Exponential of Lévy processes as a stock price - Arbitrage opportunities, completeness and derivatives valuation

Master Thesis submitted to

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in partial fulfillment of the requirements

for the degree of **Master of Sciences in Statistics**

Berlin, July 25, 2006

Declaration of Authorship

I hereby confirm that I have authored this master thesis independently and without use of others than the indicated sources. All passages which are literally or in general matter taken out of publications or other sources are marked as such.

Berlin, July 25, 2006

Marc Tisserand

I would like to thank Pr. Föllmer and Pr. Imkeller for all the lectures in Stochastic and Mathematical Finance I attended at the Humboldt-Universität. During my stay in Berlin, I have really appreciated their teaching skills.

I would like to thank Pr. Härdle who allows me to participate to the Master of Statistics program for his encouragement to improve my statistical and programming skills.

I would also like to thank Anne Gundel, Irina Penner, Stefan Weber and Thomas Knispel for helping me understand Mathematical Finance.

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1 The imperfections of the Black & Scholes Model

The beginning of modern Mathematical Finance can be attributed to Louis Bachelier, who in 1900 suggested to describe the price of an asset with the following process :

$$S_t = S_0 + \sigma W_t \quad 0 \leq t \leq T \tag{1}$$

Where W_t is a standard brownian motion.

The main drawback of this model is the fact that it allows prices to become negative. It is only 65 years later, that Samuelson suggested another model where the instantaneous returns of the stocks have a gaussian distribution. Later, this process will be one of the foundations of the famous Black and Scholes model.

$$S_t = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right] \tag{2}$$

If we want to stress the gaussian feature of the returns, the last equation can be written :

$$\frac{\mathrm{d}S_t}{S_t} = \mu \mathrm{d}t + \sigma \mathrm{d}W_t \tag{3}$$

In 1973, the articles from Fisher Black and Myron Scholes followed by the one of Robert Merton a few months later have changed the world of market finance and are considered as the starting point of the exponential growth of the derivatives markets. The main idea of this model is to replicate the european call option with the underlying and a riskless bond using a self financing strategy.

As every model in economics or mathematics, this last one is based on some hypothesis, which can be considered as an idealization of the real world :

1. Transactions occur in continuous time,
2. The market is frictionless, there are no transaction costs and no taxes,
3. Short selling is allowed, borrowing rate is the same as the lending one and is considered as a constant. There exists a bond with no risk and which gains interest at a constant rate
4. There are no dividends
5. The market does not allow arbitrage opportunities (Later, we will discuss quite a lot about this point),
6. The stock price process follows the diffusion equation given by (2), i.e. that instantaneous returns are gaussian.

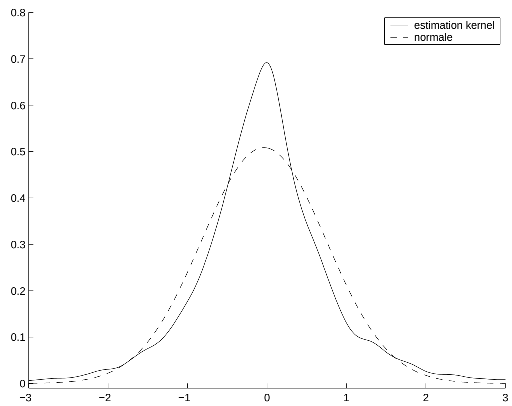

Fig. 1: Density of Log - returns S&P500 - Kernel Estimator and Gaussian fit

In our study we will keep only the five first hypotheses. The release of the sixth hypothesis is mainly motivated by the stylized facts of financial time series which are briefly developed in this next part.

1.1 Stylized facts of financial times series

Skewness and Kurtosis
Many studies about financial times series have proved that skewness and kurtosis of stocks returns are very different from the one of the Gaussian distribution. Skewness is usually slightly negative and kurtosis is statistically far different from 3, the value that was expected in the gaussian scenario.
As an example, we have studied the S&P500 Index between years 1980 and 2004, the skewness of the daily returns is $-1,92$ and the kurtosis is $45,02$. Excess kurtosis compared to the normal one (also called leptokurtic feature) shows that the tails of the historical distribution of returns are thicker than the ones of the gaussian distribution. This observation has many consequences in option pricing as it means that extreme daily returns occurs more often in real financial markets than in the Black and Scholes framework. (see figure (1.1).

Continuity of trajectories
The trajectories of the geometric brownian motion are continuous. Is continuity a good approximation for the curves of stock prices ? If we plot a stock quote at a large scale,(one year), everything will seem to be continuous. At a smaller scale, like the intraday one, it is difficult to observe any continuity at all for the same stock, discontinuities may occur at some transaction dates. Since 1976, Merton [22] has extended his model and has given stock prices the possibility to jump at random times. Rare events can be seen as the cause of jumps, but some authors have gone one step further and have even

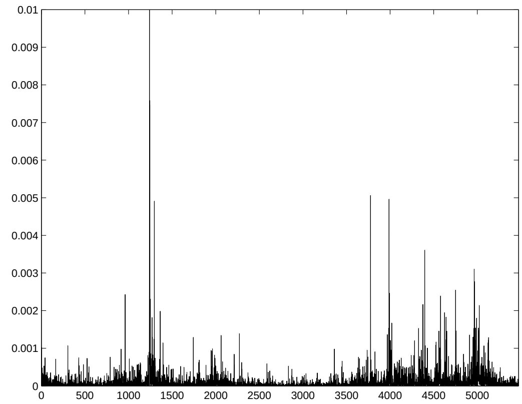

Fig. 2: Square of daily log returns - S&P 500 1981-2004

considered pure jumps models with no brownian continuous part (Carr, Geman, Madan and Yor Model).

Volatility smile

The volatility parameter of the Black and Scholes model is supposed to be constant, independent of time and independent of option features like moneyness. Empirical facts show that it is not the case and that it is impossible to find a unique volatility parameter which matches all the derivatives products on the same underlying. Thus, volatility cannot be seen as constant and is strongly dependent of time to maturity τ and moneyness $\frac{S_t}{K}$ of the option :

$$\sigma_t = \sigma_t \left(\tau = T - t, \frac{S_t}{K} \right)$$

The existence of volatility surfaces $\sigma_t \left(\tau, \frac{S_t}{K}\right)$ indicates that the model is not a perfect one and does not match exactly the reality and complexity of market finance. As the volatility parameter is the only one estimated in the Black and Scholes formula it "collects all the imperfection of the model".

Volatility clustering

One stylized fact of financial times series as shown by [11] is the existence of volatility clustering.

As we can see on graph 2, there are periods of time where volatility is high followed by periods of low volatility. This feature is neither taken into account by the geometric brownian motion, nor by the processes that we will encounter later in this thesis. It would have required to take into account volatility not any more as a deterministic process but as a stochastic process to reproduce this common feature of the financial time series.

1.2 Solutions to overcome those statistical facts

Modeling of the volatility surfaces
One way to solve the previous problems is to consider implied volatility surface $\sigma_t\left(\tau, \frac{K}{S_t}\right)$ and to extend in some way the Black and Scholes Merton model to calculate numerically the price of an european option according to its maturity and moneyness. Nevertheless, modeling the volatility surface is complex and is strongly time dependent. Haerdle and Fengler even speak about dynamic of the volatilty surface.
Stochastic volatility
Now, if the volatility parameter is driven by some stochastic process, we can have some volatility clustering effect. For example, the Heston model is a model where the volatility process follows a Cox Ingersoll and Ross process. The equity derivatives industry widely uses this model.

Enlarge the class of stochastic process to model the underlying
An other way to proceed is to give up the geometric brownian motion as a starting point to describe the price process. Similarly, we could choose other distribution than the normal one to model the instantaneous returns.

Nevertheless, we would like to keep some interesting features of the geometric brownian motion process : we would like to keep independence and stationarity properties of increments of instantaneous returns. We also would like to keep the Markov property, as it is linked in some way to a form of market efficiency. With a slight abuse of language, using the filtration $(\mathcal{F}_t)$ generated by the stock process (S_t), the Markov property can be written as : $\forall s \leq T$,

$$\mathrm{E}\left[f\left(S_T\right)|\mathcal{F}_s\right] = \mathrm{E}\left[f\left(S_T\right)|S_s\right] \text{ for all bounded measurable function } f$$

The class of processes with independent and stationary increments, also called Lévy processes in honor of the french mathematician Paul Lévy appears as a natural candidate with all the required properties to describe the price process of a stock price. Moreover, as the brownian motion is naturally a Lévy process we will find again in our study many well known features encountered in the Black and Scholes setting.

Before studying price processes driven by the exponential of a Lévy Process, we have to define more precisely the mathematical settings for incomplete markets in continuous time (Chapter 2). Chapter 3 is about a few basic definitions and properties of Lévy processes which are required to understand the following chapters. We will study two major properties of those models in chapter 4 : The existence of arbitrage opportunities and the completeness of the model. Chapter 5 is about one example of equivalent martingale measure called the Esscher transform. The next chapter moves on to the numerical part and describes the method used to calculate prices with the help of the

Fast Fourier transform. As an example we will calibrate one of the studied models (the variance gamma) on market price in order to price non liquid derivatives like barrier options and to stress the difference with the results of the Black Scholes and Merton model.

2 Asset valuation in incomplete markets

In the case of an incomplete market, there is not always a self financing strategy for hedging derivatives even if continuous time trading was possible. The pricing model cannot be built as in the Black and Scholes framework on the main idea of replication. Thus, the fair price of a simple option in those market models is not as clear and well defined as in the case of a complete market. We will even show later that many different prices can exist and each of them are compatible with the no arbitrage opportunity assumption. The goal of this chapter is to define in a non fully but enough rigorous setting the main assumptions required to describe an incomplete market model.

2.1 The financial market

Here, we consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ and assume that $\mathcal{F}_0$ is trivial, ie $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$ for $A \in \mathcal{F}_0$. We also consider $d+1$ assets described as càdlàg processes :

$$\begin{aligned} \tilde{S} : [0,T] \times \Omega &\longmapsto \mathbb{R}^{d+1} \\ (t,\omega) &\longmapsto \left(\tilde{S}_t^0(\omega), \cdots, \tilde{S}_t^d(\omega)\right) \end{aligned}$$

where, $\tilde{S^i}_t(\omega)$ is the quote of asset i at time $t \leq T$ in case of scenario ω. The "0" asset is a special one called the riskless asset. In our study we will not investigate changing interest rates and we choose to model the risk free asset as an asset gaining constant interest r per unit of time :

$$S_t^0 = \exp(rt)$$

The actualized price process is defined as :

$$\begin{aligned} S : [0,T] \times \Omega &\longmapsto \mathbb{R}^{d} \\ (t,\omega) &\longmapsto \left(\frac{\tilde{S}_t^0(\omega)}{S_t^0}, \cdots, \frac{\tilde{S}_t^d(\omega)}{S_t^0}\right) \end{aligned}$$

2.2 Arbitrage opportunity

Conversely to the discrete time setting, it is difficult to give here an universal definition of an arbitrage opportunity (also called a "free lunch") in continuous time. The following definition is a first approach which will allow us to understand the main features of this idea, but it should be refined later in a less intuitive definition. For the moment, we will work in a general setting assuming that the discounted price process is a positive semimartingale in $\mathbf{R}^d$.

Definition 2.1: A self-financing strategy is a couple $\pi = (V_0, \Phi)$ where $V_0 \in \mathbb{R}$ and $\Phi = \left(\Phi_1^t, \Phi_2^t, \ldots, \Phi_d^t\right)_{0 \leq t \leq T}$ is a previsible process with respect to the filtration $(\mathcal{F}_t)$ such that the stochastic integral $\int_0^t \Phi_u \cdot dS_u$ is well defined. See for instance [23].

Definition 2.2: The actualized value of a self-financing strategy $\pi = (V_0, \Phi)$ is the process $(V_t^\pi)_{0 \leq t \leq T}$

$$V_t^\pi = V_0 + \int_0^t \Phi_u \cdot dS_u$$

Definition 2.3: A self-financing strategy with null initial value $(V_0 = 0)$ Φ is an arbitrage opportunity if the following conditions are fulfilled :

1. $\exists\ a \in \mathbb{R}$ such that $\mathbb{P}\left(\forall t \in [0,T],\ V_t^\pi \geq a\right) = 1$
2. $V_T^\pi \geq 0$ $\mathbb{P}$ p.s
3. $\mathbb{P}\left(V_T^\pi > 0\right) > 0$

The first condition states the existence of a lower bound for the price process in order to avoid some strategies like the suicide strategy described by Harrison and Pliska. The next one defines free lunch as an opportunity to win without any downside risk : we are sure that we will not loose ($V_T^\pi \geq 0$ $\mathbb{P}$ p.s) and we have a chance (with a strictly positive probability) to have a positive income $\mathbb{P}\left(V_T^\pi > 0\right) > 0$.

The absence of arbitrage opportunity is simply defined as a model where such a free lunch does not exist, ie a market does not admit arbitrage opportunities if there is no self-financing strategy which allows profit with positive probability without any downside risk.
The absence of arbitrage opportunity is a widely accepted concept among practitioners. In the following, we will always stay in this framework.

2.3 Equivalent martingale measures

Classically, in order to get a convenient pricing rule, we will use a change of probability measure to move from the historical or objective probability measure $\mathbb{P}$ to a probability measure $\mathbb{Q}$ which is equivalent to $\mathbb{P}$ on the σ - field $(\mathcal{F}_T)$. If the actualized price processes are $\mathbb{Q}$ - martingales, we say that $\mathbb{Q}$ is a martingale measure. The set of equivalent martingale measures is the set :

$$\mathcal{Q} = \{\mathbb{Q} \sim \mathbb{P} \mid \mathbb{Q} \ \text{ is a martingale measure}\}$$

2.4 Fundamental Theorems of asset pricing

There is a link between the non arbitrage opportunity property and the existence of an equivalent martingale measure. In a discrete time setting the equivalence has been established by Harrison and Kreps [17] with the previous definition of an arbitrage opportunity. In a continuous time setting, we have only one of the implication :

$$\mathcal{Q} \neq 0 \quad \Longrightarrow \text{Absence of Arbitrage Opportunity}$$

Nevertheless, it is possible if we change the definition of an arbitrage opportunity to get the other implication. The following definition of an arbitrage opportunity "with vanishing risk" is due to F.Delbaen and W.Schachermayer.

2.5 Arbitrage opportunity with vanishing risk

In all this part we will consider a financial market $(\Omega, (\mathcal{F})_t, \mathbb{P}, (S_t)_{t \leq T})$ with only one risky asset with positive value and the riskless asset. S is the actualized price process of an asset.
The actualized value process of a self-financing strategy $\pi = (x, H)$ is the process $V_t = x + \int_0^t H_u.dS_u$.
The following definition of arbitrage is less intuitive, but allows us to state the two fundamental theorems of asset pricing in continuous time.

Definition 2.4: Arbitrage opportunity with vanishing risk
A sequence of strategy $(x^n, H^n)_{n \in \mathbb{N}}$ is an arbitrage opportunity with vanishing risk if :

1. for all $n \in \mathbb{N}$, $x^n = 0$
2. for all $n \in \mathbb{N}$, it exists $a_n \in \mathbb{R}$ such that
$P\left(\forall t \leq T, x^n + \int_0^t H_u^n \mathrm{d}S_u \geq a_n\right) = 1$
3. For all $n \in \mathbb{N}$,
$\int_0^t H_u \mathrm{d}S_u \geq -\frac{1}{n}$ $p.s.$
4. It exists $\delta > 0$ such that for all $n \in \mathbb{N}$,
$P\left(\int_0^t H_u^n \mathrm{d}S_u > 0\right) = \delta$

A market model where such an arbitrage opportunity with vanishing risk does not exist (NFLVR) is a model where such a sequence of strategy (x^n, H^n) with properties 1 to 4 does not exist.

Definition 2.5: Completeness

A market model is said to be complete if for all bounded $\mathcal{F}_T$ measurable functions f (f is typically the payoff of an european derivative), there is a self-financing strategy (x, H) such that :

1. $f = x + \int_0^T H_s \mathrm{d}S_s$
2. $\exists a$ and b constants such that $\mathcal{P}\left(\forall t \leq T, a \leq x + \int_0^t H_s \mathrm{d}S_s \leq b\right) = 1$

The first condition tells us that the payoff f can be replicated by a self-financing strategy in the riskless asset and S. The second one states that the replicating portfolio has to remain bounded.
With those more rigorous definition, one can state the following theorems in a continuous time setting :

Theorem 2.1: **First fundamental theorem of asset pricing - FTAP I**

A market model does not allow any arbitrage opportunity with vanishing risk if and only if there is a martingale measure (local) $\tilde{\mathbb{P}} \sim \mathbb{P}$ such that S_t is a $\tilde{\mathbb{P}}$ local martingale.

Theorem 2.2: **Second fundamental theorem of asset pricing - FTAP II**
If a model does not allow any arbitrage opportunities with vanishing risk then the following assumptions are equivalents :

1. The market model is complete
2. The martingale measure $\tilde{\mathbb{P}}$ (local) is unique .
3. There is a local martingale measure such that every $\tilde{\mathbb{P}}$ local martingale M_t can be represented as a stochastic integral with respect to S (martingale representation property)i.e. , $\forall t \leq T$,
$M_t = \int_0^t k_s \mathrm{d}S_s$ where k_s is a previsible process.

Remark 2.1: As we are working with strictly positive processes, there is an equivalence between local martingale and the more difficult concept of σ-martingale used by Delbaen and Schachermayer. This equivalence is due to Ansel and Stricker [1]).

Remark 2.2: Completeness for a market model is a topic of high importance for option valuation. If the market does not allow any arbitrage opportunity, then second fundamental theorem of asset pricing implies uniqueness of the martingale measure equivalent to the historical one. This leads to a unique price for derivatives. Conversely, in an incomplete market model, the set of martingale measure is infinite and so may also be the set of prices for a single derivative! (and this without breaking the non arbitrage opportunity rule). In this case one needs to specify in some way one equivalent martingale measure in order to get a pricing rule and uniqueness of prices for derivatives.

3 Lévy Processes

We will develop in this part only the definitions, properties and theorems required for the next chapters to describe the models where the asset is driven by the exponential of a Lévy Process.

We consider a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with filtration $(\mathcal{F}_t)$.

3.1 Définition

Definition 3.1: A stochastic process $\{X(t), t \geq 0\}$ with values in $\mathbb{R}^d$ is a Lévy process if the following conditions are satisfied :

1. $X_0 = 0$ $\mathcal{P}$ - p.s.
2. $\forall n \in \mathbb{N}^*$ et $\forall 0 \leq t_0 < t_1 < \cdots < t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \cdots, X_{t_n} - X_{t_{n-1}}$ are independents (independence of increments).
3. The distribution of $X_{s+t} - X_s$ does not depend on s. (Stationarity of increments property) ;
4. The process $\{X(t), t \geq 0\}$ is stochastic continuous, i.e. :
$\forall t \geq 0,\ \forall \epsilon > 0,\ \lim_{s \to t} \mathcal{P}\left(|X(s) - X(t)| > \epsilon\right) = 0$
5. There is $\Omega_0 \in \mathcal{F}$ with $\mathcal{P}[\Omega_0] = 1$ such that for all $\omega \in \Omega_0,\ X_t(\omega)$ is right continuous in $t \geq 0$ and admit a left limit for all $t > 0$.

Proposition 3.1: **Lévy Khintchine representation**
Let X be a Lévy Process with value in $\mathbb{R}^d$. Then, there is :
– $b \in \mathbb{R}^d$,
– a non negative and semi defined quadratic form C ,
– a Lévy measure ν defined on $\mathbb{R}^d$ with

$$\nu(\{0\}) = 0$$
$$\text{and} \int_{\mathbb{R}^d} \left(|x|^2 \wedge 1\right) \nu(\mathrm{d}x) < \infty$$

such that for all $\lambda \in \mathbb{R}^d$,

$$\mathrm{E}\left[\exp(i\lambda X_t)\right] = \exp\left\{ i\lambda bt - \langle \lambda, C\lambda \rangle \frac{t}{2} + t \int_{\mathbb{R}^d} \left(e^{i\lambda x} - 1 - i\lambda x 1_{\{|x| \leq 1\}} \right) \nu(\mathrm{d}x) \right\} \quad (4)$$

Moreover, this representation is unique and the triplet (b, C, ν) fully characterizes X.

Remark 3.1: The goal of the truncation function $1_{\{|x|\leq 1\}}$ is to ensure convergence of the integral in zero. We could also have defined another truncation function. For example we will widely use the following one defined for all $a \in \mathbb{R}^+$:

$$\mathrm{H}_a(x) = xI(|x| \leq a)$$

In this case, the Lévy Khintchine representation can be written for $\mathbb{R}^d$ valued processes :

$$\mathrm{E}\left[\exp\left(i\lambda X_t\right)\right] = \exp\left\{i\lambda bt - \langle\lambda, C\lambda\rangle\frac{t}{2} + t\int_{\mathbb{R}^d}\left(e^{i\lambda x} - 1 - i\lambda H_a(x)\right)\nu'(\mathrm{d}x)\right\} \quad (5)$$

The characteristic triplet of X with respect to this truncation function is now written $(b, C, \nu')_{H_a}$

For $\mathbb{R}$ valued processes, the Lévy Khintchine representation takes the following simplified form :

$$\mathrm{E}\left[\exp\left(i\lambda X_t\right)\right] = \exp\left\{t\left[i\lambda b - \lambda^2\frac{c}{2} + \int_{\mathbb{R}}\left(e^{i\lambda x} - 1 - i\lambda H_a(x)\right)\nu'(\mathrm{d}x)\right]\right\} \quad (6)$$

with characteristic triplet $(b, c > 0, \nu')_{H_a}$

Remark 3.2: The Lévy Khintchine representation shows that a Lévy processes is "made of three parts" :

- a deterministic trend (b is analogous to a drift, but depends on the chosen truncation function),
- a brownian part,
- a jump part. The Lévy measure $\nu(\mathrm{d}x)$ rules the jumps. The jumps with value in a set $A \in \mathbb{R}$ follow a Poisson process with intensity $\int_A \nu(\mathrm{d}x)$. Let $\{\Delta Z_t, t \geq 0\}$ be a point Poisson process with measure ν. We denote T_n the successive times of this Poisson process in a Borel set B such that $0 < \nu(B) < +\infty$. Let $S_n = T_n - T_{n-1}$ (with $T_0 = 0$). Then, if we call $X_n = \Delta Z_{T_n}$, (cf [5]) $\{(X_n, S_n), \quad n \geq 0\}$ is a sequence of independent random variable with respectively uniform law on A and exponential law with parameter $\nu(A)$.

