A global L^κ-gradient estimate on weak solutions to nonlinear stationary Navier-Stokes equations under mixed boundary conditions

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Abstract

In this paper, we prove the integrability of the gradient ∇u to an exponent $\kappa > 2$ near the boundary, u being a weak solution of a nonlinear stationary Navier-Stokes equation under general mixed boundary conditions. As a consequence of this the pressure p belongs to the Campanato space $\mathcal{L}^{2,\lambda}(\Omega)$ with $\lambda := n \cdot \frac{\kappa-2}{\kappa}$. Our method of proof relies on an adaption of a technique by Gehring-Giaquinta-Modica (higher integrability by reverse Hölder inequality) to cubes which possible intersect a hyperplane.

Contents

1 Introduction

Let $\Omega \in \mathbb{R}^n$ $(2 \leq n \leq 4)$ be a bounded domain with Lipschitz boundary $\partial\Omega$, which is assumed to be decomposed into two parts:

$$\partial\Omega = \Gamma_0 \cup \Gamma_1; \Gamma_0 \cap \Gamma_1 = \emptyset,\ \Gamma_0 \ \text{ closed }, \text{int}(\Gamma_0) \neq \emptyset.$$

We consider the following system of nonhomogeneous Navier-Stokes equations in Ω:

$$(1.1) \qquad -\frac{\partial}{\partial x_k}(\mu D_{ik}(u)) \qquad + \text{grad } p + u_k u_{i,k} = f_i - g_{ik,k},\ i,k = 1,\dots,n$$ [1]

$$(1.2) \qquad \text{div } u = m$$

where $u := \{u_1,\dots,u_n\}$-vector of velocity, $u_{i,k} := \frac{\partial u_i}{\partial x_k}$, p - undetermined pressure, grad $p := \{p_{i1},\dots,p_{in}\}$, div $u := u_{k,k}$.
And $f := \{f_1,\dots,f_n\}$, $g := \{g_{ik}\}$ $(i,k = 1,\dots,n)$ and m are given functions in Ω.

As usual we denote by $D_{ij} := \frac{1}{2}(u_{i,j} + u_{j,i})$ the rate of strain tensor and by μ the viscosity.

We complete the system (1.1), (1.2) by the following boundary conditions.

$$(1.3) \qquad u = u^0 \quad \text{on} \quad \Gamma_0,$$

$$(1.4) \qquad (u|\nu) = u^1 \quad \text{on} \quad \Gamma_1$$

(Here $(\cdot|\cdot)$ denotes the usual scalar product in $\mathbb{R}^n$)

$$(1.5) \qquad -\mu D_{ik}(u)\nu_i\tau_k = b_\ell(x,u)\tau_\ell, \quad \forall\tau\perp\nu, \text{ on } \Gamma_1.$$

Where $\nu := \{\nu_1,\dots,\nu_n\}$ and $\tau := \{\tau_1,\dots,\tau_n\}$ are the unit outward normal along Γ_1 and a unit tangent vector along Γ_1, respectively.

Problems like (1.1) - (1.5) (often with d, u^0, u^1 and b equal to zero) arise when studying the stationary motion of a viscous incompessible fluid in a domain with a partially free surface represented by Γ_1. We do not go into hydromechanical details and refer to the various literature [3,8,11,12]. In some situations it is necessary to deal with the whole nonhomogenious conditions. There is a pure mathematical interest in studying the above problem, too.

The aim of the present paper is to show that the gradient ∇u of any weak solution to (1.1) - (1.5) really belongs to $L^\kappa(\Omega;\mathbb{R}^{n^2})$ with some $\kappa > 2$, if there are imposed some weak additional assumptions. And in this case the pressure p belongs to some Campanato space $\mathcal{L}^{2,\lambda}(\Omega)$ $(\lambda > 0)$. The corresponding local results can be found in [6].

We shall use the technique of reverse Hölder inequalities in the elliptic case, which was developed in [4,5] for inner estimates and in [1,2,4,9] for estimates up to the boundary, whereby in [9] under mixed boundary conditions. An in the "common" elliptic case the higher integrability will be achieved by a higher integrability of the data.

We note, that there are similar results for weak solutions of nonlinear parabolic systems under mixed boundary conditons (cf. [10]).

[1]Throughout a repeated index implies summation over $1,\dots,n$.

Now let us impose the needed conditions on the data in order to define a weak solution. The usual notations for Lebesgue and Sobolev spaces are assumed.

Let be fulfilled:

$$(1.6)\quad \mu = \text{const } > 0 \ ^{2},$$

$$(1.7)\quad \begin{cases} f \in L^{\frac{s}{s-1}}(\Omega; I\!R^n), g \in L^2(\Omega; I\!R^{n^2}), m \in L^2(\Omega), \text{ where} \\ s := \begin{cases} \frac{2n}{n-2}, & \text{if } n > 2 \\ 1 \le s < +\infty, & \text{if } n = 2, \end{cases} \end{cases}$$

$$(1.8)\quad u^0 \in W^{1/2,2}(\Gamma_0; I\!R^n),$$

$$(1.9)\quad u^1 \in W^{1/2,2}(\Gamma_1).$$

We define

$$(1.10)\quad \begin{cases} \alpha := \begin{cases} (u^0|\nu)\nu & \text{a.e. on } \Gamma_0 \\ u^1\nu & \text{a.e. on } \Gamma_1 \end{cases} \text{ and assume} \\ \alpha \in W^{1/2,2}(\partial\Omega; I\!R^n) \end{cases}$$

Further, we impose the following compatibility condition

$$(1.11)\quad \int\limits_{\Omega} m dx = \int\limits_{\Gamma_0} (u^0|\nu) d\gamma + \int\limits_{\Gamma_1} u^1 d\gamma.$$

Let b_i be Carathéodory functions satisfying

$$(1.12)\quad \begin{cases} |b_i(x,u)| \le c_0(\varphi(x) + |u|^{l-1}), \text{ f.a.a. } x \in \Gamma_1, \forall_u \in I\!R^n, \\ \text{where } l := \begin{cases} \frac{2(n-1)}{n-2}, & \text{if } n > 2 \\ 1 \le l < +\infty, & \text{if } n = 2 \end{cases} \\ \text{and } \varphi \in L^{\frac{\ell}{\ell-1}}(\Gamma_1), c_0 = \text{const} > 0. \end{cases}$$

We define the space V of test-function by

$$V := \{v \in W^{1,2}(\Omega; I\!R^n) : v = 0 \text{ a.e. on } \Gamma_0, (v|\nu) = 0 \text{ a.e. on } \Gamma_1\}.$$

Definition 1.1: *A pair $\{u,p\} \in W^{1,2}(\Omega; I\!R^n) \times L^2(\Omega)$ is called a weak solution to (1.1) - (1.5), if (1.2), (1.3) and (1.4) are fulfilled and*

$$(1.13)\quad \begin{cases} \mu \int\limits_\Omega D_{ij}(u)\varphi_{i,j} dx - \int\limits_\Omega p\varphi_{i,i} dx + \int\limits_\Omega u_k u_{i,k}\varphi_i dx + \int\limits_{\Gamma_1} b_i(x,u)\varphi_i d\gamma = \\ = \int\limits_\Omega (f_i\varphi_i + g_{ik}\varphi_{i,k}) dx, \ \forall \varphi \in V. \end{cases}$$

[2] Our main result will continue to hold, if condition (1.6) will be substituted by the more general one: $\mu \in L^\infty(I\!R)$, $0 < \mu_0 \le \mu(s) \le \mu_1 < +\infty$ f.a.a. $s \in I\!R$. In this case the above problem (1.1) - (1.5) describes a stationary motion of a general viscous fluid (see, e.g. [3]). For convenience we only deal with (1.6) and there will be no loss in mathematical generality.

Now let us homogenize the problem (1.1) - (1.5) with respect to (1.3) and (1.4). For a better clearness and convenience we shall only prove the result for this new homogenized problem (1.16) - (1.20) stated below.

Besides this, a careful inspection of the proof shows that there is no substancial loss in generality.

Homogenization of (1.3) *and* (1.4)

Firstly, we extend the function u^0 from Γ_0 onto Γ_1 in the same class $W^{1/2,2}$. After this, we set $u^2 := u^0 - (u^0|\nu)\nu$ on $\partial\Omega$. Clearly, there exists an extension of u^2 into Ω, therefore $u^2 \in W^{1,2}(\Omega; I\!R^n)$, keeping the same notation for the extension.

In the same way, the function α defined in (1.10) has a continuation into $W^{1,2}(\Omega, I\!R^n)$, too.

Obviously, the function $w = u^2 + \alpha$ satisfies the boundary conditions (1.3) and (1.4), and so we homogenize the problem (1.1) - (1.5) setting $u := v + w$.

Thus $v \in V$, and we get from (1.2) and (1.13)

$$(1.14)\quad \operatorname{div} v = m - \operatorname{div} w,$$

$$(1.15)\quad \begin{cases} \mu \int\limits_\Omega D_{ij}(v)\varphi_{i,j}dx + \mu \int\limits_\Omega D_{ij}(w)\varphi_{i,j}dx - \int\limits_\Omega p\varphi_{i,i}dx + \\ + \int\limits_\Omega v_k v_{i,k}\varphi_i dx + \int\limits_\Omega v_k w_{i,k}\varphi_i dx + \int\limits_\Omega w_k v_{i,k}\varphi_i dx + \\ + \int\limits_\Omega w_k w_{i,k}\varphi_i dx + \int\limits_{\Gamma_1} b_i(x, v+w)\varphi, d\gamma \\ = \int\limits_\Omega (f_i\varphi_i + g_{ik}\varphi_{i,k}dx \ \ \forall\varphi \in V. \end{cases}$$

As a consequence of the above discussion we state the new (partially) homogeneous problem:

$$(1.16)\quad -\frac{\partial}{\partial x_k}(\mu D_{ik}(u)) + \operatorname{grad} p + u_k u_{i,k} = f_i - g_{ik,k}$$

$$(1.17)\quad \operatorname{div} u = m,$$

$$(1.18)\quad u = 0 \text{ on } \Gamma_0,$$

$$(1.19)\quad (u|\nu) = 0 \text{ on } \Gamma_1,$$

$$(1.20)\quad -\mu D_{ik}(u)\nu_i\tau_k = b_\ell(x,u)\tau_\ell \ \forall\tau \perp \nu \text{ on } \Gamma_1.$$

Definition 1.2: *A pair $\{u, p\} \in V \times L^2(\Omega)$ is called a weak solution to (1.16) - (1.20), if (1.17) is satisfied and*

$$(1.21)\quad \begin{cases} \mu \int\limits_\Omega D_{ij}(u)\varphi_{i,j}dx - \int\limits_\Omega p\varphi_{i,i}dx + \int\limits_\Omega u_k u_{i,k}\varphi_i dx + \\ + \int\limits_{\Gamma_1} b_i(x,u)\varphi_i d\gamma = \int\limits_\Omega (f_i\varphi_i + g_{ik}\varphi_{i,k})dx, \ \forall\varphi \in V. \end{cases}$$

In the sequel we will use the well-known facts

$$D_{ij}(u)\varphi_{i,j} = D_{ij}(u)D_{ij}(\varphi) \qquad \text{and}$$

$$(1.22) \quad \mu \int_\Omega D_{ij}(u)D_{ij}(u)dx \geq \mu_0 \int_\Omega |\nabla u|^2 dx, \ \mu_0 = \text{const } > 0,$$

as a consequence of (1.6), $\lambda_{n-1}(\Gamma_0) > 0$ and Korn's inequality [12].

