Models for Interest Rates and Interest Rate Derivatives

A Diplomarbeit and Master Thesis presented

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Declaration of Authorship

We hereby confirm that we have authored this Diplomarbeit, respectivley master thesis, independently and without use of others than the indicated resources. All passages, which are literally or in general matter taken out of publications or other resources, are marked as such.

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Berlin, May 12, 2009

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- The sections "Short Rate Models" and "The Bond Valuation Equation" were written solely by Li Sun.

- The remaining sections were written in cooperation. Lasse Groth has mainly worked on the theoretical background and Li Sun has mainly worked on calibration and implementation.

Abstract

This thesis gives an introduction to the principles of modern interest rate theory. After covering the basic tools for working in an environment with stochastic interest rates, we introduce different models for the term structure. The principals of risk neutral pricing are introduced and the Black model is derived. Closed form bond valuation equations are derived for the Cox, Ingersoll and Ross (CIR) model. Short examples of calibration of the Vasicek, CIR and LIBOR market model are given.

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1 Introduction

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Pricing interest rate derivatives fundamentally depends on the term structure of interest rates. Until now we have assumed:

- constant risk free domestic interest rate
- independence of the price of the option from the possibly stochastic interest rate r.

When considering interest rate derivatives in practice both of these assumptions will not be fulfilled. Just as the dynamics of a stock price are unknown and have to be modeled via a stochastic process, the stochastics of interest rates are derived by modeling their dynamics. Being able to model the term structure of interest rates adequately is vital when it comes to valuation and trading of interest rate derivatives. As interest rate derivatives have become increasingly popular, especially among institutional investors, the standard models for the term structure have become a core part of financial engineering.

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In this thesis we will first introduce the basic tools for working in an environment with stochastic interest rates. After a quick look at the basic interest rate derivatives and the standard market model to value these, we will move on to outline the basic theoretical background in interest rate theory. Unlike in the world of equities, there is no standard model for interest rates. We will begin our overview of the different interest rate models with one-factor and two-factor short rate models. We will cover the Heath Jarrow Morton framework and introduce the LIBOR Market Model. We conclude the thesis by outlining the basic application of the above mentioned theories and models to real data.

2 Interest Rates and Prices

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DEFINITION 2.1 *A bond $V\{r(t), t, T\}$ produces at the time of maturity T a fixed amount Z, the nominal value, and if applicable, at predetermined dates before T coupon payments.*

For simplicity of notation we will write $V\{r(t), t, T\}$ as $V(t, T)$. If there are no coupons, the bond is referred to as a zero coupon bond or zero bond for short. We will be considering $V(t, T)$ as a unit principal (i.e. $V(T, T) = 1$) zero coupon bond in the following sections. We will further restrict ourselves to default free government debt.

The simple rate of return $R(t, T)$ from holding a bond over the time interval $\tau(t, T) = T - t$ equals:

$$R(t, T) = \frac{1 - V(t, T)}{\tau(t, T)V(t, T)} = \frac{1}{\tau(t, T)} \left\{ \frac{1}{V(t, T)} - 1 \right\}. \tag{2.1}$$

The equivalent rate of return, with continuous compounding, is referred to as the yield to maturity on a bond.

DEFINITION 2.2 *The yield to maturity $Y(t, T)$ is the internal rate of return at time t on a bond with maturity date T.*

$$Y(t, T) = -\frac{1}{\tau(t, T)} \log V(t, T). \tag{2.2}$$

The rates $Y(t, T)$ considered as a function of time T will be referred to as the term structure of interest rates at time t. The straightforward relationship between the yield to maturity and the bond price is given by:

$$V(t, T) = \exp\{-Y(t, T)\tau(t, T)\}.$$

In order to transform $R(t,T)$ into a continuous compounding rate calculate the relationship between $R(t,T)$ and the continuous compounded rate $Y(t,T)$ as follows:

$$Y(t,T) = \frac{1}{\tau(t,T)} \log\{1 + R(t,T)\tau(t,T)\}. \tag{2.3}$$

We have just seen that the yield to maturity applies to a bond $V(t,T)$. The forward rate $f(t,T,S)$ corresponds to the internal rate of return of a bond $V(T,S)$.

DEFINITION 2.3 *The forward rate $f(t,T,S)$ is the internal rate of return at time t on a bond lasting from time T to the maturity date S, with $t < T < S$.*

$$f(t,T,S) = \frac{\log V(t,T) - \log V(t,S)}{\tau(T,S)}. \tag{2.4}$$

This shows that the forward rate can be thought of as the yield to maturity of a bond lasting from time T to time S, i.e. $f(t,T,S) = Y(T,S)$.

An intuitive approach to the forward rate is by considering the forward rate in terms of arbitrage free investments and in a simple compounding manner. If one would invest 1 EUR in a bond $V(t,T)$ and at maturity T re-invest the received amount in a bond $V(T,S)$, by no arbitrage this has to be equal to an investment of 1 EUR at time t in a bond $V(t,S)$. Therefore due to the no arbitrage condition and following the same path as equation (2.1):

$$V(T,S) = \frac{V(t,S)}{V(t,T)}$$

and therefore

$$\begin{aligned} F(t,T,S) &= \frac{1}{\tau(T,S)} \left\{ \frac{V(t,T) - V(t,S)}{V(t,S)} \right\} \\ &= \frac{1}{\tau(T,S)} \left\{ \frac{V(t,T)}{V(t,S)} - 1 \right\}, \end{aligned} \tag{2.5}$$

where $F(t,T,S)$ is the simple compounded forward rate.

By applying equation (2.3) to the forward rate we achieve equality with equation (2.4):

$$f(t,T,S) = \frac{1}{\tau(T,S)} \log\{1 + F(t,T,S)\tau(T,S)\}.$$

The instantaneous forward rate $f(t,T)$ is the limiting case of the forward rate $f(t,T,S)$. The instantaneous forward rate is the forward rate which lasts from time t for some infinitesimal time period ds. For $S \to T$:

$$f(t,T) = \lim_{S \to T} f(t,T,S).$$

The application of l'Hospital's rule gives:

DEFINITION 2.4 *The instantaneous forward rate $f(t,T)$ is the forward interest rate at time t for instantaneous risk free borrowing or lending at time T.*

$$f(t,T) = -\frac{\partial \log V(t,T)}{\partial T}. \tag{2.6}$$

The existence of $f(t,T)$ assumes that the continuum of bond prices is differentiable w.r.t T.

It holds that:

$$V(t,T) = \exp\left\{ -\int_t^T f(t,s)\, ds \right\}.$$

2.1 Money Market Account

One of the most basic instruments related to interest rates is the money market (or savings) account. The money market account represents a risk less investment at the prevailing instantaneous interest rate $r(t)$, where

$r(t) =$ spot rate $=$ interest rate for the shortest possible investment.

DEFINITION 2.5 *Define $A(t)$ as the value of the money market account at time t. We assume $A(t) = 1$ and that the account develops according to the following differential equation:*

$$dA(t) = r(t)A(t)dt,$$

with $r(t)$ as a positive function of time.

As a consequence:

$$A(T) = \exp\left\{\int_t^T r(s)\,ds\right\}. \tag{2.7}$$

At any time t, the current value $r(t)$ of the spot rate is the instantaneous rate of increase of the money market account value. The subsequent values of the spot rate, however, are unknown. In fact, it will be assumed that $r(t)$ is a stochastic process. The general form for the process of $r(t)$ is given by the following Itô process:

$$dr(t) = \mu\{r(t), t\}dt + \sigma\{r(t), t\}dW_t \tag{2.8}$$

with W_t being a Wiener process. For the moment we will restrict ourselves to this basic set up. The stochastic process for interest rates is covered in more detail in section 5.

2.2 Forward Rate Agreement

DEFINITION 2.6 *A forward rate agreement $FRA_{R_K,S}\{r(t), t, T\}$ is an agreement at time t that a certain interest rate R_K will apply to a principal amount (for simplicity again equal to 1) for a certain period of time $\tau(T, S)$, in exchange for an interest rate payment at the future interest rates $R(T, S)$, with $t < T < S$.*

The current value of a FRA paid-in-arrear is the discounted value of the payoff received at time S.

$$\begin{aligned}
\mathrm{FRA}_{R_K,S}\{r(t), t, T\} &= \{1 + R(t, S)\tau(t, S)\}^{-1}\tau(T, S)\{R_K - R(T, S)\} \\
&= V(t, S)\{1 + R_K\tau(T, S)\} - \{1 + R(t, T)\tau(t, T)\}^{-1} \\
&= V(t, S)\tau(T, S)R_K + V(t, S) - V(t, T)
\end{aligned}$$

The payoff will be negative if the floating rate is above the fixed rate and in the opposite case the payoff will be positive. When valuing a FRA we are considering three different time instants, namely the current time t, the time at which the FRA will come into place T and the maturity of the FRA S. However, all relevant interest rates can be observed at time t, so no knowledge of the future term structure of interest

rates is needed.

Often the strike rate R_K is chosen so that the $\mathrm{FRA}_{R_K,S}\{r(t),t,T\}$ is *fair*, i.e. $\mathrm{FRA}_{R_K,S}\{r(t),t,T\} = 0$, at time t. In this case the strike rate will be equal to $F(t,T,S)$.

2.3 Interest Rate Swap

DEFINITION 2.7 *An Interest Rate Swap $IRS_{R_K,T}\{r(t),t\}$ is an agreement to exchange payments of a fixed rate R_K against a variable rate $R(t,t_i)$ over a period $\tau(t,T)$ at certain time points t_i, with $t \leq t_i \leq T$.*

There are two basic types of IRS: a payer IRS and a receiver IRS. In the case of a payer IRS the fixed rate is payed and the floating rate is received. A receiver IRS functions exactly the other way around. The two parts of an IRS can also referred to as "floating rate leg" and "fixed rate leg".