3.2 Examples

We will recall here only the main definitions and properties of some simple Lévy processes such as Poisson processes and Compound Poisson processes.

3.2.1 Poisson processes

Let $(T_n, n \geq 0)$ be a family of random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathcal{P})$ such that :

$$T_0 = 0,$$
$$T_n < T_{n+1} \text{ for all n such that } T_n < \infty$$

First, we define the counting process $(N_t)_{t \geq 0}$:

$$\begin{aligned} N_t &= n \text{ if } t \in [T_n, T_{n+1}[\\ &= +\infty \text{ otherwise} \end{aligned}$$

Remark 3.3: A counting process is a process which jump of one unit at random times and which is constant between jumps. The counting process counts the numbers of random times (T_n) smaller than t.

Definition 3.2: **Poisson process**
A Poisson process with intensity λ is a counting process $(N_t)_{t \geq 0}$ such that the family of random variables $(T_{n+1} - T_n, n \geq 0)$ is independent and identically distributed with exponential law with parameter λ.
We have,

$$\forall n \in \mathbb{N} \;\; \mathcal{P}(N_t = n) = \exp(-\lambda t) \frac{(\lambda t)^n}{n!} \tag{7}$$

3.2.2 Compound Poisson process

Definition 3.3: **Compound Poisson process**
A compound Poisson process with intensity $\lambda > 0$ is a stochastic process $(X_t)_{t \geq 0}$ such that :

$$\forall t \in \mathbb{R}^+, \;\; X_t = \sum_{i=0}^{N_t} Y_i \tag{8}$$

Where $(Y_i)_{i \in \mathbb{N}}$ is a family of random variables $\mathbb{R}^d$ independent and identically distributed with distribution f and $(N_t)_{t \geq 0}$ is a Poisson process with intensity λ independent from the family $(Y_i)_{i \in \mathbb{N}}$.

Remark 3.4: A compound Poisson process is a process whose jump size is not any more one like the simple Poisson process but is random according to the distribution f.

Proposition 3.2: **Characteristic function of a compound Poisson Process**
Let $(X_t)_{t\geq 0}$ be a compound Poisson process with values in $\mathbb{R}$ with intensity λ and with jumps distribution f. The characteristic function of X_t is for all t $\in \mathbb{R}^+$:

$$\forall u \in \mathbb{R},\ \ \mathrm{E}\left[\exp\left(iuX_t\right)\right] = \exp\left\{\lambda t \int_{\mathbb{R}} \left(e^{iux} - 1\right) f\left(\mathrm{d}x\right)\right\} \tag{9}$$

Proof :
Let X_t be a compound Poisson process : $\forall t \in \mathbb{R}^+,\quad X_t = \sum_{i=0}^{N_t} Y_i$ where $(Y_i)_{i\in\mathbb{N}}$ is a family of random variables in $\mathbb{R}^d$ independent and identically distributed with distribution f and $(N_t)_{t\geq 0}$ is a Poisson process with intensity λ independent from the family $(Y_i)_{i\in\mathbb{N}}$.

$$\begin{aligned}
u \longrightarrow \mathrm{E}\left[\exp\left(iuX_t\right)\right] &= \mathrm{E}\left[\exp\left(iu\sum_{i=0}^{N_t} Y_i\right)\right] \\
&= \sum_{n=0}^{\infty} P(N_t = n)\left(\mathrm{E}\left[\exp\left(iuY_1\right)\right]\right)^n \\
&= \sum_{n=0}^{\infty} \frac{\mathrm{e}^{-\lambda t}\left(\lambda t\right)^n}{n!}\left(\mathrm{E}\left[\exp\left(iuY_1\right)\right]\right)^n \\
&= \mathrm{e}^{-\lambda t}\exp\left(\lambda t \mathrm{E}\left[\exp\left(iuY_1\right)\right]\right) \\
&= \exp\left\{\lambda t \int_{\mathbb{R}} \left(\mathrm{e}^{iux} - 1\right) f\left(\mathrm{d}x\right)\right\}
\end{aligned}$$

Remark 3.5: If we define the new measure ν such that for all $A \in B_{\mathbb{R}},\ \nu\left(A\right) = \lambda f\left(A\right)$, the previous result can be written :

$$\mathrm{E}\left[\exp\left(iuX_t\right)\right] = \exp\left\{t \int_{\mathbb{R}} \left(\mathrm{e}^{iux} - 1\right) \nu\left(\mathrm{d}x\right)\right\}$$

According to the Lévy Kintchine formula, the characteristic triplet of the compound Poisson process is $(0, 0, \nu)_0$ (To simplify the notation, we will replace in the following H_0 by the subscript 0). The converse is also true : a process with characteristic triplet $(0, 0, \nu)_0$ where ν is a finite measure is a compound Poisson process.

3.3 Change of measure

Proposition 3.3: **Change of measure for compound Poisson Processes**
Let $(X_t)_{0\leq t\ \leq T}$ be a compound Poisson process. Let $\tilde{\nu}$ be a finite measure absolutely continuous with respect to ν. We define for all $t \leq T$

$$M_t \quad = \quad \exp\left\{t\left(\nu\left(\mathbb{R}\right)-\tilde{\nu}\left(\mathbb{R}\right)\right)+\sum_{s\leq t}\ln\rho\left(\Delta X_s\right)\right\} \tag{10}$$

with $\rho = \frac{\mathrm{d}\tilde{\nu}}{\mathrm{d}\nu}$, $\rho(0) = 0$ and $\Delta X_s = X_s - X_{s_-}$
(M_t) is a P-martingale with unit expectation. Now, we can define the new probability measure :

$$\tilde{\mathcal{P}} = M_T\mathcal{P}$$

Then X is a $\tilde{\mathcal{P}}$ -Lévy process with characteristic triplet $(0,0,\tilde{\nu})_0$

Remark 3.6: We have defined a change of measure for the compound Poisson process which changes the Lévy measure. Later, we will extend this result with the Cameron Martin formula in the next chapter to change a Lévy process with triplet $(b,c,\nu)_{H_a}$ into a Lévy process with triplet $(b',c,\nu')_{H_a}$

Proof :

First, we have to show that (M_t) is a $\mathcal{P}$-martingale.

$\nu(\mathbb{R}) < \infty$, thus the process (X_t) can only admit a finite number of jumps on the interval $[0,T]$. ((M_t) is fully defined)
Let $s \leq t \leq T$,

$$\mathrm{E}\left[M_t|\mathcal{F}_s\right] \quad = \quad \mathrm{E}\left[M_s\exp\left\{\left(t-s\right)\left(\nu\left(\mathbb{R}\right)-\tilde{\nu}\left(\mathbb{R}\right)\right)+\sum_{s\leq t}\ln\rho\left(\Delta X_s\right)\right\}\middle|\mathcal{F}_s\right]$$

M_s is $\mathcal{F}_s$ measurable, then,

$$\mathrm{E}\left[M_t|\mathcal{F}_s\right] \quad = \quad M_s\exp\left\{\left(t-s\right)\left(\nu\left(\mathbb{R}\right)-\tilde{\nu}\left(\mathbb{R}\right)\right)\right\}\times\mathrm{E}\left[\exp\left\{\sum_{s\leq r\leq t}\ln\rho\left(\Delta X_r\right)\right\}\middle|\mathcal{F}_s\right]$$

The jumps between s and t are independent from $\mathcal{F}_s$, thus can suppress the conditioning in the previous expectation.

$$\mathrm{E}\left[\exp\left\{\sum_{s\leq r\leq t}\ln\rho\left(\Delta X_r\right)\right\}\middle|\mathcal{F}_s\right] = \mathrm{E}\left[\exp\left\{\sum_{s\leq r\leq t}\ln\rho\left(\Delta X_r\right)\right\}\right]$$

$$= \mathrm{E}\left[\prod_{s\leq r\leq t}\rho\left(\Delta X_r\right)\right]$$
$$= \prod_{s\leq r\leq t}\mathrm{E}\left[\rho\left(\Delta X_r\right)\right]$$

by independence of jumps

Finally, if we condition with respect to the number of jumps, we get

$$= \sum_{k=0}^{\infty}\frac{\mathrm{e}^{\{-(t-s)\nu(\mathbb{R})\}}\left(\left(t-s\right)\nu\left(\mathbb{R}\right)\right)^k}{k!}\left(\int_{\mathbb{R}}\rho\left(x\right)\frac{\nu\left(\mathrm{d}x\right)}{\nu\left(\mathbb{R}\right)}\right)^k$$

Then,

$$\begin{aligned}\mathrm{E}\left[M_t|\mathcal{F}_s\right] &= M_s\exp\left\{\left(t-s\right)\left(\nu\left(\mathbb{R}\right)-\tilde{\nu}\left(\mathbb{R}\right)\right)\right\}\times\\ &\quad\sum_{k=0}^{\infty}\frac{\exp\left(\left\{-\left(t-s\right)\nu\left(\mathbb{R}\right)\right\}\right)\left(\left(t-s\right)\nu\left(\mathbb{R}\right)\right)^k}{k!}\left(\int_{\mathbb{R}}\rho\left(x\right)\frac{\nu\left(\mathrm{d}x\right)}{\nu\left(\mathbb{R}\right)}\right)^k\end{aligned}$$

Moreover, $\rho = \frac{\mathrm{d}\tilde{\nu}}{\mathrm{d}\nu}$ and,

$$\int_{\mathbb{R}}\rho\left(x\right)\frac{\nu\left(\mathrm{d}x\right)}{\nu\left(\mathbb{R}\right)} = \frac{\tilde{\nu}\left(\mathbb{R}\right)}{\nu\left(\mathbb{R}\right)}$$

$$\mathrm{E}\left[M_t|\mathcal{F}_s\right] = M_s\exp\left\{-\left(t-s\right)\tilde{\nu}\left(\mathbb{R}\right)\right\}\sum_{k=0}^{\infty}\frac{\left(\left(t-s\right)\nu\left(\mathbb{R}\right)\right)^k}{k!}\left(\frac{\tilde{\nu}\left(\mathbb{R}\right)}{\nu\left(\mathbb{R}\right)}\right)^k$$

If we use the serial expansion of the exponential function,

$$\begin{aligned}\mathrm{E}\left[M_t|\mathcal{F}_s\right] &= M_s\exp\left\{-\left(t-s\right)\tilde{\nu}\left(\mathbb{R}\right)\right\}\exp\left\{\left(t-s\right)\tilde{\nu}\left(\mathbb{R}\right)\right\}\\ &= M_s\end{aligned}$$

We now have to show that (X_t) is a $\tilde{\mathcal{P}}$ Lévy process by proving that increments are independent and stationary under $\tilde{\mathcal{P}}$.
Let $\lambda \in \mathbb{R}$,

$$\mathrm{E}_{\tilde{\mathcal{P}}}\left[\mathrm{e}^{i\ \lambda(X_t-X_s)}\big|\mathcal{F}_s\right] = \mathrm{E}_{\mathcal{P}}\left[\mathrm{e}^{i\ \lambda(X_t-X_s)}\frac{M_t}{M_s}\big|\mathcal{F}_s\right]$$
$$= \mathrm{E}_{\mathcal{P}}\left[\mathrm{e}^{i\ \lambda(X_t-X_s)}\exp\left\{\left(t-s\right)\left(\nu\left(\mathbb{R}\right)-\tilde{\nu}\left(\mathbb{R}\right)\right)+\sum_{s\leq r\leq t}\ln\rho\left(\Delta X_r\right)\right\}\right]$$

We have canceled conditioning in the last equation, because ΔX_r depends on jumps only after time s and then is independent from $\mathcal{F}_s$.
As $\mathrm{e}^{i\ \lambda(X_t-X_s)} = \mathrm{e}^{\sum_{s\leq r\leq t} i\ \lambda \Delta X_r}$ and if we condition with respect to the number of jumps, we can find (the calculation is very similar) :

$$\begin{aligned}
\mathrm{E}_{\tilde{\mathcal{P}}}\left[\mathrm{e}^{i\ \lambda(X_t-X_s)}\middle|\mathcal{F}_s\right] &= \exp\left\{-(t-s)\,\tilde{\nu}(\mathbb{R})\right\}\times \\
&\quad \sum_{k=0}^{\infty}\frac{((t-s)\,\nu(\mathbb{R}))^k}{k!}\left(\int_{\mathbb{R}}\mathrm{e}^{i\lambda x+\ln\rho(x)}\frac{\nu(\mathrm{d}x)}{\nu(\mathbb{R})}\right)^k \\
&= \exp\left\{-(t-s)\,\tilde{\nu}(\mathbb{R})+(t-s)\int_{\mathbb{R}}\mathrm{e}^{i\lambda x+\ln\rho(x)}\nu(\mathrm{d}x)\right\} \\
&= \exp\left\{-(t-s)\,\tilde{\nu}(\mathbb{R})+(t-s)\int_{\mathbb{R}}\mathrm{e}^{i\lambda x}\rho(x)\,\nu(\mathrm{d}x)\right\} \\
&= \exp\left\{-(t-s)\,\tilde{\nu}(\mathbb{R})+(t-s)\int_{\mathbb{R}}\mathrm{e}^{i\lambda x}\tilde{\nu}(\mathrm{d}x)\right\} \\
&= \exp\left\{(t-s)\left[-\int_{\mathbb{R}}\tilde{\nu}(\mathrm{d}x)+\int_{\mathbb{R}}\mathrm{e}^{i\lambda x}\tilde{\nu}(\mathrm{d}x)\right]\right\} \\
&= \exp\left\{(t-s)\int_{\mathbb{R}}\left(\mathrm{e}^{i\lambda x}-1\right)\tilde{\nu}(\mathrm{d}x)\right\}
\end{aligned}$$

Then, increments are stationary because they only depend on the difference $t-s$ and they are also independent of $(\mathcal{F}_s)$. Moreover with the Lévy Khintchine representation (6), the characteristic triplet under $\tilde{\mathcal{P}}$ is $(0,0,\tilde{\nu})$.

Proposition 3.4: Let $a>0$ and $(X_t)_{t\leq T}$ be a Lévy process with triplet $(b,c,\nu)_{H_a}$.
Let $\tilde{\nu}$ be a new measure such that $\tilde{\nu}(\{|x|\geq a\})\leq\infty$

$$\begin{cases}\tilde{\nu}=\nu & \text{if } \{|x|\leq a\}\\ \tilde{\nu}\sim\nu & \text{otherwise}\end{cases}$$

Then, there is a probability measure $\tilde{\mathcal{P}}\sim\mathcal{P}$ such that $(X_t)_{t\leq T}$ is a Lévy process with characteristic triplet $(b,c,\tilde{\nu})_{H_a}$.

Proof :

Let $a>0$ and $\tilde{\nu}$ a measure following the conditions of the statement. Note that such a measure exists, for example we can choose :

$$\tilde{\nu}(x)=\mathrm{e}^{-x^2}1_{\{|x|>a\}}\nu(x)+1_{\{|x|\leq a\}}\nu(x)$$

Let $\left(X_t^1\right)_{t\leq T}$ and $\left(X_t^2\right)_{t\leq T}$ defined by :

$$\begin{cases}X_t^1=\sum_{s\leq t}\Delta X_s 1_{\{|\Delta X_s|\geq a\}}\\ X_t^2=X_t-X_t^1\end{cases}$$

X_t^1 is a pure jump process, with jumps greater than a. X_t^2 is the process X without jumps greater than a. The processes $\left(X_t^1\right)_{t\leq T}$ and $\left(X_t^2\right)_{t\leq T}$ are independent processes because their jumps occur in different sets and X^1 is a pure jump process. Their characteristic triplets are : $\left(0,0,\nu|_{\{|x|>a\}}\right)_0$ et $\left(b,c,\nu|_{\{|x|\leq a\}}\right)_{H_a}$.

Let M_t be the change of probability $\mathcal{P}$-martingale associated with the compound Poisson process $\left(X_t^1\right)_{t\leq T}$. (We can check, using the same proof as in proposition 3.3 that M_t defined here is a $\mathcal{P}$-martingale)

$$\forall t \leq T,\ \ M_t = \exp\left\{ t\left(\nu\left\{|x|>a\right\} - \tilde{\nu}\left\{|x|>a\right\}\right) + \sum_{s\leq t} \ln\rho\left(\Delta X_s^1\right)\right\}$$

Let $\tilde{\mathcal{P}} = M_t\mathcal{P}$, by proposition 3.3, $\left(X_t^1\right)_{t\leq T}$ is a $\tilde{\mathcal{P}}$ Lévy process with characteristic triplet $\left(0,0,\tilde{\nu}|_{\{|x|>a\}}\right)_0$
Let $s \leq t \leq T$ and $\lambda \in \mathbb{R}$,

$$\mathrm{E}_{\tilde{\mathcal{P}}}\left[\mathrm{e}^{i\lambda(X_t - X_s)}\middle|\mathcal{F}_s\right] = \mathrm{E}_{\mathcal{P}}\left[\mathrm{e}^{i\lambda(X_t-X_s)}\frac{M_t}{M_s}\middle|\mathcal{F}_s\right]$$

Using the stationarity of increments of (X_t), we have :

$$\begin{aligned}\mathrm{E}_{\tilde{\mathcal{P}}}\left[\mathrm{e}^{i\lambda(X_t - X_s)}\middle|\mathcal{F}_s\right] &= \mathrm{E}_{\mathcal{P}}\left[\mathrm{e}^{i\lambda X_{t-s}}M_{t-s}\right] \\ &= \mathrm{E}_{\mathcal{P}}\left[\mathrm{e}^{i\lambda\left(X_{t-s}^1+X_{t-s}^2\right)}M_{t-s}\right] \\ &= \mathrm{E}_{\mathcal{P}}\left[\mathrm{e}^{i\lambda X_{t-s}^1}M_{t-s}\right] \times \mathrm{E}_{\mathcal{P}}\left[\mathrm{e}^{i\lambda X_{t-s}^2}\right]\end{aligned}$$

due to the independence of X^1 and X_2. (M depends only of X^1). Then,

$$\mathrm{E}_{\tilde{\mathcal{P}}}\left[\mathrm{e}^{i\lambda(X_{t-s} - X_s)}\middle|\mathcal{F}_s\right] = \mathrm{E}_{\tilde{\mathcal{P}}}\left[\mathrm{e}^{i\lambda X_{t-s}^1}\right] \times \mathrm{E}_{\mathcal{P}}\left[\mathrm{e}^{i\lambda X_{t-s}^2}\right]$$

Proposition 9 and the Lévy Khintchine decomposition (3.1) imply that :

$$\begin{aligned}\mathrm{E}_{\tilde{\mathcal{P}}}\left[\mathrm{e}^{i\lambda(X_t - X_s)}\middle|\mathcal{F}_s\right] &= \exp\left\{(t-s)\int_{\{|x|>a\}}\left(\mathrm{e}^{i\lambda x}-1\right)\tilde{\nu}\left(\mathrm{d}x\right)\right\}\times \\ &\quad \exp\left\{(t-s)\left[i\lambda b - \lambda^2\frac{c}{2} + \int_{\{|x|\leq a\}}\left(e^{i\lambda x} - 1 - i\lambda H_a\left(x\right)\right)\nu\left(\mathrm{d}x\right)\right]\right\}\end{aligned}$$

Moreover, $\tilde{\nu}$ et ν are the same on $\{|x|\leq a\}$. We have :

$$\mathrm{E}_{\tilde{\mathcal{P}}}\left[\mathrm{e}^{i\lambda(X_t - X_s)}\middle|\mathcal{F}_s\right] = \exp\left\{(t-s)\left[i\lambda b - \lambda^2\frac{c}{2} + \int_{\mathbb{R}}\left(e^{i\lambda x} - 1 - i\lambda H_a\left(x\right)\right)\tilde{\nu}\left(\mathrm{d}x\right)\right]\right\}$$

Then, $X_t - X_s$ is independent of $\mathcal{F}_s$ and stationary (it depends only on the difference $t-s$). We can conclude that $(X_t)_{t\leq T}$ is a $\tilde{\mathcal{P}}$ - Lévy process with characteristic triplet $(b,c,\tilde{\nu})_{H_a}$.