Conditions on the boundary

We are interested in results up to the boundary. And so we have to give further assumptions on $\partial\Omega, \Gamma_0$ and Γ_1. We use the notations

$$\begin{aligned} x &= \{x_1, \ldots, x_n\} =: \{x', x_n\} \text{ with } x' := \{x_1, \ldots, x_{n-1}\}, \\ x &= \{x_1, \ldots, x_n\} =: \{x_1, x'', x_n\} \text{ with } x'' := \{x_2, \ldots, x_{n-1}\} \text{ if } n > 2 \end{aligned}$$

and

$$\begin{aligned} B_r(x^0) &:= \{x \in I\!R^n : |x - x^0| < r\} \\ \Delta_\varphi(x^{0'}) &:= \{x' \in I\!R^{n-1} : |x_i - x_i^0| < e, i = 1, \ldots, n-1\} \\ \Delta_\varphi(x^{0''}) &:= \{x'' \in I\!R^{n-2} : |x_i - x_i^0| < \varphi, i = 2, \ldots, n-1\} \end{aligned}$$

where $|\cdot|$ denotes the Euklidean norm in $I\!R^n$ and the modules in $I\!R$, respectively, with respect to the context.
Then the conditions on $\partial\Omega$ are as follows:

$$(1.23) \quad \begin{cases} \textit{for every } x_0 \in \partial\Omega \textit{ there exist a ball } B_{R_0} = B_{R_0}(x_0) \\ \textit{and a cube } \Delta_{\rho_0} = \Delta_{\rho_0}(0') \subset I\!R^{n-1} \textit{ such that} \\ \quad \mathcal{T}_1(x_0) = 0, \\ \quad \mathcal{T}_1(\Omega \cap B_{R_0}) = \{z \in I\!R^n : z' \in \Delta_{\rho 0}, z_n > H(z')\}, \\ \quad \mathcal{T}_1(\partial\Omega \cap B_{R_0}) = \{z \in I\!R^n : z' \in \Delta_{\rho 0}, z_n = H(z')\}, \\ \textit{where}: \\ \quad z = \mathcal{T}_1(x) = \mathcal{O}x + \zeta_0, \mathcal{O} \textit{ orthogonal matrix}, \ \zeta_0 \in R^n \textit{ fixed}, \\ \quad H \textit{ Lipschitz continuous on } \Delta_{\rho 0}. \end{cases}$$

Without any loss of generality, in the sequel we may assume that

$$\partial(\Omega \cap B_{R_0}) \textit{ is Lipschitzian.}$$

Let $x_0 \in \Gamma_0 \cap \bar{\Gamma}_1$ be arbitrary. We complete (1.23) by the following conditions:

$$(1.24\text{a}) \quad \begin{cases} n \geq 3: \textit{ there exists a cube } \Delta_{\rho_1}(0'') \subset I\!R^{n-2} \textit{ such that} \\ \quad \mathcal{T}_1(\Gamma_0 \cap \bar{\Gamma}_1 \cap B_{R_0}) = \{z \in I\!R^n : z_n = H(z'), z_1 = h(z'') \\ \qquad\qquad (z' \in \Delta_{\rho 0}, z'' \in \Delta_{\rho 1})\}, \\ \quad \mathcal{T}_1(\Gamma_0 \cap B_{R_0}) = \{z \in I\!R^n : z_n = H(z'), z_1 \leq h(z'') \\ \qquad\qquad (z' \in \Delta_{\rho 0}, z'' \in \Delta_{\rho 1})\}, \\ \quad \mathcal{T}_1(\Gamma_1 \cap B_{R_0}) = \{z \in I\!R^n : z_n = H(z'), z_1 > h(z'') \\ \qquad\qquad (z' \in \Delta_{\rho 0}, z'' \in \Delta_{\rho 1})\}, \\ \textit{where}: \\ \quad H \textit{ according to } (1.23), \\ \quad h \textit{ Lipschitz continuous on } \Delta_{\rho 1}; \end{cases}$$

$$
(1.24\mathrm{b})\quad \begin{cases} n=2: \textit{ there exists an interval } \Delta_{\rho 1}=(-\rho_1,\rho_1)\subset I\!R^1 \\ \textit{such that} \\ \quad \mathcal{T}_1(\Gamma_0\cap\bar\Gamma_1\cap B_{R_0}) \quad =\{0\}, \\ \quad \mathcal{T}_1(\Gamma_0\cap B_{R_0}) \qquad =\{z\in I\!R^2: z_2=H(z_1),-\rho_1<z_1\le 0\}, \\ \quad \mathcal{T}_1(\Gamma_1\cap B_{R_0}) \qquad =\{z\in I\!R^2: z_2=H(z_1), 0<z_1<\rho_1\} \\ \quad (H \textit{ according to } (1.23)). \end{cases}
$$

Throughout the whole paper, conditions (1.23) and (1.24a) ($n\ge 2$) are assumed to hold. Besides this, for technical reason we assume

$$
(1.25)\quad h\in W^{2,\infty}(\Delta_{\rho 1}),\qquad H\in W^{2,\infty}(\bar\Delta) \quad \text{where } \bar\Delta:=\{z'\in\Delta\rho_0: z_1>h(z'')\}.
$$

Coming to the end, let us remember the so-called 'pressure estimate'

$$
(1.26)\quad \int\limits_{B_r(x^0)}|p-p_r|^2dx\le c\int\limits_{B_r(x^0)}(1+|\nabla u|^2+|u|^s \;+\; |f|^{\frac{s}{s-1}}+|g|^2)dx \qquad \forall B_r(x^0)\subset\subset\Omega.
$$

Here $B_r(x^0):=\{x\in I\!R^n:\|x-x^0\|<r\}$ is the open ball and $p_r:=(\text{meas } B_r(x^0))^{-1}\int\limits_{B_r(x^0)}p dx$. The positive constant c does not depend on r (cf. [6,8]).

2 Statement of the result

Theorem 2.1: *Let (1.6), (1.7), (1.12), (1.25), (1.24a) or (1.24b)*

$\int\limits_\Omega m\, dx=0$ *be satisfied.*

In the case $n=2$ we assume for the exponents s and ℓ in (1.7) and (1.12), respectively,

$$
(2.1)\quad s>2,\quad \ell:=\frac{s}{2}+1.
$$

Further, let

$$
(2.2)\quad \begin{cases} f\in L^{s_1}(\Omega;I\!R^n) & \text{with } s_1>\frac{s}{s-1} \\ g\in L^{s_2}(\Omega;I\!R^{n^2}), m\in L^{s_2}(\Omega) & \text{with } s_2>2; \end{cases}
$$

$$
(2.3)\quad \varphi\in L^{s_3}(\Gamma_1) \text{ with } s_3>\frac{l}{l-1}.
$$

Let $\{u,p\}\in V\times L^2(\Omega)$ be any weak solution to (1.16) - (1.20) in the sense of Definition 1.2.

Then there exists a $\kappa>2$, such that

$$(2.4)\quad (|\nabla u| + |u|^{\frac{s}{2}}) \in L^{\kappa}(\Omega),\ p \in \mathcal{L}^{2,\lambda}(\Omega) \text{ with } \lambda = n \cdot \frac{\kappa-2}{\kappa}$$

and

$$(2.5a)\qquad \int_\Omega (|\nabla u| + |u|^{\frac{n}{n-2}})^\kappa dy \le c_1 \Big\{ \big(\int_\Omega (|\nabla u|^2 + |u|^{\frac{2n}{n-2}})dy\big)^{\kappa/2} + $$
$$+ \int_\Omega (1 + |f|^{\frac{n}{n+2}\kappa} + |g|^\kappa + |m|^\kappa)dy + \big(\int_{\Gamma_1} \varphi^{\frac{\kappa}{2}-\frac{1}{n}} d\gamma\big)^{\frac{n}{(n+2)(\frac{\kappa}{2}-\frac{1}{n})}} \Big\}$$

if $n > 2$, and

$$(2.5b)\qquad \int_\Omega (|\nabla u| + |u|^{s/2})^\kappa dy \le c_1 \Big\{ \big(\int_\Omega (|\nabla u|^2 + |u|^s)dy\big)^{\kappa/2} + $$
$$+ \int_\Omega (1 + |f|^{\frac{s}{s-1}\cdot\frac{\kappa}{2}} + |g|^\kappa + |m|^\kappa)dy + \big(\int_{\Gamma_1} \varphi^{1+\frac{s}{2}(\frac{\kappa}{2}-1)} d\gamma\big)^{\frac{s\kappa}{4+s(\kappa-2)}} \Big\}$$

if $n = 2$, where the constant c_1 depends only on μ, c_0 (from (1.12), s (in the case $n = 2$), s_1, s_2, s_3 and the geometric properties of $\partial\Omega$, and $\bar\Gamma_0 \cap \bar\Gamma_1$.

We remark that an analogous result can be proved for the whole nonhomogeneous problem (1.1)-(1.5), when additionally assuming

$$\begin{cases} u^0 \in W^{1-\frac{1}{s_4},s_4}(\Gamma_0; I\!R^n), u^1 \in W^{1-\frac{1}{s_4},s_4}(\Gamma_1), \\ \alpha \in W^{1-\frac{1}{s_4},s_4}(\partial\Omega; I\!R^n) \text{ with } s_4 > 2. \end{cases}$$

The proof of the above theorem is essentially based on "theorems on higher integrability via reverse Hölder inequality". But as a consequence of the nonhomogeneous boundary condition (1.20) there arise terms involving integrals over $(n-1)$ dimensional surface. Thus we have to apply a variant of these theorems, which was proved in [1,2]. For clearness we note this theorem with a small variation made for our purpose.

