## Isospectral Orbifolds with Different Isotropy Orders

Diplom Thesis

by

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## 1 Introduction

If M is a compact Riemannian manifold, the eigenvalue spectrum of the Laplace operator  $\Delta = d^*d$  acting on  $C^{\infty}(M)$  (or, equivalently, on  $C^{\infty}(M, \mathbb{C})$ ) is a sequence of real numbers

$$0 \leqslant \lambda_0 \leqslant \lambda_1 \leqslant \lambda_2, \dots,$$

each repeated according to the finite dimension of the corresponding eigenspace. Spectral geometry deals with the question to what extent the spectrum of a manifold determines its geometry (cf. [Gor00]).

More generally, one can introduce the Laplace operator on a so-called Riemannian orbifold, a notion which generalizes manifolds and which has first been introduced in [Sat56]. Orbifolds appear naturally (but not exclusively) in the context of properly discontinuous group actions on manifolds that are not necessarily free. The crucial difference is that a Riemannian orbifold is not assumed to be locally euclidean but instead one requires that each point has a neighbourhood which is homeomorphic to a quotient  $\tilde{U}/\Gamma$  of a Riemannian manifold  $\tilde{U}$  by a finite group  $\Gamma$  of isometries. It can be shown that, as in the manifold case, the eigenspaces of the Laplace operator on a compact Riemannian orbifold are finite-dimensional and nonnegative and one obtains the same spectrum for the Laplacian on real-valued and on complex-valued functions. Orbifolds having the same spectrum on functions are called *isospectral*.

Under certain compatibility conditions the local charts described above lead to the notion of *isotropy*: At each point the isotropy is the isomorphism class of the smallest group  $\Gamma$  appearing in any orbifold chart around this point. An orbifold on which all points have trivial isotropy carries a canonical manifold structure. Probably the most interesting open question in this context is whether an orbifold containing points with nontrivial isotropy can have the same spectrum as a manifold. On the path to a conceivable affirmative answer one is naturally led to the problem of finding isospectral orbifolds  $\mathcal{O}_1$ ,  $\mathcal{O}_2$  such that a certain isotropy on  $\mathcal{O}_1$  does not occur on  $\mathcal{O}_2$ . The first example of this kind has been given in [SSW06]. However, it did not rule out the possibility that the spectrum determines the *orders* of the isotropies.

The work at hand contains an extensive study of a pair of isospectral orbifolds whose respective maximal isotropy orders are different (4 and 2, respectively) and which has recently been found by Juan Pablo Rossetti. These two orbifolds are quotients of euclidean space  $\mathbb{R}^3$  by crystallographic groups, i.e., by discrete subgroups of the isometry group of  $\mathbb{R}^3$  such that the obtained quotient is compact. Such orbifolds are called flat. In addition, we are going to examine two more pairs of crystallographic groups acting on  $\mathbb{R}^3$  such that the respective quotient orbifolds are isospectral but not isometric. The first pair is a particularly simple example of isospectral orbifolds whose maximal

isotropies are different (but have the same order) whereas the second is a pair of flat Sunada-isospectral orbifolds.

The thesis is organized as follows. Chapter 2 gives a short self-contained introduction to Riemannian orbifolds which comprises generalizations of basic concepts familiar from manifold theory, some words on isotropy and a separate section on good orbifolds, which are quotients M/G of a Riemannian manifold M by a group G of isometries acting properly discontinuously on M. Although all our examples are good orbifolds, we try to elucidate the general theory and point out how it translates into the special setting of good orbifolds, in which isotropy on M/G in the orbifold sense corresponds to the usual isotropy groups  $\{g \in G; gp = p\}, p \in M$ .

We follow the same credo in Chapter 3, where we first demonstrate how the Laplacian carries over from manifolds to orbifolds and summarize some basic properties which are direct consequences of the corresponding results on manifolds. However, in the following chapters we will only need the elementary description of the Laplacian on good orbifolds: If M/G is a good orbifold then the Laplacian on M/G is given by the restriction of  $\Delta: C^{\infty}(M) \to C^{\infty}(M)$  to the vector space of G-invariant functions on M, which is canonically identified with the space of smooth functions on the orbifold M/G. Similarly, we define the Laplace operator on k-forms on a good orbifold M/G by the restriction of  $\Delta = dd^* + d^*d$  to G-invariant k-forms on M. We point out that for compact M the corresponding eigenspaces are again finite-dimensional, and for each k the spectrum is the same whether the Laplacian acts on real-valued or on complex-valued k-forms. Note that this applies to our examples, where we can choose M to be a certain three-dimensional torus depending on the given crystallographic group.

In Chapter 4 we come to the example which gave this thesis its title. It consists of two crystallographic groups  $G_1$ ,  $G_2$  acting on  $\mathbb{R}^3$  such that the respective quotients are isospectral. Instead of relying on our imagination, we first give for each  $G_i$  a rigorous calculation of the fundamental domain and of the identifications on its boundary given by  $G_i$ . This leads both to pictures of the orbifolds  $\mathbb{R}^3/G_1$ ,  $\mathbb{R}^3/G_2$  and to the determination of the isotropy in each orbifold point. In particular, we verify that the maximal isotropy orders are different: There are points on  $\mathbb{R}^3/G_1$  with isotropy  $\mathbb{Z}_4$  whereas all points on  $\mathbb{R}^3/G_2$  have isotropy of order  $\leq 2$ .

In Chapter 5 we show that the two orbifolds  $\mathbb{R}^3/G_1$  and  $\mathbb{R}^3/G_2$  from Chapter 4 are indeed isospectral. Thanks to the simple characterization of the Laplacian on  $C^{\infty}(\mathbb{R}^3/G_i,\mathbb{C})$ , we can apply the methods from [DR04] and [MR01], which are based on the well-known eigenfunctions on a torus. Moreover, we are going to demonstrate a third method to verify isospectrality, which uses the so-called heat kernel ([Don79]), which in our case can be calculated directly from the usual heat kernel on  $\mathbb{R}^3$ .

Chapter 6 gives two more pairs of nonisometric isospectral orbifolds which we hope to be interesting in their own right: The first is a pair  $\mathbb{R}^3/G_1$ ,  $\mathbb{R}^3/G_2$  of orbifolds with different maximal isotropies: All isotropy groups of order  $\geq 4$  on  $\mathbb{R}^3/G_1$  are isomorphic to  $\mathbb{Z}_4$  whereas  $\mathbb{Z}_2 \times \mathbb{Z}_2$  is the only isotropy of order  $\geq 4$  occurring on  $\mathbb{R}^3/G_2$ . The second is easily seen to be isospectral on k-forms for all k and turns out to be an example of a pair of Sunada-isospectral orbifolds of dimension three. Note that all our examples are

#### 1 Introduction

three-dimensional flat orbifolds. In contrast to the orbifold case, there is - up to scaling - only one pair of isospectral compact three-dimensional flat manifolds ([RC06]), and these manifolds are not isospectral on 1-forms. Isospectral flat orbifolds of dimension three have not been classified yet.

I would like to thank my supervisor Professor Dorothee Schüth for her patience, continuous encouragement and numerous productive suggestions. Moreover, I am indebted to Dr. Juan Pablo Rossetti for providing an indispensable incentive for this work in the form of his example. Finally, I appreciate the financial support by the SFB 647.

## 2 Orbifold Preliminaries

## 2.1 General Concepts

In this work we deal with examples of orbifolds, which are a generalization of manifolds introduced by Satake ([Sat56]) and popularized by Thurston ([Thu81]). In this section we give a slightly different but essentially equivalent definition (cf. the appendix of [CR02]) and a few basic statements. All our orbifolds will be oriented and Riemannian, though some of the theorems which we are going to cite also hold in a more general setting.

Before we come to the definition of an orbifold, we need to define what we mean by charts on these structures.

**Definition 2.1.** Let X be a topological Hausdorff space which is second countable and let  $U \subset X$  be a connected open subset. An n-dimensional  $orbifold\ chart$  is a triple  $(U, \tilde{U}/\Gamma, \pi)$ , where

- 1.  $\tilde{U}$  is an oriented connected *n*-dimensional Riemannian manifold.
- 2.  $\Gamma$  is a finite group of orientation-preserving isometries acting effectively on  $\tilde{U}$ .
- 3.  $\pi: \tilde{U} \to U$  is a continuous map invariant under  $\Gamma$  such that the induced map  $\tilde{U}/\Gamma \to U$  is a homeomorphism.

Two charts  $(U, \tilde{U}_i/\Gamma_i, \pi_i)$ , i = 1, 2, over the same domain U are called *isomorphic* if there is an orientation-preserving isometry  $\lambda : \tilde{U}_1 \to \tilde{U}_2$  and an isomorphism  $\Theta : \Gamma_1 \to \Gamma_2$  such that  $\pi_2 \circ \lambda = \pi_1$  and  $\lambda \circ \gamma = \Theta(\gamma) \circ \lambda \ \forall \gamma \in \Gamma_1$ .

A chart isomorphism is a special case of a so-called injection:

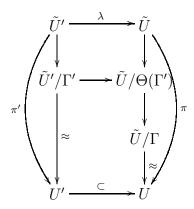
**Definition 2.2.** Let X be a topological Hausdorff space which is second countable. Let  $U' \subset U$  be open and connected subsets of X and let  $(U', \tilde{U}'/\Gamma', \pi')$ ,  $(U, \tilde{U}/\Gamma, \pi)$  be two orbifold charts. An *injection* 

$$(\lambda,\Theta):\ (U',\tilde{U'}/\Gamma',\pi')\to (U,\tilde{U}/\Gamma,\pi)$$

is a pair consisting of an open smooth isometric and orientation-preserving embedding  $\lambda: \tilde{U}' \to \tilde{U}$  and an injective homomorphism  $\Theta: \Gamma' \to \Gamma$  such that

$$\pi' = \pi \circ \lambda,$$
$$\lambda \circ \gamma = \Theta(\gamma) \circ \lambda \ \forall \gamma \in \Gamma'.$$

The conditions above lead to the following commutative diagram.



We now give a few lemmas which are convenient for the work with orbifold charts.

**Lemma 2.3.** Let  $U' \subset U$  be open connected subsets of X and let  $(\lambda, \Theta)$  be an injection-from a chart  $(U', \tilde{U}'/\Gamma', \pi')$  into a chart  $(U, \tilde{U}/\Gamma, \pi)$ . If  $\hat{\Theta}$  is an injective homomorphism from  $\Gamma'$  to  $\Gamma$  such that  $(\lambda, \hat{\Theta})$  is an injection between the two given charts, then  $\hat{\Theta} = \Theta$ . In other words, the homomorphism  $\Theta$  of an injection  $(\lambda, \Theta)$  is uniquely determined by  $\lambda$  and we can unambiguously write  $\bar{\lambda} := \Theta$ .

Proof. Assume  $\hat{\Theta}$  is a homomorphism such that  $(\lambda, \hat{\Theta})$  is an injection between the charts above and let  $\gamma \in \Gamma'$ . The definition of an injection implies that  $\Theta(\gamma) \circ \hat{\Theta}(\gamma^{-1})_{|\lambda(\tilde{U}')}$  is the identity on the open set  $\lambda(\tilde{U}') \subset \tilde{U}$ . Since an isometry on the connected Riemannian manifold  $\tilde{U}$  is uniquely determined by its differential in any given point, we deduce that  $\Theta(\gamma) \circ \hat{\Theta}(\gamma)^{-1} = \Theta(\gamma) \circ \hat{\Theta}(\gamma^{-1}) = \mathrm{id}_{\tilde{U}}$ .

In the proofs of later statements we will need the following elementary lemma.

**Lemma 2.4.** Let M be a connected smooth manifold, let  $\Gamma_1 \subset \Gamma_2$  be finite subgroups of the group of diffeomorphisms on M and let  $\pi_i : M \to M/\Gamma_i$  denote the quotient map. If there is a homeomorphism  $f : M/\Gamma_1 \to M/\Gamma_2$  such that  $\pi_2 = f \circ \pi_1$  then  $\Gamma_1 = \Gamma_2$ .

Proof. Let  $\gamma \in \Gamma_2$ . Then  $f \circ \pi_1 \circ \gamma = \pi_2 \circ \gamma = \pi_2 = f \circ \pi_1$ , hence  $\pi_1 \circ \gamma = \pi_1$ . Since  $\Gamma_2$  is finite, the set  $M^r := \{x \in M; \gamma_2 x \neq x \ \forall \gamma_2 \in \Gamma_2 \setminus \{e\}\}$  of regular points of the  $\Gamma_2$ -action is dense in M. Let  $x \in M^r$ . Since  $\pi_1(\gamma x) = \pi_1(x)$ , there is  $\gamma_1 \in \Gamma_1$  such that  $\gamma_1 x = \gamma x$ , which implies  $\gamma = \gamma_1 \in \Gamma_1$ .

We now return to our original setting of charts over a countable Hausdorff space X.

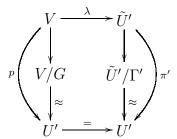
**Theorem 2.5.** Let  $(U, \tilde{U}/\Gamma, \pi)$  be a chart, and let U' be a connected open subset of  $U \subset X$ . Then there is a chart  $(U', \tilde{U}'/\Gamma', \pi')$  over U' such that there exists an injection from  $(U', \tilde{U}'/\Gamma', \pi')$  into  $(U, \tilde{U}/\Gamma, \pi)$ . Any two charts over U' from which there is an injection into  $(U, \tilde{U}/\Gamma, \pi)$  are isomorphic. This isomorphism class is called the class of charts over U' induced by  $(U, \tilde{U}/\Gamma, \pi)$ .

*Proof.* (see [CR02] 4.1.) For the existence note that (by continuity) every element of  $\Gamma$  permutes the connected components of  $\pi^{-1}(U')$  and let  $\tilde{U}'$  be one of those components.

Set  $\Gamma' := \{ \gamma \in \Gamma; \ \gamma \tilde{U}' = \tilde{U}' \}, \ \pi' := \pi_{|\tilde{U}'}$ . Since  $\gamma \tilde{U}' \cap \tilde{U}' = \emptyset \ \forall \gamma \in \Gamma \setminus \Gamma'$ , this gives a chart  $(U', \tilde{U}'/\Gamma', \pi')$  with the injection into  $(U, \tilde{U}/\Gamma, \pi)$  given by the canonical inclusions.

For the uniqueness of the induced chart up to isomorphism first note that the isomorphism class of the chart constructed above does not depend on the choice of the connected component  $\tilde{U}'$ : If  $(U', \tilde{U}'_i/\Gamma'_i, \pi'_i)$ , i=1,2, are two such charts, there is  $\gamma \in \Gamma$  such that  $\gamma \tilde{U}'_1 = \tilde{U}'_2$  (for otherwise  $\pi(\tilde{U}'_1)$  and  $\pi(\pi^{-1}(U' \setminus \tilde{U}'_1))$  would be two nonempty disjoint open sets whose union is U'). Then  $(\gamma, \Theta)$  with  $\Theta : \Gamma'_1 \ni \gamma_1 \to \gamma \gamma_1 \gamma^{-1} \in \Gamma'_2$  is a chart isomorphism from  $(U', \tilde{U}'_1/\Gamma'_1, \pi'_1)$  to  $(U', \tilde{U}'_2/\Gamma'_2, \pi'_2)$ .

Next, let (U',V/G,p) be an arbitrary chart over U' with an injection  $(\lambda,\bar{\lambda})$  into  $(U,\tilde{U}/\Gamma,\pi)$ . Then  $\lambda(V)$  lies in a connected component  $\tilde{U}'$  of  $\pi^{-1}(U')$ . This component yields a chart  $(U',\tilde{U}'/\Gamma',\pi')$  as defined in the first paragraph of this proof. We will show that  $\lambda$  is an isometry between V and  $\tilde{U}'$  and that  $\bar{\lambda}$  is a group isomorphism from G to  $\Gamma'$ . Together these observations will imply that  $(\lambda,\bar{\lambda})$  gives a chart isomorphism from (U',V/G,p) to  $(U',\tilde{U}'/\Gamma',\pi')$ .



First, we need to show that  $\lambda(V) = \tilde{U}'$ .  $\lambda(V)$  is closed in  $\tilde{U}'$  by the following argument: Let  $y_0 \in \overline{\lambda(V)} \subset \tilde{U}'$ . There is  $(x_n)_{n \in \mathbb{N}} \subset V$  such that  $y_0 = \lim_{n \to \infty} \lambda(x_n)$ . Choose  $z_0 \in p^{-1}(\pi'(y_0)) \subset V$ . Set n(0) := 0. For  $k = 1, 2, \ldots$  we successively define n(k) and  $z_k$  as follows. Since p is open, the set  ${\pi'}^{-1}(p(B_{1/k}(z_0)))$  is an open neighbourhood of  $y_0$ . Thus there is n(k) > n(k-1) such that  $\lambda(x_{n(k)}) \in {\pi'}^{-1}(p(B_{1/k}(z_0)))$  and we can find  $z_k \in B_{1/k}(z_0)$  such that

$$p(z_k) = \pi'(\lambda(x_{n(k)})).$$

Next define a new sequence  $(x'_k) \subset V$  by  $x'_k := x_{n(k)}$  and note that  $\lim_{n\to\infty} \lambda(x'_k) = y_0$ . By construction, for every k:

$$p(z_k) = \pi'(\lambda(x_k')) = p(x_k'),$$

hence there is  $a_k \in G$  such that  $a_k z_k = x_k'$ . Since G is finite, we can assume - by passing to a subsequence if necessary - that  $(a_k)$  is constant. Then  $\lim x_k' = \lim a_1 z_k = a_1 z_0 \in V$ , which implies that

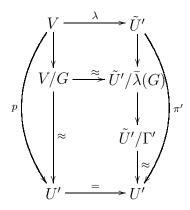
$$y_0 = \lim \lambda(x'_k) = \lambda(\lim x'_k) = \lambda(a_1 z_0) \in \lambda(V).$$

Since  $y_0 \in \overline{\lambda(V)}$  was arbitrary, we deduce that  $\lambda(V)$  is closed in  $\tilde{U}'$ . As  $\lambda$  is an injection,  $\lambda(V)$  is also open; i.e., we have  $\lambda(V) = \tilde{U}'$ .

To see that  $\bar{\lambda}(G) = \Gamma'$ , let  $x \in V$  and  $g \in G$ . Then

$$\bar{\lambda}(g)\lambda(x) = \lambda(gx) \in \lambda(V).$$

In other words,  $\bar{\lambda}(G)$  leaves  $\tilde{U}' = \lambda(V)$  invariant. By the definition of  $\Gamma'$ , this implies  $\bar{\lambda}(G) \subset \Gamma'$ . The diffeomorphism  $\lambda$  induces a homeomorphism  $V/G \to \tilde{U}'/\bar{\lambda}(G)$  and we have the following commutative diagram.



Lemma 2.4 applied to  $M = \tilde{U}'$  implies  $\bar{\lambda}(G) = \Gamma'$ .

In the following corollary we sum up a few observations from the preceding proof.

Corollary 2.6. Let  $U' \subset U$  be open connected sets in X, let  $(U, \tilde{U}/\Gamma, \pi)$  be a chart over U and let  $(U', \tilde{U}'/\Gamma', \pi')$  be an element of the isomorphism class of charts over U' induced by  $(U, \tilde{U}/\Gamma, \pi)$ .

1. If  $(\lambda, \bar{\lambda})$  is an injection from  $(U', \tilde{U}'/\Gamma', \pi')$  into  $(U, \tilde{U}/\Gamma, \pi)$ , then  $\lambda(\tilde{U}')$  is a connected component of  $\pi^{-1}(U')$  and

$$\bar{\lambda}(\Gamma') = \{ \gamma \in \Gamma; \gamma \lambda(\tilde{U}') = \lambda(\tilde{U}') \} = \{ \gamma \in \Gamma; \gamma \lambda(\tilde{U}') \cap \lambda(\tilde{U}') \neq \emptyset \}.$$

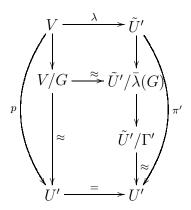
2. If V is a connected component of  $\pi^{-1}(U')$  then there is an injection  $(\lambda, \bar{\lambda})$  from  $(U', \tilde{U}'/\Gamma', \pi')$  into  $(U, \tilde{U}/\Gamma, \pi)$  such that  $\lambda(\tilde{U}') = V$ .

*Proof.* The first statement has been shown in the proof of the theorem above. As for the second, we saw that there is an isomorphism  $(\mu, \bar{\mu})$  from  $(U', \tilde{U}'/\Gamma', \pi')$  into  $(U', V/G, \pi_{|V})$ , where  $G := \{ \gamma \in \Gamma; \gamma V = V \}$ . If  $\iota : V \to \tilde{U}, \bar{\iota} : G \to \Gamma$  denote the canonical inclusions, then  $\lambda = \iota \circ \mu, \bar{\lambda} = \bar{\iota} \circ \bar{\mu}$  gives the desired injection.

**Corollary 2.7.** With  $U' \subset U$  as in the theorem above, isomorphic charts over U induce the same isomorphism class of charts over U'.

*Proof.* Let  $(U, \tilde{U}_i/\Gamma_i, \pi_i)$  be two isomorphic charts over U and for i = 1, 2 let  $(U', \tilde{U}'_i/\Gamma'_i, \pi'_i)$  be a chart in the isomorphism class over U' induced by the respective  $\pi_i$ . By the first statement of the Corollary 2.6, we can assume that  $\tilde{U}'_i$  is a connected

component of  $\pi_i^{-1}(U')$  and  $\Gamma_i'$  is the subgroup of  $\Gamma_i$  leaving  $\tilde{U}_i'$  invariant. Thus our commutative diagram looks as follows, where  $\iota_i: \tilde{U}_i' \to \tilde{U}_i$  and  $\bar{\iota}_i: \Gamma_i' \to \Gamma_i$  are the canonical inclusions and  $\lambda$  is a chart isomorphism.



To define the isomorphism  $\lambda'$ , note that  $\lambda(\tilde{U}'_1)$  is connected and  $\Gamma_2$  permutes the connected components of  $\pi_2^{-1}(U')$ ; i.e., there is  $\gamma \in \Gamma_2$  such that  $\lambda(\tilde{U}'_1) \subset \gamma \tilde{U}'_2$ . Replacing the injection  $(\iota_2, \bar{\iota}_2)$  by the injection  $(\gamma \circ \iota_2, \gamma \circ \bar{\iota}_2(\cdot) \circ \gamma^{-1})$  if necessary, we can assume that  $\lambda(\tilde{U}'_1) \subset \tilde{U}'_2$ . Then actually  $\lambda(\tilde{U}'_1) = \tilde{U}'_2$ , because  $\lambda^{-1}(\tilde{U}'_2)$  is a connected subset of  $\pi_1^{-1}(U')$  containing the component  $\tilde{U}'_1$ .

Then we can set  $\lambda' := \lambda \circ \iota_1$  and  $\bar{\lambda'}(\gamma_1) := \lambda' \circ \gamma_1 \circ \lambda'^{-1}$  for  $\gamma_1 \in \Gamma'_1$ . Note that  $\bar{\lambda'}(\gamma_1)$  is the restriction to  $\tilde{U}'_2$  of  $\bar{\lambda}(\gamma_1) \in \Gamma_2$  and maps  $\tilde{U}'_2$  to itself, hence  $\bar{\lambda}(\gamma_1) \in \Gamma'_2$ . Analogously, one has  ${\lambda'}^{-1} \circ \gamma_2 \circ \lambda' \in \Gamma'_1$  for  $\gamma_2 \in \Gamma'_2$ ; i.e.,  $\bar{\lambda'} : \Gamma'_1 \to \Gamma'_2$  is a group isomorphism. One easily verifies that  $(\lambda', \bar{\lambda}')$  is the desired chart isomorphism.