3.4 Useful results

Theorem 3.1: **Itô Formula**

Let $(X_t)_{t\geq 0}$ be a Lévy process with characteristic triplet $(b, c > 0, \nu)$ and f be a C^2 map from $\mathbb{R}$ to $\mathbb{R}$.
Then,

$$\begin{aligned} f(X_t) &= f(0) + \int_0^t \frac{c}{2} f''(X_s)\,\mathrm{d}s + \int_0^t f'(X_{s-})\,\mathrm{d}X_s \\ &\quad + \sum_{\substack{0\leq s\leq t \\ \Delta X_s \neq 0}} \left[f(X_{s-} + \Delta X_s) - f(X_{s-}) - \Delta X_s f'(X_{s-}) \right] \end{aligned} \quad (11)$$

Proof : see [15]

Remark 3.7: when X is of infinite variation (IV), the quantities $\sum_{s\leq t} \Delta X_s f'(X_s)$ and $\sum_{s\leq t} f(X_s) - f(X_{s-})$ taken separately have no sense. But, we can consider the sum

$$\sum_{s\leq t} (f(X_s) - f(X_{s-}) - \Delta X_s f'(X_s))$$

(We can find a constant K such that $|f(X_s) - f(X_{s-}) - \Delta X_s f'(X_s)| \leq K|\Delta X_s|^2$).
When X is of finite variation, Itô formula for a regular function f can be written :

$$f(X_t) = f(0) + \int_0^t f'(X_s)\mathrm{d}X_s^c + \sum_{0\leq s\leq t} f(X_t) - f(X_{t-})$$

with X^c the continuous part of the process X. Note that in this last formula the integrals are Stieltjes one.

Theorem 3.2: **Lévy-Itô Decomposition**
Let X be a Lévy process with characteristic triplet $(b, c, \nu)_h$ where h is a truncation function. Then,

$$X_t^1 = \lim_{\epsilon \downarrow 0} \left(\Delta X_s(\omega) 1_{\{|\Delta X_s| > \epsilon\}} - t \int_{|x|>\epsilon} h(x)\, \nu(\mathrm{d}x) \right), \ \forall t \geq 0 \quad (12)$$

is defined with probability one and the convergence is uniform in t on all bounded interval.
Define for all $t \geq 0$, $X_2^t = X - X_t^1$. Then X^1 and X^2 are independent Lévy processes with respective characteristic triplet $(0, 0, \nu)_h$ *and* $(b, c, 0)_h$.

Proof : see [25] Theorem 19.2

Proposition 3.5: **Exponential moments of a Lévy process**.
Let $a \geq 0$ and X be a Lévy process with characteristic triplet $(b, c, \nu)_{H_a}$.
If,

$$\int_{\{|x|>a\}} \mathrm{e}^x \nu\left(\mathrm{d}x\right) < \infty$$

then, $\forall t \geq 0$,

$$\mathrm{E}\left[\mathrm{e}^{X_t}\right] = \exp\left\{t\left(b + \frac{c}{2} + \int_{\mathbb{R}} \left(\mathrm{e}^x - 1 - H_a\left(x\right)\right)\left(\mathrm{d}x\right)\right)\right\}$$

Proof : See [25] p165

4 Exponential of Lévy processes models

Let $(\Omega, (\mathcal{F})_{t\leq T}, \mathbb{P})$ be a probability space with filtration $(\mathcal{F}_t)_{t\leq T}$. Let $(X_t)_{t\leq T}$ be a $(\mathcal{F})_{t\leq T}$ Lévy process, we can model the path of an asset S with the following process :

$$S_t = S_0 \exp\left(rt + X_t\right) \quad \forall t \leq T \tag{13}$$

with $S_0 \geq 0$, and r the constant free rate.

$\tilde{S}$ is the actualized price process, i.e. :

$$\tilde{S}_t = S_0 \exp\left(X_t\right) \quad \forall t \leq T \tag{14}$$

4.1 Absence of arbitrage opportunities

We are looking for the possible arbitrage opportunities (with vanishing risk) in the previously defined class of model.

Theorem 4.1: **Absence of arbitrage opportunities**
The model allows arbitrage opportunities only if S is monotone. Moreover, if the model does not allow arbitrage opportunities, then there is a measure $\tilde{\mathcal{P}} \sim \mathcal{P}$ such that S is a $\tilde{\mathcal{P}}$-martingale and X a $\tilde{\mathcal{P}}$ Lévy process.

Remark 4.1: Set $\hat{\mathcal{P}} \sim P$. It is important to note that a $(\mathcal{P})$ Lévy process is not necessarily a $\hat{\mathcal{P}}$ Lévy process. The class of Lévy process is not stable under change of measure.

Proof :
Let $(X_t)_{t\leq T}$ be a $\mathcal{P}$ Lévy process and $\alpha > 0$. We call $(b(\alpha), c, \nu)$ the characteristic triplet of $(X_t)_{t\leq T}$ with respect to the truncation function H_α. Let $\mathcal{F}_t$ be the natural completed filtration of X, ie
$\mathcal{F}_t = \mathcal{F}_t^1 \vee \mathcal{F}_t^2$ where $\mathcal{F}_t^1$ et $\mathcal{F}_t^2$ are respectively the filtration generated by the brownian part and the Poisson part of X. Moreover, we have, $\mathcal{F}_t^1 \perp \mathcal{F}_t^2$

The proof of this theorem is due to Cherny and Shiriaev [9] and is based on the decomposition of the following cases :

1. $\exists a \in \mathbb{R}^+$ such that $\nu\left((a, +\infty)\right) > 0$ and $\nu\left((-\infty, -a)\right) > 0$. The process has both negative and positive jumps.
2. $\nu\left((-\infty, 0)\right) = 0$ et $\int_0^1 x\nu\left(\mathrm{d}x\right) = \infty$. (The jumps are positive and X is not of finite variation)
3. $\nu\left((-\infty, 0)\right) = 0$, $\int_0^1 x\nu\left(\mathrm{d}x\right) < \infty$ and $c > 0$. (The jumps are positives , the Poisson part is of finite variation and the brownian part is different from 0)

4. $\nu((-\infty,0)) = 0$, $\int_0^1 x\nu(\mathrm{d}x) < \infty, c = 0$. (There is no brownian component, the jumps are positive and the process is of finite variation)
5. $\nu = 0$ (It is the case of the geometric brownian motion)

we should also consider 3 other cases, which are the symetrics of the cases 2,3 and 4 obtained by changing the measure support on $(-\infty, 0)$ ie, $\nu((-\infty,0)) > 0$ et $\nu((0,+\infty)) = 0$.

Case 1
$\exists a \in \mathbb{R}^+$ such that $\nu((a,+\infty)) > 0$ and $\nu((-\infty,-a)) > 0$.
Let b be the first characteristic of $(X_t)_{t\leq T}$ with respect to the truncation function H_a.

We are looking for a probability measure $\tilde{\mathcal{P}} \sim \mathcal{P}$ such that $(X_t)_{t\leq T}$ has a zero drift under $\tilde{\mathcal{P}}$ and such that $(X_t)_{t\leq T}$ is a $\tilde{\mathcal{P}}$ martingale.

We begin by the construction of a measure $\bar{\nu}$ with the following properties :

$$\text{Property } (\bigstar) \begin{cases} 1) & \bar{\nu} = \nu \text{ on } \{|x| \leq a\} \\ 2) & \bar{\nu} \sim \nu \text{ on } \{|x| > a\} \\ 3) & \tilde{\nu}(\{|x| > a\}) < \infty \\ 4) & \int_{\{|x|>a\}} \exp(x)\tilde{\nu}(\mathrm{d}x) < \infty \end{cases}$$

It is always possible to define such a measure, because we can define the real function $\bar{\rho}$:

$$\bar{\rho}(x) = \begin{cases} 1 & \forall x \in [-a,+a] \\ \mathrm{e}^{-x^2} & \forall x \in \,]-\infty,-a\,[\bigcup\,]\,a,+\infty\,[\end{cases}$$

and $\bar{\nu} = \bar{\rho}\nu$

We have defined a measure $\bar{\nu}$ equivalent to ν ($x \to \mathrm{e}^{-x^2}$ is a strictly positive function) following the $(\bigstar)$ property.

We could have also chosen on $]-\infty,-a\,[\bigcup\,]\,a,+\infty\,[$ all enough decreasing in $+\infty$ and $-\infty$ strictly positive function to ensure properties 3 and 4 of $(\bigstar)$.

Let $\tilde{\rho}_{\eta,\xi}$ be the real positive function defined for all strictly positive η and ξ :

$$\tilde{\rho}_{\eta,\xi} = \eta.1_{\{x<-a\}} + 1_{\{-a\leq x\leq a\}} + \xi.1_{\{x>a\}} \tag{15}$$

Let $\tilde{\nu}_{\eta,\xi}$ be the positive measure defined for all strictly positive η and ξ by : $\tilde{\nu}_{\eta,\xi} = \tilde{\rho}_{\eta,\xi}\bar{\nu}$. This new measure fulfills the $(\bigstar)$ property.

Moreover, for all η and ξ strictly positives, (from theorem 3.4), there is a measure $\tilde{\mathcal{P}}_{\eta,\xi} \sim \mathcal{P}$ such that X is a $\tilde{\mathcal{P}}_{\eta,\xi}$-Lévy process with characteristic triplet $(b, c, \tilde{\nu}_{\eta,\xi})$.

from proposition 3.5, we have for all $t > 0$,

$$\mathrm{E}_{\tilde{\mathcal{P}}_{\eta,\xi}}\left[\mathrm{e}^{X_t}\right] = \exp\left\{t\left[b + \frac{c}{2} + \int_{\mathbb{R}} \left(\mathrm{e}^x - 1 - H_a(x)\right)\tilde{\nu}_{\eta,\xi}(\mathrm{d}x)\right]\right\} \tag{16}$$

In order to get a $\tilde{\mathcal{P}}_{\eta,\xi}$-martingale we have first to cancel the drift of e^{X_t} :

$$b + \frac{c}{2} + \int_{\mathbb{R}} \left(\mathrm{e}^x - 1 - H_a(x)\right)\tilde{\nu}_{\eta,\xi}(\mathrm{d}x) = 0 \iff$$
$$b + \frac{c}{2} + \eta.\int_{\{x<-a\}} \left(\mathrm{e}^x - 1\right)\bar{\nu}(\mathrm{d}x) + \int_{\{-a\leq x\leq a\}} \left(\mathrm{e}^x - 1 - x\right)\bar{\nu}(\mathrm{d}x) +$$
$$\xi.\int_{\{x>a\}} \left(\mathrm{e}^x - 1\right)\bar{\nu}(\mathrm{d}x) = 0 \iff$$

We have to solve the previous equation in η and ξ with the constraints $\eta > 0$ and $\xi > 0$. This equation is :

$$\eta C_1 + C_2 + \xi C_3 = 0 \tag{17}$$

with,

$$\begin{cases} C_1 = \int_{\{x<-a\}} \left(\mathrm{e}^x - 1\right)\bar{\nu}(\mathrm{d}x) < 0 \\ C_2 = b + \frac{c}{2} + \int_{\{-a\leq x\leq a\}} \left(\mathrm{e}^x - 1 - x\right)\bar{\nu}(\mathrm{d}x) \text{ plays the role of a constant} \\ C_3 = \int_{\{x>a\}} \left(\mathrm{e}^x - 1\right)\bar{\nu}(\mathrm{d}x) > 0 \end{cases} \tag{18}$$

This equation is linear in η and ξ. C_3 is strictly positive and C_1 strictly negative. Then, there is an infinity of couple (η, ξ) which solves the equation. Let(η, ξ) be such a solution. X is then a $\tilde{\mathcal{P}}_{\eta,\xi}$-Lévy process with characteristic triplet $(b, c, \tilde{\nu}_{\eta,\xi})$. There is still to prove that $\left(\mathrm{e}^{X_t}\right)_{t\leq T}$ is a $\tilde{\mathcal{P}}_{\eta,\xi}$- martingale :

Let $s \leq t \leq T$,

$$\mathrm{E}_{\tilde{\mathcal{P}}_{\eta,\xi}}\left[\mathrm{e}^{X_t - X_s} \middle| \mathcal{F}_s\right] = \mathrm{E}_{\mathcal{P}}\left[\mathrm{e}^{X_t - X_s}\frac{M_t}{M_s} \middle| \mathcal{F}_s\right]$$

where M is the martingale from theorem 3.4 :

$$\forall t \leq T,\ \ M_t = \exp\left\{t\left(\nu\{|x| > a\} - \tilde{\nu}\{|x| > a\}\right) + \sum_{s\leq t} \ln\rho\left(\Delta X_s^1\right)\right\}$$

With $\left(X_t^1\right)_{t\leq T}$ and $\left(X_t^2\right)_{t\leq T}$ defined by :

$$\left\{\begin{array}{l} X_t^1 = \sum_{s\leq t} \Delta X_s 1_{\{|\Delta X_s|\geq a\}} \\ X_t^2 = X_t - X_t^1 \end{array}\right.$$

M is a $\mathcal{F}^1$-martingale and $M \perp \mathcal{F}^2$. The ratio $\frac{M_t}{M_s}$ depends only on the jumps of X^1 between time s and t and the increments of X^1 between t and s are independent from $\mathcal{F}_s$,

$$\begin{aligned} \mathrm{E}_{\tilde{\mathcal{P}}_{\eta,\xi}}\left[\mathrm{e}^{X_t-X_s}\middle|\mathcal{F}_s\right] &= \mathrm{E}_{\mathcal{P}}\left[\mathrm{e}^{X_t-X_s}\frac{M_t}{M_s}\middle|\mathcal{F}_s^1 \vee \mathcal{F}_s^2\right] \\ &= \mathrm{E}_{\mathcal{P}}\left[\mathrm{e}^{X_t^1+X_t^2-\left(X_s^1+X_s^2\right)}\frac{M_t}{M_s}\middle|\mathcal{F}_s^1 \vee \mathcal{F}_s^2\right] \end{aligned}$$

X^1 and X^2 are independent processes, and M depends only on X^1,

$$\begin{aligned} \mathrm{E}_{\tilde{\mathcal{P}}_{\eta,\xi}}\left[\mathrm{e}^{X_t-X_s}\middle|\mathcal{F}_s\right] &= \mathrm{E}_{\mathcal{P}}\left[\mathrm{e}^{X_t^1-X_s^1}\frac{M_t}{M_s}\middle|\mathcal{F}_s^1\right] \times \mathrm{E}_{\mathcal{P}}\left[\mathrm{e}^{X_t^2-X_s^2}\middle|\mathcal{F}_s^2\right] \\ &= \mathrm{E}_{\tilde{\mathcal{P}}_{\eta,\xi}}\left[\mathrm{e}^{X_{t-s}^1}\right] \times \mathrm{E}_{\mathcal{P}}\left[\mathrm{e}^{X_{t-s}^2}\right] \\ &= \exp\left\{(t-s)\left[\int_{\{|x|>a\}}\left(\mathrm{e}^x-1\right)\tilde{\nu}_{\eta,\xi}\left(\mathrm{d}x\right)\right]\right\} \\ &\quad \times \exp\left\{(t-s)\left[b+\frac{c}{2}+\int_{\{|x|\leq a\}}\left(\mathrm{e}^x-1-x\right)\nu\left(\mathrm{d}x\right)\right]\right\} \end{aligned}$$

Because $\tilde{\nu}_{\eta,\xi}$ and ν are the same (by construction) on $\{|x| \leq a\}$, we have

$$\mathrm{E}_{\tilde{\mathcal{P}}_{\eta,\xi}}\left[\mathrm{e}^{X_t-X_s}\middle|\mathcal{F}_s\right] = \exp\left\{(t-s)\left[b+\frac{c}{2}+\int_{\mathbb{R}}\left(\mathrm{e}^x-1-H_a\left(x\right)\right)\tilde{\nu}_{\eta,\xi}\left(\mathrm{d}x\right)\right]\right\}$$

And the couple (η,ξ) is by definition such that

$$b+\frac{c}{2}+\int_{\mathbb{R}}\left(\mathrm{e}^x-1-H_a\left(x\right)\right)\tilde{\nu}_{\eta,\xi}\left(\mathrm{d}x\right)=0$$

Then,

$$\begin{aligned} \mathrm{E}_{\tilde{\mathcal{P}}_{\eta,\xi}}\left[\mathrm{e}^{X_t-X_s}\middle|\mathcal{F}_s\right] &= 1 \\ \mathrm{E}_{\tilde{\mathcal{P}}_{\eta,\xi}}\left[\mathrm{e}^{X_t}\middle|\mathcal{F}_s\right] &= X_s \end{aligned}$$

Thus, e^{X_t} is a $\tilde{\mathcal{P}}_{\eta,\xi}$-martingale.

Remark 4.2: We have shown that in this case the couple (η, ξ) is never unique and that it exists an infinity of martingale measure such that $\left(\mathrm{e}^{X_t}\right)$ is a $\tilde{\mathcal{P}}_{\eta,\xi}$-martingale.

Case 2
$\nu\left((-\infty, 0)\right) = 0$ and $\int_0^1 x\nu\left(\mathrm{d}x\right) = \infty$.

Let $b\left(a\right)$ be the first element of the characteristic triplet of X associated with the truncation function H_a.

$\forall a \in (0,1)\,,\ \ b\left(a\right) = b\left(1\right) - \int_{\{a<x\leq 1\}} x\nu\left(\mathrm{d}x\right)$

With our hypothesis, $\lim_{a\downarrow 0} \int_{\{a<x\leq 1\}} x\nu\left(\mathrm{d}x\right) = +\infty$.
Moreover , $\lim_{a\downarrow 0} \int_{\{0\leq x\leq a\}} \left(\mathrm{e}^x - 1 - x\right)\nu\left(\mathrm{d}x\right) = 0$. It implies that there is $a \in (0,1)$ such that the two following conditions are fulfilled :

$$\begin{cases} 1) & b\left(a\right) + \frac{c}{2} + \int_{\{0\leq x\leq a\}} \left(\mathrm{e}^x - 1 - x\right)\nu\left(\mathrm{d}x\right) < 0 \\ 2) & \nu\left(\{x > a\}\right) \end{cases}$$

as $\int_{\{0\leq x\leq a\}} \left(\mathrm{e}^x - 1 - x\right)\nu\left(\mathrm{d}x\right)$ is finite because $x \to \mathrm{e}^x - 1 - x \approx x^2$ in 0.

Let$\xi \in \mathbb{R}^+$. We define the measure $\tilde{\nu}_\xi$ equivalent to ν and which agrees with ν on $[-a, a]$ by :

$$\begin{cases} \tilde{\nu}_\xi = & \nu & \text{on } \{0 \leq x \leq a\} \\ \tilde{\nu}_\xi = & \xi \mathrm{e}^{-x^2}.\nu & \text{otherwise} \end{cases}$$

As in case one, $(\tilde{\nu})$ fulfills the ($\bigstar$) property. Moreover, from theorem 3.4, for all strictly positive ξ, there is a measure $\tilde{\mathcal{P}}_\xi \sim \mathcal{P}$ such that X is a $\tilde{\mathcal{P}}_\xi$ Lévy process with characteristic triplet $(b, c, \tilde{\nu}_\xi)$.

From proposition 3.5, we have for all $t > 0$,

$$\mathrm{E}_{\tilde{\mathcal{P}}_\xi}\left[\mathrm{e}^{X_t}\right] = \exp\left\{t\left[b + \frac{c}{2} + \int_{\mathbb{R}} \left(\mathrm{e}^x - 1 - H_a\left(x\right)\right)\tilde{\nu}_\xi\left(\mathrm{d}x\right)\right]\right\} \tag{19}$$

To cancel the drift of e^{X_t} it is necessary that :

$$b\left(a\right) + \frac{c}{2} + \int_{\mathbb{R}} \left(\mathrm{e}^x - 1 - H_a\left(x\right)\right)\tilde{\nu}_\xi\left(\mathrm{d}x\right) = 0 \iff$$

$$b(a) + \frac{c}{2} + \int_{\{0<x\leq a\}} \left(\mathrm{e}^x - 1 - x\right)\tilde{\nu}_\xi\left(\mathrm{d}x\right) + \xi.\int_{\{x>a\}} \left(\mathrm{e}^x - 1\right)\mathrm{e}^{-x^2}\nu\left(\mathrm{d}x\right) = 0 \iff$$

$$b(a) + \frac{c}{2} + \int_{\{0<x\leq a\}} \left(\mathrm{e}^x - 1 - x\right)\nu\left(\mathrm{d}x\right) + \xi.\int_{\{x>a\}} \left(\mathrm{e}^x - 1\right)\mathrm{e}^{-x^2}\nu\left(\mathrm{d}x\right) = 0$$

because ν et $\tilde{\nu}_\xi$ are the same on $\{0 < x \leq a\}$.

From the first constraint equation for the choice of a,

$b + \frac{c}{2} + \int_{\{0<x\leq a\}} (\mathrm{e}^x - 1 - x)\, \nu(\mathrm{d}x) < 0.$

The second constraint equation for a ensures that
$\int_{\{x>a\}} (\mathrm{e}^x - 1)\, \mathrm{e}^{-x^2} \tilde{\nu}_\xi(\mathrm{d}x)$ is strictly positive. Then there is $\xi > 0$ such that :

$$b(a) + \frac{c}{2} + \int_{\mathbb{R}} (\mathrm{e}^x - 1 - H_a(x))\, \tilde{\nu}_\xi(\mathrm{d}x) = 0$$

As shown before (case one), with the use of theorem 3.4, there is a measure $\tilde{\mathcal{P}}_\xi \sim \mathcal{P}$ such that X is a $\tilde{\mathcal{P}}_\xi$ Lévy process with characteristic triplet $(b, c, \tilde{\nu}_\xi)$. The proof that $\left(\mathrm{e}^{X_t}\right)$ is a $\tilde{\mathcal{P}}_\xi$-martingale, is exactly the same as in case number one.

Remark 4.3: In this case, the positive real ξ is never unique and there is an infinity of martingale measure such that $\left(\mathrm{e}^{X_t}\right)$ is a $\tilde{\mathcal{P}}_\xi$-martingale. We can see this because the support of the measure ν is different from a single point and the choice of a different real a leads us to build a different martingale measure.