At first, we give some notations
Let $C := C_R(x^0) \subset I\!R^n$ $(n \ge 2)$ be the n-cube $C_R(x^0) := \{x \in I\!R^n : |x_i - x_i^0| < R, i = 1, \dots, n\}$, where $R > 0$ and $x^0 \in I\!R$ are arbitrarily given. Further, we define

$$\Gamma := C_R(x^0) \cap \{x \in I\!R^n : x_n = x_n^0\},\ \Gamma_r(x) := \Gamma \cap C_r(x)$$

for all $C_r(x) \subset C_R(x^0)$. ($\Gamma_r(x)$ may be empty, of course.)
As usual we notice

$$\fint_\Omega f\, dx := (\text{meas } \Omega)^{-1} \int_\Omega f\, dx.$$

Theorem 2.2: *Let $g \in L^\sigma(C), f \in L^{\sigma_1}(c), \varphi \in L^{\sigma_2}(\Gamma)$ be non-negative functions with $\sigma, \sigma_1, \sigma_2 > 1$.*

Assume that f.a.a. $x \in C$ there holds the estimate

$$\text{(2.6)}\qquad \fint_{C_r(x)} g^\sigma dy \le \Theta \fint_{C_{ar}(x)} g^\sigma dy + c_0\{\Big(\fint_{C_{ar}(x)} g dy\Big)^\sigma + r^{-n}\Big(\int_{C_{ar}(x)} f dy\Big)^m + $$
$$+ r^{-n}\big(\int_{\Gamma_{ar}(x)} \varphi dy'\big)^t\}$$

$\forall r \le \min\{\frac{1}{a}\mathrm{dist}(x, \partial C), r_0\}$ *with some constants*

$$\Theta \in [0,1),\ a \ge 2,\ c_0 > 0,\ m \ge 1,\ t \ge \frac{n}{n-1},\ r_0 > 0.$$

Then there exists κ_0, c_1, c_2 depending only on $\Theta, c_0, \sigma, n, m, t, \sigma_1, \sigma_2, r_0$ and R with

$$\sigma < \kappa_0 \le \sigma + \sigma \min\{\frac{1}{n}(\sigma_1 - 1), \frac{1}{t}(\sigma_2 - 1)\}$$

such that $\forall \kappa \in [\sigma, \kappa_0) : g \in L^\kappa(C_{R/2}(x^o))$ and the following estimates are true

$$\text{(2.7)}\qquad \|g\|_{\kappa, C_{R/2}(x^0)} \le c_1\Big\{\|g\|_{\sigma,C} + \|f\|^{m/\sigma}_{1+\frac{m}{\sigma}(\kappa-\sigma),C} + \|\varphi\|^{\varphi/\sigma}_{1+\frac{t}{\sigma}(\kappa-\sigma),\Gamma}\Big\}$$

and

$$\text{(2.8)}\qquad \Big(\fint_{C_r(x)} g^\kappa dy\Big)^{1/\kappa} \ \le\ c_2\Big\{\Big(\fint_{C_{ar}(x)} g^\sigma dy\Big)^{1/\sigma} + \Big(\fint_{C_{ar}(x)} f^{1+\frac{m}{\sigma}(\kappa-\sigma)} dy\Big)^{\frac{m}{\sigma}\cdot\frac{1}{1+\frac{m}{\sigma}(\kappa-\sigma)}} +$$
$$+\ \Big(\fint_{\Gamma_{ar}(x)} \varphi^{1+\frac{t}{\sigma}(\kappa-\sigma)} dy'\Big)^{\frac{t}{\sigma}\cdot\frac{1}{1+\frac{t}{\sigma}(\kappa-\sigma)}}\Big\}$$

$\forall r \le \min\{\frac{1}{a}\mathrm{dist}(x, \partial C), r_0\}$.

Besides this, in the case $n = 2$ there also holds a more specialized variant of the Theorem 2.3 needed when assuming the condition (1.12).

Theorem 2.3. *Let $n = 2$ and the same assumption on g, f and φ given as in Theorem 2.2.*
Instead of (2.6) assume that f.a.a. $x \in C$ there holds the estimate

$$\text{(2.6}'\text{)}\qquad \fint_{C_r(x)} g^\sigma dy \ \le\ \Theta \fint_{C_{ar}(x)} g^\sigma dy + c_0\Big\{\Big(\fint_{C_{ar}(x)} g dy\Big)^\sigma + r^{-2}\Big(\int_{C_{ar}(x)} f dy\Big)^m +$$
$$+\ r^{-2+\frac{2}{t}}\Big(\int_{\gamma_{ar}(x)} \varphi dy'\Big)^{\frac{2(t-1)}{t}}\Big\}$$

$\forall r \le \min\{\frac{1}{a}$ *dist* $(x, \partial c), r_0\}$ *with constants as in Theorem 2.3. Setting $t^* := \frac{2(t-1)}{t}$, then there exist κ_0, c_1, c_2 positive constants depending only on $\Theta, c_0, \sigma, n, m, t, \sigma_1, \sigma_2, r_0$ and R with*

$\sigma < \kappa_0 \le \sigma + \sigma \min\{\frac{1}{m}(\sigma_1 - 1), \frac{1}{t^*}(\sigma_2 - 1)\}$

such that $\forall \kappa \in [\sigma, \kappa_0) : g \in L^\kappa(C_{R/2}(x^0))$ *and the following estimates are true*

$$(2.7') \quad \|g\|_{\kappa, C_{R/2}(x^0)} \le c_1 \Big\{ \|g\|_{\sigma, C} + \|f\|^{\frac{m}{\sigma}}_{1+\frac{m}{\sigma}(\kappa-\sigma), C} + \|\varphi\|^{t^*/\sigma}_{1+\frac{t^*}{\sigma}(\kappa-\sigma), \Gamma} \Big\}$$

and

$$(2.8') \quad \Big(\fint_{C_r(x)} g^\kappa \, dy \Big)^{1/\kappa} \le c_1 \Big\{ \Big(\fint_{C_{ar}(x)} g^\sigma \, dy \Big)^{1/\sigma} + \Big(\fint_{C_{ar}(x)} f^{1+\frac{m}{\sigma}(\kappa-\sigma)} dy \Big)^{\frac{m}{\sigma} \cdot \frac{1}{1+\frac{m}{\sigma}(\kappa-\sigma)}} + \\ + \Big(\fint_{\Gamma_{ar}(x)} \varphi^{1+\frac{t^*}{\sigma}(\kappa-\sigma)} dy' \Big)^{\frac{t^*}{\sigma} \cdot \frac{1}{1+\frac{t^*}{\sigma}(\kappa-\sigma)}} \Big\}$$

$\forall r \le \min\{\frac{1}{a}$ *dist* $(x, \partial c), r_0\}$.

Remarks.

1^o The analogon of the Theorem 2.1 for "common" nonlinear elliptic systems of PDE in divergence form can be proved by the same method in an easier way. We refer to [9]. In [7] higher integrability is proved by an entirely different method.

2^o The convection term $u_k u_{i,k}$ in (1.1) and (1.16), respectively, leads to the upper bound 4 for the dimension n. Without this term the result is valid for arbitrary dimensions $n \ge 2$.

3 Localization and Change of Variables

Let $x^0 \in \Gamma_0 \cap \bar{\Gamma}_1$ be arbitrary. By (1.23), there exists an $R_0 := R_0(x^0) > 0$, such that $\Omega_{R_0} := \Omega \cap B_{R_0}(x^0) \subset \Omega$ has a Lipschitz boundary. We prove the statement of the theorem with $\Omega \cap B_{R_1}$ in place of Ω, where $0 < R_1 < R_0$.

Our reasoning shows that an analogous argument obviously applies to the interior of Γ_0 and Γ_1, respectively (relative to $\partial\Omega$). Thus, by a finite covering of $\partial\Omega$ by appropriately chosen balls, we obtain the statement of the theorem with a boundary strip in place of Ω. Combining this result with the well-known higher integrability of ∇u at the interior (cf. [6]), we get the full statement.

Let $\Omega_{R_0} := \Omega \cap B_{R_0}(x^0)$ be as above and $\Gamma_{1,R_0} := \Gamma_1 \cap B_{R_0}(x^0)$. Then we can specialize (1.21) to

$$(3.1) \quad \begin{cases} \mu \int_{\Omega_{R_0}} D_{ij}(u) D_{ij}(\varphi) dx - \int_{\Omega_{R_0}} p \varphi_{i,i} dx + \int_{\Omega_{R_0}} u_k u_{i,k} \varphi_i dx + \\ \quad + \int_{\Gamma_{1,R_0}} b_i(x, u) \varphi_i d\gamma = \int_{\Omega_{R_0}} (f_i \varphi_i + g_{ik} \varphi_{i,k}) dx \\ \forall \varphi \in W^{1,2}(\Omega_{R_0}, \mathbb{R}^n) \text{ with } \varphi = 0 \text{ a.e. on } \{\Gamma_0 \cap B_{R_0}(x^0)\} \cup \\ \cup \{\Omega \cap S_{R_0}(x^0)\}, \ (\varphi|\nu) = 0 \text{ a.e. on } \Gamma_1 \cap B_{R_0}(x^0). \end{cases}$$

We pass from the variables $x \in \Omega \cap B_{R_0}$ to new rectangular coordinates

$$y = \mathcal{T} x,$$

where

$$\begin{array}{ll} \mathcal{T} = \mathcal{T}_3 \circ \mathcal{T}_2 \circ \mathcal{T}_1 & \text{if} \quad n \geq 3, \\ \mathcal{T} = \mathcal{T}_2 \circ \mathcal{T}_1 & \text{if} \quad n = 2, \end{array}$$

and

$$\begin{array}{rl} \mathcal{T}_1 & \text{according to } (1.23), (1.24\text{a}), \\ \mathcal{T}_2(z) & = w = \{z', z_n - H(z')\}, \\ \mathcal{T}_3(w) & = y = \{-w_1 + h(w''), w_2, \dots, w_n\}. \end{array}$$