**Definition 2.8.** Let  $(U, \tilde{U}/\Gamma, \pi)$  and  $(U', \tilde{U}'/\Gamma', \pi')$  be orbifold charts and let  $x \in U \cap U'$ . The two charts are called *equivalent at* x if there is an open connected subset  $U'' \subset U \cap U'$  containing x such that the two isomorphism classes of charts on U'' induced by  $(U, \tilde{U}/\Gamma, \pi)$  and  $(U', \tilde{U}'/\Gamma', \pi')$  are identical. In this case we write  $\pi \sim_x \pi'$ .

Remark. When we refer to "the chart  $\pi$ " as e.g. in the notation introduced above, we use  $\pi$  as an abbreviation for the whole tuple  $(U, \tilde{U}/\Gamma, \pi)$  and not just to denote the map  $\pi: \tilde{U} \to U$ .

**Proposition 2.9.** For every  $x \in X$  the relation  $\sim_x$  is an equivalence relation on the set of all orbifold charts around x.

Proof. Reflexivity and symmetry are obvious. To see that the relation is transitive let  $(U_i, \tilde{U}_i/\Gamma_i, \pi_i)$ , i=1,2,3, be charts such that  $x \in U_1 \cap U_2 \cap U_3$  and  $\pi_1 \sim_x \pi_2$ ,  $\pi_2 \sim_x \pi_3$ . By definition, there are open connected sets  $U' \subset U_1 \cap U_2$ ,  $U'' \subset U_2 \cap U_3$  containing x, isomorphic charts  $\pi'_1$ ,  $\pi'_2$  over U' induced by  $\pi_1$ ,  $\pi_2$ , respectively, and isomorphic charts  $\pi''_2$ ,  $\pi''_3$  over U'' induced by  $\pi_2$ ,  $\pi_3$ , respectively. Let U be the connected component of  $U' \cap U''$  containing x. If  $p_i$  denotes the chart over U induced by  $\pi_i$ , we need to show that  $p_1$  is isomorphic to  $p_3$ . But, as the composition of two injections is an injection,  $p_1$  is induced by  $\pi'_1$  and  $p_2$  is induced by  $\pi'_2$ . Since  $\pi'_1$  and  $\pi'_2$  are isomorphic,  $p_1$  and  $p_2$  must

be isomorphic by Corollary 2.7. Analogously,  $p_2$  and  $p_3$  are isomorphic, hence  $p_1$  are  $p_3$  are isomorphic as charts over  $U \ni x$ , i.e.,  $p_1 \sim_x p_3$ .

**Definition 2.10.** An orbifold atlas  $\mathfrak{A}$  of dimension n on a second countable Hausdorff space X is a set  $\mathfrak{A} = \{(U_{\alpha}, \tilde{U}_{\alpha}/\Gamma_{\alpha}, \pi_{\alpha})\}_{\alpha \in I(\mathfrak{A})}$  of n-dimensional orbifold charts such that

- 1.  $\bigcup_{\alpha} U_{\alpha} = X$
- 2. If  $x \in U_{\alpha} \cap U_{\beta}$  then  $(U_{\alpha}, \tilde{U}_{\alpha}/\Gamma_{\alpha}, \pi_{\alpha})$  and  $(U_{\beta}, \tilde{U}_{\beta}/\Gamma_{\beta}, \pi_{\beta})$  are equivalent at x.

Two orbifold atlases are called equivalent if their union is again an orbifold atlas.

**Lemma 2.11.** Let X be a second countable Hausdorff space with an orbifold atlas  $\mathfrak{A} = \{(U_{\alpha}, \tilde{U}_{\alpha}/\Gamma_{\alpha}, \pi_{\alpha})\}_{\alpha \in I(\mathfrak{A})}$ . Then there is a unique maximal atlas on X containing  $\mathfrak{A}$ .

Proof. Let  $\bar{\mathfrak{A}}$  be the set of all charts  $(U, \tilde{U}/\Gamma, \pi)$  on X such that for every  $\alpha \in I(\mathfrak{A})$  and every  $x \in U \cap U_{\alpha}$  the charts  $\pi$  and  $\pi_{\alpha}$  are equivalent at x. To show that this is an atlas, let  $(U, \tilde{U}/\Gamma, \pi), (U', \tilde{U}'/\Gamma', \pi') \in \bar{\mathfrak{A}}$  and  $x \in U \cap U'$ . There is  $\alpha \in I(\mathfrak{A})$  such that  $x \in U_{\alpha}$ . By definition of  $\bar{\mathfrak{A}}$ , we have  $\pi \sim_x \pi_{\alpha}$  and  $\pi' \sim_x \pi_{\alpha}$ , hence  $\pi \sim_x \pi'$ . By the choice of  $\bar{\mathfrak{A}}$ , every atlas containing  $\mathfrak{A}$  is contained in  $\bar{\mathfrak{A}}$ .

**Definition 2.12.** An *n*-dimensional *orbifold* is a pair  $\mathcal{O} = (X, \mathfrak{A})$  of a second countable Hausdorff space X (called the *underlying space*) and a maximal *n*-dimensional orbifold atlas (called the *orbifold structure*) on X.

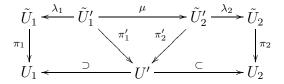
Given an orbifold  $\mathcal{O}$ , a chart  $(U, U/\Gamma, \pi)$  (as in Definition 2.1) is called an  $\mathcal{O}$ -chart if it is contained in the orbifold structure on  $\mathcal{O}$ .

Remark. If, in the situation of Theorem 2.5,  $(U, \tilde{U}/\Gamma, \pi)$  is an  $\mathcal{O}$ -chart, then so is  $(U', \tilde{U}'/\Gamma', \pi')$ . From now on, the term chart will always refer to an  $\mathcal{O}$ -chart. The given orbifold  $\mathcal{O}$  should be clear from the context.

Now let  $\mathcal{O}$  be an orbifold,  $x \in \mathcal{O}$  and let  $(U, \tilde{U}/\Gamma, \pi)$  be a chart with  $x \in U$ . For  $\tilde{x} \in \pi^{-1}(x) \subset \tilde{U}$  let  $\Gamma_{\tilde{x}} = \{\gamma \in \Gamma; \ \gamma \tilde{x} = \tilde{x}\}$  be the isotropy group (or stabilizer) of  $\tilde{x}$  under the action of  $\Gamma$ . For another  $\tilde{x}' \in \pi^{-1}(x)$  there is  $\gamma \in \Gamma$  such that  $\gamma \tilde{x} = \tilde{x}'$ . Then  $\Gamma_{\tilde{x}'} = \gamma \Gamma_{\tilde{x}} \gamma^{-1}$ ; i.e., the isotropy groups over x in this fixed chart form a well-defined conjugacy class of subgroups of  $\Gamma$ . More generally, one has

**Proposition 2.13.** Let  $x \in \mathcal{O}$  and let  $(U_i, \tilde{U}_i/\Gamma_i, \pi_i)$ , i = 1, 2, be charts with  $x \in U_i$ . If  $\tilde{x}_i \in \pi_i^{-1}(x)$ , then the groups  $\Gamma_{1\tilde{x}_1}$  and  $\Gamma_{2\tilde{x}_2}$  are isomorphic.

*Proof.* Since  $\pi_1 \sim_x \pi_2$ , there is an open connected set U' containing x, charts  $(U'_i, \tilde{U}'_i/\Gamma'_i, \pi'_i)$ , i=1,2, over U' with injections  $\lambda_i$  into  $\pi_i$  and a chart isomorphism  $\mu$  between  $\pi'_1$  and  $\pi'_2$ .



By Corollary 2.6, we can assume that  $\tilde{x}_1 \in \lambda_1(\tilde{U}'_1)$ . Write  $\tilde{x}'_1$  for the unique preimage of  $\tilde{x}_1$  under  $\lambda_1$  and set  $\tilde{x}'_2 := \mu(\tilde{x}'_1)$ . By composing  $\lambda_2$  with a suitable element of  $\Gamma_2$ , we can assume that  $\tilde{x}_2 = \lambda_2(\tilde{x}'_2)$ .

If  $\gamma \in \Gamma'_{1\tilde{x}'_1} \subset \Gamma'_1$ , then

$$\bar{\mu}(\gamma)\tilde{x}_2' = \bar{\mu}(\gamma)\mu(\tilde{x}_1') = \mu(\gamma\tilde{x}_1') = \mu(\tilde{x}_1') = \tilde{x}_2'.$$

Together with an analogous calculation for  $\mu^{-1}$  this shows that the restriction of  $\bar{\mu}$  to  $\Gamma'_{1x'_1}$  gives an isomorphism  $G_1 := \Gamma'_{1x'_1} \to \Gamma'_{2x'_2} =: G_2$ .

Moreover, for each i=1,2, restricting  $\bar{\lambda}_i$  gives an isomorphism  $G_i \to \Gamma_{i\tilde{x}_i}$ : The inclusion  $\bar{\lambda}_i(G_i) \subset \Gamma_{i\tilde{x}_i}$  is easily verified. For the opposite inclusion let  $\gamma \in \Gamma_{i\tilde{x}_i}$ . By Corollary 2.6,  $\gamma \in \bar{\lambda}_i(\Gamma_i')$ . If  $\gamma' \in \Gamma_i'$  denotes the unique preimage of  $\gamma$  under  $\bar{\lambda}_i$ , we need to show that  $\gamma' \in G_i$ . But this follows from

$$\lambda_i(\gamma'\tilde{x}_i') = \bar{\lambda}_i(\gamma')\lambda_i(\tilde{x}_i') = \gamma\tilde{x}_i = \tilde{x}_i = \lambda_i(\tilde{x}_i').$$

All in all we obtain an isomorphism between  $\Gamma_{1\tilde{x}_1}$  and  $\Gamma_{2\tilde{x}_2}$ .

**Definition 2.14.** Let  $\mathcal{O}$  be an orbifold,  $x \in \mathcal{O}$ , let  $(U, \tilde{U}/\Gamma, \pi)$  be a chart around x and  $\tilde{x} \in \pi^{-1}(x)$ . The isomorphism class of  $\Gamma_{\tilde{x}}$  is called the *isotropy of* x and is denoted by  $\operatorname{Iso}(x)$ . If  $\operatorname{Iso}(x)$  is non-trivial then x is called *singular*.

By definition, one always has that if  $(U, \tilde{U}/\Gamma, \pi)$  is a chart around x then Iso(x) is the isomorphism class of some subgroup of  $\Gamma$ . The following proposition (cf. [Bor92] Prop. 24) shows that we can obtain equality by choosing U sufficiently small.

**Proposition 2.15.** Let  $\mathcal{O}$  be an orbifold,  $x \in \mathcal{O}$  and let U be an open connected neighbourhood of x. There is a chart  $(U', \tilde{U}'/\Gamma', \pi')$  such that  $x \in U' \subset U$ ,  $\tilde{U}'$  is an open subset of  $\mathbb{R}^n$  equipped with the orientation induced by the canonical orientation of  $\mathbb{R}^n$  and  $[\Gamma'] = \mathrm{Iso}(x)$ , where  $[\Gamma']$  denotes the isomorphism class of  $\Gamma'$ .

Proof. Without loss of generality we can assume that U is a chart domain; i.e., there is a chart  $(U, \tilde{U}/\Gamma, \pi)$  around x. Choose  $\tilde{x} \in \pi^{-1}(x)$ . Since  $\Gamma$  is finite, there is  $\varepsilon \in (0, \frac{1}{2}\operatorname{dist}(\tilde{x}, \Gamma \tilde{x} \setminus \{\tilde{x}\}))$ . For  $\tilde{U}' := B_{\varepsilon}(\tilde{x})$  we have  $\Gamma_{\tilde{x}}\tilde{U}' = \tilde{U}'$  and  $\gamma \tilde{U}' \cap \tilde{U}' = \emptyset \ \forall \gamma \in \Gamma \setminus \Gamma_{\tilde{x}}$ . By choosing  $\varepsilon$  sufficiently small and composing with an orientation-preserving manifold chart for  $\tilde{U}$ , we can assume that  $\tilde{U}'$  is an open connected subset of  $\mathbb{R}^n$ . Then  $U' := \pi(\tilde{U}')$ ,  $\pi' := \pi_{|\tilde{U}'}$  and  $\Gamma' := \Gamma_{\tilde{x}} \subset \Gamma$  yield the desired chart.

In particular, we have

Corollary 2.16. Let  $\mathcal{O}$  be an n-dimensional oriented Riemannian orbifold. There is an atlas  $\{(U_x, \tilde{U}_x/\Gamma_x, \pi_x), x \in X\}$  of  $\mathcal{O}$  such that for every  $x \in X$ 

- $\tilde{U}_x$  is an open subset of  $\mathbb{R}^n$  equipped with the orientation induced by the canonical orientation on  $\mathbb{R}^n$ ,
- $x \in U_x$ ,

• 
$$[\Gamma_x] = \operatorname{Iso}(x)$$
.

A chart with the properties of Proposition 2.15 is called a fundamental chart around x. Accordingly, an atlas as in Corollary 2.16 is called a fundamental atlas of  $\mathcal{O}$ .

The following lemma implies that an orbifold without singular points can be considered as a Riemannian manifold (and vice versa, of course).

**Lemma 2.17.** If  $\mathcal{O}$  is an oriented Riemannian orbifold on which every point is non-singular, then any fundamental orbifold atlas on  $\mathcal{O}$  is a manifold atlas. The manifold structure on  $\mathcal{O}$  is independent of the choice of the fundamental orbifold atlas. Moreover, the manifold  $\mathcal{O}$  is orientable and the Riemannian metrics in the fundamental charts define a Riemannian metric on  $\mathcal{O}$  as a manifold.

Proof. Let  $\{(U_x, \tilde{U}_x/\Gamma_x, \pi_x)\}$  be a fundamental atlas of  $\mathcal{O}$ . By assumption,  $\Gamma_x = \{e\}$ , and the  $\pi_x$  are homeomorphisms. To observe that the atlas is a manifold atlas, we need to show that, for  $x, y \in X$  with  $U_x \cap U_y \neq \emptyset$  the map  $\pi_y^{-1} \circ \pi_x$  is smooth on  $\pi_x^{-1}(U_x \cap U_y) \subset \tilde{U}_x$ . Let  $z \in \pi_x^{-1}(U_x \cap U_y)$ . Since  $\pi_x \sim_z \pi_y$ , there is a chart  $(W, \tilde{W}/G, \pi)$  such that  $z \in W \subset U_x \cap U_y$  and there are injections  $\lambda_x, \lambda_y$  from  $(W, \tilde{W}/G, \pi)$  into  $(U_x, \tilde{U}_x/\Gamma_x, \pi_x)$  and  $(U_y, \tilde{U}_y/\Gamma_y, \pi_y)$ , respectively. Since  $\Gamma_x$  is trivial, so is G and  $\pi$  is a homeomorphism.

$$\begin{array}{c|c}
\tilde{U}_x & \stackrel{\lambda_x}{\longleftarrow} \tilde{W} & \stackrel{\lambda_y}{\longrightarrow} \tilde{U}_y \\
\pi_x & & \downarrow \pi & \downarrow \pi_y \\
U_x & \stackrel{\supset}{\longleftarrow} W & \stackrel{\subset}{\longrightarrow} U_y
\end{array}$$

Now  $\lambda_x(\tilde{W})$  is an open neighbourhood of  $\pi_x^{-1}(z)$  and  $\pi_y^{-1} \circ \pi_x = \lambda_y \circ \lambda_x^{-1}$  on  $\lambda_x(\tilde{W})$ . Since z was arbitrary,  $\pi_y^{-1} \circ \pi_x$  is smooth, and moreover our definition of a fundamental orbifold atlas implies that all coordinate changes  $\pi_y \circ \pi_x^{-1}$  are orientation-preserving. For the uniqueness, note that an analogous argument shows that the charts in equivalent fundamental orbifold atlases are compatible in the sense of manifold atlases.

For each x we obtain a Riemannian metric on  $U_x$  by pulling back the metric on  $\tilde{U}_x$  via  $\pi_x^{-1}$ . Since injections are local isometries, this gives a Riemannian metric on the manifold  $\mathcal{O}$ .

**Definition 2.18.** Let  $\mathcal{O}_1$ ,  $\mathcal{O}_2$  be orbifolds. A *smooth map* is a continuous map  $f: \mathcal{O}_1 \to \mathcal{O}_2$  between the underlying spaces such that for every  $x \in \mathcal{O}_1$  there is a chart  $(U_1, \tilde{U}_1/\Gamma_1, \pi_1)$  around x, a chart  $(U_2, \tilde{U}_2/\Gamma_2, \pi_2)$  around f(x), a smooth map  $\tilde{f} \in C^{\infty}(\tilde{U}_1, \tilde{U}_2)$  and a homomorphism  $\Theta: \Gamma_1 \to \Gamma_2$  such that  $f \circ \pi_1 = \pi_2 \circ \tilde{f}$  and  $\tilde{f} \circ \gamma = \Theta(\gamma) \circ \tilde{f} \ \forall \gamma \in \Gamma_1$ ; i.e., the following diagram commutes.

$$\tilde{U}_{x} \stackrel{\lambda_{x}}{\longleftarrow} \tilde{W} \stackrel{\lambda_{y}}{\longrightarrow} \tilde{U}_{y}$$

$$\downarrow^{\pi_{x}} \qquad \downarrow^{\pi_{y}}$$

$$U_{x} \stackrel{\supset}{\longleftarrow} W \stackrel{\subset}{\longrightarrow} U_{y}$$

Remark. By the definition above, a smooth function on an orbifold  $\mathcal{O}$  (with the standard structure given by the atlas  $\{(\mathbb{R}, \mathbb{R}, \mathrm{id}_{\mathbb{R}})\}$ ) is a continuous map  $f: \mathcal{O} \to \mathbb{R}$  such that  $f \circ \pi$  is smooth for every chart  $\pi$  of  $\mathcal{O}$ .

Moreover, it follows trivially from the definitions that for every chart  $(U, \tilde{U}/\Gamma, \pi)$  on  $\mathcal{O}$  the map  $\pi: \tilde{U} \to U \subset \mathcal{O}$  is smooth as a map between the orbifolds  $\tilde{U}$  and  $\mathcal{O}$ .

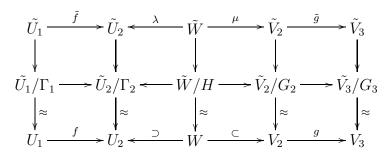
#### **Lemma 2.19.** The composition of two smooth orbifold maps is smooth.

Proof. Let  $f \in C^{\infty}(\mathcal{O}_1, \mathcal{O}_2)$ ,  $g \in C^{\infty}(\mathcal{O}_2, \mathcal{O}_3)$  and  $x \in \mathcal{O}_1$ . There is a chart  $(U_1, \tilde{U}_1/\Gamma_1, \pi_1)$  around x, charts  $(U_2, \tilde{U}_2/\Gamma_2, \pi_2)$  and  $(V_2, \tilde{V}_2/G_2, p_2)$  around y := f(x) and a chart  $(V_3, \tilde{V}_3/G_3, p_3)$  around z = g(y), smooth maps  $\tilde{f}$ ,  $\tilde{g}$  and homomorphisms  $\Theta_f : \Gamma_1 \to \Gamma_2$ ,  $\Theta_g : G_2 \to G_3$  satisfying the conditions of Definition 2.18; i.e., the following two diagrams commute (where we omit the corresponding homomorphisms  $\Theta_f$ ,  $\Theta_g$ ).

$$\begin{array}{c|c}
\tilde{U}_x & \stackrel{\lambda_x}{\longleftarrow} \tilde{W} & \stackrel{\lambda_y}{\longrightarrow} \tilde{U}_y \\
\pi_x & & \downarrow \pi & \downarrow \pi_y \\
U_x & \stackrel{\frown}{\longleftarrow} W & \stackrel{\longleftarrow}{\longrightarrow} U_y
\end{array}$$

By composing  $(\tilde{f}, \Theta_f)$  with an injection if necessary, we can assume that  $(U_1, \tilde{U}_1/\Gamma_1, \pi_1)$  is a fundamental chart around x (Prop. 2.15). Since  $\pi_2 \sim_y p_2$ , there is an open connected set  $W \subset U_2 \cap V_2$  containing y such that  $\pi_2$  and  $p_2$  induce isomorphic charts over W. Let  $\tilde{x}$  be the unique preimage of x under  $\pi_1$  and let  $\tilde{U}'_2$  be the connected component of  $\pi_2^{-1}(W)$  containing  $\tilde{f}(\tilde{x})$ .

By Corollary 2.6, there is a chart  $(W, \tilde{W}/H, \pi)$  with an injection  $(\lambda, \bar{\lambda})$  into  $(U_2, \tilde{U}_2/\Gamma_2, \pi_2)$  such that  $\lambda(\tilde{W}) = \tilde{U}_2'$  and  $\bar{\lambda}(H) = \{\gamma \in \Gamma_2; \gamma \tilde{U}_2' \subset \tilde{U}_2'\}$ . By our choice of W, there also is an injection  $\mu$  from  $(W, \tilde{W}/H, \pi)$  to  $(V_2, \tilde{V}_2/G_2, p_2)$ .



We have the following invariance properties.

- (1)  $\Gamma_2(\pi_2^{-1}(W)) \subset \pi_2^{-1}(W)$
- (2)  $\Theta_f(\Gamma_1)\tilde{U}_2' \subset \tilde{U}_2'$
- (3)  $\Gamma_1(\tilde{f}^{-1}(\tilde{U}_2')) \subset \tilde{f}^{-1}(\tilde{U}_2')$

The first equation is easily verified (and has already been used in earlier proofs):

$$p \in \pi_2^{-1}(W), \gamma \in \Gamma_2 \Rightarrow \pi_2(\gamma p) = \pi_2(p) \in W.$$

As for the second, let  $\gamma \in \Gamma_1$  and note that, by (1), we have  $\Theta_f(\gamma)(\tilde{U}_2') \subset \pi_2^{-1}(W)$ . Moreover,

$$\Theta_f(\gamma)\tilde{f}(\tilde{x}) = \tilde{f}(\gamma \tilde{x}) = \tilde{f}(\tilde{x}),$$

which implies that  $\Theta_f(\gamma)$  maps  $\tilde{U}_2'$  into the connected component of  $\pi_2^{-1}(W)$  containing  $\tilde{f}(\tilde{x})$ . By definition, this is just  $\tilde{U}_2'$ , i.e., (2) holds. To establish (3), note that for every  $p \in \tilde{f}^{-1}(\tilde{U}_2')$ ,  $\gamma \in \Gamma_1$  one has

$$\tilde{f}(\gamma p) = \Theta_f(\gamma)\tilde{f}(p) \in \Theta_f(\gamma)\tilde{U}_2' \subset \tilde{U}_2'.$$

To complete the proof of the lemma, let  $\tilde{U}_1'$  be the connected component of  $\tilde{f}^{-1}(\tilde{U}_2')$  containing  $\tilde{x}$ . From (3) and the fact that  $\Gamma_1$  fixes  $\tilde{x}$  we deduce that  $\gamma \tilde{U}_1' = \tilde{U}_1'$  for all  $\gamma \in \Gamma_1$ . If we set  $U_1' := \pi_1(\tilde{U}_1')$ , then  $(U_1', \tilde{U}_1'/\Gamma_1, \pi_{1|\tilde{U}_1'})$  is a chart around x.