Case 3
$\nu((-\infty, 0)) = 0$, $\int_0^1 x\nu(\mathrm{d}x) < \infty$ and $c > 0$

We use the Lévy-Itô decomposition of X.
$\forall t \leq T$,

$$\begin{cases} X_t^1 &= \lim_{\epsilon\downarrow 0}\left(\sum_{s\leq t} \Delta X_s 1_{\{|\Delta X_s|>\epsilon\}} - t\int_{\{|x|>\epsilon\}} H_1(x)\,\mathrm{d}(x)\right) \\ &= \sum_{s\leq t} X_s - t\int_0^1 x\nu(\mathrm{d}x) \\ X_t^2 &= X_t - X_t^1 \end{cases}$$

from proposition 3.2, X^1 et X^2 are two Lévy processes with respective characteristic triplet $(0, 0, \nu)_{H_1}$ and $(b, c, 0)_{H_1}$. We have decomposed X in a continuous process (drifted brownian motion) and in a pure jump process.

As in the previous cases, we can build a measure $\tilde{\nu}$ equivalent to ν with the ($\bigstar$) property. For example, we can choose $a = 1$.

We define for all $t \leq T$

$$M_t^1 = \exp\left\{ t\Big(\nu(\{|x| > 1\}) - \tilde{\nu}(\{|x| > 1\})\Big) + \sum_{s\leq t} \ln\rho\left(\Delta X_s^1\right) \right\} \tag{20}$$

with $\rho = \frac{\mathrm{d}\tilde{\nu}}{\mathrm{d}\nu}$

We have shown before that M_t^1 is a $\mathcal{P}$-martingale with respect to $\mathcal{F}_1$.

Let M_2^t be the $\mathcal{P}$-martingale change of measure given by the Cameron-Martin formula and associated with the drifted brownian motion X^2 and with the real θ.
$\forall t \leq T$,

$$M_t^2 = \exp\left\{\theta X_2^t - \frac{\theta^2}{2} t\right\} \tag{21}$$

We define for all $t \leq T$, $M_t = M_t^1 . M_t^2$.
We have to show that M_t is a $\mathcal{P}$-martingale : X^1 and X^2 are two independent processes and M_1 and M_2 are two independent martingales.

Let $s \leq t \leq T$,

$$\begin{aligned}
\mathrm{E}_{\mathcal{P}}\left[M_t \middle| \mathcal{F}_s\right] &= \mathrm{E}_{\mathcal{P}}\left[M_t^1 . M_t^2 \middle| \mathcal{F}_s^1 \vee \mathcal{F}_s^2\right] \\
&= \mathrm{E}_{\mathcal{P}}\left[M_t^1 \middle| \mathcal{F}_s^1\right] \times \mathrm{E}_{\mathcal{P}}\left[M_t^2 \middle| \mathcal{F}_s^2\right] \\
&= M_s^1 \times M_s^2 \\
&= M_s
\end{aligned}$$

We have to find θ such that the process $(\exp X_t)$ is a martingale under the probability measure $\tilde{\mathcal{P}}_\theta$ defined by $\tilde{\mathcal{P}}_\theta = M_t . \mathcal{P}$.

Let $s \leq t \leq T$,

$$\begin{aligned}
\mathrm{E}_{\tilde{\mathcal{P}}_\theta}\left[\mathrm{e}^{\{X_t - X_s\}} \middle| \mathcal{F}_s\right] &= \mathrm{E}_{\mathcal{P}}\left[\mathrm{e}^{\{X_t - X_s\}} \frac{M_t}{M_s} \middle| \mathcal{F}_s\right] \\
&= \mathrm{E}_{\mathcal{P}}\left[\mathrm{e}^{\left\{X_t^1 - X_s^1\right\}} \frac{M_t^1}{M_s^1} \mathrm{e}^{\left\{X_t^2 - X_s^2\right\}} \frac{M_t^2}{M_s^2} \middle| \mathcal{F}_s\right]
\end{aligned}$$

We define $\tilde{\mathcal{P}}^1 = M_t^1 . \mathcal{P}$. Under $\tilde{\mathcal{P}}^1$, X_t^1 is a Lévy process with characteristic triplet $(0, 0, \tilde{\nu})_{H_1}$ (Proposition 12). By independence of X_1 and X_2, we can write the last equation :

$$\begin{aligned}
\mathrm{E}_{\tilde{\mathcal{P}}_\theta}\left[\mathrm{e}^{\{X_t - X_s\}} \middle| \mathcal{F}_s\right] &= \mathrm{E}_{\mathcal{P}}^1\left[\mathrm{e}^{\left\{X_t^1 - X_s^1\right\}} \middle| \mathcal{F}_s^1\right] \times \mathrm{E}_{\mathcal{P}}\left[\mathrm{e}^{\left\{X_t^2 - X_s^2\right\}} \frac{M_t^2}{M_s^2} \middle| \mathcal{F}_s^2\right] \\
&= \exp\left\{(t-s) \int_{\mathbb{R}} \left(\mathrm{e}^x - 1 - H_1(x)\right) \tilde{\nu}(\mathrm{d}x)\right\} \\
&\quad \times \mathrm{E}_{\mathcal{P}}\left[\exp\left\{b(t-s) + \sigma(W_t - W_s)\right\} \frac{M_t^2}{M_s^2} \middle| \mathcal{F}_s^2\right]
\end{aligned}$$

With $\sigma = \sqrt{c}$ and $(W_t)_{t \leq T}$ is a $(\mathcal{F}_t, \mathcal{P})$ standard brownian motion.

We have,

$$\begin{aligned}
&\mathrm{E}_{\mathcal{P}}\left[\exp\left\{b(t-s)+\sigma(W_t-W_s)\right\}\frac{M_t^2}{M_s^2}\Big|\mathcal{F}_s^2\right] \\
&\quad= \mathrm{E}_{\mathcal{P}}\left[\exp\left\{b(t-s)+\sigma(W_t-W_s)+\theta(W_t-W_s)-\frac{\theta^2}{2}(t-s)\right\}\Big|\mathcal{F}_s^2\right] \\
&\quad= \exp\left\{b(t-s)-\frac{\theta^2}{2}(t-s)\right\}\mathrm{E}_{\mathcal{P}}\left[\exp\left\{(\theta+\sigma).(W_{t-s})\right\}\right] \\
&\quad= \exp\left\{\left(b-\frac{\theta^2}{2}\right)(t-s)\right\}.\exp\left\{\frac{(\sigma+\theta)^2}{2}(t-s)\right\} \\
&\quad= \exp\left\{\left(b+\frac{\sigma^2}{2}+\sigma\theta\right)(t-s)\right\}
\end{aligned}$$

Then,

$$\mathrm{E}_{\tilde{\mathcal{P}}_\theta}\left[\mathrm{e}^{\{X_t-X_s\}}\Big|\mathcal{F}_s\right] = \exp\left\{(t-s)\left(\int_{\mathbb{R}}\left(\mathrm{e}^x-1-H_1(x)\right)\tilde{\nu}(\mathrm{d}x)+b+\frac{\sigma^2}{2}+\sigma\theta\right)\right\}$$

For X to be a $\tilde{\mathcal{P}}_\theta$-martingale, it is necessary and enough that :

$$\int_{\mathbb{R}}\left(\mathrm{e}^x-1-H_1(x)\right)\tilde{\nu}(\mathrm{d}x)+b+\frac{\sigma^2}{2}+\sigma\theta=0$$

Thus, there is a unique real θ (called θ_0 in the following) :

$$\theta_0=-\frac{1}{\sigma}\left(\int_{\mathbb{R}}\left(\mathrm{e}^x-1-H_1(x)\right)\tilde{\nu}(\mathrm{d}x)+b+\frac{\sigma^2}{2}\right) \tag{22}$$

We check that X is a$\left(\mathcal{F}_t, \tilde{\mathcal{P}}_{\theta_0}\right)$ Lévy process. To simplify the notation, we will call $\tilde{\mathcal{P}} = \tilde{\mathcal{P}}_{\theta_0}$

Let $\lambda \in \mathbb{R}$ and $s \leq t \leq T$,

$$\begin{aligned}
&\mathrm{E}_{\tilde{\mathcal{P}}}\left[\exp\left\{i\lambda(X_t-X_s)\right\}\Big|\mathcal{F}_s\right] \\
&\quad= \mathrm{E}_{\mathcal{P}}\left[\exp\left\{i\lambda(X_t-X_s)\right\}\frac{M_t}{M_s}\Big|\mathcal{F}_s\right] \\
&\quad= \mathrm{E}_{\tilde{\mathcal{P}}}^1\left[\exp\left\{i\lambda\left(X_t^1-X_s^1\right)\right\}\Big|\mathcal{F}_s^1\right]\times\mathrm{E}_{\mathcal{P}}\left[\exp\left\{i\lambda\left(X_t^2-X_s^2\right)\right\}\frac{M_t^2}{M_s^2}\Big|\mathcal{F}_s^2\right] \\
&\quad= \exp\left\{(t-s)\int_{\mathbb{R}}\left(\mathrm{e}^{i\lambda x}-1-i\lambda H_1(x)\right)\tilde{\nu}(\mathrm{d}x)\right\} \\
&\qquad\times\mathrm{E}_{\mathcal{P}}\left[\exp\left\{(t-s)ib\lambda+i\lambda\sigma(W_t-W_s)+\theta_0(W_t-W_s)-\frac{\theta_0^2}{2}(t-s)\right\}\Big|\mathcal{F}_s\right]
\end{aligned}$$

Let $k_\lambda = \int_{\mathbb{R}} \left(e^{i\lambda x} - 1 - i\lambda H_1(x)\right) \tilde{\nu}(dx)$,

$$= \exp\{(t-s)k_\lambda\} \exp\left\{ib\lambda(t-s) - \frac{\theta_0^2}{2}(t-s)\right\} \mathrm{E}_{\mathcal{P}}\left[\exp\{(i\lambda\sigma+\theta_0)(W_t - W_s)\}|\mathcal{F}_s\right]$$

$$= \exp\{(t-s)k_\lambda\} \exp\left\{ib\lambda(t-s) - \frac{\theta_0^2}{2}(t-s)\right\} \exp\left\{(t-s)\frac{(i\lambda\sigma+\theta_0)^2}{2}\right\}$$

$$= \exp\left\{(t-s)\left[k_\lambda + i\lambda b + i\lambda\theta_0\sigma - \frac{\lambda^2\sigma^2}{2}\right]\right\}$$

If we replace θ_0 by its value,

$$\mathrm{E}_{\tilde{\mathcal{P}}}\left[\exp\{i\lambda(X_t - X_s)\}\Big|\mathcal{F}_s\right] = \exp\left\{(t-s)\left[k_\lambda - i\lambda\left(k + \frac{\sigma^2}{2}\right) - \frac{\lambda^2\sigma^2}{2}\right]\right\}$$

We call,

$$\begin{aligned} \tilde{b} &= -k_\lambda - \frac{\sigma^2}{2} \\ &= -\frac{\sigma^2}{2} - \int_{\mathbb{R}} \left(e^x - 1 - H_1(x)\right) \tilde{\nu}(dx) \end{aligned}$$

Thus, we have proved that X is a $\tilde{\mathcal{P}}$ Lévy process with characteristic triplet $\left(\tilde{b}, \sigma^2, \tilde{\nu}\right)_{H_1}$.

Remark 4.4: We have changed both the drift b and the jump measure ν with the change of probability measure defined by the martingale M_t.

Remark 4.5: If the support of ν is not reduced to a single point, we can build many measures $\tilde{\nu}$ equivalent to ν. Those measures will lead us to different martingale measures, because we affect different multiplicative transforms to the different elements of the support of ν. The case where ν is reduced to a single point is considered in the next theorem.

Case 4
$\nu((-\infty, 0)) = 0$, $\int_0^1 x\nu(dx) < \infty, c = 0$

We separate the case where $b < 0$ and where $b \geq 0$.

b<0
For all real number $a > 0$, we call $b(a)$ the first element of the characteristic triplet of X with respect to the truncation function H_a.

$$b(a) = b + \int_{\{0<x\leq a\}} x\nu(dx)$$

We can find $a > 0$ such that we simultaneously have :

$$\begin{cases} \nu\{(a, +\infty)\} > 0 \\ b(a) + \int_{\{0<x\leq a\}} (\mathrm{e}^x - 1 - x)\,\nu(\mathrm{d}x) < 0 \end{cases}$$

This last inequality is possible because :

$$\begin{aligned} &\lim_{a\downarrow 0} \quad b(a) = b < 0, \text{ and then,} \\ &\lim_{a\downarrow 0} \quad \left(b(a) + \int_{\{0<x\leq a\}} (\mathrm{e}^x - 1 - x)\,\nu(\mathrm{d}x) \right) = b < 0 \end{aligned}$$

We are exactly in the same setting as case number 2 by creating a measure $\tilde{\nu}$ equivalent to ν. We can also note that in the general case, we use this construction method to build many different measures $\tilde{\nu} \sim \nu$ that will once again lead us to many different martingale measures. (Unless if ν is reduced to a point).

b>0

X is an increasing process. (Both the drift and the jumps are positive). (X is called a subordinator). The process $\tilde{S}$ is an increasing process. A long position in the asset with actualized price $\tilde{S} = \mathrm{e}^{(X_t)}$ is obviously an arbitrage opportunity. (and also an arbitrage opportunity with vanishing risk).

Case 5 $\nu = 0$

In this case, we are exactly in the Black and Scholes setting. The Cameron Martin formula allows us to find a unique martingale measure equivalent to $\tilde{\mathcal{P}}$ such that the actualized price process $\tilde{S}$ is a $\tilde{\mathcal{P}}$-martingale.

Other cases

The other three cases are the symmetric of cases number 2,3 and 5 if we swap the measure support on $(-\infty, 0)$. More precisely we have $\nu((-\infty, 0)) > 0$ and $\nu((0, +\infty)) = 0$.

4.2 Completeness

As shown before, the previous model includes the Black and Scholes model. This model is known as a complete one, ie the european style derivatives are all reachable and the price is naturally defined as the cost of the hedge. We will see in this part that it is more an exception than the rule. We will prove that the only case which shares this property is the pure Poisson model.

Theorem 4.2: **Completeness of the Lévy exponentials models**
Consider a model defined by equation (14) and which satisfies the non arbitrage property. Moreover, we impose that the considered filtration is exactly the one generated by the stock S.
The model is complete only in the two following cases :

1. $X_t = \mu t + \sigma W_t$
   with W$\mathcal{P}$ a standard brownian motion and $\sigma > 0$
2. $X_t = bt + \delta N_t$
   where N_t is a Poisson process with intensity λ and $\delta b < 0$

Proof :
1) Due to the previsible representation theorem for the brownian motion, the first model is a complete one (moreover, it is well known that the Black and Scholes model is a complete one).

2) We apply the Itô formula (11) to $x \to \mathrm{e}^x$ and to the process $X_t = bt + \delta N_t$ $\forall u \leq T$,

$$\mathrm{d}S_u = \mathrm{e}^{X_{u-}}\mathrm{d}X_u + \mathrm{e}^{X_{u-}+\Delta X_u} - \mathrm{e}^{X_{u-}} - \Delta X_u \mathrm{e}^{X_{u-}}$$

We integrate,

$$\begin{aligned}
S_t - S_0 &= \int_0^t \mathrm{e}^{X_{s-}}\mathrm{d}X_s + \sum_{0\leq s\leq t} \mathrm{e}^{X_{s-}}\left(\mathrm{e}^{\Delta X_s} - 1 - \Delta X_s\right) \\
S_t - S_0 &= b\int_0^t \mathrm{e}^{X_{s-}}\mathrm{d}s + \delta\int_0^t \mathrm{e}^{X_{s-}}\mathrm{d}N_s + \sum_{0\leq s\leq t} \mathrm{e}^{X_{s-}}\left(\mathrm{e}^{\Delta X_s} - 1 - \delta\Delta N_s\right) \\
S_t - S_0 &= b\int_0^t \mathrm{e}^{X_{s-}}\mathrm{d}s + \delta\int_0^t \mathrm{e}^{X_{s-}}\mathrm{d}N_s + \int_0^t \mathrm{e}^{X_{s-}}\left(\mathrm{e}^{\delta} - 1 - \delta\right)\mathrm{d}N_s \\
S_t - S_0 &= \left(\mathrm{e}^{\delta} - 1\right)\int_0^t \mathrm{e}^{X_{s-}}\mathrm{d}\left(N_s - \frac{b}{1-\mathrm{e}^{\delta}}s\right)
\end{aligned}$$

$b\delta < 0$, this implies that $\frac{b}{1-\mathrm{e}^{\delta}} > 0$. With the help of proposition 3.4, we know that there is one probability measure $\tilde{\mathcal{P}} \sim \mathcal{P}$ such that N_t is a Poisson process with intensity $\frac{b}{1-\mathrm{e}^{\delta}} > 0$.

Then, $M_t = N_t - \frac{b}{1-\mathrm{e}^{\delta}}t$ is a $\tilde{\mathcal{P}}$-martingale, because :

$$\begin{aligned}
\mathrm{E}_{\tilde{\mathcal{P}}}\left[M_t - M_s|\mathcal{F}_s\right] &= \mathrm{E}_{\tilde{\mathcal{P}}}\left[N_t - N_s|\mathcal{F}_s\right] - \frac{b}{1-\mathrm{e}^{\delta}}(t-s) \\
&= \mathrm{E}_{\tilde{\mathcal{P}}}\left[N_{t-s}\right] - \frac{b}{1-\mathrm{e}^{\delta}}(t-s) \\
&= 0
\end{aligned}$$

We will admit here that the Poisson process also admits the previsible reprentation, ie that every random variable $H \in \mathcal{L}^2\left(\tilde{\mathcal{P}}\right)$ and $\mathcal{F}_T$ -measurable could be written :

$$H_T \quad = \quad \mathrm{E}\left[H_T\right] + \int_0^T a_s \mathrm{d}M_s$$

with $(a_s)_{s\leq T}$ a previsible process. Moreover, if H is a local martingale starting from 0 :

$$\begin{aligned} H_t &= \int_0^t a_s \mathrm{d}M_s \\ &= \int_0^t \frac{a_s}{\left(\mathrm{e}^{\delta}-1\right)\mathrm{e}^{X_{s-}}} \mathrm{d}S_s \end{aligned}$$

The second fundamental theorem of asset pricing allows us to conclude that the Poisson model is complete.
We will now have a closer look at the other models. In the proof of the previous theorem, we have built in an explicit way for the different cases a martingale measure . In particular, we have shown that when the support of the Lévy measure ν was not reduced to a point, it was possible to build many equivalent martingale measures under which the actualized price process was a martingale. The second Theorem of asset pricing allows us to conclude the incompleteness of the model when the support of ν is not a point.

If now, the support of ν is reduced to a point :

$$\forall t \leq T,\ X_t \quad = \quad bt + \sigma W_t + \delta N_t$$

with W_t a $\mathcal{P}$-brownian motion and N_t a Poisson process with intensity λ.

We apply the Itô formula to $x \rightarrow \mathrm{e}^x$ and to the process $X_t = bt + \sigma W_t + \delta N_t$:

$$\begin{aligned} S_t &= S_0 + \int_0^t \mathrm{e}^{X_{s-}} \mathrm{d}X_s + \int_0^t \frac{1}{2}\sigma^2 \mathrm{e}^{X_{s-}} \mathrm{d}s + \sum_{0\leq s\leq t} \mathrm{e}^{X_{s-}+\Delta X_s} - \mathrm{e}^{X_{s-}} - \Delta X_s \mathrm{e}^{X_{s-}} \\ &= S_0 + \int_0^t \mathrm{e}^{X_{s-}} \mathrm{d}\Big(bs + \frac{1}{2}\sigma^2 s + \sigma W_s + \left(\mathrm{e}^{\delta}-1\right) N_s \Big) \end{aligned}$$

If we consider the change of measure from $\mathcal{P}$ to $\tilde{\mathcal{P}}$ used in the third case of the previous thereom ((4.1) with X_1 and X_2 the two part of the Lévy-Itô decomposition of X.

$$M_t = \exp\left\{ t\Big(\nu\left(\{|x|>1\}\right) - \tilde{\nu}\left(\{|x|>1\}\right)\Big) + \sum_{s\leq t} \ln\rho\left(\Delta X_s^1\right) \right\} \times \exp\left\{ \theta X_2^t - \frac{\theta^2}{2} t \right\} \tag{23}$$

X is a $\tilde{\mathcal{P}}$ martingale if :

$$\int_{\mathbb{R}} \left(e^{x} - 1 - H_1(x)\right) \tilde{\nu}(dx) + b + \frac{\sigma^2}{2} + \sigma\theta = 0$$

When the support of ν is reduced to a point this equation can be written :

$$\alpha \left(e^{\delta} - 1\right) \lambda + b + \frac{\sigma^2}{2} + \sigma\theta = 0$$

Where α is a coefficient depending on $\tilde{\nu}$ and θ is the Cameron-Martin transformation parameter.
We have supposed $\delta \neq 0$ and $\sigma \neq 0$. This equation with unknown α, θ has the form :

$$\alpha C_1 + C_2 + \theta C_3 = 0$$

Where C_1, C_2, C_3 are constants independent from α and θ, $C_1 > 0$ and $C_3 > 0$.
We conclude that we can find many solution sets (α, θ) corresponding to the initial choice of $\tilde{\nu}$ and the coefficient of Cameron Martin transform. Again, the second fundamental theorem of asset pricing allows us to conclude to the incompleteness of the model.

5 Choice of an equivalent martingale measure : the Esscher transform

In this part, we will suppose that there are two assets available in the market : a riskless asset which price is given by the equation $S_t^0 = \exp(rt)$ and a risky asset modeled by the exponential of a Lévy process, i.e. :

$$S_t = S_0 \exp(rt) \exp(X_t)$$

Where $(X_t)_{0 \le t \le T}$ is a Lévy process.
We have proved in the last part that in the general case where $(X_t)_{0 \le t \le T}$ is not a pure brownian motion or a pure Poisson process, the model is an incomplete one and that we have to make a choice in one way or another for the equivalent martingale measure. The following is a simple way to define and to characterize an equivalent martingale measure called the Esscher transform.