We have

$$\begin{array}{ll} \mathcal{T}'(x) = \mathcal{T}_3'(\mathcal{T}_2 \circ \mathcal{T}_1(x)) \circ \mathcal{T}_2'(\mathcal{T}_1(x)) \circ \mathcal{O}x & \text{if} \quad n \geq 3, \\ \mathcal{T}'(x) = \mathcal{T}_2'(\mathcal{T}_1(x)) \circ \mathcal{O}x & \text{if} \quad n = 2 \end{array}$$

for a.a. $x \in \Omega \cap B_{R_0}$, where

$$\mathcal{T}_2'(z) \;=\; \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -\frac{\partial H}{\partial z_1} & -\frac{\partial H}{\partial z_2} & \dots & -\frac{\partial H}{\partial z_{n-1}} & 1 \end{pmatrix}$$

$$\mathcal{T}_3'(w) \;=\; \begin{pmatrix} -1 & \frac{\partial h}{\partial w_2} & \dots & \frac{\partial h}{\partial w_{n-1}} & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

It follows that

$$(3.2) \qquad \begin{cases} \mathcal{T}, \mathcal{T}^{-1} \text{ are Lipschitz mappings,} \\ |\det \mathcal{T}'(x)| = 1 \text{ for a.a. } x \in \Omega \cap B_{R_0} \end{cases} \qquad \blacksquare$$

For the subsequent discussion, we introduce the following notations:

$$\begin{array}{rcl} C_r(y_0) & = & \{y \in \mathbb{R}^n : |y_i - y_{0i}| < r \ (i, \dots, n)\}, \\ C_r^+(y_0) & = & \{y \in C_r(y_0) : y_n > 0\}, \\ C_r^-(y_0) & = & \{y \in C_r(y_0) : y_n < 0\}. \end{array}$$

Let $x_0 \in \Gamma_0 \cap \bar{\Gamma}_1$. Let $B_{R_0} = B_{R_0}(x_0)$ be a ball according to (1.23). From (1.23), (1.24a) or (1.24b) we obtain the existence of a $\rho_0 > 0$, such that

$$\mathcal{T}^{-1} : \overline{C_{\rho_0}^+(0)} \to \bar{\Omega} \cap B_{R_0},$$

where

$$
(3.3) \qquad \begin{cases} \mathcal{T}^{-1}(C^+_{\rho_0}(0)) \subset \Omega \cap B_{R_0}, \\ \mathcal{T}^{-1}(\{y \in \overline{C^+_{\rho_0}(0)} : y_1 \leq 0, y_n = 0\}) \subset \Gamma_0 \cap B_{R_0}, \\ \mathcal{T}^{-1}(\{y \in \overline{C^+_{\rho_0}(0)} : y_1 > 0, y_n = 0\}) \subset \Gamma_1 \cap B_{R_0}, \end{cases}
$$

We note that both R_0 and ρ_0 possibly depend on x_0.
Further, form (1.25) it follows that

$$
(3.4) \qquad \mathcal{T}^{-1} \in W^{2,\infty}(\{y \in C^+_{\rho_0}(0) : y_1 > 0\}).
$$

Now let $\{u, p\} \in V \times L^2(\Omega)$ fulfills (3.1) and (1.17). Define $v(y) := u(\mathcal{T}^{-1}y)$, $q(y) := p(\mathcal{T}^{-1}y)$ a.e. on $C^+_{\rho_0}(0)$. Then $v \in W^{1,2}(C^+_{\rho_0}(0), I\!R^n)$, $q \in L^2(C^+_{\rho_0}(0))$ and for any Lipschitzian subdomain $w \in C^+_{\rho_0}(0)$ there exist constants C_1, C_2 (possibly depending on w) such that

$$
(3.5) \qquad c_1 \int\limits_{T^{-1}(w)} |\nabla u|^2 dx \leq \int\limits_{w} |\nabla v|^2 dy \leq c_2 \int\limits_{T^{-1}(w)} |\nabla u|^2 dx.
$$

Next, we define the form

$$
(3.6) \qquad \begin{array}{l} a(y, \xi, \eta) := \mu(\xi_{ie} \frac{\partial ye}{\partial x_j} + \xi_{je} \frac{\partial y_e}{\partial x_i})(\eta_{ie} \frac{\partial y_e}{\partial x_j} + \eta_{je} \frac{\partial y_e}{\partial x_i}) \\ \text{f.a.a. } y \in C^+_{\rho_0}(0) \text{ and } \forall \xi, \eta \in I\!R^{n^2}. \end{array}
$$

Further, we set

$$
\begin{aligned}
F_i(y) &:= f_i(\mathcal{T}_y^{-1}), G_{ik}(y) := g_{i\beta}(\mathcal{T}^{-1}y) \frac{\partial y_k}{\partial x_\beta}, \\
M(y) &:= m(\mathcal{T}^{-1}y) \ \text{ a.e. on } C^+_{\rho_0}(0), \\
B_i(y, v) &:= 2b_i(\mathcal{T}^{-1}y, v), \ \Phi(y) := \varphi(\mathcal{T}^{-1}y), \\
& \qquad \text{a.e. on } C_{\rho_0}(0) \cap \{y : y_n = 0, y_1 > 0\} \ \forall v \in I\!R^n.
\end{aligned}
$$

Clearly, the new functions F_i, G_{ik}, M_i, B_i and Φ have the same integrability and growth properties as f_{ie}, g_{ik}, m, b_i and φ, respectively. The form a satisfies an estimate similar to (1.22).

The field of unit outward normals ν along $\Gamma_1 \cap B_{R_0}$ will be transformed into the new field $\tilde{\nu}(y) := \nu(\mathcal{T}^{-1}y)$. We remark that $\tilde{\nu}$ is not normal to the plane $\{y_n = 0\}$, in general, and $\tilde{\nu}$ lies not in this plane f.a.a. y.
We set $V(C^+_{\rho_0}(0)) := \{v \in W^{1,2}(C^+_{\rho_0}(0); I\!R^n) : v = 0$ a.e. on
$\{y : |y_i| \leq \rho_0, \ y_1 \leq 0, \ y_n = 0\} \cup \{y : |y| = \rho_0, y_n > 0\}$, and
$(v|\tilde{\nu}) = 0$ a.e. on $\{y : |y_i| \leq \rho_0, \ y_1 > 0, y_n = 0\}\}$ $(i = \overline{1, n})$.
Clearly, from (1.18), (1.19) we get $v \in V(C^+_{\rho_0}(0))$ $(v(y) := u(\mathcal{T}^{-1}y))$.

A simple standard procedure leads to a new variational equation which is equivalent to (3.1).

$$
(3.7) \qquad \begin{cases} \int\limits_{C^+_{\rho_0}(0)} a(y, \nabla v, \nabla \psi) dy - \int\limits_{C^+_{\rho_0}(0)} q \frac{\partial y_\beta}{\partial x_\alpha} \psi_{\alpha,\beta} dy + \int\limits_{C^+_{\rho_0}(0)} v_u, v_{i,e} \frac{\partial y_e}{\partial x_u} \psi_i dy + \\ + \frac{1}{2} \int\limits_{\gamma_{1,\rho_0}(0)} B_i(y, v) \psi_i dy' = \int\limits_{C^+_{\rho_0}(0)} (F_i \psi_i + G_{ik} \psi_{i,k}) dy \\ \forall \psi \in V(C^+_{\rho_0}(0)). \end{cases}
$$

Then we define the operator

$$L(y,\xi) := \frac{\partial y_\beta}{\partial x_\alpha}\xi_{\alpha\beta} \text{ a.e. in } C^+_{\rho_0}(0),\ \forall \xi \in I\!R^{n^2} \tag{3.8}$$

and from (1.17) we obtain

$$L(y, \nabla v) = M(y) \text{ a.e. in } C^+_{\rho_0}(0) \tag{3.9}$$

where $v(y) := u(\mathcal{T}^{-1}y)$.

4 Continuation of the problem onto $C^-_{\rho_0}(0)$

Now we extend v, q, the functions arising in (3.7), M and the operator L from $C^+_{\rho_0}(0)$ into $C^-_{\rho_0}(0)$. We remember $y = \{y', y_n\}$ for $y \in I\!R^n$ and define

$$\hat{v}(y) := \begin{cases} v(y', y_n), & \text{a.e. on } C^+_{\rho_0}(0) \\ v(y', -y_n), & \text{a.e. on } C^-_{\rho_0}(0) \end{cases} \tag{4.1}$$

And in the same way we extend q, M and F getting $\hat{q}, \hat{M}$ and $\hat{F}$, respectively.
Further, we set

$$\hat{G}_{ik}(y) := \begin{cases} G_{ik}(y', y_n), & \text{a.e. on } C^- +_{\rho_0} (0) \\ G_{ik}(y', -y_n), & \text{a.e. on } C^-_{\rho_0}(0) \text{ and for } k < n \\ -G_{ik}(y', -y_n) & \text{a.e. on } C^-_{\rho_0}(0) \text{ and for } k = n \end{cases}$$

Let $\xi \in I\!R^{n^2}$ be a matrix, $\xi = \{\xi_{ij}\}$, we define the matrix $\hat{\xi}$ by $\hat{\xi}_{ij} := \xi_{ij}$ for all pairs (i,j) with $j < n$ and $\hat{\xi}_{in} := -\xi_{in}$ for all $i = \overline{1,n}$.