The value of a receiver IRS $\mathrm{RIRS}_{R_K,T}\{r(t),t\}$ on the rate R_K starting at t and maturing at T with n payments between t and T is given by:

$$\begin{aligned}
\mathrm{RIRS}_{R_K,T}\{r(t),t\} &= \sum_{i=0}^{n-1} V(t,t_{i+1})\tau_i \left\{R_K - R(t_i,t_{i+1})\right\} \\
&= \sum_{i=0}^{n-1} V(t,t_{i+1})\tau_i \left\{R_K - F(t,t_i,t_{i+1})\right\} \\
&= \sum_{i=0}^{n-1} V(t,t_{i+1})\tau_i R_K - 1 + V(t,T).
\end{aligned} \tag{2.9}$$

where $t_0 = t$ and $t_n = T$, $i = 1,\ldots,n-1$, $t \leq t_i \leq T$, $\tau_i = \tau(t_i,t_{i+1})$ and $V(t,t) = 1$.

To simplify the valuation of an IRS, an IRS can be thought of as a portfolio of FRAs. By decomposing the IRS into a series of FRAs:

$$\mathrm{RIRS}_{R_K,T}\{r(t),t\} = \sum_{i=0}^{n-1} \mathrm{FRA}_{R_K,t_{i+1}}\{r(t),t,t_i\}$$

Alternatively an IRS can also be valued by considering the fixed and floating rate leg separately. This would correspond to thinking of an IRS as an agreement to exchange a coupon-bearing bond for a floating rate note.

The coupon payments c_i of a coupon bond paying n coupons at a rate of R_K would be:

$$c_i = \tau_i R_K$$

for $i = 1, \ldots, n$. The principal amount is repaid at the maturity of the bond. The value of the fixed leg is therefore the discounted value of the coupon payments plus the value of the principal amount received at time T:

$$\begin{aligned} \text{FixedLeg}_{R_K}\{r(t), t\} &= \sum_{i=0}^{n-1} \{1 + R(t, t_{i+1})\tau_i\}^{-1} c_i + V(t, T) \\ &= \sum_{i=0}^{n-1} V(t, t_{i+1}) R_K \tau_i + V(t, T) \end{aligned} \tag{2.10}$$

For the floating leg we can use the fact that a floating rate note will always be traded at par at the reset dates. So far we have considered time t as the first reset date, therefore if the principal amount is repaid at maturity it follows that:

$$\text{FloatingLeg}\{r(t), t\} = 1. \tag{2.11}$$

The difference between equation (2.10) and (2.11) is the value of the IRS.

$$\text{RIRS}_{R_K,T}\{r(t), t\} = \text{FixedLeg}\{r(t), t\} - \text{FloatingLeg}\{r(t), t\}$$

and we see that is equal to equation (2.9). In both valuation methods presented above no knowledge of the future term structure is needed because all relevant interest rates are known at time t. As with FRA the strike rate R_K, now referred to as the swap rate R_S, is often chosen so that the IRS has at time t a value of zero.

DEFINITION 2.8 *The swap rate $R_S(t, T)$ is the rate that makes the value of an IRS equal to zero at time t.*

By setting equation (2.9) equal to zero and rearranging:

$$R_S(t,T) = \{1 - V(t,T)\} / \left\{ \sum_{i=0}^{n-1} V(t,t_{i+1})\tau_i \right\}. \tag{2.12}$$

We have now covered the basic interest rates and we have seen the first (albeit very simple) interest rate derivatives. Before we move on, we will give a quick introduction to risk neutral pricing and equivalent martingale measures. We will need these tools in order to be able to understand the following pricing of more complex derivatives such as caps, floors, swap options and bond options.

3 Risk Neutral Valuation and Numeraire Measures

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It is vital to understand the principles of risk neutral valuation in a world with stochastic interest rates. If interest rates are stochastic the market price of risk becomes an essential factor in valuation of derivatives. Likewise, the numeraire measure needs to be introduced. In this section we will show how a convenient choice of numeraire and a corresponding choice for the market price of risk can greatly simplify the valuation of certain interest rate derivatives. We will first give a brief introduction to the principles of risk neutral valuation and the market price of risk, followed by a first look at techniques of measure change and a series of different possible numeraire measures are considered. In the next section we will see how the methods introduced here can be combined with the Black model to value interest rate derivatives.

3.1 Principles of Risk Neutral Valuation

So far we have assumed that the market price of risk is equal to zero.

DEFINITION 3.1 *The market price of risk defines a value above the risk free return for an asset.*

Following Franke, Härdle and Hafner (2008) and Hull (2006) we assume that the process θ_t is a geometric Brownian motion:

$$d\theta_t = m\theta_t dt + s\theta_t dW_t \tag{3.1}$$

where dW_t is a Wiener Process.

Assume further that V_{1t} and V_{2t} are the prices of two derivatives dependent only on θ_t and t. As a simplification, no payments are allowed during the observation time period. The processes followed by $V_{jt} = V_j(\theta, t)$, $j = 1{,}2$ are:

$$dV_{jt} = \mu_{jt} V_{jt} dt + \sigma_{jt} V_{jt} dW_t, \tag{3.2}$$

where μ_{jt}, σ_{jt} could be functions of θ_t and t. The random process W_t in equations (3.1) and (3.2) is the same, as there is only one source of uncertainty.

If we would construct a portfolio Π_t of $\sigma_{2t} V_{2t}$ units of V_{1t} and short sell $-\sigma_{1t} V_{1t}$ units of V_{2t} this portfolio is instantaneously risk less.

$$\begin{aligned}
\Pi_t &= (\sigma_{2t} V_{2t}) V_{1t} - (\sigma_{1t} V_{1t}) V_{2t} \\
&= (\sigma_{2t} V_{2t})(\mu_{1t} V_{1t} dt + \sigma_{1t} V_{1t} dW_t) - (\sigma_{1t} V_{1t})(\mu_{2t} V_{2t} dt + \sigma_{2t} V_{2t} dW_t) \\
&= (\sigma_{2t} V_{2t} V_{1t} \mu_{1t} - \sigma_{1t} V_{1t} V_{2t} \mu_{2t}) dt. \qquad (3.3)
\end{aligned}$$

This portfolio would be risk free and thus in the time period dt is must produce the risk free profit $r(t)dt$:

$$d\Pi_t = r(t) \Pi_t dt. \tag{3.4}$$

Together with equation (3.3) and equation (3.4) this produces:

$$\begin{aligned}
(\sigma_{2t} V_{2t} V_{1t} \mu_{1t} - \sigma_{1t} V_{1t} V_{2t} \mu_{2t}) dt &= (\sigma_{2t} V_{2t} V_{1t} - \sigma_{1t} V_{1t} V_{2t}) r(t) dt \\
\sigma_{2t} \mu_{1t} - \sigma_{1t} \mu_{2t} &= r(t) \sigma_{2t} - r(t) \sigma_{1t} \\
\frac{\mu_{1t} - r(t)}{\sigma_{1t}} &= \frac{\mu_{2t} - r(t)}{\sigma_{2t}}
\end{aligned}$$

The quantity:

$$\lambda_t = \frac{\mu_{1t} - r(t)}{\sigma_{1t}} = \frac{\mu_{2t} - r(t)}{\sigma_{2t}}$$

is called the market price of risk, dependent on both θ_t and t but not on the nature of the derivative V_t.

The general form for λ_t is:

$$\lambda_t = \frac{\mu(\theta_t, t) - r(t)}{\sigma(\theta_t, t)}. \tag{3.5}$$

We can rewrite this as:

$$\mu_t - r(t) = \lambda_t \sigma_t.$$

We can interpret σ_t, which in this interpretation can also be negative, as the level of the θ_t-risk in V_t.

In the risk neutral world we consider $\lambda_t = 0$, i.e. $\mu_t = r(t)$. The process for V_t was given by:

$$dV_t = r(t)V_t dt + \sigma_t V_t dW_t.$$

By making other assumptions about the market price of risk we define other "risk" worlds that are internally consistent. If the market price of risk is λ_t, in combination with (3.5), it can be shown that:

$$dV_t = \{r(t) + \lambda_t \sigma_t\} V_t dt + \sigma_t V_t dW_t.$$

3.2 Change of Numeraire

We already know that a martingale is a zero drift stochastic process for which it holds that:

$$\mathsf{E}_t(\theta_T) = \theta_t.$$

Suppose now that V_t and Z_t are the prices of traded securities dependent on the same source of uncertainty where both securities produce no income during the time under consideration. Define the relative price of V_t w.r.t Z_t as:

$$\phi_t = \frac{V_t}{Z_t}.$$

We refer to Z_t as the numeraire.

DEFINITION 3.2 *A numeraire is any non-dividend paying asset.*

A numeraire is chosen as to normalize all other asset prices with respect to it. Instead of considering the prices of V_t we are considering the relative prices $\phi_t = V_t/Z_t$.

A convenient choice of the market price of risk in combination with a particular numeraire can lead to a simplification of the valuation problems we encounter in a setting with stochastic interest rates. Combining the market price of risk with a

numeraire measure leads to the equivalent martingale measure result.