Next we construct appropriate lifts of  $f_{|U_1'}$  and  $g_{|W}$ . Let  $\lambda^{-1}: \tilde{U}_2' \to \tilde{W}$  denote the inverse of the diffeomorphism  $\lambda: \tilde{W} \to \tilde{U}_2'$  and set

$$\tilde{f}' = \lambda^{-1} \circ \tilde{f}_{|\tilde{U}_1'}: \tilde{U}_1' \to \tilde{W}.$$

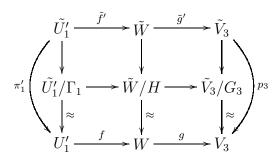
Let  $\bar{\lambda}^{-1}$ :  $\bar{\lambda}(H) \to H$  denote the inverse of the isomorphism  $\bar{\lambda}$ :  $H \to \bar{\lambda}(H)$  and note that (2) together with the first statement of Corollary 2.6 implies that  $\Theta_f(\Gamma_1) \subset \bar{\lambda}(H)$ . Then set

$$\Theta'_f: \ \Gamma_1 \ni \gamma \mapsto \bar{\lambda}^{-1}(\Theta_f(\gamma)) \in H.$$

If, moreover, we set

$$\tilde{g}' := \tilde{g} \circ \mu : \tilde{W} \to \tilde{V}_3,$$
  
 $\Theta'_g := \Theta_g \circ \bar{\mu} : H \to G_3,$ 

we obtain the following commutative diagram, where  $(\tilde{f}', \Theta'_f)$  and  $(\tilde{g}', \Theta'_g)$  satisfy the conditions of Definition 2.18 for lifts of f and g, respectively, with respect to the charts  $\pi'_1$ ,  $\pi$  and  $p_3$ .



Finally, the pair consisting of  $\widetilde{g \circ f} := \widetilde{g}' \circ \widetilde{f}' : \widetilde{U}_1' \to \widetilde{V}_3$  and  $\Theta_{g \circ f} := \Theta_g' \circ \Theta_f' : \Gamma_1 \to G_3$ 

satisfies the conditions of Definition 2.18 with respect to  $g \circ f$  and the charts  $\pi'_1$ ,  $p_3$ . Note that  $x \in U'_1$ . Since x was an arbitrary point in  $\mathcal{O}_1$ , we deduce that  $g \circ f$  is smooth.  $\square$ 

**Definition 2.20.** A diffeomorphism between two orbifolds  $\mathcal{O}_1$  and  $\mathcal{O}_2$  is a homeomorphism  $f: \mathcal{O}_1 \to \mathcal{O}_2$  between the underlying spaces such that the mappings  $\tilde{f}$  in Definition 2.18 can be chosen to be diffeomorphisms. If the  $\tilde{f}$  can be chosen to be isometries then f is called an *isometry*.

**Lemma 2.21.** Let  $f: \mathcal{O}_1 \to \mathcal{O}_2$  be a diffeomorphism and  $x \in \mathcal{O}_1$ . Then Iso(x) = Iso(f(x)).

Proof. Let  $(U_1, \tilde{U}_1/\Gamma_1, \pi_1)$ ,  $(U_2, \tilde{U}_2/\Gamma_2, \pi_2)$  be charts around x and f(x), respectively, and let  $\tilde{f}$  be the diffeomorphism according to the definition above. Note that the corresponding homomorphism  $\Theta: \Gamma_1 \to \Gamma_2$  is given by  $\Theta(\gamma) = \tilde{f} \circ \gamma \circ \tilde{f}^{-1}$ , hence it is an isomorphism. If  $\gamma \in \Gamma_{1\tilde{x}}$  then  $\Theta(\gamma)\tilde{f}(\tilde{x}) = \tilde{f}(\gamma\tilde{x}) = \tilde{f}(\tilde{x})$ , i.e.,

$$\Theta(\Gamma_{1\tilde{x}}) \subset \Gamma_{2\tilde{f}(\tilde{x})}$$
.

The opposite inclusion follows from the analogous reasoning for  $f^{-1}$ . Thus  $\Theta(\Gamma_{1\tilde{x}}) = \Gamma_{2\tilde{f}(\tilde{x})}$ , which implies that  $\Gamma_{1\tilde{x}}$  and  $\Gamma_{2\tilde{f}(\tilde{x})}$  are isomorphic, hence  $\mathrm{Iso}(x) = \mathrm{Iso}(f(x))$ .

**Definition 2.22.** Let  $\mathcal{O}_1$  and  $\mathcal{O}_2$  be two smooth orbifolds with underlying spaces  $X_1$ ,  $X_2$  and with atlases  $\{(U_{\alpha}, \tilde{U}_{\alpha}/\Gamma_{\alpha}, \pi_{\alpha})\}$  and  $\{(U_{\beta}, \tilde{U}_{\beta}/\Gamma_{\beta}, \pi_{\beta})\}$ , respectively. A smooth atlas on the *product orbifold*  $\mathcal{O}_1 \times \mathcal{O}_2$  with underlying space  $X_1 \times X_2$  is given by

$$\left\{ \left( U_{\alpha} \times U_{\beta}, (\tilde{U}_{\alpha} \times \tilde{U}_{\beta})/(\Gamma_{\alpha} \times \Gamma_{\beta}), \pi_{\alpha} \times \pi_{\beta} \right) \right\},\,$$

where  $\tilde{U}_{\alpha} \times \tilde{U}_{\beta}$  carries the product metric and the canonical orientation.

*Remark.* One easily checks that this is indeed an orbifold atlas. To see this, one uses injections of the form  $(\lambda_{\alpha} \times \lambda_{\beta}, \bar{\lambda}_{\alpha} \times \bar{\lambda}_{\beta})$ .

Moreover, note that  $\operatorname{Iso}^{\mathcal{O}_1 \times \mathcal{O}_2}(x,y) = \operatorname{Iso}^{\mathcal{O}_1}(x) \times \operatorname{Iso}^{\mathcal{O}_2}(y)$  for all  $(x,y) \in \mathcal{O}_1 \times \mathcal{O}_2$ .

If  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are two orbifolds, the respective projections  $p_i: \mathcal{O}_1 \times \mathcal{O}_2 \to \mathcal{O}_i$  are easily seen to be smooth. A map f on  $\mathcal{O}_i$  is smooth if and only if  $f \circ p_i: \mathcal{O}_1 \times \mathcal{O}_2 \to \mathbb{R}$  is smooth.

### 2.2 Integration

Let  $\mathcal{O}$  be an oriented n-dimensional Riemannian orbifold with maximal atlas

$$\{(U_{\alpha}, \tilde{U}_{\alpha}, /\Gamma_{\alpha}, \pi_{\alpha})\}.$$

For each  $\alpha$  let  $dvol_{\alpha} \in \Omega^n(\tilde{U}_{\alpha})$  denote the volume form on the oriented Riemannian manifold  $\tilde{U}_{\alpha}$ . We define the integral of a smooth function  $f \in C^{\infty}(\mathcal{O})$  with compact support on  $\mathcal{O}$  in the following way. First, assume that there is  $\alpha$  such that supp  $f \subset U_{\alpha}$ . In this case set

$$\int_{\mathcal{O}} f(x) dx := \frac{1}{|\Gamma_{\alpha}|} \int_{\tilde{U}_{\alpha}} f \circ \pi_{\alpha}(\tilde{x}) \operatorname{dvol}_{\alpha}(\tilde{x})$$

To show that this definition does not depend on the chosen chart, assume supp  $f \subset U_{\alpha_1} \cap U_{\alpha_2} =: W$ . There is a chart  $(W, \tilde{W}/\Gamma, \pi)$  with corresponding injections  $\lambda_{\alpha_1}, \lambda_{\alpha_2}$  into the two charts  $(U_{\alpha_i}, \tilde{U}_{\alpha_i}, /\Gamma_{\alpha_i}, \pi_{\alpha_i}), i = 1, 2$ . First note that for  $\alpha \in \{\alpha_1, \alpha_2\}$  we have

(\*) 
$$\int_{\tilde{U}_{\alpha}} f \circ \pi_{\alpha}(\tilde{x}) \operatorname{dvol}_{\alpha}(\tilde{x}) = [\Gamma_{\alpha} : \Gamma] \int_{\lambda_{\alpha}(\tilde{W})} f \circ \pi_{\alpha}(\tilde{x}) \operatorname{dvol}_{\alpha}(\tilde{x}),$$

where  $[\Gamma_{\alpha}:\Gamma]=|\Gamma_{\alpha}|/|\Gamma|$  denotes the index of  $\bar{\lambda}_{\alpha}(\Gamma)$  in  $\Gamma_{\alpha}$ . To establish (\*), set  $k=[\Gamma_{\alpha}:\Gamma]$  and choose representatives  $\{\gamma_{1},\ldots,\gamma_{k}\}$  of  $\Gamma_{\alpha}/\bar{\lambda}_{\alpha}(\Gamma)$ . Then  $\bigcup_{i=1}^{k}\gamma_{i}\lambda_{\alpha}(\tilde{W})\subset \tilde{U}_{\alpha}$  is a disjoint union containing  $\operatorname{supp}(f\circ\pi_{\alpha})$  and therefore

$$\int_{\tilde{U}_{\alpha}} f \circ \pi_{\alpha} = \int_{\bigcup \gamma_{i} \lambda_{\alpha}(\tilde{W})} f \circ \pi_{\alpha} = \sum_{i=1}^{k} \int_{\gamma_{i} \lambda_{\alpha}(\tilde{W})} f \circ \pi_{\alpha}$$
$$= \sum_{i=1}^{k} \int_{\lambda_{\alpha}(\tilde{W})} \gamma_{i}^{*}(f \circ \pi_{\alpha}) = k \int_{\lambda_{\alpha}(\tilde{W})} f \circ \pi_{\alpha}$$

Using (\*), we calculate

$$\frac{1}{|\Gamma_{\alpha_1}|} \int_{\tilde{U}_{\alpha_1}} f \circ \pi_{\alpha_1} = \frac{[\Gamma_{\alpha_1} : \Gamma]}{|\Gamma_{\alpha_1}|} \int_{\lambda_{\alpha_1}(\tilde{W})} f \circ \pi_{\alpha_1} = \frac{1}{|\Gamma|} \int_{\tilde{W}} f \circ \pi_{\alpha_1} \circ \lambda_{\alpha_1} \\
= \frac{1}{|\Gamma|} \int_{\tilde{W}} f \circ \pi = \frac{1}{|\Gamma|} \int_{\tilde{W}} f \circ \pi_{\alpha_2} \circ \lambda_{\alpha_2} \\
= \frac{1}{|\Gamma_{\alpha_2}|} \int_{\tilde{U}_{\alpha_2}} f \circ \pi_{\alpha_2}.$$

If  $f \in C_0^{\infty}(\mathcal{O})$  is an arbitrary function with compact support, choose a finite covering  $\{U_{\alpha(i)}\}\subset\{U_{\alpha}\}$  of supp f and a smooth partition of unity  $\{\psi_i\}$  such that each  $\psi_i$  has support in  $U_{\alpha(i)}$  (cf. [Chi90]). Then set

$$\int_{\mathcal{O}} f \, dx := \sum_{i} \int_{\mathcal{O}} (\psi_{i} f)(x) \, dx = \sum_{i} \frac{1}{|\Gamma_{i}|} \int_{\tilde{U}_{\alpha(i)}} (\psi_{i} f) \circ \pi_{i}(\tilde{x}) \, d\text{vol}_{\alpha(i)}(\tilde{x})$$

The proof that this definition is independent of the chosen covering and the partition of unity is literally the same as in the manifold case: If  $\{U_{\beta(j)}\}$  is another finite covering of supp f and  $\{\phi_j\}$  a partition of unity with supp  $\phi_j \subset U_{\beta(j)}$ , we have

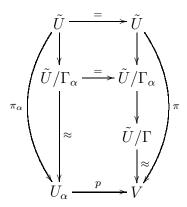
$$\sum_{i} \int_{\mathcal{O}} (\psi_{i} f)(x) dx = \sum_{i} \int_{\mathcal{O}} (\psi_{i} \sum_{j} \phi_{j} f)(x) dx = \sum_{i,j} \int_{\mathcal{O}} (\psi_{i} \phi_{j} f)(x) dx$$
$$= \sum_{j} \int_{\mathcal{O}} (\phi_{j} f)(x) dx.$$

As usual, we set  $\operatorname{vol}(\mathcal{O}) = \int_{\mathcal{O}} dx$  for a compact orbifold  $\mathcal{O}$ , and the Hilbert space  $L^2(\mathcal{O})$  is the completion of  $C_0^{\infty}(\mathcal{O})$  with respect to the scalar product  $(f_1, f_2) = \int_{\mathcal{O}} f_1 f_2 dx$ .

### 2.3 Good Orbifolds

From now on, all our orbifolds are assumed to be connected. The following definition goes back to [Thu81], compare also [Cho04].

**Definition 2.23.** Let  $\mathcal{O}_1$ ,  $\mathcal{O}_2$  be Riemannian orbifolds. A Riemannian orbifold covering is a surjective continuous map  $p: \mathcal{O}_1 \to \mathcal{O}_2$  such that for every  $y \in \mathcal{O}_2$  there is a chart  $(V, \tilde{U}/\Gamma, \pi)$  around y such that  $p^{-1}(V)$  is a disjoint union  $\bigcup_{\alpha} U_{\alpha}$  and over each connected component  $U_{\alpha}$  there is a chart  $(U_{\alpha}, \tilde{U}/\Gamma_{\alpha}, \pi_{\alpha})$  such that  $\Gamma_{\alpha} \subset \Gamma$  and the following diagram commutes.



Remark. Obviously, an orbifold covering is a smooth orbifold map in the sense of Definition 2.18. It is not hard to see that, with p as above, a function  $f: \mathcal{O}_2 \to \mathbb{R}$  is smooth if and only if  $f \circ p: \mathcal{O}_1 \to \mathbb{R}$  is smooth: The "only if"-part is the content of Lemma 2.19. For the "if"-part assume that  $f \circ p$  is smooth. Given an arbitrary point  $y \in \mathcal{O}_2$ , we can assume - by choosing a neighbourhood V of y sufficiently small - that we have an  $\alpha$  and a commutative diagram as in Definition 2.23 such that there is a smooth map  $f \circ p: \tilde{U} \to \mathbb{R}$  lifting  $f \circ p: U_{\alpha} \to \mathbb{R}$ , i.e., such that  $f \circ p \circ \pi_{\alpha} = f \circ p$ . But  $f \circ p \circ \pi_{\alpha} = f \circ \pi$ , hence  $f \circ p$  also is a lift of f over g. The corresponding group homomorphism is given by the constant map into the trivial group.

Apart from manifolds, the principal example of an orbifold is given by the following theorem going back to [Sat56] and [Thu81]. Recall that a group G is said to act properly discontinuously on a topological space M if  $\{g \in G; gK \cap K \neq \emptyset\}$  is finite for every compact  $K \subset M$ .

**Theorem 2.24.** Let M be an oriented Riemannian manifold and let  $G \subset \text{Isom}(M)$  be a group of orientation-preserving isometries acting properly discontinuously on M. Then the quotient space M/G carries a canonical oriented Riemannian orbifold structure such that the projection  $p: M \to M/G$  is a Riemannian orbifold covering. If  $x \in M/G$  and  $\tilde{x} \in p^{-1}(x)$ , the isomorphism class of the isotropy group  $G_{\tilde{x}} = \{g \in G; g\tilde{x} = \tilde{x}\}$  is Iso(x).

*Proof.* Since G acts properly discontinuously, the topology on M/G is induced by the metric

$$d(x,y) := \inf\{d(\tilde{x},\tilde{y}); \ \tilde{x} \in p^{-1}(x), \tilde{y} \in p^{-1}(y)\},\$$

hence M/G is Hausdorff. Moreover one has  $p(B_r(\tilde{x})) = B_r(p(\tilde{x}))$  for this metric. In particular, p is open and a countable basis on M is mapped onto a countable basis of M/G, i.e., M/G is second-countable.

Now let  $x \in M/G$  and choose  $\tilde{x} \in p^{-1}(x)$ . Since G acts properly discontinuously,  $G_{\tilde{x}}$  is finite and  $G\tilde{x} \setminus \{\tilde{x}\}$  is closed. Choose  $0 < r < \frac{1}{2}\operatorname{dist}(\tilde{x}, G\tilde{x} \setminus \{\tilde{x}\})$ . Then

$$G_{\tilde{x}}B_r(\tilde{x}) = B_r(\tilde{x}) \text{ and } gB_r(\tilde{x}) \cap B_r(\tilde{x}) = \emptyset \ \forall g \in G \setminus G_{\tilde{x}},$$

thus p induces a homeomorphism from  $B_r(\tilde{x})/G_{\tilde{x}}$  onto its image  $p(B_r(\tilde{x})) = B_r(x) \subset M/G$ . An orbifold chart is then given by  $(B_r(x), B_r(\tilde{x})/G_{\tilde{x}}, p_{|B_r(\tilde{x})})$ . All those charts together, i.e., all charts

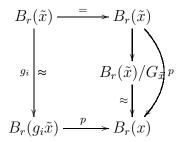
$$\{(B_{r_x}(x), B_{r_x}(\tilde{x})/G_{\tilde{x}}, p_{|B_{r_x}(\tilde{x})}); x \in M/G\},$$

form an orbifold atlas on M/G: Given two charts  $(B_{r_i}(x_i), B_{r_i}(\tilde{x}_i)/G_{\tilde{x}_i}, p_{|B_{r_i}(\tilde{x}_i)}), i = 1, 2,$  satisfying  $p(\tilde{x}_i) = x_i, r_i \in (0, \operatorname{dist}(\tilde{x}_i, G\tilde{x}_i \setminus \{\tilde{x}_i\})/2)$  and a point  $y \in B_{r_1}(x_1) \cap B_{r_2}(x_2)$  there are  $\tilde{x}'_i \in p^{-1}(x_i), \ \tilde{y} \in p^{-1}(y)$  and  $r \in (0, \operatorname{dist}(\tilde{y}, G\tilde{y} \setminus \{\tilde{y}\})/2)$  such that  $B_r(\tilde{y}) \subset B_{r_1}(\tilde{x}'_i) \cap B_{r_2}(\tilde{x}'_i)$ . Then  $(B_r(y), B_r(\tilde{y})/G_{\tilde{y}}, p_{|B_r(\tilde{y})})$  is a chart around y. Because of

$$\tilde{y} \in gB_{r_i}(\tilde{x}_i') \cap B_{r_i}(\tilde{x}_i') \ \forall g \in G_{\tilde{y}}$$

we have  $G_{\tilde{y}} \subset G_{\tilde{x}'_1} \cap G_{\tilde{x}'_2}$ . This implies that, for each i=1,2, the canonical inclusion  $B_r(\tilde{y}) \to B_r(\tilde{x}_i)$  followed by an element of G sending  $\tilde{x}'$  to  $\tilde{x}$  gives an injection from  $(B_r(y), B_r(\tilde{y})/G_{\tilde{y}}, p_{|B_r(\tilde{y})})$  into  $(B_r(x_i), B_r(\tilde{x}_i)/G_{\tilde{x}_i}, p_{|B_r(\tilde{x}_i)})$  and therefore  $p_{|B_{r_1}(\tilde{x}_1)} \sim_y p_{|B_{r_2}(\tilde{x}_2)}$ . Since M is equipped with an orientation and G is orientation-preserving, then the injections given above are also orientation-preserving and the Riemannian orbifold M/G is oriented.

To see that p is an orbifold covering let  $x \in M/G$ , fix  $\tilde{x} \in p^{-1}(x)$  and choose a chart as above. If  $\{g_1, \ldots, g_k\}$  is a set of representatives of the left cosets  $G/G_{\tilde{x}}$ , then  $p^{-1}(B_r(x)) = \bigcup_{i=1}^k B_r(g_i\tilde{x})$  is a disjoint union and for each i we have the following commutative diagram as required in Definition 2.23.



Note that the statement about the isotropy groups is a direct consequence of the definition of our charts.  $\Box$ 

Remark. If the Riemannian manifold M is complete and we equip  $\operatorname{Isom}(M)$  with the compact-open topology, then a topological subgroup G of  $\operatorname{Isom}(M)$  acts properly discontinuously if and only if it is discrete. For this and the statements about quotients of metric spaces used in the proof above see [Rat06] Chapter 5 and [Bor92].

**Definition 2.25.** An orbifold as above is called *good*. If G is finite, M/G is called *very good*. If an orbifold is not isometric to a quotient as above, it is called *bad*.

Remark. In the special case that M/G is a very good oriented Riemannian orbifold, note that an atlas is given by the single chart (M/G, M/G, p) and the integral of a function  $f \in C^{\infty}(M/G)$  is just  $\int_{M/G} f = \frac{1}{|G|} \int_{M} f \circ p$ .

We say that a Riemannian orbifold  $\mathcal{O}$  has constant sectional curvature c if, for every chart  $(U, \tilde{U}/\Gamma, \pi)$  on  $\mathcal{O}$ , the Riemannian manifold  $\tilde{U}$  has constant sectional curvature c. An orbifold of constant sectional curvature zero is called flat. In the case of sectional curvature the situation becomes particularly simple, as the following lemma shows ([Thu81] 13.3, cf. [Rat06] 13.3).

**Lemma 2.26.** Every compact Riemannian orbifold of constant sectional curvature is good.

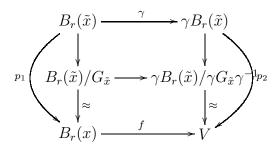
In analogy to the manifold case, we have the following sufficient criterion for the existence of an isometry between two good orbifolds which are covered by the same manifold.

**Lemma 2.27.** Let M be Riemannian manifold and let  $G_1, G_2$  be discrete subgroups of Isom(M). If  $G_1$  and  $G_2$  are conjugate in Isom(M), then the Riemannian orbifolds  $M/G_1$ ,  $M/G_2$  are isometric.

*Proof.* Let  $p_i: M \to M/G_i$  denote the quotient maps and let  $\gamma$  be an isometry on M such that  $\gamma G_1 \gamma^{-1} = G_2$ . Then the map

$$f: M/G_1 \ni [\tilde{x}] \mapsto [\gamma \tilde{x}] \in M/G_2$$

is a well-defined homeomorphism. To see that it is an isometry, let  $x \in M/G_1$ , choose  $\tilde{x} \in p_1^{-1}(x)$  and a chart  $(B_r(x), B_r(\tilde{x})/G_{\tilde{x}}, p_1)$  around x as in Theorem 2.24. Then f(x) lies in  $V := p_2(\gamma B_r(\tilde{x}))$ , the group  $\gamma G_{\tilde{x}} \gamma^{-1}$  acts isometrically on  $\gamma B_r(\tilde{x})$  and  $(V, (\gamma B_r(\tilde{x}))/\gamma G_{\tilde{x}} \gamma^{-1}, p_2)$  is a chart around f(x). We have the following commutative diagram.



Since  $\gamma$  is an isometry and  $x \in M/G_1$  has been arbitrary, f is an isometry.  $\square$ 

# 3 The Isospectrality Problem on Orbifolds

As in the preceding section on good orbifolds, all orbifolds are assumed to be connected. Let  $\mathcal{O}$  be an *n*-dimensional Riemannian orbifold. The Laplace operator on  $C^{\infty}(\mathcal{O})$  is defined via local charts.

**Definition 3.1.** For  $f \in C^{\infty}(\mathcal{O})$  and  $x \in \mathcal{O}$  let  $(U, \tilde{U}/\Gamma, \pi)$  be a chart around x. Moreover, let  $\tilde{\Delta}$  denote the Laplacian on the Riemannian manifold  $\tilde{U}$  and choose  $\tilde{x} \in \pi^{-1}(x)$ . Then set

$$\Delta f(x) := \tilde{\Delta}(f \circ \pi)(\tilde{x}).$$

Remark. Since the Laplacian on manifolds commutes with the pullback by isometries, our compatibility condition on orbifold charts shows that this definition does not depend on the choice of chart around x.