Then we extend the form a defined in (3.6) setting

$$\hat{a}(y,\xi,\eta) := \begin{cases} a(y', y_n, \xi, \eta) & \text{a.e. on } C^+_{\rho_0}(0), \forall \xi, y \in I\!R^{n^2}, \\ \mu(\hat{\xi}_{ie}\frac{\partial y_e}{\partial x_j}(y', -y_n) + \hat{\xi}_{je}\frac{\partial y_e}{\partial x_i}(y', -y_n)) \cdot & \\ \cdot(\hat{\eta}_{ie}\frac{\partial y_e}{\partial x_j}(y', -y_n) + \hat{\eta}_{je}\frac{\partial y_e}{\partial x_i}(y', -y_n)) & \text{a.e. on } C^-_{\rho_0}(0),\ \forall \xi, \eta \in I\!R^{n^2} \end{cases} \tag{4.2}$$

To proceed, we extend the operator L defined in (3.8) setting

$$\hat{L}(y,\xi) := \begin{cases} L(y', y_n, \xi) & \text{a.e. on } C^+_{\rho_0}(0),\ \forall \xi \in I\!R^{n^2}, \\ L(y', -y_n, \hat{\xi}) & \text{a.e. on } C^-_{\rho_0}(0),\ \forall \xi \in I\!R^{n^2}. \end{cases} \tag{4.3}$$

Obviously, the function v satisfies

$$\hat{L}(y, \nabla \hat{v}) = \hat{M}(y) \quad \text{a.e. on } C_{\rho_0}(0) \tag{4.4}$$

$(v(y) := u(\mathcal{T}^{-1}y), u$ fulfills (1.17)).
In the above sense we extend the convection term, too.
We define

$$K(y,w,\xi) := w_k \xi_{ie} \frac{\partial y_e}{\partial x_k} \quad \text{f.a.a.} \quad y \in C^+_{\rho_0}(0),\ \forall w \in I\!R^n, \forall \xi \in I\!R^{n^2},$$

and set

$$(4.5) \qquad \hat{K}(y,w,\xi) := \begin{cases} K(y', y_n, w, \xi) & \text{a.e. on} \quad C^+_{\rho_0}(0) \\ & \qquad \forall w \in I\!R^n,\ \forall \xi \in I\!R^{n^2} \\ K(y', -y_n, w, \hat{\xi}) & \text{a.e. on} \quad C^-_{\rho_0}(0) \quad . \end{cases}$$

Taking into account symmetry arguments, we obtain by simple calculations from (3.6), (3.7), (4.1), (4.2) and (4.5)

$$(4.6) \qquad \begin{cases} \int\limits_{C_{\rho_0}(0)} \hat{a}(y, \nabla\hat{v}, \nabla\psi)dy - \int\limits_{C_{\rho_0}(0)} \hat{q}\hat{L}(y,\nabla\psi)dy + \int\limits_{C_{\rho_0}} \hat{K}(y,\hat{v},\nabla\hat{v})\psi dy + \\ + \int\limits_{\gamma_{1,\rho_0}(0)} \hat{B}(y,\hat{v})\psi_i dy' = \int\limits_{C_{\rho_0}(0)} (\hat{F}_i\psi_i + \hat{G}_{ik}\psi_{i,k})dy \\ \forall \psi \in W^{1,2}(C_{\rho_0}(0); I\!R^n) \ \text{ with } \ \psi = 0 \ \text{ a.e. on} \\ \{y : |y|_\infty = \rho_0\} \cup \{y : |y|_\infty < \rho_0, y_1 \le 0, y_n = 0\} \\ \text{and } (\psi|\tilde{\nu}) = 0 \ \text{ a.e. on } \ \{y : |y|_\infty < \rho_0, y_1 > 0, y_n = 0\}. \end{cases}$$

The integral identity (4.6) and the generalized divergence relation (4.4) are the point of departure in the forthcoming investigations.

5 Properties of the operator $\hat{L}$

Let $C_r \subset C_{\rho_0}(0)$ be an n-cube as above and let $\omega_x := \mathcal{T}^{-1}(C_r)$ (note that $\mathcal{T}^{-1}$ is formally defined for $y_n < 0$, too).

Further, we set

$$\hat{C}^-_r := \{x \in C_{\rho_0}(0) : (x', -x_n) \in C^-_r\},$$

and define

$$\begin{aligned} V(C_r) \ &:= \ \{v \in W^{1,2}(C_r, I\!R^n) : v = 0 \ \text{ a.e. on } \ \partial C_r \cup (\{y : y_n = 0\} \cap C_r)\}, \\ &\quad\ \text{if } \bar{C}_r \cap \{y : y_1 > 0, y_n = 0\} = \emptyset, \end{aligned}$$

and

$$\begin{aligned} V(C_r) \ &:= \ \{v \in W^{1,2}(C_r; I\!R^n) : v = 0 \text{ a.e. on } \partial C_r \cup (\{y : y_n = 0, y_1 \le 0\} \cap C_r), \\ &\quad\ (v|\tilde{\nu}) = 0 \text{ a.e. on } \bar{C}_r \cap \{y : y_n = 0, y_1 > 0\}\}, \\ &\quad\ \text{if } \bar{C}_r \cap \{y : y_1 > 0, y_n = 0\} \neq \emptyset. \end{aligned}$$

Then we get the following equations by simple calculations

$$\int\limits_{C_r} \hat{q}\hat{L}\psi dy = \int\limits_{C_r^+} qL\psi dy + \int\limits_{C_r^-} \hat{q}\hat{L}\psi dy = \int\limits_{C_r^+} qL\psi dy + \int\limits_{\hat{C}_r^-} qL\psi dy =$$

$$\int\limits_{\mathcal{T}^{-1}(C_r^+)} p\psi_{i,i}dx + \int\limits_{\mathcal{T}^{-1}(\hat{C}_r^-)} p\psi_{i,i}dx = -\int\limits_{\mathcal{T}^{-1}(C_r)} \text{grad } p\ \psi dx - \int\limits_{\mathcal{T}^{-1}(\hat{C}_r^-)} \text{grad } p\ \psi dx =$$

$$-\int\limits_{C_r^+} \frac{\partial y_k}{\partial x_i}\ \frac{\partial q}{\partial y_k}\psi_i dy - \int\limits_{\hat{C}_r^-} \frac{\partial y_k}{\partial x_i}\ \frac{\partial q}{\partial y_k}\psi_i dy = \int\limits_{C_r} \hat{L}^*\hat{q}\ \psi dy\ \forall\psi \in V(C_r)$$

q smooth, where $\varphi(x) := \psi(\mathcal{T}^{-1}y)$, and $\hat{L}^*\hat{q} := -N(\nabla\hat{q})$, where

$$N(\xi) := \begin{cases} \mathcal{T}'(y', y_n)\xi, & \text{a.e. in } \ C_{\rho_0}^+(0), \\ \mathcal{T}'(y', -y_n)\hat{\xi}, & \text{a.e. in } \ C_{\rho_0}^-(0). \end{cases}$$

Indeed, $\hat{L}^*$ is the adjoint to $\hat{L}$.

Further, we get Im $(\hat{L}) = \{f \in L^2(C_r) := \int\limits_{C_r} fdy = 0\}$ and Im $(\hat{L})$ is closed. Finally, arguing as in [6,8] we obtain the existence of a constant $c = c(C_r)$ with

(5.1) $\qquad \|u\|_{V(C_r)} \leq c\ \|\hat{L}\|_{L^2(C_r)}\ \forall u \in (\text{Ker } \hat{L})^\perp$

and

(5.2) $\qquad \|q - q_{C_r}\| \leq c\ \|\hat{L}^*q\|_{V^*(C_r)}\ \forall q \in L^2(C_r)$

(note $q_{C_r} := (\text{meas } C_r)^{-1} \int\limits_{C_r} qdy$).

By a homothetical argument it follows, that the *constant* c in (5.1) and so in (5.2) *does not depend on* r.

As a consequence of (5.2) and (4.6) and standard arguments we get a generalization of the pressure estimate (1.26)

(5.3) $$\int\limits_{C_r} |\hat{q} - \hat{q}_r|^2 dy \leq c(\int\limits_{C_r} (1 + |\nabla\hat{v}|^2 + |\hat{v}|^s + |\hat{F}|^{\frac{s}{s-1}} + |\hat{G}|^2)dy + \int\limits_{\gamma_{1,r}} |\hat{v}|^\ell dy' + \int\limits_{\gamma_{1,r}} |\Phi|^{\frac{\ell}{\ell-1}} dy')$$

with $c > 0$ independently of $r, \forall C_r \subset\subset C_{\rho_0}(0)$.

6 Caccioppoli inequalities

The integral identity (4.6) is the point of departure for deriving the needed Caccioppoli inequalities.

For convenience we shall write all functions arising in (4.6) and being defined on $C_{\rho_0}(0)$ without the sign "$\wedge$".

At first, let us have some auxillary considerations for dealing with the boundary condition on $\gamma_{1,r}$.

As mentioned above, the vector field $\tilde{\nu}$ ($\tilde{\nu}(y) := \nu(\mathcal{T}^{-1}y)$) is defined on $\{y : |y|_\infty < \rho_0, y_1 > 0, y_n = 0\}$ and depends on $y_1, \ldots, y_{n-1}$, in general. Let e_k ($k = \overline{1,n}$) be the natural orthonormal basis in $I\!R^n$, i.e., the e_k are directed in accordance with y_k.

Now we complete the vector $\tilde{\nu}$ (defined at a.a. points $y \in \{x : |x|_\infty < \rho_0, x_1 > 0, x_n = 0\}$) to an orthonormal basis $\tilde{\nu}^{(1)}, \tilde{\nu}^{(2)}, \ldots, \tilde{\nu}^{(n)}$ where $\tilde{\nu}^{(n)} := \tilde{\nu}$.

By setting

$$\tilde{\nu}^{(k)}(y', y_n) := \tilde{\nu}^{(k)}(y', 0)$$

we get a basis at a.a. points of $C_{\rho_0}(0) \cap \{y_1 > 0\}$. Clearly, we have the representations of v:

$$v = v_k e_k = \tilde{v}_i \tilde{\nu}^{(i)} \text{ with } \tilde{v}_n = 0 \text{ a.e. on } \{y : |y|_\infty < \rho_0, y_1 > 0, y_n = 0\}$$

as a consequence of $(v|\tilde{\nu}) = 0$.
Coming to the end, we define for all $C_r(y^0) \subset C_{\rho_0}(0)$

$$\begin{aligned} \gamma_r^k &:= (\text{meas } C_r(y^0))^{-1} \int\limits_{C_r(y^0)} \tilde{v}_k \, dy, \; v_r^k := (\text{meas } C_r(y^0))^{-1} \int\limits_{C_r(y^0)} v_k \, dy, \\ v_r &:= \{v_r^1, \ldots, v_r^n\}. \end{aligned}$$

Now let $C_r(y^0) \subset C_{3r}(y^0) \subset C_{\rho_0}(0)$. We distinguish three cases concerning the position of $C_r(y^0)$, namely

(A) $C_r(y^0) \cap \{y : y_n = 0\} = \emptyset$
(B) $C_r(y^0) \cap \{y : y_n = 0\} \neq \emptyset$ and $C_{2r}(y^0) \cap \{y : y_n = 0, y_1 \leq 0\} = \emptyset$
(C) $C_r(y^0) \cap \{y : y_n = 0\} \neq \emptyset$ and $C_{2r}(y^0) \cap \{y : y_n = 0, y_1 \leq 0\} \neq \emptyset$

