have that
$$E_{P_1}[Z_t^{\uparrow}] \geq \lim_{i\to\infty} E_{P_1}[Z_{t_i}^{\uparrow}],$$
   since $Z_t^{\uparrow}$ is a P_1-supermartingale.

   Now we show the opposite inequality. From lemma 4.15 we know that for any $\epsilon > 0$, there exists a stopping time τ with $t \leq \tau \leq T$ and a probability measure $P_2 \in \mathcal{P}$ with $P_2 = P_1$ in $\mathcal{F}_t$ such that
$$E_{P_1}[Z_t^{\uparrow}] \leq \epsilon + E_{P_1}[E_{P_2}[H_\tau \mid \mathcal{F}_t]] = \epsilon + E_{P_2}[H_\tau]. \tag{4.19}$$
   Now we define
$$\tau^{(i)} := \tau 1_{\{\tau\geq t_i\}} + T 1_{\{\tau<t_i\}} \in \mathcal{T}[t_i, T], \tag{4.20}$$
   and let P_i be the pasting of P_1 and P_2 in $\mathcal{F}_{t_i}$. Then according to lemma 4.16 we get that
$$E_{P_i}[H_{\tau^{(i)}}] \leq E_{P_1}[Z_{t_i}^{\uparrow}]. \tag{4.21}$$
   so that $\liminf_{i\to\infty} E_{P_i}[H_{\tau^{(i)}}] \leq \liminf_{i\to\infty} E_{P_1}[Z_{t_i}^{\uparrow}]$. Now in order to obtain the inequality $E_{P_1}[Z_t^{\uparrow}] \geq \lim_{i\to\infty} E_{P_1}[Z_{t_i}^{\uparrow}]$ it only remains to show that $E_{P_2}[H_\tau] \leq \liminf_{i\to\infty} E_{P_i}[H_{\tau^{(i)}}]$.

   Let F denote the density process of P_2 with respect to P_1, notice that $\lim_{s\searrow t} F_s = F_t = 1$, R-a.s.. According to lemma 4.8, the density of P_i with respect to P_1 is equal to $\frac{F_T}{F_{t_i}}$, then
$$\begin{aligned} E_{P_2}[H_\tau] = E_{P_1}[F_T H_\tau] = E_{P_1}\left[\lim_{i\to\infty} \frac{F_T}{F_{t_i}} H_{\tau^{(i)}}\right] \\ \leq \liminf_{i\to\infty} E_{P_1}\left[\frac{F_T}{F_{t_i}} H_{\tau^{(i)}}\right] \\ = \liminf_{i\to\infty} E_{P_i}\left[H_{\tau^{(i)}}\right], \end{aligned} \tag{4.22}$$

where in the inequality we have applied Fatou's lemma. From (4.19) and (4.21) we conclude the opposite inequality $E_{P_1}[Z_t^{\uparrow}] \leq \lim_{i\to\infty} E_{P_1}[Z_{t_i}^{\uparrow}]$.

2. Let $\{U_t^{\uparrow}\}_{0\leq t\leq T}$ be a càdlàg modification of the process $\{Z_t^{\uparrow}\}_{0\leq t\leq T}$, and let $\tau \in \mathcal{T}$ be a fixed stopping time. We now show that

$$U_\tau^{\uparrow} = \operatorname{ess\,sup}_{P\in\mathcal{P}} \operatorname{ess\,sup}_{\theta\in\mathcal{T}[\tau,T]} E_P[H_\theta \mid \mathcal{F}_\tau].$$

For an arbitrary stopping time $\theta \in \mathcal{T}$, let us define the usual dyadic discretizations

$$\theta^i = \sum_{j=0}^{2^iT-1} \frac{j+1}{2^i} 1_{\{\frac{j}{2^i}\leq\theta<\frac{j+1}{2^i}\}} + T1_{\{\theta=T\}}. \tag{4.23}$$

Clearly $\{\theta^i\}_{i=1}^{\infty}$ is a decreasing sequence of stopping times converging to θ, R-a.s. Note also that $U_{\theta^i}^{\uparrow} = Z_{\theta^i}^{\uparrow}$ R-a.s. since the stopping time θ^i takes only a finite number of values.

Let $\tau \in \mathcal{T}$ be an arbitrary stopping time. In order to prove that $Z_\tau^{\uparrow} \leq U_\tau^{\uparrow}$ we have to show that $E_P[H_\theta \mid \mathcal{F}_\tau] \leq U_\tau^{\uparrow}$ for $\theta \in \mathcal{T}[\tau,T]$ and $P \in \mathcal{P}$, i.e., $E_P[1_A H_\theta] \leq E_P[1_A U_\tau^{\uparrow}]$ for any $A \in \mathcal{F}_\tau$. Indeed:

$$\begin{aligned} E_P[1_A H_\theta] &= E_P[\lim_{i\to\infty} 1_A H_{\theta^i}] \leq E_P[\liminf_{i\to\infty} 1_A Z_{\theta^i}^{\uparrow}] \\ &= E_P[\liminf_{i\to\infty} 1_A U_{\theta^i}^{\uparrow}] \leq \liminf_{i\to\infty} E_P[1_A U_{\theta^i}^{\uparrow}] \\ &= \liminf_{i\to\infty} E_P[1_A E_P[U_{\theta^i}^{\uparrow} \mid \mathcal{F}_\tau]] \leq E_P[1_A U_\tau^{\uparrow}], \end{aligned}$$

where in the first equality we used the fact that H is right continuous, in the following inequality the definition of $Z_{\theta^i}^{\uparrow}$, in the next equality that $U_{\theta^i}^{\uparrow} = Z_{\theta^i}^{\uparrow}$, in the following inequality Fatou's lemma, and in the last inequality the P-supermartingale property of $U^{\uparrow}$.

In order to prove equality of these variables, it suffices to demonstrate that $E_P[U_\tau^{\uparrow}] \leq E_P[Z_\tau^{\uparrow}]$ for $P \in \mathcal{P}$ fixed, since we now know that $Z_\tau^{\uparrow} \leq U_\tau^{\uparrow}$. Using again the usual dyadic discretisations $\{\tau^i\}_{i=1}^{\infty}$ of τ we get the following inequalities:

$$E_P[Z_\tau^{\uparrow}] \geq \liminf_{i\to\infty} E_P[Z_{\tau^i}^{\uparrow}] = \liminf_{i\to\infty} E_P[U_{\tau^i}^{\uparrow}] \geq E_P[U_\tau^{\uparrow}],$$

where we have applied the P-supermartingale property of $Z^{\uparrow}$, and where the last inequality follows by Fatou's lemma since $U^{\uparrow}$ is right continuous.

3. We now prove the last part of the theorem. Let S be a càdlàg $\mathcal{P}$-supermartingale such that $S \geq H$. Then it follows that $E_P[H_\theta \mid \mathcal{F}_t] \leq E_P[S_\theta \mid \mathcal{F}_t] \leq S_t$ P-a.s. for $\theta \in \mathcal{T}[t,T]$. This implies that $S_t \geq Z_t^P$ for any $P \in \mathcal{P}$, and thus $S_t \geq Z_t^{\uparrow}$. Since the processes $U^{\uparrow}$ and S are right continuous we obtain that $S \geq U^{\uparrow}$.□

4.2.4 Existence of t-optimal times for the upper Snell envelope in continuous time

In this subsection we construct t-optimal stopping times for the upper Snell envelope of H in continuous time, for a class of processes satisfying the regularity condition of definition 4.19 below. The upper Snell envelope of theorem 4.3 will be crucial in the construction. We start with the next proposition which provides a characterization of t-optimal stopping times for the upper Snell envelope of H. Since the stochastic process H and the family of equivalent probability measures $\mathcal{P}$ are fixed, we simply write $U^{\uparrow}$ for the upper Snell envelope of H with respect to $\mathcal{P}$.

Proposition 4.18 *A stopping time $\tau^* \geq t$ is t-optimal for the upper Snell envelope of H if and only if the following two properties are satisfied:*

1. $$\operatorname{ess\,sup}_{P\in\mathcal{P}} E_P[U^{\uparrow}_{\tau^*} \mid \mathcal{F}_t] = \operatorname{ess\,sup}_{P\in\mathcal{P}} E_P[H_{\tau^*} \mid \mathcal{F}_t], \tag{4.24}$$

2. *for $s \geq t$, the stopped process $U^{\uparrow}_{\tau^*\wedge s}$ has the following property*
$$U^{\uparrow}_t = \operatorname{ess\,sup}_{P\in\mathcal{P}} E_P[U^{\uparrow}_{\tau^*\wedge s} \mid \mathcal{F}_t]. \tag{4.25}$$

Proof. Sufficiency. If we take $s = T$ on the second condition (4.25) then $U^{\uparrow}_t = \operatorname{ess\,sup}_{P\in\mathcal{P}} E_P[U^{\uparrow}_{\tau^*} \mid \mathcal{F}_t]$, and we conclude that τ^* is t-optimal for the upper Snell envelope from the first condition (4.24).

Necessity. Let τ^* be a t-optimal stopping time for the upper Snell envelope and consider the stopped process $U^{\uparrow}_{\tau^*\wedge s}$. In the following relationships, lemma 4.17 justifies the first and last equalities, while the second and third equalities follows from the t-optimality of the stopping time τ^*:

$$\begin{aligned}
\operatorname{ess\,sup}_{P\in\mathcal{P}} E_P[U^{\uparrow}_{\tau^*\wedge s} \mid \mathcal{F}_t] &= \operatorname{ess\,sup}_{P\in\mathcal{P}} \operatorname{ess\,sup}_{\theta\in\mathcal{T}[\tau^*\wedge s,T]} E_P[H_\theta \mid \mathcal{F}_t] \\
&= \operatorname{ess\,sup}_{P\in\mathcal{P}} E_P[H_{\tau^*} \mid \mathcal{F}_t] \\
&= \operatorname{ess\,sup}_{P\in\mathcal{P}} \operatorname{ess\,sup}_{\theta\in\mathcal{T}[t,T]} E_P[H_\theta \mid \mathcal{F}_t] \\
&= U^{\uparrow}_t.
\end{aligned}$$

This proves property (4.25), and if we take $s = T$ then (4.24) results.□

Let λ be a constant in $(0,1)$ and let θ be a stopping time in $\mathcal{T}$. We define

$$\tau_\theta^\lambda := \inf\{u \geq \theta \mid \lambda U_u^\uparrow \leq H_u\}. \tag{4.26}$$

These stopping times will be important for the construction of t-optimal stopping times for the upper Snell envelope of H. Our motivation to consider the family of stopping times $\{\tau_\theta^\lambda\}_{0<\lambda<1}$ are the stopping times D_T^λ in proposition 2.32 p. 130 of El Karoui[18].

In the theorem below we prove that τ_t^λ is a lower bound for any t-optimal stopping time and that the limit

$$\tau_t^\uparrow := \lim_{\lambda \to 1} \tau_t^\lambda$$

is a t-optimal stopping time. In theorem (4.20) we will assume the following regularity condition on H.

Definition 4.19 *Let $Y := \{Y_t\}_{0\leq t\leq T}$ be a positive $\mathbb{F}$-adapted process and let $t \in [0,T]$ be fixed. We say that Y is t-upper semicontinuous in expectation from the left with respect to the family $\mathcal{P}$ if for any increasing sequence of stopping times $\{\theta_i\}_{i=1}^\infty \subset \mathcal{T}[t,T]$ converging to θ, the following inequality holds*

$$\limsup_{i\to\infty} \text{ess sup}_{P\in\mathcal{P}} E_P[Y_{\theta_i} \mid \mathcal{F}_t] \leq \text{ess sup}_{P\in\mathcal{P}} E_P[Y_\theta \mid \mathcal{F}_t], \quad R-a.s.\Box \tag{4.27}$$

Theorem 4.20 *The upper Snell envelope $U^\uparrow(H,\mathcal{P})$ and the stopping time τ_t^λ defined in (4.26) are related by the identity*

$$U_t^\uparrow = \text{ess sup}_{P\in\mathcal{P}} E_P\left[U_{\tau_t^\lambda}^\uparrow \mid \mathcal{F}_t\right]. \tag{4.28}$$

Moreover, if H is t-upper semicontinuous in expectation from the left with respect to $\mathcal{P}$, then $\tau_t^\uparrow$ is a t-optimal stopping time for the upper Snell envelope of H in the sense of definition 4.5:

$$\text{ess sup}_{P\in\mathcal{P}} E_P[H_{\tau_t^\uparrow} \mid \mathcal{F}_t] = \text{ess sup}_{\theta\in\mathcal{T}[t,T]} \text{ess sup}_{P\in\mathcal{P}} E_P[H_\theta \mid \mathcal{F}_t].$$

Proof. Formula (4.18) in lemma 4.17 gives the inequality

$$\text{ess sup}_{P\in\mathcal{P}} E_P\left[U_{\tau_t^\lambda}^\uparrow \mid \mathcal{F}_t\right] \leq U_t^\uparrow.$$

In order to prove the opposite inequality, let $\theta \in \mathcal{T}[t,T]$. From the definition of the stopping time τ_θ^λ, and considering the events $\{\theta = \tau_\theta^\lambda\}$ and $\{\theta < \tau_\theta^\lambda\}$ separately, we obtain the estimate

$$H_\theta \leq \lambda U_\theta^\uparrow + (1-\lambda) U_{\tau_\theta^\lambda}^\uparrow.$$

If we take conditional expectation with respect to $P \in \mathcal{P}$, then we get

$$E_P\left[H_\theta \mid \mathcal{F}_t\right] \leq \lambda U_t^\uparrow + (1-\lambda) E_P[U_{\tau_t^\lambda}^\uparrow \mid \mathcal{F}_t],$$

since

$$\begin{aligned} E_P[U_{\tau_\theta^\lambda}^\uparrow \mid \mathcal{F}_t] &= E_P\left[E_P[1_{\{\tau_\theta^\lambda \leq \tau_t^\lambda\}} U_{\tau_\theta^\lambda}^\uparrow \mid \mathcal{F}_{\tau_t^\lambda}] + E_P[1_{\{\tau_\theta^\lambda > \tau_t^\lambda\}} U_{\tau_\theta^\lambda}^\uparrow \mid \mathcal{F}_{\tau_t^\lambda}] \mid \mathcal{F}_t\right] \\ &\leq E_P[1_{\{\tau_\theta^\lambda \leq \tau_t^\lambda\}} U_{\tau_t^\lambda}^\uparrow + 1_{\{\tau_\theta^\lambda > \tau_t^\lambda\}} U_{\tau_t^\lambda}^\uparrow \mid \mathcal{F}_t] \\ &= E_P[U_{\tau_t^\lambda}^\uparrow \mid \mathcal{F}_t]. \end{aligned}$$

Here we have used the fact that on the event $\{\tau_\theta^\lambda \leq \tau_t^\lambda\}$ the stopping times τ_θ^λ and τ_t^λ coincide, that the event $\{\tau_\theta^\lambda > \tau_t^\lambda\}$ is $\mathcal{F}_{\tau_t^\lambda}$-measurable, and that the process $\{U_t^\uparrow\}_{0 \leq t \leq T}$ is a $\mathcal{P}$-supermartingale. We have proved (4.28).

The second part of the theorem follows now from the previous step and the upper-semicontinuity in expectation of $\{H_t\}_{0 \leq t \leq T}$ with respect to $\mathcal{P}$. In fact, let $\{\lambda_i\}_{i=1}^\infty$ be an increasing sequence of numbers converging to 1. Then

$$\begin{aligned} U_t^\uparrow &= \limsup_{i\to\infty} \operatorname{ess\,sup}_{P\in\mathcal{P}} E_P\left[U_{\tau_t^{\lambda_i}}^\uparrow \mid \mathcal{F}_t\right] \\ &\leq \limsup_{i\to\infty} \frac{1}{\lambda_i} \operatorname{ess\,sup}_{P\in\mathcal{P}} E_P\left[H_{\tau_t^{\lambda_i}} \mid \mathcal{F}_t\right] \\ &\leq \operatorname{ess\,sup}_{P\in\mathcal{P}} E_P\left[H_{\tau_t^\uparrow} \mid \mathcal{F}_t\right]. \end{aligned}$$

To conclude we only have to notice that

$$U_t^\uparrow = Z_t^\uparrow = \operatorname{ess\,sup}_{\theta\in\mathcal{T}[t,T]} \operatorname{ess\,sup}_{P\in\mathcal{P}} E_P[H_\theta \mid \mathcal{F}_t]. \square$$

Remark 4.21 *Additional to the conditions of theorem 4.20, let us assume that H is upper semicontinuous in expectation from the left with respect to some $P_0 \in \mathcal{P}$. Then P_0-a.s. $U_{\tau_t^\uparrow}^\uparrow = H_{\tau_t^\uparrow}$. Indeed, for $\lambda \in (0,1)$ we know from (4.26) that $U_{\tau_t^\lambda}^\uparrow \leq (\lambda)^{-1} H_{\tau_t^\lambda}$. This inequality develops into*

$$E_{P_0}[U_{\tau_t^\uparrow}^\uparrow] \leq E_{P_0}[U_{\tau_t^\lambda}^\uparrow] \leq (\lambda)^{-1} E_{P_0}[H_{\tau_t^\lambda}],$$

where we have used the fact that $U^{\uparrow}$ is a P_0-supermartingal. But then

$$E_{P_0}[U^{\uparrow}_{\tau^{\uparrow}_t}] \leq \limsup_{\lambda \nearrow 1} (\lambda)^{-1} E_{P_0}[H_{\tau^{\lambda}_t}] \leq E_{P_0}[H_{\tau^{\uparrow}_t}],$$

since we have assumed that H is upper semicontinuous in expectation from the left with respect P_0. Since we know that $U^{\uparrow}_{\tau^{\uparrow}_t} \geq H_{\tau^{\uparrow}_t}$, then we conclude equality R-a.s.$\diamondsuit$

The condition of upper semicontinuity in expectation from the left with respect to the stable family $\mathcal{P}$ is rather strong and its verification may be hard. Without this condition, and considering only the case $t = 0$, the stopping times τ^{λ}_0 still provide ϵ-optimal stopping times as stated in the following corollary.