Let  $\tilde{f}$  be a smooth function on the Riemannian manifold  $\tilde{U}$ . Recall that the Laplacian  $\tilde{\Delta}$  of  $\tilde{f}$  is given by

$$\tilde{\Delta}\tilde{f} = d^*d\tilde{f} = -\operatorname{tr}(\operatorname{Hess}\tilde{f}).$$

Moreover, one has the following two local characterizations (cf. [BGM71]): First, if y is a manifold chart on  $\tilde{U}$  around  $\tilde{x}$ ,  $\tilde{g}$  denotes the Riemannian metric on  $\tilde{U}$ ,  $\tilde{g}_{ij} := \tilde{g}(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})$ ,  $\rho := \det(\tilde{g}_{ij})$  and  $(\tilde{g}^{ij}) := (\tilde{g}_{ij})^{-1}$ , then

$$\tilde{\Delta}\tilde{f}(\tilde{x}) = -\left(\frac{1}{\sqrt{\rho}}\sum_{i,j=1}^{n}\frac{\partial}{\partial y^{i}}\left(\tilde{g}^{ij}\frac{\partial\tilde{f}}{\partial y^{j}}\sqrt{\rho}\right)\right)(\tilde{x}).$$

Second, if  $\{X_i\}_{i=1}^n$  is an orthonormal basis of the tangent space  $T_{\tilde{x}}(\tilde{U})$  and  $\gamma_i$  denotes the unique geodesic with  $\dot{\gamma}_i(0) = X_i$  defined on an open interval around 0, then

$$\tilde{\Delta}\tilde{f}(\tilde{x}) = -\sum_{i=1}^{n} (\tilde{f} \circ \gamma_i)''(0).$$

The following properties are direct consequences of the respective statements for the Laplacian on manifolds ([BGM71]).

**Lemma 3.2.** 1.  $\Delta: C^{\infty}(\mathcal{O}) \to C^{\infty}(\mathcal{O})$  is linear.

2. Let  $f_1, f_2 \in C^{\infty}(\mathcal{O})$ ,  $x \in \mathcal{O}$ . If  $\pi$ ,  $\tilde{x}$  are chosen as in Definition 3.1 and  $\tilde{g}$  denotes the metric on  $\tilde{U}$ , then

$$\Delta(f_1 f_2)(x) = (f_1 \Delta f_2)(x) - 2 \langle \operatorname{grad}_{\tilde{g}}(f_1 \circ \pi), \operatorname{grad}_{\tilde{g}}(f_2 \circ \pi) \rangle (\tilde{x}) + (f_2 \Delta f_1)(x).$$

3. Let  $\psi: \mathcal{O}_1 \to \mathcal{O}_2$  be an orbifold isometry and  $f \in C^{\infty}(\mathcal{O}_2)$ . Then

$$(\Delta_2 f) \circ \psi = \Delta_1(f \circ \psi).$$

4. Let  $\mathcal{O}_1 \times \mathcal{O}_2$  be a product orbifold. For a fixed  $i \in \{1,2\}$  let  $p_i : \mathcal{O}_1 \times \mathcal{O}_2 \to \mathcal{O}_i$  be the projection and let  $f \in C^{\infty}(\mathcal{O}_i)$ . Then

$$\Delta^{\mathcal{O}_1 \times \mathcal{O}_2}(f \circ p_i) = (\Delta_i f) \circ p_i.$$

*Proof.* 1 is clear and 2 follows directly from the respective formula on the Riemannian manifold  $\tilde{U}$ .

As for 3, let  $x \in \mathcal{O}_1$ . There is a commutative diagram as in Definition 2.18 with an isometry  $\tilde{\psi}$  in the top row. Choose  $\tilde{x} \in \pi_1^{-1}(x)$ . Since relation 3 holds for  $\tilde{\psi}: \tilde{U}_1 \to \tilde{U}_2$ ,  $f \circ \pi_2 \in C^{\infty}(\tilde{U}_2)$  and the local Laplacians  $\tilde{\Delta}_i$  on  $\tilde{U}_i$ , we have

$$\Delta_1(f \circ \psi)(x) = \tilde{\Delta}_1(f \circ \psi \circ \pi_1)(\tilde{x}) = \tilde{\Delta}_1(f \circ \pi_2 \circ \tilde{\psi})(\tilde{x}) = \tilde{\Delta}_2(f \circ \pi_2)(\tilde{\psi}(\tilde{x}))$$
$$= (\Delta_2 f) \circ \psi(x).$$

Next, we prove 4: Without loss of generality, consider i = 1 and let  $(x_1, x_2) \in \mathcal{O}_1 \times \mathcal{O}_2$ . For j = 1, 2 let  $\pi_j$  be an  $\mathcal{O}_j$ -chart around  $x_j$  and  $\tilde{x}_j \in \pi_j^{-1}(x_j)$ . Then

$$\Delta^{\mathcal{O}_1 \times \mathcal{O}_2}(f \circ p_1)(x_1, x_2) = \tilde{\Delta}(f \circ p_1 \circ (\pi_1 \times \pi_2))(\tilde{x}_1, \tilde{x}_2) = \tilde{\Delta}_1(f \circ \pi_1)(\tilde{x}_1)$$
$$= (\Delta_1 f) \circ p_1(x_1, x_2).$$

Now we come to the principal object of our investigations. From now on all our Riemannian orbifolds are assumed to be compact (and connected).

**Definition 3.3.** Let  $\mathcal{O}$  be a compact Riemannian orbifold. The spectrum  $spec(\mathcal{O})$  is the set of eigenvalues of  $\Delta$  with multiplicities, i.e.,  $spec(\mathcal{O}) \subset \mathbb{R}$  is a multiset, where the multiplicity of  $\lambda \in spec(\mathcal{O})$  is the dimension of the eigenspace

$$E_{\lambda}(\mathcal{O}) := \{ f \in C^{\infty}(\mathcal{O}); \ \Delta f = \lambda f \}$$

of  $\Delta$  to the eigenvalue  $\lambda$ . Moreover, we write

$$E(\mathcal{O}) := \bigoplus_{\lambda \in \operatorname{spec}(\mathcal{O})} E_{\lambda}(\mathcal{O})$$

for the space of finite sums of eigenfunctions on  $\mathcal{O}$ .

Two compact Riemannian orbifolds  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are called *isospectral* if  $\operatorname{spec}(\mathcal{O}_1) = \operatorname{spec}(\mathcal{O}_2)$  with multiplicities.

Obviously, Lemma 3.2.3 implies that two isometric orbifolds are isospectral. For overviews over the relationship between the spectrum and the geometry of a manifold see [Gor00] or [Bro88].

The spectrum of the Laplacian on compact orbifolds was first investigated by Donnelly ([Don79]). He proved the following theorem for good orbifolds which was later generalized to arbitrary orbifolds by Chiang ([Chi90]):

**Theorem 3.4.** Let  $\mathcal{O}$  be a compact Riemannian orbifold. Then every eigenvalue of  $\Delta$  on  $C^{\infty}(\mathcal{O})$  has finite multiplicity and  $\operatorname{spec}(\mathcal{O})$  consists of a sequence  $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$ , where  $\lambda_i \to \infty$ . Moreover there is an orthonormal basis  $\{\phi_i\} \subset C^{\infty}(\mathcal{O})$  of the Hilbert space  $L^2(\mathcal{O})$  such that  $\Delta \phi_i = \lambda_i \phi_i$ .

In light of our examples in the following sections, we should note that we obtain the same spectrum for real- and for complex-valued functions: For a function  $f = u + iv \in C^{\infty}(\mathcal{O}, \mathbb{C})$  with  $u, v \in C^{\infty}(\mathcal{O}) = C^{\infty}(\mathcal{O}, \mathbb{R})$  set  $\Delta f := \Delta u + i\Delta v$ . Then we have

**Lemma 3.5.** Let  $\lambda \in \mathbb{C}$  and set

$$E_{\lambda}^{\mathbb{C}}(\mathcal{O}):=\{f\in C^{\infty}(\mathcal{O},\mathbb{C});\ \Delta f=\lambda f\}.$$

Then

$$E_{\lambda}^{\mathbb{C}}(\mathcal{O}) = \begin{cases} E_{\lambda}(\mathcal{O}) \otimes_{\mathbb{R}} \mathbb{C} & \text{for } \lambda \in \mathbb{R} \\ \{0\} & \text{for } \lambda \in \mathbb{C} \setminus \mathbb{R} \end{cases}.$$

In particular,

$$\dim_{\mathbb{C}} E_{\lambda}^{\mathbb{C}}(\mathcal{O}) = \begin{cases} \dim_{\mathbb{R}} E_{\lambda}(\mathcal{O}), & \lambda \in \mathbb{R} \\ 0, & \lambda \in \mathbb{C} \setminus \mathbb{R} \end{cases}.$$

Before proving the lemma above, we note that Green's Identity for manifolds carries over to orbifolds.

**Proposition 3.6** (Green's Identity). Let  $\mathcal{O}$  be a compact orientable Riemannian orbifold and let  $f_i \in C^{\infty}(\mathcal{O})$ , i = 1, 2. Then

$$\int_{\mathcal{O}} f_1 \Delta f_2 = \int_{\mathcal{O}} f_2 \Delta f_1$$

*Proof.* Let  $\{U_i\}$  be a finite covering of  $\mathcal{O}$  with associated charts  $\{(U_i, \tilde{U}_i/\Gamma_i, \pi_i)\}$  and let  $\{\psi_i\}$  be a subordinate partition of unity. Then, by the definition of an integral and the Laplacian,

$$\int_{\mathcal{O}} f_1 \Delta f_2 = \sum_{i} \int_{\mathcal{O}} \psi_i f_1 \Delta f_2 = \sum_{i} \frac{1}{|\Gamma_i|} \int_{\tilde{U}_i} (\psi_i f_1 \Delta f_2) \circ \pi_i$$
$$= \sum_{i} \frac{1}{|\Gamma_i|} \int_{\tilde{U}_i} (\psi_i f_1) \circ \pi_i \cdot \tilde{\Delta}_i (f_2 \circ \pi_i).$$

Since  $(\psi_i f_1) \circ \pi_i$  has compact support in  $\tilde{U}_i$ , Green's Identity on the Riemannian manifold  $\tilde{U}_i$  implies that the last integral is equal to  $\int_{\tilde{U}_i} f_2 \circ \pi_i \cdot \tilde{\Delta}_i((\psi_i f_1) \circ \pi_i)$ , hence

$$\int_{\mathcal{O}} f_1 \Delta f_2 = \sum_{i} \frac{1}{|\Gamma_i|} \int_{\tilde{U}_i} f_2 \circ \pi_i \cdot \tilde{\Delta}_i((\psi_i f_1) \circ \pi_i)$$

$$= \sum_{i} \frac{1}{|\Gamma_i|} \int_{\tilde{U}_i} f_2 \circ \pi_i \cdot \Delta(\psi_i f_1) \circ \pi_i = \sum_{i} \int_{\mathcal{O}} f_2 \Delta(\psi_i f_1)$$

$$= \int_{\mathcal{O}} f_2 \Delta \left(\sum_{i} \psi_i f_1\right) = \int_{\mathcal{O}} f_2 \Delta f_1$$

Note that the third equality is merely the definition of the integral of the function  $f_2\Delta(\psi_i f_1)$  which has compact support in  $U_i$ .

Extending the integral to complex-valued functions in the usual way, one easily verifies that  $\Delta$  is also symmetric on  $C^{\infty}(\mathcal{O}, \mathbb{C})$  with respect to the Hermitian form  $\langle f_1, f_2 \rangle = \int_{\mathcal{O}} f_1 \overline{f_2}$ .

Corollary 3.7. Let  $\mathcal{O}$  be a compact orbifold and let  $f_1, f_2 \in C^{\infty}(\mathcal{O}, \mathbb{C})$ . Then

$$\int_{\mathcal{O}} f_1 \overline{\Delta^{\mathbb{C}} f_2} = \int_{\mathcal{O}} \overline{f_2} \Delta^{\mathbb{C}} f_1$$

Now we will prove Lemma 3.5. First note that the corollary above implies that the eigenvalues of  $\Delta^{\mathbb{C}}$  are real. Next let  $\lambda \in \mathbb{R}$ . If  $u \in E_{\lambda}$  and  $z \in \mathbb{C}$ , then  $\Delta^{\mathbb{C}}(zu) = z\Delta u = \lambda(zu)$ , i.e.,  $E_{\lambda} \otimes_{\mathbb{R}} \mathbb{C} \subset E_{\lambda}^{\mathbb{C}}$ . For the opposite inclusion, observe that  $\Delta^{\mathbb{C}}(u+iv) = \lambda(u+iv)$  implies  $u, v \in E_{\lambda}$ , and the proof of Lemma 3.5 is complete. From now on, we shall omit the superscript  $\mathbb{C}$  for the Laplacian on complex-valued functions.

To determine the spectrum of a product orbifold, we follow the proof for the manifold setting given in [BGM71]. We will need the following lemma which is a simple consequence of Proposition 3.6 (see [BGM71] III.A.II.1 for the manifold case).

**Lemma 3.8.** Let  $\mathcal{O}$  be a compact oriented Riemannian orbifold and for each  $i \in \mathbb{N}$  let  $V_i$  be a subspace of  $C^{\infty}(\mathcal{O})$  such that:

- 1. For every  $i \in \mathbb{N}$  there is  $\lambda_i \in \mathbb{R}$  such that  $\Delta \phi = \lambda_i \phi \ \forall \phi \in V_i$ .
- 2. The sum  $\bigoplus_{i\in\mathbb{N}} V_i$  is dense in  $C^{\infty}(\mathcal{O})$  with respect to the  $L^2$ -norm.

Then the spectrum of  $\mathcal{O}$  (as as set) consists of the numbers  $\lambda_i$  and for every i the space  $V_i$  is the eigenspace of  $\Delta$  associated with the eigenvalue  $\lambda_i$ .

Given an orbifold  $\mathcal{O}$  and subspaces V,W of the algebra  $C^{\infty}(\mathcal{O})$  let  $V\otimes W$  denote the span of  $\{fg;\ f\in V,g\in W\}$ . Moreover, for a smooth orbifold map  $\phi:\mathcal{O}\to\mathcal{O}'$  and a subspace  $V\subset C^{\infty}(\mathcal{O}')$  set  $\phi^*V:=\{f\circ\phi;\ f\in V\}\subset C^{\infty}(\mathcal{O})$ . Using this notation, we have the following lemma, whose proof in [BGM71] for the manifold case carries over to the orbifold setting.

**Lemma 3.9.** Let  $\mathcal{O}_1$  and  $\mathcal{O}_2$  be two compact oriented Riemannian orbifolds and let  $p_i: \mathcal{O}_1 \times \mathcal{O}_2 \to \mathcal{O}_i$  denote the projection. Then

$$E(\mathcal{O}_1 \times \mathcal{O}_2) = p_1^* E(\mathcal{O}_1) \otimes p_2^* E(\mathcal{O}_2).$$

Moreover, for  $\nu \geqslant 0$ :

$$E_{\nu}(\mathcal{O}_1 \times \mathcal{O}_2) = \bigoplus_{\lambda + \mu = \nu} p_1^* E_{\lambda}(\mathcal{O}_1) \otimes p_2^* E_{\mu}(\mathcal{O}_2).$$

In particular, if  $\mathcal{O}_1$  and  $\mathcal{O}_2$  have spectrum  $0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots$  and  $0 = \mu_0 < \mu_1 \leqslant \mu_2 \leqslant \ldots$ , respectively, then  $\mathcal{O}_1 \times \mathcal{O}_2$  has spectrum  $(\lambda_i + \mu_j)_{i,j=0}^{\infty}$ .

Proof. For each i = 1, 2 let  $f_i \in C^{\infty}(\mathcal{O}_i)$ , let  $x_i \in \mathcal{O}_i$  and choose a chart  $(U_i, \tilde{U}_i/\Gamma_i, \pi_i)$  around  $x_i$ . Consider the chart  $(U_1 \times U_2, \tilde{U}_1 \times \tilde{U}_2/\Gamma_1 \times \Gamma_2, \pi := \pi_1 \times \pi_2)$  on  $\mathcal{O}_1 \times \mathcal{O}_2$ . Then each  $f_i \circ p_i \circ \pi = f_i \circ \pi_i \in C^{\infty}(\tilde{U}_1 \times \tilde{U}_2)$  depends only on the *i*-th component. Hence the middle term on the right hand side of Lemma 3.2.2 for  $\Delta((f_1 \circ p_1)(f_2 \circ p_2))(x_1, x_2)$  vanishes and (since the  $x_i$  were arbitrary) we have

$$\Delta((f_1 \circ p_1)(f_2 \circ p_2)) = (f_1 \circ p_1)\Delta(f_2 \circ p_2) + (f_2 \circ p_2)\Delta(f_1 \circ p_1)$$

on  $\mathcal{O}_1 \times \mathcal{O}_2$ . Lemma 3.2.4 now implies that if  $f_1 \in E_{\lambda}(\mathcal{O}_1)$ ,  $f_2 \in E_{\mu}(\mathcal{O}_2)$  are eigenfunctions to the eigenvalues  $\lambda$  and  $\mu$ , then  $(f_1 \circ p_1)(f_2 \circ p_2) \in E_{\lambda+\mu}(\mathcal{O}_1 \times \mathcal{O}_2)$ .

Moreover, if  $\{f_{1i}\}_i$ ,  $\{f_{2j}\}_j$  are linearly independent functions on  $\mathcal{O}_1$  and  $\mathcal{O}_2$ , respectively, then the  $\{(f_{1i} \circ p_1)(f_{2j} \circ p_2)\}_{i,j}$  are linearly independent functions on  $\mathcal{O}_1 \times \mathcal{O}_2$ . These two observations imply

$$(*) \quad E_{\nu}(\mathcal{O}_1 \times \mathcal{O}_2) \supset \bigoplus_{\lambda + \mu = \nu} p_1^* E_{\lambda}(\mathcal{O}_1) \otimes p_2^* E_{\mu}(\mathcal{O}_2).$$

For the opposite inclusion, note that Theorem 3.4 implies that  $E(\mathcal{O}_i)$  is dense in  $C^{\infty}(\mathcal{O}_i)$  with respect to the  $L^2$ -norm, thus  $p_1^*E(\mathcal{O}_1)\otimes p_2^*E(\mathcal{O}_2)$  is dense in  $p_1^*C^{\infty}(\mathcal{O}_1)\otimes p_2^*C^{\infty}(\mathcal{O}_2)$  with respect to the  $L^2$ -norm on  $\mathcal{O}_1\times\mathcal{O}_2$ . Applying the Theorem of Stone-Weierstrass to the compact topological space  $\mathcal{O}_1\times\mathcal{O}_2$ , we observe that  $p_1^*C^{\infty}(\mathcal{O}_1)\otimes p_2^*C^{\infty}(\mathcal{O}_2)$  is dense in  $C^{\infty}(\mathcal{O}_1\times\mathcal{O}_2)$  with respect to the supremum-norm, hence also with respect to the  $L^2$ -norm. These two observations imply that  $p_1^*E(\mathcal{O}_1)\otimes p_2^*E(\mathcal{O}_2)$  is dense in  $C^{\infty}(\mathcal{O}_1\times\mathcal{O}_2)$  with respect to the  $L^2$ -norm. An application of Lemma 3.8 in connection with (\*) shows that

$$E(\mathcal{O}_1 \times \mathcal{O}_2) = p_1^* E(\mathcal{O}_1) \otimes p_2^* E(\mathcal{O}_2).$$

This equation in turn implies that we have equality in (\*).

The following theorem, which is a generalization of Weyl's formula, implies that the spectrum of a compact orientable orbifold determines its dimension and volume.

**Theorem 3.10** ([Far01]). Let  $\mathcal{O}$  be a compact orientable n-dimensional Riemannian orbifold with spectrum  $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$  Then for  $N(\lambda) = \#\{j; \lambda_j \leq \lambda\}$  we have

$$N(\lambda) \sim (2\pi)^{-n} \omega_n \operatorname{vol}(\mathcal{O}) \lambda^{n/2}$$

as  $\lambda \to \infty$  (i.e., the quotient of the two terms above converges to 1).  $\omega_n$  denotes the volume of the ball of radius 1 in  $\mathbb{R}^n$ .

The question whether an orbifold with singular points can be isospectral to a manifold is still open. However, [GR03] contains the following obstruction.

**Proposition 3.11.** Let  $\mathcal{O}$  be a compact good Riemannian orbifold with singularities and let M be a compact Riemannian manifold. If  $\mathcal{O}$  and M have a common Riemannian covering manifold, then they are not isospectral.

If M/G is a good Riemannian orbifold, we obtain the Laplacian on smooth k-forms on M/G by restricting  $\Delta = dd^* + d^*d$  on the space  $\Omega^k(M)$  of k-forms on M to

$$\Omega^k(M/G) := (\Omega^k(M))^G := \{ \omega \in \Omega^k(M); \ g^*\omega = \omega \ \forall g \in G \}.$$

Since  $g^*\Delta\omega = \Delta g^*\omega = \Delta\omega$  for any  $\omega \in \Omega^k(M/G)$  and  $g \in G$ , we indeed have  $\Delta(\Omega^k(M/G)) \subset \Omega^k(M/G)$ . The corresponding eigenspaces are then given by

$$E^k_\lambda(M/G) := E^k_\lambda(M) \cap \Omega^k(M/G) = E^k_\lambda(M)^G$$

**Definition 3.12.** If M is a compact Riemannian manifold and G is a subgroup of Isom(M) acting properly discontinuously, then the eigenvalues of  $\Delta: \Omega^k(M/G) \to \Omega^k(M/G)$  with multiplicities are called the k-spectrum of M/G. Two good orbifolds  $M_1/G_1$  and  $M_2/G_2$  (with  $M_i$  compact) are called k-isospectral if they have the same k-spectrum with multiplicities.

The fact that these eigenvalues are nonnegative and have finite multiplicity follows directly from the respective statement for the compact manifold M (cf. the appendix of [Cha84]). Since the Laplacian is symmetric on forms, too, we observe that we again obtain the same spectrum for the real- and the complex-valued case. Note that two good orbifolds are 0-isospectral if and only if they are isospectral in the sense of Definition 3.3.

# 4 Two Flat Orbifolds with Different Isotropy Orders

In this section we examine a pair of orbifolds with different isotropy orders. The fact that they are isospectral will be shown in the next section. These examples have recently been found by Juan Pablo Rossetti.

Before giving the definitions of our orbifold pair we recall some facts from the theory of quotients of euclidean space  $\mathbb{R}^n$  by groups of isometries (compare [Wol74] Chapter 3). The isometry group of  $\mathbb{R}^n$  is given by the semidirect product  $I(\mathbb{R}^n) = O(n) \ltimes \mathbb{R}^n$ ; i.e.,  $I(\mathbb{R}^n)$  consists of all transformations  $BL_b$ , where  $B \in O(n)$ ,  $b \in \mathbb{R}^n$  and  $L_b(x) = x + b$  for  $x \in \mathbb{R}^n$ . Note that

(\*) 
$$L_bB = BL_{B^{-1}b}, BL_bB^{-1} = L_{Bb} \text{ and } (BL_b)^{-1} = B^{-1}L_{-Bb}$$

The following holds for subgroups of  $I(\mathbb{R}^n)$  equipped with the compact-open topology.

**Theorem 4.1.** Let G be a subgroup of  $I(\mathbb{R}^n)$ .

- 1. G acts properly discontinuously if and only if G is discrete in  $I(\mathbb{R}^n)$ .
- 2. Let G be discrete in  $I(\mathbb{R}^n)$ .
  - a)  $\mathbb{R}^n/G$  is compact if and only if  $I(\mathbb{R}^n)/G$  is compact.
  - b) G acts freely on  $\mathbb{R}^n$  if and only if G is torsion-free.

*Proof.* cf. [Wol74] 3.1.3

A subgroup G of  $I(\mathbb{R}^n)$  is called *cocompact* if  $I(\mathbb{R}^n)/G$  is compact. If G is discrete, cocompact and torsion-free, it is called a *Bieberbach group*, and the theorem above implies that  $\mathbb{R}^n/G$  is a compact flat Riemannian manifold. Conversely, every compact flat Riemannian manifold is isometric to such a quotient. The spectrum of the Laplacian on such manifolds has been examined in [MR01] and [MR03]. In [RC06] it is shown that in dimension three there is, up to scaling, exactly one pair of isospectral non-isometric compact flat manifolds.