Lemma 6.1: *Let the assumptions of Theorem 2.1 be given.*
Assume (A) holds. Then

$$(6.1) \quad \left\{ \begin{aligned} &\int\limits_{C\,r/2(y^0)} |\nabla v|^2 dy \leq \tfrac{c}{r^2} \int\limits_{C_r(y^0)} |v - v_r|^2 dy + c \int\limits_{C_r(y^0)} (1 + |F|^{\frac{s}{s-1}} + |G|^2 + |M|^2 + |v|^s) dy \\ &\quad + c \int\limits_{C_r} |v - v_r|^s dy + \varepsilon \int\limits_{C_r} |\nabla v|^2 dy. \end{aligned} \right.$$

Assume (B) holds. Then

$$(6.2) \quad \left\{ \begin{aligned} &\int\limits_{C\,r/2(y^0)} |\nabla v|^2 dy \leq \tfrac{c}{r^2} \{ \int\limits_{C_r(y^0)} |\tilde{v}_n|^2 dy + \int\limits_{C_r(y^0)} \sum_{i=1}^{n-1} |\tilde{v}_i - \gamma_{2r}^i|^2 dy \} + \\ &+ c \int\limits_{C_{2r}(y^0)} (1 + |F|^{\frac{s}{s-1}} + |G|^2 + |M|^2 + |v|^s) dy + \varepsilon \int\limits_{C_r(y^0)} |\nabla v|^2 dy \\ &\quad + c \int\limits_{C_r(y^0)} (|\tilde{v}_n|^s + \sum_{i=1}^{n-1} |\tilde{v}_i - \gamma_{2r}^i|^2) dy + c \int\limits_{\gamma_{1,r}(y^0)} |v|^\ell dy' + c \int\limits_{\gamma_{1,r}(y^0)} \Phi^{\frac{\ell}{\ell-1}} dy' \\ &\quad\quad + c \int\limits_{\gamma_{1,r}(y^0)} |B(y,v)| (|\tilde{v}_n| + \sum_{i=1}^{n-1} |\tilde{v}_i - \gamma_{2r}^i|) dy' \end{aligned} \right.$$

Assume (C) holds. Then

$$(6.3)\quad \begin{cases} \int\limits_{C_{r/2}(y^0)} |\nabla v|^2 dy \le \frac{c}{\gamma^2} \int\limits_{C_r(y^0)} |v|^2 dy + c \int\limits_{C_r(y^0)} (1+|F|^{\frac{s}{s-1}} + |G|^2 + |M|^2 + |v|^s) dy + \\ +\varepsilon \int\limits_{C_r(y^0)} |\nabla v|^2 dy + c \int\limits_{\gamma_{1,r}(y^0)} |B(y,v)|\,|v| dy' + c \int\limits_{\gamma_{1,r}} |v|^\ell dy' + c \int\limits_{\gamma_{1,r}} \Phi^{\frac{\ell}{\ell-1}} dy' \end{cases}$$

$\forall C_r(y^0) \subset C_{2r}(y^0) \subset C_{\rho_0}(0)$. The constants c do not depend on r, $\gamma_{1,r}$ may be empty in the case (C) and $0 < \varepsilon < 1$ can be chosen arbitrarily.

Proof. Let $\zeta \in C^\infty(I\!R^n)$ be a cut-off function as follows

$$\begin{cases} \zeta \equiv 1 \text{ on } C_{r/2}(y^0), \zeta \equiv 0 \text{ on } I\!R^n \backslash C_r(y^0),\ 0 \le \zeta \le 1, \\ |\nabla \zeta| \le \frac{c_0}{r} \text{ on } I\!R^n,\ c_0 = \text{ const } > 0, \text{ independent of } r. \end{cases}$$

Case (A): The test-function $\psi := (v - v_r)\zeta^2$ is admissible in (4.6). We get (C_r-means $C_r(y^0)$)

$$\int\limits_{C_r} a(y, \nabla v, \nabla((v-v_r)\zeta^2))dy - \int\limits_{C_r} (q-q_r) L(y, \nabla((v-v_r)\zeta^2))dy +$$
$$+ \int\limits_{C_r} K(y,v,\nabla v)(v-v_r)\zeta^2 dy = \int\limits_{C_r} F(v-v_r)\zeta^2 dy + \int\limits_{C_r} G_{ik}((v_i - v_r^i)\zeta^2),_k dy,$$

shortly

$$\mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 = \mathcal{J}_4 + \mathcal{J}_4.$$

The estimates of $\mathcal{J}_1, \mathcal{J}_3, \mathcal{J}_4$ and $\mathcal{J}_5$ are standard. When estimating the integral $\mathcal{J}_2$, we use the "pressure estimate" (5.3). Finally, we get (6.1).

Case (B). We use the admissible test-function

$$\psi := (v - \sum_{i=1}^{n-1} \gamma_{2r}^i \tilde{\nu}^{(i)})\zeta^2 = (\tilde{v}_n \tilde{\nu}^{(n)} + \sum_{i=1}^{n-1} (\tilde{v}_i - \gamma_{2r}^i)\tilde{\nu}^{(i)})\zeta^2$$

and get

$$\int\limits_{C_r} a(y, \nabla v, \nabla((v - \sum_{i=1}^{n-1}\gamma_{2r}^i \tilde{\nu}^{(i)})\zeta^2))dy - \int\limits_{C_r} (q-q_r) L(y, \nabla((v - \sum_{i=1}^{n-1}\gamma_{2r}^i \tilde{\nu}^{(i)})\zeta^2) dy +$$

$$+ \int\limits_{C_r} K(y,v,\nabla v)(v - \sum_{i=1}^{n-1}\gamma_{2r}^i \tilde{\nu}^{(i)})\zeta^2 dy + \int\limits_{\gamma_{1,r}} B(y,v)(\tilde{v}_n \tilde{\nu}^{(n)} + \sum_{i=1}^{n-1}(\tilde{v}_i - \gamma_{2r}^i)\tilde{\nu}^{(i)})\zeta^2 dy' =$$

$$= \int\limits_{C_r} F(\tilde{v}_n \tilde{\nu}^{(n)} + \sum_{i=1}^{n-1}(\tilde{v}_i - \gamma_{2r}^i)\tilde{\nu}^{(i)})\zeta^2 dy + \int\limits_{C_r} G\ \nabla((v - \sum_{i=1}^{n-1}\gamma_{2r}^i \tilde{v}^{(i)})\zeta^2)dy$$

again shorted by $\mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4 = \mathcal{J}_5 + \mathcal{J}_6$.
We use the relation

$$\nabla((v - \sum_{n=1}^{n-1} \gamma_{2r}^i \tilde{\nu}^{(i)})\zeta^2) = (\nabla v - \nabla(\sum_{i=1}^{n-1} \gamma_{2r}^i \tilde{\nu}^{(i)}))\zeta^2 +$$

$$+2(\tilde{v}_n \tilde{\nu}^{(n)} + \sum_{i=1}^{n-1} (\tilde{v}_i - \gamma_{2r}^i)\tilde{\nu}^{(i)})\zeta \nabla \zeta$$

and estimate the integrals in a standard way. Taking into account (3.4) and using Hölder's inequality, we have

$$\int\limits_{C_r} |\nabla(\sum_{i=1}^{n-1} \gamma_{2r}^i \tilde{\nu}^{(i)}|^2 dy = \int\limits_{C_r} |\sum_{i=1}^{n-1} \gamma_{2r}^i \nabla \tilde{\nu}^{(i)}|^2 dy \le c\, r^{-n} \sum_{i=1}^{n-1} |\gamma_{2r}^i|^2 \le$$

$$\le c\, r^{-2n} (\int\limits_{C_{2r}} |v| dy)^2 \le c \int\limits_{C_{2r}} |v|^s dy + c \int\limits_{C_{2r}} 1 dy.$$

Then (6.2) follows after simple calculations.

Case (C): Clearly, $\psi := v\zeta^2$ is admissible in (4.6) and we get the assertion (6.3) as above, but in a simpler way.

7 Proof of Theorem 2.1

At first we recall some embedding inequalities needed for later use. We shall write its scalar variants for convenience.

The well-known Sobolev-Poincaré inequality reads as

$$(7.1)\quad (\int\limits_{C_r(y^0)} |u - u_r|^s dy)^{1/s} \le c_0 r^{1+\frac{n}{s}-\frac{n}{t}} (\int\limits_{C_r(y^0)} |\nabla u|^t dy)^{1/t}$$

$\forall u \in W^{1,t}(C_r(y^0))$, where

$$(7.2)\quad \begin{cases} 1 \le s \le \frac{nt}{n-t}, & \text{if } \ 1 \le t < n \\ 1 \le s < +\infty, & \text{if } \ t = n \end{cases}$$

$C_r(y^0) \subset I\!R^n$ is the cube defined above, $u_r := (\text{meas}\, C_r)^{-1} \int u dy$ and the constant c_0 arising in (7.1) and the forthcoming inequalities is independent of r.

Further, the Sobolev-Friedrichs inequality holds

$$(7.3)\quad (\int\limits_{C_r(0)} |u|^s dy)^{1/2} \le c_0 r^{1+\frac{n}{s}-\frac{n}{t}} (\int\limits_{C_r(0)} |\nabla u|^t dy)^{1/t}$$

$\forall u \in W^{1,t}(C_r(0))$ with $u = 0$ a.e. on $\gamma_{0,r} := \{y \in C_r(0) : y_n = 0, y_1 \le 0\}$ and s, t are satisfying (7.2).