5.1 Esscher transform

Definition 5.1: Set $(X_t)_{0 \le t \le T}$ a $\mathbb{P}$ - Lévy process, we define the Esscher transform as a change of probability from $\mathbb{P}$ to $\mathbb{Q}$ with $\mathbb{Q} \overset{loc}{\sim} \mathbb{P}$ and which admits a density $Z_t = \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t}$ such that :

$$Z_t = \frac{\exp(\theta X_t)}{E_{\mathbb{P}}[\exp(\theta X_t)]}$$

with $\theta \in \mathbb{R}$ such that $E_{\mathbb{P}}[\exp(\theta X_t)] < \infty$

Remark 5.1: If we use the generating function of the moments of the distribution of $(X_t)_{0 \le t \le T}$ defined by :

$$\begin{aligned} \text{mgf}(\theta) &= \Phi(-i\theta) \\ &= E_{\mathbb{P}}[\exp \theta X_1] \end{aligned}$$

where Φ is the Fourier transform of X_1.
$\forall s < t$, we have

$$\begin{aligned} E_{\mathbb{P}}[\exp \theta X_t] &= E_{\mathbb{P}}[\exp\{\theta(X_t - X_s + X_s)\}] \\ &= E_{\mathbb{P}}[\exp\{\theta X_{t-s}\}] \cdot E_{\mathbb{P}}[\exp\{\theta X_s\}] \end{aligned}$$

Here we have used the independence and the stationarity of increments of $(X_t)_{0 \le t \le T}$.

In particular, we have ∀ t :

$$\begin{aligned} \text{mgf}\,(\theta)^t &= E_{\mathbb{P}}\left[\exp\theta X_1\right]^t \\ &= E_{\mathbb{P}}\left[\exp\theta X_t\right] \end{aligned}$$

then, the Radon Nykodim density of the Esscher transform can be written :

$$Z_t = \frac{\exp(\theta X_t)}{\text{mgf}\,(\theta)^t}$$

Remark 5.2: We have to prove that we have defined a change of measure , i.e that Z_t is a $\mathbb{P}$-Martingale with unit expectation 1.

Z_t is integrable from the hypothesis on θ and X.

The martingale property of Z_t could be proved the following way : for all $t > s$,

$$\begin{aligned} E_{\mathbb{P}}\left[Z_t|\mathcal{F}_s\right] &= E_{\mathbb{P}}\left[\frac{\exp(\theta X_t)}{E_{\mathbb{P}}\left[\exp(\theta X_t)\right]}\big|\mathcal{F}_s\right] \\ &= \frac{1}{E_{\mathbb{P}}\left[\exp(\theta X_t)\right]}E_{\mathbb{P}}\left[\exp\{\theta(X_t - X_s) + \theta X_s\}\big|\mathcal{F}_s\right] \\ &= \frac{\exp(\theta X_s)E_{\mathbb{P}}\left[\exp\{\theta(X_t - X_s)\}\right]}{E_{\mathbb{P}}\left[\exp(\theta X_1)\right]^t} \\ &= \frac{\exp(\theta X_s)E_{\mathbb{P}}\left[\exp\{\theta X_{t-s}\}\right]}{E_{\mathbb{P}}\left[\exp(\theta X_1)\right]^t} \\ &\quad \text{(due to independence and stationarity of increments)} \\ &= \exp(\theta X_s)\frac{E_{\mathbb{P}}\left[\exp(\theta X_1)\right]^{t-s}}{E_{\mathbb{P}}\left[\exp(\theta X_1)\right]^t} \\ &= \frac{\exp(\theta X_s)}{\text{mgf}\,(\theta)^s} \\ &= Z_s \end{aligned}$$

Proposition 5.1: Set $(X_t)_{0\leq t\leq T}$ a Lévy process with characteristic triplet $(\gamma, \sigma, \nu(\mathrm{d}x))$ such that the existence conditions of the Esscher transform are fulfilled. Set $\mathbb{Q}$ defined by the Esscher transform of $\mathbb{P}$.
Then, $(X_t)_{0\leq t\leq T}$ is a Lévy process under $\mathbb{Q}$ and its characteristic triplet $(\tilde{\gamma}, \tilde{\sigma}, \tilde{\nu}(\mathrm{d}x))$ is :

$$\begin{cases} \tilde{\gamma} &= \gamma + \sigma^2\theta + \int_{-1}^{+1}(\exp\theta x - 1)\,\nu(\mathrm{d}x) \\ \tilde{\sigma} &= \sigma \\ \tilde{\nu} &= \exp(\theta x)\nu(\mathrm{d}x) \end{cases}$$

Proof :
First, we have to show that the increments of $(X_t)_{0\le t\le T}$ are independent and stationary under $\mathbb{Q}$:
Given $s \le t \le T$ and let $\wedge_s$ be a set in $\mathcal{F}_s$
$\forall\ \lambda \in \mathbb{R}$,

$$E_{\mathbb{Q}}\left[\exp i\lambda\,(X_t - X_s)1_{\wedge_s}\right] = \frac{1}{\mathrm{mgf}\,(\theta)^t} E_{\mathbb{P}}\left[\exp i\lambda\,(X_t - X_s)1_{\wedge_s}\exp\theta X_t\right]$$

$$= \frac{1}{\mathrm{mgf}\,(\theta)^t} E_{\mathbb{P}}\left[\exp\,(i\lambda+\theta)\,(X_t - X_s)1_{\wedge_s}\exp\,(\theta X_s)\right]$$

$$= \frac{1}{\mathrm{mgf}\,(\theta)^t} E_{\mathbb{P}}\left[\exp\,(i\lambda+\theta)\,(X_t - X_s)\right] E_{\mathbb{P}}\left[1_{\wedge_s}\exp\,(\theta X_s)\right]$$

(by independence and stationarity of the increments of the Lévy process X)

$$= \frac{E_{\mathbb{P}}\left[\exp\,(i\lambda X_{t-s})\exp\,(\theta X_{t-s})\right]}{\mathrm{mgf}\,(\theta)^{t-s}}\,\frac{E_{\mathbb{P}}\left[1_{\wedge_s}\exp\,(\theta X_s)\right]}{\mathrm{mgf}\,(\theta)^{s}}$$

$$= E_{\mathbb{Q}}\left[\exp\,(i\lambda X_{t-s})\right]\cdot E_{\mathbb{Q}}\left[1_{\wedge_s}\right]$$

Then, the increments are independent and stationary.

$$\mathrm{E}_{\mathbb{Q}}\left[\exp\,(\imath\lambda X_1)\right] = \mathrm{E}_{\mathbb{P}}\left[\exp\,(\imath\lambda X_1)\frac{\exp\,(\theta X_1)}{\Phi(-\imath)}\right]$$

$$= \frac{1}{\Phi(-\imath)}\mathrm{E}_{\mathbb{P}}\left[\exp\,(\imath(\lambda - \imath\theta)X_1)\right]$$

$$= \frac{1}{\Phi(-\imath)}\exp\Big\{\imath\gamma(\lambda-\imath\theta) - \frac{\sigma}{2}(\lambda-\imath\theta)^2$$
$$+ \int_{\mathbb{R}-0}\exp\,(\imath(\lambda-\imath\theta)x) - 1 - \imath(\lambda-\imath\theta)H(x)\nu(\mathrm{d}x)\Big\}$$

$$= \exp\Big(\imath\gamma\lambda + \gamma\theta - \frac{\sigma}{2}\lambda^2 + \imath\lambda\theta\sigma + \frac{\sigma}{2}\theta^2$$
$$+ \int_{\mathbb{R}-0}\mathrm{e}^{\imath\lambda x+\theta x} - 1 - \imath\lambda H(x) - \theta H(x)\nu(\mathrm{d}x)$$
$$-\theta\gamma - \frac{\theta^2}{2}\sigma - \int_{\mathbb{R}-0}\mathrm{e}^{\theta x} - 1 - \theta H(x)\nu(\mathrm{d}x)\Big)$$

$$= \exp\Big(\imath\lambda(\gamma + \theta\sigma + \int_{\mathbb{R}-0}(\exp\,(\theta x) - 1)H(x)\nu(\mathrm{d}x)) - \frac{\lambda^2\sigma}{2}$$
$$+ \int_{\mathbb{R}-0}(\exp\,(\imath\lambda x) - 1 - \imath\lambda H(x))\exp\,(\theta x)\nu(\mathrm{d}x)\Big)$$

And so we have the announced characteristic triplet from the Lévy Kintchine formula.
The valuation of derivatives requires the search for a probability measure $\mathbb{Q}$ such that the actualized price of the asset is a $\mathbb{Q}$ martingale. The following proposition gives a condition on the real θ in order to guarantee that the Esscher transform is an equivalent martingale measure.

Proposition 5.2: Let $\mathbb{Q}$ be a probability defined with respect to $\mathbb{P}$ by its Esscher transform associated with the real θ. One necessary and sufficient condition for $\mathbb{Q}$ to be an equivalent martingale measure is :

$$\mathrm{mgf}\,(\theta) = \mathrm{mgf}\,(\theta + 1) \tag{24}$$

Moreover, there is at best only one equivalent martingale measure defined by the Esscher transform (ie there is at the most only one real θ fulfilling the previous equality).

Proof :
Let Z_t be the Radon Nikodym derivative defined by $Z_t = \frac{d\mathbb{Q}}{d\mathbb{P}}$
$S_t \exp(-rt)$ is a $\mathbb{Q}$ martingale if and only if $S_t \exp(-rt) Z_t$ is a $\mathbb{P}$ martingale.
Given $s < t \leq T$,

$$E_{\mathbb{P}}\left[S_t \exp(-rt) Z_t | \mathcal{F}_s\right] = E_{\mathbb{P}}\left[\exp X_t \frac{\exp \theta X_t}{\mathrm{mgf}\,(\theta)^t} | \mathcal{F}_s\right]$$

$$= \frac{S_0}{\mathrm{mgf}(\theta)^t} \mathrm{E}_{\mathbb{P}}\left[\exp\{(\theta+1)(X_t - X_s) + (\theta+1) X_s\} | \mathcal{F}_s\right]$$

$$= \frac{S_0}{\mathrm{mgf}\,(\theta)^t} \exp\{(\theta+1) X_s\} \mathrm{E}_{\mathbb{P}}\left[\exp(\theta+1) X_{t-s} | \mathcal{F}_s\right]$$

(by independence and stationarity of the increments of X)
or, $X_t - X_s$ is independent of $\mathcal{F}_s$

$$= \frac{\mathrm{mgf}\,(\theta+1)^{t-s}}{\mathrm{mgf}\,(\theta)^t} S_0 \exp\{(\theta+1) X_s\}$$

$$= \left[\frac{\mathrm{mgf}\,(\theta+1)^q}{\mathrm{mgf}\,(\theta)^q}\right]^{t-s} \cdot S_0 \mathrm{mgf}\,(\theta)^{-s} \exp\{(\theta+1) X_s\}$$

$$= \left[\frac{\mathrm{mgf}\,(\theta+1)^q}{\mathrm{mgf}\,(\theta)^q}\right]^{t-s} \cdot S_0 Z_s \exp(X_s)$$

$$= \left[\frac{\mathrm{mgf}\,(\theta+1)^q}{\mathrm{mgf}\,(\theta)^q}\right]^{t-s} \cdot S_s \exp(-rs) Z_s$$

And then, $S_t \exp(-rt) Z_t$ is a $\mathbb{P}$ martingale if and only if :

$$\mathrm{mgf}\,(\theta) = \mathrm{mgf}\,(\theta + 1) \tag{25}$$

Moreover, as the moment generating function is convex on its definition domain, there could be at the most only one minimum on this domain. If there is no minimum then the moment generating function is increasing and there is no θ fulfilling the previous equation. In the case where the function admits a minimum, graphically there is only one unit segment with both extremities on the curve surrounding this minimum.
In the following, we will call this (unique) real matching equation (25) θ_m.

5.2 Valuation of European options with the help of the Esscher transform

Let $(X_t)_{0\leq t\leq T}$ be a Lévy process such that the real (θ_m) defined by equation (25) exists. We have defined an equivalent martingale measure $\mathbb{Q}$ such that the actualized price process is a $\mathbb{Q}$-martingale.

Remark 5.3: It is important to note at this point that nothing could allow us to say that the martingale measure $\mathbb{Q}$ defined by the Esscher transform will give us results in accordance with the market. This choice is an arbitrary one and is equivalent to defining a pricing kernel.

The price of an european option with maturity T at time $t \leq T$ is :

$$V(t) = \mathrm{E}_{\mathbb{Q}}\left[\mathrm{e}^{-r(T-t)} p(S_T) | \mathcal{F}_t\right] \tag{26}$$

where p is the payoff function of the considered european option.

Proposition 5.3: [24] The price of the european option (26) at time $t \leq T$ is given by the expectation under the historical probability $\mathbb{P}$ of another option with the following modified payoff :

$$\tilde{p}(S_t) = p(S_t)\left(\frac{S_T}{S_t}\right)^{\theta} \tag{27}$$

with the actualization factor $\tilde{r} = r(\theta+1) + \ln(\mathrm{mgf}(\theta))$

Proof :

$$\begin{aligned} V(t) &= \mathrm{E}_{\mathbb{Q}}\left[\mathrm{e}^{-r(T-t)} p(S_T) | \mathcal{F}_t\right] \\ V(t) Z_t &= \mathrm{E}_{\mathbb{P}}\left[\mathrm{e}^{-r(T-t)} p(S_T) Z_T | \mathcal{F}_t\right] \end{aligned}$$

Where $(Z_t)_{0\leq tT}$ is the Radon Nikodym density associated with the Esscher transform with parameter θ_m :

$$\begin{aligned} \frac{Z_T}{Z_t} &= \frac{\exp(\theta X_T)}{\mathrm{mgf}(\theta)^T} \cdot \frac{\mathrm{mgf}(\theta)^t}{\exp(\theta X_t)} \\ &= \frac{\exp(\theta X_T + \theta r T)}{\exp(\theta X_t + \theta r t)} \cdot \mathrm{mgf}(\theta)^{t-T} \exp\{\theta r t - \theta r T\} \\ &= \left(\frac{S_T}{S_t}\right)^{\theta} \exp\{-(T-t)(r\theta + \ln \mathrm{mgf}(\theta))\} \end{aligned}$$

Then,

$$\begin{aligned} V(t) &= \mathrm{E}_{\mathbb{P}}\left[\mathrm{e}^{-r(T-t)} p\left(S_T\right) \mathrm{e}^{-(T-t)(r\theta+\ln \operatorname{mgf}\theta)} \left(\frac{S_T}{S_t}\right)^{\theta} \Big| \mathcal{F}_t\right] \\ &= \mathrm{e}^{-(T-t)(r(1+\theta)+\ln \operatorname{mgf}\theta)} \cdot \mathrm{E}_{\mathbb{P}}\left[p\left(S_T\right) \left(\frac{S_T}{S_t}\right)^{\theta} \Big| \mathcal{F}_t\right] \end{aligned}$$

Remark 5.4: Links with the Girsanov formula.
We investigate for the real θ_m of the Esscher transform when the underlying X follows the geometric brownian motion process from the Black and Scholes model :

$$S_t = S_0 \mathrm{e}^{rt} \mathrm{e}^{\left[\left(\mu - r - \frac{\sigma^2}{2}\right)t + \sigma W_t\right]}$$

with $(W_t)_{0 \le t \le T}$ *a* $\mathbb{P}$- brownian motion.

$$\begin{aligned} \operatorname{mgf}_t(\theta) &= \mathrm{E}\left[\mathrm{e}^{\theta\left(\mu - r - \frac{\sigma^2}{2}\right) + \sigma\theta W_1}\right]^t \\ & \quad \text{With } W_1 \sim \mathcal{N}(0,1) \\ &= \left(\mathrm{e}^{\theta\left(\mu - r - \frac{\sigma^2}{2}\right)} \mathrm{E}\left[\mathrm{e}^{\sigma\theta W_1}\right]\right)^t \\ &= \exp\left\{\theta\left(\mu - r - \frac{\sigma^2}{2}\right)t + \frac{1}{2}\sigma^2\theta^2 t\right\} \end{aligned}$$

If we solve (24) :

$$\begin{aligned} \operatorname{mgf}_t(\theta_m) &= \operatorname{mgf}_t(\theta_m + 1) \Longleftrightarrow \\ (\theta_m + 1 - \theta_m)\left(\mu - r - \frac{\sigma^2}{2}\right) &= \frac{\sigma^2}{2}\left(\theta_m^2 - (\theta_m + 1)^2\right) \Longleftrightarrow \\ \theta_m &= -\frac{\mu - r}{\sigma^2} \end{aligned}$$

And then,

$$\begin{aligned} Z_t &= \frac{\exp \theta X_t}{\mathrm{E}\left[\exp \theta X_t\right]} \\ &= \frac{\exp\left(-\frac{\mu - r}{\sigma} W_t\right)}{\mathrm{E}\left[\exp\left(-\frac{\mu - r}{\sigma} W_t\right)\right]} \\ &= \exp\left(-\frac{\mu - r}{\sigma} W_t - \frac{t}{2}\left(\frac{\mu - r}{\sigma}\right)^2\right) \end{aligned}$$

The last formula appears to be the one given by the Girsanov theorem. The parameter is in the case the opposite of the well known market price of risk

$-\frac{\mu-r}{\sigma}$. This result was previsible, because as soon as we have proved the existence of the real θ_m, as the Black and Scholes model is complete, there is only one equivalent martingale measure.

6 Methods to valuate european derivatives

For most of the models, it is impossible to find a closed form solution even in simple cases like plain vanilla derivatives. Again, the Black and Scholes model appears as an exception. In the following, we suppose that the difficult choice of a martingale measure has been made (one can use the Escher transform for example).

6.1 Risk neutral density valuation

The knowledge of the density $f_{\mathbb{Q}}$ of S_T under the equivalent risk neutral measure $\mathbb{Q}$ allows us to have a closed form for the price of the european call with strike K and maturity T :

$$\begin{aligned} C(K,T) &= \mathrm{E}_{\mathbb{Q}}\left[\exp\{-rt\}\left(S_T - K\right)^+\right] \\ &= \exp\{-rt\}\int_0^\infty f_{\mathbb{Q}}(v,T)\,(v-K)^+\,\mathrm{d}v \\ &= \exp\{-rt\}\int_K^\infty v f_{\mathbb{Q}}(v,T)\,\mathrm{d}v - K\exp\{-rt\}\Pi_2 \end{aligned}$$

Where Π_2 is the probability for the call option to be in the money at expiration. But for most of the Lévy distribution used we have to consider this equation numerically. Moreover, we do not always know explicitly the density of S_t and so this method is of a limited interest in practice.

We will also remark that the calculation of the integral of the previous equation can be computationally very demanding and will not allow us to evaluate the position of a large book of option or to perform a calibration procedure. One solution for the pricing of european options with an underlying driven by the exponential of a Lévy process is to consider the Fourier transform theory.

6.2 Valuation with Fourier transform

This method has the following advantages :

1. The risk neutral density is rarely known, nevertheless we know from the Lévy Khintchine representation the equation for the Fourier transform of S_t
2. The algorithms used for the inversion of the Fourier transform are well known, fast and optimized, because of the huge use of Fourier theory in many fields like signal theory
3. If we use the algorithm named Fast Fourier Transform, then we will be able to valuate options with different strikes in a single calculation.

6.3 Method

The method described here is issued from [7]. We try to valuate an european call with underlying (S_t) and with strike K.
We call,

$$\begin{aligned} k &= \ln(K) \text{ and} \\ s_T &= \ln(S_T) \end{aligned}$$

Let $\Phi_T(u)$ be the characteristic function of the density of the logarithm of the underlying in T. $\forall\ u\ \in\ \mathbb{R}$,

$$\Phi(u) = \int_{-\infty}^{\infty} \mathrm{e}^{\imath us} q_T(s)\,\mathrm{d}s \tag{28}$$

with $q_T(s)$ the density of (s_T) under the risk neutral probability.

$$\begin{aligned} C_T &= \mathrm{e}^{-rt}\mathrm{E}_{\mathbb{Q}}\left[(S_T-K)^+\right] \\ &= \int_K^{\infty} \mathrm{e}^{-rt}\left(\mathrm{e}^s - \mathrm{e}^k\right) q_T(s)\,\mathrm{d}s \end{aligned}$$

Nevertheless, the C_T function is not square integrable in k which is a required condition to calculate the inverse Fourier transform. Because if $k \to -\infty$ i.e. $(K \to 0)$, $C_T \to S_0$. Carr and Madan suggested to calculate the Fourier transform of a modified call price $c_T(k) = \exp(\alpha k) C_T(k)$ for $\alpha > 0$ to ensure integrability in $-\infty$.

The Fourier transform of the modified price of the call is :

$$\Psi_T(v) = \int_{-\infty}^{+\infty} \mathrm{e}^{\imath vk} c_T(k)\,\mathrm{d}k \tag{29}$$

As $c_T = C_T(k)\exp(\alpha k) \underset{k\to-\infty}{\approx} S_0\exp(\alpha k)$ we have the integrability of the square of c_T in $-\infty$. Nevertheless, we may accentuate the problem in $+\infty$. We will come back on this problem once Ψ derivation done. For the moment, we suppose that $\Psi(0)$ is defined and then that c_T is integrable in $+\infty$.