Corollary 4.22 *Let $\{H_t\}_{0\leq t\leq T}$ be a process satisfying the conditions of theorem 4.3. Then for any $\epsilon > 0$ there exists a stopping time $\tau^{\epsilon} \in \mathcal{T}$ such that*

$$U^{\uparrow}_0 \leq \epsilon + \sup_{P\in\mathcal{P}} E_P[H_{\tau^{\epsilon}}]. \tag{4.29}$$

Proof. We may assume $\epsilon < U^{\uparrow}_0$, and we take $\lambda = 1 - \epsilon(U^{\uparrow}_0)^{-1}$. We have the following relationships

$$U^{\uparrow}_0 = \sup_{P\in\mathcal{P}} E_P[U^{\uparrow}_{\tau^{\lambda}_0}] \leq \sup_{P\in\mathcal{P}} E_P[U^{\uparrow}_{\tau^{\lambda}_0} - H_{\tau^{\lambda}_0}] + \sup_{P\in\mathcal{P}} E_P[H_{\tau^{\lambda}_0}], \tag{4.30}$$

where in the first equality we have applied (4.28). By the definition of the stopping time τ^{λ}_0 we know that $\lambda U^{\uparrow}_{\tau^{\lambda}_0} \leq H_{\tau^{\lambda}_0}$, so that

$$U^{\uparrow}_{\tau^{\lambda}_0} - H_{\tau^{\lambda}_0} \leq (1-\lambda) U^{\uparrow}_{\tau^{\lambda}_0}.$$

We next apply P-expectation and then take the supremum over $P \in \mathcal{P}$ to obtain that

$$\sup_{P\in\mathcal{P}} E_P[U^{\uparrow}_{\tau^{\lambda}_0} - H_{\tau^{\lambda}_0}] \leq (1-\lambda) \sup_{P\in\mathcal{P}} E_P[U^{\uparrow}_{\tau^{\lambda}_0}] \leq (1-\lambda) U^{\uparrow}_0 = \epsilon. \tag{4.31}$$

Equations (4.30) and (4.31) imply (4.29).$\square$

4.3 Special cases

4.3.1 A study case based on compactness

In this subsection we specialize theorem 4.20 to the case $t = 0$ and a particular class of stable families of probability measures. The construction of

the stopping time in theorem 4.20 depended on upper semicontinuity in expectation from the left for the process H with respect to a stable family $\mathcal{P}$. When $\mathcal{P}$ reduces to a singleton this is the well-known condition which we introduced in definition 1.9.
Here we check upper semicontinuity from the left under the condition that the stable family is weakly compact, and that the process H is *quasi-left continuous* in the following sense: For any increasing sequence of stopping times $\{\theta_i\}_{i=1}^{\infty}$ converging to a stopping time $\theta \in \mathcal{T}$, we have

$$\lim_{i\to\infty} H_{\theta_i} = H_\theta, \quad R - a.s.$$

For equivalent formulations of this property see, e.g., theorem 32 p. 84 in Dellacherie[11].

Lemma 4.23 *Let $\mathcal{Q}$ be a convex family of probability measures absolutely continuous with respect to R. Assume that the family of densities of $\mathcal{Q}$ with respect to R is $\sigma(L^1(R), L^\infty(R))$-compact.*

Let $\{Y^i\}_{i=1}^{\infty}$ be a sequence of random variables such that there exists a constant $K \geq 0$ with

$$-K \leq Y^i \leq K, \quad R - a.s.$$

for $i \in \mathbb{N}$. Then, $\lim_{i\to\infty} Y^i = 0 \quad R - a.s.$ implies that

$$\lim_{i\to\infty} \sup_{Q\in\mathcal{Q}} E_Q[Y^i] = 0.$$

Proof. We first show that $\liminf_{i\to\infty} \sup_{Q\in\mathcal{Q}} E_Q[Y^i] \geq 0$. Indeed,

$$\liminf_{i\to\infty} \sup_{Q\in\mathcal{Q}} E_Q[Y^i] \geq \liminf_{i\to\infty} E_{Q_0}[Y^i] \geq E_{Q_0}[\liminf_{i\to\infty} Y^i] = 0,$$

where $Q_0 \in \mathcal{Q}$ is fixed, and in the last inequality we have applied Fatou's lemma.

We now show that $l = 0$, where

$$l := \limsup_{i\to\infty} \sup_{Q\in\mathcal{Q}} E_Q[Y^i].$$

For $\delta > 0$ arbitrary we have that

$$\begin{aligned} l &= \limsup_{i\to\infty} \sup_{Q\in\mathcal{Q}} \left\{ E_Q[Y^i; Y^i \le \delta] + E_Q[Y^i; Y^i > \delta] \right\} \\ &\le \delta + K \limsup_{i\to\infty} \sup_{Q\in\mathcal{Q}} Q[Y^i > \delta]. \end{aligned}$$

By assumption, the set of densities $\left\{\frac{dQ}{dR}\right\}_{Q\in\mathcal{Q}}$ is weakly compact, and by the Dunford-Pettis criterion of uniform integrability (see, e.g., theorem IV.8.9 in Dunford and Schwartz[16]), $\left\{\frac{dQ}{dR}\right\}_{Q\in\mathcal{Q}}$ is uniformly integrable with respect to R. But $\lim_{i\to\infty} R[Y^i > \delta] = 0$, and then uniform integrability implies the following uniform absolutely continuous property for the family $\mathcal{Q}$ (see e.g., Meyer[44] theorem 19 part (b))

$$\limsup_{i\to\infty} \sup_{Q\in\mathcal{Q}} Q[Y^i > \delta] = 0.$$

We infer that $l \le \delta$, and because δ was arbitrary we conclude that $l = 0$.□

Proposition 4.24 *Assume that the stable family of probability measures $\mathcal{P}$ satisfies the condition of lemma 4.23. Furthermore, assume that the process H is bounded in the sense that $H \le K$ for some constant $K > 0$. If H is quasi-left continuous, then it is 0-upper semicontinuous in expectation from the left with respect to $\mathcal{P}$.*

Proof. Let $\{\theta^i\}_{i=1}^{\infty} \subset \mathcal{T}$ be an increasing sequence of stopping times converging to θ. We want to show that

$$\limsup_{i\to\infty} \sup_{P\in\mathcal{P}} E_P\left[H_{\theta^i}\right] \le \sup_{P\in\mathcal{P}} E_P\left[H_\theta\right].$$

It is enough to prove that

$$\limsup_{i\to\infty} \sup_{P\in\mathcal{P}} E_P\left[H_{\theta^i} - H_\theta\right] = 0,$$

since

$$\limsup_{i\to\infty} \sup_{P\in\mathcal{P}} E_P\left[H_{\theta^i}\right] \le \limsup_{i\to\infty} \left\{ \sup_{P\in\mathcal{P}} E_P\left[H_{\theta^i} - H_\theta\right] + \sup_{P\in\mathcal{P}} E_P\left[H_\theta\right] \right\}.$$

Let us consider the random variable defined by

$$Y^i := H_{\theta^i} - H_\theta.$$

Then Y^i is a $\mathcal{F}_\theta$-measurable random variable with $-K \le Y^i \le K$. Furthermore, $R(\lim_{i\to\infty} Y^i = 0) = 1$ since the process H is quasi-left continuous.

We can apply lemma 4.23 to the sequence $\{Y^i\}_{i=1}^\infty$ to conclude that

$$\lim_{i\to\infty} \sup_{P\in\mathcal{P}} E_P[Y^i] = 0.$$

Thus

$$\lim_{i\to\infty} \sup_{P\in\mathcal{P}} E_P[H_{\theta^i} - H_\theta] = 0,$$

as needed to be proved.□

Theorem 4.20 and proposition 4.24 imply the following corollary.

Corollary 4.25 *Assume the conditions of proposition 4.24. Then there exists a* 0*-optimal stopping time for the upper Snell envelope of* H *in the sense of definition 4.5.*

Proof. According to proposition 4.24, H is $0-$upper semicontinuous in expectation from the left with respect to $\mathcal{P}$. In this case, theorem 4.20 says that the stopping time $\tau_0^\uparrow$ is a 0-optimal stopping time for the upper Snell envelope of H.□

4.3.2 Absolutely continuous martingale measures

In the special case where $\mathcal{P}$ is the set of equivalent martingale measures for the price process X, we may consider the family of martingale measures which are absolutely continuous with respect to R

$$\mathcal{M}^a := \{P \ll R \mid X \text{ is a local martingale with respect to } P\},$$

to define an upper envelope. In many interesting models the inclusion $\mathcal{M} \subset \mathcal{M}^a$ will be strict; see for example theorem 5.4 in Delbaen Schachermayer[9]. The next proposition shows that the upper Snell envelope remains unchanged if we replace the class $\mathcal{M}$ used in (1.4), resp. (4.10), by $\mathcal{M}^a$.

Proposition 4.26 *The upper Snell envelope has the following property. For any* $P^a \in \mathcal{M}^a$ *and any stopping time* $\tau \in \mathcal{T}[t,T]$*, the following inequality holds* P^a*-a.s.*

$$E_{P^a}[H_\tau \mid \mathcal{F}_t] \le U_t^\uparrow.$$

Proof. Let $P^a \in \mathcal{M}^a$ be an absolutely continuous martingale measure and $P \in \mathcal{M}$ be any equivalent martingale probability measure. For $i \in \mathbb{N}$ define

$$P^i := (1 - \frac{1}{i})P^a + \frac{1}{i}P.$$

Let $\tau \in \mathcal{T}[t, T]$; then the conditional expectation with respect to P^i is

$$E_{P^i}[H_\tau \mid \mathcal{F}_t] = (1 - \frac{1}{i})E_{P^a}[H_\tau \mid \mathcal{F}_t] + \frac{1}{i}E_P[H_\tau \mid \mathcal{F}_t], \tag{4.32}$$

in particular setting $t = 0$ implies that H_τ is P^a-integrable and condition (1.3) implies the apparently stronger condition:

$$\sup_{P \in \mathcal{M}^a} \sup_{\theta \in \mathcal{T}} E_P[H_\theta] < \infty.$$

In the right-hand term at (4.32), the first conditional expectation is P^a integrable and the second is P integrable, from this we get that

$$\lim_{i \to \infty} E_{P^i}[H_\tau \mid \mathcal{F}_t] = E_{P^a}[H_\tau \mid \mathcal{F}_t],\ P^a - \text{a.s.},$$

and the proof concludes with the inequality

$$E_{P^a}[H_\tau \mid \mathcal{F}_t] \leq \limsup\nolimits_{i \to \infty} E_{P^i}[H_\tau \mid \mathcal{F}_t] \leq U_t^{\uparrow}. \square$$

4.3.3 Existence of t-optimal times for the upper Snell envelope in discrete time

In this section we show how the problem 4.5 can be solved in discrete time following the analysis in section 6.5 of [27]. Here we consider a stochastic base in discrete time of the form

$$(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t \in \{0, \ldots, T\}}, R).$$

The family of stopping times is again denoted by $\mathcal{T}$.

Theorem 4.27 *Let $\mathcal{P}$ be a stable family of probability measures defined in the probability space $(\Omega, \mathcal{F}, R)$. Let $K := \{K_t\}_{t \in \{0, \cdots T\}}$ be a positive $\mathbb{F}$-adapted process satisfying*

$$\sup_{P \in \mathcal{P}} E_P[K_t] < \infty, \tag{4.33}$$

for any $t \in \{0, \cdots, T\}$. Let the upper Snell envelope of K with respect to $\mathcal{P}$ be defined by

$$U_t^{\uparrow} = \operatorname{ess\,sup}_{P \in \mathcal{P}} \operatorname{ess\,sup}_{\theta \in \mathcal{T}[t, T]} E_P[K_\theta \mid \mathcal{F}_t].$$

Then, for $t \in \{0, \cdots, T\}$, the stopping time defined by

$$\tau_t^{\uparrow} := \inf\{s \geq t \mid K_s = U_s^{\uparrow}\},$$

is a t-optimal stopping time for the upper Snell envelope of K with respect to $\mathcal{P}$ in the discrete-time formulation of definition 4.5.

Proof. Clearly we have $R(\tau_t^{\uparrow} \leq T) = 1$, since $U_T^{\uparrow} = K_T$. The first part of theorem 4.29 below says that the upper Snell envelope can be computed in a recursive way as

$$U_t^{\uparrow} = K_t \vee \operatorname{ess\,sup}_{P \in \mathcal{P}} E_P[U_{t+1}^{\uparrow} \mid \mathcal{F}_t]. \tag{4.34}$$

Through an induction argument we can see that

$$U_t^{\uparrow} = K_t \vee \operatorname{ess\,sup}_{P \in \mathcal{P}} E_P[\operatorname{ess\,sup}_{P \in \mathcal{P}} E_P[K_{\tau_{t+1}^{\uparrow}} \mid \mathcal{F}_{t+1}] \mid \mathcal{F}_t].$$

The martingale property as stated in the second part of theorem 4.29 implies that

$$U_t^{\uparrow} = K_t \vee \operatorname{ess\,sup}_{P \in \mathcal{P}} E_P[K_{\tau_{t+1}^{\uparrow}} \mid \mathcal{F}_t]. \tag{4.35}$$

The obvious relationships

$$\tau_t^{\uparrow} = 1_{\{\tau_t^{\uparrow} = t\}} t + 1_{\{\tau_t^{\uparrow} > t\}} \tau_{t+1}^{\uparrow}$$

combined with (4.35) lead to

$$U_t^{\uparrow} = \operatorname{ess\,sup}_{P \in \mathcal{P}} E_P[K_{\tau_t^{\uparrow}} \mid \mathcal{F}_t].$$

In fact, the inequality

$$K_t \vee \operatorname{ess\,sup}_{P \in \mathcal{P}} E_P[K_{\tau_{t+1}^{\uparrow}} \mid \mathcal{F}_t] \geq \operatorname{ess\,sup}_{P \in \mathcal{P}} E_P[K_{\tau_t^{\uparrow}} \mid \mathcal{F}_t],$$

follows easily from the identity

$$\operatorname{ess\,sup}_{P \in \mathcal{P}} E_P[K_{\tau_t^{\uparrow}} \mid \mathcal{F}_t] = 1_{\{\tau_t^{\uparrow} = t\}} K_t + 1_{\{\tau_t^{\uparrow} > t\}} \operatorname{ess\,sup}_{P \in \mathcal{P}} E_P[K_{\tau_{t+1}^{\uparrow}} \mid \mathcal{F}_t].$$

For the opposite inequality we must prove that

$$1_{\{\tau_t^{\uparrow} = t\}} \left\{ K_t \vee \operatorname{ess\,sup}_{P \in \mathcal{P}} E_P[K_{\tau_{t+1}^{\uparrow}} \mid \mathcal{F}_t] \right\} \leq 1_{\{\tau_t^{\uparrow} = t\}} K_t,$$
$$1_{\{\tau_t^{\uparrow} > t\}} \left\{ K_t \vee \operatorname{ess\,sup}_{P \in \mathcal{P}} E_P[K_{\tau_{t+1}^{\uparrow}} \mid \mathcal{F}_t] \right\} \leq 1_{\{\tau_t^{\uparrow} > t\}} \operatorname{ess\,sup}_{P \in \mathcal{P}} E_P[K_{\tau_{t+1}^{\uparrow}} \mid \mathcal{F}_t].$$

Both inequalities follow from the set inclusion

$$\left\{ K_t > \operatorname{ess\,sup}_{P \in \mathcal{P}} E_P[K_{\tau_{t+1}^{\uparrow}} \mid \mathcal{F}_t] \right\} \subset \left\{ \tau_t^{\uparrow} = t \right\},$$

which is justified by (4.34) and the definition of the stopping time $\tau_t^{\uparrow}$. We have proved the theorem.□

Remark 4.28 *If the stopping time $\tau_t^{\uparrow}$ is adopted as an optimal exercise rule, then the optimality criterion does require a patient attitude. To explain this, notice that if U^P denotes the Snell envelope of K with respect to $P \in \mathcal{P}$, then $K_{\tau_t^{\uparrow}} = U^P_{\tau_t^{\uparrow}}$. If τ_t^P denotes the minimal optimal stopping time of H with respect to P, then the discrete-time version of theorem 1.10 (see [27]) implies that $\tau_t^P \leq \tau_t^{\uparrow}$. This means that the option will be exercised after each minimal optimal stopping time corresponding to any probability measure $P \in \mathcal{P}$.*$\diamondsuit$

The next theorem taken from Föllmer and Schied[27] states that the upper Snell envelope in discrete time can be computed in a recursive way, and that the operator $\operatorname{ess\,sup}_{P\in\mathcal{P}} E_P[\cdot]$ satisfies an analogous property to the martingale property.

Theorem 4.29 *Let $\{K_t\}_{t\in\{0,\cdots T\}}$ be as in theorem 4.27, and let $U^{\uparrow}$ denote its upper Snell envelope. The following assertions hold true:*

1. *The upper Snell envelope $U^{\uparrow}$ can be computed in the following recursive way:*
$$U_t^{\uparrow} = K_t \vee \operatorname{ess\,sup}_{P\in\mathcal{P}} E_P[U_{t+1}^{\uparrow} \mid \mathcal{F}_t].$$

2. *Let $Y \geq 0$ be an $\mathcal{F}_T$-measurable random variable such that $V_0 < \infty$, where*
$$V_t := \operatorname{ess\,sup}_{P\in\mathcal{P}} E_P[Y \mid \mathcal{F}_t].$$
The nonlinear operator $\operatorname{ess\,sup}_{P\in\mathcal{P}} E_P[\cdot \mid \cdot]$ satisfies the following property:
$$V_\sigma = \operatorname{ess\,sup}_{P\in\mathcal{P}} E_P[V_\tau \mid \mathcal{F}_\sigma] \tag{4.36}$$
for any stopping times $\sigma, \tau \in \mathcal{T}$ with $\sigma \leq \tau$.

Proof. See theorems 6.52 and 6.53 in Föllmer and Schied[27].□

4.3.4 Stopping times of maximal risk

In this section we motivate the robust optimal stopping problem formulated in definition 4.5 from the point of view of convex risk measures. Let us first recall from Föllmer and Schied[28] the definition and some basic properties of convex risk measures.

Definition 4.30 *Let $\mathcal{X}$ be a linear space of bounded functions containing the constants. A mapping $\rho : \mathcal{X} \to \mathbb{R}$ is called a monetary measure of risk if it satisfies the following conditions for all $X, Y \in \mathcal{X}$.*

1. *Monotonicity: If* $X \leq Y$*, then* $\rho(X) \geq \rho(Y)$*.*

2. *Cash invariance: If* $m \in \mathbb{R}$*, then* $\rho(X+m) = \rho(X) - m$*.*□

Definition 4.31 *A monetary risk measure* $\rho : \mathcal{X} \to \mathbb{R}$ *is called a convex measure of risk if it satisfies*

- *Convexity:* $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$*, for* $0 \leq \lambda \leq 1$*.*□

Definition 4.32 *A monetary measure of risk* $\rho : \mathcal{X} \to \mathbb{R}$ *is called a coherent measure of risk if it satisfies*

- *Positive homogeneity: If* $\lambda \geq 0$*, then* $\rho(\lambda X) = \lambda\rho(X)$*.*□

We now introduce a probability space $(\Omega, \mathcal{F}, R)$ and consider convex risk measures defined on the Banach space $\mathcal{X} = L^\infty(R)$.

Remark 4.33 *A convex risk measure* ρ *on the Banach space of bounded measurable function on* $(\Omega, \mathcal{F})$ *may be viewed as a convex risk measure on* $L^\infty(R)$ *if it respects the* R*-null sets, i.e.,*

$$\rho(X) = \rho(Y), \qquad \text{if } X = Y \quad R - a.s. \diamondsuit$$

The next theorem clarifies the structure of a convex risk measure on $L^\infty(R)$, by extending Delbaen's representation theorem for coherent measures of risk to the general convex case; see [8] theorem 3.2.

Theorem 4.34 *Suppose* $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, R)$*,* $\mathcal{P}$ *is the set of probability measures* $P \ll R$*, and* $\rho : \mathcal{X} \to \mathbb{R}$ *is a convex measure of risk. Then the following properties are equivalent.*

1. *There is a "penalty function"* $\alpha : \mathcal{P} \to (-\infty, \infty]$ *such that*

$$\rho(X) = \sup_{P \in \mathcal{P}} \left(E_P[-X] - \alpha(P) \right), \text{ for all } X \in \mathcal{X}, \tag{4.37}$$

with

$$\alpha(P) := \sup_{X \in \mathcal{X}} \left(E_P[-X] - \rho(X) \right).$$

2. ρ *possesses the Fatou property: If the sequence* $(X_n)_{n\in\mathbb{N}} \subset \mathcal{X}$ *is uniformly bounded, and* X_n *converges to some* $X \in \mathcal{X}$ *in probability, then* $\rho(X) \leq \liminf_n \rho(X_n)$*.*

3. *If the sequence* $(X_n)_{n\in\mathbb{N}} \subset \mathcal{X}$ *decreases to* $X \in \mathcal{X}$*, then* $\rho(X_n) \to \rho(X)$*.*

In the coherent case, the representation (4.37) reduces to the representation

$$\rho(X) = \sup_{Q\in\mathcal{Q}} E_Q[-X], \text{ for all } X \in \mathcal{X}. \tag{4.38}$$

for the family $\mathcal{Q} = \{Q \in \mathcal{P} \mid \alpha(Q) = 0\}$.

