Adaptive copula estimation: sensitivity analysis and applications

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DECLARATION OF AUTHORSHIP

I hereby confirm that I have authored this master thesis independently and without use of others than the indicated sources. All passages which are literally or in general matter taken out of publications or other sources are marked as such.

Berlin, February 28, 2007,

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CONTENTS

LIST OF FIGURES

LIST OF TABLES

1. INTRODUCTION

In the world of financial econometrics when dealing, for example, with assets and their derivatives behavior or when working in the sphere of risk management, one of the main issues is to define properly the structure of the conditional covariance matrix of innovations.

Different authors usually solved this problem by using multivariate normal distribution or multivariate Student distribution. These assumptions are very easy to use. However, it is shown in lots of empirical works that such multivariate distributions cannot provide adequate results due to the presence of asymmetry and excess in financial data. In many finance applications it is obvious that there exist stronger dependence between big losses than between big gains. Such asymmetries cannot be modelled with symmetric distributions.

The usage of *copula* functions is a recently new approach in financial econometrics. With a help of *copula* one can easily combine univariate marginals into a multivariate distribution. That is why the modelling problem consists of two steps:

1. identify the marginal distributions;
2. define the appropriate *copula* function.

A copula is simply a multivariate cumulative distribution function defined on the d-dimensional unit cube $[0,1]^d$ such that every marginal distribution is uniform on the interval $[0,1]$.

Copulae contain all information on the nature of the dependence between the random variables that can be given without marginal distributions, but at the same time, they give no information on marginal distributions. Another point to mention about copulae is that they are not limited to such dependence measure as linear correlation coefficient and can model different, more complex relationships. Copulae provide an ideal means of modelling multivariate distributions, such as the returns on different assets in a portfolio. Therefore now we have a tool which enables us to model correctly return distribution that differ substantially from normal ones.

Copulae can be applied to estimate Value-at-Risk (VaR), which is a common measure to quantify the risk of a portfolio. As by definition VaR is simply a quantile of the distribution of portfolio losses, it can easily be estimated given that we know the dependence structure between the data.

This work contains the following parts. Firstly, basic definitions and theorems on copulae are introduced. Secondly, we discuss methods to estimate *Value-at-Risk*. One method is the *RiskMetrics* approach developed by ***Mordnan/Rueters (1996).*** The other method is based on the assumption of *copula*-distributed innovations. Thirdly, we use the *adaptive copula estimation* procedure and test this procedure assuming that the copula parameter changes over time gradually. Finally, we apply *RiskMetrics* approach and *copula-based* approach to real data and compare the results.

2. INTRODUCTION TO COPULAE

In this chapter a brief introduction to copulae is given. Apart from basic definition and properties, methods to measure dependence are also presented. Also an example of copula from the Archimedean family is considered more closely. For more details see ***Nelsen (1998)*** and ***Embrechts, Lindskog, and McNeil (2001)***.

2.1 Definition and Properties

Definition 2.1.1: (Copula) A d-dimensional copula is a function $C : [0,1]^d \to [0,1]$ that for every $u = (u_1, \ldots, u_d)^T \in [0,1]^d$ and $j \in \{1, \ldots, d\}$ satisfies the following properties:

1. if $u_j = 0$ then $C(u_1, \ldots, u_d) = 0$;
2. $C(1, \ldots, 1, u_j, 1, \ldots, 1) = u_j$;
3. for every $v = (v_1, \ldots, v_d)^T \in [0,1]^d, v_j \le u_j$

$$V_C(u, v) \ge 0,$$

   where
   $V_C(u,v) = \sum_{i_1=1}^{2} \cdots \sum_{i_d=1}^{2} (-1)^{i_1+\ldots+i_d} C(g_{1i_1}, \ldots, g_{di_d})$
   and
   $g_{j1} = v_j,\ g_{j2} = u_j$

It follows from the definition that as $F_1, \ldots, F_d$ are univariate distribution functions, then the constructed copula $C\{F_1(x_1), \ldots, F_d(x_d)\}$ is a multivariate distribution function with margins $F_1, \ldots, F_d$, and $u_i = F_i(x_i)$ is a uniform random variable.

In order to analyze the dependence structure of multivariate distribution without studying marginal distributions, Sklar's theorem should be introduced.

Theorem 2.1.1: (Sklar's theorem [1959]) Let F be a d-dimensional distribution function with margins $F_1 \dots, F_d$. Then there exists a copula C such that for all $x_1, \dots x_d$ in R

$$F(x_1, \dots, x_d) = C\left\{F_1(x_1), \dots, F_d(x_d)\right\}. \tag{2.1}$$

Conversely, if C is a copula and $F_1 \dots, F_d$ are distribution functions, then the function F defined above is a joint distribution function with margins $F_1 \dots, F_d$.

Definition 2.1.2: (Copula density): For an absolutely continuous copula C, the *copula density* is given by

$$c(u_1, \dots, u_d) = \frac{\partial^d C(u_1, \dots, u_d)}{\partial u_1 \dots \partial u_d}. \tag{2.2}$$

Given that distribution function F_X and copula C_X for random variable $X = (X_1, \dots, X_d)^T$ are absolutely continuous, the following expression holds

$$f\{F_{X_1}^{-1}(u_1), \dots, F_{X_d}^{-1}(u_d)\} = c_X(u_1, \dots, u_d) \prod_{i=1}^{d} f_i\{F_{X_i}^{-1}(u_i)\}, \tag{2.3}$$

where f is the joint density of F_X, f_i is the joint density of F_{X_i} and c_X is the density of the copula C_X.

2.1.1 The Fréchet-Hoeffding Bounds

Because of existence of extreme cases of dependency copula function always lies in between certain bounds. One of the cases is when two random variables U and V are equal, i.e. $U = V$, so they are extremely dependent. In this case the copula takes the form

$$C^+(u, v) = P(U \leq u, U \leq v) = min(u, v) \tag{2.4}$$

The other extreme case is when $V = 1 - U$. The copula then takes the form of

$$\begin{aligned} C^-(u,v) &= P(U \leq u, 1 - U \leq v) \\ &= P(U \leq u, 1 - v \leq U) = u + v - 1. \end{aligned} \tag{2.5}$$

Then the following theorem takes place.

Theorem 2.1.2: (Fréchet-Hoeffding Bounds) Every copula $C = C(u_1, \ldots, u_d)$ is bounded by the minimum and maximum copulae

$$C^-(u_1, \ldots, u_d) \leq C \leq C^+(u_1, \ldots, u_d), \tag{2.6}$$

where $C^-(u_1, \ldots, u_d) = max\{\sum_{i=1}^d u_i + 1 - d, 0\}$ and $C^+(u_1, \ldots, u_d) = min\{u_1, \ldots, u_d\}$.

Another extreme case to be mentioned is when two variables are independent. Then the copula function can be expressed as

$$C^\perp(u,v) = C(u,v) = u \cdot v \tag{2.7}$$

and is usually called *product* or *independent* copula. Fréchet-Hoeffding bounds and product copulae are presented in figure 2.1.

2.2 Measures of Dependence

Dealing with jointly distributed variables calls for appropriate methods to measure dependence. There are three main approaches: classical Pearson correlation coefficient, rank correlation coefficients and coefficients of tail dependence. In contrast to the first coefficient, the last two are sensible enough to give good results for any dependence structure.

2.2.1 Linear correlation coefficient

Pearson correlation between a pair of random vectors $(X, Y)^T$ is given by

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X) \cdot Var(Y)}}, \tag{2.8}$$

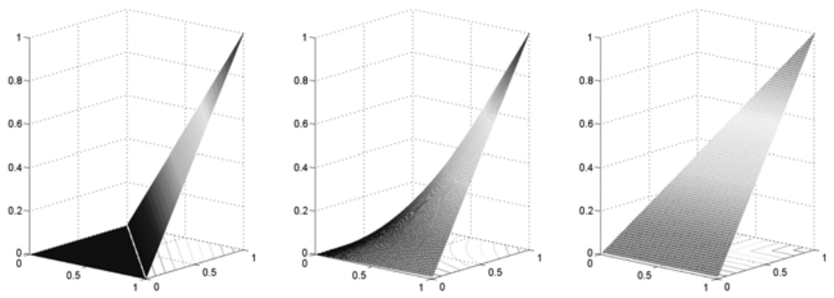


Fig. 2.1: Lower Fréchet-Hoeffding bound (left panel), product copula (middle panel) and upper Fréchet-Hoeffding bound (right panel)

where $Cov(X, Y) = E(XY) - E(X)E(Y)$. It is a linear correlation coefficient and it can measure only linear dependence. That is why it can be used for elliptically distributed random variables (i.e. multivariate normal or multivariate t-distribution). For other distributions Pearson correlation coefficient might give misleading results. See ***Embrechts, Lindskog, and McNeil (2001)*** for more details.

2.2.2 Rank correlation coefficients

Sperman's rho and *Kendall's tau* are two widely used rank correlation coefficients. Both of these coefficients can be presented directly for copulae. The idea is simply to take the ranks of the observed variables and calculate the correlation of the ranks. More precisely, for two random variables X and Y with marginals F and G, respectively, *Spearman's rho* is given by

$$\rho_S = Corr(F(X), G(Y)). \tag{2.9}$$

In a multivariate case the correlation matrix has to be estimated.

For *Kendall's tau* lets consider two pairs of equally and jointly distributed, but independent from each other variables (X, Y) and $(\widetilde{X}, \widetilde{Y})$. Then, *Kendall's tau* is given by

$$\rho_\tau(X,Y) = E\left[sign\left((X-\widetilde{X})(Y-\widetilde{Y})\right)\right]. \tag{2.10}$$

Both coefficients can be directly presented for copula C that describes dependence between X and Y

$$\rho_S(X,Y) = 12\int_0^1\int_0^1 (C(u,v)-uv)\,dudv \tag{2.11}$$

$$\rho_\tau(X,Y) = 4\int_0^1\int_0^1 C(u,v)dC(u,v)-1 \tag{2.12}$$

2.2.3 Tail dependence

For the tail dependence of a copula we consider two cases: upper and lower tail dependence. The intuition of these properties is very simple, i.e. for a pair of random variables U and V *upper tail dependence* means that for high values of U we expect also high values of V. More precisely, for uniform U and V upper tail dependence is defined as

$$\delta = \lim_{\theta\to 1-} P(U>u|V>v) \tag{2.13}$$

$$= \lim_{\theta\to 1-} \frac{1-2u+C(u,u)}{1-u}, \tag{2.14}$$

provided that the limit exists and $\delta \in [0,1]$. If $\delta = 0$, then U and V are asymptotically independent in the upper tail.

Analogously, the coefficient of lower tail dependence is defined by

$$\gamma = \lim_{\theta\to 0+} P(U\le u|V\le v) \tag{2.15}$$

$$= \lim_{\theta\to 0+} \frac{C(u,u)}{u}, \tag{2.16}$$

provided that the limit exists and $\gamma \in [0,1]$. Similary, if $\gamma = 0$, then U and V are asymptotically independent in lower tail.

2.3 Archimedean copulae

In this section Archimedean copulae are introduced. One of the main reasons why these copulae are of interest is that they are not elliptical copulae and allow to model a big variety of different dependence structures. First, the basic definition of Archimedean copulae is given. Then, an example of Clayton copula is presented. Clayton copula was chosen because of its property to model lower tail dependence, which is essential for financial data analysis. This type of copula will be used in simulations (section 6.1) as well as in the empirical analysis (section 6.2).

Definition 2.3.1: Let φ be a continuous, strictly decreasing function from $[0,1]$ to $[0,\infty]$ such that $\varphi(1) = 0$. The pseudo-inverse of φ is function $\varphi^{[-1]} : [0,\infty] \to [0,1]$ given by

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{[-1]}(t), & 0 \le t \le \varphi(0) \\ 0, & \varphi(0) \le t \le \infty \end{cases} \tag{2.17}$$

φ possesses the following properties:

- $\varphi^{[-1]}$ is continuous and decreasing on $[0,\infty]$ and strictly decreasing on $[0,\varphi(0)]$
- $\varphi^{[-1]}(\varphi(u)) = u$ on $[0,1]$
- if $\varphi(0) = \infty$, then $\varphi^{[-1]} = \varphi^{-1}$

Theorem 2.3.1: Let φ be a continuous, strictly decreasing function from $[0,1]$ to $[0,\infty]$ such that $\varphi(1) = 0$, and let $\varphi^{[-1]}$ be the pseudo-inverse of φ. Let C be the function from $[0,1]^2$ to $[0,1]$ given by

$$C(u,v) = \varphi^{[-1]}(\varphi(u) + \varphi(v)). \tag{2.18}$$

Then C is a copula if and only if φ is convex.

Such copulae are called Archimedean and function φ is called a generator of the copula.

One of the types of *Copula* of the *Archimedean family* is *Copula* discussed by Clayton (1978) and others.

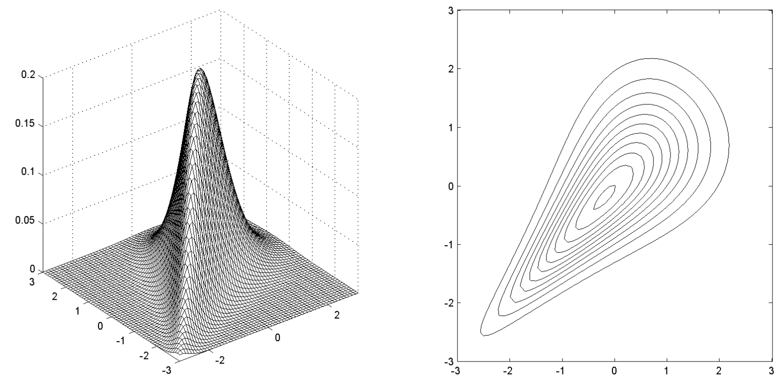


Fig. 2.2: Density (left panel) and contour plot (right panel) of the Clayton Copula with Gaussian marginals, $\theta = 1.5$

2.3.1 Bivariate Clayton copula

If we use $\varphi_\theta(t) = (t^\theta - 1)$ as generator in (2.18), then we get the Clayton copula

$$C(u,v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-\frac{1}{\theta}}. \tag{2.19}$$

This is a *copula* only if $\theta > 0$.
The density function of the Clayton copula takes the form of

$$c_\theta(u,v) = (1+\theta)u^{-(\theta+1)}v^{-(\theta+1)}(u^{-\theta} + v^{-\theta} - 1)^{-\theta^{-1}-2}. \tag{2.20}$$

Density and counter plot of the Clayton copula with $\theta = 1.5$ are presented in figure 2.2.

If θ tends to 0, variables become independent and $\lim_{\theta\to 0} C_\theta = C^\perp(u,v)$. If θ tends to infinity, then the dependence becomes maximal and $\lim_{\theta\to\infty} C_\theta = C^+$.

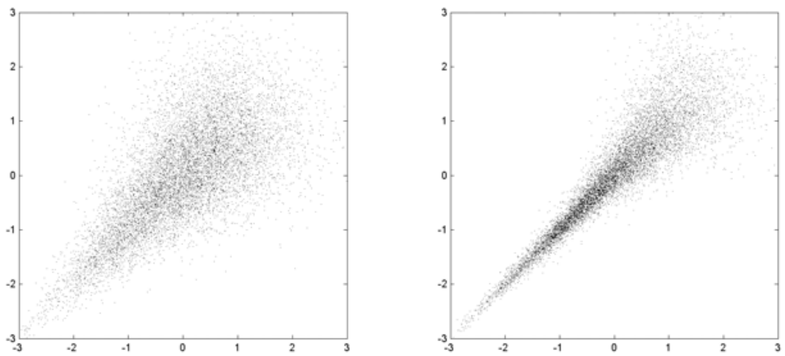


Fig. 2.3: Random variables simulated from Clayton copula with Gaussian marginals for $\theta = 2$ (left panel) and $\theta = 8$ (right panel). 10000 samples.

Clayton copula has no upper tail dependence

$$\delta = \lim_{\theta \to 1-} \frac{1 - 2u + C(u,u)}{1-u} = 0. \tag{2.21}$$

Lower tail dependence for Clayton copula is equal to

$$\gamma = \lim_{\theta \to 0+} \frac{C(u,u)}{u} = 2^{-\theta^{-1}}. \tag{2.22}$$

In figure 2.3 random variables simulated from Clayton copula with Gaussian margins for $\theta = 2$ and $\theta = 8$ are presented. As it is seen, the higher θ is, the tighter is the dependence between data.

2.3.2 Multivariate Clayton copula

In a multivariate case Clayton copula can be defined by the following equations:

Definition 2.3.2: (Multivariate Clayton copula)

$$C_\theta\left(u_1,\ldots,u_d\right)=\left\{\left(\sum_{j=1}^{d}u_j^{-\theta}\right)-d+1\right\}^{-\theta^{-1}},\theta>0$$

with density:

$$c_\theta\left(u_1,\ldots,u_d\right)=\prod_{j=1}^{d}\left\{1+\left(j-1\right)\theta\right\}u_j^{-(\theta+1)}\left\{\left(\sum_{j=1}^{d}u_j^{-\theta}\right)-d+1\right\}^{-\left(\theta^{-1}+d\right)}$$

2.4 Kullback-Leibler Divergence

The Kullback-Leibler divergence is a distance measure from "true" probability distribution P to casual probability distribution Q. Typically P represents data, observations, or precisely calculated probability distribution. The measure Q typically represents a theory, a model, a description or an approximation of P.

Here Kullback-Leibler divergence is used as a criterion to choose a proper copula, when we observe a jump in the dependence parameter θ for the simulations in section 6.1.

Denote random variable $X\sim C_\theta\{F_{X_1}(x_1),\ldots,F_{X_d}(x_d)\}$ with density function given by

$$f_\theta(x_1,\ldots,x_d)=c_\theta(F_{X_1}(x_1),\ldots,F_{X_d}(x_d))\prod_{i=1}^{d}f_i(x_i) \tag{2.23}$$

where c_θ is copula density. The log-likelihood function is then given by

$$\ell(\theta)=\log c_\theta(F_{X_1}(x_1),\ldots,F_{X_d}(x_d))+\sum_{i=1}^{d}\log f_i(x_i). \tag{2.24}$$

Only the first term in (2.24) depends on the copula, and it follows that the larger $E_{\theta_0}[c_{\theta_0}(F_{X_1}(X_1),\ldots,F_{X_d}(X_d))]$ is, the closer is this model to the true model. The Kullback-Leiber divergence is then denoted as

$$\mathcal{K}(C_{\theta_0}, C_{\theta_1}) = E_{\theta_0}\left[\log\left\{\frac{c_{\theta_0}(F_{X_1}(X_1), \ldots, F_{X_d}(X_d))}{c_{\theta_1}(F_{X_1}(X_1), \ldots, F_{X_d}(X_d))}\right\}\right]. \tag{2.25}$$

Therefore, the following hypothesis can be tested: the copula model with copula C_{θ_1} is not worse than the copula model with copula C_{θ_0} if

$$H_0 : \mathcal{K}(C_{\theta_0}, C_{\theta_1}) \leq 0 \tag{2.26}$$

3. VALUE AT RISK ESTIMATION

Value-at-Risk (VaR) is a category of risk metrics that describes probabilistically the market risk of a trading portfolio. *Value-at-Risk* is widely used by banks, securities firms, commodity merchants, energy merchants, and other trading organizations. First, we make a brief overview of what is VaR of a portfolio. Then, in section 3.2 we discuss two main approaches to measure *Value-at-Risk* of a portfolio. We introduce the *RiskMetrics* approach and the *Copula-based* approach. To compare these methods both of them will be used to analyze real portfolios in section 6.2 empirical analysis.