The inversion formula of the Fourier transform gives :

$$c_T(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-\imath vk}\Psi_T(v)\,\mathrm{d}v \tag{30}$$

$$C_T(k) = \frac{1}{2\pi}\exp(-\alpha k)\int_{-\infty}^{+\infty} \mathrm{e}^{-\imath vk}\Psi_T(v)\,\mathrm{d}v \tag{31}$$

$C_T(k)$ is real and then $\forall k \in \mathbb{R}$,

$$\mathrm{Im}\left(\int_{-\infty}^{+\infty} \mathrm{e}^{-\imath vk}\Psi_T(v)\,\mathrm{d}v\right) = 0 \tag{32}$$

We call respectively $a(v)$ and $b(v)$ the real part and the imaginary part of $\Psi_T(v)$.

$$\begin{aligned} a &: v \longrightarrow \int_{-\infty}^{\infty} \cos(vk)C_T(k)\mathrm{d}k \\ b &: v \longrightarrow \int_{-\infty}^{\infty} \sin(vk)C_T(k)\mathrm{d}k \end{aligned}$$

a is even and b is odd. Then, $\forall v \in \mathbb{R},\ \Psi(-v) = a(v) - \imath b(v)$

Let, A and B be the functions defined for all $k \in \mathbb{R}$ by :

$$\begin{aligned} A(k) &= \int_{-\infty}^{0} \mathrm{e}^{-\imath vk}\Psi(v)\mathrm{d}v \\ B(k) &= 2\pi\exp(\alpha k)C_T(k) - A(k) \\ &= \int_{0}^{+\infty} \mathrm{e}^{-\imath vk}\Psi(v)\mathrm{d}v \end{aligned}$$

With the change of variable $v \to -v$,

$$\begin{aligned} A(k) &= \int_{+\infty}^{0} -\mathrm{e}^{\imath vk}\Psi_T(-v)\mathrm{d}v \\ &= \int_{0}^{+\infty} \Big[\cos(vk)a(v) + \sin(vk)b(v) + \imath\Big(\sin(vk)a(v) - b(v)\cos(vk)\Big)\Big]\,\mathrm{d}v \end{aligned}$$

If we compare this last equation with :

$$\begin{aligned} B(k) &= \int_{0}^{+\infty} \mathrm{e}^{-\imath vk}\Psi_T(v)\mathrm{d}v \\ &= \int_{0}^{+\infty} \Big[\cos(vk)a(v) + \sin(vk)b(v) - \imath\Big(\sin(vk)a(v) - b(v)\cos(vk)\Big)\Big]\,\mathrm{d}v \end{aligned}$$

Then,

$$\mathrm{Re}\,[A(k)] = \mathrm{Re}\,[B(k)] \tag{33}$$

$$\mathrm{Im}\,[A(k)] = -\mathrm{Im}\,[B(k)] \tag{34}$$

$$\begin{aligned} 2\pi\exp(\alpha k)C_T(k) &= A(k) + B(k) \\ &= \mathrm{Re}\,[A(k)] + \mathrm{Re}\,[B(k)] \quad \text{given (34)} \\ &= 2\mathrm{Re}\,[B(k)] \quad \text{with (33)} \end{aligned}$$

finally,

$$C_T(k) = \frac{\exp(-\alpha k)}{\pi}\mathrm{Re}\left[\int_0^{+\infty} \mathrm{e}^{-\imath v k}\Psi(v)\mathrm{d}v\right]$$

We try to find Ψ_T as a function of Φ_T. From (29),

$$\Psi_T(v) = \mathrm{e}^{-rt}\int_{-\infty}^{+\infty}\int_k^{+\infty} \mathrm{e}^{\alpha k}\mathrm{e}^{\imath v k}\left(\mathrm{e}^s - \mathrm{e}^k\right) q_T(s)\mathrm{d}s\mathrm{d}k \tag{35}$$

The integration domain is defined by the upper half plane defined by equation $s = k$. With the use of the Fubini theorem,

$$\begin{aligned}
\Psi_T(v) &= \mathrm{e}^{-rt}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{s} \mathrm{e}^{\imath v\alpha k+\alpha k+s} - \mathrm{e}^{\imath v k+k(\alpha+1)}\right) q_T(s)\mathrm{d}k\mathrm{d}s \\
&= \mathrm{e}^{-rt}\int_{-\infty}^{+\infty} q_T(s)\left[\frac{\mathrm{e}^{\imath v\alpha k+\alpha k+s}}{\imath v+\alpha} - \frac{\mathrm{e}^{\imath v k+k(\alpha+1)}}{\imath v+\alpha+1}\right]_{\infty}^{s}\mathrm{d}s \\
&= \mathrm{e}^{-rt}\int_{-\infty}^{+\infty} q_T(s)\left(\frac{\mathrm{e}^{\imath v\alpha k+\alpha k+s}}{\imath v+\alpha} - \frac{\mathrm{e}^{\imath v k+k(\alpha+1)}}{\imath v+\alpha+1}\right)\mathrm{d}s \\
&= \mathrm{e}^{-rt}\int_{-\infty}^{+\infty} q_T(s)\frac{\mathrm{e}^{(\imath v+\alpha+1)s}}{(\imath v+\alpha)(\imath v+\alpha+1)}\mathrm{d}s \\
&= \frac{\mathrm{e}^{-rt}\Phi_T(v-\imath(1+\alpha))}{\alpha^2+\alpha-v^2+\imath v(2\alpha+1)}
\end{aligned}$$

Remark 6.1: The integrability condition in $+\infty$ on α which was $\Psi_T(0) < \infty$ becomes $\Phi_T(0-\imath(1+\alpha)) < \infty$, then :

$$\int_{-\infty}^{+\infty} \mathrm{e}^{(1+\alpha)s} q_T(s)\mathrm{d}s < \infty$$

and then we should have that, $\mathrm{E}_{\mathbb{Q}}\left[S_T^{\alpha+1}\right] < \infty$.

we get,

$$C_T(k) = \frac{\mathrm{e}^{-\alpha k}\mathrm{e}^{-rt}}{\pi}\mathrm{Re}\left[\int_0^{+\infty}\frac{\mathrm{e}^{-\imath v k}\Phi_T(v-\imath(1+\alpha))}{\alpha^2+\alpha-v^2+\imath v(2\alpha+1)}\mathrm{d}v\right] \tag{36}$$

Remark 6.2: We check that if $\alpha = 0$, ie that we consider the non modified price of the call, we have valuation problem under the sum sign in zero.

Remark 6.3: Carr and Madan suggest to choose α close to $0,25$. W. Schoutens proposes 0.75. The choice of α plays a role on the convergence speed.

6.4 Discretisation and FFT

We try to discretize the following integral :

$$\begin{aligned} C_T(k) &= \frac{\exp(-\alpha k)}{\pi}\mathrm{Re}\left[\int_0^{+\infty} \mathrm{e}^{-\imath vk}\Psi(v)\mathrm{d}v\right] \\ &\approx \frac{\exp(-\alpha k)}{\pi}\mathrm{Re}\left[\int_0^{(N-1)\eta} \mathrm{e}^{-\imath vk}\Psi(v)\mathrm{d}v\right] \end{aligned}$$

Where η is the integration step and $N \in \mathbb{N}$ is an typically large integer. By using the trapeze method (with a one half coefficient for the first and the last terms of the sum)

$$C_T(k) \approx \frac{\exp(-\alpha k)}{\pi}\mathrm{Re}\left[\sum_{j=0}^{N-1} \mathrm{e}^{-\imath v_j k}\Psi_T(v_j)\cdot\eta\cdot w_j\right]$$

With $v_j = \eta \cdot j$ for $j = 0 \cdots N-1$

$$\text{and } w_j = \begin{cases} \frac{1}{2} & j=0 \text{ or } j=N-1 \\ 1 & otherwise \end{cases}$$

We center our options list on those which are at the money (K=1, ie k=0).

$$\begin{aligned} k_u &= -b+\lambda u \quad u=0\cdots N-1 \\ \text{with } \lambda &= \frac{2b}{N-1} \end{aligned}$$

And then,

$$\begin{aligned} C_T(k_u) &\approx \frac{\exp(-\alpha k)}{\pi}\mathrm{Re}\left[\sum_{j=0}^{N-1} \mathrm{e}^{-\imath\eta j(-b+\lambda u)}\Psi_T(v_j)\cdot\eta\cdot w_j\right] \\ &\approx \frac{\exp(-\alpha k)}{\pi}\eta\cdot\mathrm{Re}\left[\sum_{j=0}^{N-1} \mathrm{e}^{-\imath\eta j\lambda u}\left(\Psi_T(v_j)\mathrm{e}^{\imath\eta bj}\cdot w_j\right)\right] \end{aligned}$$

The Fast Fourier Transform algorithm allows us to calculate the N values of the sum :

$$w(u) = \sum_{j=0}^{N-1} \mathrm{e}^{-\imath\frac{2\pi}{N}u\cdot j}x(j) \qquad u=0\cdots N-1 \tag{37}$$

with not N^2 multiplication but only $N \ln N$.
In order to use this algorithm with (37), it is required to choose λ, η and N such that :

$$\eta \cdot \lambda = \frac{2\pi}{N} \tag{38}$$

Remark 6.4: This constraint is one of the drawback of this method. If we choose η (the discretisation step of the integral) small, then the strikes grid k will admit a larger step.

7 Valuation of path dependents options

We will develop here some numerical examples to show in practice how to price exotic path dependent options (an up and in call and an up and out call on the Standard and Poors 500 index) when the underlying is supposed to follow the exponential of a particular Lévy process called the Variance Gamma. This study uses some market data, like the prices of european calls on the S&P 500 at some fixed date. We have chosen to adjust the model parameters on the market value of those european calls. This method fully agrees with some form of market efficiency and the Markov property as we use only present values to develop our pricing model. In particular, we will not use in the following any statistics about time series or data from the past to price those exotic derivatives.

The first step is to adjust or calibrate the pricing model on the data. For this, we will choose the parameters of the Variance Gamma in order to minimize the quadratic error between the market prices of the call options and the call options prices given by the model. We will show that the Variance Gamma process as many other models like the Normal Inverse Gaussian (NIG), the Meixner model and the Carr, Madan, Geman, Yor model (CGMY) give a quite good fit of the the market prices, far better than the Black and Scholes one with a unique volatility parameter. Next, we will simulate with the help of Monte Carlo techniques a large number of paths of the Variance Gammma process with the optimized parameters to price the previously described barrier options. The prices are given for many value of the barrier and the results are compared with the Black and Scholes ones where some closed formula are available.

7.1 Model calibration

7.1.1 Data

The call options prices on the S&P 500 at 12.41 pm the 27th of June 2006 are given in the Appendix one. The prices are available from July 2006 to December 2007. At that time, the S&P 500 Index quote was 1243.73.

7.1.2 Variance Gamma process

The characteristic function of the Variance Gamma $VG(\sigma,\nu,\theta)$ process is :

$$\Phi_{VG(\sigma,\nu,\theta)}(u) = (1 - \imath u\theta\nu + \frac{1}{2}\sigma^2\nu u^2)^{-1/\nu} \qquad (39)$$

As this distribution is infinitely divisible, we can give the following definition for the Variance Gamma process :

Definition 7.1: A stochastic process $(X_t)_{t\geq 0}$ is a Variance Gamma process if :

1. $X_0 = 0$ p.s.,
2. the increments are independent and stationary,
3. for all $t, s \geq 0$, the increments $X_{t+s} - X_s$ are distributed according to the Variance Gamma law with parameters $(\sigma\sqrt{s}, \nu/\sqrt{s}, s\theta)$.

The Variance Gamma process is a process with finite variation (It also implies that it has no brownian component).
One can easily simulate on a regular grid time the Variance Gamma process with the procedures of Johnk and Best (see [10])

7.1.3 Normal Inverse Gaussian Process

The Normal Inverse Gaussian law $NIG(\alpha, \beta, \delta)$ with parameters $\alpha > 0$, $-\alpha < \beta < \alpha$ and $\delta > 0$ is defined by its characteristic function :

$$\Phi_{NIG(\alpha,\beta,\delta)}(u) = \exp\left(-\delta(\sqrt{\alpha^2 - (\beta + \imath u)^2} - \sqrt{\alpha^2 - \beta^2})\right) \tag{40}$$

We define the Normal Inverse Gaussian process from the characteristic function like the Variance Gamma process : $(X_t)_{t\geq 0}$ is a Normal Inverse Gaussian process with parameters (α, β, δ) if $X_0 = 0$ and if the increments $X_{t+s} - X_t$ are independent, stationary and distributed according to the $NIG(\alpha, \beta, s\delta)$ distribution.
The NIG process is an infinite variation process without any brownian component.

7.1.4 Meixner process

The Meixner distribution(α, β, δ) is defined for $\alpha > 0$, $-\pi < \beta < \pi$ et $\delta > 0$
Its characteristic function is :

$$\Phi_{Meixner(\alpha,\beta,\delta)}(u) = \left(\frac{\cos\left(\frac{\beta}{2}\right)}{\cosh\left(\frac{\alpha u - \imath\beta}{2}\right)}\right)^{2\delta} \tag{41}$$

A Meixner process is a process $(X_t)_{t\geq 0}$ starting from zero in $t = 0$, with independent and stationary increments distributed according the $Meixner(\alpha, \beta, s\delta)$ distribution.
The Meixner process like the NIG one is an infinite variation process without any brownian part.

7.1.5 CGMY process

The four parameters C,G,M and Y of this distribution are named according to the first letters of P. Carr, H. Geman, D.B. Madan and M. Yor. The characteristic function is :

$$\Phi_{C,G,M,Y}(u) = \exp\left(C\Gamma(-Y)\left((M-\imath u)^Y - M^Y + (G+\imath u)^Y - G^Y\right)\right) \quad (42)$$

with $C, G, M > 0$ and $-\infty < Y < 2$.
the Γ function is defined for all strictly positive real a by :

$$\Gamma(a) : a \longrightarrow \int_0^\infty \exp(-t) t^{a-1} \mathrm{d}t$$

and extended to the negative non integer by the formula : $\Gamma(a+1) = a\Gamma(a)$. Again, the CGMY process is a process starting from zero, with independent and stationary increments distributed according to the $CGMY(Cs, G, M, Y)$ law. The CGMY is of infinite variation if $1 \leq Y < 2$ and it has no brownian component.

Remark 7.1: The four previously defined processes are all Lévy processes because the considered distributions are all infinitely divisible. (See [25])

7.1.6 Modeling of the S&P 500 Index

Our study follows the method initially suggested by Madan and Schoutens ([21], [26]) :
Under the historical probability the price process can be written :

$$S_t = S_0 \exp(mt + X_t(\sigma_s, \nu_s, \theta_s) + \omega_s t) \quad (43)$$

where the subscript s is to keep in mind that the parameters are the one under the historical probability. ω_s is chosen such that it cancel the drift of the Variance Gamma process $X_t(\sigma_s, \nu_s, \theta_s)$: $\omega_s = \frac{1}{\nu_s} \ln\left(1 - \theta_s\nu_s - \sigma_s^2\nu_s/2\right)$ and m is the expected rate of return under the historical probability measure and is more complicated to estimate.

Thus, we choose to estimate the parameters directly under one risk neutral probability :

$$S_t = S_0 \exp(rt + X_t(\sigma_{RN}, \nu_{RN}, \theta_{RN}) + \omega_{RN} t) \quad (44)$$

The subscript RN is for the risk neutral parameters. Thus, we choose ω_{RN} such that the actualized risk neutral price process is a martingale. From a practical point of view, we evaluate the characteristic function of the Variance Gamma process $X_t(\sigma_{RN}, \nu_{RN}, \theta_{RN})$ in $1/\imath$.

$$\omega_{RN} = \frac{1}{\nu_{RN}} \ln\left(1 - \theta_{RN}\nu_{RN} - \sigma_{RN}^2\nu_{RN}/2\right) \quad (45)$$

Models	R	Parameters			
Black Scholes		σ			
	6,73	0,1806			
VG		C	G	M	
	3,57	1,6008	7,6897	29,2489	
		α	β	δ	
		0,1193089	0,6246870	-0,15344	
NIG		α	β	δ	
	4,81	3,688	-3,681	0,0409	
Meixner		α	β	δ	
	4,08	0,3428	-1,492	0,299	
CGMY		C	G	M	Y
	3,98	0,017	0,1087	7,55	1,2955

Tab. 1: Estimate of the parameters

7.1.7 Algorithm

The Matlab Code used to calibrate the different models is provided in the appendix.

For the set of market prices of N calls, we choose the risk neutral parameters for which the sum of the quadratic error between market prices and the prices given by the model of the call options is minimum. Thus, we minimize the quantity :

$$R_{\sigma_{RN},\nu_{RN},\theta_{RN}} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\Big(\text{market price}_i - \quad \text{calculated price}_i(\sigma_{RN},\nu_{RN},\theta_{RN})\Big)^2} \tag{46}$$

over the parameters σ_{RN}, ν_{RN} and θ_{RN}
Calls prices are calculated according to the method of P.Carr and D.B. Madan developed in the previous chapter. The grid of the logarithm of the strike is chosen in order to interpolate with an acceptable error the prices of options for the strikes which are really traded on the market.

7.1.8 Results

The results are given table 7.1.8. Black and Scholes model is the one which give the worst fit, but we should keep in mind that this model is the only one with only one degree of freedom. To judge the quality of the calibration for each model it is useful to consider the graph 7.1.8. Those graphs allow us to check if all the prices given by the model are acceptable compared to the market prices. For example, the Black and Scholes model seems to under evaluate the calls out of the money and to over evaluate the calls in the

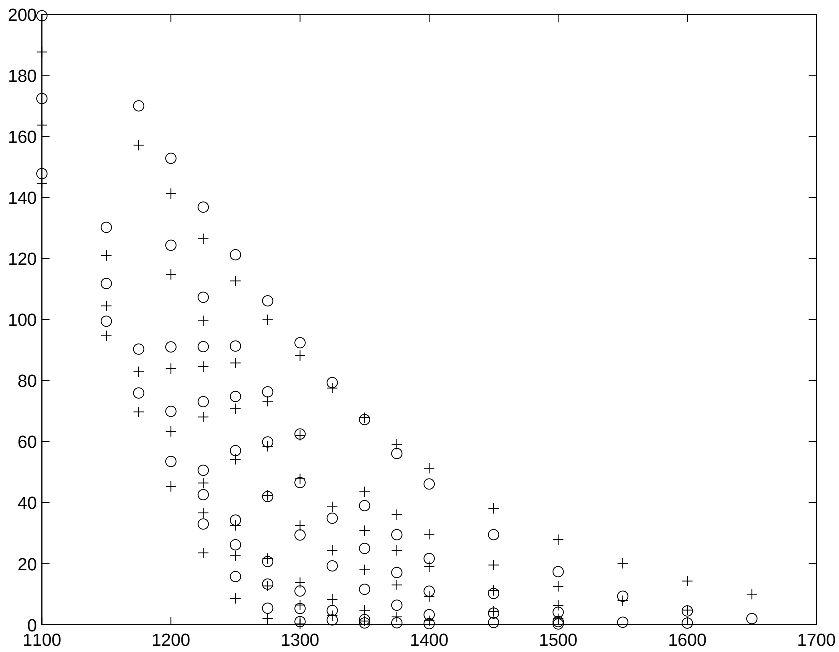

Fig. 3: Calibration of the Black Scholes Merton model

money (according to the well known phenomena of smile, which in the case of the S&P 500 could be approximated by a decreasing affine function of the strike). The fit is better for the others models, but it seems that the smile effect is not fully taken into account and that we still have some valuation problem for the call options far out of the money.

The implied volatility surface of the S&P 500 the 27th of june 2006 is given figure 8. It is interesting to compare this graphs with the one given by the fitted models as practitioners are always discussing not about prices but about the corresponding Black and Scholes volatilities. If the main shape of those graph are the same, there could be some differences especially at the frontier of the surface. That seems to be due to the calibration procedure itself : we have carefully selected the prices of the call options which are significant in terms of liquidity and we have some holes in the grid of call prices due to the bid ask spread which was so high compared to the price itself that the Put Call parity was difficult to apply. As a conclusion, it seems that the use of the exponential of some Lévy processes models allow to have a better fit of the market liquid prices. We should remark that our study is static in the sense that we have not checked the stability of the parameters over time.