Proof. See theorem 6 in Föllmer and Schied[29], parts *1, 3, 4*.□

It is the robust representation (4.37) resp. (4.38) which will allow us to motivate the robust stopping problem 4.5 in the context of convex risk measures.

To this end, consider a filtration $\mathbb{F}$ of the probability space $(\Omega, \mathcal{F}, R)$ satisfying the usual conditions of right continuity and completeness. Let $H := \{H_t\}_{0\le t\le T}$ be a positive càdlàg $\mathbb{F}$-adapted stochastic process. In order to avoid technical difficulties we assume that $H \le K$ for some constant $K > 0$.

Typically, the stochastic process H represents the evolution of a financial position with an uncertain liquidation date. An important example would be an American put option, seen from the point of view of the seller.

We can now consider the maximal risk defined as

$$\sup_{\theta\in\mathcal{T}} \rho(-H_\theta) = \sup_{\theta\in\mathcal{T}} \sup_{Q\in\mathcal{Q}} \left\{E_Q[H_\theta] - \alpha(Q)\right\}.$$

Our discussion of the robust problem covers the *coherent* case (4.38) under the assumption that the class $\mathcal{Q}$ is stable under pasting. Note that this version of the robust stopping problem is no longer preference free since the risk measure involves the investor's preference in the face of uncertainty.

With this interpretation, a 0-optimal stopping time for the upper Snell envelope is a stopping time τ_0^* with the property that

$$\rho(-H_{\tau_0^*}) = \sup_{\theta\in\mathcal{T}} \rho(-H_\theta).$$

We may thus say that τ_0^* is a stopping time of maximal risk for the process H, if risk is quantified by the risk measure ρ.

From this point of view, it would be interesting to extend our analysis of the robust stopping problem in such a way that it covers the *convex* case. This would involve an additional penalization in the formulation of the stopping problem.

Chapter 5

The lower Snell envelope and stopping times of maximal utility

In this chapter we study the lower Snell envelope $U^{\downarrow}(H, \mathcal{M})$ introduced in equation (1.16) of chapter 1. This envelope was associated to a process $H := \{H_t\}_{0 \le t \le T}$ representing an American option and to the family of equivalent martingale measures $\mathcal{M}$. In section 1.5 we explained how this envelope allows us to characterize the infimum $\pi_{\inf}(H)$ of the arbitrage free prices interval; see theorem 1.20.
The origin of this envelope goes back to El Karoui and Quenez[19] for European options in an incomplete geometric Brownian motion model. Föllmer and Schied[27] extended the lower envelope to American options for a general *stable* family of equivalent probability measures $\mathcal{P}$ in discrete time; see section 6.5 in [27]. Karatzas and Kou[35] constructed lower envelopes for American options in an incomplete model driven by a multidimensional Brownian motion.

In this chapter, we fix a stable family $\mathcal{P}$ of equivalent probability measures and a stochastic process $H := \{H_t\}_{0 \le t \le T}$. Our first goal is to solve a robust version of the optimal stopping problem which is motivated by model ambiguity. In a given time t, we want to maximize

$$\operatorname{ess\,sup}_{\theta \in \mathcal{T}[t,T]} \operatorname{ess\,inf}_{P \in \mathcal{P}} E_P[H_\theta \mid \mathcal{F}_t],$$

where the essential supremum is taken over stopping times $\theta \ge t$, and the essential infimum is taken over the class $\mathcal{P}$.

Our second goal in this chapter is to construct a good version of the value process corresponding to our robust optimal stopping problem. In our construction we use a result from Dellacherie[12] on the essential infimum of a class of stochastic processes.

5.1 Problem formulation

We fix some notation to be used throughout this chapter. We are given a stochastic base $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}, R)$. The probability measure R is a reference measure. We assume that the filtration $\mathbb{F}$ satisfies the usual assumptions of right continuity and completeness and $\mathcal{F}_T = \mathcal{F}$. $\mathcal{P}$ will denote a fixed *stable* family of equivalent probability measures. We fix a positive càdlàg $\mathbb{F}$-adapted process $H := \{H_t\}_{0\le t\le T}$ with

$$\sup_{\theta\in\mathcal{T}} E_P[H_\theta] < \infty, \tag{5.1}$$

for any $P \in \mathcal{P}$.

Let us recall that for τ a stopping time and $P \in \mathcal{P}$ we have defined

$$Z^P_\tau = \text{ess sup}_{\theta\in\mathcal{T}[\tau,T]} E_P\left[H_\theta \mid \mathcal{F}_\tau\right]$$

in (4.9), and let us introduce the random variable

$$Z^\downarrow_\tau = Z^\downarrow_\tau(H,\mathcal{P}) := \text{ess inf}_{P\in\mathcal{P}} Z^P_\tau = \text{ess inf}_{P\in\mathcal{P}} \text{ess sup}_{\theta\in\mathcal{T}[\tau,T]} E_P\left[H_\theta \mid \mathcal{F}_\tau\right]. \tag{5.2}$$

In a first step, we define the lower Snell envelope associated to H with respect to the family $\mathcal{P}$ to be the collection

$$Z^\downarrow = Z^\downarrow(H,\mathcal{P}) := \{Z^\downarrow_\tau\}_{\tau\in\mathcal{T}}. \tag{5.3}$$

We now introduce t-optimal stopping times for the lower Snell envelope.

Definition 5.1 *Let $t \in [0,T]$ be a constant time. We say that a stopping time $\sigma^*_t \in \mathcal{T}[t,T]$ is t-optimal for the lower Snell envelope of H if*

$$Z^\downarrow_t = \text{ess inf}_{P\in\mathcal{P}} E_P[H_{\sigma^*_t} \mid \mathcal{F}_t].\square$$

In theorem 5.6 we show how to construct t-optimal stopping times. The existence of a t-optimal stopping times for the lower Snell envelope has two important consequences. It implies an identity of the form

$$\text{ess inf}_{P\in\mathcal{P}} \text{ess sup}_{\theta\in\mathcal{T}[t,T]} E_P\left[H_\theta \mid \mathcal{F}_t\right] = \text{ess sup}_{\theta\in\mathcal{T}[t,T]} \text{ess inf}_{P\in\mathcal{P}} E_P\left[H_\theta \mid \mathcal{F}_t\right],$$

and that the robust stopping problem

$$\text{ess}\sup_{\theta\in\mathcal{T}[t,T]}\text{ess}\inf_{P\in\mathcal{P}} E_P\left[H_\theta \mid \mathcal{F}_t\right],$$

has a solution. We show this facts in corollary 5.8. For the case $t = 0$, we motivate this robust stopping problem from the point of view of robust utility functionals in subsection 5.3.4. This is a robust stopping problem for continuous time. For the case $t = 0$, it is natural to ask whether the robust stopping problem can be reduced to a classical stopping problem with respect to a worst-case probability measure in the sense that there exists a measure $P^* \in \mathcal{P}$ with

$$\sup_{\theta\in\mathcal{T}}\inf_{P\in\mathcal{P}} E_P[H_\theta] = \sup_{\theta\in\mathcal{T}} E_{P^*}[H_\theta].$$

Based on compactness arguments and the results of corollary 5.8, we will prove in corollaries 5.15 and 5.17 the existence of a worst-case probability measure.

Remark 5.2 *Zamfirescu[55] studies a robust stopping problem similar to the problem of definition 5.1. The setting she considers is the same as we explained in remark 4.6, without assumptions involving the stability under pasting, which is crucial for our approach. Riedel[49] studied a robust stopping problem in discrete time and finite horizon similar to our problem 5.1 for* $t = 0$. *His formulation is analogous to our problem since he works with a family of equivalent probability measures having a property equivalent to stability under pasting. For the discrete-time case, he solves the robust stopping problem in part (iii) of his theorem 3.7, where he constructs an optimal stopping time for the case* $t = 0$.$\diamondsuit$

Now, looking back to the construction of the upper Snell envelope, it is natural to ask for an optional process $\{U_t^{\downarrow}\}_{0\leq t\leq T}$ such that

$$U_t^{\downarrow} = Z_t^{\downarrow}, \quad R - a.s.,$$

for all $t \in [0, T]$. We give an affirmative answer to this existence problem in theorem 5.21 as follows. We consider the collection of Snell envelopes of the process H with respect to every probability measure $P \in \mathcal{P}$ and then construct the essential infimum of this collection in a sense we define below. From this procedure we obtain an optional process. We cannot prove the existence of a càdlàg version, the main difficulty being that in general this process is not a submartingale nor a supermartingale and the well known procedure to regularize trajectories does not apply.

5.2 Solution

5.2.1 Existence of t-optimal stopping times for the lower Snell envelope

We prepare the construction of a t-optimal stopping time for the lower Snell envelope with two lemmas. The first one is similar to lemma 4.15, the main difference is that lemma 5.3 is stated in terms of the random variables $\left\{Z_\tau^P\right\}_{P\in\mathcal{P}}$ instead of the family $\{E_P\left[H_\theta \mid \mathcal{F}_\tau\right] \mid \theta \in \mathcal{T}[\tau,T]\}$. It extends the first part of lemma 6.50 in [27].

Lemma 5.3 *Let $\tau \in \mathcal{T}$ be a fixed stopping time and $P_0 \in \mathcal{P}$ be a fixed probability measure.*

1. *The family $\{Z_\tau^P \mid P \in \mathcal{P}\}$ is directed downwards, that is, for $P_1, P_2 \in \mathcal{P}$ there exists $P_3 \in \mathcal{P}$ such that*
$$Z_\tau^{P_3} = Z_\tau^{P_1} \wedge Z_\tau^{P_2}. \tag{5.4}$$
2. *There exists a sequence $\{P^i\}_{i=1}^\infty \subset \mathcal{P}(P_0,\tau)$ such that*
$$Z_\tau^{P^i} \searrow Z_\tau^{\downarrow}. \tag{5.5}$$

Proof.

1. Let $P_1, P_2 \in \mathcal{P}$ and $B := \{Z_\tau^{P_1} \geq Z_\tau^{P_2}\}$. We define the stopping time
$$\sigma := \tau 1_B + T 1_{B^c},$$
and let P_3 be the pasting of P_1 and P_2 in σ. Now we show that
$$Z_\tau^{P_3} = Z_\tau^{P_1} \wedge Z_\tau^{P_2}.$$
The following formula holds
$$E_{P_2}[H_\theta \mid \mathcal{F}_{\tau\vee\sigma}] = 1_B E_{P_2}[H_\theta \mid \mathcal{F}_\tau] + 1_{B^c} H_\theta,$$
and from lemma 4.11 we deduce that
$$E_{P_3}[H_\theta \mid \mathcal{F}_\tau] = 1_B E_{P_2}[H_\theta \mid \mathcal{F}_\tau] + 1_{B^c} E_{P_1}[H_\theta \mid \mathcal{F}_\tau].$$
This equality together with the obvious fact that $B \cap B^c = \emptyset$ implies
$$\begin{aligned}\operatorname{ess\,inf}_{\theta\in\mathcal{T}[\tau,T]} E_{P_3}[H_\theta \mid \mathcal{F}_\tau] = {} & 1_B \operatorname{ess\,inf}_{\theta\in\mathcal{T}[\tau,T]} E_{P_2}[H_\theta \mid \mathcal{F}_\tau] \\ & + 1_{B^c} \operatorname{ess\,inf}_{\theta\in\mathcal{T}[\tau,T]} E_{P_1}[H_\theta \mid \mathcal{F}_\tau],\end{aligned}$$
which proves (5.4).

2. The first part of the lemma implies the existence of a sequence $\{\widetilde{P}^i\}_{i=1}^{\infty} \subset \mathcal{P}$ such that $Z_\tau^{\widetilde{P}^i} \searrow Z_\tau^{\downarrow}$. Now, let $P^0 := P_0$ and by way of induction define the following elements
$$\begin{aligned} B_i &:= \{Z_\tau^{P^{i-1}} \geq Z_\tau^{\widetilde{P}^i}\}, \\ \sigma_i &:= 1_{B_i}\tau + 1_{B_i^c}T, \\ P^i &:= \text{ the pasting of } P^{i-1} \text{ and } \widetilde{P}^i \text{ in } \sigma_i. \end{aligned}$$
Note that $P^i = P^{i-1}$ in $\mathcal{F}_{\sigma_i}$ and this implies that $P^i = P_0$ in $\mathcal{F}_\tau$. A computation as in the first step shows that $Z_\tau^{P^i} = Z_\tau^{P^{i-1}} \wedge Z_\tau^{\widetilde{P}^i}$; thus, the sequence $\{P^i\}_{i=1}^{\infty}$ has the desired property.□

Lemma 5.4 *Let $P_0 \in \mathcal{P}$ be arbitrary but fixed. Then*
$$E_{P_0}[Z_t^{\downarrow}] = \inf_{P \in \mathcal{P}} E_{P_0}[Z_t^P]. \tag{5.6}$$
Moreover, for any pair of stopping times $\tau, \theta \in \mathcal{T}$ with $\theta \in \mathcal{T}[\tau, T]$ we have
$$E_{P_0}[\operatorname{ess\,inf}_{P \in \mathcal{P}} E_P\left[H_\theta \mid \mathcal{F}_\tau\right]] = \inf_{P \in \mathcal{P}} E_{P_0}[E_P\left[H_\theta \mid \mathcal{F}_\tau\right]] = \inf_{P \in \mathcal{P}(P_0, \tau)} E_P[H_\theta]. \tag{5.7}$$

Proof. The proof is analogous to the one in lemma 4.16, for the case $\tau = 0$.□

We next introduce the concept of *class*(D) with respect to a family of probability measures $\mathcal{P}$ equivalent to R.

Definition 5.5 *Let $\mathcal{P}$ be a family of probability measures equivalent to R. A process is said to be of class(D) with respect to $\mathcal{P}$ if it is of class(D) with respect to every $P \in \mathcal{P}$.*□

The next theorem is the main result of this section.

Theorem 5.6 *Assume that the process H is of class(D) with respect to $\mathcal{P}$ and is upper semicontinuous in expectation from the left with respect to all $P \in \mathcal{P}$. Then, the stopping time τ_t^P defined by*
$$\tau_t^P := \inf\{u \geq t \mid U_u^P \leq H_u\},$$
is P-optimal in the following sense:
$$E_P[H_{\tau_t^P}] = \sup_{\theta \in \mathcal{T}[t,T]} E_P[H_\theta]. \tag{5.8}$$
Moreover, we have the following assertions:

1. *The random time*
$$\tau_t^{\downarrow} := \operatorname{ess\,inf}_{P \in \mathcal{P}} \tau_t^P, \tag{5.9}$$
is a stopping time and is a t-optimal stopping time for the lower Snell envelope of H:
$$Z_t^{\downarrow} = \operatorname{ess\,inf}_{P \in \mathcal{P}} E_P[H_{\tau_t^{\downarrow}} \mid \mathcal{F}_t]. \tag{5.10}$$

2. *The stopped process $\{Z^{\downarrow}_{\tau_t^{\downarrow} \wedge s}\}_{s \in [t,T]}$, is a $\mathcal{P}$-submartingale from time t on.*

Proof. The optimality of τ_t^P in the sense of the equality (5.8) follows from theorem 1.10.

1. First we show that the family $\{\tau_t^P\}_{P \in \mathcal{P}}$ is directed downwards. Let $P_1, P_2 \in \mathcal{P}$ and let $A := \{\tau_t^{P_1} \geq \tau_t^{P_2}\}$, $\sigma := 1_A \tau_t^{P_2} + 1_{A^c} T$ and P_3 the pasting of P_1 and P_2 in σ. Then
$$Z^{P_3}_{\tau_t^{P_1} \wedge \tau_t^{P_2}} = Z^{P_2}_{\tau_t^{P_2}} \, 1_A + Z^{P_1}_{\tau_t^{P_1}} 1_{A^c},$$
and this implies that $\tau_t^{P_3} \leq \tau_t^{P_1} \wedge \tau_t^{P_2}$. We conclude the existence of a sequence $\{P_i\}_{i=1}^{\infty} \subset \mathcal{P}$ such that
$$\tau_t^{P_i} \searrow \operatorname{ess\,inf}_{P \in \mathcal{P}} \tau_t^P, \tag{5.11}$$
so that $\tau_t^{\downarrow}$ is in fact a stopping time.

   Moreover, let $P_0 \in \mathcal{P}$ be arbitrary but fixed. Then there exists a sequence $\{\widetilde{P}_i\}_{i=1}^{\infty}$ constructed in the same way as $\{P_i\}_{i=1}^{\infty}$ and such that $\tau_t^{\widetilde{P}_i} \leq \tau_t^{P_i} \wedge \tau_t^{P_0}$ with the further property that
$$\widetilde{P}_i = P_0 \text{ in } \mathcal{F}_{\tau_t^{\widetilde{P}_i}}.$$

2. Now we prove (5.10). Only the inequality
$$Z_t^{\downarrow} \leq \operatorname{ess\,inf}_{P \in \mathcal{P}} E_P[H_{\tau_t^{\downarrow}} \mid \mathcal{F}_t],$$
needs a proof. We first note that for any $P \in \mathcal{P}$ the inequality $\tau_t^{\downarrow} \leq \tau_t^P$ holds P-a.s. and infer that
$$Z_t^P = E_P\left[Z^P_{\tau_t^{\downarrow}} \mid \mathcal{F}_t\right] \geq E_P\left[H_{\tau_t^{\downarrow}} \mid \mathcal{F}_t\right], \tag{5.12}$$
where we have used the fact that the random variable $Z^P_{\tau_t^{\downarrow}}$ is equal P-a.s. to the Snell envelope of H with respect to P stopped in $\tau_t^{\downarrow}$, and

the fact that the stopped process $\{U^P_{\tau_t^\downarrow \wedge s}\}_{s\in[t,T]}$ is a P-martingale from time t on; see theorem 1.10.

Let $P_0 \in \mathcal{P}$ be fixed but arbitrary and let $\{P_i\}_{i=1}^\infty \subset \mathcal{P}$ be a sequence of probability measures such that

$$\tau_t^{P_i} \searrow \operatorname{ess\,inf}_{P\in\mathcal{P}} \tau_t^P \text{ and } P_i = P_0 \text{ in } \mathcal{F}_{\tau_t^{P_i}}, \tag{5.13}$$

as constructed in the previous step.

By definition of the stopping time $\tau_t^{P_i}$, we have that

$$Z^{P_i}_{\tau_t^{P_i}} = H_{\tau_t^{P_i}}. \tag{5.14}$$

If we take limits on both sides of this identity, then we obtain:

$$H_{\tau_t^\downarrow} = \lim_{i\to\infty} H_{\tau_t^{P_i}} = \lim_{i\to\infty} Z^{P_i}_{\tau_t^{P_i}}. \tag{5.15}$$

In the first equality we have used the fact that the process H is right-continuous, and in the second equality we have used (5.14).