3.1 *Profit and Loss Function*

If we have a portfolio, constructed of d assets with $S_t = (S_{1,t}, \dots, S_{d,t})^T$ denoting the prices of the assets at time t, and the allocation of the assets are given by vector $\omega = (\omega_1, \dots, \omega_d)^T \in \mathbb{R}^d$, then the portfolio value is given by

$$V_t = \sum_{j=1}^{d} \omega_j S_{j,t}. \tag{3.1}$$

Assets prices in financial econometrics are considered as random variables, so the value of a portfolio V_t at time t is also a random variable.

The meaning of *profit and loss (P & L) function* is to measure changes in the value of a portfolio over time. The (P & L) function is given by

$$L_t = (V_t - V_{t-1}) = \sum_{j=1}^{d} \omega_j S_{j,t-1}\{exp(X_{j,t}) - 1\}, \tag{3.2}$$

where $X_t = \log S_t - \log S_{t-1}$ are the *log-returns* of a portfolio. As (P & L) function is a random variable, the corresponding distribution function is denoted by $F_{t,L_t}(x) = P_t(L_t \leq x)$, and VaR is just an α-quantile of F_{t,L_t}:

$$VaR_t(\alpha) = F^{-1}_{t,L_t}(\alpha). \tag{3.3}$$

The task is to find proper methods to estimate VaR, and this will be discussed in following section.

3.2 Methods to measure risk

3.2.1 Modelling VaR with RiskMetrics

The *Exponentially Weighted Moving Average* (EWMA) approach for characterizing volatility is an example of exponential smoothing techniques that employ one or more exponential smoothing parameters to give more weight to recent observations and less weight to older observations, in an attempt to respond "dynamically" to the changing value of the time series.

EWMA is an example of the simplest form of the exponential smoothing method which employs a single smoothing parameter. Following assumptions should be made on the nature of the data of the underlying time series, so that *EWMA* is appropriate:

- the process generating the data is "stationary";
- variance around the mean remains constant over time, and no systematic trend exists in the day-to-day changes in the time series.

Let risk factor X_t have a conditional multivariate normal distribution. Then, in order to estimate the conditional distribution of log-returns $\sim N\left(0, \hat{\Sigma}_t\right)$ we use *EWMA*:

$$\hat{\Sigma}_t = \left(e^{\lambda} - 1\right) \sum_{s<t} e^{-\lambda(t-s)} X_s X_s^T.$$

The model requires a "decay factor" parameter $\lambda \in (0, 1)$ to be specified. λ is used in order to determine the rate at which the weights of past observations diminish. Here decay factor λ according to ***Morgan/Reuters (1996)*** is set to be equal to 0.05.

3.2.2 Modelling VaR with Copula

In the *Copula-based* approach in order to calculate *Value-at-Risk* we assume that log-returns follow the given process:

$$X_{j,t} = \sigma_{j,t}\epsilon_{j,t}, \tag{3.4}$$

where $\epsilon_t = (\epsilon_{1,t}, \dots, \epsilon_{d,t})^T$ are standardized innovations for $j = 1, \dots, d$, and

$$\sigma^2_{j,t} = E\left[X^2_{j,t}|\mathcal{F}_{t-1}\right] \tag{3.5}$$

is the conditional variance, given information set $\mathcal{F}_{t-1}$. The innovations $\epsilon = (\epsilon_1, \dots, \epsilon_d)^T$ have joint distribution F_ϵ, and ϵ_j have continuous marginal distributions $F_j, j = 1, \dots, d$. The distribution function of innovations ϵ is described by

$$F_\epsilon(\epsilon_1, \dots, \epsilon_d) = C_\theta\left\{F_1(\epsilon_1), \dots, F_d(\epsilon_d)\right\}, \tag{3.6}$$

where C_θ is a *copula* belonging to a parametric family $\mathcal{C} = \{C_\theta, \theta \in \Theta\}$.

Analytical methods to calculate *Value-at-Risk* assume multivariate normal distribution. In case of the *Copula-based* approach we use estimated parameters of a *Copula* to generate Monte Carlo simulations for the computation of the *Value-at-Risk*.

The following procedure is described in ***Giacomini and Härdle (2005)***. For a portfolio $\omega \in \mathbb{R}^d$ and sample $\{x_{j,t}\}_{t=1}^T, j = 1, \dots, d$ of log-returns the *Value-at-Risk* is estimated according to the following steps:

1. determine innovations $\{\hat{\epsilon}_t\}_{t=1}^T$ by deGARCHing;
2. specify and estimate marginal distributions $F_j(\hat{\epsilon}_j))$;
3. specify parametric copula family $\mathcal{C}$ and estimate dependence parameter θ;
4. generate innovations ϵ via Monte Carlo method;
5. estimate the empirical $\alpha-$quantile $V\hat{a}R_t(\alpha)$.

4. COPULA ESTIMATION

There are several ways to estimate parameter of a *copula*-function. Most popular methods deal with maximization of log-likelihood function of the copula density with respect to the parameters. ***Joe (1997)***, ***Durrleman (2000)*** suggest three different ways: the *full maximum likelihood* (FML) method, the *inference for margins* (IFM) method and the *canonical maximum likelihood* (CML) method. All three methods are briefly introduced further.

For a random vector $X = X\left(X_1, \ldots, X_d\right)^T$ with parametric univariate marginal distributions $F_{X_j}\left(x_j, \delta_j\right), j = 1, \ldots, d$ the conditional distribution of X_t can be written as:

$$F\left(\alpha; x_1, \ldots, x_T\right) = C\left(F_1\left(x_1; \delta_1\right), \ldots, F_d\left(x_d; \delta_d\right); \theta\right), \tag{4.1}$$

where C is from parametric copula family with dependence parameter θ.

Assuming that c is density of C we get the conditional density of X_t

$$f\left(\alpha; x_1, \ldots, x_T\right) = c\left\{F_1\left(x_1; \delta_1\right), \ldots, F_d\left(x_d; \delta_d\right); \theta_t\right\} \prod_{i=1}^{d} f_i\left(x_i; \delta_i\right). \tag{4.2}$$

Then the log-likelihood function is given by

$$\begin{aligned} \ell\left(\alpha; x_1, \ldots, x_T\right) = \sum_{t=1}^{T} \log c\left\{F_{X_1}\left(x_{1,t}; \delta_1\right), \ldots, F_{X_d}\left(x_{d,t}; \delta_d\right); \theta\right\} \\ + \sum_{t=1}^{T} \sum_{j=1}^{d} \log f_j\left(x_{j,t}; \delta_j\right), \end{aligned} \tag{4.3}$$

where $c\left\{u_1,\ldots,u_d\right\} = \frac{\partial^d C(u_1,\ldots,u_d)}{\partial u_1 \ldots u_d}$ is a copula density, and $\alpha = (\theta, \delta_1, \ldots, \delta_d)^T$ are the parameters to be estimated.

4.1 Full maximum likelihood

The full maximum likelihood (FML) method estimates the parameters exactly from (4.3), maximizing the likelihood function with respect to the unknown parameter α:

$$\widetilde{\alpha}_{FML} = \arg\underset{\alpha}{max}\, l(\alpha). \tag{4.4}$$

But the maximization of such log-likelihood function is complicated as it involves estimation of all the parameters simultaneously, and the increase of scale problem makes the algorithm too cumbersome.

4.2 Inference for margins

The inference for margins (IFM) method assumes that each univariate margin of a log-likelihood function takes the following form:

$$\ell_j(\delta_j) = \sum_{t=1}^{T} \log f_j(x_{j,t}; \delta_j), j = 1, \ldots, d \tag{4.5}$$

and the log-likelihood for the joint distribution looks like:

$$\ell(\theta, \delta_1, \ldots, \delta_T) = \sum_{t=1}^{T} \log f(\alpha; x_1, \ldots, x_T). \tag{4.6}$$

The log-likelihoods $\ell_j(\delta_j), j = 1, \ldots, d$ are separately maximized with respect to δ

$$\hat{\delta}_j = \arg\underset{\delta}{max}\, \ell_j(\delta_j). \tag{4.7}$$

Then the log-likelihood function $\ell(\theta, \hat{\delta}_1, \ldots, \hat{\delta}_d)$ is maximized with respect to θ given $\hat{\delta}_1, \ldots, \hat{\delta}_d$

$$\hat{\theta} = \arg\underset{\theta}{max}\, \ell(\theta, \hat{\delta}_1, \ldots, \hat{\delta}_d) \tag{4.8}$$

Compared to the full maximum likelihood method, this procedure is much simpler and less time-consuming as it involves maximizing several numerical optimizations, each with few parameters. In ***Joe and Xu (1996)*** it is shown that IFM estimators are asymptotically efficient and some numerical examples are provided.

4.3 Canonical maximum Likelihood

In the canonical maximum likelihood (CML) method proposed by ***Mashal and Zeevi (2002)*** no assumptions on the distributions of the marginals are needed. Instead, empirical marginal distributions are estimated and the approximation of the unknown marginal distribution $F_n(\cdot)$ is given by

$$\widehat{F}_{X_j}(x) = \frac{1}{T+1}\sum_{t=1}^{T}\mathbb{I}\left\{X_{j,t} \leq x\right\}, \tag{4.9}$$

where $\mathbb{I}\left\{X_{j,t} \leq x\right\}$ is an indicator function. Thus, pseudo-samples of uniform variates are obtained. Then we can maximize the pseudo log-likelihood function given by

$$\ell(\theta) = \sum_{t=1}^{T}\log c\left\{\widehat{F}_{X_1}(x_{1,t}), \ldots, \widehat{F}_{X_d}(x_{d,t}); \theta\right\} \tag{4.10}$$

and obtain the desired estimate

$$\widehat{\vartheta}_{CML} = \arg\underset{\theta}{max}\,\ell(\theta). \tag{4.11}$$

This method is computationally more advantageous than the two other described above as it is numerically more stable.

5. INHOMOGENEOUS DEPENDENCY MODELING

In the previous section methods how to estimate dependence parameter θ of a copula for some interval of observations are discussed. For these methods an assumption of invariability of the parameter θ is used. However, what if during some period of time the dependence structure between data changes? Then θ is not constant and depends on time $\theta(t) = \theta_t$. In this case, the joint distribution for $X_t = (X_{t,1}, \ldots, X_{t,d})$ can be modified, and it takes the form $F_{t,X_t} = C_{\theta_t}\{F_{t,1}(X_1), \ldots, F_{t,d}(X_d)\}$ with probability measure P_{θ_t}.

In this work a procedure to capture local changes in dependency structure is used. The procedure adaptively estimates intervals of homogeneity, and for each interval a copula parameter can be estimated. This local parametric fitting approach was introduced by ***Mercurio and Spokoiny (2004)*** and ***Härdle et al. (2003)***. The Local Change Point (LCP) procedure tests the hypothesis $H_0 : \theta_t = \theta$ within some interval I. Therefore, for the largest possible interval $I = [t_0 - m_{k^*}, t_0]$ the *small modelling bias condition* (SMB) should be fulfilled:

$$\Delta_I(\theta) = \sum_{t \in I} \mathcal{K}(P_\theta, P_{\theta_t}) \leq \Delta \tag{5.1}$$

where θ is constant and

$$\mathcal{K}(P_\vartheta, P_{\vartheta'}) = E_\vartheta \log \frac{p(y, \vartheta)}{p(y, \vartheta')} \tag{5.2}$$

is a *Kullback-Leibler divergence.*

5.1 LCP procedure

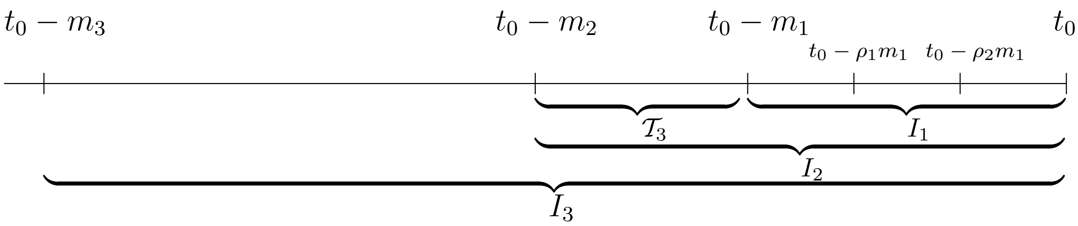
With the Local change point (LCP) procedure the intervals of homogeneity are determined (the following approach is from ***Giacomini et al. (2007)***).

The procedure is adaptive and it tests for the sudden jump in the dependence parameter on the interval-candidate. The following notations are needed to describe the steps of the procedure:

- $\mathcal{I} = \{I_k, k = -1, 0, 1, \ldots\}$ is a family of intervals such that $I_k = [t_0 - m_k, t_0]$ with $m_k : m_{-1} < m_0 < \ldots \leq t_0$ and $m_{-1} = \rho_2 m_1$, $m_0 = \rho_1 m_1$ from $\rho_1 > \rho_2 \in (0, 1)$;
- $\mathcal{T}_k \subset I_k$ for $k = 1, 2, \ldots$ are sets of internal points of the form $\mathcal{T}_k = [t_0 - m_{k-1}, t_0 - m_{k-2}]$.

The procedure starts from $k = 1$, for which we

1. test the $H_{0,k}$ hypothesis of homogeneity within I_k on $\mathcal{T}_k$;
2. if $H_{0,k}$ is not rejected, take the next larger interval I_{k+1} and repeat the previous step until homogeneity is rejected or the largest possible interval $[0, t_0]$ is reached;
3. if $H_{0,k}$ is rejected within I_k, the estimated interval of homogeneity is the last accepted interval $\widehat{I} = I_{k-2}$;
4. if the largest possible interval is reached, we take $\widehat{I} = [0, t_0]$.



The copula dependence parameter θ is estimated from the observations that belong to estimated interval of homogeneity $\widehat{I}$, i.e. $\widehat{\theta}_{t_0} = \widetilde{\theta}_{\widehat{I}}$.

In the following paragraph a local homogeneity test is described.

5.1.1 *Test of Homogeneity against a change point alternative*

For a set of internal points $\mathcal{T}_I$ within an interval-candidate $I = [t_0 - m, t_0]$ the null hypothesis takes the following form:

$$H_0 : \forall \tau \in \mathcal{T}_I, \theta_t = \theta,$$

i.e. on some interval I the observations follow the model with dependence parameter θ. This hypothesis leads to the log-likelihood $\ell_I(\theta)$. The alternative hypothesis claims that

$$H_1 : \exists \tau \in \mathcal{T}_I : \begin{array}{ll} \theta_t = \theta_1 & \text{for} \quad t \in J = [\tau, t_0] \\ \theta_t = \theta_2 \neq \theta_1 & \text{for} \quad t \in J^c = [t_0 - m, \tau) , \end{array}$$

i.e. the parameter θ changes spontaneously in some internal point τ of the interval I. This corresponds to the log-likelihood $\ell_J(\theta_1) + \ell_{J^c}(\theta_2)$.

Then, the likelihood ratio test for the single change point with known fixed location τ can be written as:

$$\begin{aligned} T_{I,\tau} &= \max_{\theta_1,\theta_2} \{\ell_J(\theta_1) + \ell_{J^c}(\theta_2)\} - \max_{\theta} \ell_I(\theta) \\ &= \ell_J\left(\widehat{\theta}_J\right) + \ell_{J^c}\left(\widehat{\theta}_{J^c}\right) - \ell_I\left(\widehat{\theta}_I\right) \\ &= \widehat{\ell}_J + \widehat{\ell}_{J^c} - \widehat{\ell}_I. \end{aligned} \tag{5.3}$$

The change-point test for the interval I is defined as the maximum of (5.3) over $\tau \in \mathcal{T}_I$:

$$T_I = \max_{\tau \in \mathcal{T}_I} T_{I,\tau}. \tag{5.4}$$

The change point test compares this test statistics with a critical value λ_I which may depend on the interval I and the nominal first kind error probability α. One rejects the hypothesis of homogeneity if $T_I > \lambda_I$. The estimator of the change point is then defined as

$$\widehat{\tau} = \arg\,\underset{\tau \in \mathcal{T}_\mathcal{I}}{max}\, T_{I,\tau}. \tag{5.5}$$

For more details, i.e. how to choose the parameters of the procedure and etc., refer to ***Mercurio and Spokoiny (2004)***.

6. TESTING LCP PROCEDURE AND APPLICATIONS

In the previous chapters methods to estimate dependence parameter of a copula function were presented. First, a Canonical maximum Likelihood method (section 4.3) was chosen to estimate θ on the interval of homogeneity. Then, Local Change Point (LCP) procedure which estimates intervals of homogeneity was discussed.

In this chapter the quality of the LCP procedure is tested. For this, different sets of simulations were carried out. The results are presented in the following section 6.1. The conducted simulations are expansion of the analysis presented in ***Giacomini et al (2007)***. Further, in section 6.2 the LCP procedure is applied to the real financial data. The results from applying copula based approach for analysing financial data are then compared to the existing *RiskMetrics* method, discussed in paragraph 3.2.1.