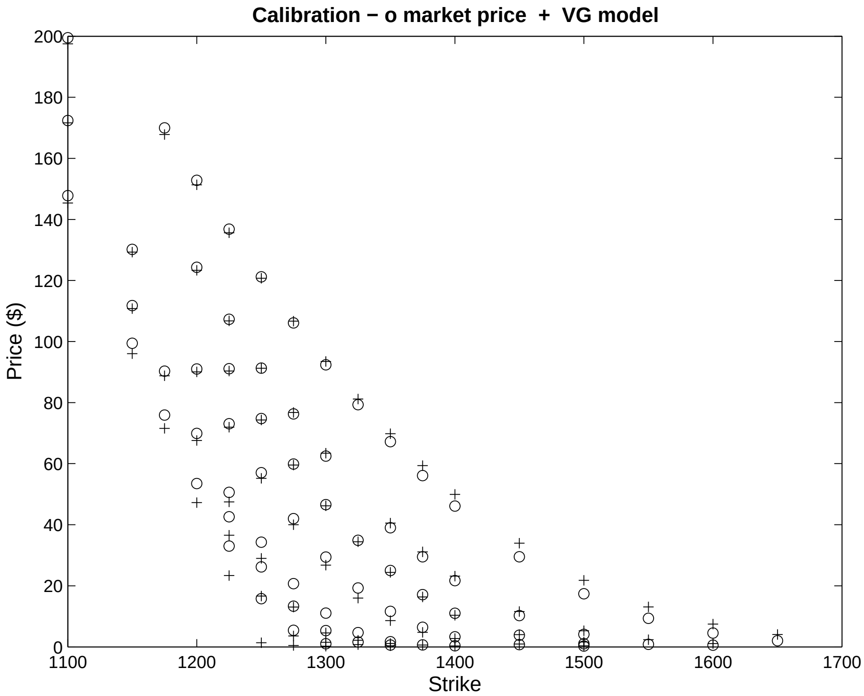

Fig. 4: Calibration of the Variance Gamma model

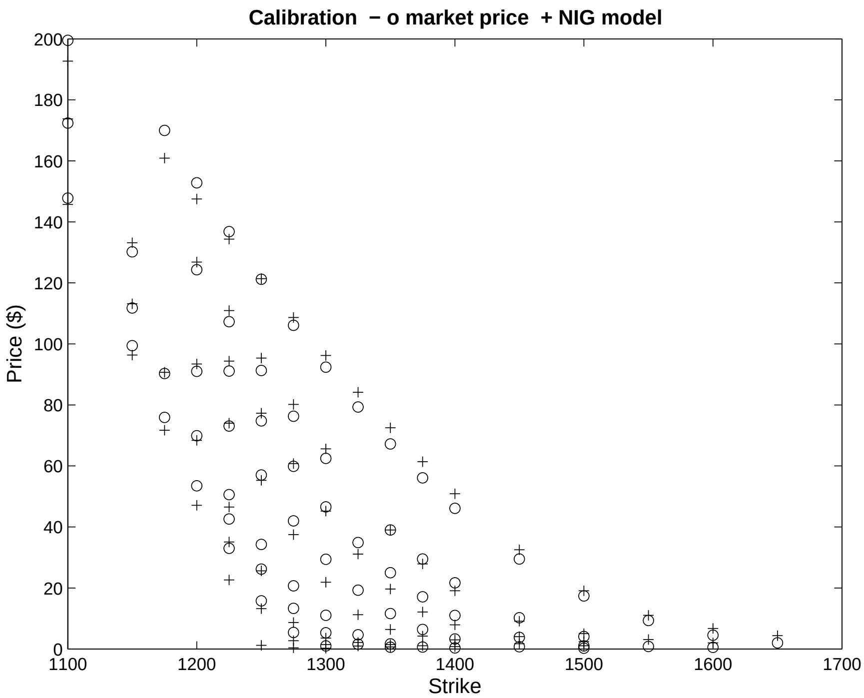

Fig. 5: Calibration of the NIG model

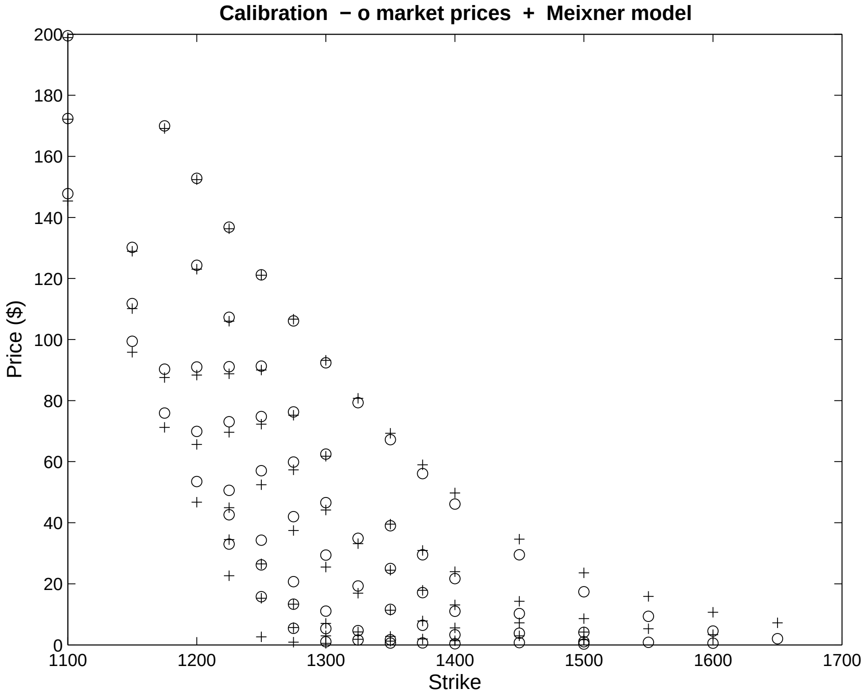

Fig. 6: Calibration of the Meixner model

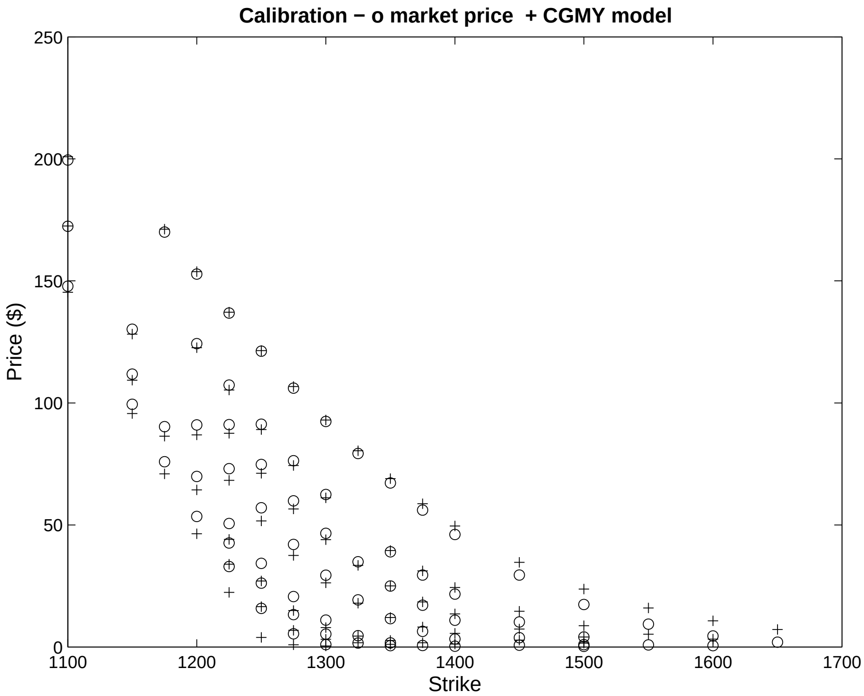

Fig. 7: Calibration of the CGMY model

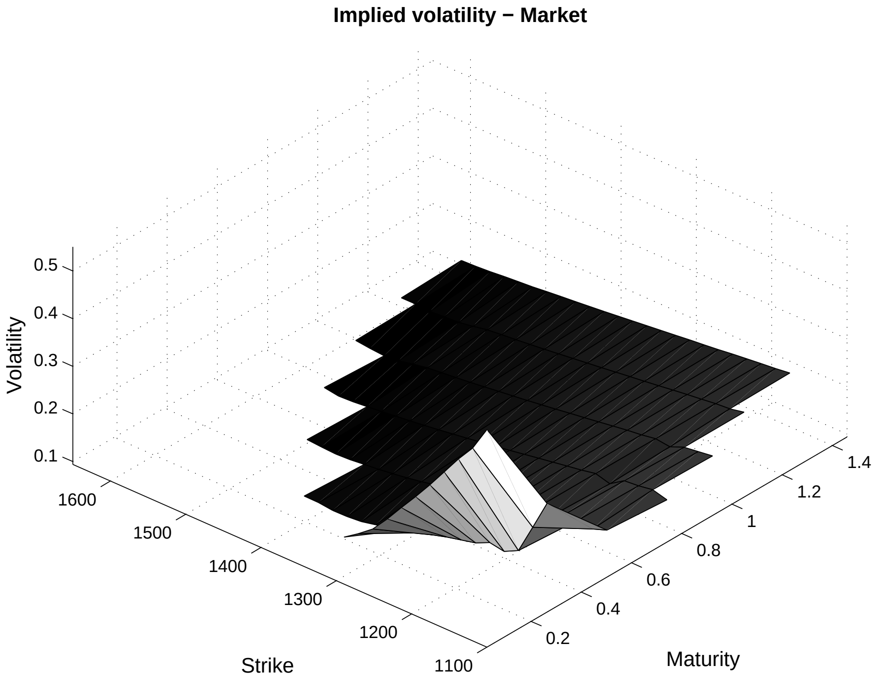

Fig. 8: implied volatility- S&P 500 June the 27th 2006

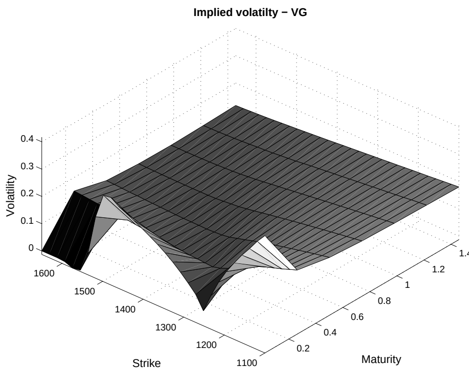

Fig. 9: Variance Gamma

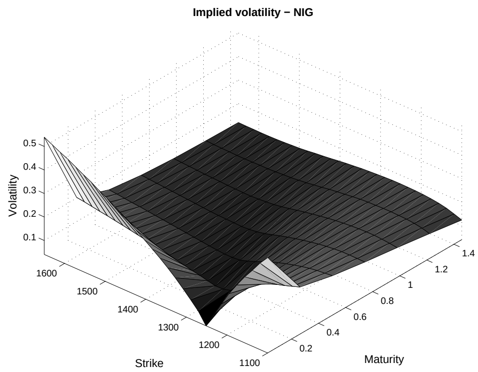

Fig. 10: NIG

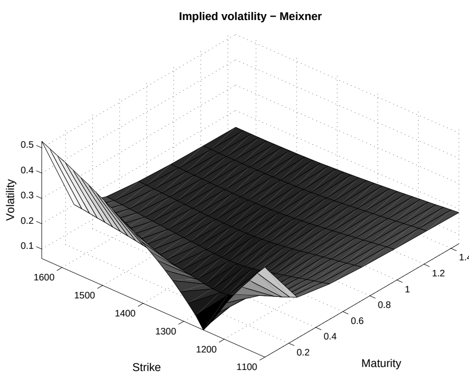

Fig. 11: Meixner

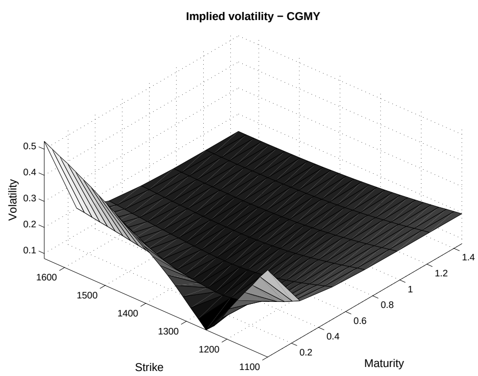

Fig. 12: CGMY

7.2 Valuation of the Up and In Call and of the Up and Out Call

7.2.1 Barrier options

The payoff of a barrier option depends on the fact that the underlying has reached or not a certain level H called the barrier. It is possible to conceive barrier options with two or more barriers, but in the following, we will concentrate on the most simple barrier option with only one barrier and a call option payoff.

Up an in call
The payoff of an up and in call with strike K and barrier H is equal to the payoff of the european call, if the underlying reached or crossed between time zero and T the barrier H. If the barrier has not been reached, then the payoff is zero.
The price in $t = 0$ of the up and in call is simply given by the expectation under the (or one chosen) risk neutral probability measure of the actualized payoff :

$$C_{UI} = \exp(-rT)\mathrm{E}_{\mathbb{Q}}\left[(S_T - K)^+ 1_{M_T^S \geq H}\right] \tag{47}$$

where M_T^S is the maximum of the asset S between time 0 and T.
If $H \leq K$, the up and in call and the european call with strike K and maturity T have the same value. ($S_T - K > 0$ implies that the barrier $H \leq K$ has been reached before T).

Up and out call
Conversely to the up and in call, the up and out call looses its value if the maximum of the underlying S between 0 and T is greater than the barrier H. The value in $t = 0$ of the up and out call is :

$$C_{UO} = \exp(-rT)\mathrm{E}_{\mathbb{Q}}\left[(S_T - K)^+ 1_{M_T^S < H}\right] \tag{48}$$

If $H \leq K$, then the value of the up and out call is simply zero.

Remark 7.2: Consider an up and in call and an up and out call with same strike and maturity on the same underlying S. We have,

$$\begin{aligned}
C_{UI} + C_{UO} &= \exp(-rT)\mathrm{E}_{\mathbb{Q}}\left[(S_T - K)^+ 1_{M_T^S \geq H}\right] + \\
&\quad \exp(-rT)\mathrm{E}_{\mathbb{Q}}\left[(S_T - K)^+ 1_{M_T^S < H}\right] \\
&= \exp(-rT)\mathrm{E}_{\mathbb{Q}}\left[(S_T - K)^+ \left(1_{M_T^S \geq H} + 1_{M_T^S < H}\right)\right] \\
&= \exp(-rT)\mathrm{E}_{\mathbb{Q}}\left[(S_T - K)^+\right]
\end{aligned}$$

Thus, the sum of the up and in call and of the up and out call is equal to the european option with the same maturity T and strike K.

7.2.2 Pricing in the Black and Scholes framework

In the Black and Scholes framework, we can find a closed form solution for the price of the up an in call and the up and out call (see [16]) :

$$\begin{aligned} C_{UI}^{BS} &= S_0 N(x_1)\exp(-qT) - K\exp(-rT)N(x_1 - \sigma\sqrt{(T)}) \\ &\quad -S_0\exp(-qT)\left(\frac{H}{S_0}\right)^{2\lambda}(N(-y) - N(-y_1)) + \\ &\quad K\exp(-rT)\left(\frac{H}{S_0}\right)^{2\lambda-2}(N(-y+\sigma\sqrt{t}) - N(-y_1+\sigma\sqrt{t})) \end{aligned}$$

With,

$$\begin{aligned} \lambda &= \frac{r - q + \frac{1}{2}\sigma^2}{\sigma^2} \\ y &= \frac{\log\left(\frac{H^2}{S_0 K}\right) + \lambda\sigma\sqrt{T}}{\sigma\sqrt{T}} \\ y_1 &= \frac{\log\left(\frac{H}{S_0}\right) + \lambda\sigma\sqrt{T}}{\sigma\sqrt{T}} \\ x_1 &= \frac{\log\left(\frac{S_0}{H}\right) + \lambda\sigma\sqrt{T}}{\sigma\sqrt{T}} \end{aligned}$$

If we use remark 7.2,

$$C_{UO}^{BS} = C^{BS} - C_{UI}^{BS}$$

7.2.3 Monte Carlo simulations

We valuate both the up and in call and the up an out call with strike $K = 1250$ on the S&P 500 for different values of the barrier H between 1260 and 1200. The maturity of the considered options is one year ($T = 1$). We consider as in the previous chapter that the S&P500 is supposed to follow the exponential of a Variance Gamma process with the previous estimated parameters.
The classical Monte Carlo procedure to evaluate those derivatives is the following :

1. The parameters of the risk neutral process are calibrated on the market prices of european calls according to the previous part.

Strike	Barrier						
	1260	1280	1300	1320	1350	1380	1400
CUI	94.32	93.26	92.5	91.91	88.71	81.12	72.75
CUO	0.03	0.3	0.96	2.43	6.63	13.00	20.07
Strike	Barrier						
	1420	1440	1460	1480	1500	1550	1600
CUI	68.56	60.16	47.51	40.27	32.68	16.34	8.41
CUO	27.01	35.46	45.84	54.61	62.66	78.28	86.26

Tab. 2: Up and In, Up and Out call options prices - VG model

2. A large number N of trajectories of this process is simulated on a regular time grid .
3. For each of the trajectories, the payoff is evaluated. For example, the payoff of the up and in call for the trajectories i is defined by :

$$P_i = \exp(-rT)\mathrm{E}_{\mathbb{Q}}\left[(S_T - K)^+ 1_{M_S^i \geq H_a}\right] \tag{49}$$

   where M_S^i is the maximum of S evaluated on the considered time grid. H_a is the adapted barrier level to take into account the fact that we only estimate the maximum on a discrete time grid and can be seen as a correction parameter. We have chosen to use the formula given by Broadie (1997) in the Black and Scholes framework for the up and out call.

$$H_a = H \exp(0.582\sigma\sqrt{\delta_t}) \tag{50}$$

4. One estimate is given by the actualized mean of the payoff corresponding to the N trajectories :

$$\widehat{C}_{UO} = \exp(-rT)\frac{1}{N}\sum_{i=1}^{N} P_i \tag{51}$$

7.2.4 Results

The results (table 7.2.4) from the VG model are very different from the results of the Black and Scholes model. Other studies like [26] show that for other exotic derivatives the prices issued from the Lévy model CGMY, Meixner, Variance Gamma and NIG are quite close but are very different from the one given by the Black and Scholes model with constant volatility.

Remark 7.3: We have used in this procedure the volatility issued from the calibration of the Black and Scholes model (0.18). If we use the implied volatility parameter corresponding to the strike and maturity of the considered

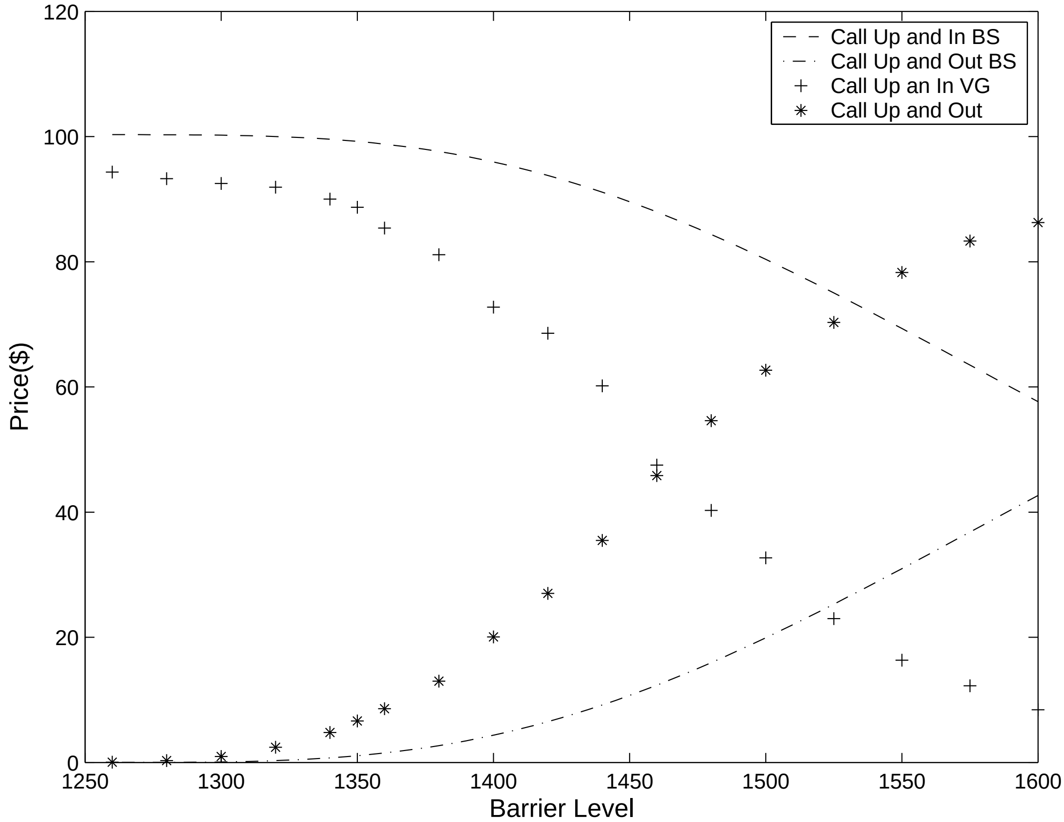

Fig. 13: Evaluation of Up and out and up and in call with VG model and BS model ($\sigma = 0.1489$) for different barrier levels

options, we still have some values for the barrier options which are very different from the variance gamma model.

8 Conclusion

We have shown that when we describe the price of a risky asset with the exponential of a Lévy process, in many cases we can keep the absence of arbitrage property. Nevertheless, except for the brownian and the pure Poisson models, we have to deal with incomplete markets and to choose in one way or another an equivalent martingale measure to get our pricing rule.

In our opinion, incompleteness of the model should not be considered as a drawback of the model but as something positive. Clearly, real markets are incomplete, as the number of relevant uncertainty sources is probably more important than the number of available assets on the market.

The numerical part gives interesting results : first, the calibration accuracy is better for the considered models than the Black Scholes one and the volatility smile is quite well reproduced. But, we have to keep in mind that in some sense it is normal as we use up to four degrees of freedom in those models and just one (the volatility) in the Black and Scholes one. It could also have been interesting to check if the results issued from the calibration procedure are stable : if we try to reproduce those results one week later or even one day later, it is not obvious that we will get approximately the same set of parameters for the models.

Our study has just focused on the pricing of exotics derivatives. and we have not discussed how to hedge them. Nevertheless, pricing and hedging are tightly related and we could also have defined the price of the derivative in a more conservative way either as the price to cover all the risks (superhedging price) or to cover just one part of the risks (quantile hedging and mean variance hedging).
Another possible extension of this thesis is the case with more than one risky underlying. In particular, it seems interesting to specify the correlation between the primary assets in a multi dimensional Lévy measure and to extend the pricing method to derivatives which depend on more than one asset. Surprisingly, the number of publications on this topic is very limited compared to the single underlying case.