If we drop the condition that G be torsion-free, then G can have fixed points. A discrete, cocompact subgroup of  $I(\mathbb{R}^n)$  is called a *crystallographic group*. The corresponding quotient  $\mathbb{R}^n/G$  is a compact good Riemannian orbifold by Theorem 2.24. Conversely, by Lemma 2.26, if  $\mathcal{O}$  is a compact Riemannian orbifold of constant curvature zero then there is a crystallographic group  $G \subset I(\mathbb{R}^n)$  such that  $\mathcal{O}$  is isometric to  $\mathbb{R}^n/G$ .

In the orbifold setting we have the isotropy groups as an additional structure (unlike in the manifold setting). [SSW06] gives a construction of an arbitrarily large number of pairwise isospectral orbifolds each of which contains a point whose isotropy group is not isomorphic to an isotropy group occurring in any other orbifold of this set. In other words, the spectrum does not determine the isotropy types on an orbifold. These examples have been exhibited using a technique by Sunada (see Theorem 6.1 below). However, they did not rule out the possibility that the orbifold spectrum might determine the *order* of the isotropy groups. We examine a pair of crystallographic groups such that the orders of the respective largest isotropy groups are different.

Let  $\Lambda$  be the lattice  $2\mathbb{Z} \times 2\mathbb{Z} \times \mathbb{Z}$  in  $\mathbb{R}^3$ . Let  $\tau$  be the quarter-rotation around the  $x_3$ -axis in the mathematically positive direction.

Then

$$\tau = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tau^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tau^3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tau^4 = I_3$$

The rotation  $\tau$  leaves  $\Lambda$  invariant, and the relations

$$(\tau^i L_{\lambda_1})(\tau^j L_{\lambda_2}) = \tau^{i+j} L_{\tau^{-j}\lambda_1 + \lambda_2}$$
 and  $(\tau^i L_{\lambda})^{-1} = \tau^{-i} L_{-\tau^i \lambda_1}$ 

(which one easily checks using (\*)) imply that

$$G_1 := \{ \tau^i L_{\lambda}; i \in \{0, 1, 2, 3\}, \lambda \in \Lambda \}$$

is a subgroup of  $I(\mathbb{R}^3)$ .

Moreover, set

$$\rho_0 := I_3$$

$$\rho_1 := \chi_1 \circ L_{b_1} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \circ L_{(1,0,0)}$$

$$\rho_2 := \chi_2 \circ L_{b_2} := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \circ L_{(0,0,0)}$$

$$\rho_3 := \chi_3 \circ L_{b_3} := \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \circ L_{(-1,0,0)}$$

Note that  $\Lambda$  is invariant under every  $\chi_i$ . Using (\*), one verifies the relations

$$(\rho_1 L_{\lambda})^{-1} = \rho_1 L_{-\chi_1 \lambda - (2,0,0)}, \ (\rho_2 L_{\lambda})^{-1} = \rho_2 L_{-\chi_2 \lambda}, \ (\rho_3 L_{\lambda})^{-1} = \rho_3 L_{-\chi_3 \lambda}.$$

Thus

$$G_2 := \{ \rho_i L_{\lambda}; i \in \{0, 1, 2, 3\}, \lambda \in \Lambda \}$$

is closed under inversion. (\*) also implies that  $G_2$  is closed under composition, hence it is another subgroup of  $I(\mathbb{R}^3)$ .

For a subgroup G of  $I(\mathbb{R}^n)$  let  $\eta: G \to O(3)$  denote the natural projection given by  $\eta(BL_b) = B$ . The image F of  $\eta$  is called the *point group* of G. The kernel of  $\eta$  is the subgroup  $\Lambda$  of translations in G, and we have an exact sequence of groups

$$I_3 \to \Lambda \to G \to F \to I_3$$

and an isomorphism  $F \simeq G/\Lambda$ .

Note that, in our examples,

$$F_1 = \{I_3, \tau, \tau^2, \tau^3\}, F_2 = \{I_3, \chi_1, \chi_2, \chi_3\}$$

are finite, which together with the fact that the given lattice  $\Lambda = 2\mathbb{Z} \times 2\mathbb{Z} \times \mathbb{Z}$  is discrete and cocompact implies that the  $G_i$  are crystallographic ([Rat06] 7.5). However, the  $G_i$  are not torsion-free, since, e.g.,  $\tau^4 = I_3$  and  $\rho_2^2 = I_3$ .

Let  $\mathcal{O}_i = \mathbb{R}^3/G_i$  denote the corresponding compact good Riemannian orbifolds. They are orientable, because all transformations in  $G_i$  are orientation-preserving.

### 4.1 The Fundamental Domains

On the following pages we give an exhaustive calculation of the fundamental domains of the actions of  $G_i$  on  $\mathbb{R}^3$  and the identifications on their boundaries given by the group actions. The impatient reader may skip these tables and focus on the images. Section 4.2 contains the interpretation of the calculations with regard to the isotropy groups.

The cuboid  $[-1,1] \times [-1,1] \times [0,1]$  is a fundamental domain for the action of the lattice  $\Lambda$  on  $\mathbb{R}^3$ , and the torus  $\mathbb{R}^3/\Lambda$  can be regarded as this cuboid with opposite sides identified by the canonical translations.

### **4.1.1** The Orbifold $\mathcal{O}_1$

Since  $\tau$  is a quarter-rotation around the  $x_3$ -axis, every point in  $[-1,1] \times [-1,1] \times [0,1]$  is equivalent to a point in  $[0,1]^3$  under a rotation  $\tau^i$ . Next we examine the identifications and fixed points on  $[0,1]^3$  by the  $G_1$ -action: For each i=0,1,2,3 we determine all  $\lambda \in \Lambda$  for which there is an  $x \in [0,1]^3$  such that  $\tau^i L_{\lambda} x \in [0,1]^3$ . We call such  $\lambda$  relevant and list all pairs  $(x,\tau^i L_{\lambda} x) \in ([0,1]^3)^2$  corresponding to each relevant  $\lambda$ .

• i = 0:

$$\tau^0 L_{\lambda} x = L_{\lambda} x = (x_1 + \lambda_1, x_2 + \lambda_2, x_3 + \lambda_3)$$

The only non-zero relevant  $\lambda$  are  $\lambda = (0, 0, \pm 1)$ , which lead to the identification  $(x_1, x_2, 0) \sim (x_1, x_2, 1)$ .

• i = 1:

$$\tau^{1}L_{\lambda}x = (-x_{2} - \lambda_{2}, x_{1} + \lambda_{1}, x_{3} + \lambda_{3})$$

Only  $\lambda \in \{0\} \times \{-2,0\} \times \{-1,0,1\}$  are relevant.

$\lambda_1$	$\lambda_2$	$\lambda_3$	x	$\tau L_{\lambda} x$	
0	-2	-1	$(x_1, 1, 1)$	$(1, x_1, 0)$	
0	-2	0	$(x_1,1,x_3)$	$(1, x_1, x_3)$	F
0	-2	1	$(x_1, 1, 0)$	$(1, x_1, 1)$	
0	0	-1	$(x_1,0,1)$	$(0, x_1, 0)$	
0	0	0	$(x_1,0,x_3)$	$(0, x_1, x_3)$	Р
0	0	1	$(x_1,0,0)$	$(0, x_1, 1)$	

(The letters in the last column correspond to the sides in Figure 4.1.)

• i = 2:

$$\tau^{2}L_{\lambda}x = (-x_{1} - \lambda_{1}, -x_{2} - \lambda_{2}, x_{3} + \lambda_{3})$$

Only  $\lambda \in \{-2,0\} \times \{-2,0\} \times \{-1,0,1\}$  are relevant. Since

$$(\tau^2 L_{\lambda})^{-1} = L_{-\lambda} \tau^2 = \tau^2 L_{(\lambda_1, \lambda_2, -\lambda_3)},$$

we can omit the case  $\lambda_3 = -1$ .

$\lambda_1$	$\lambda_2$	$\lambda_3$	x	$\tau^2 L_{\lambda} x$
-2	-2	0	$(1,1,x_3)$	$(1,1,x_3)$
-2	-2	1	(1, 1, 0)	(1, 1, 1)
-2	0	0	$(1,0,x_3)$	$(1,0,x_3)$
-2	0	1	(1,0,0)	(1,0,1)
0	-2	0	$(0,1,x_3)$	$(0,1,x_3)$
0	-2	1	(0, 1, 0)	(0, 1, 1)
0	0	0	$(0,0,x_3)$	$(0,0,x_3)$
0	0	1	(0, 0, 0)	(0, 0, 1)

• i = 3: Since

$$(\tau L_{\lambda})^{-1} = L_{-\lambda}\tau^3 = \tau^3 L_{(\lambda_2, -\lambda_1, \lambda_3)},$$

we obtain the same identifications and fixed points as in the case i = 1.

All in all,  $[0,1]^3$  is a fundamental domain for the action of  $G_1$  on  $\mathbb{R}^3$  and  $\mathcal{O}_1$  is just  $[0,1]^3$  with (closed) sides identified as indicated in the following picture, where we omit the identifications of the top and bottom side given by the vertical translation  $L_{e_3}$ .

## 4.1.2 The Orbifold $\mathcal{O}_2$

 $\mathcal{O}_2$  is also just the cube  $[0,1]^3$ , but with different identifications: First note that every point in  $[-1,1] \times [-1,1] \times [0,1]$  is the image of a point in  $[0,1]^3$  under one of the following

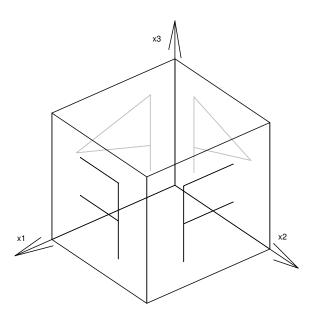


Figure 4.1: The underlying space of  $\mathcal{O}_1$  as a quotient of the unit cube

(restrictions of) transformations in  $G_2$ .

$$\rho_1 L_{(-2,0,-1)} : [0,1]^3 \to [-1,0] \times [-1,0] \times [0,1]$$

$$\rho_2 L_{(0,0,-1)} : [0,1]^3 \to [-1,0] \times [0,1] \times [0,1]$$

$$\rho_3 : [0,1]^3 \to [0,1] \times [-1,0] \times [0,1]$$

Since  $[-1,1] \times [-1,1] \times [0,1]$  is a fundamental domain for the action of  $\Lambda$  on  $\mathbb{R}^3$ , this implies that every point in  $\mathbb{R}^3$  is equivalent to a point in  $[0,1]^3$  under the action of  $G_2$ . For the identifications and fixed points in  $[0,1]^3$  we follow the algorithm above and, for every i=0,1,2,3, determine all  $\lambda \in \Lambda$  for which there is an  $x \in [0,1]^3$  such that  $\rho_i L_{\lambda} x \in [0,1]^3$ .

• i = 0:

$$\rho_0 L_{\lambda} x = L_{\lambda} x = (x_1 + \lambda_1, x_2 + \lambda_2, x_3 + \lambda_3)$$

The only non-zero relevant  $\lambda$  are  $\lambda = (0, 0, \pm 1)$ , which lead to the identification  $(x_1, x_2, 0) \sim (x_1, x_2, 1)$ .

• i = 1:

$$\rho_1 L_{\lambda} x = (x_1 + \lambda_1 + 1, -x_2 - \lambda_2, -x_3 - \lambda_3)$$

The only relevant  $\lambda$  are  $\lambda \in \{-2,0\} \times \{-2,0\} \times \{-2,-1,0\}$ .

Since

$$(\rho_1 L_{\lambda})^{-1} = L_{-\lambda} \rho_1^{-1} = L_{(-\lambda_1 - 1, -\lambda_2, -\lambda_3)} \chi_1 = \chi_1 L_{(-\lambda_1 - 1, \lambda_2, \lambda_3)}$$
  
=  $\rho_1 L_{(-\lambda_1 - 2, \lambda_2, \lambda_3)}$ ,

the value  $\lambda_1 = 0$  gives the same identifications as  $\lambda_1 = -2$  and we can omit the case  $\lambda_1 = 0$ .

$\lambda_1$	$\lambda_2$	$\lambda_3$	x	$\rho_1 L_{\lambda} x$
-2	-2	-2	(1, 1, 1)	(0, 1, 1)
-2	-2	-1	$(1,1,x_3)$	$(0,1,1-x_3)$
-2	-2	0	(1, 1, 0)	(0, 1, 0)
-2	0	-2	(1,0,1)	(0, 0, 1)
-2	0	-1	$(1,0,x_3)$	$(0,0,1-x_3)$
-2	0	0	(1,0,0)	(0, 0, 0)

• i = 2:

$$\rho_2 L_{\lambda} x = (-x_1 - \lambda_1, x_2 + \lambda_2, -x_3 - \lambda_3)$$

The only relevant  $\lambda$  are  $\lambda \in \{-2,0\} \times \{0\} \times \{-2,-1,0\}$ .

$\lambda_1$	$\lambda_2$	$\lambda_3$	x	$\rho_2 L_{\lambda} x$	
-2	0	-2	$(1, x_2, 1)$	$(1, x_2, 1)$	
-2	0	-1	$(1, x_2, x_3)$	$(1, x_2, 1 - x_3)$	Р
-2	0	0	$(1, x_2, 0)$	$(1, x_2, 0)$	
0	0	-2	$(0, x_2, 1)$	$(0, x_2, 1)$	
0	0	-1	$(0, x_2, x_3)$	$(0, x_2, 1 - x_3)$	$\Delta$
0	0	0	$(0, x_2, 0)$	$(0, x_2, 0)$	

(The letters in the last column correspond to the sides in Figure 4.2.)

• i = 3:

$$\rho_3 L_{\lambda} x = (-x_1 - \lambda_1 + 1, -x_2 - \lambda_2, x_3 + \lambda_3)$$

The only relevant  $\lambda$  are  $\lambda \in \{0\} \times \{-2, 0\} \times \{-1, 0, 1\}$ .

Since  $(\rho_3 L_{\lambda})^{-1} = \rho_3 L_{(\lambda_1, \lambda_2, -\lambda_3)}$ , we obtain the same identifications for  $\lambda_3 = -1$  and  $\lambda_3 = 1$ , and we can omit the case  $\lambda_3 = 1$  in the following table.

$\lambda_1$	$\lambda_2$	$\lambda_3$	x	$\rho_3 L_{\lambda} x$	
			/ /	$(1-x_1,1,0)$	
0	-2	0	$(x_1,1,x_3)$	$(1-x_1,1,x_3)$	L
0	0	-1	$(x_1,0,1)$	$(1-x_1,0,0)$	
0	0	0	$(x_1,0,x_3)$	$(1-x_1,0,x_3)$	F

(Again, the letters in the last column correspond to the sides in Figure 4.2.)

All in all,  $[0,1]^3$  is a fundamental domain for the action of  $G_2$  on  $\mathbb{R}^3$ , and we obtain the following picture for  $\mathcal{O}_2$ , where we again omit the identifications by the vertical translation  $L_{e_3}$ .

## 4.2 The Isotropy Groups

Next, we examine the tables from the preceding section to find the singular points on the orbifolds and determine their isotropy groups. We write  $\sim$  for the equivalence of points

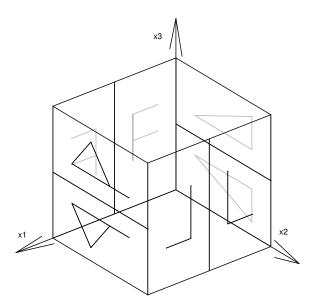


Figure 4.2: The underlying space of  $\mathcal{O}_2$  as a quotient of the unit cube

under the action of the respective  $G_i$ . We explicitly mention all equivalences between fixed points on  $[0,1]^3$  apart from those arising from  $L_{e_3}$ .

### **4.2.1** The Isotropy Groups on $\mathcal{O}_1$

The only fixed points in the table for i=1 are  $(0,0,x_3)$  and  $(1,1,x_3)$ . The table for i=2 contains the same fixed points and the additional fixed points  $(0,1,x_3) \sim (1,0,x_3)$ . The corresponding isotropy groups are also easily read off from the tables (where we use the notation of Theorem 2.24):

$$G_{1(0,0,x_3)} = \left\{ I_3, \tau, \tau^2, \tau^3 \right\} \simeq \mathbb{Z}_4$$

$$G_{1(1,1,x_3)} = \left\{ I_3, \tau L_{(0,-2,0)}, \tau^2 L_{(-2,-2,0)}, \tau^3 L_{(-2,0,0)} \right\} \simeq \mathbb{Z}_4$$

$$G_{1(0,1,x_3)} = \left\{ I_3, \tau^2 L_{(0,-2,0)} \right\} \simeq \mathbb{Z}_2$$

Note that  $\mathcal{O}_1$  is just the product of a 442-orbifold (in the notation of [Con92]) with  $\mathbb{R}/\mathbb{Z}$ .

## 4.2.2 The Isotropy Groups on $\mathcal{O}_2$

In the table for i=2 we find the fixed points  $(1,x_2,\frac{1}{2}), (0,x_2,\frac{1}{2})$  (where  $(1,1,\frac{1}{2}) \sim (0,1,\frac{1}{2})$  and  $(1,0,\frac{1}{2}) \sim (0,0,\frac{1}{2})$ ),  $(1,x_2,1) \sim (1,x_2,0)$  and  $(0,x_2,1) \sim (0,x_2,0)$ . Besides, i=3 gives the fixed points  $(\frac{1}{2},1,x_3), (\frac{1}{2},0,x_3)$ .

Each of these fixed points appears in exactly one row in the tables above, hence their

isotropy always is  $\mathbb{Z}_2$ :

$$G_{2(1,x_2,\frac{1}{2})} = \left\{ I_3, \rho_2 L_{(-2,0,-1)} \right\}$$

$$G_{2(0,x_2,\frac{1}{2})} = \left\{ I_3, \rho_2 L_{(0,0,-1)} \right\}$$

$$G_{2(1,x_2,1)} = \left\{ I_3, \rho_2 L_{(-2,0,-2)} \right\}$$

$$G_{2(0,x_2,1)} = \left\{ I_3, \rho_2 L_{(0,0,-2)} \right\}$$

$$G_{2(\frac{1}{2},1,x_3)} = \left\{ I_3, \rho_3 L_{(0,-2,0)} \right\}$$

$$G_{2(\frac{1}{2},0,x_3)} = \left\{ I_3, \rho_3 \right\}$$

In contrast to  $\mathcal{O}_1$ , all isotropy groups on  $\mathcal{O}_2$  have order  $\leq 2$ . In particular, by Lemma 2.21,  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are not diffeomorphic.

The existence of singular points on the orbifolds  $\mathcal{O}_i$  implies that none of them is isospectral to a flat Riemannian manifold: Such a manifold would have dimension three by Theorem 3.10 and thus be a quotient of euclidean space  $\mathbb{R}^3$  by Lemma 2.26. By Proposition 3.11, this is impossible, since the  $\mathcal{O}_i$  themselves are also covered by  $\mathbb{R}^3$ .

## 5 Verification of Isospectrality

In this chapter we are going to show in three different ways that the orbifolds  $\mathcal{O}_1$  and  $\mathcal{O}_2$  constructed in Chapter 4 are isospectral; see Sections 5.1, 5.2, 5.3. Moreover, in 5.2 we are going to show that  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are not 1-isospectral.

We first summarize a few basic facts from [BGM71], [Cha84] about the spectrum of a torus  $T_{\Lambda} = \mathbb{R}^n/\Lambda$ , where  $\Lambda$  is an *n*-dimensional lattice in  $\mathbb{R}^n$ . For convenience, we let  $H_{\mu}(\mathcal{O}) := E_{4\pi^2\mu}^{\mathbb{C}}(\mathcal{O})$  denote the eigenspace associated with the eigenvalue  $4\pi^2\mu$  of a flat orbifold  $\mathcal{O}$  and set  $H_{\mu} := H_{\mu}(\mathbb{R}^n)$ . Since  $T_{\Lambda}$  is compact, the space

$$H_{\mu}(T_{\Lambda}) = H_{\mu}^{\Lambda} = \{ \phi \in H_{\mu}; \phi \circ L_{\lambda} = \phi \ \forall \lambda \in \Lambda \}$$

is finite-dimensional. A basis for  $H_{\mu}(T_{\Lambda})$  is given by

$$\{\phi_v; \ v \in \Lambda^*, \|v\|^2 = \mu\},\$$

where  $\phi_v(x) = e^{2\pi i \langle v, x \rangle}$ , and  $\Lambda^* = \{v \in \mathbb{R}^n; \langle v, w \rangle \in \mathbb{Z} \ \forall w \in \Lambda\}$  is the dual lattice of  $\Lambda$ .

## 5.1 Matching up Eigenfunctions

In the rare cases that explicit bases of the eigenspaces on a compact Riemannian manifold M are known, one can try to verify the isospectrality of two good orbifolds  $M/G_1$ ,  $M/G_2$  by matching up the eigenfunctions of the respective eigenspaces on the two quotients. The general procedure for manifolds which are compact quotients of  $\mathbb{R}^n$  has been outlined in [DR04], where the authors examine the unique pair of compact three-dimensional flat manifolds which are isospectral but not isometric.