6.1 Simulations

In the article of ***Giacomini et al (2007)*** the LCP procedure is applied to different sets of initial parameters for the case of sudden jump in dependence θ_t, and for each set simulations are conducted. Here sensitivity of the LCP procedure to linear change in dependence parameter is presented. The LCP procedure is tested with respect to changes in data dimension ($d = 2, 6, 10$), to changes in the height of an increase in dependence parameter θ_t ($\Delta = \vartheta - 0.1$, where $\vartheta = 1.5, 3$ and 6) and to changes of the angle, at which the dependence parameter θ_t increases and decreases ($\tan \alpha_{upward} = -\Delta/\Delta t_{upward} = -\Delta/10, -\Delta/30$ and $-\Delta/50$, $\tan \alpha_{downward} = \Delta/\Delta t_{downward} = \Delta/10, \Delta/30$ and $\Delta/50$, where Δt_{upward} and $\Delta t_{downward}$ are lengths of intervals of increase and decrease respectively, from right to left) which makes in total 27 different sets. For each set 200 distinct simulations from Clayton copula were generated and then estimated via LCP procedure. The dynamics structure of θ_t parameter for the cases of different angles of increase and decrease in θ_t are given by

$$\theta_t = \begin{cases} 0.1 & \text{if } 1 \leq t \leq 100 \\ 0.1 + \frac{1}{10}\Delta(t-100) & \text{if } 101 \leq t \leq 110 \\ \vartheta & \text{if } 111 \leq t \leq 210 \\ \vartheta - \frac{1}{10}\Delta(t-210) & \text{if } 211 \leq t \leq 220 \\ 0.1 & \text{if } 221 \leq t \leq 320 \end{cases} \tag{6.1}$$

$$\theta_t = \begin{cases} 0.1 & \text{if } 1 \leq t \leq 100 \\ 0.1 + \frac{1}{30}\Delta(t-100) & \text{if } 101 \leq t \leq 130 \\ \vartheta & \text{if } 131 \leq t \leq 230 \\ \vartheta - \frac{1}{30}\Delta(t-230) & \text{if } 231 \leq t \leq 260 \\ 0.1 & \text{if } 261 \leq t \leq 360 \end{cases} \tag{6.2}$$

$$\theta_t = \begin{cases} 0.1 & \text{if } 1 \leq t \leq 100 \\ 0.1 + \frac{1}{50}\Delta(t-100) & \text{if } 101 \leq t \leq 150 \\ \vartheta & \text{if } 151 \leq t \leq 250 \\ \vartheta - \frac{1}{50}\Delta(t-250) & \text{if } 251 \leq t \leq 300 \\ 0.1 & \text{if } 301 \leq t \leq 400, \end{cases} \tag{6.3}$$

where $\vartheta = 1.5, 3$ and 6 and $\Delta = \vartheta - 0.1$. Figures 6.1, 6.2 and 6.3 present the results of simulations and show the true simulated parameter θ_t, the pointwise median and quantiles of the estimated parameter $\hat{\theta}_t$.

As it is seen from figures 6.1, 6.2 and 6.3 the higher the increase in dependence parameter θ_t is, or the higher the dimension of the data set is, the faster the LCP procedure captures the dependence structure and the estimate $\hat{\theta}_t$ is closer to the true parameter θ_t.

In ***Giacomini et al. (2007)*** for the case of sudden jump in dependence parameter θ_t descriptive statistics for detection delay are presented. From Kullback-Leibler divergence, calculated for upward ($\mathcal{K}_d(0.1, \vartheta)$) and downward ($\mathcal{K}_d(\vartheta, 0.1)$) jumps it is shown that delay for downward jump is higher than for the upward jump, which is directly proportional to the probability of error of type *II*.

Here the detection delay cannot be calculated the way it was in ***Giacomini et al.(2007)*** due to the linear change in the dependency structure. In order to compare the speed at which the LCP procedure captures the behavior of true parameter θ_t the following measures are estimated: relative area of exceedance of θ_t over the medium of estimated $\hat{\theta}_t$ for the interval of decrease in θ_t (see auxiliary picture, S_{ABE}/S_{ABCD}) and relative area of exceedance

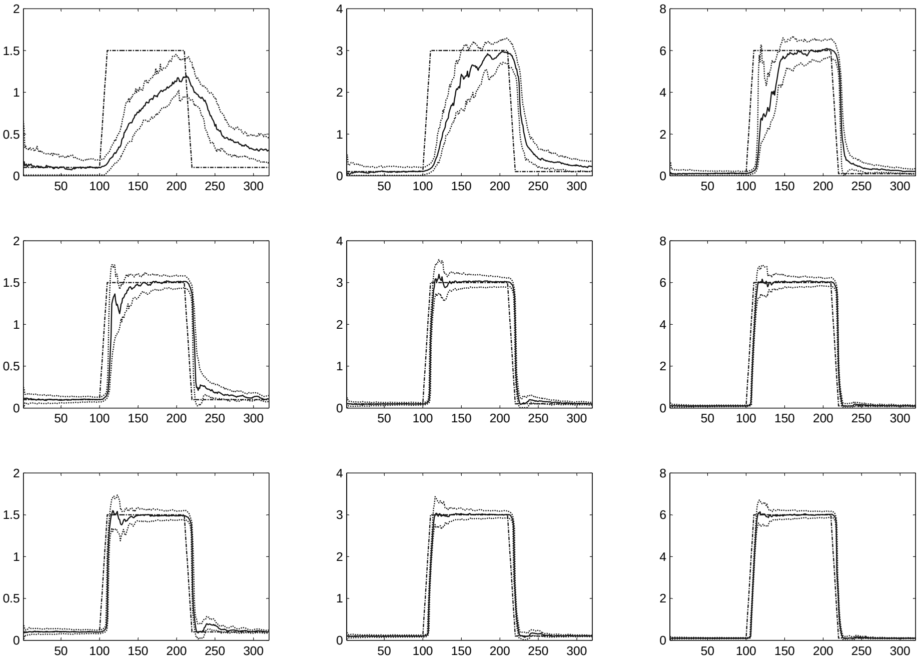


Fig. 6.1: Pointwise median (full), 0.25, 0.75 quantiles (dotted) of estimated parameter $\hat{\theta}_t$, true parameter θ_t (dash-dot), from left to right $\vartheta = 1.5, 3, 6$, from top to bottom $d = 2, 6, 10$. Based on 200 simulations from Clayton copula. Angles of increase and decrease for true parameter are $-\Delta/10$ and $\Delta/10$.

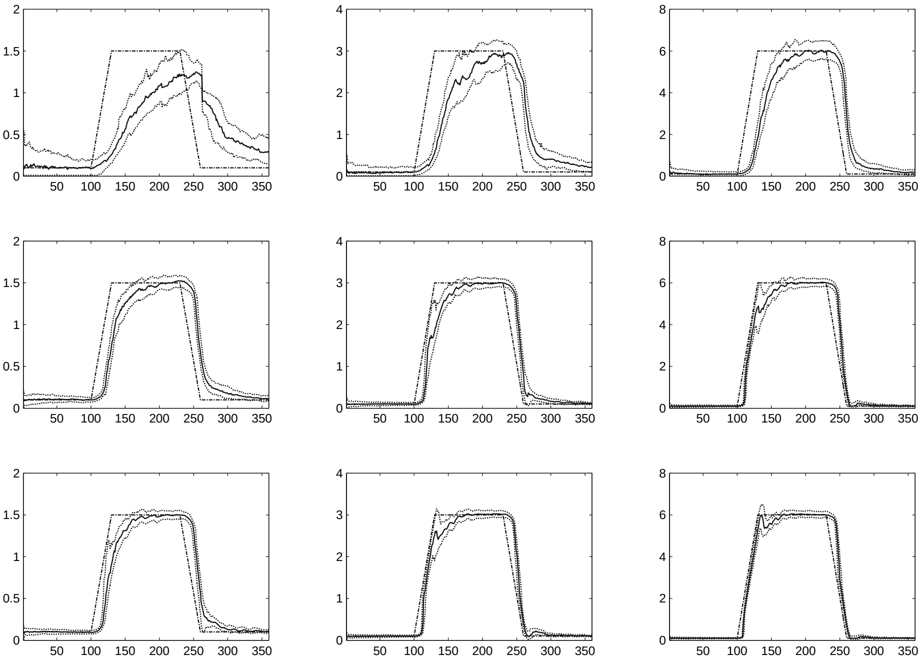


Fig. 6.2: Pointwise median (full), 0.25, 0.75 quantiles (dotted) of estimated parameter $\hat{\theta}_t$, true parameter θ_t (dash-dot), from left to right $\vartheta = 1.5, 3, 6$, from top to bottom $d = 2, 6, 10$. Based on 200 simulations from Clayton copula. Angles of increase and decrease for true parameter are $-\Delta/30$ and $\Delta/30$.

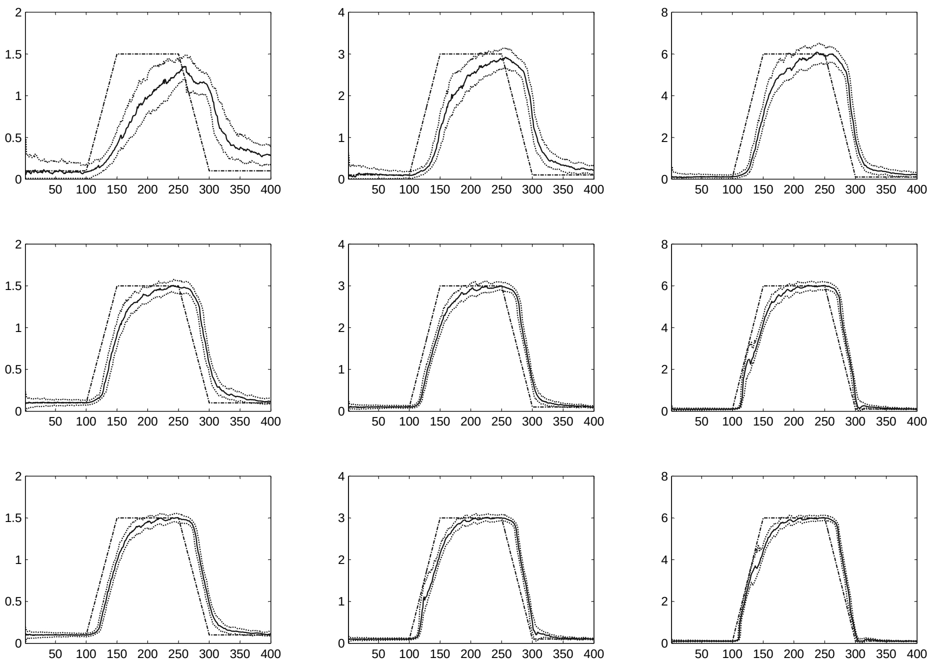
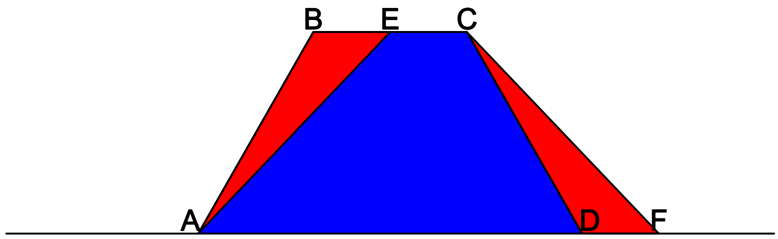


Fig. 6.3: Pointwise median (full), 0.25, 0.75 quantiles (dotted) of estimated parameter $\hat{\theta}_t$, true parameter θ_t (dash-dot), from left to right $\vartheta = 1.5, 3, 6$, from top to bottom $d = 2, 6, 10$. Based on 200 simulations from Clayton copula. Angles of increase and decrease for true parameter are $-\Delta/50$ and $\Delta/50$.



of the medium of estimated $\hat{\theta}_t$ over θ_t for the interval of increase in θ_t (see auxiliary picture, S_{CFD}/S_{ABCD}). Here "relative" denotes that both areas are divided by the area of trapezium $ABCD$.

From tables 6.1-6.6 it is seen that the higher is the height of increase in parameter, or the higher the dimension is, the faster the LCP procedure captures the dependence structure of the data set. Tables 6.1-6.2, 6.3-6.4 and 6.5-6.6 also show that the procedure estimates the parameter better on the intervals of increase than on the intervals of decrease of the parameter. On the intervals of decrease the parameter is estimated from large intervals, for which homogeneity is not verified. That is why the estimate of the dependence parameter is below the true value and the variability is very high.

In the next section the LCP procedure is applied to the real data set.

	$\vartheta = 1.5$	$\vartheta = 3$	$\vartheta = 6$
d=2	0.52	0.32	0.22
d=6	0.12	0.05	0.04
d=10	0.05	0.05	0.04

Tab. 6.1: Relative area of exceedance of θ_{true} over the medium of estimated $\hat{\theta}_t$ for the interval of decrease in θ_{true}.$\tan \alpha_{downward} = \Delta/10$.

	$\vartheta = 1.5$	$\vartheta = 3$	$\vartheta = 6$
d=2	0.31	0.18	0.11
d=6	0.11	0.05	0.04
d=10	0.05	0.04	0.04

Tab. 6.2: Relative area of exceedance of the medium of estimated $\hat{\theta}_t$ over θ_{true} for the interval of increase in θ_{true}.$\tan \alpha_{upward} = -\Delta/10$.

	$\vartheta = 1.5$	$\vartheta = 3$	$\vartheta = 6$
d=2	0.43	0.27	0.19
d=6	0.15	0.10	0.07
d=10	0.12	0.06	0.04

Tab. 6.3: Relative area of exceedance of θ_{true} over the medium of estimated $\hat{\theta}_t$ for the interval of decrease in θ_{true}.$\tan \alpha_{downward} = \Delta/30$.

	$\vartheta = 1.5$	$\vartheta = 3$	$\vartheta = 6$
d=2	0.28	0.18	0.13
d=6	0.13	0.09	0.06
d=10	0.10	0.06	0.05

Tab. 6.4: Relative area of exceedance of the medium of estimated $\hat{\theta}_t$ over θ_{true} for the interval of increase in θ_{true}.$\tan \alpha_{upward} = -\Delta/30$.

	$\vartheta = 1.5$	$\vartheta = 3$	$\vartheta = 6$
d=2	0.45	0.31	0.22
d=6	0.19	0.15	0.11
d=10	0.16	0.11	0.08

Tab. 6.5: Relative area of exceedance of θ_{true} over the medium of estimated $\hat{\theta}_t$ for the interval of decrease in θ_{true}. $\tan \alpha_{downward} = \Delta/50$.

	$\vartheta = 1.5$	$\vartheta = 3$	$\vartheta = 6$
d=2	0.29	0.21	0.17
d=6	0.16	0.12	0.09
d=10	0.13	0.10	0.07

Tab. 6.6: Relative area of exceedance of the medium of estimated $\hat{\theta}_t$ over θ_{true} for the interval of increase in θ_{true}.$\tan \alpha_{upward} = -\Delta/50$.

6.2 Empirical analysis

In chapter 3 two methods to estimate Value-at-Risk of a portfolio were discussed:

- RiskMetrics approach developed by ***Morgan/Reuters (1996)*** and based on Exponential Weighted Moving Average method and
- adaptive estimation procedure based on the assumption of copula distributed innovations and estimated via Local Change Point procedure (LCP).

In this section both methods are applied to the real data set and the quality of these methods is tested via backtesting method.

The analysis is carried out for such financial assets as Siemens (SIE), E.on AG (EOA), ThyssenKrupp (THY), Schering (SCH), Henkel (HEN) and Lufthansa (LHA) for the period from 1 January 2001 till 30 December 2005.

In paragraph 6.2.1 the analysis is performed for 10 randomly constructed 6-dimensional portfolios: SIE, EOA, THY, SCH, HEN and LHA. In paragraph 6.2.2 both methods are applied to all possible combinations of 2-dimensional portfolios: SIE-THY, SIE-EOA, SIE-SCH, SIE-HEN, SIE-LHA, THY-EOA, THY-SCH, THY-HEN, THY-LHA, EOA-SCH, EOA-HEN, EOA-LHA, SCH-HEN, SCH-LHA, HEN-LHA.

For the *RiskMetrics* approach the decay factor λ from 6.4 is set to the level of 1/20

$$\widehat{\Sigma}_t = (e^{\lambda} - 1) \sum_{s<t} e^{-\lambda(t-s)} X_s X_s^T, \tag{6.4}$$

where $X_t = (X_{1,t}, \dots, X_{6,t})$ are the log returns of SIE, EOA, THY, SCH, HEN and LHA.

For the adaptive estimation procedure Clayton copula is chosen, as it possesses the property of dependence in lower tail. The parameters of the procedure are set to be $\alpha = 0.05$, $c = 1.25$ and $m_1 = 21$ (for the choice of the parameters refer to ***Giacomini et al. (2007)***).

For every combination of stock returns dependence parameter θ is estimated via LCP and is presented in the figures as well as the estimated intervals of time homogeneity.

For *backtesting* procedure VaR is estimated using both methods (RiskMetrics and adaptive estimation approach). Then VaR is compared to the true realizations $\{l_t\}$ of the P&L function of portfolios. Exceedance ratios are estimated from 6.5 for levels of $\alpha = 5\%, 4\%, 3\%, 2\%$ and 1% and are presented in the tables.

$$\hat{\alpha} = \frac{1}{T-w} \sum_{t=w}^{T} \mathbb{I}_{\{l_t < \widehat{VaR}_t(\alpha)\}} \tag{6.5}$$

Average exceedance, standard deviation, squared deviation ($\sum_{w \in W} (\hat{\alpha} - \alpha)^2$) and relative squared deviation ($\sum_{w \in W} (\hat{\alpha} - \alpha)^2 / \alpha$) are estimated and given in tables as well.

6.2.1 6-dimensional portfolio

First, the analysis is conducted for portfolios that consist of all 6 assets: SIE, EOA, THY, SCH, HEN and LHA. Dependence parameter θ is estimated from Clayton copula via LCP and is presented in figure 6.4. In the lower panel of figure 6.4 the intervals of time homogeneity are displayed. As one can see from the figure, in the beginning the dependence between data was small and rather stable (intervals of time homogeneity are quite smooth), but then after 2002 the behavior of the estimated parameter changes. The observed peaks refer to changes in the structure of dependence between the data.

In tables 6.7 and 6.8 the summary of exceedance ratios $\hat{\alpha}$ for estimated VaR via adaptive estimation procedure and *RiskMetrics* are presented, respectively. Exceedance ratios are calculated for 10 randomly generated portfolios. From the tables it is seen that for the case of 6-dimensional portfolio Adaptive estimation procedure underestimates VaR in average as, for example, for the level of $\alpha = 5\%$ estimated $E[\hat{\alpha}] = 2.93\%$. On the contrary, the RiskMetrics procedure overestimates VaR and gives the result of $E[\hat{\alpha}] = 5.37\%$.

In figures 6.5 and 6.6 log returns are plotted versus $\widehat{VaR}(\alpha)$ estimated via adaptive estimation procedure and *RiskMetrics* for 10 different portfolios, respectively. In figure 6.7 results for Portfolio 1 are presented as a case in point. As it is seen, the exceedance occurs more often for the case of *RiskMetrics* approach. The other thing to mention here is that for the *RiskMetrics* approach we observe clusters of exceedance.