Appendix

A Prices of Standard&Poors 500 european call options - June, 27th 2006 at 12.51pm

Strike	Maturité						
	Jul.06	Aug.06	Sep.06	Dec06	Mar07	Jun07	Dec07
1100	147.8			172.4		199.5	
1150	99.45		111.8	130.2			
1175	75.9		90.3				170.0
1200	53.5		69.9	91.0		124.3	152.8
1225	33.0	42.6	50.6	73.1	91.1	124.3	152.8
1250	15.8	26.2	34.3	57.05	74.8	91.3	121.2
1260	11.1	28.5					
1275	5.4	13.3	20.7	42.0	59.9	76.3	106.1
1300	1.0	5.3	11.05	29.4	46.6	62.5	92.4
1325	0.25	1.6	4.65	19.3	34.9		79.3
1350	0.15	0.575	1.675	11.6	25	39.0	67.2
1375	0.1	0.25	0.65	6.4	17.1	29.5	56.1
1400			0.35	3.275	11.0	21.7	46.1
1450			0.3	0.725	3.8	10.25	29.5
1500				0.275	1.0	4.1	17.4
1550						0.85	9.35
1600						0.55	4.5
1250							2.0

The 27th of June 2006 at 12.51pm, The Standard and Poors 500 index quote was 1243.73. The estimated continuous dividend rate was 1.65% and the risk free rate estimated with the future quotes was 5.0%.

B Matlab code

Calibration program : example for the CGMY process

```
function principale= CGMYcalibration clear;
%format compact;

%Definition of the global variable
global ma strike t S_0 size_S size_T lstrike pmarchenorm r q a N
eta b lambda alpha0 u x T B pma sma tma maturg strikeg A

%matrix of the prices of european call options
ma = [ 147.80 0 0 172.4 0 199.5 0; 99.45 0 111.8 130.2 0 0 0 ;
75.9 0 90.3 0 0 0 170.0 ; 53.5 0 69.9 91.0 0 124.3 152.8 ;33.0
42.6 50.6 73.1 91.1 107.3 136.8  ...
        ;15.8 26.2 34.3 57.05 74.8 91.3 121.2; 11.1 0 28.5 0 0 0 0;
        5.4 13.3 20.7 42 59.9 76.3 106.1; ...
        1.0 5.3 11.05 29.4 46.6 62.5 92.4; 0.25 1.6 4.65 19.3 34.9 0 79.3
         ; 0.15 0.575 1.675 11.6 25.0 39.0 67.2;
         0.1 0.25 0.65 6.4 17.1 29.5 56.1; 0 0 0.35 3.275 11.0 21.7 46.1;...
        0 0 0.3 0.725 3.8 10.25 29.5 ; 0 0 0 0.275 1 4.1 17.4; ...
        0 0 0 0 0  0.85 9.35;0 0 0 0 0 0.55 4.5;0 0 0 0 0 0 2];
%Strikes
strike= [ 1100 1150 1175 1200 1225 1250  1260 1275 1300 1325 1350
1375 1400 1450 1500 1550 1600 1650];

%Maturity of the considered options
T = [0.023904 0.10757 0.19522 0.4502 0.6932 0.9442 1.45817];

% Market price of the risky underlying a time 0
S_0= 1243.73;

% parameters of the model
    %risk free rate
    r=0.05;
    %dividends
    q=0.0165;

%preliminary calculus  (normed by S_0)
    size_S=length(strike);
    size_T=length(T);
%log-strike normalize
    lstrike= log(strike/S_0);
```

```
%normalized matrix of prices
    pmarchenorm=ma'/S_0;

%integration grid - log strike grid
%sommecarre=[];
a = 600*2;             % integration between  0
%and  a for the inverse Fourier transform
N =  4096*2;           % number of  strike pour for FFT
A=zeros(size_T,N); B=zeros(size_T,size_S);
eta = a/N;           % integration grid
b = pi/eta;          % limits of the log-strike (-b,+b)
lambda = 2*pi/a;     % step of the log strike

%  Carr and Madan parmater (to avoid integration problem in 0)
alpha0 = 0.75;
%alpha0=1.25;

% integration grid
u = (0:N-1) * eta;
% log strike grid
x = -b + (0:N-1) * lambda;

%initialization of the C,G,M,Y parameters
v0=[0.026 0.0765 7.55 1.3 ]

%Activate the following lines may improve precision for the minima search
%options=optimset('LargeScale','on','display','iter','TolFun',1e-8,'TolX',1e-8);

%search for the minima. The two vectors represents the range of the search
sigma_m=fmincon(@sommecarre,v0,[],[],[],[],[0 0 0 0],[15 15 15
15])

%root mean square error
sc=sommecarre(sigma_m); Az=zeros(size_T,size_S);
nboptions=sum(sum(not(Az==pmarchenorm)));
roomsq=S_0*sqrt(sc/nboptions)

%graph market -prices - calibrated prices
%plot(strike,ma, 'o',strike, B*S_0,'+')



%graph procedure
maturg=[]; strikeg=[]; marcheg=[]; calculg=[];
```

```
for compteur8=1:size_S
    for compteur7=1:size_T
        if ma(compteur8,compteur7)~=0
      strikeg=[ strikeg, strike(compteur8)];
            marcheg=[ marcheg, ma(compteur8,compteur7)];
            calculg=[calculg , S_0*B(compteur7,compteur8)];
            maturg=[maturg,T(compteur7)];
        end;
       end;
   end;

plot(strikeg,marcheg,'o',strikeg,calculg,'+')
xlabel('Strike','FontSize',12) ylabel('Price ($)','FontSize',12)
titre='Calibration - o market prices  +  CGMY model'
title(titre,'FontSize',12,'FontWeight','bold')
%print  -dps2 'calibrationCGMY2006.eps'

%implied volatility surface (data);
bbb=[]; sizegg=size(strikeg);
%for compteur9=1:sizegg(2)
 %   pma=marcheg(compteur9);tma=maturg(compteur9);sma=strikeg(compteur9);
  %  aaa=fzero(@bsm,0.2);
%bbb=[bbb aaa];
%end;

%model implied volatility surface;
ccc=[] for compteur9=1:sizegg(2)
    pma=calculg(compteur9);tma=maturg(compteur9);sma=strikeg(compteur9);
    aaa=fzero(@bsm,0.2);
ccc=[ccc aaa]; end;


graphique(bbb,1,'Volatilité implicite') print  -dps2
'volimpCGMY2006.eps'
%graphique(ccc,2,'Implied volatility CGMY')
%print  -dps2 'volmodelCGMY2006.eps'

%function to plot implied volatlilty
function y=graphique(ddd,nbg,titre) global maturg strikeg
xlin=linspace(min(maturg),max(maturg),7);
ylin=linspace(min(strikeg),max(strikeg),30); [XX,
YY]=meshgrid(xlin,ylin);
ZZ=griddata(maturg,strikeg,ddd,XX,YY,'cubic'); figure
```

```
%subplot(1,2,nbg)
y = surf(XX,YY,ZZ) view(-49,52) axis tight colormap gray
xlabel('Maturity','FontSize',12) ylabel('Strike','FontSize',12)
zlabel('Volatility','FontSize',12)
title(titre,'FontSize',12,'FontWeight','bold')


%shading interp
%set(h,'EdgeColor','k')


%function to calculate the sum of square of errors
function y=sommecarre(v0) global ma strike t S_0 size_S size_T
lstrike pmarchenorm r q a N eta b lambda alpha0 u x T B A C=v0(1);
G=v0(2); M=v0(3); Y=v0(4);
%m=v0(4);

    for compteur=1:size_T;
        t = T(compteur);
        h = cfn(u, r,q, C, t, alpha0,G,M,Y);
        h2 = exp(i*b*u) .* h * eta;
        g   = fft(h2);
        % prix calculé par la transformée de Fourier
        A(compteur,:)  = real( g .* exp(-alpha0*x) / pi);
        % prix Black Scholes Merton (pour vérification)
        %A(compteur,:) = bs(exp(x), r,q, sigma, t);
    end;

    %interpolation
    for compteur2=1:size_T;
        B(compteur2,:)=interp1(x,A(compteur2,:),lstrike,'cubic');
    end;

Azero=zeros(size_T,size_S); Bzeroo=not(Azero==pmarchenorm);
y=sum(sum(Bzeroo.*((B-pmarchenorm).^2)))


%characteristic function of the  C,G,M,Y process
function y = cfn(th, r,q, C, t, alpha0,G,M,Y) th1 = th -
(alpha0+1)*i;
%Carr Madan parameter to adjust the drift
w  = t*(r-q) -t*(C*gamma(-Y)*((M-1)^Y-M^Y+(G+1)^Y-G^Y)); y0 =
exp(i*w*th1); y1 =
exp(-r*t)*exp(t*(C*gamma(-Y)*((M-i*th1).^Y-M^Y+(G+i*th1).^Y-G^Y)));
```

```
y2  = y0 .* y1; f2 = alpha0^2 + alpha0 - (th.^2) +
i*(2*alpha0+1)*th; y3 = y2./f2;
%f1 = exp(-r*t) * exp(i*th1*mnew*t).
%*((1-i*th1*theta*nu+0.5*sig^2*nu*th1.^2).^(-t/nu));
%f2 = alpha0^2 + alpha0 - (th.^2) + i*(2*alpha0+1)*th;
%y  = f1 ./ f2;
% Simpson algorithm for integration
N  = size(th,2); q1 = (-1).^(1:N); q2 = eye(1,N); S  = ( 3 + q1 -
q2 )/3; y  = y3 .* S;

% Black Scholes Merton formula (for checking)
function y = bs(x, r,q, sig, t) d1 = ( -log(x)+( r -q + .5*sig^2
)*t ) / sig/sqrt(t); d2 = d1 - sig*sqrt(t); n1 = normcdf(d1); n2 =
normcdf(d2); y = 10/10*(exp(-q*t)*n1 - x.*n2*exp(-r*t));

% Black Scholes Merton formula
function y=bsm(sig) global S_0 pma sma tma r q x=sma/S_0; d1 = (
-log(x)+( r -q + .5*sig^2 )*tma ) / sig/sqrt(tma); d2 = d1 -
sig*sqrt(tma); n1 = normcdf(d1); n2 = normcdf(d2); y =
S_0*(exp(-q*tma)*n1 - x.*n2*exp(-r*tma))-pma;
```

Monte Carlo simulation program

```
function second = exotic clear; format compact;

 % Market price of S in zero
S_0= 1243.73;

% model parameters
    %risk free rate
    r=0.05;
    %continuous dividends
    q=0.0165;

sigma=0.11930897392999 nu=0.62468767776107 theta=-0.15344383047117


%BS volatility parameters
sig=0.1806 %(vol BSM)

%CGMY parameters
C=1/nu
G=(sqrt(0.25*theta^2*nu^2+0.5*sigma^2*nu)-0.5*theta*nu)^(-1)
M=(sqrt(0.25*theta^2*nu^2+0.5*sigma^2*nu)+0.5*theta*nu)^(-1)

%features of the up and out call
K=1150; %strike
H=1200;  %Barrier
C=48.6; %market price of the european call with same strike and maturity

%Monte carlo parmaeters
n=250; %number of subdivision (time)
T=0.731; %maturity
nbsim=50000; %number of simulations

%preliminary calculus
%subdivision time
delta_t=T/n;
%adjusted barrier
Ha=H*exp(0.582*sig*sqrt(delta_t))
%timegrid
x=(0:n-1)*delta_t;

%initialization of variables
result=zeros(nbsim,3); CUI=0 CUO=0 ERUI=[]; ERUO=[];
```

```
%Monte Carlo
for compteur3=1:nbsim
result(compteur3,:)=path(S_0,r,q,x,sigma,nu,theta,n,delta_t); end;

%results
for compteur5=1:nbsim
    CUI=CUI+max(result(compteur5,3)-K,0)*(result(compteur5,1)>Ha);

    ERUI=[ERUI max(result(compteur5,3)-K,0)*(result(compteur5,1)>Ha)];

    CUO=CUO+max(result(compteur5,3)-K,0)*(result(compteur5,1)<Ha);
    ERUO=[ERUO max(result(compteur5,3)-K,0)*(result(compteur5,1)<Ha)];
end; CUI=CUI*exp(-r*T)/nbsim CUO=CUO*exp(-r*T)/nbsim

%Standard errors
ERSUI=sqrt(sum((ERUI-mean(ERUI)).^2)/(nbsim-1)^2)
ERSUI=sqrt(sum((ERUO-mean(ERUO)).^2)/(nbsim-1)^2)

%CUIBS(K,H,T,q,r,S_0,sig)
%bs(S_0,K, r,q, sig, T);
%ans-CUIBS(K,H,T,q,r,S_0,sig)

%VG trajectories
function y=path(S_0,r,q,x,sigma,nu,theta,n,delta_t)
%Risk neutral parametrization
m=x*(r-q) + x/nu*log(1 - theta*nu - .5*nu*sigma^2);
vg=nu* gamrnd(delta_t/nu,1,[1 n]); %VG number generator
norma=normrnd(0,1,[1 n]);
deltax=sigma*norma.*(vg.^(1/2))+theta*vg; vecteur2=zeros(1,n); for
compteur2=2:n;
    vecteur2(compteur2)=vecteur2(compteur2-1)+deltax(compteur2);
end; vecteur3=zeros(1,n); size(m); size(vecteur2);
vecteur3=S_0*exp(vecteur2).*(exp(m));
%exponential of  VG with  drift to remain risk neutral
y=[max(vecteur3), min(vecteur3), vecteur3(n)];


%up and In call - BS framework formula
function y = CUIBS(K,H,T,q,r,S_0,sig)
lambda=sig^(-2)*(r-q+0.5*sig^2);
y=1/(sig*sqrt(T))*log(H^2/(S_0*K))+lambda*sig*sqrt(T);
x1=1/(sig*sqrt(T))*log(S_0/H)+lambda*sig*sqrt(T);
y1=1/(sig*sqrt(T))*log(H/S_0)+lambda*sig*sqrt(T); y =
```

```
S_0*normcdf(x1)*exp(-q*T)-K*exp(-r*T)*...
    normcdf(x1-sig*sqrt(T))-S_0*exp(-q*T)*(H/S_0)^(2*lambda)*...
    (normcdf(-y)-normcdf(-y1))+K*exp(-r*T)*(H/S_0)^(2*lambda-2)...
    *(normcdf(-y+sig*sqrt(T))-normcdf(-y1+sig*sqrt(T)));

%european call option  BS prices
function y = bs(S_0,K, r,q, sig, t) d1 = ( log(S_0./K)+( r -q +
.5*sig^2 )*t ) / sig/sqrt(t); d2 = d1 - sig*sqrt(t); n1 =
normcdf(d1); n2 = normcdf(d2); y = (exp(-q*t)*S_0*n1 -
K*n2*exp(-r*t));
```

comparison graph between Black Scholes model and Variance Gamma model

```
function graphicVG= gr clear; format compact;

% Market price of S in zero
S_0= 1128.91; S_0= 1243.73;

% model parameters
    %risk free rate
    r=0.05;
    %continuous dividends
    q=0.0165;

sigma=0.11930897392999 nu=0.62468767776107 theta=-0.15344383047117


%prices obtained with the Monte Carlo program
Barr=[1260 1280 1300 1320 1340 1350 1360 1380 1400 1420 1440 1460
1480 1500 1525 1550 1575 1600]; CUIVG=[94.32 93.26 92.5 91.91 90.0
88.71 85.37 81.12 72.75 68.56 60.16 47.51 40.27 32.68 22.99 16.34
12.25 8.41]; CUOVG=[0.0314 0.3 0.96 2.434 4.79 6.63 8.6 13.0 20.07
27.01 35.46 45.84 54.61 62.66 70.3 78.28 83.3 86.26];

SOMM=[47.44 48.01 48.94 48.11 48.35 48.28]
%parameters BS
%sig=0.14890625000000 %(volatiliy)
sig=0.1706
% VG parameters
C=1/nu
G=(sqrt(0.25*theta^2*nu^2+0.5*sigma^2*nu)-0.5*theta*nu)^(-1)
M=(sqrt(0.25*theta^2*nu^2+0.5*sigma^2*nu)+0.5*theta*nu)^(-1)

%Features of the up and in call
K=1250; %strike
H=1260:5:1600 %Barrier


n=250; %number of time subdivisions
T=1.0; %maturity
nbsim=1000; %number of simulation
```

```
delta_t=T/n; Ha=H*exp(0.582*sig*sqrt(delta_t)) x=(0:n-1)*delta_t;
result=zeros(nbsim,3); CUI=0 CUO=0 ERUI=[]; ERUO=[];

CUIB=CUIBS(K,H,T,q,r,S_0,sig) bscho= bs(S_0,K, r,q, sig, T); CUOB=
bscho-CUIB plot(H,CUIB,'--k',H,CUOB,'-.k') hold on
plot(Barr,CUIVG,'+k',Barr,CUOVG,'*k') hold off xlabel('Barrier
Level','FontSize',12) ylabel('Price($)','FontSize',12)
legend('Call Up and In BS','Call Up and Out BS','Call Up an In
VG','Call Up and Out') print  -dps2 'barriere0142006.eps'


function y = CUIBS(K,H,T,q,r,S_0,sig)
lambda=sig^(-2)*(r-q+0.5*sig^2);
y=1/(sig*sqrt(T))*log(H.^2/(S_0*K))+lambda*sig*sqrt(T);
x1=1/(sig*sqrt(T))*log(S_0./H)+lambda*sig*sqrt(T);
y1=1/(sig*sqrt(T))*log(H./S_0)+lambda*sig*sqrt(T); y =
S_0*normcdf(x1)*exp(-q*T)-K*exp(-r*T).*normcdf(x1-sig*sqrt(T))
-S_0*exp(-q*T)*(H/S_0).^(2*lambda).*(normcdf(-y)-normcdf(-y1))+...
K*exp(-r*T)*(H./S_0).^(2*lambda-2).*
(normcdf(-y+sig*sqrt(T))-normcdf(-y1+sig*sqrt(T)));

function y = bs(S_0,K, r,q, sig, t)
% bs price
d1 = ( log(S_0./K)+( r -q + .5*sig^2 )*t ) / sig/sqrt(t); d2 = d1
- sig*sqrt(t); n1 = normcdf(d1); n2 = normcdf(d2); y =
(exp(-q*t)*S_0*n1 - K*n2*exp(-r*t));
```

C Bibliography

[1] Ansel, C. et Stricker, C. *Couverture des actifs contingents et prix maximum*, Annales de l'Institut Henri Poincaré, Numéro 2, pp. 303-315, 1994

[2] Bachelier, L., *Théorie de la spéculation*, Annales de l'Ecole Normale Supérieure III, 17, pp 21-86

[3] Bakshi, G. et Madan, D.B. *Spanning and derivative security valuation*, Financial economics, 55, pp 205-238

[4] Bates, D.S., *Jumps and Stochastic Volatility : Exchange Rate Processes Implicit in Deutsche Mark Options*, The Review of Financial Studies, vol 9, No 1, 1996

[5] Bertoin, J., *Lévy Processes*, Cambridge University Press, 1998

[6] Black, F. et Scholes, M., *The pricing of options and corporate liabilities*, Journal of Political Economy, vol. 81, 637-659, 1973

[7] Carr, P. et Madan D.B. *Option valuation using the Fast Fourier Transform*, Journal of Computational Finance, 2, pp 61-73

[8] Cherny, A.S., *General arbitrage pricing model : Probability and possibility approaches*, Moscow State University, 2004

[9] Cherny, A.S., et Shiryaev, A.N. *Change of time and measure for Lévy processes*, Lectures for the summer schools "From levy Processes to Semimartingales" Recent Theoretical Developments and Apllications to Finance", August 2002

[10] Cont, R. et Tankov, P., *Financial modeling with jump processes*, Chapman et Hall/CRC, 2004.

[11] Cont, R., *Empirical properties of asset returns : Stylized facts and statistical issues*, Quantitative Finance, pp. 1-14, 2001.

[12] Dana, R.A. et Jeanblanc-Picqué M., *Marchés financiers en temps continu*, Economica, 2éme édition, 1998

[13] Delbaen, F., Schachermayer, W. *The fundamental theorem of asset pricing for unbounded stochastic processes*, Mathematische Annalen, 312, pp. 215-260

[14] Föllmer, H. et Schied, A. *Stochastic Finance. An introduction in disctrete time*, Walter de Gruyter, 2002

[15] Jeanblanc, M., *Cours de formation par la recherche*, Notes extraites d'un ouvrage à paraître chez Springer, 2004.

[16] Hull, J.C., *Options,Futures,&other derivatives*, 4th edition, Prenctice Hall, 1997.

[17] Harrison, J.M. et Kreps D.M., *Martingales and arbitrage in multiperiod securities market*, Journal of Economic Theory, 20, pp 381-408, 1979

[18] Harrison, J.M. et Pliska S.R., *Martingales and stochastic integrals in the theory of continuous trading*, Stochastic processes and their applications, 11, No 3, pp 215-260, 1981

[19] Lamberton, D. et Lapeyre, B., *Introduction au calcul stochastique appliqué à la finance*, Ellipses, 1997

[20] Lewis, A., *A simple option formula for general jump-diffusion and other exponential Lévy processes.*, 2001

[21] Madam, D.B., Carr, P. et Chang, E.C. The Variance Gamma process and option pricing, European Finance Review 2,79-105

[22] Merton, R., *Option pricing when underlying stock returns are discontinuous.*, J. Financial economics, 3(1976), pp. 125,144

[23] Protter, P., *Stochastic Integration and Differential Equations.*, Springer Verlag, 1990.

[24] Raible, S., *Lévy processes in finance : Theory, numerics and empirical facts*, Thèse, Freiburg Universität, 1998

[25] Sato, K.I.,*Lévy Processes and Infinitely Divisible Distribution*, Cambridge University Press, 1999

[26] Schoutens, W., *Lévy processes in Finance : Pricing Financial Derivatives*, Wiley 2003

[27] Selivanov, A.V., *On the martingale measures in exponential Lévy models*, Moscow State University