We apply this procedure to our orbifold examples viewed as quotients of  $T_{\Lambda}$  with  $\Lambda = 2\mathbb{Z} \times 2\mathbb{Z} \times \mathbb{Z}$ . Note that  $\Lambda^* = \frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z} \times \mathbb{Z}$ . Let  $G_1$ ,  $G_2$  be as in Chapter 4. Then  $H_{\mu}(\mathcal{O}_i) = H_{\mu}^{G_i}$ , which can be identified with certain invariant eigenfunctions on the torus:

$$\begin{split} H^{G_1}_{\mu} &= \{ \phi \in H^{\Lambda}_{\mu}; \ \phi \circ \tau^i = \phi \ \forall i \} \\ H^{G_2}_{\mu} &= \{ \phi \in H^{\Lambda}_{\mu}; \ \phi \circ \rho_i = \phi \ \forall i \} \end{split}$$

Define linear endomorphisms of  $C^{\infty}(T_{\Lambda}, \mathbb{C}) = C^{\infty}(\mathbb{R}^{3}, \mathbb{C})^{\Lambda}$  by

$$\sigma^{1}(\phi) = \frac{1}{4}(\phi + \phi \circ \tau + \phi \circ \tau^{2} + \phi \circ \tau^{3})$$
$$\sigma^{2}(\phi) = \frac{1}{4}(\phi + \phi \circ \rho_{1} + \phi \circ \rho_{2} + \phi \circ \rho_{3})$$

Then each  $\sigma^i$  is a projection which commutes with  $\Delta$  because the  $\tau^j$ ,  $\rho_j$  are isometries of  $\mathbb{R}^3$ . Moreover,  $\sigma^i$  has image  $C^{\infty}(\mathcal{O}_i, \mathbb{C}) = C^{\infty}(\mathbb{R}^3, \mathbb{C})^{G_i}$ , hence  $\sigma^i(H^{\Lambda}_{\mu}) = H^{G_i}_{\mu}$ . Isospectrality of  $\mathcal{O}_1$  and  $\mathcal{O}_2$  is thus equivalent to

$$\dim \sigma^1(H^{\Lambda}_{\mu}) = \dim \sigma^2(H^{\Lambda}_{\mu}) \ \forall \mu \in [0, \infty).$$

For  $(a,b,c) \in \Lambda^*$  write  $\phi_{a,b,c}(x) := \phi_{(a,b,c)}(x) = \exp(2\pi i(ax_1 + bx_2 + cx_3))$  and observe that

$$\phi_{a,b,c} \circ \tau = \phi_{b,-a,c}, \qquad \phi_{a,b,c} \circ \tau^2 = \phi_{-a,-b,c}, \quad \phi_{a,b,c} \circ \tau^3 = \phi_{-b,a,c}$$
 (5.1)

$$\phi_{a,b,c} \circ \rho_1 = e^{2\pi i a} \phi_{a,-b,-c}, \quad \phi_{a,b,c} \circ \rho_2 = \phi_{-a,b,-c}, \quad \phi_{a,b,c} \circ \rho_3 = e^{2\pi i a} \phi_{-a,-b,c}$$
 (5.2)

If  $a^2 + b^2 + c^2 = \mu$  then

$$V_{a,b,c} := \operatorname{span}_{\mathbb{C}} \{ \phi_{\pm a,\pm b,\pm c}, \phi_{\pm b,\pm a,\pm c} \}$$

is a subspace of  $H^{\Lambda}_{\mu}$  which, by the relations above, is invariant under each  $\sigma^i$ . For  $\mu \in [0, \infty)$  we set

$$\Lambda_{\mu}^{*} := \{(a,b,c) \in \Lambda^{*}; \ 0 \leqslant a \leqslant b, \ c \geqslant 0, \ a^{2} + b^{2} + c^{2} = \mu\}.$$

Then  $H^{\Lambda}_{\mu} = \bigoplus_{(a,b,c) \in \Lambda^*_{\mu}} V_{a,b,c}$  is a (possibly empty) direct sum and

$$H^{G_i}_{\mu} = \sigma^i(H^{\Lambda}_{\mu}) = \sigma^i\left(\bigoplus_{(a,b,c)\in\Lambda^*_{\mu}} V_{a,b,c}\right) = \bigoplus_{(a,b,c)\in\Lambda^*_{\mu}} \sigma^i(V_{a,b,c}).$$

Therefore, isospectrality of  $\mathcal{O}_1$  and  $\mathcal{O}_2$  is equivalent to

$$\forall \mu \in [0, \infty) : \sum_{(a,b,c) \in \Lambda_{\mu}^*} \dim \sigma^1(V_{a,b,c}) = \sum_{(a,b,c) \in \Lambda_{\mu}^*} \dim \sigma^2(V_{a,b,c})$$

We observe that none of the summands above can be greater than 4:

 $\mathcal{O}_1$ : Relation (5.1) implies that for every  $(a, b, c) \in \Lambda^*$ :

$$\sigma^{1}(\phi_{a,b,c}) = \sigma^{1}(\phi_{b,-a,c}) = \sigma^{1}(\phi_{-a,-b,c}) = \sigma^{1}(\phi_{-b,a,c}),$$

hence  $\sigma^1(V_{a,b,c})$  is spanned by the (not necessarily distinct) functions

$$\Phi_1 := \{ \sigma^1(\phi_{a,b,c}), \sigma^1(\phi_{-a,b,c}), \sigma^1(\phi_{a,b,-c}), \sigma^1(\phi_{-a,b,-c}) \}.$$

 $\mathcal{O}_2$ : Relation (5.2) implies that for every  $(a, b, c) \in \Lambda^*$ :

$$\sigma^2(\phi_{a,b,c}) = e^{2\pi i a} \sigma^2(\phi_{a,-b,-c}) = \sigma^2(\phi_{-a,b,-c}) = e^{2\pi i a} \sigma^2(\phi_{-a,-b,c}),$$

hence  $\sigma^2(V_{a,b,c})$  is spanned by the (not necessarily distinct) functions

$$\Phi_2 := \{ \sigma^2(\phi_{a,b,c}), \sigma^2(\phi_{-a,b,c}), \sigma^2(\phi_{b,a,c}), \sigma^2(\phi_{-b,a,c}) \}.$$

First we assume that either none, one or three of the numbers a, b, c are zero. Under this assumption  $\dim \sigma^1(V_{a,b,c}) = \dim \sigma^2(V_{a,b,c})$ , as can be read off from the following tables which cover all possible cases: For each line one easily verifies that every element of the corresponding  $\Phi_i$  either coincides with one of the given functions or vanishes completely. Moreover, writing out the definition of  $\sigma^i$  (and using the well-known fact that the set  $\{\phi_{r,s,t}; (r,s,t) \in \mathbb{R}^3\}$  is linearly independent in  $C^{\infty}(\mathbb{R},\mathbb{C})$ ) shows that each of the sets below is indeed linearly independent.

	basis of $\sigma^1(V_{a,b,c})$	$\dim \sigma^1(V_{a,b,c})$
0 < a < b, c > 0	$\sigma^{1}(\phi_{a,b,c}), \sigma^{1}(\phi_{-a,b,c}), \sigma^{1}(\phi_{a,b,-c}), \sigma^{1}(\phi_{-a,b,-c})$	4
0 < a = b, c > 0	$\sigma^1(\phi_{a,a,c}),\sigma^1(\phi_{a,a,-c})$	2
0 < a < b, c = 0	$\sigma^{1}(\phi_{a,b,0}), \sigma^{1}(\phi_{-a,b,0})$	2
0 < a = b, c = 0	$\sigma^1(\phi_{a,a,0})$	1
0 = a < b, c > 0	$\sigma^{1}(\phi_{0,b,c}), \sigma^{1}(\phi_{0,b,-c})$	2
0 = a = b, c = 0	$\sigma^1(\phi_{0,0,0})$	1

	basis of $\sigma^2(V_{a,b,c})$	$\dim \sigma^2(V_{a,b,c})$
0 < a < b, c > 0	$\sigma^{2}(\phi_{a,b,c}), \sigma^{2}(\phi_{-a,b,c}), \sigma^{2}(\phi_{b,a,c}), \sigma^{2}(\phi_{-b,a,c})$	4
0 < a = b, c > 0	$\sigma^2(\phi_{a,a,c}), \sigma^2(\phi_{-a,a,c})$	2
0 < a < b, c = 0	$\sigma^2(\phi_{a,b,0}), \sigma^2(\phi_{b,a,0})$	2
0 < a = b, c = 0	$\sigma^2(\phi_{a,a,0})$	1
0 = a < b, c > 0	$\sigma^2(\phi_{0,b,c}),\sigma^2(\phi_{b,0,c})$	2
0 = a = b, c = 0	$\sigma^1(\phi_{0,0,0})$	1

The case that exactly two of the numbers a, b, c are zero requires more care. Since  $a \leq b$  by assumption, we have a = 0 in this case. Recall that  $b \in \frac{1}{2}\mathbb{N}_0$  whereas  $c \in \mathbb{N}_0$ .

• If  $b \in \frac{1}{2} + \mathbb{N}_0$  and c = 0, we still have equality: The functions in  $\Phi_1$  coincide (and are non-zero). As for  $\Phi_2$ , we observe that the first two functions coincide and  $\sigma^2(\phi_{b,0,0}) = \sigma^2(\phi_{-b,0,0}) = 0$  by (5.2) (note that  $e^{2\pi i b} = -1$ ), i.e.,  $\dim \sigma^1(V_{a,b,c}) = \dim \sigma^2(V_{a,b,c}) = 1$ .

• To complete the proof of isospectrality we need to show that for  $n \in \mathbb{N}$ :

$$\dim \sigma^1(V_{0,0,n}) + \dim \sigma^1(V_{0,n,0}) = \dim \sigma^2(V_{0,0,n}) + \dim \sigma^2(V_{0,n,0}).$$

But this follows from the following two tables:

#### 5.2 A Dimension Formula

Using the explicit basis of  $H^{\Lambda}_{\mu}$  given at the beginning of this chapter, Miatello and Rossetti ([MR01]) derived a formula for the dimension of the eigenspace corresponding to the eigenvalue  $4\pi^2\mu$  on a quotient of  $\mathbb{R}^n$  by a Bieberbach group. A closer investigation of the proof shows that the formula remains true for crystallographic groups, i.e., in the setting of compact good orbifolds (cf. [MR02] Remark 2.6).

**Theorem 5.1.** Let G be a crystallographic group acting on  $\mathbb{R}^n$  and let F be the point group of G. Then the dimension of the eigenspace of  $\Delta$  on  $C^{\infty}(\mathbb{R}^n/G)$  corresponding to the eigenvalue  $4\pi^2\mu$  is given by

$$d_{\mu}(G) := \dim H_{\mu}^{G} = |F|^{-1} \sum_{B \in F} e_{\mu,B}(G)$$

where

$$e_{\mu,B}(G) := \sum_{v \in \Lambda^*: \|v\|^2 = \mu} e^{2\pi i \langle v, b \rangle}, \text{ with } b \text{ chosen such that } BL_b \in G.$$

We are now going to apply Theorem 5.1 in order to give a second proof of isospectrality for the orbifolds  $\mathcal{O}_1$ ,  $\mathcal{O}_2$  from Chapter 4.

First, we observe that

$$e_{\mu,I_3}(G_i) = \sum_{\substack{v \in \Lambda^* \\ \|v\|^2 = \mu}} e^{2\pi i \langle v, 0 \rangle} = \#\{v \in \Lambda^*; \|v\|^2 = \mu\} =: e_{\mu,I_3}$$

depends only on the lattice  $\Lambda$ , i.e., it is the same for both  $G_i$ . Moreover  $e_{0,B}(G_i) = 1 \ \forall B \in F$  and therefore  $d_0(G_i) = \frac{1}{4} \cdot 4 = 1$ , as also follows directly from Theorem 3.4. Now let  $\mu > 0$ . Then the only  $v \in \mathbb{R}^3$  with  $||v||^2 = \mu$  which are fixed by any element

Now let  $\mu > 0$ . Then the only  $v \in \mathbb{R}^3$  with  $||v||^2 = \mu$  which are fixed by any element of  $F_1 \setminus \{I_3\} = \{\tau, \tau^2, \tau^3\}$  are  $(0, 0, \pm \sqrt{\mu})$ . As for  $F_2$ , observe that for i = 1, 2, 3 the only two  $v \in \mathbb{R}^3$  with  $||v||^2 = \mu$  and  $\chi_i v = v$  are given by  $v = \pm \sqrt{\mu}e_i$ , where  $e_i$  denotes the *i*-th standard unit vector.

We now distinguish the following three cases.

**1st case**  $\sqrt{\mu} \notin \frac{1}{2}\mathbb{N}$ : In this case the considerations above show that there are no vectors of length  $\sqrt{\mu}$  in  $\Lambda^*$  which are fixed by a non-trivial element of  $F_1$  or  $F_2$ , i.e.,  $e_{\mu,B}(G_i) = 0 \ \forall B \in F_i \setminus \{I_3\}$  and therefore  $d_{\mu}(G_1) = \frac{1}{4}e_{\mu,I_3} = d_{\mu}(G_2)$ .

2nd case  $\sqrt{\mu} \in \mathbb{N}$ :

$$e_{\mu,\tau^i}(G_1) = \exp(2\pi i \langle (0,0,\sqrt{\mu}),0\rangle) + \exp(2\pi i \langle (0,0,-\sqrt{\mu}),0\rangle) = 2 \text{ for } i \in \{1,2,3\},$$

i.e.,  $d_{\mu}(G_1) = \frac{1}{4}(e_{\mu,I_3} + 6)$ . As for  $\mathcal{O}_2$ , we have

$$e_{\mu,\chi_1}(G_2) = \exp(2\pi i \langle (\sqrt{\mu}, 0, 0), (1, 0, 0) \rangle) + \exp(2\pi i \langle (-\sqrt{\mu}, 0, 0), (1, 0, 0) \rangle) = 2$$

$$e_{\mu,\chi_2}(G_2) = \exp(2\pi i \langle (0, \sqrt{\mu}, 0), (0, 0, 0) \rangle) + \exp(2\pi i \langle (0, -\sqrt{\mu}, 0), (0, 0, 0) \rangle) = 2$$

$$e_{\mu,\chi_3}(G_2) = \exp(2\pi i \langle (0, 0, \sqrt{\mu}), (-1, 0, 0) \rangle) + \exp(2\pi i \langle (0, 0, -\sqrt{\mu}), (-1, 0, 0) \rangle) = 2$$

and therefore

$$d_{\mu}(G_2) = \frac{1}{4}(e_{\mu,I_3} + 6) = d_{\mu}(G_1).$$

**3rd case:**  $\sqrt{\mu} \in \mathbb{N}_0 + \frac{1}{2}$ :  $e_{\mu,\tau^i}(G_1) = 0$  for all  $i \in \{1,2,3\}$ , since  $(0,0,\pm\sqrt{\mu}) \notin \Lambda^*$ . Thus  $d_{\mu}(G_1) = \frac{1}{4}e_{\mu,I_3}$ . As for  $\mathcal{O}_2$ , we calculate:

$$e_{\mu,\chi_1}(G_2) = \exp(2\pi i \langle (\sqrt{\mu}, 0, 0), (1, 0, 0) \rangle) + \exp(2\pi i \langle (-\sqrt{\mu}, 0, 0), (1, 0, 0) \rangle) = -2$$

$$e_{\mu,\chi_2}(G_2) = \exp(2\pi i \langle (0, \sqrt{\mu}, 0), (0, 0, 0) \rangle) + \exp(2\pi i \langle (0, -\sqrt{\mu}, 0), (0, 0, 0) \rangle) = 2$$

$$e_{\mu,\chi_3}(G_2) = 0, \text{ since } (0, 0, \pm \sqrt{\mu}) \notin \Lambda^*$$

Therefore  $d_{\mu}(G_2) = \frac{1}{4}e_{\mu,I_3} = d_{\mu}(G_1)$  in this case. All in all we have  $d_{\mu}(G_1) = d_{\mu}(G_2)$  for every  $\mu \geq 0$ , i.e.,  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are (0-)isospectral.

Remark. Note that alternatively we also could have used Theorem 5.1 in connection with Lemma 3.9 regarding the spectrum of a product orbifold to determine the spectrum of  $\mathcal{O}_1$ . Moreover, the same lemma (in connection with the remark after Definition 2.22) gives a simple method to construct isospectral pairs of higher-dimensional orbifolds with isotropy groups of different orders.

We are now going to show that, however,  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are *not* isospectral on 1-forms. In fact, [MR01] also contains a formula for the dimension of the eigenspaces for the Laplace-Beltrami operator acting on k-forms. Let G be a crystallographic group acting on  $\mathbb{R}^n$ . For  $k \in \mathbb{N}$  and  $\mu \in [0, \infty)$  set  $H_{k,\mu} = \{\omega \in \Omega^k(\mathbb{R}^n); \Delta\omega = 4\pi^2\mu\omega\}$  and

$$H_{k,\mu}^G = \{ \omega \in H_{k,\mu}; \ g^*\omega = \omega \ \forall g \in G \}$$

Let  $\tau_k: O(n) \to \operatorname{End}(\Lambda^k(\mathbb{R}^n))$  be the canonical representation of O(n) on the finite-dimensional space  $\Lambda^k(\mathbb{R}^n)$  given by

$$\tau_k(B)\omega[X_1,\ldots,X_k] = (B^{-1})^*\omega(X_1,\ldots,X_k) = \omega[B^{-1}X_1,\ldots,B^{-1}X_k]$$

and write  $\operatorname{tr}_k(B) := \operatorname{tr} \tau_k(B)$ . As a generalization of Theorem 5.1, the following formula holds, whose purely algebraic proof in [MR01] for Bieberbach groups again carries over to good orbifolds (as remarked in [MR02]).

**Theorem 5.2.** Let G be a crystallographic group and let F be the point group of G. Then the dimension of the eigenspace of the Laplace-Beltrami operator on  $\Omega^k(\mathbb{R}^n/G)$  corresponding to the eigenvalue  $4\pi^2\mu$  is given by

$$d_{k,\mu}(G) := \dim H_{k,\mu}^G = |F|^{-1} \sum_{B \in F} \operatorname{tr}_k(B) e_{\mu,B}(G),$$

where  $e_{\mu,B}(G)$  is defined as in Theorem 5.1.

The coefficients  $\tau_k(B)$  can easily be read off from the characteristic polynomial of B: Let  $A = (a_{ij})_{i,j=1}^n$  be an arbitrary  $n \times n$ -matrix. For a multi-index  $I = \{i_1 < \ldots < i_k\}$  in  $\{1,\ldots,n\}$  let  $A_I$  denote the  $k \times k$ -matrix  $(a_{i_{\alpha}i_{\beta}})_{\alpha,\beta=1}^k$ . Recall that the coefficients of the characteristic polynomial

$$\det(\lambda I_n - A) = \sum_{k=0}^n s_k \lambda^{n-k}$$

are given by  $s_k = (-1)^k \sum_I \det(A_I)$ , where the sum runs over all multi-indices I of order k in  $\{1, \ldots, n\}$ . Moreover, a simple calculation shows

$$A^*(dx^{i_1} \wedge \ldots \wedge dx^{i_k}) (e_{i_1}, \ldots, e_{i_k}) = \det(A_I),$$

hence

$$\operatorname{tr}(A^*|\Lambda^k(\mathbb{R}^n)) = \sum_I \det(A_I) = (-1)^k s_k$$

or

$$\det(\lambda I_n - A) = \sum_{k=0}^n (-1)^k \operatorname{tr}(A^* | \Lambda^k(\mathbb{R}^n)) \lambda^{n-k}.$$

For  $B \in O(n)$  we finally have

$$\det(\lambda I_n - B) = \det(\lambda I_n - B^t) = \det(\lambda I_n - B^{-1}) = \sum_{k=0}^n (-1)^k \operatorname{tr}((B^{-1})^* | \Lambda^k(\mathbb{R}^n)) \lambda^{n-k}$$
$$= \sum_{k=0}^n (-1)^k \operatorname{tr}_k(B) \lambda^{n-k}$$

In particular,  $tr_0(B) = 1$  and  $tr_1(B) = tr B$ .

In order to apply the theorem above to our three-dimensional examples, observe that

$$\operatorname{tr}(\tau) = 1, \operatorname{tr}(\tau^2) = -1, \operatorname{tr}(\tau^3) = 1$$
  
 $\operatorname{tr}(\chi_1) = \operatorname{tr}(\chi_2) = \operatorname{tr}(\chi_3) = -1.$ 

To show that  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are not isospectral on 1-forms, we choose  $\mu$  such that  $\sqrt{\mu} \in \mathbb{N}$ . Using the results for  $e_{\mu,B}(G_i)$  from the second case above, we obtain

$$\begin{split} d_{1,\mu}(G_1) &= \frac{1}{4}(3e_{\mu,I_3} + 2 - 2 + 2) = \frac{3}{4}e_{\mu,I_3} + \frac{1}{2} \\ d_{1,\mu}(G_2) &= \frac{1}{4}(3e_{\mu,I_3} - 2 - 2 - 2) = \frac{3}{4}e_{\mu,I_3} - \frac{3}{2} = d_{1,\mu}(G_1) - 2. \end{split}$$

Hence,  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are not 1-isospectral. Since both our point groups  $F_i$  lie in SO(3) and, as one easily shows,  $\operatorname{tr}_k(B) = \det(B) \operatorname{tr}_{n-k}(B)$ , we have  $d_{k,\mu}(G_i) = d_{n-k,\mu}(G_i)$ ; i.e.,  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are 0- and 3-isospectral, but neither 1- nor 2-isospectral.

#### 5.3 The Heat Kernel

Another way to verify the isospectrality of compact good Riemannian orbifolds is the use of the so-called heat kernel.

**Definition 5.3.** Let  $\mathcal{O}$  be an oriented Riemannian orbifold. A smooth function  $K:(0,\infty)\times\mathcal{O}\times\mathcal{O}\to(0,\infty)$  is called a *heat kernel* (or a fundamental solution of the heat equation) if it has the following properties:

- (K1)  $(\frac{\partial}{\partial t} + \Delta_x)K(t, x, y) = 0$ , where  $\Delta_x$  is the Laplacian acting on the second variable.
- (K2)  $\lim_{t\to 0^+} \int_{\mathcal{O}} K(t,x,y) f(y) dy = f(x)$  for every smooth function f with compact support on  $\mathcal{O}$ .

Remark. For more on the heat kernel on manifolds see [Cha84].

**Theorem 5.4** ([Don79]). Let M be a connected Riemannian manifold and let G be a subgroup of the isometry group  $\operatorname{Isom}(M)$  acting properly discontinuously on M such that M/G is compact. Then there is a unique heat kernel K on the Riemannian manifold M and a unique heat kernel  $\bar{K}$  on the Riemannian orbifold M/G. Moreover, for t > 0 and  $x, y \in \mathcal{O}$  one has

$$\bar{K}(t, x, y) = \sum_{g \in G} K(t, \tilde{x}, g\tilde{y}),$$

where  $\tilde{x}$ ,  $\tilde{y}$  are preimages of x, y under the projection  $M \to M/G$ . The convergence of the sum ist uniform on each subset of the form  $[t_1, t_2] \times M \times M$ .

Remark. Note that the sum on the right hand side is well-defined because for all  $h, k \in G$  we have  $K(t, h\tilde{x}, gk\tilde{y}) = K(t, \tilde{x}, h^{-1}gk\tilde{y})$  and  $g \mapsto h^{-1}gk$  is a bijection of G.

The relation between the eigenfunctions of the Laplacian and the heat kernel  $\bar{K}$  on a compact good orbifold M/G is now contained in the following theorem (cf. Theorem 3.4).

**Theorem 5.5** ([Don79]). Let M/G be a compact good Riemannian orbifold with heat kernel K and  $\operatorname{spec}(\mathcal{O}) = (0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots)$ , and let  $\{\phi_i\}_{i \geq 0} \subset C^{\infty}(\mathcal{O})$  be an orthonormal basis of  $L^2(\mathcal{O})$  such that  $\Delta \phi_i = \lambda_i \phi_i$ . Then

$$K(t, x, y) = \sum_{i>0} e^{-\lambda_i t} \phi_i(x) \phi_i(y),$$

$$\sum_{i \ge 0} e^{-t\lambda_i} = \int_{M/G} K(t, x, x) \, dx.$$

We set  $\operatorname{tr} K_t = \int_{M/G} K(t, x, x) dx$ , the trace of the heat kernel on M/G. The following corollary allows us to verify isospectrality by calculating the respective heat kernel traces.

Corollary 5.6. Two compact good orbifolds are isospectral if and only if their heat traces coincide.

*Proof.* Let M/G be a compact good orbifold. By Theorem 5.5, the spectrum  $\lambda_i$  determines the heat trace  $\operatorname{tr} K_t$  on M/G. Conversely,  $\operatorname{tr} K_t$  determines the spectrum: We know  $\lambda_0 = 0$ . Inductively, assume that  $\lambda_0, \ldots, \lambda_k$  have been found. Then  $\lambda_{k+1}$  is the largest value  $\lambda$  such that

$$\lim_{t \to \infty} \frac{\operatorname{tr} K_t - \sum_{i=0}^k e^{-\lambda_i t}}{e^{-\lambda t}}$$

is finite.  $\Box$ 

We are now going to apply the preceding statements in order to give a third proof of isospectrality for  $\mathcal{O}_1$ ,  $\mathcal{O}_2$ . It is well-known that the function

$$H(t, x, y) = (4\pi t)^{-3/2} e^{-|x-y|^2/4t}$$

is the heat kernel on  $\mathbb{R}^3$ . Let K denote the heat kernel on the torus  $T_{\Lambda} = \mathbb{R}^3/\Lambda$ . Theorem 5.4 gives

$$K(t, x, y) = (4\pi t)^{-\frac{3}{2}} \sum_{\lambda \in \Lambda} e^{-|x - (y + \lambda)|^2/4t}$$

(where we use the same letter for a point in  $\mathbb{R}^3$  and its image in  $T_{\Lambda}$ ). Let  $K^i$  denote the heat kernel on the orbifold  $\mathcal{O}_i = \mathbb{R}^3/G_i$ . Then, again by Theorem 5.4, if  $\tilde{x}$  denotes the preimage of  $x \in \mathcal{O}_i$  in  $T_{\Lambda}$ , and  $\overline{\tau}^i$ ,  $\overline{\rho}_i$  denote the isometries of  $T_{\Lambda}$  induced by  $\tau^i$ ,  $\rho_i$ :

$$\operatorname{tr}(K_t^1) = \int_{\mathcal{O}_1} K^1(t, x, x) \, dx = \int_{\mathcal{O}_1} \sum_{i=0}^3 K(t, \tilde{x}, \overline{\tau}^i \tilde{x}) \, dx = \sum_{i=0}^3 \int_{\mathcal{O}_1} K(t, \tilde{x}, \overline{\tau}^i \tilde{x}) \, dx$$

$$\operatorname{tr}(K_t^2) = \int_{\mathcal{O}_2} K^2(t, x, x) \, dx = \int_{\mathcal{O}_2} \sum_{i=0}^3 K(t, \tilde{x}, \overline{\rho}_i \tilde{x}) \, dx = \sum_{i=0}^3 \int_{\mathcal{O}_2} K(t, \tilde{x}, \overline{\rho}_i \tilde{x}) \, dx$$

The last expression in each line makes sense here because  $\mathcal{O}_1$ ,  $\mathcal{O}_2$  happen to be quotients of  $T_{\Lambda}$  by abelian groups (of order four). Hence, not only  $x \mapsto \sum_{i=0}^{3} K(t, \tilde{x}, \overline{\tau}^{i} \tilde{x})$  and  $x \mapsto \sum_{i=0}^{3} K(t, \tilde{x}, \overline{\rho}_{i} \tilde{x})$  are well-defined functions on  $\mathcal{O}_1$ , resp.  $\mathcal{O}_2$ , but already the individual terms in the sum are well-defined.