Further, let $\gamma_{1,r} := \{y \in C_r(0) : y_n = 0, y_1 > 0\}$. Then we get the inequality

$$(7.4)\qquad \Big(\int\limits_{\gamma_{1,r}(0)} |u-u_r|^\ell dy'\Big)^{1/\ell} \le c_0\, r^{1-\frac{n}{t}+\frac{n-1}{\ell}}\Big(\int\limits_{C_r(0)} |\nabla u|^t dy\Big)^{1/t}$$

$\forall u \in W^{1,t}(C_r(0))$, where

$$(7.5)\qquad \begin{cases} 1 \le \ell \le \frac{t(n-1)}{n-t}, & \text{if } \ 1 \le t < n \\ 1 \le \ell < +\infty, & \text{if } \ t = n \end{cases} .$$

And finally, we have

$$(7.6)\qquad \Big(\int\limits_{\gamma_{u,r}} |u|^\ell dy'\Big)^{1/\ell} \le c_0\, r^{1-\frac{n}{t}+\frac{n-1}{\ell}}\Big(\int\limits_{C_0(0)} |\nabla u|^t dy\Big)^{1/t}$$

$\forall u \in W^{1,t}(C_r(0))$ with $u = 0$ a.e. on $\gamma_{0,r}$ and ℓ, t are satisfying (7.5).
Now we are in a position to prove

Lemma 7.1: *Besides the assumptions of Theorem 2.1 assume* (A), (B) *and* (C), *respectively.*
Then

$$(7.7\text{a})\qquad \begin{cases} \int\limits_{C_{r/2}(y^0)} |\nabla v|^2 dy \le \frac{c}{r^2}\Big(\int\limits_{C_{6r}(y^0)} |\nabla v|^{\frac{2n}{n+2}} dy\Big)^{\frac{n+2}{n}} + (\varepsilon+\sigma(r)) \int\limits_{C_{6r}(y^0)} |\nabla v|^2 dy + \\ +c \int\limits_{C_{6r}(y^0)} (1+|v|^s+|F|^{\frac{s}{s-1}}+|G|^2+|M|^2)dy + c\Big(\int\limits_{\gamma_{1,6r}(y^0)} \Phi^{\frac{\ell}{\ell-1}} dy'\Big)^{\frac{n}{n-1}} \end{cases}$$

in the case $n > 2$, and

$$(7.7\text{b})\qquad \begin{cases} \int\limits_{C_{r/2}(y^0)} |\nabla v|^2 dy \le \frac{c}{r^2}\Big(\int\limits_{C_{6r}(y^0)} |\nabla v| dy\Big)^{2} + (\varepsilon+\sigma(r)) \int\limits_{C_{6r}(y^0)} |\nabla v|^2 dy + \\ +c \int\limits_{C_{6r}(y^0)} (1+|v|^s+|F|^{\frac{s}{s-1}}+|G|^2+|M|^2)dy + c\, r^{\frac{2}{\ell}}\Big(\int\limits_{\gamma_{1,6r}(y^0)} \Phi^{\frac{\ell}{\ell-1}} dy'\Big)^{\frac{2(\ell-1)}{\ell}} \end{cases}$$

in the case $n = 2$,
where $0 < \varepsilon < 1$, $\sigma(r) \ge 0$, $\lim\limits_{r\to 0}\sigma(r) = 0$, $\forall\, C_{6r}(y^0) \subset C_{\rho_0}(0)$, c does not depend on r.

Proof: Firstly, assume (A). Applying the vectorial variant of (7.1) we get

$$\int\limits_{C_r(y^0)} |v-v_r|^s dy \le c_0\Big(\int\limits_{C_r(y^0)} |\nabla v|^2 dy\Big)^{s/2} \text{ if } n > 2$$

and

$$\int\limits_{C_r(y^0)} |v-v_r|^s dy \le c_0\, r^{\frac{2}{s}}\Big(\int\limits_{C_r(y^0)} |\nabla v|^2 dy\Big)^{s/2} \text{ if } n = 2.$$

In both cases we have $s > 2$ and so

$$(7.8)\qquad \int\limits_{C_r(y^0)} |v-v_r|^s dy \le c_1\Big(\int\limits_{C_r(y^0)} |\nabla v|^2 dy\Big)^{\frac{s}{2}-1} \int\limits_{C_r(y^0)} |\nabla v|^2 dy.$$

Further, we apply (7.1) again and obtain

$$(7.9)\qquad \int\limits_{C_r(y^0)} |v-v_r|^2 dy \le c_0\Big(\int\limits_{C_r(y^0)} |\nabla v|^{\frac{2n}{n+2}} dy\Big)^{\frac{n+2}{2}}.$$

Hence, (7.7) holds with $\sigma(r) := c_1\Big(\int\limits_{C_r(y^0)} |\nabla v|^2 dy)^{\frac{s}{2}-1}$ fulfilling the assertion via Lebesque's theorem.

Secondly, assume (B). Obviously,

$$C_r(y^0) \subset C_{2r}(y^{0'},0) \subset C_{3r}(y^0).$$

And so we apply the slightly changed inequality (7.1) and (7.3) to the functions $\tilde{v}_i - \gamma_{i,2r}$ and $\tilde{v}_n$, respectively, with respect to the cube $C_{2r}(y^{0'},0)$. Thus, in an analogous way to the case (A) we get estimates being similar to (7.8), (7.9).

Now, let us deal with the integral over $\gamma_{1,r}(y^0)$. Via Hölder's inequality and (1.12), (3.4) we obtain

$$(7.10)\qquad \begin{cases} \int\limits_{\gamma_{1,r}} |B(y,v)||\tilde{v}_n\tilde{\nu}^{(n)} - \sum\limits_{i=1}^{n-1}(\tilde{v}_i-\gamma_{i,2r})\tilde{\nu}^{(i)}|\xi^2 dy' \le \\ \le c\Big(\int\limits_{\gamma_{1,r}(y^0)} |\Phi|^{\frac{\ell}{\ell-1}} dy'\Big)^{\frac{\ell-1}{\ell}} \Big\{\Big(\int\limits_{\gamma_{1,r}} |\tilde{v}_n|^\ell dy'\Big)^{1/\ell} + \sum\limits_{i=1}^{n-1}\Big(\int\limits_{\gamma_{1,r}(y^0)} |\tilde{v}_i - \gamma_{i+2r}|^\ell dy'\Big)^{1/\ell}\Big\} \\ +c\Big(\int\limits_{\gamma_{1,r}} |v|^\ell dy'\Big)^{\frac{\ell-1}{\ell}} \Big\{\Big(\int\limits_{\gamma_{1,r}} |\tilde{v}_n|^\ell dy'\Big)^{1/\ell} + \sum\limits_{i=1}^{n-1}\Big(\int\limits_{\gamma_{1,r}} |\tilde{v}_i - \gamma_{i,2r}|^\ell dy'\Big)^{1/\ell}\Big\} \\ =: T_1 + T_2 \end{cases}$$

Using the inclusions $\gamma_{1,r}(y^0) \subset \gamma_{1,2r}(y^{0'},0) \subset \gamma_{1,3r}(y^0)$ [Φ will be extended by zero, if neccessary] and the inqualities (7.4), (7.6), we get

$$(7.11)\quad T_1 \le c\Big(\int\limits_{\gamma_{1,3r}} |\Phi|^{\frac{\ell}{\ell-1}} dy'\Big)^{\frac{n}{n-1}} + \varepsilon \int\limits_{C_{3r}(y^0)} |\nabla v|^2 dy \text{ in the case } n>2,$$

and

$$(7.12)\quad T_1 \le c\, r^{\frac{2}{\ell}}\Big(\int\limits_{\gamma_{1,3r}(y^0)} \Phi^{\frac{\ell}{\ell-1}} dy'\Big)^{\frac{2(\ell-1)}{\ell}} + \varepsilon \int\limits_{C_{3r}(y^0)} |\nabla v|^2 dy \text{ if } n=2.$$

In a similar way it follows

$$(7.13)\quad T_2 \;\le\; \varepsilon \int\limits_{C_{3r}(y^0)} |\nabla v|^2 dy + c\, r^{2(1-\frac{n}{2}+\frac{n-1}{\ell})}\Big(\int\limits_{\gamma_{1,r}(y^0)} |v|^\ell dy'\Big)^{\frac{2(\ell-1)}{\ell}} \le$$

$$\leq \ \varepsilon \int\limits_{C_{3r}(y^0)} |\nabla v|^2 dy + c\, r^{2(1-\frac{n}{2}+\frac{n-1}{\ell})} \Big\{ \Big(\int\limits_{\gamma_{1,r}(y^0)} |v - v_{2r}|^\ell dy' \Big)^{\frac{2(\ell-1)}{\ell}}$$
$$+ \Big(\int\limits_{\gamma_{1,r}(y^0)} |v_{2r}|^\ell dy' \Big)^{\frac{2(\ell-1)}{\ell}} \Big\}.$$

Applying $\gamma_{1,r}(y^0) \subset \gamma_{1,2r}(y^{0'},0) \subset \gamma_{1,3r}(y^0)$ again and the inequality (7.4), we obtain

$$(7.14)\quad T_2 \ \leq \ \varepsilon \int\limits_{C_{3r}(y^0)} |\nabla v|^2 dy + c\Big(\int\limits_{C_{3r}(y^0)} |\nabla v|^2 dy \Big)^{\ell-2} \int\limits_{C_{3r}(y^0)} |\nabla v|^2 dy +$$
$$+ c\, r^{2(1-\frac{n}{2}+\frac{n-1}{\ell})} \Big(\int\limits_{\gamma_{1,r}} |v_{2r}|^\ell dy' \Big)^{\frac{2}{\ell}(\ell-1)}.$$

By straighforward calculations we get

$$(7.15)\quad c\, r^{2(1-\frac{n}{2}+\frac{n-1}{\ell})} \Big(\int\limits_{\gamma_{1,r}} |v_{2r}|^\ell dy' \Big)^{\frac{2}{\ell}(\ell-1)} \leq c\, r^n |v_{2r}|^{2(\ell-1)} \leq c \int\limits_{C_{2r}(y^0)} |v|^s dy$$

in the case $n > 2$ (Note, $2(\ell-1) = s$) and in the case $n = 2$, when chosing $\ell := \frac{s}{2} + 1$ according to (2.1),

$$(7.16)\quad c\, r^{\frac{2}{\ell}} \Big(\int\limits_{\gamma_{1,r}} |v_{2r}|^\ell dy' \Big)^{\frac{2}{\ell}(\ell-1)} \ \leq \ c\, r^2 |v_{2r}|^{2(\ell-1)} \leq$$
$$\leq \ c\, r^{\frac{4s-2}{s-1}} \int\limits_{C_{2r}(y^0)} |v|^s dy \leq \bar{c} \int\limits_{C_{2r}} |v|^s dy.$$

In order to estimate integrals like $\int\limits_{C_r(y^0)} |v - v_r|^2 dy$ we use the inequalities (7.1), (7.3), respectively, chosing $s := 2$ and $t := \frac{2n}{n+2}$.

Combining the estimates of this point with (6.2) we derive the assertion (7.7) with the function

$$\sigma(r) := c\Big\{ \Big(\int\limits_{C_{3r}(y^0)} |\nabla v|^2 dy \Big)^{\frac{s}{2}-1} + \Big(\int\limits_{C_{3r}(y^0)} |\nabla v|^2 dy \Big)^{\ell-2} \Big\}$$

(c being positive and independently of r).