3.3 Equivalent Martingale Measure

For a certain choice of the market price of risk ϕ_t will be a martingale. If we put $\sigma_{Z_t} = \lambda_t$, in combination with (3.5) and under the usual assumptions:

$$dV_t = \{r(t) + \sigma_{V_t}\sigma_{Z_t}\}V_t dt + \sigma_{V_t} V_t dW_t$$

and

$$dZ_t = \{r(t) + \sigma_{Z_t}^2\}Z_t dt + \sigma_{Z_t} Z_t dW_t.$$

Itô's lemma gives:

$$d\log V_t = \left\{r(t) + \sigma_{Z_t}\sigma_{V_t} - \frac{\sigma_{V_t}^2}{2}\right\} dt + \sigma_{V_t} dW_t$$

and

$$d\log Z_t = \left\{r(t) + \frac{\sigma_{Z_t}^2}{2}\right\} dt + \sigma_{Z_t} dW_t$$

so that

$$d\left(\log V_t - \log Z_t\right) = \left(\sigma_{Z_t}\sigma_{V_t} - \frac{\sigma_{V_t}^2}{2} - \frac{\sigma_{Z_t}^2}{2}\right) dt + (\sigma_{V_t} - \sigma_{Z_t})dW_t$$

or

$$d\left(\log \frac{V_t}{Z_t}\right) = -\frac{(\sigma_{V_t} - \sigma_{Z_t})^2}{2} dt + (\sigma_{V_t} - \sigma_{Z_t})dW_t.$$

Again using Itô's Lemma:

$$d\frac{V_t}{Z_t} = (\sigma_{V_t} - \sigma_{Z_t})\frac{V_t}{Z_t} dW_t,$$

showing that ϕ_t is a martingale. A stochastic system where the market price of risk is σ_{Z_t} is referred to as a world that is forward risk neutral w.r.t. Z_t. Because ϕ_t is martingale it follows that:

$$\frac{V_t}{Z_t} = \mathsf{E}_{Z_t}\left(\frac{V_T}{Z_T}\right)$$

or

$$V_t = Z_t \mathsf{E}_{Z_t}\left(\frac{V_T}{Z_T}\right) \tag{3.6}$$

where E_{Z_t} denotes the expected value in a world that is forward risk neutral w.r.t Z_t.

We can generalize this reasoning to different numeraires. Choosing an appropriate numeraire can be helpful when valuing derivatives in the context of stochastic interest rates. In the following we will see the basic choices for numeraire and the corresponding choice of the market price of risk, which will lead to an equivalent martingale measure result.

3.4 Traditional Risk Neutral Numeraire

A world that is forward risk neutral w.r.t. to the money market account $A(t)$ is a world where the market price of risk is equal to zero. This is the traditional risk neutral world we have considered so far. Denoting A_t as $A(t)$ It follows that:

$$V_t = A_t \mathsf{E}_t\left(\frac{V_T}{A_T}\right)$$

where E_t denotes the expectation in the traditional risk neutral world. In the case of $A_t = 1$ and formula (2.7) this reduces to

$$V_t = \mathsf{E}_t\left[\exp\left\{-\int_t^T r(s)\,ds\right\}V_T\right].$$

This is a crucial result. If we consider V_t as a bond $V(t,T)$ it shows that there is a clear relationship, as one might have assumed, between the price of a bond and the term structure of interest rates. However, a different choice of numeraire might be more practical when dealing with interest rate derivatives.

3.5 Other Choices of Numeraire

In order to ease the valuation of interest rate derivatives it can be helpful to deviate from the traditional risk neutral world. We will give an overview of the basic numeraire

measures that are used in the valuation of interest rate derivatives.

3.5.1 Zero Bond as Numeraire

Setting Z_t equal to $V(t,T)$ will yield another martingale measure. To not confuse notations we will write S_t instead of V_t, where S_t now refers to some traded asset. Because $Z_T = V(T,T) = 1$ and $Z_t = V(t,T)$ we get:

$$S_t = V(t,T)\mathsf{E}_{V_t}(S_T), \tag{3.7}$$

where we use E_{V_t} to denote the expectation in a world that is forward risk neutral w.r.t. $V(t,T)$.

This result will be helpful when considering the valuation of different interest rate derivatives. It shows that we can value any security that provides a payoff at time T by calculating the expected payoff in a world that is forward risk neutral w.r.t to a bond maturing at time T and discount it by multiplying by the value of $V(t,T)$. It is correct to assume that the expected value of the underlying asset equals its forward value when computing the expected payoff. These results will be critical to our understanding of the standard market model for bond options.

3.5.2 Interest Rates with a Bond as Numeraire

Remember that $F(t,T,S)$ is the forward interest rate for the period between T and S as seen at time t. The forward price, as seen at time t, of a zero-coupon bond lasting between times T and S is:

$$V(T,S) = \frac{V(t,S)}{V(t,T)}.$$

It follows that:

$$F(t,T,S) = \frac{1}{\tau(T,S)}\left\{\frac{V(t,T) - V(t,S)}{V(t,S)}\right\}.$$

Set

$$S_t = \frac{1}{\tau(T,S)}\{V(t,T) - V(t,S)\}$$

and $Z_t = V(t, S)$. The equivalent martingale measure result shows that $F(t, T, S)$ is a martingale in a world that is forward risk neutral w.r.t. $V(t, S)$. This means that:

$$F(t, T, S) = \mathsf{E}_{V_S}\{F(T, T, S)\}$$

where E_{V_S} denotes the expectation in a world that is forward risk neutral w.r.t. $V(t, S)$. Combining this result with equation (3.6) will be decisive to understand the pricing of caps and floors.

3.5.3 Annuity Factor as Numeraire

We can also consider the annuity factor from equation (2.12) as a numeraire. This can be helpful when pricing swap options. We can rewrite the solution for the swap rate R_S from equation (2.12) as:

$$R_S(t, T) = \frac{1 - V(t, t_n)}{U(t, T)},$$

where

$$U(t, T) = \sum_{i=0}^{n-1} V(t, t_{i+1})\tau_i.$$

If we now set S_t equal to $1 - V(t, t_n)$ and Z_t equal to $U(t, T)$ this leads to:

$$R_S(t, T) = \mathsf{E}_{U_t}\{R_S(T, S)\} \tag{3.8}$$

where E_{U_t} denotes the expectation in a world that is forward risk neutral w.r.t. to $U(t, T)$. This result in combination with equation (3.6) will be critical to our understanding of the standard market model for European swap options.

This concludes our overview of the choices of numeraire measures and risk neutral pricing. We will now move on to see how these methods can be used in the valuation of interest rate derivatives with help of the Black model.

4 Interest Rate Derivatives

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We have presented the basic tools for an analysis of stochastic interest rates and can now move on to interest rate derivatives. The standard market model to price interest rate derivatives is the Black model.

4.1 The Black Model

A large number of the commonly traded derivatives are priced via the Black model. If the future term structure of interest rates is needed in order to price the derivative we apply the Black (1975) model. This includes caps, floors, swap options and bond options.

Consider a European Call $C_{K,T}(V,t)$ with payoff $\max(V_T - K, 0)$ at time T and K being the strike price.

Assuming:

- The value of the option today is its discounted expected payoff.
- V_T has a lognormal distribution with the standard deviation of $\log V_T$ being $\sigma\sqrt{\tau}$.
- The expected value of V_T at time t is the forward price F_t

By means of the Black Scholes framework this implies that:

$$C_{K,T}(V,T) = \mathsf{E}(V_T)\Phi(y + \sigma\sqrt{\tau}) - K\Phi(y)$$

with

$$y = \frac{\log\{\mathsf{E}(V_T)/K\} - \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}.$$

Because interest rates now considered to be stochastic, we discount the expected payoff by multiplying with $V(t,T)$. With the $\mathsf{E}(V_T) = F_t$ the value of the option at time t is:

$$C_{K,T}(V,t) = V(t,T)\left\{F_t\Phi(y+\sigma\sqrt{\tau}) - K\Phi(y)\right\} \tag{4.1}$$

with

$$y = \frac{\log\left(\frac{F_t}{K}\right) - \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}.$$

4.2 Bond Option

DEFINITION 4.1 *A bond options is an agreement which gives the holder the right to buy or sell a particular bond at a specified time T for a specified strike price K.*

As an example consider a European call with a strike price K and a maturity T on a zero bond with a maturity of $S > T$, i.e., the option is to buy the bond at time T at a price K. Therefore:

$$C_{K,T}\{r(t),t\} = \max\{V(T,S) - K, 0\}$$

To value a bond option we will again apply Black's model. We change the numeraire from the current cash amount to a bond $V(t,T)$ and thus we can, in combination with the result from equation (3.7), consider that the current value of any security as its expected future value at time T multiplied by $V(t,T)$. It can shown that the expected value of any traded security at time T is equal to its forward price. Thus the price of an option with maturity T on a bond $V(t,S)$ with $S > T$ is given by:

$$C_{K,T}\{r(t),t\} = V(t,T)\mathsf{E}_T\left[\max\{V(T,S) - K, 0\}\right]$$

and

$$\mathsf{E}_T\{V(T,S)\} = F_t, \tag{4.2}$$

with E_T denoting the forward risk neutral expectation w.r.t to $V(t,T)$ as the numeraire. Again applying the same assumptions as above we find that

$$C_{K,T}\{r(t),t\} = V(t,T)\left[\mathsf{E}_T\{V(T,S)\}\Phi(y+\sigma\sqrt{\tau}) - K\Phi(y)\right]$$

with

$$y = \frac{\log\left[\mathsf{E}_T\{V(T,S)\}/K\right] - \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}.$$

By replacing $\mathsf{E}_T\{V(T,S)\}$ in the above equation with (4.2) we achieve equality with equation (4.1).

4.3 Caps and Floors

DEFINITION 4.2 *A Cap contract $Cap_{R_K}, T\{r(t), t\}$ gives the holder the right to receive the floating interest rate $R(t_i, t_{i+1})$ at certain time points t_i in exchange for the fixed rate R_K, with $i = 1, \ldots, n$.*

DEFINITION 4.3 *A floor contract $Flr_{R_K}, T\{r(t), t\}$ gives the holder the right to receive the fixed rate R_K at certain time points t_i in exchange for the floating rate $R(t_i, t_{i+1})$, with $i = 1, \ldots, n$.*

Caps and Floors are derivatives which can be used to insure the holder against interest rates rising above or dropping below a certain level, the cap rate or floor rate. Therefore these interest rate derivatives can be used to hedge against increasing or decreasing interest rates.