Now, for $A \in \mathcal{F}_t$ the equality (5.15) develops into

$$\begin{aligned}
\int_A Z_t^\downarrow dP_0 &\leq \int_A \liminf_{i\to\infty} Z_t^{P_i} dP_0 \\
&\leq \liminf_{i\to\infty} \int_A Z_t^{P_i} dP_0 && (5.16)\\
&= \liminf_{i\to\infty} \int_A E_{P_i}[Z^{P_i}_{\tau_t^{P_i}} \mid \mathcal{F}_t] dP_0 && (5.17)\\
&= \liminf_{i\to\infty} \int_A E_{P_0}[Z^{P_i}_{\tau_t^{P_i}} \mid \mathcal{F}_t] dP_0 && (5.18)\\
&= \liminf_{i\to\infty} \int_A Z^{P_i}_{\tau_t^{P_i}} dP_0 && (5.19)\\
&= \liminf_{i\to\infty} \int_A H_{\tau_t^{P_i}} dP_0 && (5.20)\\
&= \int_A H_{\tau_t^\downarrow} dP_0 && (5.21)\\
&= \int_A E_{P_0}[H_{\tau_t^\downarrow} \mid \mathcal{F}_t] dP_0, && (5.22)
\end{aligned}$$

where the inequality in (5.16) is an application of Fatou's lemma. The identity in (5.17) follows from the first part of (5.12) and (5.13). The identity (5.18) is justified from the fact that $P_i = P_0$ in $\mathcal{F}_{\tau_t^{P_i}}$. The equality (5.19) follows because A is $\mathcal{F}_t$ measurable. The equality (5.20) follows from (5.14). In the equality (5.21) we have applied Lebesgue's convergence theorem, which we are allowed to do justified by (5.15) and the fact that the process H is of $class(D)$ with respect to P_0. The last equality (5.22) follows because A is $\mathcal{F}_t$ measurable. Since $P_0 \in \mathcal{P}$ was arbitrary we conclude (5.10).

3. Now we prove that $\{Z^{\downarrow}_{\tau_t^{\downarrow}\wedge s}\}_{s\in[t,T]}$ is a $\mathcal{P}$-submartingale from time t on. Let θ be a stopping time with $t \le \theta \le \tau_t^{\downarrow}$ and let us define

$$\tau_\theta^{\downarrow} := \operatorname{ess\,inf}_{P\in\mathcal{P}} \tau_\theta^P.$$

The same argument of the first step proves that this is a stopping time and from the fact that $\tau_\theta^P = \tau_t^P$ it follows that $\tau_\theta^{\downarrow} = \tau_t^{\downarrow}$. We can conclude as in the second step that

$$Z_\theta^{\downarrow} = \operatorname{ess\,inf}_{P\in\mathcal{P}} E_P\left[H_{\tau_t^{\downarrow}} \mid \mathcal{F}_\theta\right].$$

For $P_0 \in \mathcal{P}$ we have

$$\begin{aligned} E_{P_0}[Z_\theta^{\downarrow}] &= \inf_{P\in\mathcal{P}} E_{P_0}\left[E_P\left[H_{\tau_t^{\downarrow}} \mid \mathcal{F}_\theta\right]\right] \\ &\ge E_{P_0}[\operatorname{ess\,inf}_{P\in\mathcal{P}} E_P[H_{\tau_t^{\downarrow}} \mid \mathcal{F}_t]] = E_{P_0}[Z_t^{\downarrow}]. \end{aligned} \tag{5.23}$$

The first equality is justified by (5.7), the second from the fact that there exists a sequence of probabilities $\{P_i\}_{i=1}^{\infty}$ such that $P_i = P_0$ in $\mathcal{F}_\theta$ and

$$E_{P_i}[H_{\tau_t^{\downarrow}} \mid \mathcal{F}_\theta] \searrow \operatorname{ess\,inf}_{P\in\mathcal{P}} E_P\left[H_{\tau_t^{\downarrow}} \mid \mathcal{F}_\theta\right].$$

We conclude that $Z_t^{\downarrow}$ is a $\mathcal{P}$-submartingale on the interval $[t, \tau_t^{\downarrow}]$, since P_0 was arbitrary.□

Remark 5.7 *As we mentioned in remark 5.2, in Riedel[49] and Zamfirescu[55], a robust optimal stopping problem similar to our problem 5.1 was studied. In terms of our notation, they showed that*

$$\inf\{s \ge t \mid H_s \ge Z_s^{\downarrow}\},$$

is t-optimal in the sense of our definition 5.1. For $t \ge 0$, continuous-time and infinite horizon in [55], and for $t = 0$, discrete-time and finite horizon in [49]. In theorem 5.6 we have constructed a different solution in the form of the stopping time $\tau_t^{\downarrow}$.◇

The next corollary establishes a minimax identity and that t-optimal stopping times for the lower Snell envelope solves a robust stopping problem. It will be convenient to recall (5.2):

$$Z_t^{\downarrow} = \text{ess inf}_{P\in\mathcal{P}}\text{ess sup}_{\theta\in\mathcal{T}[t,T]} E_P\left[H_\theta \mid \mathcal{F}_t\right].$$

Corollary 5.8 *Let $t \in [0,T]$ and assume the conditions of theorem 5.6. Then*

$$Z_t^{\downarrow} = \text{ess sup}_{\theta\in\mathcal{T}[t,T]}\text{ess inf}_{P\in\mathcal{P}} E_P\left[H_\theta \mid \mathcal{F}_t\right], \quad R-a.s. \tag{5.24}$$

The stopping time $\tau_t^{\downarrow}$ solves the following robust stopping problem

$$\text{ess sup}_{\theta\in\mathcal{T}[t,T]}\text{ess inf}_{P\in\mathcal{P}} E_P\left[H_\theta \mid \mathcal{F}_t\right]. \tag{5.25}$$

In particular, for $t = 0$, $\tau_0^{\downarrow}$ solves the robust stopping problem

$$\sup_{\theta\in\mathcal{T}} \inf_{P\in\mathcal{P}} E_P[H_\theta], \tag{5.26}$$

and

$$\sup_{\theta\in\mathcal{T}} \inf_{P\in\mathcal{P}} E_P[H_\theta] = \inf_{P\in\mathcal{P}} \sup_{\theta\in\mathcal{T}} E_P[H_\theta]. \tag{5.27}$$

Proof. We show (5.24). The inequality $\geq$ is obvious.

For the converse, note that we have the obvious inequality

$$\text{ess inf}_{P\in\mathcal{P}} E_P[H_{\tau_t^{\downarrow}} \mid \mathcal{F}_t] \leq \text{ess sup}_{\theta\in\mathcal{T}[t,T]}\text{ess inf}_{P\in\mathcal{P}} E_P\left[H_\theta \mid \mathcal{F}_t\right],$$

which together with (5.10) implies that

$$Z_t^{\downarrow} = \text{ess inf}_{P\in\mathcal{P}} E_P[H_{\tau_t^{\downarrow}} \mid \mathcal{F}_t] \leq \text{ess sup}_{\theta\in\mathcal{T}[t,T]}\text{ess inf}_{P\in\mathcal{P}} E_P\left[H_\theta \mid \mathcal{F}_t\right] \leq Z_t^{\downarrow}.$$

This establishes (5.24) and at the same time (5.25).

The second part of the corollary follows by setting $t = 0$ in (5.24) and (5.25).□

Remark 5.9 *The second part of corollary 5.8 completes the proof of theorem 1.20 on arbitrage free prices. There we stated that*

$$\pi_{\inf}(H) = \inf_{P\in\mathcal{M}} \sup_{\theta\in\mathcal{T}} E_P\left[H_\theta\right] = \sup_{\theta\in\mathcal{T}} \inf_{P\in\mathcal{M}} E_P\left[H_\theta\right],$$

where H represents an American option as in definition 1.2, and $\mathcal{M}$ is the family of martingale measures for the price process X of section 1.1. This identity goes back to Karatzas and Kou[35] in a model driven by a multi-dimensional Brownian motion and Föllmer and Schied[27] in a discrete-time model for a general stable family $\mathcal{P}$. This identity was also obtained in Zamfirescu[55] and Riedel[49] in their respective setups.$\diamondsuit$

Remark 5.10 *Theorem 5.6 implies the minimax identity of corollary 5.8, and moreover it constructs a one-sided saddle point given by the stopping time $\tau_t^{\downarrow}$. If we are only interested in the minimax identity, then corollary 5.8 holds even if we drop the requirement of H being upper semicontinuous in expectation from the left with respect to any $P \in \mathcal{P}$. The proof of this claim would involve for $\lambda \in (0,1)$ the stopping time defined by*

$$\tau_t^{P,\lambda} := \inf\{u \geq t \mid H_u \geq \lambda U_u^P\},$$

and similar steps as in the proof of theorem 5.6 to obtain an "ϵ-minimax identity" for arbitrary $\epsilon > 0$.$\diamondsuit$

5.2.2 Existence of a worst-case probability measure

In this section we assume the conditions of theorem 5.6, and study further the problem 5.1 in the case $t = 0$. We are interested in a worst-case probability measure $P^* \in \mathcal{P}$ in the sense of the following definition.

Definition 5.11 *A probability measure $P^* \in \mathcal{P}$ is said to be a worst-case probability measure for the lower Snell envelope of H if it solves the following equality*

$$\sup_{\theta \in \mathcal{T}} E_{P^*}[H_\theta] = \sup_{\theta \in \mathcal{T}} \inf_{P \in \mathcal{P}} E_P[H_\theta]. \square$$

In the next proposition we give a sufficient condition for a probability measure $P^* \in \mathcal{P}$ to be a worst-case probability measure for the lower Snell envelope of H.

Proposition 5.12 *Assume that $\mathcal{P}$ and H satisfy the conditions of theorem 5.6 for $t = 0$. Then, $P^* \in \mathcal{P}$ is a worst-case probability measure for the lower Snell envelope of H if*

$$\sup_{\theta \in \mathcal{T}} E_{P^*}[H_\theta] = \inf_{P \in \mathcal{P}} \sup_{\theta \in \mathcal{T}} E_P[H_\theta]. \tag{5.28}$$

Proof. According to the second part of corollary 5.8 we have that

$$\inf_{P\in\mathcal{P}}\sup_{\theta\in\mathcal{T}} E_P[H_\theta] = \sup_{\theta\in\mathcal{T}}\inf_{P\in\mathcal{P}} E_P[H_\theta].$$

This equality combined with (5.28) implies that

$$\sup_{\theta\in\mathcal{T}} E_{P^*}[H_\theta] = \sup_{\theta\in\mathcal{T}}\inf_{P\in\mathcal{P}} E_P[H_\theta],$$

which is the definition of a worst-case probability measure for the lower Snell envelope of H.□

Using game theoretic language, a pair $(\tau_0^\downarrow, P^*)$ with $\tau_0^\downarrow$ the stopping time constructed in theorem 5.6 and $P^* \in \mathcal{P}$ a worst-case probability measure as in definition 5.11 is a saddle-point:

Proposition 5.13 *Let $\tau_0^\downarrow$ be the stopping time constructed in theorem 5.6 and let $P^* \in \mathcal{P}$ be a worst-case probability measure as in definition 5.11. Then the pair $(\tau_0^\downarrow, P^*)$ is a saddle-point in the following sense. For any pair $(\tau, P) \in \mathcal{T}\times\mathcal{P}$, we have that*

$$E_{P^*}[H_\tau] \le E_{P^*}[H_{\tau_0^\downarrow}] \le E_P[H_{\tau_0^\downarrow}]. \tag{5.29}$$

Proof. Let $\tau_0^\downarrow$ be the stopping time constructed in theorem 5.6 for the case $t = 0$. Then we know that

$$\inf_{P\in\mathcal{P}}\sup_{\theta\in\mathcal{T}} E_P[H_\theta] = \inf_{P\in\mathcal{P}} E_P[H_{\tau_0^\downarrow}].$$

Let $P^* \in \mathcal{P}$ be a worst-case probability measure, so that

$$\sup_{\theta\in\mathcal{T}}\inf_{P\in\mathcal{P}} E_P[H_\theta] = \sup_{\theta\in\mathcal{T}} E_{P^*}[H_\theta].$$

From corollary 5.8 we know that

$$\sup_{\theta\in\mathcal{T}}\inf_{P\in\mathcal{P}} E_P[H_\theta] = \inf_{P\in\mathcal{P}}\sup_{\theta\in\mathcal{T}} E_P[H_\theta],$$

and we infer that

$$\inf_{P\in\mathcal{P}} E_P[H_{\tau_0^\downarrow}] = \sup_{\theta\in\mathcal{T}} E_{P^*}[H_\theta].$$

This last equality easily implies (5.29).□

In the next subsections we verify the condition (5.28) of proposition 5.12 based on compactness arguments, using a weak formulation. We then in corollaries 5.16 and 5.18 conclude the existence of a worst-case probability measure.

Compactness with respect to the $\sigma(L^p(R), L^q(R))$-topology

In this subsection we show how to construct a probability measure $P^* \in \mathcal{P}$ solving the equality (5.28) of proposition 5.12. For the main result of this subsection we identify $\mathcal{P}$ with the corresponding set of densities

$$dens(\mathcal{P}) := \left\{ \frac{dP}{dR} \mid P \in \mathcal{P} \right\}. \tag{5.30}$$

We assume that

$$dens(\mathcal{P}) \subset L^p(R),$$

for some $p > 1$. Furthermore, we assume that $dens(\mathcal{P})$ is norm bounded and closed. Note that in this case the family $dens(\mathcal{P})$ is $\sigma(L^p(R), L^q(R))$-compact, with q the conjugate exponent of p. This follows from the fact that $L^p(R)$ is a reflexive Banach space. This assumption will allow us to apply the following well-known convergence theorem. We give an sketch of the proof.

Theorem 5.14 *Let E be a reflexive Banach space with norm $\|\cdot\|_E$. Let $\{f_i\}_{i=1}^{\infty}$ be a norm bounded sequence in E. Then, there exists a sequence of convex combinations*

$$\widetilde{f}_i \in \operatorname{conv}\{f_i, f_{i+1}, \ldots\}$$

and $f^* \in E$, such that $\lim_{i\to\infty} \left\| \widetilde{f}_i - f^* \right\|_E = 0$.

Proof. Since E is a normed spaced it is locally convex. Then, the closure of a convex subset is the same for the norm topology and the weak topology $\sigma(E, E^*)$; see e.g., Dunford and Schwartz[16], theorem V.3.13. Let $K > 0$ be such that $\|f_i\|_E \leq K$ for all $i \in \mathbb{N}$, and let B_K be the ball of radius K of E. Since E is reflexive then B_K is compact in the weak topology $\sigma(E, E^*)$, see e.g., Dunford and Schwartz[16], theorem V.4.7. Then, the sequence $\{f_i\}_{i=1}^{\infty}$ converges to some element $f^* \in B_K$ in the weak topology, and we conclude the proof of the theorem with corollary V.3.14 in Dunford and Schwartz[16].□

In the next proposition $\mathcal{Q}$ will be a convex family of absolutely continuous probability measures; in particular the family $\mathcal{Q}$ is not necessarily stable under pasting, and the measures are not necessarily equivalent to R.

Proposition 5.15 *Let $p > 1$ be an arbitrary but fixed number, and q be the conjugate exponent: $p^{-1} + q^{-1} = 1$. Let $\mathcal{Q}$ be a convex family of probability measures such that the family of densities $dens(\mathcal{Q})$ is a closed bounded subset of $L^p(R)$. Moreover, let $K = \{K_t\}_{0\leq t\leq T}$ be a positive càdlàg $\mathbb{F}$-adapted process with*

$$\sup_{\theta\in\mathcal{T}} E_R[(K_\theta)^q] < \infty. \tag{5.31}$$

Then, there exists a probability measure $Q^* \in \mathcal{Q}$ such that

$$\sup_{\theta \in \mathcal{T}} E_{Q^*}[K_\theta] = \inf_{Q \in \mathcal{Q}} \sup_{\theta \in \mathcal{T}} E_P[K_\theta]. \tag{5.32}$$

Proof. Let us recall that $U^Q(K)$ denotes the Snell envelope of K with respect to Q. In particular

$$U_0^Q(K) = \sup_{\theta \in \mathcal{T}} E_Q[K_\theta].$$

1. Let $Q \in \mathcal{Q}$ be a probability measure, and let $\{Q_i\}_{i=1}^\infty \subset \mathcal{Q}$ be a sequence of probability measures converging to Q in the sense that the corresponding sequence of densities with respect to R converges in $L^p(R)$ to the density of Q. We will prove that

$$U_0^Q(K) = \lim_{i \to \infty} U_0^{Q_i}(K). \tag{5.33}$$

Let $\theta \in \mathcal{T}$ be a fixed stopping time. Then the following inequalities hold

$$\left| E_R \left[\left(\frac{dQ_i}{dR} - \frac{dQ}{dR} \right) K_\theta \right] \right| \leq E_R \left[\left| \frac{dQ_i}{dR} - \frac{dQ}{dR} \right| K_\theta \right] \leq \left\| \frac{dQ_i}{dR} - \frac{dQ}{dR} \right\|_{L^p(R)} \|K_\theta\|_{L^q(R)}, \tag{5.34}$$

where we have used Hölder's inequality in the last term. This implies that

$$\lim_{i \to \infty} E_{Q_i}[K_\theta] = E_Q[K_\theta]. \tag{5.35}$$

From (5.35) we conclude that $\lim_{i \to \infty} U_0^{Q_i}(K) = U_0^Q(K)$. Indeed, let $\epsilon > 0$ and $Q, \{Q_i\}_{i=1}^\infty \subset \mathcal{Q}$ as previously fixed. There exists an ϵ-optimal stopping time $\theta^\epsilon \in \mathcal{T}$ for Q in the following sense

$$U_0^Q(K) \leq E_Q[K_{\theta^\epsilon}] + \frac{\epsilon}{3}. \tag{5.36}$$

From (5.35) we infer that

$$U_0^Q(K) \leq \lim_{i \to \infty} E_{Q_i}[K_{\theta^\epsilon}] + \frac{\epsilon}{3} \leq \liminf_{i \to \infty} U_0^{Q_i}(K) + \frac{\epsilon}{3}.$$

We conclude that

$$U_0^Q(K) \leq \liminf_{i \to \infty} U_0^{Q_i}(K). \tag{5.37}$$

Now we show that

$$\limsup_{i \to \infty} U_0^{Q_i}(K) \leq U_0^Q(K). \tag{5.38}$$

The inequalities (5.34) imply the existence of $N_0 \in \mathbb{N}$ such that

$$|E_{Q_i}[K_\theta] - E_Q[K_\theta]| \leq \frac{\epsilon}{3},$$

for any stopping time $\theta \in \mathcal{T}$ and any $i \geq N_0$.

Let now θ_i^ϵ be an ϵ-optimal stopping for Q_i in the sense that

$$U_0^{Q_i}(K) \leq E_{Q_i}[K_{\theta_i^\epsilon}] + \frac{\epsilon}{3}.$$

This inequality combined with (5.36) implies

$$U_0^{Q_i}(K) \leq E_Q[K_{\theta_i^\epsilon}] + \frac{2\epsilon}{3} \leq E_Q[K_{\theta^\epsilon}] + \epsilon,$$

for $i \geq N_0$. This proves (5.38). The inequalities (5.37) and (5.38) imply (5.33).

2. Now let $\{Q^i\}_{i=1}^\infty \subset \mathcal{Q}$ be a sequence such that

$$\lim_{i\to\infty} U_0^{Q^i} = \inf_{Q\in\mathcal{Q}} U_0^Q.$$

We can apply theorem 5.14 to conclude the existence of a sequence of convex combinations

$$Y^i \in \text{conv}\left\{\frac{dQ^i}{dR}, \frac{dQ^{i+1}}{dR}, \dots\right\}$$

converging to a random variable $Y^* \in L^p(R)$. The convexity of $dens(\mathcal{Q})$ implies that $Y^i \in dens(\mathcal{Q})$. Moreover, $Y^* \in dens(\mathcal{Q})$ since $dens(\mathcal{Q})$ is closed in $L^p(R)$. Thus, the probability measures $\widetilde{Q}^i$ and P^* with densities $\frac{d\widetilde{Q}^i}{dR} := Y^i$ and $\frac{dQ^*}{dR} := Y^*$ are elements of $\mathcal{Q}$.