Portfolio	Exceedances ratio α (in %) 5%	4%	3%	2%	1%
1	4.01	3.16	2.30	1.51	0.57
2	1.36	1.00	0.65	0.57	0.43
3	4.16	3.59	2.65	1.87	0.86
4	2.29	1.87	1.58	1.22	0.72
5	1.79	1.65	1.36	1.15	0.57
6	2.72	2.08	1.79	1.22	0.29
7	2.15	1.58	1.00	0.57	0.36
8	2.36	1.72	1.36	0.72	0.29
9	4.16	3.66	2.44	1.58	0.65
10	4.34	3.08	2.22	1.15	0.86
avg.	2.93	2.34	1.74	1.15	0.56
std.dev.	1.06	0.90	0.63	0.41	0.20
$\sum_{\omega \in W}(\hat{\alpha}-\alpha)^2$	0.54	0.36	0.20	0.09	0.02
$\sum_{\omega \in W}(\hat{\alpha}-\alpha)^2/\alpha$	10.78	8.91	6.63	4.41	2.36

Tab. 6.7: Exceedances ratio α for different portfolios. Clayton copula. Adaptive estimation procedure. 6-dim data: SIE, THY, EOA, SCH, HEN, LHA (from 1-Jan-2001 to 30-Dec-2005)

Portfolio	Exceedances ratio α (in %) 5%	4%	3%	2%	1%
1	5.14	4.50	3.86	2.96	1.67
2	5.01	4.41	3.47	2.44	1.41
3	5.14	4.50	3.21	2.31	1.41
4	6.68	5.27	4.11	2.83	1.54
5	5.40	4.37	3.60	2.31	1.41
6	5.40	4.63	3.73	2.70	1.80
7	5.14	4.37	3.08	2.19	1.54
8	5.01	4.37	3.34	2.70	1.80
9	5.66	5.14	3.86	2.83	1.67
10	5.14	4.76	3.34	2.57	1.80
avg.	5.37	4.61	3.56	2.58	1.61
std.dev.	0.48	0.33	0.31	0.25	0.15
$\sum_{\omega \in W}(\hat{\alpha}-\alpha)^2$	0.04	0.05	0.04	0.04	0.04
$\sum_{\omega \in W}(\hat{\alpha}-\alpha)^2/\alpha$	0.73	1.21	1.37	2.01	3.92

Tab. 6.8: Exceedances ratio α for different portfolios. *RiskMetrics* approach. 6-dim data: SIE, THY, EOA, SCH, HEN, LHA (from 1-Jan-2001 to 30-Dec-2005)

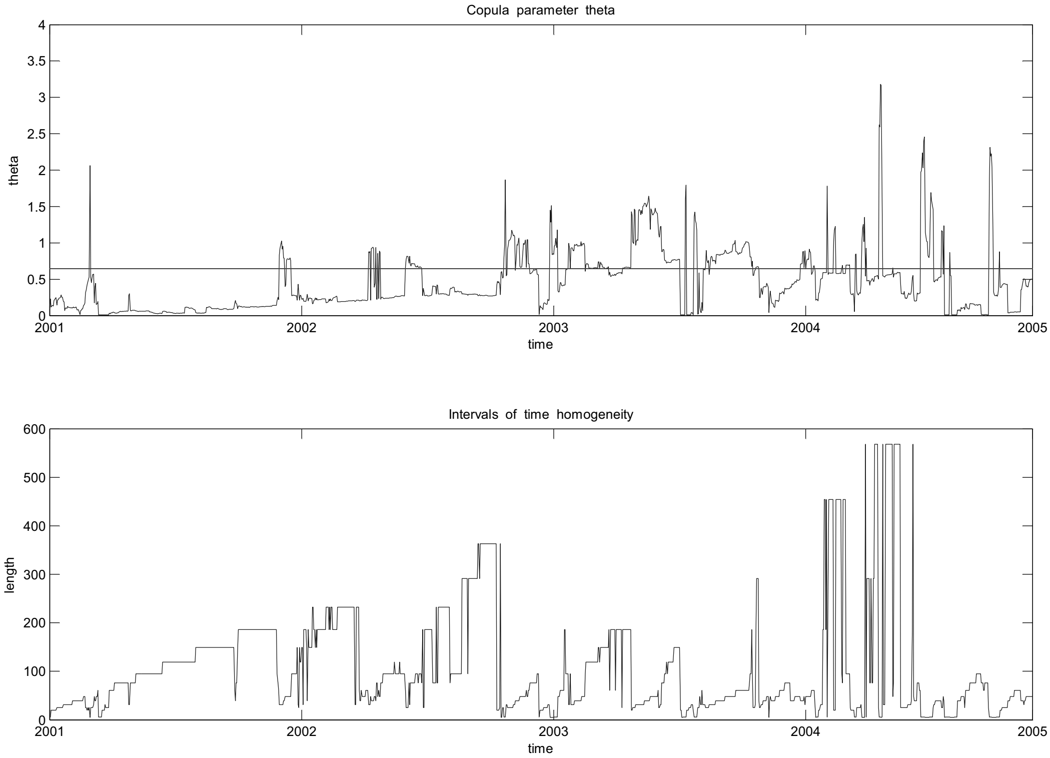


Fig. 6.4: Upper panel: estimated copula dependence parameter θ for 6-dim data: SIE, THY, EOA, SCH, HEN, LHA (from 1-Jan-2001 to 30-Dec-2005). Lower panel: estimated intervals of time homogeneity.

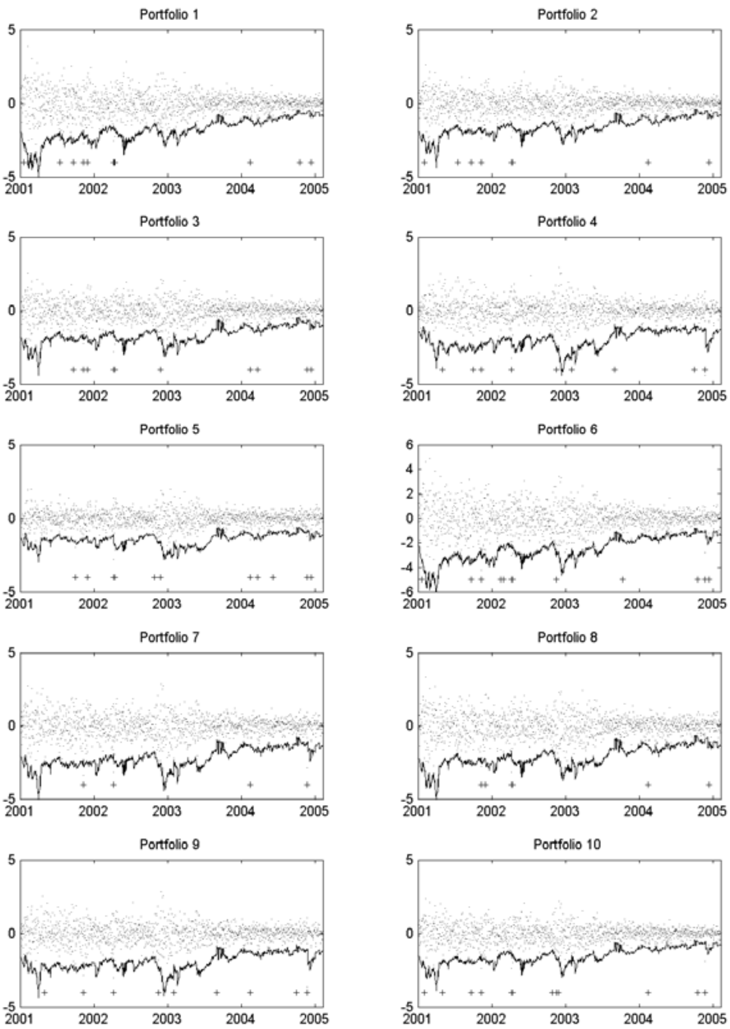


Fig. 6.5: P&L (dots) and $\widehat{VaR}(\alpha)$ at level $\alpha_1 = 0.01$ estimated using adaptive estimation procedure for 6-dim data: SIE, THY, EOA, SCH, HEN, LHA (from 1-Jan-2001 to 30-Dec-2005).

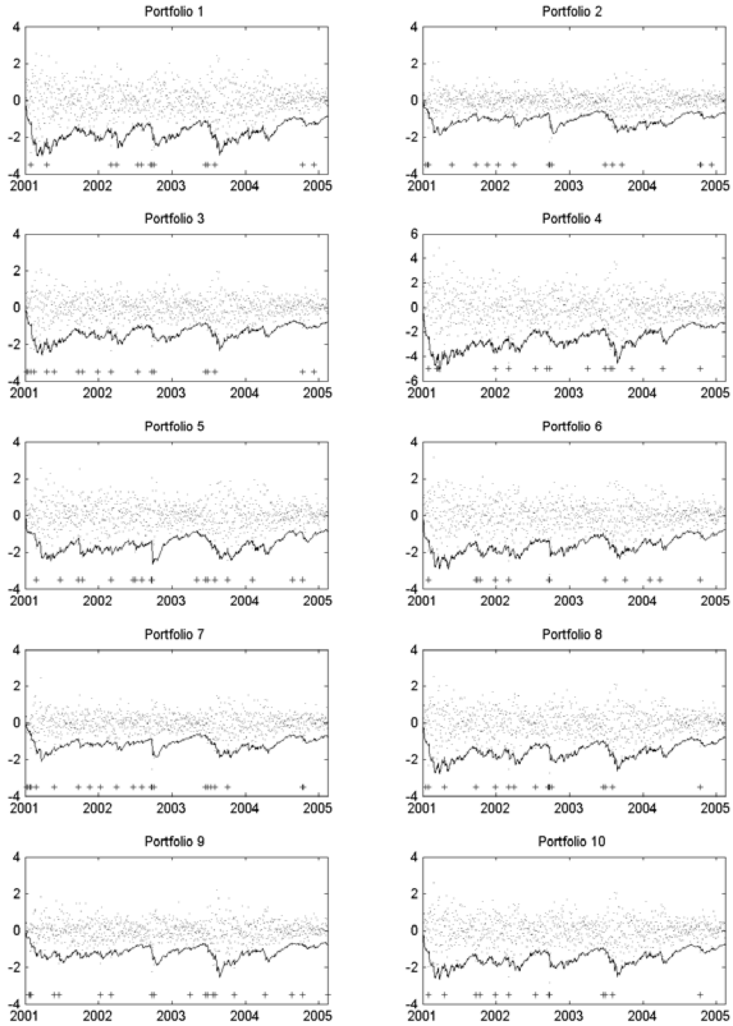


Fig. 6.6: P&L (dots) and $\widehat{VaR}(\alpha)$ at level $\alpha_1 = 0.01$ estimated using *RiskMetrics* approach for 6-dim data: SIE, THY, EOA, SCH, HEN, LHA (from 1-Jan-2001 to 30-Dec-2005).

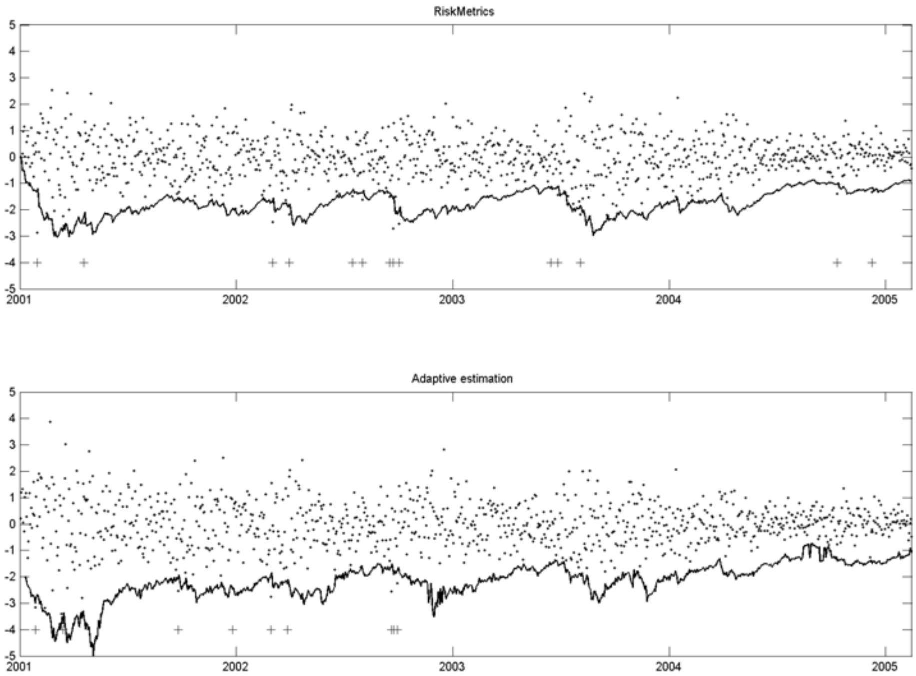


Fig. 6.7: P&L (dots) and $\widehat{VaR}(\alpha)$ at level $\alpha_1 = 0.01$ estimated using *RiskMetrics* approach (upper panel) and adaptive estimation procedure (lower panel) for 6-dim data (Portfolio 1): SIE, THY, EOA, SCH, HEN, LHA (from 1-Jan-2001 to 30-Dec-2005).

6.2.2 2-dimensional portfolios

For 6-dimensional portfolio adaptive estimation procedure was based on the assumption that the dependence structure between all data is the same, which in reality does not hold. This might be one of the reasons why the procedure underestimated VaR. In this section portfolios constructed of 2 assets are analyzed. These portfolios are constructed out of SIE, EOA, THY, SCH, HEN, and LHA, which makes 15 different combinations. Such analysis will show which stock paths are interdependent and which are not.

For each set of data 10 random portfolios were generated and for each portfolio VaR was estimated via adaptive estimation procedure and via *RiskMetrics* approach. The results for exceedance of estimated $\widehat{VaR}_t(\alpha)$ over true realization $\{l_t\}$ of P&L function are given in tables 6.9-6.38 for adaptive estimation procedure and *RiskMetrics* approach, respectively. For each set of portfolios dependence parameter $\hat{\theta}_t$ was estimated and its dynamics as well as the estimated intervals of time homogeneity are presented in figures 6.8-6.22.

From the presented figures it is seen that the dependence in case of two assets portfolio is usually constant over some period of time and intervals of time homogeneity are rather smooth. In some cases gradual increase in dependence parameter is observed, for example for THY and EOA (figure 6.13), THY and SCH (figure 6.14) and SCH and HEN (figure 6.20).

Exceedance ratios estimated via *RiskMetrics* approach show for most portfolios the same results. For the level $\alpha = 5\%$ estimated $\hat{\alpha}_{RM}$ vary from 4.68 to 5.63. At the same time, in some cases adaptive estimation procedure significantly underestimates VaR, for example for SIE and HEN (table 6.15), THY and HEN (table 6.23), THY and LHA (table 6.25), EOA and HEN (table 6.29), SCH and HEN (table 6.33) and HEN and LHA (table 6.37). For portfolio constructed of EOA and SCH (table 6.27) the adpative estimatoin procedure significantly overestimates VaR. The reason for this may lie in the incorrect choice of copula for these particular pairs of assets. That is why for the case of portfolios constructed of 6 assets the adequacy of estimated parameter is questionable.

Portfolio	Exceedances ratio α (in %)				
	5%	4%	3%	2%	1%
1	2.80	2.37	2.15	1.87	1.08
2	2.94	2.30	2.15	1.87	1.08
3	3.44	2.94	2.01	1.58	1.29
4	3.80	3.16	2.15	1.65	0.93
5	3.95	3.44	2.30	1.58	0.86
6	4.02	3.23	2.37	1.51	0.72
7	4.30	3.37	2.65	1.65	0.79
8	4.16	3.52	2.37	1.72	0.72
9	4.38	3.16	2.44	1.65	0.86
10	4.09	3.08	2.51	1.65	0.93
avg.	3.79	3.06	2.31	1.67	0.93
std.dev.	0.52	0.40	0.19	0.11	0.17
$\sum_{\omega \in W}(\hat{\alpha} - \alpha)^2$	0.17	0.10	0.05	0.01	0.00
$\sum_{\omega \in W}(\hat{\alpha} - \alpha)^2/\alpha$	3.49	2.62	1.70	0.60	0.35

Tab. 6.9: Exceedances ratio α for different portfolios. Clayton copula. Adaptive estimation procedure. 2-dim data: SIE and THY (from 1-Jan-2001 to 30-Dec-2005)

Portfolio	Exceedances ratio α (in %)				
	5%	4%	3%	2%	1%
1	4.76	4.50	3.21	2.57	1.54
2	5.40	3.98	3.34	2.19	1.29
3	5.14	3.98	3.08	2.19	1.29
4	5.01	3.98	2.96	2.06	1.16
5	4.88	3.73	3.08	1.80	1.03
6	5.14	3.73	3.08	1.80	1.16
7	5.27	3.60	2.31	1.54	1.16
8	4.76	3.21	2.19	1.54	1.16
9	4.88	3.34	2.06	1.54	1.16
10	5.14	3.34	2.06	1.54	1.16
avg.	5.04	3.74	2.74	1.88	1.21
std.dev.	0.21	0.37	0.49	0.34	0.13
$\sum_{\omega \in W}(\hat{\alpha} - \alpha)^2$	0.00	0.02	0.03	0.01	0.01
$\sum_{\omega \in W}(\hat{\alpha} - \alpha)^2/\alpha$	0.09	0.51	1.03	0.66	0.61

Tab. 6.10: Exceedances ratio α for different portfolios. *RiskMetrics* approach. 2-dim data: SIE and THY (from 1-Jan-2001 to 30-Dec-2005)

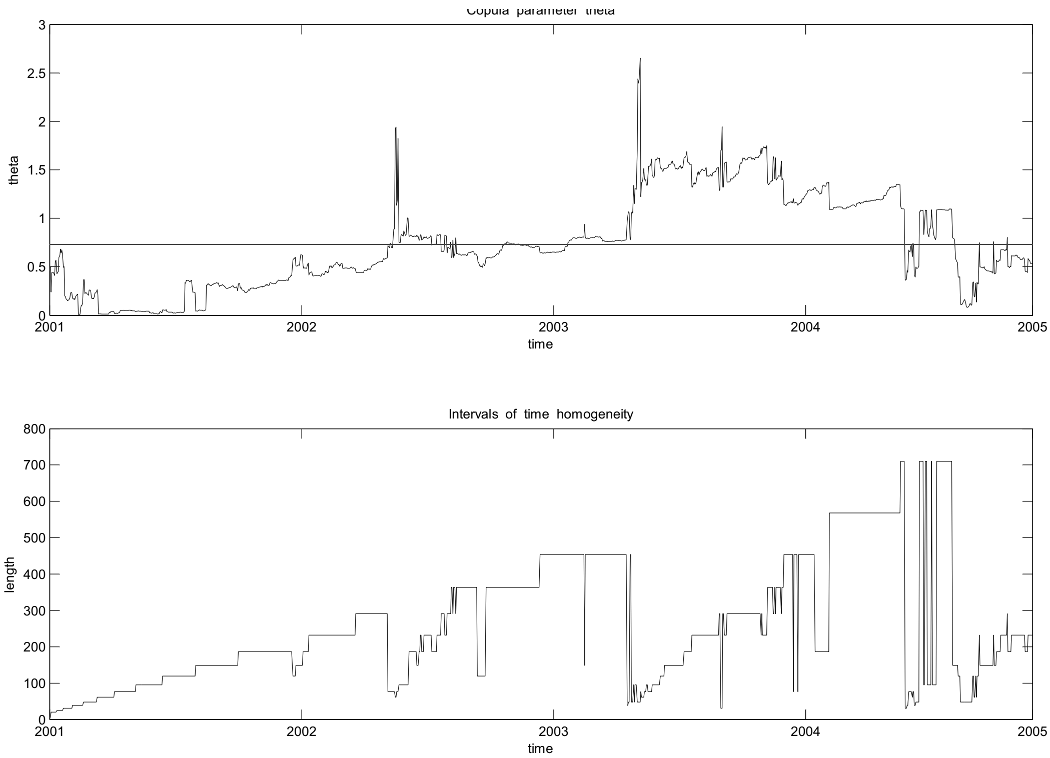


Fig. 6.8: Upper panel: estimated copula dependence parameter θ for 2-dim data:SIE and THY (from 1-Jan-2001 to 30-Dec-2005). Lower panel: estimated intervals of time homogeneity.