A cap is a contract that can be viewed as a payer IRS where each exchange payment is executed only if it has a positive value. A floor is equivalent to a receiver IRS where each exchange payment is executed only if it has a positive value.

A cap contract can be decomposed into a series of caplets. Floorlets are defined analogously. Valuing a cap contract can therefore be decomposed into the valuation of single caplets. For a cap contract with n resets, cap rate R_K and time to maturity $\tau(t, T)$ we have:

$$\mathrm{Cap}_{R_K,T}\{r(t), t\} = \sum_{i=1}^{n} \mathrm{Cpl}_i\{r(t), t\},$$

with $i = 1, \ldots, n$, $\tau_i = \tau(t_i, t_{i+1})$ and

$$\begin{aligned} \mathrm{Cpl}_i\{r(t), t\} &= \{1 + R(t, t_{i+1})\tau(t, t_{i+1})\}^{-1}\tau_i \max\{R(t_i, t_{i+1}) - R_K, 0\} \\ &= \{1 + R(t, t_{i+1})\tau(t, t_{i+1})\}^{-1}\tau_i \max\{F(t, t_i, t_{i+1}) - R_K, 0\} \end{aligned} \tag{4.3}$$

Equation (4.3) shows how caps and floors, essentially being options on interest rates, can be used to hedge against changes in the term structure. A caplet can be interpreted as a call option on the interest rate, while a floorlet would correspond to a put option. Unlike a FRA or IRS it is not possible to determine the value of a cap (or floor) by knowing only the current term structure at time t. In order to calculate a price for a cap contract we need to apply Black's model.

With the usual assumptions on the strike rate and volatility, and considering a world that is forward risk neutral w.r.t. a bond $V(t, t_{i+1})$ we can consider:

$$\mathsf{E}_{V_{t+1}}\{R(t_i, t_{i+1})\} = F(t, t_i, t_{i+1})$$

The Black model for the ith caplet becomes:

$$\mathrm{Cpl}_i\{r(t), t\} = V(t, t_{i+1})\tau_i \left\{F(t, t_i, t_{i+1})\Phi(y + \sigma\sqrt{\tau_i}) - R_K\Phi(y)\right\} \tag{4.4}$$

with

$$y = \frac{\log\left\{\frac{F(t,t_i,t_{i+1})}{K}\right\} - \frac{1}{2}\sigma_i^2\tau_i}{\sigma_i\sqrt{\tau_i}}.$$

Again equality with equation (4.1) is achieved. Floorlets can either be calculated by using the adapted Put-Call-Parity or by adapting the payoff function in the above equations.

Analogously to the Put-Call-Parity for equity options it holds that:

$$\mathrm{Cap}_{R_K,T}\{r(t), t\} = \mathrm{Flr}_{R_K,T}\{r(t), t\} + \mathrm{FRA}_{R_K,T}\{r(t), t\},$$

where $\mathrm{Flr}_{R_K,T,n}\{r(t), t\}$ is a floor contract with floor rate R_K and time to maturity $\tau(t, T)$.

4.4 Swaption

DEFINITION 4.4 *A European swap option or swaption $SWP_{R_K,T}\{r(t), t)\}$ is an option giving the right to enter a IRS at a given future time T with a specified rate R_K lasting until S.*

A market participant will only exercise this option if the market swap rate at matu-

rity of the swaption is less favorable. Therefore, a swap option is essentially an option on the forward swap rate $R_S(T,S)$.

Like with IRS, we can distinguish between payer and receiver swaptions. The holder of a payer swaption has the right, but not the obligation to pay fixed in exchange for variable interest rate. The holder of the equivalent receiver swaption has the right, but not the obligation to receive interest at a fixed rate and pay variable.

The value of a payer swaption at time t of is:

$$\mathrm{SWP}_{R_K,T}\{r(t),t\} = V(t,T)\max\left[\sum_{i=0}^{n-1} V(T,T_{i+1})\tau_i\{R_S(T,S)-R_K\},0\right],$$

with $T \leq T_i \leq S$.

To determine the value of a swaption we again use the Black model. We consider a world that is risk neutral w.r.t. the numeraire measure $U(t,T)$ from section 3.5.3. If we apply the usual assumptions on the distribution of swap rates and volatility we can show that:

$$\mathrm{SWP}_{R_K,T}\{r(t),t\} = U(t,T)\mathsf{E}_{U_t}[\max\{R_S(T,S)-R_K,0\}]$$

Thus by using the Black model the time t value is:

$$\mathrm{SWP}_{R_K,T}\{r(t),t\} = U(t,T)[\mathsf{E}_{U_t}\{R_S(T,S)\}\Phi(y+\sigma\sqrt{\tau}) - R_K\Phi(y)],$$

with

$$y = \frac{\log\left[\mathsf{E}_{U_t}\{R_S(T,S)\}/R_K\right] - \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}.$$

Applying equation (3.8) we replace the expected future swap rate with the current forward swap rate and achieve equality with equation (4.1).

Note that the different versions of the Black model we have seen above are defined by different measures. Therefore the Black model for caps is not consistent with the approach to price swap or bond options and vice versa. This is due to the assumptions on the distribution of the underlying interest and swap rates. However, this fact is neglected in practice and remains one of the critical points in applying the Black model.

It is to be noted, that the above tools are only of basic character. As with equity options, exotic interest rate derivatives exist. Covering these however is beyond the

scope of this thesis. We will now move on to discuss different models, starting with models for the short rate process.

5 Short Rate Models

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Now we move on to include stochastic elements into the dynamic of interest rates. In order to have an unambiguous, fixed interest rate, one considers the interest rate of an investment over the shortest possible time period, the short rate $r(t)$. Practice shows that $r(t)$ does not follow a geometric Brownian motion, so that the Black-Scholes approach cannot be used.

There are a number of models that are special cases of the Itô Process (2.8):

$$dr(t) = \mu\{r(t), t\}dt + \sigma\{r(t), t\}dW_t,$$

where W_t represents as usual a standard Wiener process.

By equation (2.2) we know that we can obtain the entire term structure of interest rates by defining the dynamics for the spot rate, $r(t)$. Interest rates have two main properties:

- Mean reversion: interest rates tend to return to an average level.
- $r(t)$ should be non-negative.

There are essentially two approaches to model the term structure. For the equilibrium approach today's term structure of interest rate is endogenously derived by the model. In the no-arbitrage approach today's term structure of interest rate is an input to the model.

5.1 One-Factor Short-Rate Models

One factor short rate models consider only one factor of uncertainty in the dynamics of the interest rate. There are a number of models that define the process of $r(t)$. We

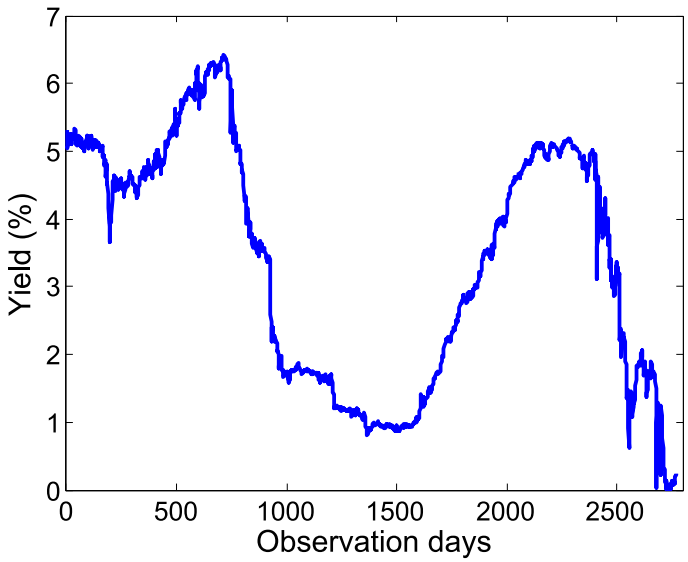

Figure 5.1: 3-month U.S Treasury bill daily yield from 1998 to 2008 as approximation for the short rate. SFEustb

introduce the most often used and discussed models.

5.1.1 Vasicek model

Vasicek (1977) introduced an interest rate model as:

$$dr(t) = a\{b - r(t)\}dt + \sigma dW_t$$

where a,b and σ are constants, W_t is a Wiener process. It is consistent with the mean reversion feature of the interest rate at a reversion rate a to the level b. However, in this model $r(t)$ can be negative.

5.1.2 Rendleman-Bartter model

In the Rendleman and Bartter (1980) model, the dynamics of $r(t)$ are:

$$dr(t) = \mu r(t)dt + \sigma r(t)dW_t$$

where μ and σ are constants, W_t is a Wiener process. In this model $r(t)$ follows geometric Brownian motion. It is not consistent with the mean reversion property and has shown in the practice to be a less ideal model.

5.1.3 Cox, Ingersoll and Ross (CIR) model

Cox, Ingersoll and Ross (1985) proposed an alternative model from Vasicek as:

$$dr(t) = a\{b - r(t)\}dt + \sigma\sqrt{r(t)}dW_t \tag{5.1}$$

where a, b and σ are constants, W_t is a standard Wiener process. The disadvantage of possible negative $r(t)$ in Vasicek is avoided here. The drift part does not differ to the Vasicek model. However, $\sqrt{r(t)}$ is included in the diffusion process as a proportion of the standard deviation. Therefore $r(t)$ has a positive impact on the standard deviation through this setting.

5.1.4 Ho-Lee model

The former three models are all equilibrium models. The equilibrium approach yields today's term structure as output and is adapted to fit the term structure by choosing proper parameters. Sometimes it is difficult to find the parameters to fit today's term structure quite well. In order to overcome this problem the no-arbitrage approach was introduced. Other than the equilibrium approach, it takes today's term structure as an input to ensure that the model fits today's term structure perfectly by imposing a time function in the drift part.