According to the previous step, we have $\lim_{i\to\infty} U_0^{\widetilde{Q}^i} = U_0^{Q^*}$. It is clear that the convexity of the correspondence $Q \to U_0^Q(H)$ implies that

$$\lim_{i\to\infty} U_0^{\widetilde{Q}^i} = \lim_{i\to\infty} U_0^{Q^i}.$$

We conclude that the probability measure Q^* satisfies (5.32).□

In the next corollary we show the existence of a worst-case probability measure in the sense of definition 5.11, under the assumption that $dens(\mathcal{P})$ is a norm bounded closed subset of $L^p(R)$ for $p > 1$. Recall that we have assumed the conditions of theorem 5.6.

Corollary 5.16 *Let $dens(\mathcal{P})$ be the family of densities of $\mathcal{P}$ with respect to R. Assume that there exists an exponent $p > 1$ such that the family of densities $dens(\mathcal{P})$ is a norm bounded closed subset of $L^p(R)$. Furthermore, let us assume that H satisfies the integrability condition (5.31). Then, there exists a worst-case probability measure $P^* \in \mathcal{P}$ for the lower Snell envelope of H, in the sense of definition 5.11.*

Proof. The conditions of proposition 5.15 are satisfied and we may conclude the existence of a probability measure $P^* \in \mathcal{P}$ with

$$\sup_{\theta \in \mathcal{T}} E_{P^*}[H_\theta] = \inf_{P \in \mathcal{P}} \sup_{\theta \in \mathcal{T}} E_P[H_\theta].$$

This equality is condition (5.28) of proposition 5.12. The conditions of proposition 5.12 are satisfied and we conclude that the probability measure P^* is a worst-case probability measure for the lower Snell envelope of H.□

Compactness with respect to the $\sigma(L^1(R), L^\infty(R))$-topology

In proposition 5.15, based on $\sigma(L^p(R), L^q(R))$-compactness of the family of densities $dens(\mathcal{P})$ (5.30), a probability measure $P^* \in \mathcal{P}$ solving the equality (5.28) of proposition 5.12 was constructed, and we then, concluded the existence of a worst-case probability measure. In this subsection we give an alternative formulation of this result. We drop the assumption of $\sigma(L^p(R), L^q(R))$-compactness, and instead, we assume compactness in the weak topology $\sigma(L^1(R), L^\infty(R))$. Moreover, we drop the integrability assumption (5.31) for the process H, and instead, we assume the following uniform integrability condition:

$$\lim_{i \to \infty} \sup_{P \in \mathcal{P}} E_P[H_\theta; H_\theta \geq i] = 0, \tag{5.39}$$

for any stopping time $\theta \in \mathcal{T}$.

In the next proposition, $\mathcal{Q}$ will be a convex family of absolutely continuous probability measures; in particular the family $\mathcal{Q}$ is not necessarily stable under pasting, and the measures are not necessarily equivalent to R.

Proposition 5.17 *Let $\mathcal{Q}$ be a convex family of absolutely continuous probability measures with respect to R. Assume that the corresponding family* *of densities dens*$(\mathcal{Q})$ *of* $\mathcal{Q}$ *with respect to* R *is* $\sigma(L^1(R), L^\infty(R))$*-compact. Moreover, let* $K := \{K_t\}_{0\leq t\leq T}$ *be a positive càdlàg* $\mathbb{F}$*-adapted process with*

$$\sup_{\theta\in\mathcal{T}} E_Q[K_\theta] < \infty,$$

for any $Q \in \mathcal{Q}$*.*

Let K *satisfy the integrability condition (5.39) with respect to* $\mathcal{Q}$ *for any stopping time* $\theta \in \mathcal{T}$*. Then, there exists a probability measure* $Q^* \in \mathcal{Q}$ *such that (5.32) is satisfied:*

$$\sup_{\theta\in\mathcal{T}} E_{Q^*}[K_\theta] = \inf_{Q\in\mathcal{Q}} \sup_{\theta\in\mathcal{T}} E_Q[K_\theta].$$

Proof. Let $\{Q^i\}_{i=1}^\infty \subset \mathcal{Q}$ be a sequence converging to $Q \in \mathcal{Q}$ in the sense that the corresponding densities converges with respect to the weak topology $\sigma(L^1(R), L^\infty(R))$. In particular this means that for any $g \in L^\infty(R)$,

$$\lim_{i\to\infty} E_{Q^i}[g] = E_Q[g].$$

But (5.39) implies that for any $\theta \in \mathcal{T}$ this identity extends to

$$\lim_{i\to\infty} E_{Q^i}[K_\theta] = E_Q[K_\theta]. \tag{5.40}$$

Now, let $\epsilon > 0$ be arbitrary, and let $\tau^\epsilon \in \mathcal{T}$ be an ϵ-optimal stopping time of K with respect to Q in the sense that

$$\sup_{\theta\in\mathcal{T}} E_Q[K_\theta] \leq E_Q[K_{\tau^\epsilon}] + \epsilon. \tag{5.41}$$

Then

$$\begin{aligned}
\liminf_{i\to\infty} \sup_{\theta\in\mathcal{T}} E_{Q^i}[K_\theta] &\geq \liminf_{i\to\infty} E_{Q^i}[K_{\tau^\epsilon}] \\
&= \lim_{i\to\infty} E_{Q^i}[K_{\tau^\epsilon}] \qquad (5.42)\\
&= E_Q[K_{\tau^\epsilon}] \qquad (5.43)\\
&\geq -\epsilon + \sup_{\theta\in\mathcal{T}} E_Q[K_\theta]. \qquad (5.44)
\end{aligned}$$

In (5.42) and (5.43) we have used (5.40). In (5.44) we have used (5.41). Since ϵ was arbitrary we infer that

$$\liminf_{i\to\infty} \sup_{\theta\in\mathcal{T}} E_{Q^i}[K_\theta] \geq \sup_{\theta\in\mathcal{T}} E_Q[K_\theta]. \tag{5.45}$$

In this concluding step, we identify the family of probability measures $\mathcal{Q}$ with the corresponding set of densities $dens(\mathcal{Q})$. The correspondence

$$\frac{dQ}{dR} \to \sup_{\theta \in \mathcal{T}} E_Q[K_\theta]$$

is clearly convex, and the inequality (5.45) implies that it is lower semi-continuous with respect to weak convergence. Since we assumed that the family of densities $dens(\mathcal{Q})$ is compact with respect to the weak topology $\sigma(L^1(R), L^\infty(R))$, through a standard argument, we can construct a probability measure $Q^* \in \mathcal{Q}$ where the infimum over $\mathcal{Q}$ is attained. This is the required probability measure in the proposition.□

In the next corollary we assume that the set of densities $dens(\mathcal{P})$ of $\mathcal{P}$ with respect to R is $\sigma(L^1, L^\infty)$-compact, and show the existence of a worst-case probability measure in the sense of definition 5.11. Recall that we have assumed the conditions of theorem 5.6.

Corollary 5.18 *Assume that the set of densities $dens(\mathcal{P})$ of $\mathcal{P}$ with respect to R is $\sigma(L^1(R), L^\infty(R))$-compact. If H satisfies the uniform integrability condition (5.39) with respect to $\mathcal{P}$ for any stopping time $\theta \in \mathcal{T}$, then there exists a worst-case probability measure $P^* \in \mathcal{P}$ for the lower Snell envelope of H, in the sense of definition 5.11.*

Proof. The conditions of proposition 5.17 are satisfied, and we conclude the existence of a probability measure $P^* \in \mathcal{P}$ with

$$\sup_{\theta \in \mathcal{T}} E_{P^*}[H_\theta] = \inf_{P \in \mathcal{P}} \sup_{\theta \in \mathcal{T}} E_P[H_\theta].$$

This equality is condition (5.28) of proposition 5.12. The hypotheses of proposition 5.12 are satisfied and we may conclude that P^* is a worst-case probability measure for the lower Snell envelope of H.□

5.2.3 Optionality of the lower Snell envelope

In section 5.1 we defined the random variables $Z^{\downarrow}_{\tau}$ and we constructed an optimal stopping time in the sense of the formula (5.10). Now we are interested in the process $\{Z^{\downarrow}_t\}_{0 \le t \le T}$, and we search for a good version of this process. The process is clearly adapted, and we will prove that it has an optional version. Recall that the optional σ-algebra in the product space $[0,T] \times \Omega$ is generated by the class of $\mathbb{F}$-adapted, càdlàg processes, viewed as function on the product space $[0,T] \times \Omega$; see e.g., Protter[48] p. 102 for the definition of the optional σ-algebra in the product space $[0,\infty)\times\Omega$.

Loosely speaking, in the definition of the lower Snell envelope as the family of random variables $\{Z_\theta^\downarrow\}_{\theta\in\mathcal{T}}$, we first fixed a stopping time θ and then used the whole family $\mathcal{P}$. Now we are going to consider the lower Snell envelope as a process. To this end, we first fix a probability measure $P\in\mathcal{P}$ and consider the Snell envelope U^P. Then, in a second step, we construct the essential infimum of all the Snell envelopes U^P.

Definition 5.19 *Let $\mathbb{X}$ be a family of càdlàg optional processes defined in the interval $[0,T]$. We say that $X^\downarrow := \{X^\downarrow\}_{0\leq t\leq T}$ is the essential infimum of the family $\mathbb{X}$ if it is an optional process and*

1. *For any stopping time $\tau\in\mathcal{T}$ and any $X\in\mathbb{X}$ the following inequality holds R-a.s.*
$$X_\tau^\downarrow \leq X_\tau. \tag{5.46}$$
2. *It is maximal for this property in the class of optional processes. This means that if an optional process Y satisfies (5.46) for any stopping time $\tau\in\mathcal{T}$, then $Y_\tau\leq X_\tau^\downarrow$ R-a.s. for any $\tau\in\mathcal{T}$.*

In this case we write
$$X^\downarrow = \operatorname{ess\,inf}\mathbb{X}.\square$$

Let us notice that the random variable $X_\tau^\downarrow$ appearing in (5.46) is well defined since we required the process $X^\downarrow$ to be optional; see definition III.19 p. 50, and theorem III.20 in [11].

Definition 5.19 is adapted from a concept introduced by Dellacherie[12]. We recall definition I in [12].

Definition 5.20 *Let $(\Omega,\mathcal{F},\mathbb{F}=\{\mathcal{F}_t\}_{t\in[0,\infty)},R)$ be a stochastic base with infinite horizon satisfying the usual conditions of right continuity and completeness.*

1. *A measurable stochastic process $\{Z_t\}_{0\leq t<\infty}$ is said to be essentially dominated by another measurable stochastic process $\{Y_t\}_{0\leq t<\infty}$ if the stochastic set*
$$A(Z,Y) := \{(t,\omega)\in[0,\infty)\times\Omega \mid Z_t(\omega) > Y_t(\omega)\}$$
is evanescent, meaning that $R(\pi(A(Z,Y)))=0$, where π denotes the projection of $[0,\infty)\times\Omega$ in Ω.

2. *Let $\mathbb{Y} = \{Y_i\}_{i\in I}$ be a family of measurable stochastic processes. A measurable stochastic process $Y := \{Y_t\}_{0\leq t\leq T}$ is said to be the essential infimum of the family $\mathbb{Y}$, if Y is essentially dominated by every element of $\mathbb{Y}$, and is maximal with this property.*□

We will work in a setup where the horizon is finite: $T < \infty$ and with càdlàg processes. This is the reason why we work with definition 5.19.

We are interested in definition 5.19 for the family of Snell envelopes:

$$\mathbb{X}(\mathcal{P}) := \{U^P(H) \mid P \in \mathcal{P}\},$$

and we write $U^{\downarrow} := \{U_t^{\downarrow}\}_{0\leq t\leq T}$ for the essential infimum of $\mathbb{X}(\mathcal{P})$ (which in theorem 5.21 we prove exists). Note that in this case, the maximality property of definition 5.19 implies that $U_t^{\downarrow} \geq H_t$ R-a.s. for all $t \in [0, T]$.

The next theorem is a special case of a general result from Dellacherie[12]. In order to verify explicitly the optionality of the process $U^{\downarrow}$ we include the detailed proof.

Theorem 5.21 *We have the following assertions*

1. *There exists a countable subset $\mathcal{P}^{\#} \subset \mathcal{P}$ such that the process $U^{\downarrow} := \{U_t^{\downarrow}\}_{0\leq t\leq T}$ defined by*
$$U^{\downarrow} := \operatorname{ess\,inf}_{P\in\mathcal{P}^{\#}} U^P \tag{5.47}$$
is an optional version of the essential infimum of the family $\mathbb{X}(\mathcal{P})$. The essential infimum in (5.47) is taken in the product space $[0, T] \times \Omega$ with respect to $\lambda \otimes R$, where λ is the Lebesgue measure in $[0, T]$.

2. *For λ-almost all $t \in [0, T)$ and any decreasing sequence $\{t_i\}_{i=1}^{\infty} \subset [0, T]$ converging to t, we have*
$$\limsup_{i\to\infty} U_{t_i}^{\downarrow} \leq U_t^{\downarrow}, \quad R-a.s. \tag{5.48}$$

3. *For λ-almost all $t \in [0, T]$*
$$U_t^{\downarrow} = Z_t^{\downarrow}, \quad R-a.s. \tag{5.49}$$

Assume furthermore that the stochastic process H satisfies the conditions of theorem 5.6. Then, for λ-almost all $t \in [0, T)$ and any decreasing sequence $\{t_i\}_{i=1}^{\infty} \subset [0, T]$ converging to t, we have

$$\liminf_{i\to\infty} U_{t_i}^{\downarrow} \geq U_t^{\downarrow}, \quad R-a.s. \tag{5.50}$$

Proof.

1. Assume $\mathcal{P}^{\#}$ is a countable subset of $\mathcal{P}$, with the following property. For any positive rational $l \in \mathbb{Q}_+$ and for the stochastic set defined by

$$A^{P,l} := \{(t,\omega) \in [0,T] \times \Omega \mid U_t^P(\omega) < l\},$$

the following equality holds

$$\bigcup_{P \in \mathcal{P}} A^{P,l} = \bigcup_{P \in \mathcal{P}^{\#}} A^{P,l}. \tag{5.51}$$

Let us fix $\mathcal{P}^{\#}$ with the property (5.51), and let us define a process $U^{\downarrow}$ through (5.47). Then

$$\{(t,\omega) \in [0,T] \times \Omega \mid U_t^{\downarrow}(\omega) < l\} = \bigcup_{P \in \mathcal{P}^{\#}} A^{P,l} = \bigcup_{P \in \mathcal{P}} A^{P,l},$$

which implies that the process $U^{\downarrow}$ is optional. Moreover, for any $P_0 \in \mathcal{P}$

$$A^{P_0,l} \subset \{U_t^{\downarrow}(\omega) < l\}.$$

This implies that for any stopping time $\tau \in \mathcal{T}$, the set

$$\{U_\tau^{P_0} < U_\tau^{\downarrow}\} = \bigcup_{l \in \mathbb{Q}_+} \{U_\tau^{P_0} < l\} \cap \{U_\tau^{\downarrow} \geq l\}$$

is contained in a set of measure zero $\mathcal{N}(P_0)$ which only depends on P_0 but not τ. This proves that $U^{\downarrow}$ satisfies (5.46). The maximality of $U^{\downarrow}$ follows from (5.47).

2. Now we construct the countable set $\mathcal{P}^{\#}$ satisfying (5.51). We follow the proof of theorem I in [12]. Let us denote the complement of $A^{P,l}$ by

$$B^{P,l} := \{(t,\omega) \mid U_t^P(\omega) \geq l\}.$$

Then, we need to show that there exists a countable subset $\mathcal{P}^{\#}$ such that $\bigcap_{P \in \mathcal{P}} B^{P,l} = \bigcap_{P \in \mathcal{P}^{\#}} B^{P,l}$. With this in mind, let us introduce the following objects

$$\begin{aligned} \overline{B^{P,l}}(\omega) &:= \{t \in [0,T] \mid \text{ such that there exists a sequence } t_n \to t \text{ and } U_{t_n}^P(\omega) \geq l\}, \\ \overline{B^{P,l}} &:= \{(t,\omega) \in [0,T] \times \Omega \mid t \in \overline{B^{P,l}}(\omega)\}, \\ T_r^{P,l} &:= \inf\{t \geq r \mid U_t^P \geq l\}, r \in \mathbb{Q}_+ \cap [0,T]. \end{aligned}$$

The following claim is easily proved. Let $\mathcal{P}' \subset \mathcal{P}$ be a countable subset and let us define $T_r^{\mathcal{P}',l} := \sup_{P\in\mathcal{P}'} T_r^{P,l}$. Then the following equality holds

$$\bigcap_{P\in\mathcal{P}'} \overline{B^{P,l}}$$
$$= \Big\{(t,\omega) \mid \text{ there exists a sequence of rationals } r_n \to t \text{ such that } T_{r_n}^{\mathcal{P}',l}(\omega) \to t, \Big\}.$$

A direct consequence of this claim is that for a countable subset $\mathcal{P}^0 \subset \mathcal{P}$ with

$$T_r^{\mathcal{P}^0,l} = \sup_{P\in\mathcal{P}^0} T_r^{P,l} = \operatorname{ess\,sup}_{P\in\mathcal{P}} T_r^{P,l},$$

then

$$K^0 := \bigcap_{P\in\mathcal{P}^0} \overline{B^{P,l}} = \bigcap_{P\in\mathcal{P}} \overline{B^{P,l}},$$

where the intersection on the right must be interpreted as an "essential intersection", in the sense that the left term is contained on each $\overline{B^{P,l}}$, except for an evanescent set (a stochastic set in $[0,T]\times\Omega$ is evanescent if the projection in Ω is a R-null set), and is maximal with this property.

To conclude the proof, this intersection property must be transfered from the closed sets $\overline{B^{P,l}}$ to the sets $B^{P,l}$ themselves. Recall that $B^{P,l}$ is closed with the right topology, and thus the difference $\overline{B^{P,l}}/B^{P,l}$ consists of points isolated from the right and approximable from the left.

Let us call $D \subset K^0$ the set of isolated points from the right. Then theorem 27 p. 137 in [11] says that there exists a sequence of positive random variables τ_n such that $D = \bigcup_{n=1}^{\infty}[\tau_n]$. Let us define the measure

$$\mu(X) := \sum_{n=1}^{\infty} \frac{1}{2^n} E_R[X_{\tau_n}; \tau_n < \infty],$$

and let $\mathcal{P}^1 \subset \mathcal{P}$ be a countable subset such that

$$\bigcap_{P\in\mathcal{P}^1} B^{P,l} = \bigcap_{P\in\mathcal{P}} B^{P,l},$$

where the intersection on the right is again interpreted to be in a generalized way and with respect to the measure μ.

Let us define $\mathcal{P}^{\#} := \mathcal{P}^0 \cup \mathcal{P}^1$ and

$$K^{\#} := \bigcap_{P\in\mathcal{P}^{\#}} B^{P,l}.$$

Then $\mathcal{P}^\#$ and $K^\#$ are the objects we were searching for. Indeed we already knew that $K^\# \subset K^0$ and $K^0 - B^{P,l} \subset D$, and $K^\# \subset B^{P,l}$ except for a set of μ-measure zero.