Finally, assume (C). Here are two cases possible, namely,

(CI) $\quad C_r(y^0) \cap \{y : y_n = 0, y_1 > 0\} = \emptyset$

or

(CII) $\quad C_r(y^0) \cap \{y : y_n = 0, y_1 > 0\} \notin \emptyset.$

Clearly, in the first case there do not arise the integral over $\gamma_{1,r}(y^0)$. Further, in a similar way as above we get

$$(7.17)\quad \int\limits_{C_r(y^0)} |v|^2 dy \le \int\limits_{C_{2r}(y^{0\prime},0)} dy \le c\Big(\int\limits_{C_{2r}(y^{0\prime},0)} |\nabla v|^{\frac{2n}{n+2}} dy\Big)^{\frac{n+2}{n}} \le c\Big(\int\limits_{C_{3r}(y^0)} |\nabla v|^{\frac{2n}{n+2}} dy\Big)^{\frac{n+2}{n}}$$

and so the proof of (7.7) can be completed.

Secondly, in the case (CII) we use the conclusion

$$C_r(y^0) \subset C_{2r}(0, y^{0''}, 0) \subset C_{3r}(y^0)$$

and obtain an assertion like (7.17).

In a similar but easyer way we deal with the integral over $\gamma_{1,r}(y^0)$. In this case we use the relations

$$\gamma_{1,r}(y^0) \subset \gamma_{1,2r}(y^0) \subset \gamma_{1,4_r}(0, y^{0''}, 0) \subset \gamma_{1,6r}(y^0),$$

applying the estimated (7.4) and (7.6) with respect to $\gamma_{1,4r}(0, y^{0''}, 0)$ when performing analogous steps as in the case (B).

8 Proof of Theorem 2.1 completed

At first, let us note the obvious inequality

$$(8.1)\quad \begin{cases} \int\limits_{C_{6r}(y^0)} |v|^s dy \le c \int\limits_{C_{6r}(y^0)} |v - v_r|^s dy + c \int\limits_{C_{6r}(y^0)} |v_r|^s dy \le \\ \le c\, r^{s[1+n(\frac{1}{s}-\frac{1}{2})]} \Big(\int\limits_{C_{6r}(y^0)} |\nabla v|^2 dy\Big)^{\frac{s}{2}-1} \int\limits_{C_{6r}(y^0)} |\nabla v|^2 dy + \\ +c\, r^n \Big(\fint\limits_{C_{6r}(y^0)} |v|^{\frac{sn}{n+2}} dy\Big)^{\frac{n}{n+2}}, \quad \forall C_{6r}(y^0) \subset C_{\rho_0}(0), \end{cases}$$

where s is defined as in (1.7) and by c are denoted several positive constants being independently of r.

To proceed, we add the term $\int\limits_{C_{r/2}(y^0)} |v|^s dy$ on both sides of (7.7) and estimate it by (8.1) on the right-hand side. After this, we chose $\varepsilon := \frac{1}{2}$. The Lebesque's theorem leads to an $r_0 > 0$, such that the (new) function $\sigma = \sigma(r)$ fulfills $\sigma(r) \le \frac{1}{4}\ \forall r : 0 < r \le r_0$ uniformly with respect to y^0.

Collecting these steps, we get the inquality

$$(8.2)\quad \begin{cases} \fint\limits_{C_{r/2}(y^0)} |\nabla v|^2 dy + \fint\limits_{C_{r/2}(y^0)} |v|^s dy \le c\Big\{\Big(\fint\limits_{C_{6r}(y^0)} |\nabla v|^{\frac{2n}{n+2}} dy\Big)^{\frac{n+2}{n}} + \Big(\fint\limits_{C_{6r}(y^0)} |v|^{\frac{sn}{n+2}} dy\Big)^{\frac{n+2}{n}}\Big\} + \\ +c \fint\limits_{C_{6r}(y^0)} (1 + |F|^{\frac{s}{s-1}} + |G|^2 + |M|^2) dy + \frac{3}{4} \fint\limits_{C_{6r}(y^0)} |\nabla v|^2 dy + \\ \quad +c\, r^{-n} \Big(\int\limits_{\gamma_{1,6r}(y^0)} \Phi^{\frac{\ell}{\ell-1}} dy'\Big)^{\frac{n}{n-1}} \end{cases}$$

in the case $n>2$, $\forall C_{6r}(y^0)\subset C_{\rho_0}(0)$, with $0<r\le r_0$, c does not depend on r.
In the case $n=2$ the last term in (8.2) will be replaced by $c\,r^{-n+\frac{2}{\ell}}\Big(\int\limits_{\gamma_{1,6r}(y^0)}\Phi^{\frac{\ell}{\ell-1}}dy'\Big)^{\frac{2(\ell-1)}{\ell}}$.

Now we are in a position to apply Theorem 2.2 and Theorem 2.3, respectively. To proceed, we set

$$\begin{aligned} g &:= |\nabla v|^{\frac{2n}{n+2}}+|v|^{\frac{2n}{n+2}},\ f:=1+|F|^{\frac{s}{s-1}}+|G|^2+|M|^2,\\ \varphi &:= \Phi^{\frac{\ell}{\ell-1}}, m:=1, \Theta:=3/4, a:=12, \sigma:=\frac{n+2}{n}, t:=\frac{n}{n-1}. \end{aligned}$$

And it is not difficult to derive the assertion (2.6) of Theorem 2.3, using the inequality (8.2) (in the case $n>2$). Clearly, the assumptions of Theorem 2.1 ensure the integrability of the new functions f and φ by an exponent $\sigma_1:=\sigma>1$.
Thus, there exists an exponent τ with

$$\frac{n+2}{n}<\tau<\tau_0$$

and τ_0 is determined by the constants involved in the theorem. We rewrite the assertion (2.7) as

$$(8.3)\quad \Big(\int\limits_{C_{\rho_{0/2}}}\Big(|\nabla v|^{\frac{2n}{n+2}}+|v|^{\frac{sn}{n+2}}\Big)^{\tau}dy\Big)^{1/\tau}\le c_1\Big\{\Big(\int\limits_{C_{\rho_0}(0)}\Big(|\nabla v|^2+|v|^s\Big)dy\Big)^{\frac{n}{n+2}}+ \\ +\Big(\int\limits_{C_{\rho_0}(0)}f^{\frac{n\tau}{n+2}}dy\Big)^{1/\tau}+\Big(\int\limits_{\gamma_{1,\rho_0}}\varphi^{\frac{n^2\tau-(n+2)}{(n+2)(n-1)}}dy'\Big)^{\frac{1}{\tau-\frac{n+2}{n^2}}}\Big\}$$

in the case $n>2$.
We set $\kappa:=\frac{2n\tau}{n+2}$. Then $\kappa>2$, and the estimate (8.3) implies

$$(8.4)\quad \int\limits_{C_{\rho_{0/2}}(0)}\Big(|\nabla v|+|v|^{\frac{n}{n-2}}\Big)^{\kappa}dy\le c_1\Big\{\Big(\int\limits_{C_{\rho_0}(0)}\Big(|\nabla v|^2+|v|^{\frac{2n}{n-2}}\Big)dy\Big)^{\kappa/2}+ \\ +\int\limits_{C_{\rho_0}(0)}\Big(1+|F|^{\frac{n\kappa}{n+2}}+|G|^{\kappa}+|M|^{\kappa}\Big)dy+\Big(\int\limits_{\gamma_{1,\rho_0}}\Phi^{\frac{\kappa}{2}-\frac{1}{n}}dy'\Big)^{\frac{n}{(n+2)(\frac{\kappa}{2}-\frac{1}{n})}}\Big\}$$

in the case $n>2$, and

$$(8.5)\quad \int\limits_{C_{\rho_{0/2}}(0)}\Big(|\nabla v|+|v|^{s/2}\Big)^{\kappa}dy\le c_1\Big\{\Big(\int\limits_{C_{\rho_0}(0)}\Big(|\nabla v|^2+|v|^s\Big)dy\Big)^{\frac{\kappa}{2}}+ \\ +\int\limits_{C_{\rho_0}(0)}\Big(1+|F|^{\frac{s}{s-1}\cdot\frac{\kappa}{2}}+|G|^{\kappa}+|M|^{\kappa}\Big)dy+ \\ +\Big(\int\limits_{\gamma_{1,\rho_0}}\Phi^{1+\frac{(\ell-1)}{2}(\kappa-2)}dy'\Big)^{\frac{(\ell-1)\kappa}{2+(\ell-1)(\kappa-2)}}\Big\}$$

in the case $n = 2$.
Take into account the relation (2.1) $\ell := \frac{s}{2} + 1$, if $n = 2$.
The construction of the transform $\mathcal{T}$ implies the existence of an $0 < R_1 < R_0$ such that

$$\mathcal{T}^{-1}(C^+_{\rho_0/2}(0)) \supset \Omega \cap B_{R_1}(x^0).$$

Passing back to the variables x and using (3.5) we get the assertion of Theorem 2.1 with $\Omega \cap B_{R_1}(x^0)$ in place of Ω.

Via compactness arguments we obtain the result for a boundary strip. Combining this with the well-known interior estimates, we deduce $\nabla u \in L^\kappa(\Omega)$ and (2.5a) with an exponent $\kappa > 2$. This new κ may be different from the above, of course.

Finally, let us deal with the pressure.

Firstly, assume $B_r(x^0) \subset \Omega$. Then we may apply the Hölder inequality to the pressure estimate (1.26) with the exponent $\frac{\kappa}{2}$. This leads to

$$p \in \mathcal{L}^{2,\lambda}_{\text{loc}}(\Omega) \quad \text{with} \quad \lambda := n \cdot \frac{\kappa - 2}{\kappa}.$$

Secondly, let $x^0 \in \partial\Omega$ and R_0, ρ_0 be as in point 3.
Now using the pressure estimate (5.3) and the established higher integrability of ∇v, we obtain

$$\hat{q} \in \mathcal{L}^{2,\lambda}(C_{\rho_0/2}(0)) \quad \text{with} \quad \lambda := n \cdot \frac{\kappa - 2}{\kappa} \quad \text{again.}$$

The properties of the transform $\mathcal{T}$, and the "property (A)" of the Lipschitz boundary $\partial\Omega$ (cf. [4]) and compactness arguments ensure that $p \in \mathcal{L}^{2,\lambda}(\Omega)$.

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