By our remark under Definition 2.25 concerning integration on very good orbifolds, the equations above imply

$$\operatorname{tr}(K_t^1) = \frac{1}{4} \sum_{i=0}^3 \int_{T_\Lambda} K(t, x, \overline{\tau}^i x) dx$$
 (5.3)

$$\operatorname{tr}(K_t^2) = \frac{1}{4} \sum_{i=0}^3 \int_{T_\Lambda} K(t, x, \overline{\rho_i} x) dx$$
 (5.4)

In order to compute the latter integrals, we apply Theorem 5.4 to the covering  $\mathbb{R}^3 \to \mathbb{R}^3/\Lambda = T_{\Lambda}$ . Since there won't be any more integrals over  $\mathcal{O}_i$ , we return to the use of the letter x for points in  $T_{\Lambda}$  or in  $\mathbb{R}^3$ .

#### The heat kernel trace on $\mathcal{O}_1$

First note that

$$\int_{T_{\Lambda}} K(t, x, x) dx = \int_{[-1,1]^{2} \times [0,1]} \sum_{\lambda \in \Lambda} H(t, x, x + \lambda) dx$$

$$= (4\pi t)^{-3/2} \int_{[-1,1]^{2} \times [0,1]} \sum_{\lambda \in \Lambda} e^{-|\lambda|^{2}/4t} dx$$

$$= (4\pi t)^{-3/2} \cdot 4 \sum_{\lambda \in \Lambda} e^{-|\lambda|^{2}/4t}.$$

For  $\lambda = (2m_1, 2m_2, m_3) \in \Lambda = 2\mathbb{Z} \times 2\mathbb{Z} \times \mathbb{Z}$  we calculate:

$$|x - (\tau x + \lambda)|^2 = (x_1 + x_2 - 2m_1)^2 + (x_2 - x_1 - 2m_2)^2 + m_3^2$$

$$|x - (\tau^2 x + \lambda)|^2 = (2x_1 - 2m_1)^2 + (2x_2 - 2m_2)^2 + m_3^2$$

$$|x - (\tau^3 x + \lambda)|^2 = (x_1 - x_2 - 2m_1)^2 + (x_2 + x_1 - 2m_2)^2 + m_3^2$$

Thus

$$\begin{split} \int_{T_{\Lambda}} &K(t, x, \overline{\tau}x) \, dx = \int_{[-1, 1]^2 \times [0, 1]} \sum_{\lambda \in \Lambda} H(t, x, \tau x + \lambda) \, dx \\ &= (4\pi t)^{-3/2} \sum_{m_1, m_2, m_3 \in \mathbb{Z}} \int_{[-1, 1]^2 \times [0, 1]} e^{\left[(-(x_1 + x_2 - 2m_1)^2 - (x_2 - x_1 - 2m_2)^2 - m_3^2)/4t\right]} \, dx \\ &= (4\pi t)^{-3/2} A \sum_{m_3 \in \mathbb{Z}} e^{-m_3^2/4t} \end{split}$$

with  $A := \sum_{m_1, m_2 \in \mathbb{Z}} \int_{-1}^{1} \int_{-1}^{1} \exp\left[\left(-(x_1 + x_2 - 2m_1)^2 - (x_2 - x_1 - 2m_2)^2\right)/4t\right] dx_1 dx_2$ . Note that interchanging summation and integration was allowed by uniform convergence (see Theorem 5.4).

To obtain A, we observe, using uniform convergence and rearranging sums over posi-

tive terms:

$$\sum_{m_1 \in \mathbb{Z}} \sum_{\substack{m_2 \in \mathbb{Z} \\ 2|m_1 - m_2}} \int_{-1}^{1} \int_{-1}^{1} \exp\left[\left(-(x_1 + x_2 - 2m_1)^2 - (x_2 - x_1 - 2m_2)^2\right)/4t\right] dx_1 dx_2$$

$$= \sum_{m_1 \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \int_{-1}^{1} \int_{-1}^{1} \exp\left[\left(-(x_1 + (x_2 - 2m_1))^2 - ((x_2 - 2m_1) - x_1 - 4l)^2\right)/4t\right] dx_2 dx_1$$

$$= \sum_{l \in \mathbb{Z}} \int_{-1}^{1} \int_{-\infty}^{\infty} \exp\left[\left(-(x_1 + x_2)^2 - (x_2 - x_1 - 4l)^2\right)/4t\right] dx_2 dx_1$$

$$= \sum_{l \in \mathbb{Z}} \int_{-1}^{1} \int_{-\infty}^{\infty} \exp\left[\left(-((x_1 + 2l) + (x_2 - 2l))^2 - ((x_2 - 2l) - (x_1 + 2l))^2\right)/4t\right] dx_2 dx_1$$

$$= \sum_{l \in \mathbb{Z}} \int_{-1}^{1} \int_{-\infty}^{\infty} \exp\left[\left(-((x_1 + 2l) + x_2)^2 - (x_2 - (x_1 + 2l))^2\right)/4t\right] dx_2 dx_1$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[\left(-(x_1 + x_2)^2 - (x_2 - x_1)^2\right)/4t\right] dx_2 dx_1$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[\left(-(x_1^2 - x_2^2)/2t\right] dx_2 dx_1$$

$$= \int_{-\infty}^{\infty} e^{-x_1^2/2t} dx_1 \int_{-\infty}^{\infty} e^{-x_2^2/2t} dx_2$$

$$= \sqrt{2\pi t} \cdot \sqrt{2\pi t} = 2\pi t$$

Analogously, the sum over  $m_1 \in \mathbb{Z}$ ,  $m_2 \in \mathbb{Z}$  with  $2|m_1 - m_2 + 1$  also equals  $2\pi t$ , hence  $A = 4\pi t$  and

$$\int_{T_{\Lambda}} K(t, x, \bar{\tau}x) \, dx = (4\pi t)^{-3/2} 4\pi t \sum_{m_3 \in \mathbb{Z}} e^{-m_3^2/4t}.$$

Comparing the terms for  $|x - (\tau^3 x + \lambda)|^2$  and  $|x - (\tau x + \lambda)|^2$  and interchanging the role of  $m_1$  and  $m_2$  in the calculation above, we observe that

$$\int_{T_{\Lambda}} K(t, x, \bar{\tau}^3 x) \, dx = \int_{T_{\Lambda}} K(t, x, \bar{\tau} x) \, dx = (4\pi t)^{-3/2} 4\pi t \sum_{m \in \mathbb{Z}} e^{-m^2/4t}.$$

Moreover,

$$\int_{T_{\Lambda}} K(t, x, \overline{\tau}^{2}x) dx = \int_{[-1,1]^{2} \times [0,1]} \sum_{\lambda \in \Lambda} H(t, x, \tau^{2}x + \lambda) dx$$

$$= (4\pi t)^{-3/2} \sum_{m_{1}, m_{2}, m_{3} \in \mathbb{Z}} \int_{[-1,1]^{2} \times [0,1]} \exp[(-(2x_{1} - 2m_{1})^{2} - (2x_{2} - 2m_{2})^{2} - m_{3}^{2})/4t] dx$$

$$= (4\pi t)^{-3/2} \sum_{m_{1} \in \mathbb{Z}} \int_{-1}^{1} e^{-(x_{1} - m_{1})^{2}/t} dx_{1} \sum_{m_{2} \in \mathbb{Z}} \int_{-1}^{1} e^{-(x_{2} - m_{2})^{2}/t} dx_{2} \sum_{m_{3} \in \mathbb{Z}} e^{-m_{3}^{2}/4t}$$

$$= (4\pi t)^{-3/2} 4\pi t \sum_{m \in \mathbb{Z}} e^{-m^{2}/4t}.$$

Plugging the four terms for  $\int_{T_{\Lambda}} K(t, x, \overline{\tau}^{i}x) dx$ , i = 0, 1, 2, 3, into (5.3), we finally obtain

$$\operatorname{tr}(K_t^1) = (4\pi t)^{-3/2} \left( \sum_{\lambda \in \Lambda} e^{-|\lambda|^2/4t} + 3\pi t \sum_{m \in \mathbb{Z}} e^{-m^2/4t} \right).$$

#### The heat kernel trace on $\mathcal{O}_2$

Since all the  $\chi_i$  are diagonal, the analogous calculations for  $\mathcal{O}_2$  are a little easier: For  $\lambda = (2m_1, 2m_2, m_3) \in \Lambda = 2\mathbb{Z} \times 2\mathbb{Z} \times \mathbb{Z}$  we have

$$|x - (\rho_1 x + \lambda)|^2 = (2m_1 + 1)^2 + (2x_2 - 2m_2)^2 + (2x_3 - m_3)^2$$
  

$$|x - (\rho_2 x + \lambda)|^2 = (2x_1 - 2m_1)^2 + (2m_2)^2 + (2x_3 - m_3)^2$$
  

$$|x - (\rho_3 x + \lambda)|^2 = (2x_1 - 2m_1 - 1)^2 + (2x_2 - 2m_2)^2 + m_3^2.$$

Thus

$$\int_{T_{\Lambda}} K(t, x, \overline{\rho}_{1}x) dx = \int_{[-1,1]^{2} \times [0,1]} \sum_{\lambda \in \Lambda} H(t, x, \rho_{1}x + \lambda) dx$$

$$= (4\pi t)^{-3/2} \sum_{m_{1}, m_{2}, m_{3} \in \mathbb{Z}} \int_{[-1,1]^{2} \times [0,1]} e^{\left[(-(2m_{1}+1)^{2} - (2x_{2}-2m_{2})^{2} - (2x_{3}-m_{3})^{2})/4t\right]} dx$$

$$= (4\pi t)^{-3/2} \cdot 2 \sum_{m_{1} \in \mathbb{Z}} e^{-(2m_{1}+1)^{2}/4t} \sum_{m_{2} \in \mathbb{Z}} \int_{-1}^{1} e^{-(x_{2}-m_{2})^{2}/t} dx_{2} \sum_{m_{3} \in \mathbb{Z}} \int_{0}^{1} e^{-(x_{3}-\frac{m_{3}}{2})^{2}/t} dx_{3}$$

$$= (4\pi t)^{-3/2} 8\pi t \sum_{m \in \mathbb{Z}} e^{-(2m+1)^{2}/4t},$$

$$\begin{split} \int_{T_{\Lambda}} K(t,x,\overline{\rho}_{2}x) \, dx &= \int_{[-1,1]^{2} \times [0,1]} \sum_{\lambda \in \Lambda} H(t,x,\rho_{2}x+\lambda) \, dx \\ &= (4\pi t)^{-3/2} \sum_{m_{1},m_{2},m_{3} \in \mathbb{Z}} \int_{[-1,1]^{2} \times [0,1]} e^{\left[(-(2x_{1}-2m_{1})^{2}-(2m_{2})^{2}-(2x_{3}-m_{3})^{2})/4t\right]} \, dx \\ &= (4\pi t)^{-3/2} \sum_{m_{1} \in \mathbb{Z}} \int_{-1}^{1} e^{-(x_{1}-m_{1})^{2}/t} \, dx_{1} \cdot 2 \sum_{m_{2} \in \mathbb{Z}} e^{-(2m_{2})^{2}/4t} \sum_{m_{3} \in \mathbb{Z}} \int_{0}^{1} e^{-(x_{3}-\frac{m_{3}}{2})^{2}/t} \, dx_{3} \\ &= (4\pi t)^{-3/2} 8\pi t \sum_{m \in \mathbb{Z}} e^{-(2m)^{2}/4t} \end{split}$$

and

$$\int_{T_{\Lambda}} K(t, x, \overline{\rho}_{3}x) dx = \int_{[-1,1]^{2} \times [0,1]} \sum_{\lambda \in \Lambda} H(t, x, \rho_{3}x + \lambda) dx$$

$$= (4\pi t)^{-3/2} \sum_{m_{1}, m_{2}, m_{3} \in \mathbb{Z}} \int_{[-1,1]^{2} \times [0,1]} e^{\left[(-(2x_{1} - 2m_{1} - 1)^{2} - (2x_{2} - 2m_{2})^{2} - m_{3}^{2})/4t\right]} dx$$

$$= (4\pi t)^{-3/2} \sum_{m_{1} \in \mathbb{Z}} \int_{-1}^{1} e^{-(x_{1} - m_{1} - 1/2)^{2}/t} dx_{1} \sum_{m_{2} \in \mathbb{Z}} \int_{-1}^{1} e^{-(x_{2} - m_{2})^{2}/t} \sum_{m_{3} \in \mathbb{Z}} e^{-m_{3}^{2}/4t}$$

$$= (4\pi t)^{-3/2} 4\pi t \sum_{m \in \mathbb{Z}} e^{-m^{2}/4t}.$$

Plugging these three terms and the one for  $\int_{T_{\Lambda}} K(t, x, x) dx$  above into (5.4), we obtain

$$\operatorname{tr}(K_t^2) = (4\pi t)^{-3/2} \left( \sum_{\lambda \in \Lambda} e^{-|\lambda|^2/4t} + 2\pi t \sum_{m \in \mathbb{Z}} e^{-(2m+1)^2/4t} + 2\pi t \sum_{m \in \mathbb{Z}} e^{-(2m)^2/4t} + \pi t \sum_{m \in \mathbb{Z}} e^{-m^2/4t} \right)$$
$$= (4\pi t)^{-3/2} \left( \sum_{\lambda \in \Lambda} e^{-|\lambda|^2/4t} + 3\pi t \sum_{m \in \mathbb{Z}} e^{-m^2/4t} \right).$$

Hence,  $\operatorname{tr}(K_t^1) = \operatorname{tr}(K_t^2)$  and the two orbifolds  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are 0-isospectral by Corollary 5.6.

### 6 More Isospectral Flat Orbifolds

Although the following examples are not interesting from the point of view of the existence of isospectral orbifolds with different isotropy orders, we give two more pairs of compact flat isospectral orbifolds of dimension three. The first example resembles our main example from the two preceding chapters. Although its construction is considerably simpler, its maximal isotropy groups are again not isomorphic. Our second pair is easily seen to be k-isospectral for all  $k \ge 0$ , and we will show that the two orbifolds are indeed "Sunada-isospectral" (in a sense to be made precise in Section 6.2). Note that in all our figures we again omit the identification of the top and bottom side by the respective vertical translation.

# 6.1 Two Orbifolds with Non-isomorphic Maximal Isotropy Groups

Let  $\Lambda = 2\mathbb{Z} \times 2\mathbb{Z} \times 2\mathbb{Z}$  and

$$\tau = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly as before,

$$G_1 := \{ \tau^i L_{\lambda}; \ i = 0, 1, 2, 3, \ \lambda \in \Lambda \}$$

is a crystallographic group. Set  $\mathcal{O}_1 := \mathbb{R}^3/G_1$ .

It is easily seen that  $[0,1] \times [0,1] \times [0,2]$  is a fundamental domain for the action of  $G_1$  on  $\mathbb{R}^3$ . The top and bottom side are identified by the canonical translation  $L_{2e_3}$ , and the rest of the boundary is identified as indicated in Figure 6.1.

The isotropy groups on  $\mathcal{O}_1$  are the same as those given in section 4.2.1; i.e., the singular stratum consists of two copies of  $S^1$  with isotropy  $\mathbb{Z}_4$  and one copy of  $S^1$  with isotropy  $\mathbb{Z}_2$  (where each of these copies of  $S^1$  now has length 2).

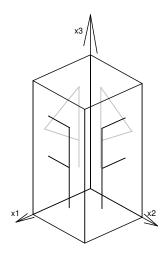


Figure 6.1: The underlying space of  $\mathcal{O}_1$  as a quotient of  $[0,1] \times [0,1] \times [0,2]$ 

Next set

$$\rho_0 = I_3$$

$$\rho_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\rho_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\rho_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and note that

$$G_2 := \{ \rho_i L_{\lambda}; \ i = 0, 1, 2, 3, \ \lambda \in \Lambda \}$$

also is a crystallographic group. Then set  $\mathcal{O}_2 := \mathbb{R}^3/G_2$ . A fundamental domain for the action of  $G_2$  on  $\mathbb{R}^3$  is given by  $[0,1] \times [0,1] \times [0,2]$ . To see this, just note that the following restrictions of transformations in  $G_2$  are bijections (and that there are no non-trivial elements in  $G_2$  identifying points within  $(0,1)\times(0,1)\times$ 

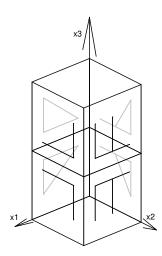


Figure 6.2: The underlying space of  $\mathcal{O}_2$  as a quotient of  $[0,1] \times [0,1] \times [0,2]$ 

(0,2):

$$\begin{split} \rho_1 L_{(0,-2,-2)}: & [0,1] \times [1,2] \times [0,2] \to [0,1] \times [0,1] \times [0,2] \\ \rho_2 L_{(-2,0,-2)}: & [1,2] \times [0,1] \times [0,2] \to [0,1] \times [0,1] \times [0,2] \\ \rho_3 L_{(-2,-2,0)}: & [1,2] \times [1,2] \times [0,2] \to [0,1] \times [0,1] \times [0,2] \end{split}$$

Regarding the identifications on the boundary, note that the top and bottom side are identified by the canonical translation  $L_{2e_3}$ . The rest of the boundary is identified as indicated in Figure 6.2, the corresponding transformations are given in the following table.

One easily verifies the following sets of fixed points of the given elements of  $G_2$ :

$$\begin{split} & \rho_1 L_{(0,\lambda_2,\lambda_3)}: \ \{(r,-\lambda_2/2,-\lambda_3/2); \ r \in \mathbb{R}\} \\ & \rho_2 L_{(\lambda_1,0,\lambda_3)}: \ \{(-\lambda_1/2,r,-\lambda_3/2); \ r \in \mathbb{R}\} \\ & \rho_3 L_{(\lambda_1,\lambda_2,0)}: \ \{(-\lambda_1/2,-\lambda_2/2,r); \ r \in \mathbb{R}\} \end{split}$$

Since  $\Lambda = 2\mathbb{Z} \times 2\mathbb{Z} \times 2\mathbb{Z}$ , the points with two coordinates in  $\mathbb{Z}$  and one in  $\mathbb{R} \setminus \mathbb{Z}$  have isotropy  $\mathbb{Z}_2$ , whereas those in  $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$  have isotropy isomorphic to  $\{I_3, \rho_1, \rho_2, \rho_3\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ , the Klein four-group. All other points have trivial isotropy. Taking the identifications within  $[0,1] \times [0,1] \times [0,2]$  into account, we observe that the singular stratum on  $\mathcal{O}_2$  consists of twelve open line segments with isotropy  $\mathbb{Z}_2$  and eight points with isotropy  $\mathbb{Z}_2 \times \mathbb{Z}_2$ .

Taking isospectrality (which we are going to verify below) for granted, we should point out that  $\mathcal{O}_1$ ,  $\mathcal{O}_2$  is a pair of isospectral orbifolds whose maximal isotropy groups have the same order but are not isomorphic ( $\mathbb{Z}_4 \not\simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ ). The first examples of such pairs have been given in [SSW06]. Note that, moreover, in our example the sets of points of maximal isotropy have different dimension as topological manifolds: dimension one in the case of  $\mathcal{O}_1$  but dimension zero in the case of  $\mathcal{O}_2$ .

Isospectrality is easily verified using Theorem 5.1: We have  $\Lambda^* = \frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z}$ . Let

$$F_1 = \{I_3, \tau, \tau^2, \tau^3\}, \quad F_2 = \{I_3, \rho_1, \rho_2, \rho_3\}$$

denote the point groups of  $G_1$  and  $G_2$ . Note that they are the same as for our example from Chapters 4 and 5. Therefore the next paragraph is merely a repetition of our reasoning in Section 5.2.

Since  $d_0(G_1) = 1 = d_0(G_2)$  (by Theorem 3.4 or Theorem 5.1), it suffices to consider  $d_{\mu}(G_i)$  for  $\mu > 0$ . The only  $v \in \mathbb{R}^3$  with  $||v||^2 = \mu$  which are fixed by any element of  $F_1 \setminus \{I_3\} = \{\tau, \tau^2, \tau^3\}$  are  $(0, 0, \pm \sqrt{\mu})$ . As for  $F_2$ , observe that for i = 1, 2, 3 the only  $v \in \mathbb{R}^3$  with  $||v||^2 = \mu$  and  $\rho_i v = v$  are given by  $v = \pm \sqrt{\mu} e_i$ , where  $e_i$  denotes the *i*-th standard unit vector. Next set

$$e_{\mu,I_3} := \#\{v \in \Lambda^*; \|v\|^2 = \mu\}.$$

We now distinguish the following two cases.

**1st case**  $\sqrt{\mu} \in (0, \infty) \setminus \frac{1}{2}\mathbb{N}$ : Since there are no vectors of length  $\sqrt{\mu}$  in  $\Lambda^*$  fixed by a non-trivial element of  $F_1$  or  $F_2$ , we have  $e_{\mu,B}(G_i) = 0$  for every i and every  $B \in F_i \setminus \{I_3\}$ . Thus

$$d_{\mu}(G_1) = \frac{1}{4}e_{\mu,I_3} = d_{\mu}(G_2).$$

**2nd case**  $\sqrt{\mu} \in \frac{1}{2}\mathbb{N}$ : The calculation of fixed points given above implies, for  $j \in \{1,2,3\}$ ,

$$e_{\mu,\tau^j}(G_1) = \exp(2\pi i \langle (0,0,\sqrt{\mu}), (0,0,0) \rangle + \exp(2\pi i \langle (0,0,-\sqrt{\mu}), (0,0,0) \rangle = 2.$$

Similarly,  $e_{\mu,\rho_j}(G_2) = 2$  for j = 1, 2, 3 and therefore

$$d_{\mu}(G_1) = \frac{1}{4}(e_{\mu,I_3} + 6) = d_{\mu}(G_2).$$

All in all, we deduce that  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are 0-isospectral. However, an application of Theorem 5.2 shows that they are not 1-isospectral.