Ho and Lee (1986) presented the first no-arbitrage model as:

$$dr(t) = \delta(t)dt + \sigma dW_t$$

where σ is constant, $\delta(t)$ is a deterministic function of time and W_t is a Wiener

process. The time dependent variable $\delta(t)$ defines the trend of $r(t)$ at time t. The Ho-Lee model lacks mean reversion and $r(t)$ can be negative.

5.1.5 Hull-White model

We have discussed before that it can be difficult to fit the Vasicek model to the initial term structure of interest rates. Hull and White (1990) proposed an extended Vasicek model to address this problem. The model is:

$$dr(t) = \{\delta(t) - ar(t)\}dt + \sigma dW_t$$

where a and σ are constants, $\delta(t)$ is a deterministic function of time, W_t is a Wiener process. Compared to the Vasicek model, it uses the time-dependent reversion level $\delta(t)/a$ instead of the constant b in Vasicek. It is also a special case of the Ho-Lee model with a mean reversion rate a.

5.1.6 Black, Derman and Toy (BDT) model

In the former two models, $r(t)$ is normally distributed and can be negative. Black, Derman and Toy (1990) gave a log-normal model, in which only positive $r(t)$ are allowed. The continuous-time limit model is:

$$d\log r(t) = \{\delta(t) - \phi(t)\log r(t)\}dt + \sigma(t)dW_t$$

where $\delta(t)$ and $\phi(t)$ are deterministic functions of time, $\sigma(t)$ depends on $\phi(t)$, W_t is a Wiener process. This model is widely used by practitioners, since it fits both the current term structure of interest rate and the current term structure of volatility.

5.1.7 Black-Karasinski model

Black and Karasinski (1991) presented another log-normal interest rate model as:

$$d\log r(t) = \delta(t)\{\log\mu(t) - \log r(t)\}dt + \sigma(t)dW_t$$

with $\delta(t)$ as a deterministic function of time, $\mu(t)$ as the "target interest rate" and W_t as a Wiener process. If $r(t)$ is above $\mu(t)$, it will have a negative drift to pull $r(t)$

Vasicek	$dr(t) = a\{b - r(t)\}dt + \sigma dW_t$
Rendleman-Bartter	$dr(t) = \mu r(t)dt + \sigma r(t)dW_t$
CIR	$dr(t) = a\{b - r(t)\}dt + \sigma\sqrt{r(t)}dW_t$
Ho-Lee	$dr(t) = \delta(t)dt + \sigma dW_t$
Hull-White one-factor	$dr(t) = \{\delta(t) - ar(t)\}dt + \sigma dW_t$
BDT	$d\log r(t) = \{\delta(t) - \phi(t)\log r(t)\}dt + \sigma(t)dW_t$
Black-Karasinski	$d\log r(t) = \delta(t)\{\log\mu(t) - \log r(t)\}dt + \sigma(t)dW_t$

Table 5.1: One-factor short rate models

to $\mu(t)$, and vise versa. The biggest drawback for this model is that no closed form of valuation formula for valuing bonds in terms of $r(t)$ can be derived by this model.

5.2 Two-Factor Short-Rate Models

One-factor models imply that the instantaneous rates for all maturities in the yield curve are perfectly correlated which means that a shock on $r(t)$ at time t will transmit rigidly to all maturities in the curve. This property is clearly unrealistic. A more satisfactory method to model the interest rate process is needed. Involving more explanatory factors into the model is an effective way to solve this problem. We will briefly introduce two representative two-factor models.

5.2.1 Longstaff-Schwartz model

Longstaff and Schwartz (1992) developed a two-factor equilibrium model that is based on the CIR framework where $r(t)$ is a linear combination of X_t and Y_t as:

$$\begin{aligned} r_t &= \alpha X_t + \beta Y_t \\ dX_t &= (a - bX_t)dt + \sqrt{X_t}dW_{1t} \\ dY_t &= (e - fY_t)dt + \sqrt{Y_t}dW_{2t} \end{aligned}$$

a, b, e, $f > 0$, X and Y are state variables, W_{1t} and W_{2t} are Wiener process. The two factors are the short-term interest rate and the volatility of the short-term interest rate increment. This feature makes the the contingent claim values to reflect both the current interest rate level and the interest rate volatility level. This model is proved to be quite tractable.

5.2.2 Hull-White two-factor model

Hull and White (1994) presented a no-arbitrage two-factor model which assumed the short rate following the process:

$$dr(t) = \{\delta(t) + u(t) - ar(t)\}dt + \sigma_1 dW_{1t}$$

where

$$du(t) = -bu(t)dt + \sigma_2 dW_{2t}, \quad u(0) = 0$$

a, b, σ_1 and σ_2 are constants, W_{1t} and W_{2t} are Wiener processes, $dW_{1t}dW_{2t} = \rho dt$. The two factors are the short-term and long-term interest rates. $\delta(t)$ is deterministic and can be properly chosen to exactly fit the initial term structure. $u(t)$ is a reversion level component which mean reverts to zero.

6 Heath Jarrow Morton Framework

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We have just seen a number of different possibilities to model the evolution of interest rates by means of modeling the short rate process. A drawback of many one factor short rate models is the difficulty to calibrate the model according to the current yield curve and an unrealistic presentation of the variance and covariance structure. Heath, Jarrow and Morton (1992) have derived an arbitrage-free framework for a stochastic evolution of the entire yield curve from an instantaneous forward rate.

6.1 HJM Approach

By equation (2.6) we know that:

$$f(t,T) = -\frac{\partial \log V(t,T)}{\partial T}.$$

In a discrete time setting with $S = T + \Delta$ this will become

$$F(t,T,T+\Delta) = \frac{\log V(t,T+\Delta) - \log V(t,T)}{\Delta},$$

which corresponds to equation (2.4). As a bond is a traded security, its price development can be expressed as a SDE.

$$dV_t = \mu(t,T,V_t)V_t dt + \sigma(t,T,V_t)V_t dW_t,$$

where $V_t = V(t,T)$, $\sigma(t,T,V_t)$ is the volatility of $V(t,T)$ and W_t is a Wiener process w.r.t. the real world measure. Other than in the Black Scholes world, the diffusion process depends on $V(t,T)$ and the Wiener process governing the SDE can be different

for every maturity. Therefore every bond with different maturity can theoretically be driven by a different diffusion process.

As in the Black Scholes framework, the drift coefficient can be modified according to Girsanov's theorem. In the risk neutral world the SDE becomes:

$$dV_t = r(t)V_t dt + \sigma(t, T, V_t)V_t dW_t^*, \tag{6.1}$$

where W_t^* is now a Wiener Process under the risk neutral measure.

In order to find the dynamics for $F(t, T, T + \Delta)$ we apply Itô's Lemma to equation (6.1) and get:

$$d \log V(t, T) = \left\{ r(t) - \frac{1}{2}\sigma(t, T, V_t)^2 \right\} dt + \sigma(t, T, V_t) dW_t^*$$

and

$$d \log V(t, T + \Delta) = \left\{ r(t) - \frac{1}{2}\sigma(t, T + \Delta, V_t)^2 \right\} dt + \sigma(t, T + \Delta, V_t) dW_t^*.$$

Thus

$$\begin{aligned} dF(t, T, T + \Delta) &= \frac{1}{2\Delta} \left[\sigma\{t, T + \Delta, V(t, T + \Delta)\}^2 - \sigma(t, T, V_t)^2 \right] dt \\ &+ \frac{1}{\Delta} \left[\sigma\{t, T + \Delta, V(t, T + \Delta)\} - \sigma(t, T, V_t) \right] dW_t^* \end{aligned}$$

For $\Delta \to 0$ we get the dynamics of the instantaneous forward rate.

$$\begin{aligned} df(t, T) &= \sigma(t, T, V_t) \left\{ \frac{\partial \sigma(t, T, V_t)}{\partial T} \right\} dt \\ &+ \left\{ \frac{\partial \sigma(t, T, V_t)}{\partial T} \right\} dW_t^*, \end{aligned}$$

where $\sigma(\cdot)$ are the bond price volatilities. This can simplified to:

$$df(t, T) = \alpha(t, T)dt + \beta(t, T)dW_t^*,$$

where

$$\alpha(t, T) = \sigma(t, T, V_t) \left\{ \frac{\partial \sigma(t, T, V_t)}{\partial T} \right\}$$

and

$$\beta(t,T) = \left\{ \frac{\partial \sigma(t,T,V_t)}{\partial T} \right\}.$$

If the volatility term structure in the form of $\sigma(t,T,V_t)$ is developed from the underlying data set, the risk neutral process for $f(t,T)$ is known. By defining the volatility term structure accordingly we can formulate all of the short rate models considered in section 5 within the HJM framework.

The advantages of the HJM framework are that the framework permits a large number of possible assumptions about the evolution of the yield curve and that the resulting models will, by definition, be consistent with the initial term structure that is observed in the market.

6.2 Short Rate Process in the HJM Framework

By integrating the process $df(t,T)$ we get:

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)\,ds + \int_0^t \beta(s,T)\,dW_s^*$$

We can now set $T = t$ to receive the short rate $r(t)$ as:

$$r(t) = f(0,t) + \int_0^t \alpha(s,T)\,ds + \int_0^t \beta(s,t)\,dW_s^*.$$

This is somewhat problematic and poses the biggest problem in the HJM framework. The process for short rate is in general no longer Markov as the drift term for the short rate process is now a function of all past volatilities. Only with a number of selected models, such as the Hull-White model or the Black, Derman and Toy model, the process for the short rate becomes a Markov process. In order to use the HJM framework in practice it requires Monte Carlo simulation.