3. Let us write $\mathcal{P}^\#$ as a sequence $\{P_i\}_{i=1}^\infty$. We define $X^1 := U^{P_1}(H)$, $X^i := X^{i-1} \wedge U^{P_i}(H)$ and observe that for almost all $t \in [0,T]$

$$\lim_{i\to\infty} X_t^i = U_t^\downarrow, \quad R - a.s.$$

Let $\{t_i\}_{i=1}^\infty \subset [0,T)$ be a decreasing sequence converging to $t \in [0,T)$. Then, for $m \in \mathbb{N}$ fixed, and any $i \in \mathbb{N}$ we have

$$U_{t_i}^\downarrow \leq X_{t_i}^m,$$

since X^m is right continuous. Then we get

$$\limsup_{i\to\infty} U_{t_i}^\downarrow \leq X_t^m.$$

Now letting $m \nearrow \infty$ we obtain that

$$\limsup_{i\to\infty} U_{t_i}^\downarrow \leq U_t^\downarrow,$$

which is (5.48).

4. We now prove (5.49). The proof is direct but non trivial since it depends on the stability of $\mathcal{P}$. We have to control R-null sets.

We first prove that $R(U_t^\downarrow \leq Z_t^\downarrow) = 1$. According to (5.5) in lemma 5.3, there exists a sequence of probability measures $\{Q^i\}_{i=1}^\infty \subset \mathcal{P}$ such that

$$Z_t^{Q^i} \searrow Z_t^\downarrow.$$

We do not distinguish between the random variable $Z_t^{Q^i}$ and the Snell envelope U^{Q^i} sampled in t. We then obtain that $R(U_t^\downarrow \leq Z_t^{Q^i}) = 1$. Since the sequence $\{Q^i\}_{i=1}^\infty$ is countable, we conclude that $R(U_t^\downarrow \leq Z_t^\downarrow) = 1$.

For the converse $R(U_t^\downarrow \geq Z_t^\downarrow) = 1$, we only have to recall (5.47) and use the fact that $\mathcal{P}^\#$ is countable. This allows to control the R-null sets involved.

5. It remains to prove (5.50). This inequality is proved in proposition 5.22.□

Proposition 5.22 *Let $U^{\downarrow}$ be the process constructed in theorem 5.21. Assume that the process H satisfies the hypotheses of theorem 5.6. Let $t \in [0, T)$ and $\{t_i\}_{i=1}^{\infty} \subset [0, T]$ be a decreasing sequence converging to t. Then*

$$\liminf_{i \to \infty} U^{\downarrow}_{t_i} \geq U^{\downarrow}_t, \quad R - a.s. \tag{5.52}$$

Proof. Justified by (5.49) we do not distinguish between the random variables $U^{\downarrow}_t$ and $Z^{\downarrow}_t$ for $t \in [0, T]$ fixed. Let $\{t_i\}_{i=1}^{\infty} \subset [0, T]$ be a decreasing sequence converging to $t \in [0, T)$. In corollary 5.8 we have proved the identity

$$Z^{\downarrow}_t = \operatorname{ess\,sup}_{\theta \geq t} \operatorname{ess\,inf}_{P \in \mathcal{P}} E_P[H_\theta \mid \mathcal{F}_t].$$

Thus, to conclude (5.52) it is enough to establish the inequality

$$\liminf_{i \to \infty} Z^{\downarrow}_{t_i} \geq \operatorname{ess\,inf}_{P \in \mathcal{P}} E_P[H_\tau \mid \mathcal{F}_t], \tag{5.53}$$

for $\tau \in \mathcal{T}[t, T]$ a fixed stopping time.

We will reduce the proof of (5.53) to (5.56) below. And then, in a second step, prove (5.56).

1. In this step we reduce the proof of (5.53) to (5.56) below. Similar to (4.20) we define

$$\tau^{(i)} := \tau 1_{\{\tau \geq t_i\}} + T 1_{\{\tau < t_i\}} \in \mathcal{T}[t_i, T].$$

   Then we get

$$Z^{\downarrow}_{t_i} \geq \operatorname{ess\,inf}_{P \in \mathcal{P}} E_P[H_{\tau^{(i)}} \mid \mathcal{F}_{t_i}],$$

   so that

$$\liminf_{i \to \infty} Z^{\downarrow}_{t_i} \geq \liminf_{i \to \infty} \operatorname{ess\,inf}_{P \in \mathcal{P}} E_P[H_{\tau^{(i)}} \mid \mathcal{F}_{t_i}].$$

   To prove (5.53) it is enough to show that

$$\liminf_{i \to \infty} \operatorname{ess\,inf}_{P \in \mathcal{P}} E_P[H_{\tau^{(i)}} \mid \mathcal{F}_{t_i}] \geq \operatorname{ess\,inf}_{P \in \mathcal{P}} E_P[H_\tau \mid \mathcal{F}_t]. \tag{5.54}$$

   We simplify (5.54) even more. Note that

$$E_P[H_{\tau^{(i)}} \mid \mathcal{F}_{t_i}] = 1_{\{\tau \geq t_i\}} E_P[H_\tau \mid \mathcal{F}_{t_i}] + 1_{\{\tau < t_i\}} E_P[H_T \mid \mathcal{F}_{t_i}],$$

so that (5.54) will follow from the next inequality

$$\liminf_{i\to\infty} \operatorname{ess\,inf}_{P\in\mathcal{P}} 1_{\{\tau\geq t_i\}} E_P[H_\tau \mid \mathcal{F}_{t_i}] \geq \operatorname{ess\,inf}_{P\in\mathcal{P}} E_P[H_\tau \mid \mathcal{F}_t]. \tag{5.55}$$

Since $\lim_{i\to\infty} R(1_{\{\tau\geq t_i\}} = 1) = 1$ monotonously, then we can simplify the proof of (5.55) into the proof of the following inequality

$$\liminf_{i\to\infty} \operatorname{ess\,inf}_{P\in\mathcal{P}} E_P[H_\tau \mid \mathcal{F}_{t_i}] \geq \operatorname{ess\,inf}_{P\in\mathcal{P}} E_P[H_\tau \mid \mathcal{F}_t]. \tag{5.56}$$

2. Now we prove (5.56). We actually first prove that the opposite inequality holds. In fact, since H is of $class(D)$ with respect to $\mathcal{P}$, in particular we have that

$$\sup_{\theta\in\mathcal{T}} E_P[H_\theta] < \infty.$$

Lemma 5.23 below, allows us to conclude the opposite inequality in (5.56), namely:

$$\liminf_{i\to\infty} \operatorname{ess\,inf}_{P\in\mathcal{P}} E_P[H_\tau \mid \mathcal{F}_{t_i}] \leq \operatorname{ess\,inf}_{P\in\mathcal{P}} E_P[H_\tau \mid \mathcal{F}_t]. \tag{5.57}$$

In fact, for $P \in \mathcal{P}$ fixed we have

$$\liminf_{i\to\infty} \operatorname{ess\,inf}_{P\in\mathcal{P}} E_P[H_\tau \mid \mathcal{F}_{t_i}] \leq \liminf_{i\to\infty} E_P[H_\tau \mid \mathcal{F}_{t_i}] = E_P[H_\tau \mid \mathcal{F}_t],$$

where the last equality follows from lemma 5.23, since the random variable H_τ is integrable with respect P, and the filtration $\mathbb{F}$ is right continuous.

The inequality (5.57) allows to reduce the proof of (5.56) in expectation:

$$E_{P_0}[\liminf_{i\to\infty} \operatorname{ess\,inf}_{P\in\mathcal{P}} E_P[H_\tau \mid \mathcal{F}_{t_i}]] \geq E_{P_0}[\operatorname{ess\,inf}_{P\in\mathcal{P}} E_P[H_\tau \mid \mathcal{F}_t]], \tag{5.58}$$

for $P_0 \in \mathcal{P}$ arbitrary but fixed. Of course we then conclude equality in (5.56), but this is irrelevant for the proof of the proposition.

According to lemma 5.24 below, the sequence of random variables

$$\{Y_i\}_{i=1}^\infty := \{\operatorname{ess\,inf}_{P\in\mathcal{P}} E_P[H_\tau \mid \mathcal{F}_{t_i}]\}_{i=1}^\infty \tag{5.59}$$

is a Backwards-submartingale for any $P \in \mathcal{P}$; see lemma 5.24. This same proposition yields that the limit inferior in (5.56) actually exists as a limit.

The inequality (5.58) will follow from

$$\limsup_{i\to\infty} E_{P_0}[\text{ess}\inf_{P\in\mathcal{P}} E_P[H_\tau \mid \mathcal{F}_{t_i}]] \geq E_{P_0}[\text{ess}\inf_{P\in\mathcal{P}} E_P[H_\tau \mid \mathcal{F}_t]]. \tag{5.60}$$

In fact, in this case we get:

$$\begin{aligned} E_{P_0}[\liminf_{i\to\infty} \text{ess}\inf_{P\in\mathcal{P}} E_P[H_\tau \mid \mathcal{F}_{t_i}]] & \\ &= E_{P_0}[\limsup_{i\to\infty} \text{ess}\inf_{P\in\mathcal{P}} E_P[H_\tau \mid \mathcal{F}_{t_i}]] \quad (5.61) \\ &\geq \limsup_{i\to\infty} E_{P_0}[\text{ess}\inf_{P\in\mathcal{P}} E_P[H_\tau \mid \mathcal{F}_{t_i}]] \quad (5.62) \\ &\geq E_{P_0}[\text{ess}\inf_{P\in\mathcal{P}} E_P[H_\tau \mid \mathcal{F}_t]]. \quad (5.63) \end{aligned}$$

In (5.61) we have used the fact that the limit exists. In (5.62) we have used Fatou's lemma, which we are allowed to apply since the sequence $\{Y_i\}_{i=1}^\infty$ (5.59) is, obviously, uniformly integrable with respect to P_0. The last part (5.63) is (5.60) which we now prove. We first observe that

$$E_{P_0}[\text{ess}\inf_{P\in\mathcal{P}} E_P[H_\tau \mid \mathcal{F}_{t_i}]] = \inf_{P\in\mathcal{P}(P_0,t_i)} E_P[H_\tau]$$

and

$$E_{P_0}[\text{ess}\inf_{P\in\mathcal{P}} E_P[H_\tau \mid \mathcal{F}_t]] = \inf_{P\in\mathcal{P}(P_0,t)} E_P[H_\tau],$$

where, we recall, $\mathcal{P}(P_0, s) = \{P \in \mathcal{P} \mid P = P_0 \text{ in } \mathcal{F}_s\}$. Note that $\mathcal{P}(P_0, t_i) \subset \mathcal{P}(P_0, t)$. Let $\epsilon > 0$ and let $P^i \in \mathcal{P}(P_0, t_i)$ be such that

$$E_{P^i}[H_\tau] - \epsilon \leq \inf_{P\in\mathcal{P}(P_0,t_i)} E_P[H_\tau].$$

Then, it is enough to show that

$$\limsup_{i\to\infty} E_{P^i}[H_\tau] \geq \inf_{P\in\mathcal{P}(P_0,t)} E_P[H_\tau], \tag{5.64}$$

but it is obvious that $P^i \in \mathcal{P}(P_0, t)$ so that

$$E_{P^i}[H_\tau] \geq \inf_{P\in\mathcal{P}(P_0,t)} E_P[H_\tau],$$

implying (5.64).□

In the proof of proposition 5.22 we have applied the following lemmas.

Let us recall that we have fixed a stochastic base $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}, R)$. However, in the next lemma we only consider the probability space $(\Omega, \mathcal{F}, R)$.

Lemma 5.23 *Let $(\Omega, \mathcal{F}, R)$ be a probability space. Let Y be a positive integrable random variable. Let $\{\mathcal{F}_i\}_{i=1}^{\infty}$ be a decreasing sequence of sub-σ-algebras of $\mathcal{F}$, that is, $\mathcal{F}_{i+1} \subset \mathcal{F}_i \subset \mathcal{F}$. Then*

$$\lim_{i\to\infty} E_R[Y \mid \mathcal{F}_i] = E_R[Y \mid \mathcal{F}_{-\infty}],$$

where $\mathcal{F}_{-\infty} = \cap_{i=1}^{\infty}\mathcal{F}_i$.

Proof. This is a special case of the backwards martingale convergence theorem; see e.g., theorem 2.I.5, or theorem 2.III.16 in Doob[15].□

Lemma 5.24 *Let Y be a positive random variable $\mathcal{F}_T$-measurable such that*

$$E_P[Y] < \infty,$$

for any $P \in \mathcal{P}$. Let $t \in [0, T)$ and let $\{t_i\}_{i=1}^{\infty}$ be a decreasing sequence converging to t. Then, the sequence of random variables $\{Y_i\}_{i=1}^{\infty}$ defined by

$$Y_i := \operatorname{ess\,inf}_{P\in\mathcal{P}} E_P[Y \mid \mathcal{F}_{t_i}],$$

is a backwards $\mathcal{P}$-submartingale in the following sense: For any $P \in \mathcal{P}$ and $i \in \mathbb{N}$

$$E_P[Y_i \mid \mathcal{F}_{t_{i+1}}] \geq Y_{i+1}, \quad P - a.s. \tag{5.65}$$

Moreover,

$$\lim_{i\to\infty} Y_i \tag{5.66}$$

exists R-a.s. and in $L^1(P)$ for any $P \in \mathcal{P}$.

Proof. The identity

$$Y_{i+1} = \operatorname{ess\,inf}_{P\in\mathcal{P}} E_P[Y_i \mid \mathcal{F}_{t_{i+1}}], \quad P - a.s. \tag{5.67}$$

follows from part three in lemma 5.30, applying an argument similar to the proof of the last part in lemma 5.30. Formula (5.67) implies (5.65). The existence of the limit (5.66) follows from the submartingale version of theorem 2.III.17 in [15].□

5.3 Illustrations and special cases

5.3.1 The lower Snell envelope for European options

In the previous subsection we have seen that there exists an optional process $U^{\downarrow}$ which is the essential infimum of the family of Snell envelopes $U^P(H)$. In this section we consider a special case where this process is in fact a submartingale.

Let H_T be a European option as in definition 1.2. The lower Snell envelope takes the form

$$U_t^{\downarrow} = \operatorname{ess\,inf}_{P\in\mathcal{M}} E_P[H_T \mid \mathcal{F}_t].$$

In a model driven by a Brownian motion, El Karoui and Quenez[19] proved that the lower envelope is a $\mathcal{M}$-submartingale; see their theorem 2.4.1. In that theorem they assumed that there exists a $\mathcal{M}$-martingale dominating $U^{\downarrow}$. In the next proposition we relax this condition.

Proposition 5.25 *Let H_T be a European option. Then, the lower Snell envelope of H_T is a $\mathcal{M}$-submartingale.*□

Proof. Indeed, if $P_0 \in \mathcal{M}$ is a fixed martingale measure, then for any pair $s, t \in [0, T]$ with $s < t$ follows that

$$\begin{aligned} E_{P_0}[U_t^{\downarrow} \mid \mathcal{F}_s] &= E_{P_0}[\operatorname{ess\,inf}_{P\in\mathcal{M}} E_P[H_T \mid \mathcal{F}_t] \mid \mathcal{F}_s] \\ &\geq \operatorname{ess\,inf}_{P\in\mathcal{M}} E_P[\operatorname{ess\,inf}_{P\in\mathcal{M}} E_P[H_T \mid \mathcal{F}_t] \mid \mathcal{F}_s] \\ &= \operatorname{ess\,inf}_{P\in\mathcal{M}} E_P[H_T \mid \mathcal{F}_s], \end{aligned}$$

where the last equality follows as a special case of part four in lemma 5.30 below.□

5.3.2 An example of a $\sigma(L^p(R), L^q(R))$-compact stable family of measures

In this subsection we construct an example of a convex family of probability measures equivalent to R and satisfying the conditions of proposition 5.15: Stability under pasting, and weak compactness in $L^p(R)$ of the set of densities with respect to the reference probability measure R.

The example is simple but the construction requires advanced results from martingale theory. We are going to proceed as follows. The family of probability measures will be defined through the set of densities (5.70) below, which involve stochastic exponentials of BMO-martingales. We then prove that this family is norm bounded in a space $L^p(R)$ for a exponent p, which will depend on the constant K of the inequality (5.68). This will involve the so-called *p-reverse Hölder inequality*, denoted R_p. In appendix A.1 we present the space BMO and collect the results we will need. See Kazamaki[38] for a more systematic presentation of continuous BMO-martingales.

We then prove that the set of densities (5.70) is closed in $L^p(R)$. To this end, we apply Doob's p-maximal inequality and the Burkholder-Davis-Gundy inequalities, two fundamentals results of martingale theory. This step is actually the hardest part of the construction.

Now we conclude the $\sigma(L^p(R), L^q(R))$-compactness of the set of densities $dens(\mathcal{P})$ as follows. There exists a constant r such that

$$dens(\mathcal{P}) \subset B_r := \{f \in L^p(R) \mid \|f\|_{L^p(R)} \leq r\},$$

since $dens(\mathcal{P})$ will be norm-bounded in $L^p(R)$. Since $dens(\mathcal{P})$ will be a norm-closed convex subset of a locally convex space, it is closed with respect to the weak topology $\sigma(L^p(R), L^q(R))$. According to the Banach-Alaoglu theorem B_r is weakly compact; see e.g., theorem A.62 in Föllmer and Schied[27]. As a closed subset of a compact set, $dens(\mathcal{P})$ is also compact.

Before the start of the construction, let us give two comments. Our example can be seen as a special case of theorem 1.3 in Delbaen[7]. In the proof of the convexity and stability we have followed [7]. However, we will explicitly verify norm boundedness and closedness. The second comment is that our goal here was exclusively to construct a non trivial example of a stable compact family of probability measures. We refer to Delbaen et al[10], to see BMO-martingales techniques and martingale inequalities applied in relation to the Föllmer-Schweizer decomposition of local-martingales.

Let us recall that in definition 4.13 we have introduced the stochastic exponential of a continuous martingale.

Example 5.26 *Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}, R)$ be a stochastic base such that every martingale admits a continuous version. Let $N := \{N_t\}_{0\leq t\leq T}$ be a square integrable martingale such that its quadratic variation process is equivalent to Lebesgue measure in $[0, T]$. Let $\xi^0 = \{\xi^0_t\}_{0\leq t\leq T}$ be a predictable process such that for a constant $K > 0$*

$$\int_0^T (\xi^0_s)^2 d\langle N\rangle_s \leq K. \tag{5.68}$$

Then

1. *If* $\xi := \{\xi_t\}_{0\le t\le T}$ *is a predictable process such that*
$$R(\{\omega \in \Omega \mid |\xi_t(\omega)| \le \left|\xi_t^0(\omega)\right| \textit{ for almost all } t \in [0,T]\}) = 1, \quad (5.69)$$
then the stochastic integral $\xi \cdot N_t := \int_0^t \xi_s dN_s$ *is well defined and is a uniformly integrable martingale in the space* BMO*.*

2. *If* ξ *satisfies (5.69), then* $\mathcal{E}_T(\xi \cdot N) > 0$ *and* $E_R[\mathcal{E}_T(\xi \cdot N)] = 1$*.*

3. *The family of probability measures* $\mathcal{P}$ *obtained from the family of densities with respect to* R
$$dens(\mathcal{P}) := \\ \{\mathcal{E}_T(\xi \cdot N) \mid \{\xi_t\}_{0\le t\le T} \textit{ is a predictable process satisfying } (5.69)\}, \quad (5.70)$$
is convex and stable under pasting.