Portfolio	Exceedances ratio α (in %)				
	5%	4%	3%	2%	1%
1	2.94	2.37	2.01	1.51	1.00
2	3.37	2.58	1.94	1.36	0.93
3	3.80	2.73	2.01	1.43	0.86
4	3.73	2.80	2.01	1.43	0.86
5	3.80	3.01	2.08	1.29	0.86
6	4.38	2.87	2.30	1.43	0.65
7	4.66	3.44	2.15	1.51	0.93
8	4.66	3.52	2.37	1.65	1.15
9	4.66	3.08	2.44	1.72	1.00
10	4.52	3.01	2.44	1.65	1.00
avg.	4.05	2.94	2.17	1.50	0.93
std.dev.	0.58	0.34	0.18	0.13	0.13
$\sum_{\omega \in W}(\hat{\alpha} - \alpha)^2$	0.12	0.12	0.07	0.03	0.00
$\sum_{\omega \in W}(\hat{\alpha} - \alpha)^2/\alpha$	2.47	3.09	2.39	1.34	0.21

Tab. 6.11: Exceedances ratio α for different portfolios. Clayton copula. Adaptive estimation procedure. 2-dim data: SIE and EOA (from 1-Jan-2001 to 30-Dec-2005)

Portfolio	Exceedances ratio α (in %)				
	5%	4%	3%	2%	1%
1	4.88	3.86	3.34	2.96	1.93
2	5.01	3.98	3.34	2.96	1.93
3	5.14	3.98	3.47	2.31	1.80
4	5.27	3.86	3.08	2.06	1.93
5	4.88	4.11	3.21	2.19	1.67
6	5.14	4.24	3.08	2.06	1.03
7	5.14	4.11	2.70	1.67	1.03
8	5.14	4.11	2.70	1.67	1.03
9	5.14	3.98	2.70	1.67	1.03
10	5.14	3.98	1.93	1.41	1.03
avg.	5.09	4.02	2.96	2.10	1.44
std.dev.	0.12	0.12	0.44	0.50	0.42
$\sum_{\omega \in W}(\hat{\alpha} - \alpha)^2$	0.00	0.00	0.02	0.03	0.04
$\sum_{\omega \in W}(\hat{\alpha} - \alpha)^2/\alpha$	0.04	0.03	0.65	1.32	3.68

Tab. 6.12: Exceedances ratio α for different portfolios. *RiskMetrics* approach. 2-dim data: SIE and EOA (from 1-Jan-2001 to 30-Dec-2005)

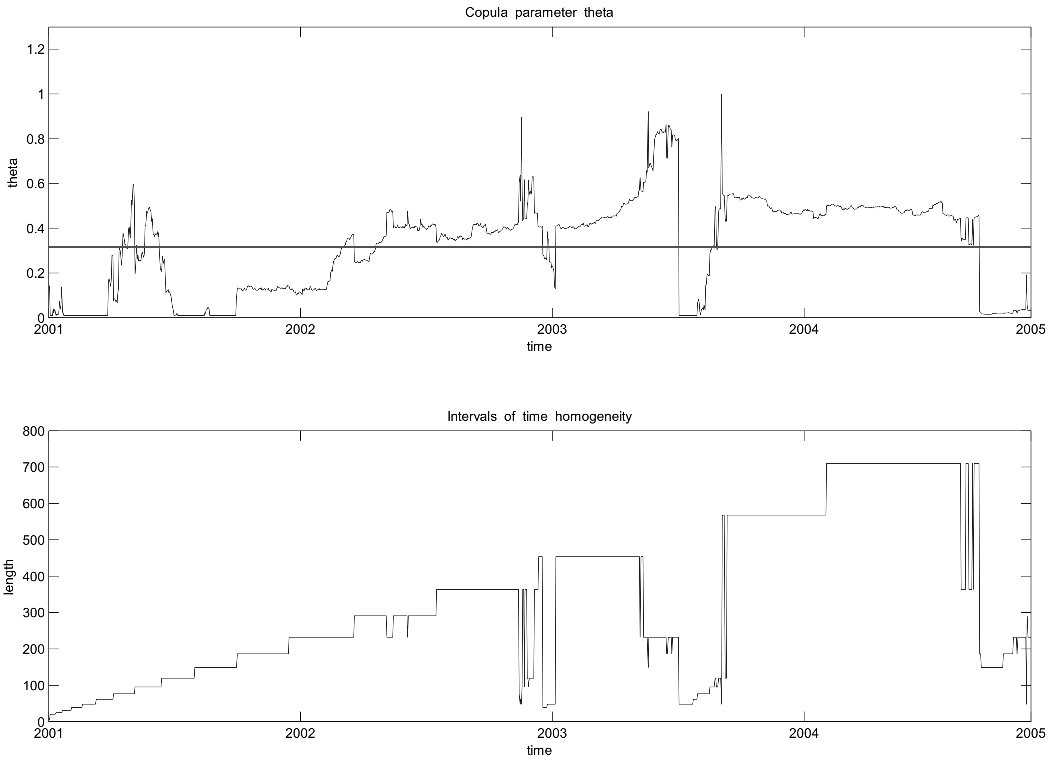


Fig. 6.9: Upper panel: estimated copula dependence parameter θ for 2-dim data:SIE and EOA (from 1-Jan-2001 to 30-Dec-2005). Lower panel: estimated intervals of time homogeneity.

Portfolio	Exceedances ratio α (in %) 5%	4%	3%	2%	1%
1	3.87	3.16	2.58	2.01	1.36
2	4.73	3.44	2.30	1.65	1.00
3	4.73	3.37	2.30	1.65	1.00
4	4.66	3.30	2.30	1.65	1.00
5	4.73	3.66	2.37	1.51	1.08
6	4.81	3.59	2.37	1.58	1.00
7	4.59	3.44	2.44	1.43	1.00
8	4.52	3.52	2.44	1.65	1.08
9	4.30	3.44	2.37	1.65	1.00
10	4.23	3.16	2.44	1.65	1.08
avg.	4.52	3.41	2.39	1.64	1.06
std.dev.	0.28	0.16	0.09	0.14	0.11
$\sum_{\omega \in W}(\hat{\alpha}-\alpha)^2$	0.03	0.04	0.04	0.01	0.00
$\sum_{\omega \in W}(\hat{\alpha}-\alpha)^2/\alpha$	0.62	0.94	1.27	0.74	0.15

Tab. 6.13: Exceedances ratio α for different portfolios. Clayton copula. Adaptive estimation procedure. 2-dim data: SIE and SCH (from 1-Jan-2001 to 30-Dec-2005)

Portfolio	Exceedances ratio α (in %) 5%	4%	3%	2%	1%
1	5.40	3.98	3.47	2.70	1.80
2	5.14	4.37	3.34	2.06	1.41
3	5.14	4.24	2.96	1.93	1.41
4	5.14	4.24	3.08	1.80	1.41
5	5.01	3.98	3.08	1.80	1.29
6	4.88	3.86	2.96	2.06	1.29
7	4.88	3.86	2.96	2.06	1.29
8	4.76	3.73	2.96	1.93	1.29
9	4.50	3.60	2.96	1.80	1.29
10	4.76	2.96	1.93	1.54	1.16
avg.	4.96	3.88	2.97	1.97	1.36
std.dev.	0.25	0.38	0.39	0.29	0.16
$\sum_{\omega \in W}(\hat{\alpha}-\alpha)^2$	0.01	0.02	0.02	0.01	0.02
$\sum_{\omega \in W}(\hat{\alpha}-\alpha)^2/\alpha$	0.12	0.41	0.50	0.42	1.58

Tab. 6.14: Exceedances ratio α for different portfolios. *RiskMetrics* approach. 2-dim data: SIE and SCH (from 1-Jan-2001 to 30-Dec-2005)

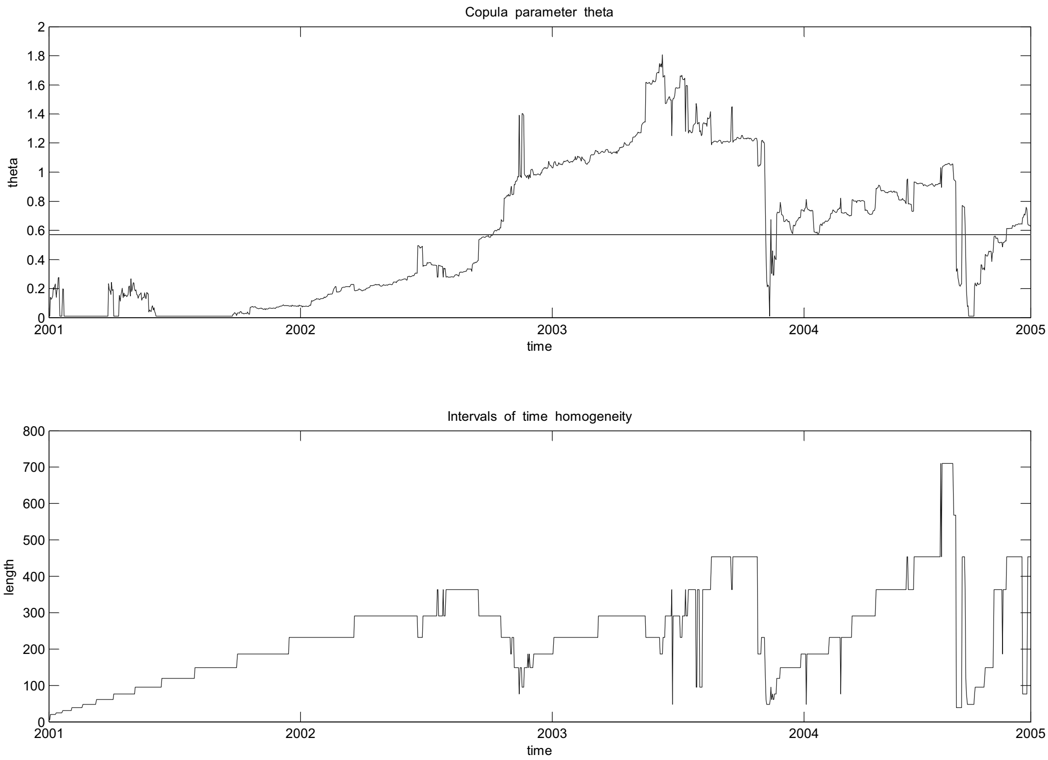


Fig. 6.10: Upper panel: estimated copula dependence parameter θ for 2-dim data:SIE and SCH (from 1-Jan-2001 to 30-Dec-2005). Lower panel: estimated intervals of time homogeneity.

Portfolio	Exceedances ratio α (in %) 5%	4%	3%	2%	1%
1	0.79	0.79	0.57	0.43	0.22
2	1.08	0.86	0.65	0.43	0.22
3	1.22	0.86	0.72	0.50	0.29
4	1.29	0.93	0.72	0.57	0.29
5	1.29	0.93	0.79	0.57	0.29
6	1.65	1.15	0.93	0.65	0.36
7	2.73	1.94	1.58	1.08	0.50
8	3.80	2.87	2.37	1.43	0.86
9	4.23	3.08	2.44	1.72	0.86
10	4.30	3.16	2.37	1.72	0.93
avg.	2.24	1.66	1.31	0.91	0.48
std.dev.	1.33	0.96	0.75	0.50	0.28
$\sum_{\omega\in W}(\hat{\alpha}-\alpha)^2$	0.94	0.64	0.34	0.14	0.03
$\sum_{\omega\in W}(\hat{\alpha}-\alpha)^2/\alpha$	18.77	16.01	11.38	7.20	3.46

Tab. 6.15: Exceedances ratio α for different portfolios. Clayton copula. Adaptive estimation procedure. 2-dim data: SIE and HEN (from 1-Jan-2001 to 30-Dec-2005)

Portfolio	Exceedances ratio α (in %) 5%	4%	3%	2%	1%
1	4.37	4.11	3.08	1.80	1.29
2	4.76	3.73	2.83	2.19	1.16
3	4.88	3.73	2.70	1.67	1.03
4	4.76	3.73	2.70	1.67	1.03
5	4.88	3.47	2.57	1.93	0.90
6	4.88	3.21	2.57	1.93	0.90
7	4.76	3.21	2.31	2.06	0.90
8	4.50	3.34	2.06	1.54	1.03
9	4.50	3.21	2.06	1.54	1.03
10	4.50	3.21	1.93	1.67	1.16
avg.	4.68	3.50	2.48	1.80	1.04
std.dev.	0.18	0.30	0.36	0.21	0.12
$\sum_{\omega\in W}(\hat{\alpha}-\alpha)^2$	0.01	0.03	0.04	0.01	0.00
$\sum_{\omega\in W}(\hat{\alpha}-\alpha)^2/\alpha$	0.27	0.86	1.33	0.42	0.16

Tab. 6.16: Exceedances ratio α for different portfolios. *RiskMetrics* approach. 2-dim data: SIE and HEN (from 1-Jan-2001 to 30-Dec-2005)

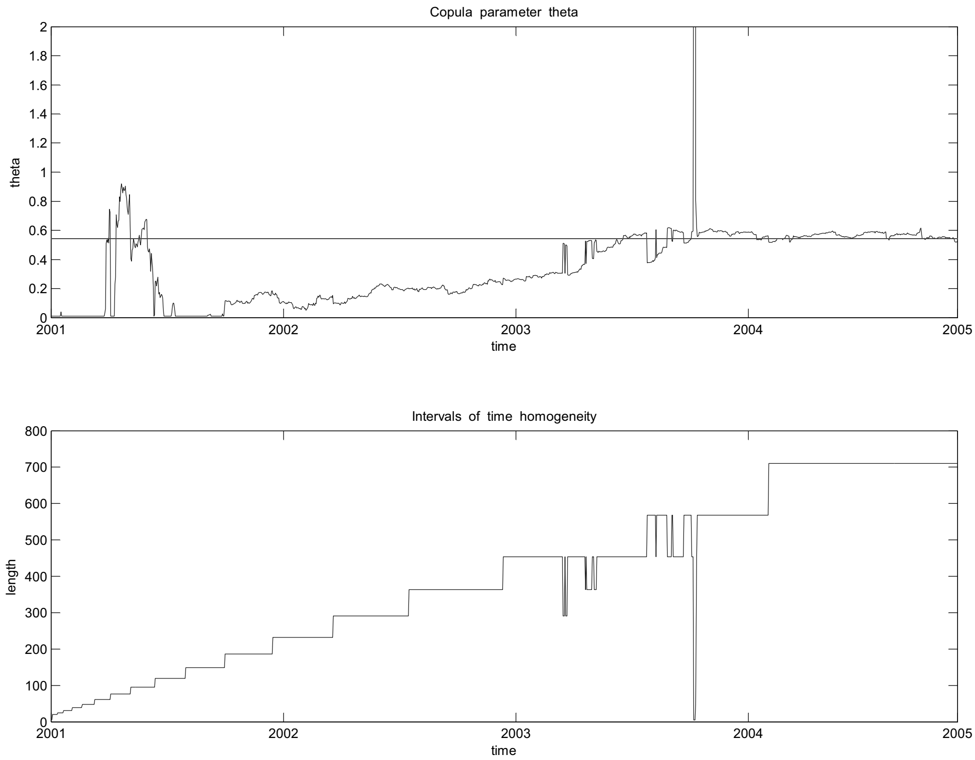


Fig. 6.11: Upper panel: estimated copula dependence parameter θ for 2-dim data:SIE and HEN (from 1-Jan-2001 to 30-Dec-2005). Lower panel: estimated intervals of time homogeneity.

Portfolio	Exceedances ratio α (in %) 5%	4%	3%	2%	1%
1	2.30	1.72	1.43	0.93	0.57
2	3.66	2.80	1.65	1.08	0.79
3	3.66	2.73	2.22	1.79	0.86
4	3.66	2.80	2.30	1.79	0.86
5	3.73	2.87	2.37	1.72	0.86
6	3.87	2.94	2.44	1.72	0.93
7	4.02	3.01	2.58	1.79	1.00
8	4.38	3.08	2.51	1.72	1.08
9	4.38	3.16	2.44	1.72	1.08
10	4.30	3.08	2.37	1.72	1.08
avg.	3.79	2.82	2.23	1.60	0.91
std.dev.	0.57	0.39	0.36	0.30	0.15
$\sum_{\omega \in W} (\hat{\alpha} - \alpha)^2$	0.18	0.15	0.07	0.03	0.00
$\sum_{\omega \in W} (\hat{\alpha} - \alpha)^2 / \alpha$	3.56	3.87	2.41	1.25	0.31

Tab. 6.17: Exceedances ratio α for different portfolios. Clayton copula. Adaptive estimation procedure. 2-dim data: SIE and LHA (from 1-Jan-2001 to 30-Dec-2005)

Portfolio	Exceedances ratio α (in %) 5%	4%	3%	2%	1%
1	5.40	4.24	3.08	1.54	0.77
2	5.27	4.24	2.96	1.93	0.64
3	5.40	3.86	3.08	1.67	0.77
4	5.40	3.86	2.83	1.67	0.90
5	5.14	3.86	2.06	1.93	1.03
6	4.63	3.21	2.19	1.54	1.16
7	4.76	3.21	1.93	1.54	1.16
8	4.76	3.21	1.93	1.54	1.16
9	4.50	3.08	1.93	1.67	1.16
10	4.50	3.08	1.93	1.67	1.16
avg.	4.97	3.59	2.39	1.67	0.99
std.dev.	0.36	0.45	0.50	0.14	0.19
$\sum_{\omega \in W} (\hat{\alpha} - \alpha)^2$	0.01	0.04	0.06	0.01	0.00
$\sum_{\omega \in W} (\hat{\alpha} - \alpha)^2 / \alpha$	0.27	0.93	2.07	0.64	0.37

Tab. 6.18: Exceedances ratio α for different portfolios. *RiskMetrics* approach. 2-dim data: SIE and LHA (from 1-Jan-2001 to 30-Dec-2005)

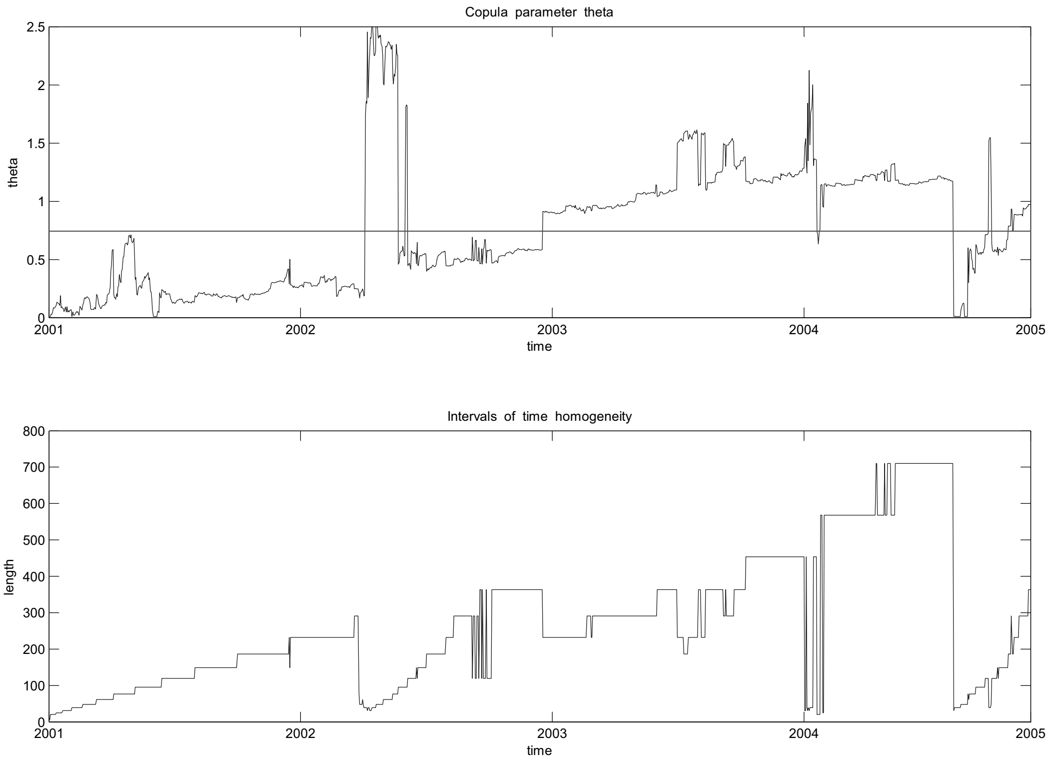


Fig. 6.12: Upper panel: estimated copula dependence parameter θ for 2-dim data:SIE and LHA (from 1-Jan-2001 to 30-Dec-2005). Lower panel: estimated intervals of time homogeneity.