Like with models for the short rate process, the HJM approach can be modified to incorporate multiple factors so to include e.g. macroeconomic variables. This is however beyond the scope of this thesis. The interested reader is referred to either Heath et al. (1992) or to chapter 5 of Brigo and Mercurio (2001).

7 LIBOR Market Model

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The term structure models we introduced before have a common drawback that neither the instantaneous spot rate nor the instantaneous forward rate can be directly observed in the market. Hence they are not compatible to price caps and swaptions with Black's formula. An alternative was proposed by Brace, Gatarek and Musiela (1997), Jamshidian (1997) and Miltersen, Sandmann and Sondermann (1997) who modeled LIBORs instead of instantaneous rates. This approach is known as the LIBOR market model (LMM).

DEFINITION 7.1 *The London Interbank Offered Rate (LIBOR), $L_n(t)$, is the forward rate over the period $[t_n, t_{n+1}]$ as observed at time t with compounding period $\tau(t_n, t_{n+1})$.*

7.1 Dynamics in the LMM

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The relation between a zero-bond and the LIBOR forward rate is defined as:

$$1 + \tau_n L_n(t) = \frac{V_n(t)}{V_{n+1}(t)}.$$

where we define t_n, $n = 0, 1, \ldots, M$ to be the times at which M assets are traded in the market, $t_0 = 0 < t_1 < t_2 \ldots < t_{M+1}$ and $\tau_n = t_{n+1} - t_n$. Consider $V_n(t)$ as a bond maturing at time $t_n > t$. It is straightforward that the LIBOR forward rate can be represented as:

$$L_n(t) = \frac{1}{\tau_n}\left\{\frac{V_n(t) - V_{n+1}(t)}{V_{n+1}(t)}\right\}.$$

This is similar to equation (2.5), with the simple difference that the forward rate $F(t, T, S)$ is replaced by the LIBOR forward rate $L_n(t)$.

It is proposed that the forward rate follows a d-dimensional Brownian motion as follows:

$$\frac{dL_n(t)}{L_n(t)} = \mu_n(t)dt + \sigma_n(t)dW_n$$

where we define:

dW_n: is a Wiener process at time t.

$\mu_n(t)$: the drift part, depends on both time t and the forward rate. The form will depend on the choice of measure.

$\sigma_n(t)$: d-dimensional volatility of $L_n(t)$ at time t.

7.2 The Numeraire Measure

Consider the equivalent martingale measure Wsq^{n+1}, $L_n(t)$ is a martingale under Q^{n+1}. The stochastic differential equation has no drift term and can be written as:

$$\frac{dL_n(t)}{L_n(t)} = \sigma_n(t)dW^{n+1}(t), \tag{7.1}$$

where $dW^{n+1}(t)$ is a Wiener process under Q^{n+1}.

In order to evolve all the LIBOR rates under the same measure, we consider the change of measure to determine the drift term for Q^n martingale $L_n(t)$ under Q^{n+1}. According to Girsanovs theorem, the relation between the Brownian Motion under Q^n and Q^{n+1} can be represented as:

$$dW^n(t) = dW^{n+1}(t) - \frac{\tau_n \sigma_n(t) L_n(t)}{1 + \tau_n L_n(t)} dt. \tag{7.2}$$

If there are M LIBOR rates in the economy, we use the so called terminal measure which takes the numeraire under the measure Q^{M+1} associated with a bond maturing at time t_{M+1}. By repeatedly applying the equation (7.2) in combination with (7.1), we can derive the process as:

$$\frac{dL_n(t)}{L_n(t)} = -\sum_{j=n+1}^{M} \frac{\tau_j L_j(t) \sigma_j(t) \sigma_n(t)}{1 + \tau_j L_j(t)} dt + \sigma_n(t) dW^{M+1}(t),$$

for all $1 \leq i \leq M$. Comparing to the HJM model, the HJM model is the limit case when τ_n in the LIBOR model tends to zero.

Considering the third measure Q^i with $i < n$ and considering the same assumptions as above, the dynamics for the LIBOR market model become:

$$\frac{dL_n(t)}{L_n(t)} = \sum_{j=n+1}^{M} \frac{\tau_j L_j(t)\sigma_j(t)\sigma_n(t)}{1+\tau_j L_j(t)} dt + \sigma_n(t) dW^{M+1}(t). \tag{7.3}$$

In the process of calibration, $\sigma_n(t)$, $n = 1, \ldots, M$ as the deterministic functions are chosen to resemble the Black implied volatilities as good as possible. To price the caplets and swaptions by using the LIBOR model, Monte Carlo simulation has proven to be the most reliable method. The implementation will be introduced in the calibration section.

8 The Bond Valuation Equation

LS

In this part, we take the simple one-factor short-rate model as an example to show how to derive a closed form of the bond valuation equation. Under the corresponding numeraire, the value of a bond can be represented as:

$$V(t,T) = \mathsf{E}_t\left[\exp\left\{-\int_t^T r(s)ds\right\}V(T,T)\right].$$

We already saw that the general Itô Process of $r(t)$ can be written as:

$$dr(t) = \mu_r dt + \sigma_r dW_t, \tag{8.1}$$

where $\mu_r = \mu\{r(t),t\}$ and $\sigma_r = \sigma\{r(t),t\}$. Under the condition that $V(T,T) = 1$ and in combination with Itô's Lemma we can write:

$$dV(t,T) = \frac{\partial V(t,T)}{\partial t}dt + \frac{1}{2}\sigma^2\frac{\partial^2 V(t,T)}{\partial r^2}dt + \frac{\partial V(t,T)}{\partial r}dr(t).$$

If we plug in $dr(t)$ from equation (8.1), we get:

$$dV(t,T) = \left\{\frac{\partial V(t,T)}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 V(t,T)}{\partial r^2} + \mu_r\frac{\partial V(t,T)}{\partial r}\right\}dt + \sigma\frac{\partial V(t,T)}{\partial r}dW_t.$$

Under the risk-neutral measure, the PDE can thus be written as:

$$r(t)V(t,T) = \frac{\partial V(t,T)}{\partial t} + \frac{1}{2}\sigma_r^2\frac{\partial^2 V(t,T)}{\partial r^2} + \mu_r\frac{\partial V(t,T)}{\partial r}.$$

8.1 Solving the Zero Bond Valuation

We take the CIR model as an example (see Figure 8.1). The PDE of the CIR model under the risk-neutral measure is

$$r(t)V(t,T) = \frac{\partial V(t,T)}{\partial t} + \frac{1}{2}r\sigma^2\frac{\partial^2 V(t,T)}{\partial r^2} + a(b-r)\frac{\partial V(t,T)}{\partial r}. \tag{8.2}$$

Assume the bond value $V(t,T) = \exp\{A(t) - rB(t)\}$ with a nominal value 1 EUR. Due to this we can consider the following:

$$\begin{aligned}
\frac{\partial V(t,T)}{\partial t} &= \{A'(t) - rB'(t)\}V(t) \\
\frac{\partial V(t,T)}{\partial r} &= -B(t)V(t) \\
\frac{\partial^2 V(t,T)}{\partial r^2} &= B^2(t)V(t)
\end{aligned}$$

If we plug in these three functions into the PDE of equation (8.2) which we obtained above, then we get:

$$\begin{aligned}
0 &= \{A'(t) - B'(t)r(t)\}V(t) + \frac{1}{2}\sigma^2 rV(t)B^2(t) \\
&\quad -a(b-r)B(t)V(t) - r(t)V(t) \\
A'(t) &= abB(t) - \sigma^2 rB^2(t)/2 \\
B'(t) &= aB(t) - 1
\end{aligned}$$

With the boundary condition $V(T,T) = 1$, $A(T,T) = B(T,T) = 0$, we can derive an explicit expression of:

$$V(t,T) = \exp\{A(t) - rB(t)\} \tag{8.3}$$

where

$$
\begin{aligned}
A(t) &= \frac{2ab}{\sigma^2} \log \frac{2\psi \exp\{(a+\psi)(T-t)/2\}}{2\psi + (a+\psi)\exp\{\psi(T-t)-1\}} \\
B(t) &= \frac{2\exp\{\psi(T-t)-1\}}{2\psi + (a+\psi)\exp\{\psi(T-t)-1\}} \\
\psi &= \sqrt{a^2 + 2\sigma^2}.
\end{aligned}
$$

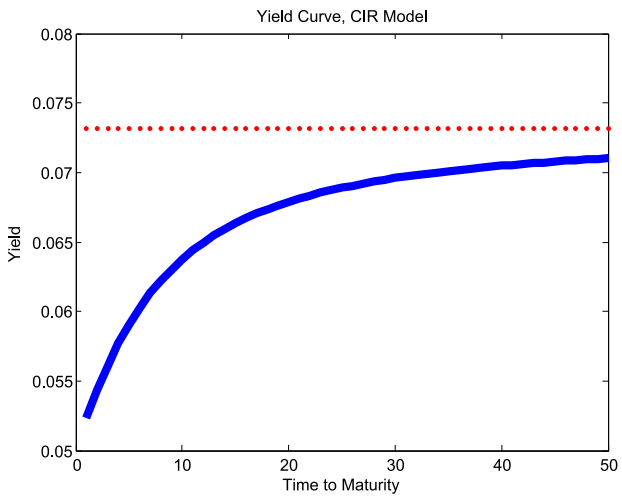

Figure 8.1: Term structure according to the CIR model with $a = b = \sigma = 0.1$, $r = 0.05$. SFEcir

9 Calibrating Short-Rate Models

LG

To conclude this thesis we will now see a basic empirical application of the CIR model and the Vasicek model to real data. We will use the maximum likelihood estimator (MLE) in order to estimate the underlying parameters of these models. To achieve convergence to the global optimum we will choose our starting values by considering the least squares estimator as proposed by Overbeck and Rydén (1997). For the implementation we refer also to Kladivko (2007). We will begin with a brief description of the data set used, followed by the estimation of the initial starting values and the MLEs.