Note that the formulas in Theorem 5.1 imply that we obtain the same spectrum if we

replace  $\mathcal{O}_1$  by  $\mathcal{O}_1' := \mathbb{R}^3/G_1'$  with  $G_1' := \{\tau^i L_\lambda; i = 0, 1, 2, 3, \lambda \in \Lambda\}$ , where

$$\tau' = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} L_{(b_1, b_2, 0)}$$

with  $b_i \in \mathbb{R}$ . However, all the  $\mathcal{O}'_1$  are isometric to  $\mathcal{O}_1$  by Lemma 2.27: For  $v = \frac{1}{2}(b_1 + b_2, b_2 - b_1, 0)$  one easily verifies that  $(L_v)^{-1}\tau^i L_v = L_{-v}\tau^i L_v = \tau'^i$ .

Similarly, one could (without changing the spectrum) replace the  $\rho_i$  by

$$\rho'_0 = I_3 
\rho'_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} L_{(0,b_2,b_3)}, 
\rho'_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} L_{(b_1,0,b_3)}, 
\rho'_3 = \rho'_1 \rho'_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} L_{(b_1,b_2,0)},$$

and set  $G'_2 := \{ \rho'_i L_\lambda; \lambda \in \Lambda \}$ . But again, observe that for  $w := \frac{1}{2}(b_1, b_2, b_3)$ , we have  $L_{-w}\rho_i L_w = \rho'_i$ , hence  $\mathcal{O}_2$  and  $\mathcal{O}'_2 := \mathbb{R}^3/G'_2$  are isometric.

#### 6.2 Two Sunada-isospectral Orbifolds

Consider the lattice  $\Lambda := \mathbb{Z} \times \mathbb{Z} \times \frac{1}{\sqrt{2}}\mathbb{Z}$ . Let

$$\tau := \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and

$$\rho := \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Both matrices are involutions in SO(3), hence they are rotations by the angle  $\pi$  around a line through the origin. Calculating the respective one-dimensional eigenspaces associated with the eigenvalue 1, we observe that  $\tau$  is a rotation by  $\pi$  around the 1-dimensional space spanned by the vector (1, -1, 0), whereas  $\rho$  is a rotation by  $\pi$  around the  $x_3$ -axis.

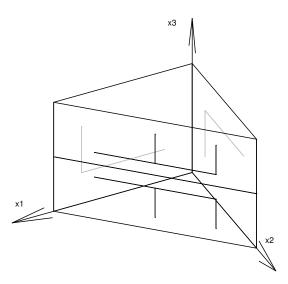


Figure 6.3: The underlying space of  $\mathcal{O}_1$ 

Since  $\Lambda$  is invariant under  $\tau$  and  $\rho$ , it easily follows that

$$G_1 := \{ \tau^j L_{\lambda}; \ j = 0, 1, \lambda \in \Lambda \}, \ G_2 := \{ \rho^j L_{\lambda}; \ j = 0, 1, \lambda \in \Lambda \}$$

are groups, which are crystallographic by a reasoning similar to the one in Chapter 4. In the following, we will show that the orbifolds  $\mathcal{O}_i := \mathbb{R}^3/G_i$  are k-isospectral for all  $k \ge 0$ .

It is not hard to see that a fundamental domain both of the actions of  $G_1$  and of  $G_2$  on  $\mathbb{R}^3$  is given by the prism of height  $\frac{1}{\sqrt{2}}$  over the triangle with vertices (0,0,0), (1,0,0), (0,1,0).

Following the same algorithm as in Chapter 4, we obtain the identifications for  $\mathcal{O}_1$  given in Figure 6.3. Instead of giving the whole calculation (which would of course automatically yield the singular points on  $\mathcal{O}_1$ ), we only show which points are fixed under non-trivial elements of  $G_1$ . Since  $L_{\lambda}$  with  $\lambda \in \Lambda \setminus \{0\}$  has no fixed points, it suffices to consider transformations of the form  $\tau L_{\lambda}$ . But

$$\tau L_{\lambda} x = (-x_2 - \lambda_2, -x_1 - \lambda_1, -x_3 - \lambda_3),$$

which (since  $\Lambda = \mathbb{Z} \times \mathbb{Z} \times \frac{1}{\sqrt{2}}$ ) implies that x is fixed by a nontrivial element of  $G_1$  if and only if  $x_1 + x_2 \in \mathbb{Z}$  and  $x_3 \in \frac{1}{2\sqrt{2}}\mathbb{Z}$ . Each such point x is fixed by  $\tau L_{(-x_1-x_2,-x_1-x_2,-2x_3)}$  but no other element of  $G_1 \setminus \{I_3\}$ . Taking the identifications within the given fundamental domain into account, we deduce that the singular stratum of  $\mathcal{O}_1$  consists of two copies of  $S^1$  represented by the horizontal line segments  $\{(t, 1-t, 0); t \in [0, 1]\}$  and  $\{(t, 1-t, 1/(2\sqrt{2})); t \in [0, 1]\}$  of length  $\sqrt{2}$  as depicted in Figure 6.3, and all singular points have isotropy  $\mathbb{Z}_2$ .

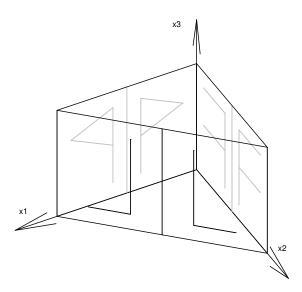


Figure 6.4: The underlying space of  $\mathcal{O}_2$ 

The orbifold  $\mathcal{O}_2$  is the product of a so-called 4-pillow (here: quadratic with side length 1/2) with  $\mathbb{R}/(\frac{1}{\sqrt{2}}\mathbb{Z})$ . Its underlying space is given in Figure 6.4. Since  $\rho L_{\lambda} x = (-x_1 - \lambda_1, -x_2 - \lambda_2, x_3 + \lambda_3)$ , the fixed points under nontrivial elements of  $G_2$  are precisely the points  $x \in \mathbb{R}^3$  satisfying  $x_1, x_2 \in \frac{1}{2}\mathbb{Z}$ . Each such point x is fixed by  $\tau L_{(-2x_1, -2x_2, 0)}$  but no other element of  $G_2 \setminus \{I_3\}$ . Taking the identifications within the given fundamental domain (whose rigorous justification we again omit) into account, we observe that the singular stratum of  $\mathcal{O}_2$  consists of four copies of  $S^1$  represented by the vertical line segments of length  $1/\sqrt{2}$  over the points (0,0,0), (0,1/2,0), (1/2,0,0), (1/2,1/2,0), and all singular points have isotropy  $\mathbb{Z}_2$ .

The structure of the singular strata in connection with Lemma 2.21 implies that  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are not diffeomorphic.

For the calculation of the spectrum note that  $\tau$  and  $\rho$  do not involve translations, hence

$$e_{\mu,\chi_i}(G_i) = \#\{v \in \Lambda^*; \ \chi_i v = v, \|v\|^2 = \mu\}$$

for  $\chi_1 = \tau$  and  $\chi_2 = \rho$  in the notation of Theorem 5.1.

 $\mathcal{O}_1$ : If  $\mu > 0$ , the only vectors in  $\mathbb{R}^3$  of length  $\sqrt{\mu}$  which are fixed by  $\tau$  are  $\left(\sqrt{\mu/2}, -\sqrt{\mu/2}, 0\right)$  and  $\left(-\sqrt{\mu/2}, \sqrt{\mu/2}, 0\right)$ . We have  $\Lambda^* = \mathbb{Z} \times \mathbb{Z} \times \sqrt{2}\mathbb{Z}$  and therefore

$$e_{\mu,\tau}(G_1) = \begin{cases} 1, & \mu = 0\\ 2, & \sqrt{\mu} \in \sqrt{2}\mathbb{N}\\ 0, & \text{otherwise} \end{cases}$$

 $\mathcal{O}_2$ : For  $\mu > 0$  the only vectors of length  $\sqrt{\mu}$  fixed by  $\rho$  are  $(0, 0, \pm \sqrt{\mu})$  and therefore we obtain the same values:  $e_{\mu,\rho}(G_2) = e_{\mu,\tau}(G_1) \ \forall \mu \geqslant 0$ . Since, moreover, the characteristic polynomials of  $\tau$  and  $\rho$  coincide, we have  $\operatorname{tr}_k \tau = \operatorname{tr}_k \rho$  for all k, hence the orbifolds  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are k-isospectral for all  $k \geqslant 0$  by Theorem 5.2.

#### **Sunada-Isospectrality**

In a certain sense (cf. [Pes97]), most pairs of isospectral quotients of a fixed compact Riemannian manifold by finite groups can be constructed using a general technique by Sunada ([Sun85]), which has first been generalized to orbifolds in [Bér92]. A common feature of isospectral manifolds or orbifolds arising from Sunada's method is that they are k-isospectral for all k. Thus one is led to suspect that our given pair  $\mathcal{O}_1$ ,  $\mathcal{O}_2$  might also arise from that technique. In fact, this will turn out to be the case.

First, we are going to summarize the generalization of Sunada's result to good orbifolds. Let  $\Gamma$  be a finite group and for each  $g \in \Gamma$  let  $[g]_{\Gamma}$  denote the conjugacy class of g in  $\Gamma$ . Two subgroups  $\Gamma_1, \Gamma_2$  of  $\Gamma$  are called almost conjugate if

$$\#([g]_{\Gamma} \cap \Gamma_1) = \#([g]_{\Gamma} \cap \Gamma_2) \ \forall g \in \Gamma.$$

This condition is easily seen to be equivalent to the existence of a bijection  $\Phi: \Gamma_1 \to \Gamma_2$  such that every  $\gamma \in \Gamma_1$  is conjugate in  $\Gamma$  to  $\Phi(\gamma) \in \Gamma_2$ .

**Theorem 6.1.** Let M be a compact Riemannian manifold and let  $\Gamma_1$  and  $\Gamma_2$  be almost conjugate in a finite subgroup  $\Gamma$  of  $\mathrm{Isom}(M)$ . Then  $\mathrm{spec}_k(M/\Gamma_1) = \mathrm{spec}_k(M/\Gamma_2) \ \forall k \geqslant 0$ .

*Proof.* Note that the eigenforms on  $M/\Gamma_i$  are given by the  $\Gamma_i$ -invariant eigenforms on M. Then follow the same steps as in the manifold case (cf. [Gor00] or the original proof in [Sun85]).

Remark. Of course, conjugate subgroups  $\Gamma_1$ ,  $\Gamma_2$  of  $\Gamma$  as in the theorem above are almost conjugate. However, the resulting orbifold quotients  $M/\Gamma_1$ ,  $M/\Gamma_2$  would be isometric by Lemma 2.27 and hence irrelevant for the construction of non-isometric isospectral orbifods. Only the fact that there are almost conjugate pairs which are not conjugate (see [Bro88] for some examples) makes the theorem above fruitful.

**Definition 6.2.** Let  $\mathcal{O}_1$ ,  $\mathcal{O}_2$  be compact Riemannian orbifolds. If there is a tuple  $(M, \Gamma, \Gamma_1, \Gamma_2)$  satisfying the conditions in the theorem above such that there are isometries  $\mathcal{O}_1 \simeq M/\Gamma_1$  and  $\mathcal{O}_2 \simeq M/\Gamma_2$ , then  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are called *Sunada-isospectral*.

Concerning our example, note that for  $\Lambda = \mathbb{Z} \times \mathbb{Z} \times \frac{1}{\sqrt{2}}\mathbb{Z}$  there is no finite subgroup  $\Gamma$  of Isom $(T_{\Lambda})$  such that the tuple  $(T_{\Lambda}, \Gamma, \langle \overline{\tau} \rangle, \langle \overline{\rho} \rangle)$  satisfies the conditions of the definition above: If the two-element subgroups  $\langle \overline{\tau} \rangle, \langle \overline{\rho} \rangle$  of Isom $(T_{\Lambda})$  were almost conjugate in  $\Gamma \subset \text{Isom}(T_{\Lambda})$ , then they would automatically be conjugate in Isom $(T_{\Lambda})$ , and the orbifolds  $\mathcal{O}_1$ ,  $\mathcal{O}_2$  would be isometric by Lemma 2.27. However, we are now going to show that there

is a certain torus  $T_{\Lambda'}$  covering  $\mathcal{O}_1$  and  $\mathcal{O}_2$  and a corresponding Sunada triple  $(\Gamma, \Gamma_1, \Gamma_2)$ , which will imply that  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are Sunada-isospectral.

First, we will determine the isometry group of a general n-dimensional torus. Let  $\Lambda$  be an arbitrary lattice in  $\mathbb{R}^n$ , denote the corresponding torus by  $T_{\Lambda}$  and write  $p: \mathbb{R}^n \to \mathbb{R}^n/\Lambda = T_{\Lambda}$  for the quotient map. If  $\psi \in C^{\infty}(\mathbb{R}^n, \mathbb{R}^n)$  satisfies

$$p \circ \psi \circ L_{\lambda} = p \circ \psi \ \forall \lambda \in \Lambda$$

we denote the induced map in  $C^{\infty}(T_{\Lambda}, T_{\Lambda})$  by  $\overline{\psi}$ . Note that, given two such maps  $\psi_1$ ,  $\psi_2$ , we have  $\overline{\psi_1 \circ \psi_2} = \overline{\psi_1} \circ \overline{\psi_2}$ .

**Lemma 6.3.** Let  $\Lambda = \operatorname{span}_{\mathbb{Z}}\{v_1, \ldots, v_n\}$  be a lattice in  $\mathbb{R}^n$ , and set

$$O(\Lambda) := \{ B \in O(n); B\Lambda \subset \Lambda \}$$

and

$$P(v_1, \dots, v_n) := \left\{ \sum_{i=1}^n t_i v_i; \ t_i \in [0, 1) \right\}.$$

Then  $O(\Lambda)$  is a finite group, and the isometry group on the torus is given by

$$\operatorname{Isom}(T_{\Lambda}) = \{ \overline{BL_b}; \ B \in O(\Lambda), b \in P(v_1, \dots, v_n) \}.$$

*Proof.* The only non-trivial part of the proof that  $O(\Lambda)$  is a subgroup of O(n) is closeness under inversion. For an arbitrary lattice  $\Gamma$  in  $\mathbb{R}^n$  let  $s_{\Gamma}$  denote the function

$$\mathbb{R}_+ \ni r \mapsto \#\{\gamma \in \Gamma; \ \|\gamma\| = r\} \in \mathbb{N}_0.$$

Note that each set on the right hand side is finite, since it is a discrete subset of the compact sphere of radius r. Now let  $B \in O(\Lambda)$ . By assumption,  $B\Lambda \subset \Lambda$ . Moreover,  $B \in O(n)$  implies  $s_{B\Lambda} = s_{\Lambda}$ . Thus  $B\Lambda = \Lambda$ , in particular  $B^{-1}\Lambda \subset \Lambda$ .

 $O(\Lambda)$  is finite by the following direct argument: For every  $i \in \{1, ..., n\}$  the set  $\{\gamma \in \Gamma; \|\gamma\| = \|v_i\|\}$  contains only a finite number of vectors. Since every element of  $O(\Lambda)$  maps each  $v_i$  to a vector of length  $\|v_i\|$  and is uniquely determined by its values on the basis vectors  $v_1, ..., v_n$ , the group  $O(\Lambda)$  is finite.

Regarding the statement on the isometry group, first note that  $BL_b$  as above indeed induces a smooth map  $T_{\Lambda} \to T_{\Lambda}$ : If  $\lambda \in \Lambda$ , then

$$p \circ BL_b \circ L_\lambda = p \circ BL_\lambda L_b = p \circ L_{B\lambda} BL_b = p \circ BL_b.$$

If we assume that  $B_1, B_2 \in O(\Lambda)$ ,  $b_1, b_2 \in P(v_1, \dots, v_n)$  and  $\overline{B_1 L_{b_1}} = \overline{B_2 L_{b_2}}$ , then

$$\operatorname{Id}_{T_{\Lambda}} = \overline{B_1 L_{b_1}} (\overline{B_2 L_{b_2}})^{-1} = \overline{B_1 L_{b_1 - b_2} B_2^{-1}} = \overline{B_1 B_2^{-1} L_{B_2(b_1 - b_2)}}$$

and therefore there is  $\lambda \in \Lambda$  such that  $B_1B_2^{-1}L_{B_2(b_1-b_2)}=L_{\lambda}$ , which implies  $B_1=B_2$  and  $B_2(b_1-b_2)\in \Lambda$ . Thus  $b_1-b_2\in \Lambda$ , which implies  $b_1=b_2$  by our choice of the  $b_i$ . In other words, the  $\overline{BL_b}$  in the lemma are pairwise distinct and form a set.

 $\supset$ : Since  $BL_b$  is an isometry,  $\overline{BL_b}$  is a local isometry. Its inverse is given by  $\overline{(BL_b)^{-1}}$ , i.e., it is an isometry on the torus.

 $\subset$ : Let  $f: T_{\Lambda} \to T_{\Lambda}$  be an isometry. Let  $\tilde{f}: \mathbb{R}^n \to \mathbb{R}^n$  be a lift of  $f \circ p$  via p. Then  $\tilde{f}$  is an isometry on  $\mathbb{R}^n$ , hence there is a unique  $B \in O(n)$  and  $a \in \mathbb{R}^n$  such that  $\tilde{f} = BL_a$ . If we denote the unique element of  $P(v_1, \ldots, v_n) \cap (a + \Lambda)$  by b, then  $f = \overline{BL_b}$ .

Now we return to our example and set

$$\Lambda' := \operatorname{span}_{\mathbb{Z}} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \sqrt{2} \end{pmatrix} \right\} \subset \Lambda.$$

Since the three vectors lie in  $\Lambda$  and are linearly independent,  $\Lambda'$  is a sublattice of  $\Lambda$ . Moreover, one easily verifies that it is invariant under  $\tau$  and  $\rho$ . To determine representatives of the abelian group  $\Lambda/\Lambda'$  note that

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}^{-1} = \begin{pmatrix} 1/2 & 1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1/\sqrt{2} \end{pmatrix}.$$

Hence, if  $(k_1, k_2, k_3/\sqrt{2})$  with  $k_i \in \mathbb{Z}$  is an arbitrary element of  $\Lambda$ , then its coordinates with respect to the basis of  $\Lambda'$  given above are

$$\begin{pmatrix} 1/2 & 1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3/\sqrt{2} \end{pmatrix} = \begin{pmatrix} (k_1 + k_2)/2 \\ (k_2 - k_1)/2 \\ k_3/2 \end{pmatrix}.$$

By requiring that each of these coordinates lie in [0,1), we obtain the following full set of representatives of  $\Lambda/\Lambda'$ :

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \right\}.$$

Now set

$$A := \begin{pmatrix} -1/2 & -1/2 & 1/\sqrt{2} \\ 1/2 & 1/2 & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}.$$

It is easily verified that A lies in O(3) and leaves  $\Lambda'$  invariant, hence  $A \in O(\Lambda')$ . Moreover, since the first two rows of A span the eigenspace of  $\tau$  corresponding to the eigenvalue -1 and the last row is a fixed point of  $\tau$ , we have the crucial relation

$$A\tau A^{-1} = A\tau A^t = \rho.$$

We consider the following subgroups of Isom $(T_{\Lambda'})$ :

$$\Gamma_1 := \{ \overline{\tau^i L_\lambda}; \ i \in \{0, 1\}, \lambda \in \Lambda/\Lambda' \},$$
  
$$\Gamma_2 := \{ \overline{\rho^i L_\lambda}; \ i \in \{0, 1\}, \lambda \in \Lambda/\Lambda' \}$$

Then we have the following isometries:

$$\mathcal{O}_1 \simeq T_{\Lambda'}/\Gamma_1$$
  
 $\mathcal{O}_2 \simeq T_{\Lambda'}/\Gamma_2$ .

As for the first isometry, we note that the covering  $T_{\Lambda'} \to T_{\Lambda}$  induces a map  $f: T_{\Lambda'}/\Gamma_1 \to T_{\Lambda}/\langle \bar{\tau} \rangle$  which is bijective and seen to be a homeomorphism by elementary topology. The isometries lifting f as required in Definition 2.20 are given by suitable restrictions of the Riemannian covering  $T_{\Lambda'} \to T_{\Lambda}$ . Hence  $T_{\Lambda'}/\Gamma_1$  is isometric to  $T_{\Lambda}/\langle \bar{\tau} \rangle$  which itself is isometric to  $\mathcal{O}_1 = \mathbb{R}^3/G_1$  by a similar argument. The proof for  $\mathcal{O}_2$  is analogous.

Let H denote the subgroup of  $O(\Lambda')$  generated by  $\tau, \rho$  and A. By Lemma 6.3, H is finite. Then set p

$$\Gamma := \{ \overline{BL_b}; B \in H, b \in (\Lambda'/4)/\Lambda' \} \subset \operatorname{Isom}(T_{\Lambda'}).$$

Note that  $\Gamma$  is a group, because  $\Lambda'/4$  is invariant under H. Since H and  $(\Lambda'/4)/\Lambda'$  are finite, so is  $\Gamma$ . Moreover,  $\Gamma_1$  and  $\Gamma_2$  are subgroups of  $\Gamma$ . To see that they are almost conjugate in  $\Gamma$ , we define a bijection  $\Phi$ :  $\Gamma_1 \to \Gamma_2$  by  $\overline{L_{\lambda}} \mapsto \overline{L_{\lambda}}$  for  $\lambda \in \Lambda/\Lambda'$  and

$$\begin{split} \frac{\overline{\tau} \mapsto \overline{\rho}}{\overline{\tau L_{(0,1,0)}}} &\mapsto \overline{\rho} \\ \frac{\overline{\tau L_{(0,0,1/\sqrt{2})}}}{\overline{\tau L_{(0,1,1/\sqrt{2})}}} &\mapsto \overline{\rho L_{(0,1,1/\sqrt{2})}} \\ \overline{\tau L_{(0,1,1/\sqrt{2})}} &\mapsto \overline{\rho L_{(0,1,1/\sqrt{2})}}. \end{split}$$

The fact that every  $\gamma \in \Gamma_1$  is  $\Gamma$ -conjugate to  $\Phi(\gamma)$  follows from the relation

$$A\tau L_{\lambda}A^{-1} = \rho L_{A\lambda}$$

in connection with

$$L_{(-1/4,1/4,0)}(\rho L_{A(0,1,0)^t})L_{(1/4,-1/4,0)} = \rho L_{(0,0,-1/\sqrt{2})} \sim \rho L_{(0,0,1/\sqrt{2})}$$

$$L_{(1/4,-1/4,0)}(\rho L_{A(0,0,1/\sqrt{2})^t})L_{(-1/4,1/4,0)} = \rho L_{(0,1,0)}$$

$$\rho L_{A(0,1,1/\sqrt{2})^t} = \rho L_{(0,1,-1/\sqrt{2})} \sim \rho L_{(0,1,1/\sqrt{2})},$$

where the sign  $\sim$  between two elements of  $I(\mathbb{R}^3)$  means that they induce the same element of  $Isom(T_{\Lambda'})$ . Eventually, note that indeed  $\overline{A}, \overline{L_{(1/4,-1/4,0)}} \in \Gamma$ .

We conclude that the tuple  $(T_{\Lambda'}, \Gamma, \Gamma_1, \Gamma_2)$  satisfies the conditions of Theorem 6.1,

#### 6 More Isospectral Flat Orbifolds

and the orbifolds  $\mathcal{O}_1 \simeq T_{\Lambda'}/\Gamma_1$  and  $\mathcal{O}_2 \simeq T_{\Lambda'}/\Gamma_2$  are Sunada-isospectral. Note that the unique nontrivial pair of (0-)isospectral compact flat *manifolds* of dimension three ([RC06]) is not 1-isospectral ([DR04]) and therefore not Sunada-isospectral.

It remains an interesting open question whether there exists a pair of flat orbifolds which are k-isospectral for all k and have maximal isotropy groups of different order.

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## Selbständigkeitserklärung

Ich erkläre, dass ich die vorliegende Arbeit selbständig und nur unter Verwendung der angegebenen Quellen und Hilfsmittel angefertigt habe.

Berlin, den 12.06.2007

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