Portfolio	Exceedances ratio α (in %) 5%	4%	3%	2%	1%
1	5.67	4.52	3.52	2.44	1.87
2	6.46	5.31	4.02	3.01	1.58
3	6.17	5.24	4.16	3.16	1.94
4	6.17	5.31	4.16	3.08	2.01
5	6.03	5.16	4.02	3.23	1.94
6	6.03	5.09	4.02	3.23	1.94
7	5.95	5.09	3.95	3.16	2.01
8	5.81	4.81	3.87	3.30	2.22
9	5.81	4.81	3.87	3.30	2.22
10	5.74	4.88	3.87	3.30	2.22
avg.	5.98	5.02	3.95	3.12	1.99
std.dev.	0.23	0.25	0.18	0.24	0.19
$\sum_{\omega \in W} (\hat{\alpha} - \alpha)^2$	0.10	0.11	0.09	0.13	0.10
$\sum_{\omega \in W} (\hat{\alpha} - \alpha)^2 / \alpha$	2.04	2.76	3.08	6.58	10.24

Tab. 6.19: Exceedances ratio α for different portfolios. Clayton copula. Adaptive estimation procedure. 2-dim data: THY and EOA (from 1-Jan-2001 to 30-Dec-2005)

Portfolio	Exceedances ratio α (in %) 5%	4%	3%	2%	1%
1	4.88	3.98	3.47	2.96	2.06
2	5.27	3.98	3.47	2.96	1.93
3	5.40	4.11	3.21	2.96	1.93
4	5.40	4.11	3.34	2.96	1.93
5	5.53	4.63	3.73	2.44	1.67
6	5.40	4.88	3.60	2.44	1.80
7	5.27	4.76	3.73	2.44	1.67
8	5.14	4.37	3.47	2.57	1.54
9	5.40	4.24	3.47	2.44	1.54
10	5.66	4.50	3.60	2.44	1.41
avg.	5.33	4.36	3.51	2.66	1.75
std.dev.	0.20	0.31	0.15	0.24	0.20
$\sum_{\omega \in W} (\hat{\alpha} - \alpha)^2$	0.02	0.02	0.03	0.05	0.06
$\sum_{\omega \in W} (\hat{\alpha} - \alpha)^2 / \alpha$	0.30	0.55	0.94	2.48	6.00

Tab. 6.20: Exceedances ratio α for different portfolios. *RiskMetrics* approach. 2-dim data: THY and EOA (from 1-Jan-2001 to 30-Dec-2005)

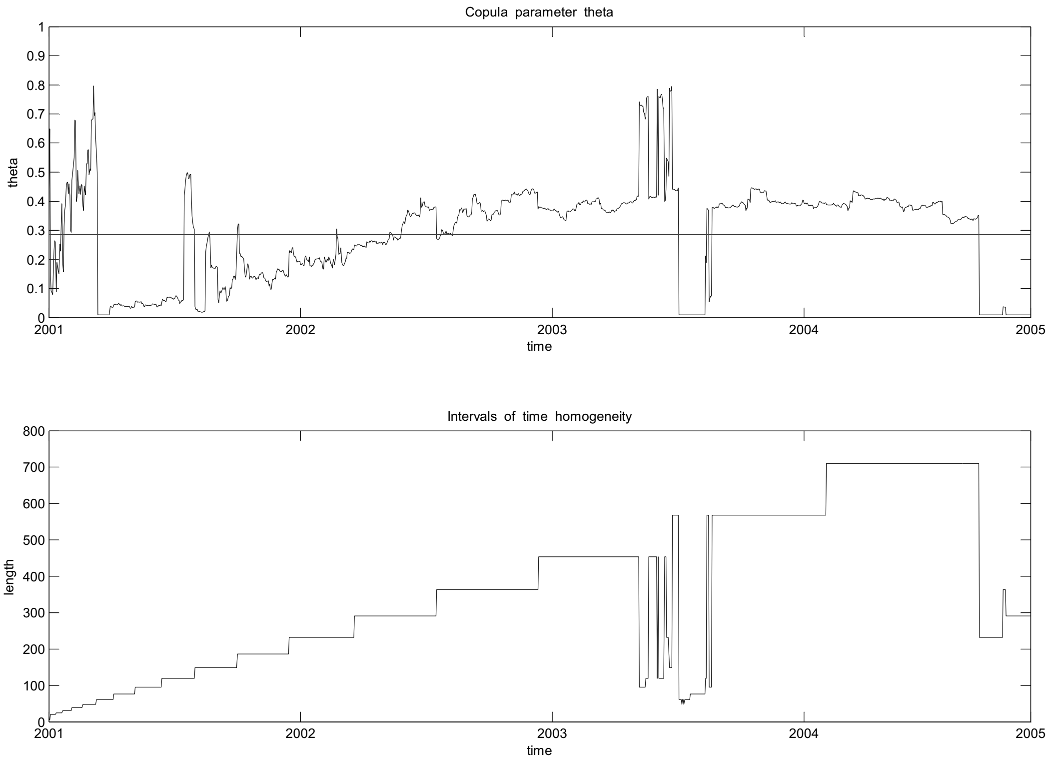


Fig. 6.13: Upper panel: estimated copula dependence parameter θ for 2-dim data: THY and EOA (from 1-Jan-2001 to 30-Dec-2005). Lower panel: estimated intervals of time homogeneity.

Portfolio	Exceedances ratio α (in %) 5%	4%	3%	2%	1%
1	4.38	3.44	2.80	2.30	1.58
2	4.45	3.37	2.87	2.51	1.36
3	4.30	3.37	2.87	2.44	1.36
4	4.38	3.37	2.87	2.51	1.36
5	4.02	3.44	2.87	2.37	1.22
6	4.02	3.52	2.87	2.37	1.22
7	4.02	3.52	2.80	2.22	1.22
8	3.87	3.23	2.73	1.72	1.08
9	3.37	2.80	2.01	1.36	1.15
10	3.30	2.51	2.15	1.65	1.00
avg.	4.01	3.26	2.68	2.14	1.26
std.dev.	0.38	0.32	0.31	0.39	0.16
$\sum_{\omega \in W}(\hat{\alpha}-\alpha)^2$	0.11	0.07	0.02	0.02	0.01
$\sum_{\omega \in W}(\hat{\alpha}-\alpha)^2/\alpha$	2.26	1.63	0.65	0.86	0.90

Tab. 6.21: Exceedances ratio α for different portfolios. Clayton copula. Adaptive estimation procedure. 2-dim data: THY and SCH (from 1-Jan-2001 to 30-Dec-2005)

Portfolio	Exceedances ratio α (in %) 5%	4%	3%	2%	1%
1	5.40	5.01	4.11	2.96	1.93
2	5.27	5.01	4.11	2.96	1.93
3	5.27	5.01	4.11	2.96	1.93
4	5.27	4.76	3.60	3.08	1.93
5	5.66	5.01	3.86	2.57	1.54
6	5.66	5.14	3.86	2.44	1.54
7	5.40	4.63	3.86	2.31	1.41
8	5.66	4.88	4.11	2.57	1.29
9	5.40	4.24	3.73	2.44	1.80
10	5.66	4.50	3.73	2.19	1.80
avg.	5.46	4.82	3.91	2.65	1.71
std.dev.	0.17	0.27	0.18	0.30	0.23
$\sum_{\omega \in W}(\hat{\alpha}-\alpha)^2$	0.02	0.07	0.09	0.05	0.06
$\sum_{\omega \in W}(\hat{\alpha}-\alpha)^2/\alpha$	0.48	1.87	2.86	2.55	5.56

Tab. 6.22: Exceedances ratio α for different portfolios. *RiskMetrics* approach. 2-dim data: THY and SCH (from 1-Jan-2001 to 30-Dec-2005)

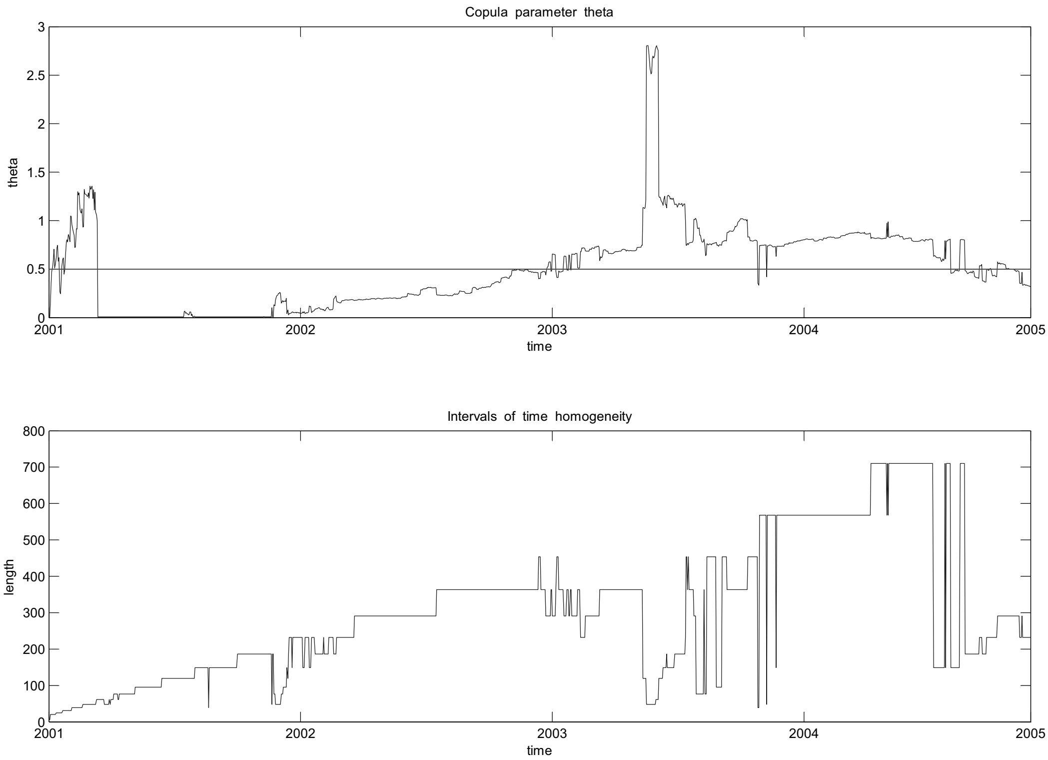


Fig. 6.14: Upper panel: estimated copula dependence parameter θ for 2-dim data: THY and SCH (from 1-Jan-2001 to 30-Dec-2005). Lower panel: estimated intervals of time homogeneity.

Portfolio	Exceedances ratio α (in %) 5%	4%	3%	2%	1%
1	0.79	0.65	0.57	0.43	0.22
2	0.79	0.65	0.57	0.43	0.22
3	0.72	0.65	0.65	0.36	0.22
4	0.72	0.65	0.65	0.36	0.22
5	0.72	0.65	0.65	0.36	0.22
6	0.93	0.72	0.65	0.36	0.22
7	0.93	0.93	0.57	0.43	0.22
8	0.93	0.93	0.57	0.43	0.29
9	1.00	0.93	0.65	0.36	0.29
10	1.15	0.86	0.57	0.36	0.29
avg.	0.87	0.76	0.61	0.39	0.24
std.dev.	0.14	0.13	0.04	0.04	0.03
$\sum_{\omega \in W}(\hat{\alpha}-\alpha)^2$	1.71	1.05	0.57	0.26	0.06
$\sum_{\omega \in W}(\hat{\alpha}-\alpha)^2/\alpha$	34.18	26.28	19.05	13.01	5.84

Tab. 6.23: Exceedances ratio α for different portfolios. Clayton copula. Adaptive estimation procedure. 2-dim data: THY and HEN (from 1-Jan-2001 to 30-Dec-2005)

Portfolio	Exceedances ratio α (in %) 5%	4%	3%	2%	1%
1	4.37	3.60	3.08	2.19	1.16
2	4.37	3.86	3.08	2.31	1.16
3	4.37	3.86	3.08	2.31	1.16
4	4.76	3.98	3.60	2.31	1.29
5	4.63	4.24	3.60	2.57	1.29
6	5.53	4.88	3.34	2.83	1.67
7	5.53	4.88	3.47	2.83	1.67
8	5.14	4.88	4.11	3.21	1.80
9	5.53	5.01	4.37	2.96	1.67
10	5.78	5.14	3.86	2.57	1.67
avg.	5.00	4.43	3.56	2.61	1.45
std.dev.	0.53	0.55	0.42	0.32	0.25
$\sum_{\omega \in W}(\hat{\alpha}-\alpha)^2$	0.03	0.05	0.05	0.05	0.03
$\sum_{\omega \in W}(\hat{\alpha}-\alpha)^2/\alpha$	0.57	1.23	1.64	2.37	2.68

Tab. 6.24: Exceedances ratio α for different portfolios. *RiskMetrics* approach. 2-dim data: THY and HEN (from 1-Jan-2001 to 30-Dec-2005)

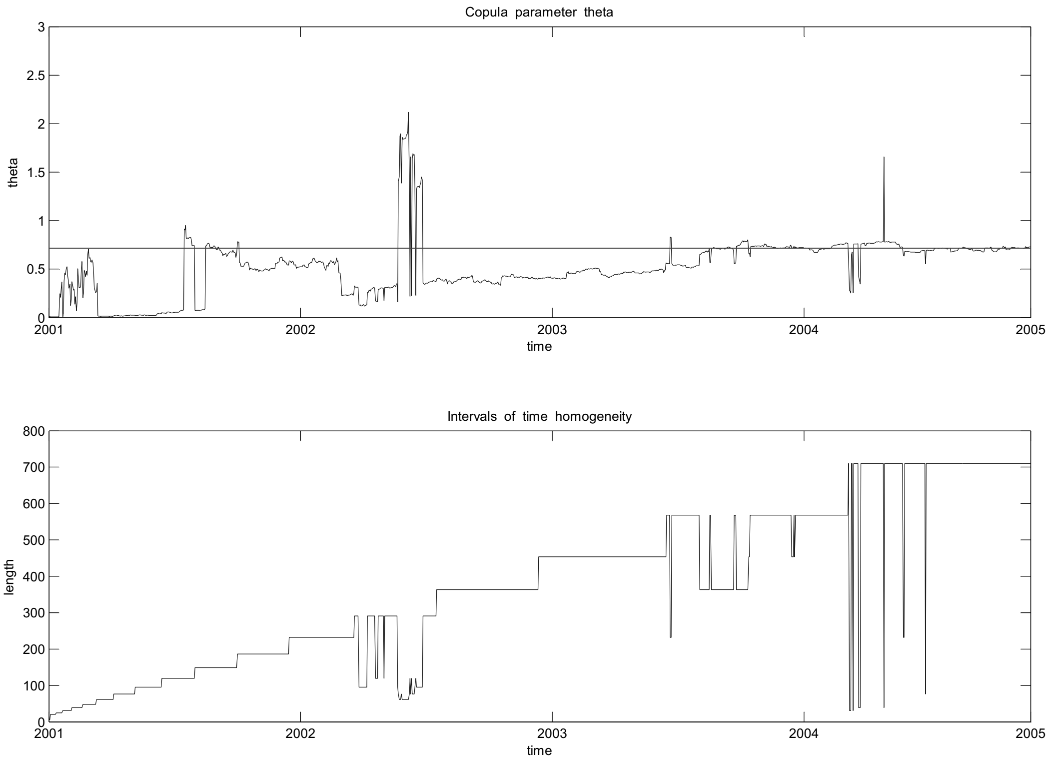


Fig. 6.15: Upper panel: estimated copula dependence parameter θ for 2-dim data: THY and HEN (from 1-Jan-2001 to 30-Dec-2005). Lower panel: estimated intervals of time homogeneity.