4. *For some* $p > 1$ *depending on* K*, the family* $dens(\mathcal{P})$ *is norm bounded and closed in* $L^p(R)$*. In particular it is* $\sigma(L^p(R), L^q(R))$*-compact, where* q *denotes the conjugate exponent of* p*.*

Proof.

1. We verify the first claim. Let ξ be a predictable process satisfying (5.69). Thus
$$\int_0^T (\xi_s)^2 d\langle N\rangle_s \le K, \quad (5.71)$$
due to (5.68) and (5.69), and so the stochastic integral with respect to N is a square integrable uniformly integrable martingale; see e.g., proposition 3.2.10 in Karatzas[36]. Itô isometry (see formula (2.22) in proposition 3.2.10 in [36]) implies that this martingale is in BMO_2:
$$E_R[|\xi \cdot N_T - \xi \cdot N_{\theta-}|^2 \mid \mathcal{F}_\theta] = E_R\left[\int_\theta^T (\xi_s)^2 d\langle N\rangle_s \mid \mathcal{F}_\theta\right] \le K.$$

2. From the previous step we conclude that the norms in BMO_1 and BMO_2 of $\xi \cdot N$ are uniformly bounded by the same constant and independent of ξ. Now, theorem A.3 implies that $\mathcal{E}_T(\xi \cdot N)$ is a uniformly integrable martingale. In particular $E_R[\mathcal{E}_T(\xi \cdot N)] = 1$. The inequality (5.71) implies that
$$R(\mathcal{E}_T(\xi \cdot N) > 0) = 1$$
by an argument similar to that of example 4.14.

3. The convexity and stability of the family $\mathcal{P}$ is proved as in example 4.14.

4. It remains to prove that $dens(\mathcal{P})$ is norm bounded and closed in a space $L^p(R)$ for some $p > 1$ depending on K. We verify that it is norm bounded. Let Φ be the function

$$\Phi(x) := \left\{1 + \frac{1}{x^2} \ln\left(\frac{2x-1}{2x-2}\right)\right\}^{\frac{1}{2}} - 1,$$

as defined in formula (A.1) of appendix A.1. Then, there exists $p = p(K) > 1$ with $K \leq \Phi(p)$. Theorem A.5 implies that there exists a constant $C > 0$ such that

$$E_R[(\mathcal{E}_T(\xi \cdot N))^p] < C. \tag{5.72}$$

This proves that $dens(\mathcal{P})$ is a norm-bounded subset of the space $L^p(R)$.

In proposition 5.27 below we show that $dens(\mathcal{P})$ is a closed subset of $L^p(R)$. This will complete the construction of the example.□

Proposition 5.27 *With the notation of example 5.26, the set of densities $dens(\mathcal{P})$ defined in (5.70) is strongly closed in $L^p(R)$.*

Proof.

1. Let $\{\xi^n\}_{n=1}^\infty$ be a sequence of predictable processes satisfying (5.69). We set

$$M^n := \{\xi^n \cdot N_t\}_{0 \leq t \leq T}, \text{ and } f^n := \{\mathcal{E}_t(M^n)\}_{0 \leq t \leq T}.$$

We assume that the sequence of random variables f_T^n converges to a random variable F in $L^p(R)$. We must prove that $F \in dens(\mathcal{P})$. It is easy to see that $F > 0$. To this end, note that

$$(f_t^n)^{-1} = \mathcal{E}_t(-M^n) \exp\left\{\int_0^t (\xi_s^n)^2 d\left\langle N\right\rangle_s\right\} \leq \mathcal{E}_t(-M^n)\exp(K), \tag{5.73}$$

so that $(f_t^n)^{-1} \in L^p(R)$ due to (5.72) and the fact that the process $-\xi^n$ satisfies (5.69). We can actually say that

$$E_R[(f_T^n)^{-p}] \leq C\exp(pK). \tag{5.74}$$

Let $A := \{F = 0\}$, and by way of contradiction assume that $R(A) > 0$. Passing to a subsequence if necessary, we may assume that f_T^n converges to F R-a.s. By Fatou's lemma

$$\liminf_{n\to\infty} E_R[1_A (f_T^n)^{-p}] \geq E_R[1_A (F)^{-p}] = \infty$$

a clear contradiction to the estimate (5.74), so that it must be the case that $R(A) = 0$.

2. Let $\widetilde{F}$ be a continuous version of the martingale $\{E_R[F \mid \mathcal{F}_t]\}_{0\leq t\leq T}$. We apply Doob's maximal inequality (see e.g., theorem 1.3.8 part (iv) in [36]) to obtain

$$E_R\left[\sup_{0\leq t\leq T}\left|\widetilde{F}_t - f_t^n\right|^p\right] \leq \left(\frac{p}{p-1}\right)^p E_R\left[|F_T - f_T^n|^p\right]. \tag{5.75}$$

The right term converges to zero as $n \to \infty$. The Burkholder-Davis-Gundy inequalities (see e.g., theorem 3.3.28 in [36]) implies that

$$E_R\left[\left\langle \widetilde{F} - f^n\right\rangle_T^{\frac{p}{2}}\right] \to 0. \tag{5.76}$$

3. The martingale $\widetilde{F}$ is locally square integrable. Through a localizing argument and the Kunita-Watanabe decomposition for square integrable martingales we can prove that there exists a predictable process $\widetilde{\eta}$ such that

$$\widetilde{F}_t = 1 + \widetilde{\eta} \cdot N_t + L_t$$

where $\{L_t\}_{0\leq t\leq T}$ is a continuous locally square integrable martingale with $L_0 = 0$, and orthogonal to N. We prove that $L = 0$. In fact:

$$E_R\left[\left(\int_0^T (\xi_s^n f_s^n - \widetilde{\eta})_s^2 d\left\langle N\right\rangle_s + \left\langle L\right\rangle_T\right)^{\frac{p}{2}}\right] = E_R\left[\left\langle \widetilde{F} - f^n\right\rangle_T^{\frac{p}{2}}\right] \to 0, \tag{5.77}$$

where we have applied (5.76) and the fact that f^n satisfies

$$f^n = 1 + f^n \cdot M^n = 1 + f^n \xi^n \cdot N.$$

Then, $R(\langle L\rangle_T = 0) = 1$, and thus, $R(\{\omega \in \Omega \mid L_t(\omega) = 0 \text{ for all } t \in [0,T]\}) = 1$.

4. In the previous step we showed that

$$F = \widetilde{F}_T = 1 + \int_0^T \widetilde{\eta}_s dN_s.$$

Now, in order to conclude the proof of the proposition, we construct a predictable process $\widetilde{\xi}$ satisfying (5.69) such that

$$\widetilde{\eta} = \widetilde{\xi}\widetilde{F}. \tag{5.78}$$

Due to (5.77), there exists a subsequence n_k which we simply denote by n, such that $\int_0^T (\xi_s^n f_s^n - \widetilde{\eta}_s)^2 d\langle N \rangle_s$ converges to zero in a measurable set $\widetilde{\Omega}$ with $R(\widetilde{\Omega}) = 1$. Since we assumed that $d\langle N \rangle$ is equivalent to Lebesgue measure in $[0, T]$, we conclude that

$$\lim_{n\to\infty} (\xi_t^n f_t^n - \widetilde{\eta}_t)^2 = 0, \text{ for almost every } t \in [0, T] \tag{5.79}$$

and for $\omega \in \widetilde{\Omega}$, after passing to a subsequence if necessary.

In the first step we proved that $R(F > 0) = 1$. This implies that $R(\widetilde{F}_t > 0) = 1$ for all $t \in [0, T]$. Since $\widetilde{F}$ has continuous trajectories we can say something stronger:

$$R(\{\omega \in \Omega \mid F_t(\omega) > 0 \quad for \quad t \in [0, T]\}) = 1.$$

So we may define

$$\widetilde{\xi} := \frac{\widetilde{\eta}}{\widetilde{F}}.$$

We certainly have that $\widetilde{\xi}$ is predictable, and in order to conclude the proof of the proposition, we must verify that it satisfies (5.69). But this follows from (5.75) and (5.79).□

5.3.3 The lower Snell envelope in discrete time

The main result of this section is the decomposition (5.86) of theorem 5.32. It describes the lower Snell envelope in discrete time. Here we consider a stochastic base in discrete time

$$(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\in\{0,\dots,T\}}, R).$$

We fix a convex stable family of probability measures $\mathcal{P}$, and an adapted positive stochastic process $H := \{H_t\}_{t\in\{0,\cdots,T\}}$ such that

$$E_P[H_t] < \infty,$$

for any $P \in \mathcal{P}$ and $t \in \{0, \cdots, T\}$.

Definition 5.28 *Let $\mathcal{P}$ be a convex stable family of probability measures. For a σ-algebra $\mathcal{G} \subset \mathcal{F}$ we set*

$$E^{\downarrow}[\cdot \mid \mathcal{G}] := \operatorname{ess\,inf}_{P\in\mathcal{P}} E_P[\cdot \mid \mathcal{G}].$$

We say that a positive $\mathbb{F}$-adapted process $Y := \{Y_t\}_{t=0,\cdots,T}$ is a $E^{\downarrow}$-supermartingale if

$$E^{\downarrow}[Y_{t+1} \mid \mathcal{F}_\theta] \le Y_t \tag{5.80}$$

for $t < T$. We say that Y is a $E^{\downarrow}$-submartingale if in (5.80) the opposite inequality holds. We say that Y is a $E^{\downarrow}$-martingale if it is both a $E^{\downarrow}$-supermartingale and a $E^{\downarrow}$-submartingale.□

Remark 5.29 *Note that any $E^{\downarrow}$-martingale is a $\mathcal{P}$-submartingale.*◇

In the next lemma we collect and prove some basic properties of the operator $E^{\downarrow}[\cdot \mid \cdot]$.

Lemma 5.30 *Let $\mathcal{G} \subset \mathcal{F}_T$ be a sub-σ-algebra. Let X^i be a positive $\mathcal{F}_T$-measurable random variable such that*

$$E_P[X^i] < \infty,$$

for $P \in \mathcal{P}$ and for $i = 1, 2$. Then, the operator $E^{\downarrow}[\cdot \mid \cdot]$ has the following properties.

1. *Superlinearity:*

$$E^{\downarrow}[X^1 + X^2 \mid \mathcal{G}] \ge E^{\downarrow}[X^1 \mid \mathcal{G}] + E^{\downarrow}[X^2 \mid \mathcal{G}].$$

2. *$\mathcal{G}$-linearity:*

$$E^{\downarrow}[X^1 + X^2 \mid \mathcal{G}] = E^{\downarrow}[X^1 \mid \mathcal{G}] + E^{\downarrow}[X^2 \mid \mathcal{G}],$$

if X^1 or X^2 is $\mathcal{G}$-measurable.

3. *Let $\mathcal{G}^1 \subset \mathcal{G}$ be a sub-σ-algebra of $\mathcal{G}$. Let $P_0 \in \mathcal{P}$ be a fixed probability measure. Let us introduce the notation*

$$\mathcal{P}(P_0, \mathcal{G}) = \{P \in \mathcal{P} \mid P = P_0 \quad in \quad \mathcal{G}\}.$$

Then

$$E_{P_0}[E^{\downarrow}[X^1 \mid \mathcal{G}] \mid \mathcal{G}^1] = \operatorname{ess\,inf}_{P\in\mathcal{P}(P_0,\mathcal{G})} E_P[X^1 \mid \mathcal{G}^1].$$

4. *Let us define*
$$V_t := E^{\downarrow}[X^1 \mid \mathcal{F}_t].$$
Then $V := \{V_t\}_{t=0,\cdots,T}$ *is a* $E^{\downarrow}$*-martingale and in particular a* $\mathcal{P}$*-submartingale.*

Proof. Let us start with the following identity

$$E^{\downarrow}[X^1 + X^2 \mid \mathcal{G}] = \operatorname{ess\,inf}_{P\in\mathcal{P}} \left\{ E_P[X^1 \mid \mathcal{G}] + E_P[X^2 \mid \mathcal{G}] \right\}. \tag{5.81}$$

1. In order to prove the first part, we only need to observe that $E_P[X^i \mid \mathcal{G}] \geq E^{\downarrow}[X^i \mid \mathcal{G}]$ and apply (5.81).

2. To prove the second part, let us assume that X^1 is $\mathcal{G}$-measurable. Then $E_P[X^1 \mid \mathcal{G}] = X^1$, and so the claim follows from the identity (5.81).

3. We now prove the third part. Let us observe that for arbitrary $P \in \mathcal{P}(P_0, \mathcal{G})$ we have

$$\begin{aligned} E_{P_0}[E^{\downarrow}[X^1 \mid \mathcal{G}] \mid \mathcal{G}^1] &\leq E_{P_0}[E_P[X^1 \mid \mathcal{G}] \mid \mathcal{G}^1] \\ &= E_P[E_P[X^1 \mid \mathcal{G}] \mid \mathcal{G}^1] \\ &= E_P[X^1 \mid \mathcal{G}^1]. \end{aligned}$$

This implies the inequality $\leq$. To prove the inequality $\geq$, is now enough to prove that

$$E_{P_0}[E^{\downarrow}[X^1 \mid \mathcal{G}]] \geq E_{P_0}[\operatorname{ess\,inf}_{P\in\mathcal{P}(P_0,\mathcal{G})} E_P[X^1 \mid \mathcal{G}^1]]. \tag{5.82}$$

Similar to the first part of lemma 5.3, we can construct a sequence of probability measures $\{P^i\}_{i=1}^{\infty} \in \mathcal{P}(P_0, \mathcal{G})$ such that

$$E_{P^i}[X^1 \mid \mathcal{G}] \searrow E^{\downarrow}[X^1 \mid \mathcal{G}],$$

and we can apply Lebesgue's dominated convergence theorem to conclude that
$$\lim_{i\to\infty} E_{P_0}[E_{P^i}[X^1 \mid \mathcal{G}]] = E_{P_0}[E^{\downarrow}[X^1 \mid \mathcal{G}]].$$
But it is clear that

$$E_{P_0}[E_{P^i}[X^1 \mid \mathcal{G}]] = E_{P_0}[E_{P^i}[X^1 \mid \mathcal{G}^1]] \geq E_{P_0}[\operatorname{ess\,inf}_{P\in\mathcal{P}(P_0,\mathcal{G})} E_P[X^1 \mid \mathcal{G}^1]],$$

and so, (5.82) follows.

4. We first prove that
$$E^{\downarrow}[V_{t+1} \mid \mathcal{F}_t] = V_t.$$
From the previous part we get that
$$E^{\downarrow}[V_{t+1} \mid \mathcal{F}_t] = \operatorname{ess\,inf}_{P_0 \in \mathcal{P}} \operatorname{ess\,inf}_{P \in \mathcal{P}(P_0, \mathcal{F}_{t+1})} E_P[X^1 \mid \mathcal{F}_t].$$
This immediately implies the desired identity. Now let $P \in \mathcal{P}$, then
$$E_P[V_{t+1} \mid \mathcal{F}_t] \geq E^{\downarrow}[V_{t+1} \mid \mathcal{F}_t] = V_t.$$
This implies that the process V is in fact a $\mathcal{P}$-submartingale.□

Remark 5.31 *The last part of lemma 5.30 establishes a consistency property similar to the martingale property for ordinary conditional expectations for the operator $E^{\downarrow}$. This property is similar to the property in the last part of theorem 4.29. A result which we have taken from Föllmer and Schied[27]; see their theorem 6.53.*◇

In the next proposition we use the notation $\Delta X_{i+1} := X_{i+1} - X_i$.

Proposition 5.32 *Let $\mathcal{P}$ be a family of probability measures and H a process as previously fixed. Then, the lower Snell envelope of H is a $E^{\downarrow}$-supermartingale and it admits the decomposition*
$$U_t^{\downarrow} = S_t - A_t, \tag{5.83}$$
where $S := \{S_t\}_{t \in \{0,\cdots,T\}}$ is a $E^{\downarrow}$-martingale, and $\{A_t\}_{t \in \{0,\cdots,T\}}$ is an increasing process with $A_0 = 0$ which is predictable in the sense that A_t is $\mathcal{F}_{t-1}$-measurable. This decomposition is unique, with
$$A_{t+1} := A_t + \left\{U_t^{\downarrow} - E^{\downarrow}[U_{t+1}^{\downarrow} \mid \mathcal{F}_t]\right\}. \tag{5.84}$$
Moreover, if $\mathcal{P}$ is the set of martingale measures for a price process $\{X_t\}_{t \in \{0,\cdots,T\}}$, then S admits the decomposition
$$S_t = U_0^{\downarrow} + \sum_{i=1}^{t} \xi_i \Delta X_i + C_t, \tag{5.85}$$
where ξ_s is $\mathcal{F}_{s-1}$-measurable and $\{C_t\}_{t \in \{0,\cdots,T\}}$ is an adapted non decreasing process with $C_0 = 0$. In this case, combining (5.83) and (5.85) we obtain the decomposition
$$U_t^{\downarrow} = U_0^{\downarrow} + \sum_{i=1}^{t} \xi_i \Delta X_i + C_t - A_t. \tag{5.86}$$

Proof.

1. Theorem 5.6 was proved in continuous time and it also holds in discrete time. So we know due to formula (5.10) and definition 5.28 that

$$U^{\downarrow}_{t+1} = E^{\downarrow}[H_{\tau^{\downarrow}_{t+1}} \mid \mathcal{F}_{t+1}].$$

This implies

$$\begin{aligned} E^{\downarrow}[U^{\downarrow}_{t+1} \mid \mathcal{F}_t] &= E^{\downarrow}[E^{\downarrow}[H_{\tau^{\downarrow}_{t+1}} \mid \mathcal{F}_{t+1}] \mid \mathcal{F}_t] \\ &= E^{\downarrow}[H_{\tau^{\downarrow}_{t+1}} \mid \mathcal{F}_t] \qquad (5.87) \\ &\leq U^{\downarrow}_t, \end{aligned}$$

where in (5.87) we have applied the last part of lemma 5.30. We conclude that $U^{\downarrow}$ is a $E^{\downarrow}$-supermartingale. Now we define a process A by $A_0 = 0$ and by formula (5.84).

The process A is clearly a predictable non decreasing process. Moreover, it is integrable in the sense that

$$E^{\downarrow}[A_T] < \infty.$$

In fact, let us assume that $E^{\downarrow}[A_{t-1}] < \infty$ and apply mathematical induction. We certainly have that $A_t \leq A_{t-1} + U^{\downarrow}_{t-1}$ and it follows that $E^{\downarrow}[A_t] \leq E^{\downarrow}[A_{t-1} + U^{\downarrow}_{t-1}]$. But for any $P \in \mathcal{M}$ the lower Snell envelope $U^{\downarrow}_t$ is integrable:

$$E_P[U^{\downarrow}_t] \leq E_P[U^P_t] \leq U^P_0 < \infty.$$

According to the mathematical induction hypothesis, A_{t-1} is integrable with respect to some $P^0 \in \mathcal{M}$. Now we have the inequality

$$E^{\downarrow}[A_t] \leq E_{P^0}[A_{t-1} + U^{\downarrow}_{t-1}] < \infty.$$

In the same way it can be proved that the process $S := \{S_t\}_{t \in \{0, \cdots, T\}}$ defined by

$$S_t := U^{\downarrow}_t + A_t \qquad (5.88)$$

is integrable in the sense that $E^{\downarrow}[S_t] < \infty$.