Our empirical work uses daily observations of the annualized yield on U.S. Treasury Bills with three months to maturity. The series was constructed from a daily series available from the Federal Reserve. We have 2769 observations, ranging from 02. January 1998 to 30. January 2009.

We consider the process for $r(t)$ to be given by equation (5.1). Our objective is to estimate the parameters a, b and σ from the observations of $r(t)$ at time intervals of Δt and to simulate a CIR process based on these results. We will denote $r(t)$ as r_t.

9.1 CIR Process Densities

For maximum likelihood estimation of the parameter vector $\theta \stackrel{\text{def}}{=} (a, b, \sigma)$ transition densities are required. The CIR process is one of the few cases where the transition density has a closed form expression. Cox et al. (1985) show that the density of $r_{t+\Delta t}$ at time $t + \Delta t$ is:

$$p(r_{t+\Delta t}|r_t, \theta, \Delta t) = c \exp(-u - v) \left(\frac{v}{u}\right)^{\frac{q}{2}} I_q(2\sqrt{uv}),$$

where

$$c = \frac{2a}{\sigma^2\{1 - \exp(-a\Delta t)\}},$$

$$u = cr_t \exp(-a\Delta t),$$

$$v = cr_{t+\Delta t},$$

$$q = 2ab/\sigma^2 - 1,$$

and $I_q(2\sqrt{uv})$ is the modified Bessel function of the first order q.

9.2 Initial Estimates

LS

The success of the MLE approach is dependent on the availability of good starting values for the numerical optimization algorithm. We choose the starting values by means of a conditional least squares estimation as applied by Overbeck and Rydén (1997). Following the notation we have used above, the conditional mean function for the CIR model is derived as:

$$m(r;\theta) = \mathsf{E}_\theta(r_t|r_{t-1} = r) = \gamma_0 + \gamma_1 r$$

with

$$\gamma_0 = -b\{\exp(-a\Delta t) - 1\}$$

and

$$\gamma_1 = \exp(-a\Delta t).$$

Overbeck and Rydén (1997) show that the conditional least squares estimators for a and b are given by:

$$\hat{a} = -\frac{1}{\Delta t}\left[\left\{n^{-1}\sum_{t=1}^{n}(r_t - \overline{r}_n)(r_{t-1} - \overline{r}'_n)\right\} / \left\{n^{-1}\sum_{t=1}^{n}(r_{t-1} - \overline{r}'_n)^2\right\}\right]$$

and

$$\hat{b} = -\frac{\overline{r}_n - \exp(-a\Delta t)\overline{r}'_n}{\exp(-a\Delta t) - 1},$$

where $\overline{r}_n = n^{-1}\sum_{t=1}^{n} r_t$ and $\overline{r}'_n = n^{-1}\sum_{t=1}^{n} r_{t-1}$.

The estimator for b is based on the conditional second moment function which is given by:

$$v(r;\theta) = \mathsf{E}_\theta[\{r_t - \mathsf{E}_\theta(r_t|r_{t-1}=r)\}^2|r_{t-1}=r] = \sigma^2(\eta_0 + \eta_1 r)$$

with

$$\eta_0 = \frac{b}{2a}\{\exp(-a\Delta t) - 1\}^2$$

and

$$\eta_1 = -\frac{1}{a}\exp(-a\Delta t)\{\exp(-a\Delta t) - 1\}.$$

As an estimator for sigma we use:

$$\hat{\sigma}^2 = n^{-1}\sum_{t=1}^{n}\frac{\{r_t - m(r_{t-1};\hat{a},\hat{b})\}^2}{\hat{\eta_0} + \hat{\eta_1} r_{t-1}}$$

where $\hat{\eta_0}$ and $\hat{\eta_1}$ are evaluated at $(\hat{a},\hat{b})$.

9.3 Maximum Likelihood Estimator

We determine our parameters with a MLE. The likelihood function for interest rate time series with n observations is:

$$L(\theta) = \prod_{t=1}^{n-1} p(r_{t+1}|r_t,\theta,\Delta t).$$

Thus the log-likelihood function is:

$$\log L(\theta) = \sum_{t=1}^{n-1}\log p(r_{t+1}|r_t,\theta,\Delta t).$$

number of observations n	2600
time step Δt	1/252
estimated $\hat{a}$	0.221
estimated $\hat{b}$	0.02
estimated $\hat{\sigma}$	0.055
LR	5.502

Table 9.1: MLE results for the CIR model with 2600 observations from 19980102 to 20080522 from the dataset SFECIRmle

number of observations n	2600
time step Δt	1/252
estimated $\hat{a}$	0.161
estimated $\hat{b}$	0.014
estimated $\hat{\sigma}$	0.009
LR	5.502

Table 9.2: MLE results for the Vasicek model with 2600 observations from 19980102 to 20080522 from the dataset SFEVasiml

The log-likelihood function of the CIR process is given by:

$$\begin{aligned} \log L(\theta) &= (n-1)\log c \\ &+ \sum_{t=1}^{n-1}\left[-u_t - v_{t+1} + 0.5q\log\frac{v_{t+1}}{u_t} + \log\{I_q(2\sqrt{u_t v_{t+1}})\}\right], \end{aligned} \tag{9.1}$$

where $u_t = cr_t\exp(-a\Delta t)$, and $v_{t_+1} = cr_{t+1}$. We find the MLE $\hat{\theta}$ by maximizing the log-likelihood function in equation (9.1) over its parameter space.

$$\hat{\theta} = (\hat{a}, \hat{b}, \hat{\sigma}) = \arg\max_{\theta} \log L(\theta).$$

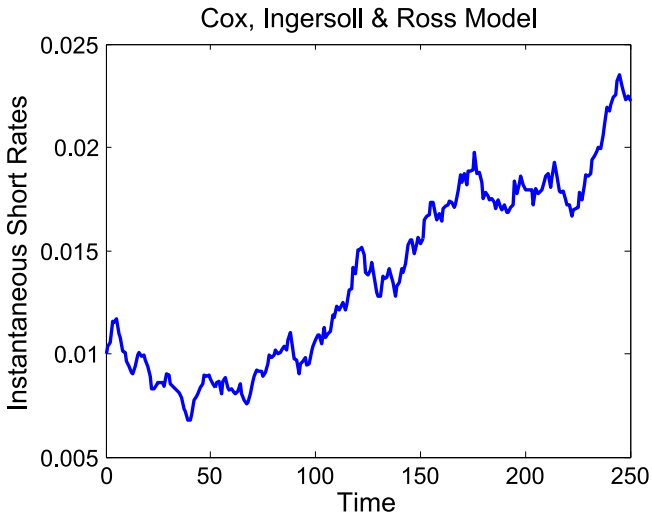

Figure 9.1: Simulated CIR process with $a = 0.221$, $b = 0.02$, $\sigma = 0.055$ and $r_0 = 0.01$.
SFEsimCIR

9.4 Implementation Results

After we have now derived the underlying factors for the CIR model, we now move on to price a theoretical bond by applying the results in table 9.1 to 8.3. The results are presented in table 9.3.

The examples of simulated CIR and Vasicek path are shown in figure 9.1 and figure 9.2. The comparison of these two models to the real data is shown in figure 9.3

We now move on to give a short introduction to the calibration of the LIBOR market model.

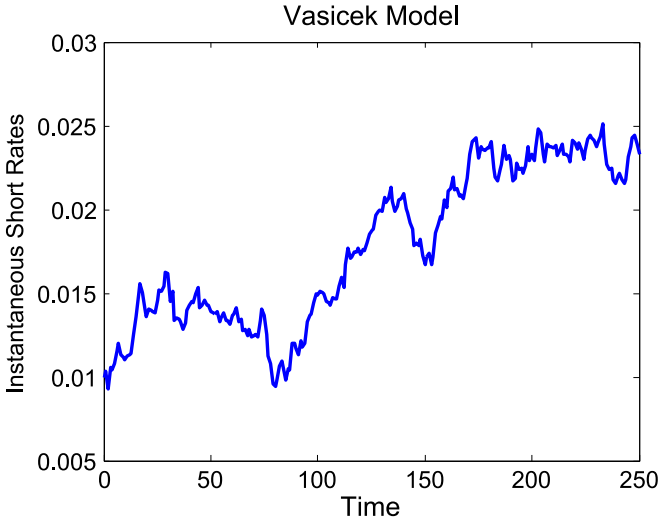

Figure 9.2: Simulated Vasicek process with $a = 0.161$, $b = 0.014$, $\sigma = 0.009$ and $r_0 = 0.01$. SFEsimVasi

face value	1 EUR
time to maturity T	3 month
short rate at time t	0.02
bond price $V(t,T)$	0.99 EUR

Table 9.3: Results of bond pricing with the CIR model SFECIRpricing

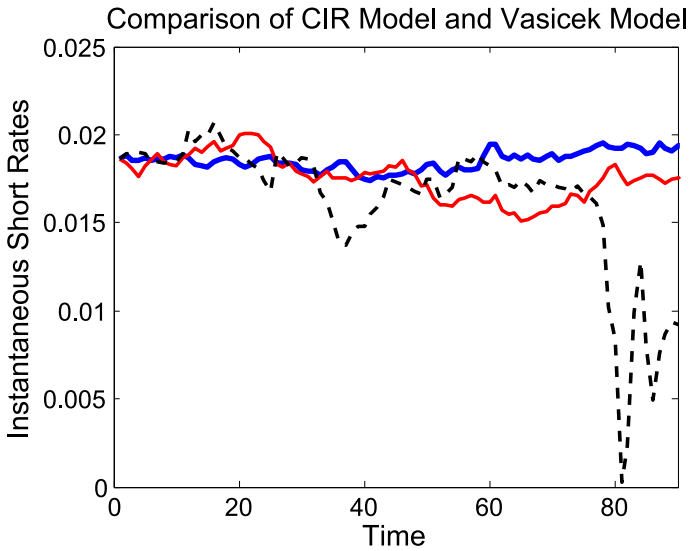

Figure 9.3: Comparison of simulated Vasicek (red) and CIR (blue) process to the real data (dotted), $r_0 = 0.0186$, parameters from table 9.1 and 9.2. SFEscomCIR

10 Calibrating the LIBOR Market Model

LG

In this section we calibrate the LMM from section 7. Implementation of the LMM is done using Monte Carlo simulation of the SDE (7.3). The first step is to discretize the forward rate equation. As the LMM is driven by the instantaneous volatility, calibration of the LMM essentially refers to the calibration of the instantaneous volatility function. The goal will be to assure that the modeled volatility resembles the Black implied volatility as good as possible. We will calibrate the LMM to caplets.