Portfolio	Exceedances ratio α (in %) 5%	4%	3%	2%	1%
	2.73	2.01	1.51	0.93	0.65
	2.87	2.30	2.01	1.29	0.86
	2.80	2.37	1.94	1.29	0.79
	2.87	2.22	1.94	1.51	1.00
	2.94	2.22	1.94	1.51	1.00
	2.94	2.37	1.94	1.51	1.00
	2.80	2.30	1.94	1.51	1.08
	2.87	2.37	2.01	1.51	1.00
	3.08	2.44	2.15	1.79	1.15
	3.08	2.44	2.15	1.79	1.22
avg.	2.90	2.30	1.95	1.46	0.98
std.dev.	0.11	0.12	0.17	0.24	0.16
$\sum_{\omega \in W}(\hat{\alpha}-\alpha)^2$	0.44	0.29	0.11	0.03	0.00
$\sum_{\omega \in W}(\hat{\alpha}-\alpha)^2/\alpha$	8.86	7.24	3.76	1.72	0.27

Tab. 6.25: Exceedances ratio α for different portfolios. Clayton copula. Adaptive estimation procedure. 2-dim data: THY and LHA (from 1-Jan-2001 to 30-Dec-2005)

Portfolio	Exceedances ratio α (in %) 5%	4%	3%	2%	1%
	4.37	3.98	3.34	2.57	1.67
	5.01	4.50	3.73	2.83	2.19
	5.14	4.50	3.86	2.83	2.06
	5.53	4.88	3.73	2.96	1.93
	5.78	5.27	3.73	2.70	1.93
	5.78	5.27	3.86	2.70	1.93
	5.91	5.27	3.47	2.44	2.06
	5.40	4.50	3.47	2.31	1.67
	5.40	4.50	3.34	2.31	1.67
	5.40	4.50	3.34	2.31	1.67
avg.	5.37	4.72	3.59	2.60	1.88
std.dev.	0.43	0.41	0.20	0.23	0.18
$\sum_{\omega \in W}(\hat{\alpha}-\alpha)^2$	0.03	0.07	0.04	0.04	0.08
$\sum_{\omega \in W}(\hat{\alpha}-\alpha)^2/\alpha$	0.65	1.72	1.28	2.04	8.02

Tab. 6.26: Exceedances ratio α for different portfolios. *RiskMetrics* approach. 2-dim data: THY and LHA (from 1-Jan-2001 to 30-Dec-2005)

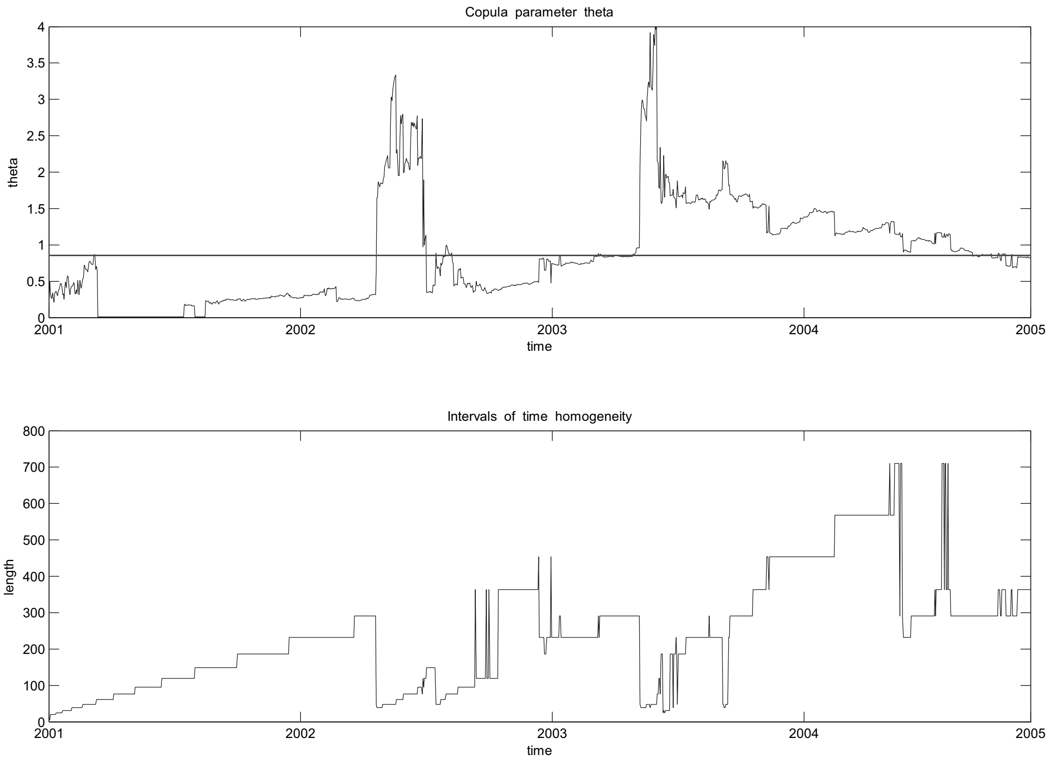


Fig. 6.16: Upper panel: estimated copula dependence parameter θ for 2-dim data: THY and LHA (from 1-Jan-2001 to 30-Dec-2005). Lower panel: estimated intervals of time homogeneity.

Portfolio	Exceedances ratio α (in %) 5%	4%	3%	2%	1%
1	7.17	6.03	5.31	3.87	2.80
2	8.39	7.17	5.81	4.02	2.94
3	8.68	7.46	5.74	4.66	3.23
4	8.32	7.75	6.53	4.88	3.08
5	8.90	7.68	6.38	4.52	2.87
6	8.61	7.39	6.31	4.45	2.94
7	8.54	7.17	5.74	4.16	3.01
8	8.25	7.25	5.45	4.23	2.94
9	8.25	7.10	5.52	4.23	2.73
10	8.25	7.10	5.52	4.16	2.73
avg.	8.34	7.21	5.83	4.32	2.93
std.dev.	0.44	0.45	0.41	0.29	0.15
$\sum_{\omega \in W} (\hat{\alpha} - \alpha)^2$	1.13	1.05	0.82	0.55	0.37
$\sum_{\omega \in W} (\hat{\alpha} - \alpha)^2 / \alpha$	22.64	26.26	27.29	27.30	37.35

Tab. 6.27: Exceedances ratio α for different portfolios. Clayton copula. Adaptive estimation procedure. 2-dim data: EOA and SCH (from 1-Jan-2001 to 30-Dec-2005)

Portfolio	Exceedances ratio α (in %) 5%	4%	3%	2%	1%
1	5.53	4.88	3.60	2.70	1.54
2	5.40	4.76	3.86	2.70	1.67
3	5.40	4.76	3.98	2.70	1.67
4	5.53	5.14	4.11	2.44	1.54
5	5.66	5.27	4.24	2.44	1.41
6	6.43	5.01	3.86	2.44	1.67
7	6.56	5.27	3.86	2.31	1.67
8	5.66	4.76	3.47	2.44	1.54
9	5.14	4.63	3.47	2.83	1.41
10	5.01	3.98	3.47	2.83	2.06
avg.	5.63	4.85	3.79	2.58	1.62
std.dev.	0.47	0.36	0.26	0.18	0.17
$\sum_{\omega \in W} (\hat{\alpha} - \alpha)^2$	0.06	0.08	0.07	0.04	0.04
$\sum_{\omega \in W} (\hat{\alpha} - \alpha)^2 / \alpha$	1.24	2.11	2.32	1.86	4.14

Tab. 6.28: Exceedances ratio α for different portfolios. *RiskMetrics* approach. 2-dim data: EOA and SCH (from 1-Jan-2001 to 30-Dec-2005)

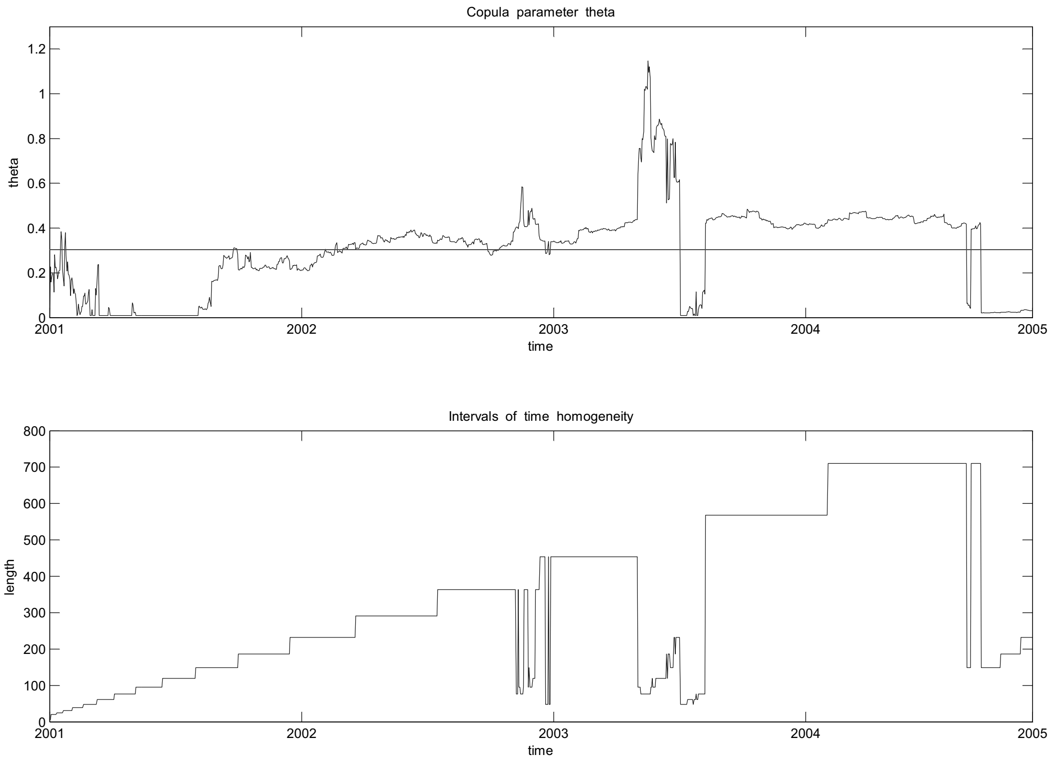


Fig. 6.17: Upper panel: estimated copula dependence parameter θ for 2-dim data: EOA and SCH (from 1-Jan-2001 to 30-Dec-2005). Lower panel: estimated intervals of time homogeneity.

Portfolio	Exceedances ratio α (in %) 5%	4%	3%	2%	1%
1	0.86	0.79	0.65	0.36	0.29
2	0.86	0.79	0.65	0.36	0.29
3	0.79	0.72	0.57	0.36	0.29
4	0.79	0.72	0.57	0.29	0.29
5	0.79	0.79	0.65	0.29	0.29
6	1.00	0.79	0.50	0.43	0.36
7	1.00	0.72	0.50	0.43	0.36
8	1.00	0.65	0.57	0.43	0.36
9	1.72	1.29	1.15	0.72	0.36
10	2.51	1.87	1.51	1.08	0.79
avg.	1.13	0.91	0.73	0.47	0.37
std.dev.	0.53	0.36	0.31	0.23	0.15
$\sum_{\omega \in W}(\hat{\alpha}-\alpha)^2$	1.52	0.97	0.52	0.24	0.04
$\sum_{\omega \in W}(\hat{\alpha}-\alpha)^2/\alpha$	30.46	24.18	17.48	11.92	4.23

Tab. 6.29: Exceedances ratio α for different portfolios. Clayton copula. Adaptive estimation procedure. 2-dim data: EOA and HEN (from 1-Jan-2001 to 30-Dec-2005)

Portfolio	Exceedances ratio α (in %) 5%	4%	3%	2%	1%
1	4.50	3.98	3.34	2.19	1.03
2	4.88	3.86	3.47	2.31	1.41
3	4.88	3.86	3.47	2.19	1.41
4	5.91	5.14	3.86	2.96	1.80
5	5.91	5.14	4.11	2.70	1.93
6	5.40	4.37	3.47	2.83	1.67
7	5.27	4.24	3.47	2.83	1.93
8	5.40	4.11	3.08	2.57	2.31
9	5.27	4.37	2.96	2.70	2.31
10	5.27	4.37	3.08	2.70	2.19
avg.	5.27	4.34	3.43	2.60	1.80
std.dev.	0.42	0.44	0.34	0.26	0.40
$\sum_{\omega \in W}(\hat{\alpha}-\alpha)^2$	0.02	0.03	0.03	0.04	0.08
$\sum_{\omega \in W}(\hat{\alpha}-\alpha)^2/\alpha$	0.50	0.78	1.00	2.12	8.01

Tab. 6.30: Exceedances ratio α for different portfolios. *RiskMetrics* approach. 2-dim data: EOA and HEN (from 1-Jan-2001 to 30-Dec-2005)

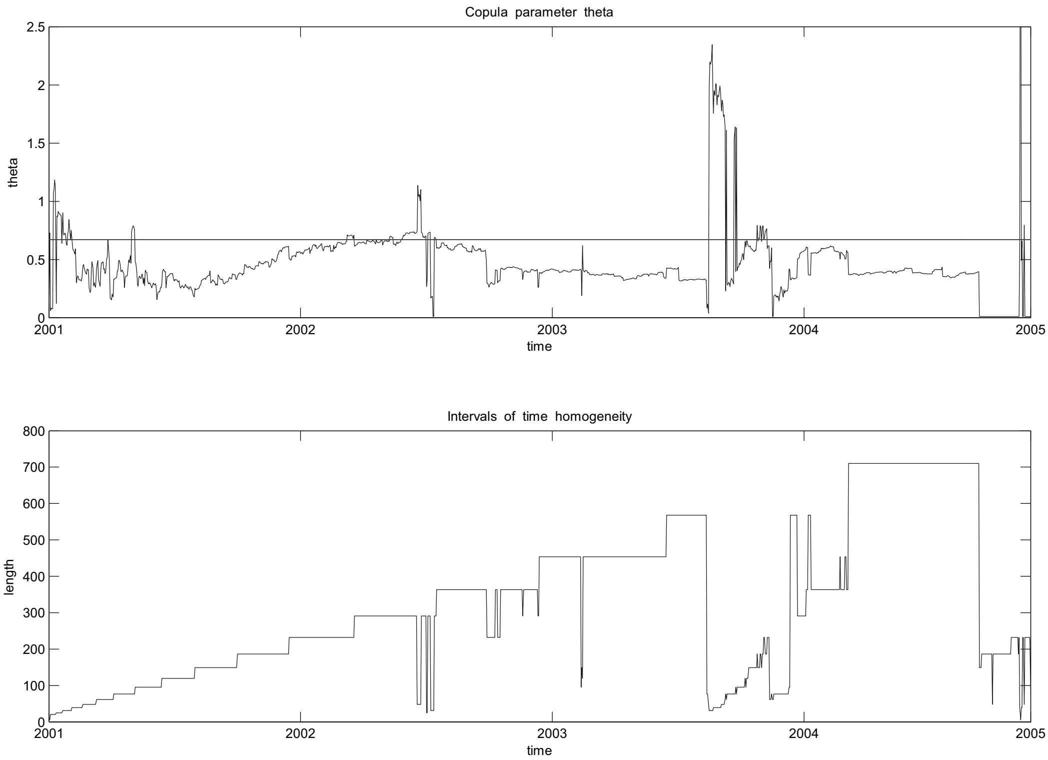


Fig. 6.18: Upper panel: estimated copula dependence parameter θ for 2-dim data: EOA and HEN (from 1-Jan-2001 to 30-Dec-2005). Lower panel: estimated intervals of time homogeneity.

Portfolio	Exceedances ratio α (in %) 5%	4%	3%	2%	1%
1	2.73	1.72	1.51	1.00	0.65
2	2.65	2.15	1.43	1.00	0.57
3	2.30	1.87	1.58	1.15	0.79
4	2.37	1.87	1.51	1.29	0.93
5	2.51	2.08	1.65	1.36	0.79
6	2.51	2.15	1.79	1.29	0.79
7	2.51	2.22	1.72	1.43	0.79
8	3.01	2.44	1.72	1.43	0.93
9	2.94	2.44	1.94	1.51	1.00
10	3.01	2.44	2.08	1.51	1.00
avg.	2.65	2.14	1.69	1.30	0.82
std.dev.	0.25	0.25	0.19	0.18	0.14
$\sum_{\omega \in W}(\hat{\alpha}-\alpha)^2$	0.56	0.35	0.17	0.05	0.00
$\sum_{\omega \in W}(\hat{\alpha}-\alpha)^2/\alpha$	11.13	8.82	5.82	2.62	0.49

Tab. 6.31: Exceedances ratio α for different portfolios. Clayton copula. Adaptive estimation procedure. 2-dim data: EOA and LHA (from 1-Jan-2001 to 30-Dec-2005)

Portfolio	Exceedances ratio α (in %) 5%	4%	3%	2%	1%
1	4.50	3.98	2.83	2.19	1.41
2	5.27	4.11	3.08	2.44	1.41
3	4.76	4.11	3.08	2.19	1.54
4	4.88	3.98	3.34	2.19	1.67
5	5.14	3.73	3.34	2.19	1.80
6	5.01	3.86	3.34	2.19	1.54
7	4.88	3.86	3.34	2.31	1.54
8	5.27	4.11	3.21	2.31	1.54
9	4.76	4.24	3.47	2.83	1.67
10	4.76	4.24	3.47	2.83	1.67
avg.	4.92	4.02	3.25	2.37	1.58
std.dev.	0.24	0.16	0.19	0.25	0.12
$\sum_{\omega \in W}(\hat{\alpha}-\alpha)^2$	0.01	0.00	0.01	0.02	0.04
$\sum_{\omega \in W}(\hat{\alpha}-\alpha)^2/\alpha$	0.12	0.07	0.33	0.97	3.51

Tab. 6.32: Exceedances ratio α for different portfolios. *RiskMetrics* approach. 2-dim data: EOA and LHA (from 1-Jan-2001 to 30-Dec-2005)

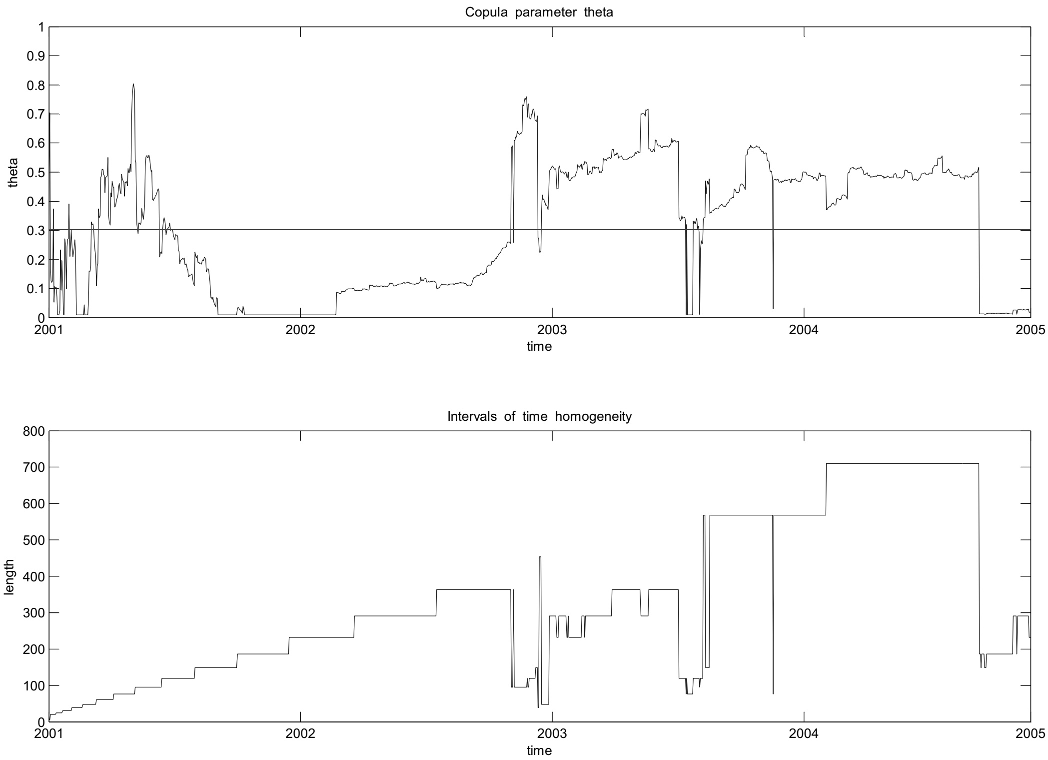


Fig. 6.19: Upper panel: estimated copula dependence parameter θ for 2-dim data: EOA and LHA (from 1-Jan-2001 to 30-Dec-2005). Lower panel: estimated intervals of time homogeneity.