Now we prove that S is a $E^{\downarrow}$-martingale: For any pair s, t with $s > t$

$$E^{\downarrow}[S_s \mid \mathcal{F}_t] = S_t. \tag{5.89}$$

In fact, we start with the pair t, $t+1$ and obtain the identities

$$\begin{aligned} E^{\downarrow}[S_{t+1} \mid \mathcal{F}_t] &= E^{\downarrow}[U^{\downarrow}_{t+1} + A_{t+1} \mid \mathcal{F}_t] \\ &= E^{\downarrow}[U^{\downarrow}_{t+1} \mid \mathcal{F}_t] + A_{t+1} \\ &= E^{\downarrow}[U^{\downarrow}_{t+1} \mid \mathcal{F}_t] + A_t + U^{\downarrow}_t - E^{\downarrow}[U^{\downarrow}_{t+1} \mid \mathcal{F}_t] \\ &= S_t. \end{aligned}$$

The first identity is justified by the definition of S_{t+1}. The second is justified by the fact that A_{t+1} is $\mathcal{F}_t$-measurable and the second part of lemma 5.30. The third identity is justified by the definition of A_{t+1}. The last is direct.

In order to compute $E^{\downarrow}[S_s \mid \mathcal{F}_t]$ for a general pair s, t with $s > t$, we compute recursively the conditional expectations

$$E^{\downarrow}[\cdot \mid \mathcal{F}_{s-1}], E^{\downarrow}[\cdot \mid \mathcal{F}_{s-2}], \cdots, E^{\downarrow}[\cdot \mid \mathcal{F}_t],$$

to obtain (5.89).

From (5.89) follows that S is a $\mathcal{P}$-submartingale: For any $P \in \mathcal{P}$

$$E_P[S_{t+1} \mid \mathcal{F}_t] \geq E^{\downarrow}[S_{t+1} \mid \mathcal{F}_t] = S_t.$$

Thus, $U^{\downarrow}_t$ admits the decomposition (5.83).

2. We now show that the predictable increasing process A is unique. So let us assume that $U^{\downarrow}$ admits a decomposition of the form

$$U^{\downarrow} = M - B \tag{5.90}$$

where $M = \{M_t\}_{t=0,\cdots,T}$ is a $E^{\downarrow}$-martingale and $B := \{M_t\}_{t=0,\cdots,T}$ is an increasing predictable process with $B_0 = 0$. We apply mathematical induction. Assume we know that $M_t = S_t$ and $B_t = A_t$. Now we want to show that $M_{t+1} = S_{t+1}$ and $B_{t+1} = A_{t+1}$. But then, we have that

$$\begin{aligned} B_{t+1} + E^{\downarrow}[U^{\downarrow}_{t+1} \mid \mathcal{F}_t] &= E^{\downarrow}[B_{t+1} + U^{\downarrow}_{t+1} \mid \mathcal{F}_t] \\ &= E^{\downarrow}[M_{t+1} \mid \mathcal{F}_t] = M_t. \end{aligned}$$

The first identity is justified from the fact that B_{t+1} is $\mathcal{F}_t$-measurable and the second part of lemma 5.30. The second identity follows from the decomposition (5.90). The last identity follows from the fact that we assumed M to be a $E^{\downarrow}$-martingale. Then, we obtain that

$$B_{t+1} = M_t - E^{\downarrow}[U^{\downarrow}_{t+1} \mid \mathcal{F}_t] = S_t - E^{\downarrow}[U^{\downarrow}_{t+1} \mid \mathcal{F}_t] = A_t + U^{\downarrow}_t - E^{\downarrow}[U^{\downarrow}_{t+1} \mid \mathcal{F}_t],$$

where we have used (5.88). This proves, according to formula (5.84), that $B_{t+1} = A_{t+1}$.

3. If $\mathcal{P}$ is the family of martingale measures for a price process $\{X_t\}_{t\in\{0,\cdots,T\}}$, similar arguments as in the proof of the optional decomposition theorem 7.5 in [27] allow us to represent the $\mathcal{P}$-submartingale S as stated in (5.85):

$$S_t = U^{\downarrow}_0 + \sum_{i=1}^{t} \xi_i \Delta X_i + C_t.$$

The proof of the proposition is complete.□

Remark 5.33 *Assume that the stable family of probability measures $\mathcal{P}$ is such that dens($\mathcal{P}$), the corresponding family of densities with respect to R, is compact in the topology $\sigma(L^1(R), L^\infty(R))$. Then we can say more about the structure of the lower Snell envelope in discrete time: It is a P^*-supermartingale for some probability measure $P^* \in \mathcal{P}$. This result was proved by Riedel[49]; see his lemma 3.4 and assumption 2.4.*

Riedel[49] also studies a robust version of supermartingales which he calls minimax supermartingales, instead of $E^{\downarrow}$-supermartingales. He also develops a robust Doob decomposition for minimax supermartingales and a robust optional sampling theorem for minimax supermartingales; see his theorems 3.5 and 3.6. Note that the first part of our proposition 5.32 extends his result on Doob's decomposition in the case of a general stable family of equivalent probability measures $\mathcal{P}$ which is not necessarily weakly compact.◇

The next proposition describes the increasing process $\{A_t\}_{t\in\{0,\cdots,T\}}$ in proposition 5.32. Recall the stopping time $\tau^{\downarrow}_t$ constructed in theorem 5.6. We use the notation $\Delta\tau^{\downarrow}_{t+1} := \tau^{\downarrow}_{t+1} - \tau^{\downarrow}_t$.

Proposition 5.34 *The increasing process $\{A_t\}_{t\in\{0,\cdots,T\}}$ of proposition 5.32 admits the representation*

$$A_t := -\sum_{s=0}^{t-1} Y_s(\tau^{\downarrow}_{s+1} - s)^{-1} \Delta\tau^{\downarrow}_{s+1}, \tag{5.91}$$

where Y is the adapted process defined by

$$Y_t := E^{\downarrow}[H_t - H_{\tau^{\downarrow}_{t+1}} \mid \mathcal{F}_t]. \tag{5.92}$$

Proof. Since theorem 5.6 also holds in discrete time, the lower Snell envelope admits the representation

$$U^{\downarrow}_t = E^{\downarrow}[H_{\tau^{\downarrow}_t} \mid \mathcal{F}_t],$$

in terms of the stopping time $\tau^{\downarrow}_t$ defined in (5.9). It follows that

$$\begin{aligned} U^{\downarrow}_t - E^{\downarrow}[U^{\downarrow}_{t+1} \mid \mathcal{F}_t] &= E^{\downarrow}[H_{\tau^{\downarrow}_t} \mid \mathcal{F}_t] - E^{\downarrow}[E^{\downarrow}[H_{\tau^{\downarrow}_{t+1}} \mid \mathcal{F}_{t+1}] \mid \mathcal{F}_t] \\ &= E^{\downarrow}[H_{\tau^{\downarrow}_t} \mid \mathcal{F}_t] - E^{\downarrow}[H_{\tau^{\downarrow}_{t+1}} \mid \mathcal{F}_t]. \end{aligned}$$

We claim that the definition of the stopping time $\tau^{\downarrow}_t$ implies the identity

$$\tau^{\downarrow}_{t+1} - \tau^{\downarrow}_t = (\tau^{\downarrow}_{t+1} - t)1_{\{\tau^{\downarrow}_t = t\}},$$

and hence

$$H_{\tau^{\downarrow}_t} - H_{\tau^{\downarrow}_{t+1}} = (H_t - H_{\tau^{\downarrow}_{t+1}})1_{\{\tau^{\downarrow}_t = t\}}.$$

We then get

$$E^{\downarrow}[H_{\tau^{\downarrow}_t} \mid \mathcal{F}_t] - E^{\downarrow}[H_{\tau^{\downarrow}_{t+1}} \mid \mathcal{F}_t] = -1_{\{\tau^{\downarrow}_t = t\}} E^{\downarrow}[H_{\tau^{\downarrow}_{t+1}} - H_t \mid \mathcal{F}_t],$$

and

$$A_t := -\sum_{s=0}^{t-1} E^{\downarrow}[H_{\tau^{\downarrow}_{s+1}} - H_s \mid \mathcal{F}_s]1_{\{\tau^{\downarrow}_s = s\}}.$$

And now we observe that $1_{\{\tau^{\downarrow}_s = s\}} = (\tau^{\downarrow}_{s+1} - s)^{-1}\Delta\tau^{\downarrow}_{s+1}$ to conclude the representation (5.91).

It remains to prove the claimed identity for $\tau^{\downarrow}_t$. The equality clearly holds on the event $\{\tau^{\downarrow}_t = t\}$. But in the event $\{\tau^{\downarrow}_t > t\}$ we know that for any $P \in \mathcal{P}$ it follows that $\tau^P_t \geq t + 1$, hence $\tau^P_t = \tau^P_{t+1}$, and this implies the claim.□

5.3.4 Stopping times of maximal utility

In this section we motivate 0-optimal stopping times for the lower Snell envelope, from the point of view of robust Savage preferences. Let us fix a probability space $(\Omega, \mathcal{F}, R)$. An interesting class of robust Savage preferences defined on $L^\infty(R)$ admits a representation of the form

$$\psi(\cdot) := \inf_{Q \in \mathcal{Q}} E_Q[u(\cdot)], \tag{5.93}$$

where $\mathcal{Q}$ is a set of probability measures defined on $(\Omega, \mathcal{F})$ and are absolutely continuous with respect to R. See theorem 2.87 part (a) in [27].

The robust representation (5.93) will allow us to give an interpretation to a 0-optimal stopping times for the lower Snell envelope. To this end, consider a filtration $\mathbb{F}$ of the probability space $(\Omega, \mathcal{F}, R)$ satisfying the usual conditions of right continuity and completeness. Let $H := \{H_t\}_{0 \leq t \leq T}$ be a positive càdlàg $\mathbb{F}$-adapted stochastic process. We assume that $H \leq K$ for some constant $K > 0$. Typically, the stochastic process H represents the evolution of a financial position giving the right to choose the liquidation date. An important example would be an American put option from the point of view of the buyer.

We can now consider the maximal robust utility by

$$\sup_{\theta \in \mathcal{T}} \psi(H_\theta).$$

This optimization problem was discussed in remark 6.51 of [27] in discrete time for the special case where $\mathcal{Q}$ is a stable family of equivalent probability measures. The axiomatic framework of this special class of preferences, and the corresponding robust representation for the preference order, is due to Epstein and Schneider[20]. Starting with a set of axioms, notably including an axiom of time-consistency, they obtained a robust representation of the form (5.93) where $\mathcal{Q}$ is a *rectangular* family of probability measures. Rectangularity is equivalent to stability under pasting; see e.g., Delbaen[7] theorem 6.2. See also lemma B.1 in [49] for a proof in discrete time.

With this interpretation, a 0-optimal stopping time for the lower Snell envelope σ_0^* attains the maximal robust utility

$$\psi(H_{\sigma_0^*}) = \sup_{\theta \in \mathcal{T}} \psi(H_\theta).$$

We may thus say that σ_0^* is a stopping time of maximal utility for the process H, if utility is quantified by the robust utility functional ψ.

Part III

Appendix

Appendix A

A.1 BMO-Martingales

We fix a stochastic base in continuous time and finite horizon

$$(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}, R).$$

In section 5.3.4 we constructed an example of a stable family of probability measure whose densities with respect to R is a norm bounded closed subset of $L^p(R)$, for some $p > 1$. The construction was based on stochastic exponentials of BMO martingales. In this appendix we collect the concepts and results that were applied in the construction of example 5.26.

Let us recall definition 4.13 where we introduced the stochastic exponential of a continuous local martingale.

Let $M := \{M_t\}_{0\leq t\leq T}$ be a continuous local martingale with $M_0 = 0$. The stochastic exponential of M, denoted by $\mathcal{E}(M)$, is defined by

$$\mathcal{E}_t(M) := \exp\left\{M_t - \frac{1}{2}\langle M\rangle_t\right\},$$

where $\{\langle M\rangle_t\}_{0\leq t\leq T}$ is the quadratic variation process of the martingale M.□

Theorem A.1 *Let $M := \{M_t\}_{0\leq t\leq T}$ be a continuous local martingale with $M_0 = 0$. Then, the stochastic exponential $\mathcal{E}(M)$ is a local martingale and $\mathcal{E}_0(M) = 1$.*

Proof. See theorem 1.2 in Kazamaki[38].□

It is important to know if the stochastic exponential of a martingale is itself a martingale, and not only a local martingale. A positive answer can be given when martingales belong to the space of BMO-martingales.

Definition A.2 *Let $p \geq 1$ be a fixed exponent. A uniformly integrable martingale M with $M_0 = 0$ belongs to the space BMO_p if and only if there exists a positive number $C > 0$ such that for any stopping time $\theta \in \mathcal{T}$*

$$E_R[|M_T - M_{\theta-}|^p \mid \mathcal{F}_\theta] < C,$$

where C is a positive constant independent of θ. The infimum over all C satisfying this inequality is defined to be the norm of M in BMO_p and is denoted by $\|M\|_{BMO_p}$.$\square$

The space BMO_p is invariant with respect to p, see corollary 2.1 in [38]. We then simply write BMO, as usual. The next theorem improves theorem A.1: The stochastic exponential of a martingale M is always a local martingale, and if additionally M belongs to BMO then the stochastic exponential $\mathcal{E}(M)$ is a martingale.

Theorem A.3 *Let $M := \{M_t\}_{0 \leq t \leq T}$ be a continuous martingale. If $N \in BMO$ then $\mathcal{E}(M)$ is a uniformly integrable martingale.*

Proof. See theorem 2.3 in [38].$\square$

In the construction of example 5.26 we applied the so-called reverse Hölder inequality.

Definition A.4 *Let $p > 1$ be a fixed exponent and M be a continuous martingale with $M_0 = 0$. Then we say that the stochastic exponential $\mathcal{E}(M)$ satisfies R_p if the reverse Hölder inequality*

$$E_R[(\mathcal{E}_T(M))^p \mid \mathcal{F}_\theta] \leq C_p(\mathcal{E}_\theta(M))^p,$$

is satisfied for every stopping time $\theta \in \mathcal{T}$, and the constant C_p depends only on p and M.$\square$

Note that setting $\theta \equiv 0$ in this definition results in $E_R[\mathcal{E}_T(M)^p] \leq C_p$. In particular

$$\mathcal{E}_T(M) \in L^p(R).$$

It happens that the stochastic exponential of a BMO_p martingale automatically satisfies a reverse Hölder inequality R_p. It is possible to relate the BMO-norm with the norm of the corresponding stochastic exponential. This is made precise in the following theorem.

We need the function $\Phi : (1, \infty) \to \mathbb{R}$ defined by

$$\Phi(x) := \left\{1 + \frac{1}{x^2} \ln\left(\frac{2x-1}{2x-2}\right)\right\}^{\frac{1}{2}} - 1. \tag{A.1}$$

Theorem A.5 *Let $p > 1$, and let M be a continuous martingale with $M_0 = 0$. Then*

1. $\|M\|_{BMO_2} < \Phi(p)$ *implies that the stochastic exponential $\mathcal{E}(M)$ satisfies R_p.*

2. *There exists a number $C > 0$ depending only on p and the norms BMO_1 and BMO_2 of M, such that*

$$E_R[\mathcal{E}_T(M)^p] < C.$$

Proof. See theorem 3.1 in [38].□

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List of symbols

Abbreviation

càdlàg right continuous with finite limits, see definition 1.2

Basics

$a := b$ b defines a

$a \vee b$ $:= \max(a, b)$

$a \wedge b$ $:= \min(a, b)$

$(a)^+$ $:= \max(a, 0)$

$\mathbb{N}$ the set of natural numbers $\{1, 2, \ldots\}$

$\mathbb{R}_+$ the set of positive real numbers.

$1_A(\cdot)$ the indicator function of the set A

Notation parts I and II

$\mathcal{A}$ the family of randomized stopping times, see definition 2.13.

$\overline{\mathcal{A}}$ the family of quasi-randomized stopping times, see definition 2.13.

Ad_c the family of admissible strategies at cost c, see definition 1.3.

BMO the space of continuous martingales of bounded mean oscillation, see definition A.2.

$class(D)$ see definition 1.8.

$class(D)$ with respect to a family $\mathcal{P}$, see definition 5.5.

$dens(\mathcal{P})$ the family of densities of $\mathcal{P}$ with respect to R, see 5.30.

$\mathcal{E}(\cdot)$ the stochastic exponential of a continuous local martingale, see definition 4.13.

$E^{\downarrow}[\cdot,\cdot]$ see definition 5.28.

$\operatorname{ess\,inf} \mathbb{X}$ the essential infimum of a family of stochastic processes $\mathbb{X}$, see definition 5.19.

Φ the function (A.1).

ϕ^{ξ} the ratio process of a strategy ξ, see definition 3.3.

$\mathbb{L}^{\infty}$ the Banach space of uniformly bounded càdlàg stochastic processes, see definition 2.17.

$\mathcal{M}$ the family of equivalent martingale measures for the price process X, see (1.1).

$\mathcal{M}^{a}$ the family of absolutely continuous martingale measures for the price process X, see section 4.3.1.

$\mathcal{P}$ a stable family of equivalent probability measures, see definition 4.2 and section 4.2.1.

$\mathcal{P}$ -martingale see definition 1.5.

$\mathcal{P}$ -submartingale see definition 1.5.

$\mathcal{P}$ -supermartingale see definition 1.5.

$\mathcal{P}(P_0,\tau)$ see (4.12).

$PH(\cdot)$ the value function of robust partial hedging, see definition 2.2.

$\pi_{\sup}(\cdot)$ the supremum of arbitrage free prices for American options, see definition 1.19.

$\pi_{\inf}(\cdot)$ the infimum of arbitrage free prices for American options, see definition 1.19.

$QH(\cdot)$ the value function of quantile hedging, see definition 3.3.

R_p the reverse Hölder inequality, see definition A.4.

$\mathcal{R}$ the family of randomized test processes, see definition 3.2.

$\mathcal{R}_c$ the family of randomized test processes satisfying the budget constraint (3.5), see definition 3.6.

$\mathcal{T}$ given a filtration $\mathbb{F}$, the family of stopping times $[0, T]$–valuated, see page 11.

$\mathcal{T}[\tau, T]$ the class of stopping times after τ, see (1.2).

$T(\cdot)$ the value function of the testing problem, see definition 3.6.

$T^+(\cdot)$ the upper value function of optimal testing, see section 3.2.2.

$\tau_t^\uparrow$ t-optimal stopping time for the upper Snell envelope, see theorem 4.20.

$\tau_t^\downarrow$ t-optimal stopping time for the lower Snell envelope, see theorem 5.6.

$\{U_t^P(H)\}_{0\leq t\leq T}$ the Snell envelope of a process H with respect to a probability measure P, see definition 1.11.

$\{U_t^\uparrow\}_{0\leq t\leq T}$ the upper Snell envelope, see (1.4) and theorem 4.3.

$\{U_t^\downarrow\}_{0\leq t\leq T}$ the lower Snell envelope, see (1.16) and section 5.2.3.

$Z^\uparrow$ the upper Snell envelope as a family of random variables, see (4.11).

$Z^\downarrow$ the lower Snell envelope as a family of random variables, see (5.3).

Z_τ^P see (4.9).

$V^{c,\xi}$ the value process of an admissible strategy ξ constructed at cost c, see definition 1.3.

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Selbständigkeitserklärung

Hiermit erkläre ich, die vorliegende Arbeit selbständig ohne fremde Hilfe verfasst und nur die angegebene Literatur und Hilfsmittel verwendet zu haben.

Erick Treviño Aguilar
Berlin, Dezember 14, 2007