10.1 Discretization of the Forward Rate

For the LMM under the spot measure it is not possible to derive known transition densities. Therefore the model dynamics have to be discretized in order to perform simulation. In line with Glasserman (2004), application of the Euler method to the SDE of LIBOR market model under the spot measure, which we have seen in equation (7.3), leads to:

$$\begin{aligned} L_n(t_{i+1}) &= L_n(t_i) + \sum_{j=i+1}^{n} \frac{\tau_j L_j(t_i)\sigma_n(t_i)\sigma_j(t_i)}{1+\tau_j L_j(t_i)} L_n(t_i)\tau_i + L_n(t_i) \\ &\times \sqrt{\tau_i}\sigma_n(t_i)\varepsilon_{i+1}, \end{aligned}$$

where ϵ_1, ϵ_2, ... are independent $\phi(0,1)$ distributed variables. Given a set of bond prices we can initialize the simulation with:

$$L_n(0) = \frac{V_n(0) - V_{n+1}(0)}{\tau_n V_{n+1}(0)},$$

where $n = 1, \ldots, M$. However, this approach can lead to negative rates. An alternative can be to apply the Euler scheme to $\log L_i$. This approach leads to the following approximates for the LIBOR rates under the spot measure, where all rates produced are positive:

$$\begin{aligned} L_n(t_{i+1}) &= L_n(t_i) \exp\left\{ \sum_{j=i+1}^{n} \frac{\tau_j L_j(t_i)\sigma_n(t_i)\sigma_j(t_i)}{1+\tau_j L_j(t_i)} - \frac{1}{2}\sigma_n(t_i)^2 \right\} \\ &\times \tau_i + \sqrt{\tau_i}\sigma_n(t_i)\epsilon_{i+1}. \end{aligned}$$

This would correspond to approximating the LIBOR rates by a geometric Brownian motion over $[t_i, t_{i+1}]$, with drift and volatility parameters fixed at t_i.

10.2 Instantaneous Volatility Function

After we have now discretized the LIBOR rate equation, we now need to determine the instantaneous volatility function. There are several suggestions for both parametric and non-parametric forms of the structure of the instantaneous volatility function in the common literature. General requirements, as suggested by Rebonato (2002), to the functional form of the instantaneous volatility function are that:

- the chosen form should be able to reproduce either a monotonically decreasing or a humped shaped volatility
- the parameters should be economically interpretable.

As shown in Rebonato (2004), the following parametric form fulfills these criteria:

$$\sigma_n(t) = g(t_n)f(t_n - t) = k_n[\{a + b(t_n - t)\} \exp\{-c(t_n - t)\} + d],$$

where $g(t_n)$ is a complete function of the individual forward rate and $f(t_n - t)$ is a time-homogeneous component. This type of instantaneous volatility function allows a humped shaped form for the instantaneous volatility, while also having included the possibility to modify the volatility for each maturity separately, thus adding flexibility to the time-homogeneous component.

For a more thorough treatment of the different possibilities for the choice of the instantaneous volatility function we refer to Brigo and Mercurio (2001), chapter 6.

When calibrating this function to the given data set, we must make sure that the parameters a, b, c and d are chosen such that the volatility function is well-behaved. Therefore the following conditions have to be given:

- $a + d > 0$
- $d > 0$
- $c > 0$.

10.3 Calibration

LS

The parameters are chosen so that the instantaneous volatility function is consistent with the implied volatilities of the Black model. We infer the implied volatilities from a series of N traded caplets. We know by equation (7.1) that under the correct measure we can express the forward rate process as a driftless diffusion process:

$$\frac{dL_n(t)}{L_n(t)} = \sigma_n(t) dW^{n+1}(t),$$

where the instantaneous volatility function is connected with the average volatility of the Black model by:

$$(\sigma^{Black})^2 T = \int_0^T \sigma(t)^2 dt.$$

We therefore choose the the instantaneous volatility function as:

$$(\sigma_n^{Black})^2 t_n = \int_0^{t_n} \sigma_n^2(s) ds,$$

where σ_n^{Black} is the Black implied volatility for the caplet associated with the forward rate L_i.

If we want to price N different caplets we start by calibrating the time-homogeneous part. To do this, first set $g(t_n) = 1$ for $n = 1, \ldots, N$. We perform a least-squares

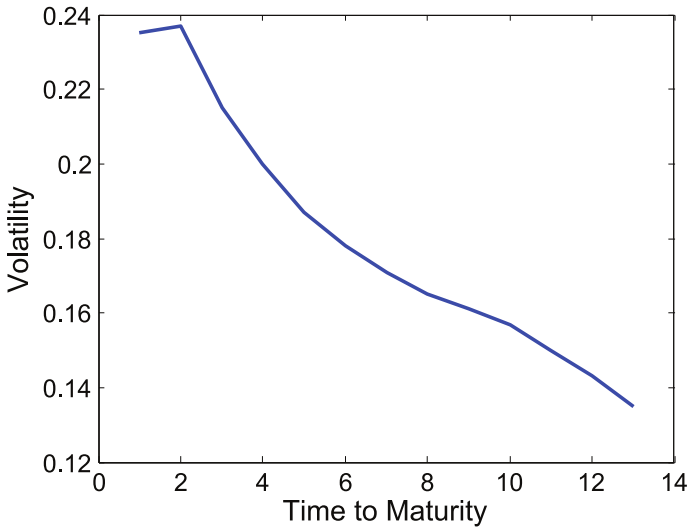

Figure 10.1: Black implied volatility structure of EUR caplets on 20020813. SFEscap

minimization of the following equation:

$$\min \sum_{i=1}^{N} \left\{ (\sigma_i^{Black})^2 t_i - \int_0^{t_i} f^2(t_i - s) ds \right\}^2 .$$

The conditions for the parameters a, b, c and d are checked at the end of the minimization process and are fulfilled.

After we have fitted the function for $f(t_n - t)$ we now turn to the forward rate specific function $g(t_n)$. By using $g(t_n) = k_n$ we can assure that the caplets are priced correctly by letting:

$$k_n^2 = \frac{(\sigma_n^{Black})^2 t_n}{\int_0^{t_n} f^2(t_n - s) ds}.$$

In order to preserve the time-homogeneous features of the instantaneous volatility function, k_n should be keep as constant as possible.

This concludes out calibration of the LIBOR market model. The results we attained can now be used to price traded caps with the Black Model (see equation 4.4) .

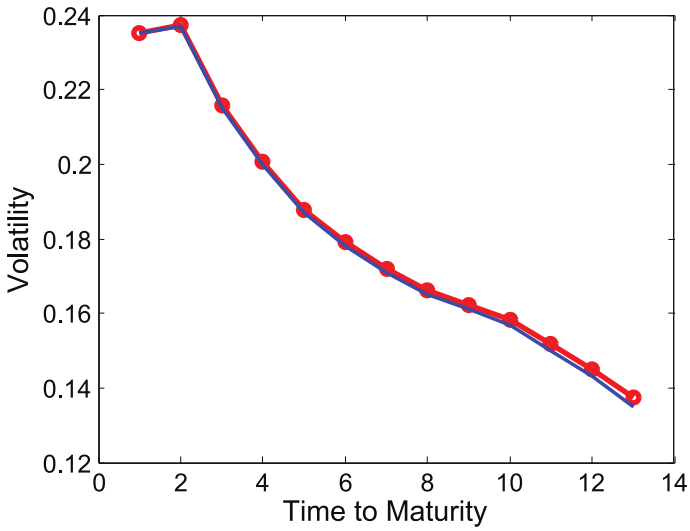

Figure 10.2: Calibrated volatility structure (red) with parameters by table 10.1 and the Black implied volatility (blue). SFEscapvplot

a	b	c	d
0.0017	1.2382	0.001	6.7578

Table 10.1: Estimated parameters for time-homogeneous component of the volatility function. SFEcapvola

11 Conclusion

LS

In this thesis we have given an introduction to the basics of modern interest rate theory. We have covered a number of interest rate derivatives and have shown how these can be valued in a setting where the development of interest rates is uncertain. We have introduced the principles of risk neutral valuation and have shown how the market price of risk is used in the application of the Black model to price caps, swap options and bond options. After having given an overview of the classic and modern theories on interest rates, covering models for the short term interest rate, the Heath-Jarrow-Morton framework and the LIBOR market model, we concluded our thesis with examples of how to calibrate the different models to real data sets.

LG

In order to demonstrate how the models for stochastic interest rates are applied in practice, we calibrated and implemented two basic models. We derived the underlying factors for the Cox, Ingersoll and Ross model and the Vasicek model. Both the CIR model and the Vasicek model yielded satisfying results, which have been used to calculate theoretical bond values. We have given a brief introduction to the calibration of the LIBOR market model. The results we retained could be used to price traded caps. We would recommend for further work a more detailed look at the calibration of the LIBOR market model with regard to the valuation and pricing of swap options.

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