Portfolio	Exceedances ratio α (in %) 5%	4%	3%	2%	1%
1	0.79	0.72	0.57	0.43	0.22
2	0.72	0.57	0.57	0.43	0.22
3	0.93	0.65	0.50	0.29	0.22
4	0.93	0.72	0.43	0.29	0.22
5	0.86	0.72	0.57	0.29	0.22
6	1.65	1.08	0.86	0.50	0.14
7	2.80	2.30	1.72	1.00	0.57
8	3.30	2.94	2.30	1.51	0.93
9	3.66	3.37	2.80	1.94	1.29
10	3.95	3.37	2.73	2.08	1.43
avg.	1.96	1.64	1.31	0.88	0.55
std.dev.	1.25	1.15	0.93	0.68	0.47
$\sum_{\omega\in W}(\hat{\alpha}-\alpha)^2$	1.08	0.69	0.37	0.17	0.04
$\sum_{\omega\in W}(\hat{\alpha}-\alpha)^2/\alpha$	21.63	17.17	12.44	8.61	4.27

Tab. 6.33: Exceedances ratio α for different portfolios. Clayton copula. Adaptive estimation procedure. 2-dim data: SCH and HEN (from 1-Jan-2001 to 30-Dec-2005)

Portfolio	Exceedances ratio α (in %) 5%	4%	3%	2%	1%
1	4.11	3.98	3.08	2.19	1.29
2	4.11	3.98	2.96	2.06	1.29
3	4.11	4.11	3.08	1.93	1.29
4	4.24	3.98	2.96	2.06	1.29
5	3.86	3.47	2.96	2.31	1.67
6	4.24	2.83	2.57	2.31	1.93
7	4.24	3.21	2.57	2.31	2.06
8	5.27	4.63	3.98	2.96	1.80
9	5.66	4.76	3.73	3.08	1.29
10	5.66	4.88	3.73	2.44	1.67
avg.	4.55	3.98	3.16	2.37	1.56
std.dev.	0.66	0.63	0.46	0.36	0.29
$\sum_{\omega\in W}(\hat{\alpha}-\alpha)^2$	0.06	0.04	0.02	0.03	0.04
$\sum_{\omega\in W}(\hat{\alpha}-\alpha)^2/\alpha$	1.27	1.01	0.81	1.31	3.92

Tab. 6.34: Exceedances ratio α for different portfolios. *RiskMetrics* approach. 2-dim data: SCH and HEN (from 1-Jan-2001 to 30-Dec-2005)

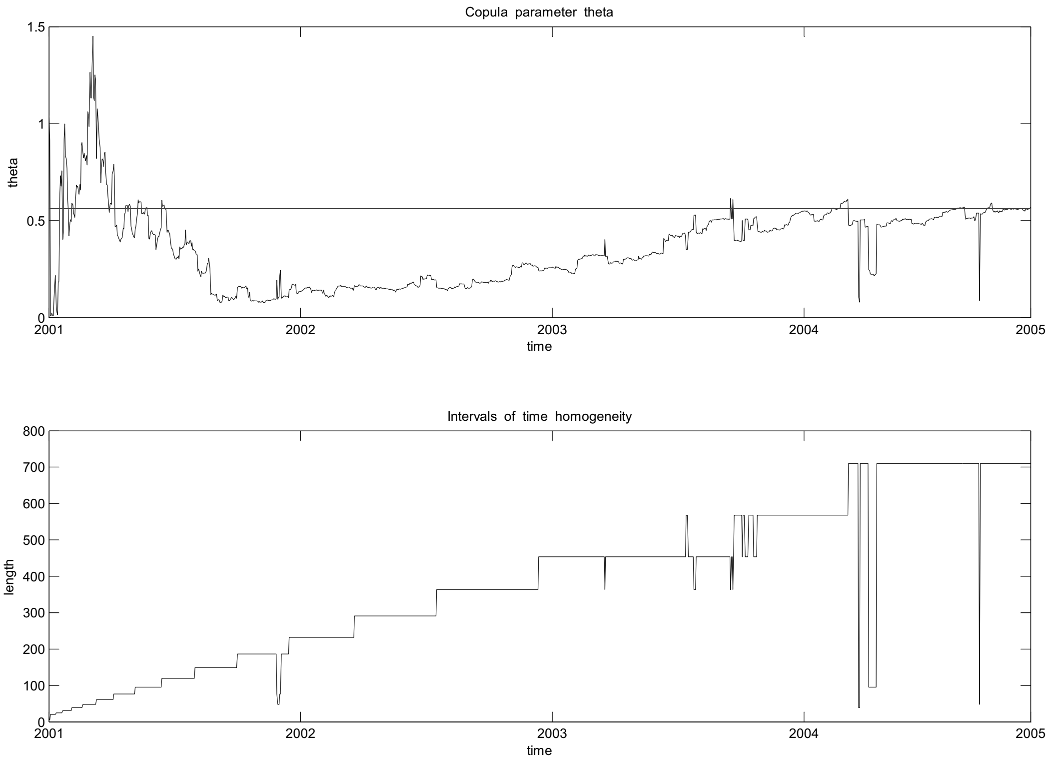


Fig. 6.20: Upper panel: estimated copula dependence parameter θ for 2-dim data: SCH and HEN (from 1-Jan-2001 to 30-Dec-2005). Lower panel: estimated intervals of time homogeneity.

Portfolio	Exceedances ratio α (in %) 5%	4%	3%	2%	1%
1	2.44	2.08	1.65	1.00	0.72
2	3.66	3.16	2.30	1.72	1.08
3	4.09	3.30	2.30	1.65	1.08
4	4.02	3.08	2.51	1.94	1.36
5	4.02	3.52	2.65	2.30	1.51
6	4.02	3.52	2.65	2.37	1.51
7	4.23	3.59	2.80	2.30	1.58
8	4.38	3.59	2.80	2.30	1.58
9	4.38	3.59	2.80	2.30	1.58
10	4.38	3.59	2.80	2.30	1.58
avg.	3.96	3.30	2.53	2.02	1.36
std.dev.	0.55	0.44	0.35	0.42	0.28
$\sum_{\omega\in W}(\hat{\alpha}-\alpha)^2$	0.14	0.07	0.03	0.02	0.02
$\sum_{\omega\in W}(\hat{\alpha}-\alpha)^2/\alpha$	2.77	1.72	1.15	0.88	2.07

Tab. 6.35: Exceedances ratio α for different portfolios. Clayton copula. Adaptive estimation procedure. 2-dim data: SCH and LHA (from 1-Jan-2001 to 30-Dec-2005)

Portfolio	Exceedances ratio α (in %) 5%	4%	3%	2%	1%
1	4.76	3.73	3.08	2.06	1.54
2	5.27	4.24	3.47	2.57	1.54
3	5.40	4.76	3.34	2.57	1.80
4	5.40	4.76	3.34	2.57	1.80
5	5.53	5.14	4.11	2.83	2.06
6	5.53	5.14	3.98	2.70	2.06
7	5.40	5.01	3.86	2.57	1.80
8	5.53	5.01	3.86	2.70	1.80
9	5.53	4.76	3.86	2.70	1.67
10	5.53	4.88	3.73	2.70	1.67
avg.	5.39	4.74	3.66	2.60	1.77
std.dev.	0.23	0.42	0.32	0.20	0.17
$\sum_{\omega\in W}(\hat{\alpha}-\alpha)^2$	0.02	0.07	0.05	0.04	0.06
$\sum_{\omega\in W}(\hat{\alpha}-\alpha)^2/\alpha$	0.40	1.82	1.80	1.97	6.28

Tab. 6.36: Exceedances ratio α for different portfolios. *RiskMetrics* approach. 2-dim data: SCH and LHA (from 1-Jan-2001 to 30-Dec-2005)

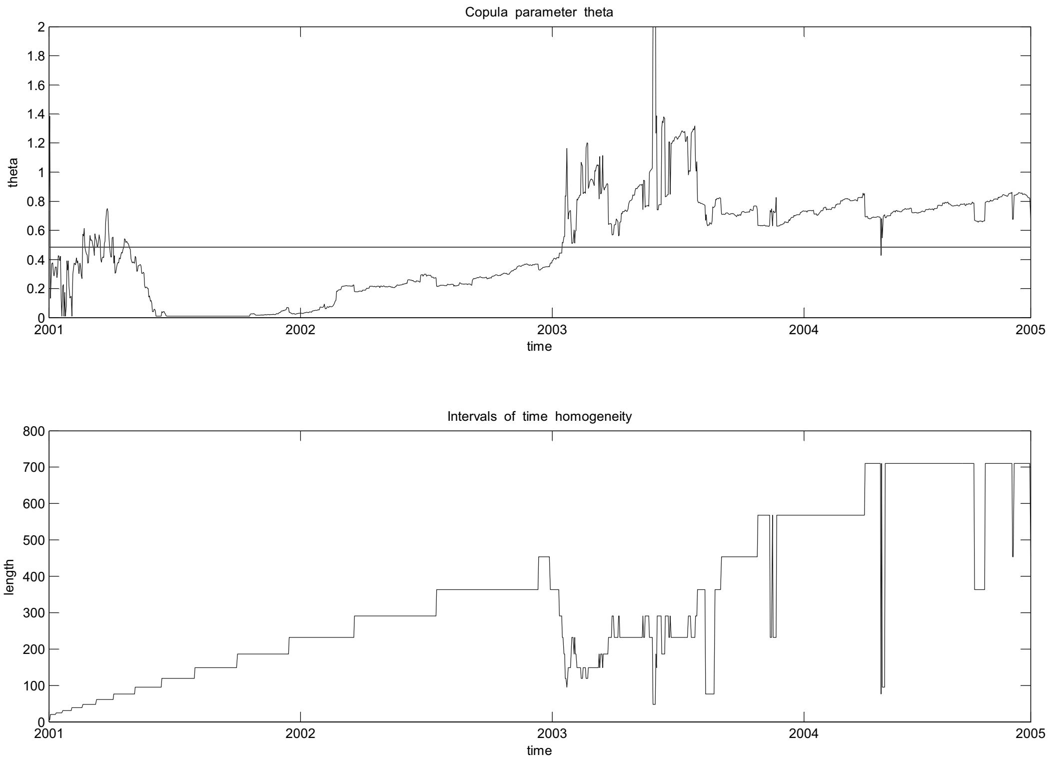


Fig. 6.21: Upper panel: estimated copula dependence parameter θ for 2-dim data: SCH and LHA (from 1-Jan-2001 to 30-Dec-2005). Lower panel: estimated intervals of time homogeneity.

Portfolio	Exceedances ratio α (in %) 5%	4%	3%	2%	1%
1	2.08	1.58	1.00	0.86	0.65
2	0.72	0.57	0.50	0.29	0.22
3	0.72	0.57	0.50	0.36	0.22
4	0.72	0.57	0.50	0.36	0.22
5	0.72	0.57	0.50	0.29	0.22
6	0.65	0.65	0.50	0.43	0.22
7	0.65	0.65	0.57	0.36	0.22
8	0.79	0.65	0.57	0.36	0.22
9	0.79	0.65	0.57	0.36	0.22
10	0.79	0.72	0.50	0.36	0.22
avg.	0.86	0.72	0.57	0.40	0.26
std.dev.	0.41	0.29	0.15	0.16	0.13
$\sum_{\omega \in W}(\hat{\alpha}-\alpha)^2$	1.73	1.09	0.59	0.26	0.06
$\sum_{\omega \in W}(\hat{\alpha}-\alpha)^2/\alpha$	34.60	27.15	19.69	12.90	5.67

Tab. 6.37: Exceedances ratio α for different portfolios. Clayton copula. Adaptive estimation procedure. 2-dim data: HEN and LHA (from 1-Jan-2001 to 30-Dec-2005)

Portfolio	Exceedances ratio α (in %) 5%	4%	3%	2%	1%
1	4.76	3.73	2.83	1.93	1.41
2	5.78	5.14	3.47	2.19	1.03
3	5.66	4.76	3.47	2.57	1.16
4	5.66	4.50	3.34	2.70	1.03
5	5.14	4.11	3.60	2.96	1.03
6	4.76	4.37	3.73	2.44	0.90
7	5.01	4.24	3.34	2.19	0.90
8	4.50	3.86	3.60	2.19	1.16
9	4.50	3.86	3.08	2.19	1.16
10	4.37	3.73	3.21	2.19	1.16
avg.	5.01	4.23	3.37	2.35	1.09
std.dev.	0.50	0.45	0.26	0.29	0.14
$\sum_{\omega \in W}(\hat{\alpha}-\alpha)^2$	0.03	0.03	0.02	0.02	0.00
$\sum_{\omega \in W}(\hat{\alpha}-\alpha)^2/\alpha$	0.50	0.63	0.67	1.05	0.29

Tab. 6.38: Exceedances ratio α for different portfolios. *RiskMetrics* approach. 2-dim data: HEN and LHA (from 1-Jan-2001 to 30-Dec-2005)

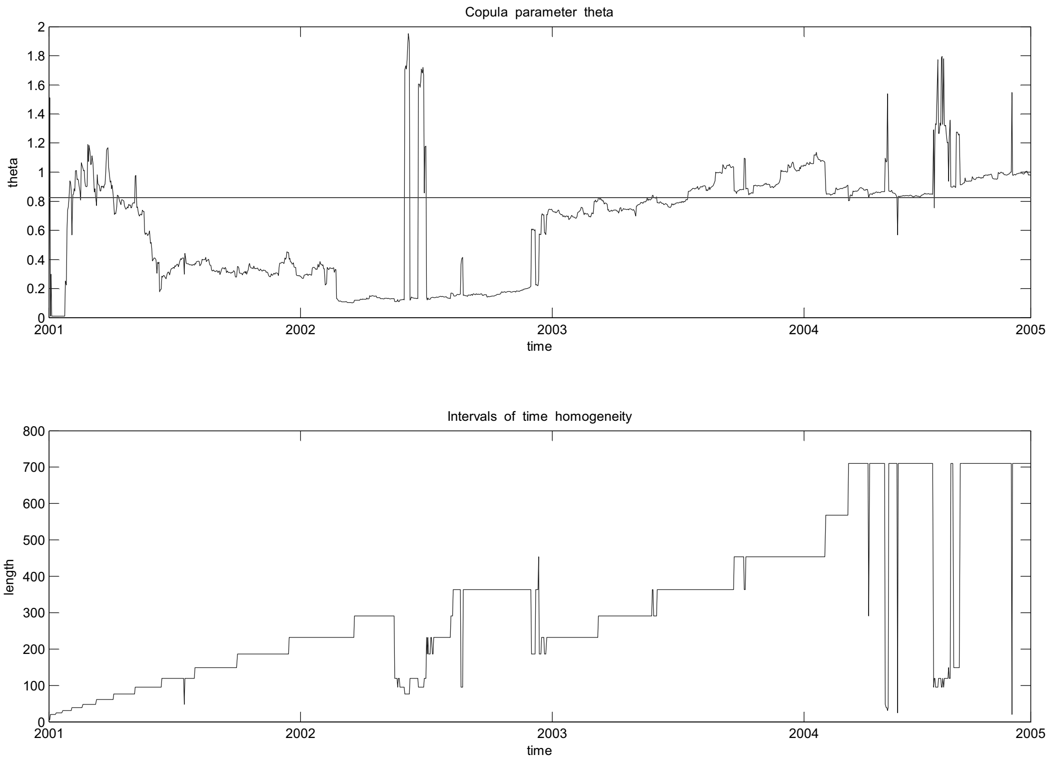


Fig. 6.22: Upper panel: estimated copula dependence parameter θ for 2-dim data: HEN and LHA (from 1-Jan-2001 to 30-Dec-2005). Lower panel: estimated intervals of time homogeneity.

7. CONCLUSION

In this work the Copula based approach was used to estimate *Value-at-Risk (VaR)* of a portfolio. The approach was based on the adaptive estimation procedure introduced by ***Mercurio and Spokoiny (2004)***. In ***Giacomini et al. (2007)*** the procedure is tested for different initial parameters as well as for the instantaneous jump in the dependence structure between data. Here, the procedure was tested for the graduate increase and decrease in the tightness of dependence. The results show that if the height of increase is not high enough, or if the increase and decrease intervals are not long enough, then the procedure performs a certain lag between the true parameter and the estimate.

Further, the Copula-based approach was implemented to model the dependence structure between the financial data in order to estimate VaR of a portfolio. The results of such approach were compared to results from a widely used *RiskMetrics* method developed by ***Morgan/Reuters (1996)*** and based on the assumption of jointly normally distributed innovations. The analysis was conducted for 6-asset portfolio and for 15 different sets of 2-asset portfolios. The performances of the methods were compared via backtesting procedure. The results are quite controversial. For the 6-asset portfolio Copula-based approach on average overestimates risk. At the same time, the *RiskMetrics* method underestimates risk. From the graph of behavior of the estimated dependence parameter for the Copula-based approach it is seen that the dependence structure between data is not very tight and the parameter on average is on the level of 0.7 (if the dependence parameter is close to zero, the data considered to be independent). Another issue to mention is that the intervals of homogeneity are very short, and we observe a lot of fluctuations in the estimator. The reason for such a result is that the dependence structure between the data is considered to be symmetric and is modeled with only one parameter. This assumption is very strong, as the dependence between different pairs of data might be different. That is why the same approach was used to analyze 2-asset portfolios. For some pairs of data the results of the copula-based approach outperformed the results from the *RiskMetrics* approach and for the others they did not. It is also seen from the given figures that the estimates of the dependence parameters for some pairs demonstrate homogeneous behavior on some intervals, while the other pairs show unhomogendous dependence structure. This can be explained by the wrong choice of copula for these particular pairs of data. That shows once again that modelling of the dependence between more than 2-dimensional data using one dependent parameter, is incorrect.

The Copula-based approach to model dependence between financial data is a new strong instrument, which gives a lot of flexibility for the researcher. It is an attractive field for experiments for experiments to create new methods for the explanation of financial instruments' behavior. In this work such an experiment was conducted to model the joint behavior of data over time. Another step to be done, is to model joint, but not symmetric behavior, which is closer to the real life.

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