

THE SEMISTATIC LIMIT FOR MAXWELL'S EQUATIONS IN AN EXTERIOR DOMAIN

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Abstract:

This paper provides a L^p -theory for the Maxwell-system in the semistatic limit case in an exterior domain. The problem under consideration is of mixed type, since the possibly nonlinear electric conductivity vanishes on a certain subset of the domain. The solution is obtained from a singular perturbation of the full Maxwell-system including the displacement current.

1 Introduction

This paper concerns the initial-boundary value problem for Maxwell's equations [6] in an exterior domain, that is the system

$$\operatorname{curl} \mathbf{h}_\varepsilon = \varepsilon \partial_t \mathbf{E}_\varepsilon + \sigma(t, x, \mathbf{E}_\varepsilon) + \mathbf{j}_0 \quad (1.1)$$

$$\operatorname{curl} \mathbf{E}_\varepsilon = -\mu \partial_t \mathbf{h}_\varepsilon \quad (1.2)$$

supplemented by the initial-boundary-conditions

$$\vec{n} \wedge \mathbf{E}_\varepsilon = 0 \text{ on } [0, \infty) \times \partial\Omega, \quad (1.3)$$

$$\mathbf{E}_\varepsilon(0) = \mathbf{E}_0 \text{ and } \mathbf{h}_\varepsilon(0) = \mathbf{h}_0. \quad (1.4)$$

Here $\Omega \subset \mathbb{R}^3$ is a domain with bounded complement, $\mathbf{E}_\varepsilon, \mathbf{h}_\varepsilon$ denote the electric and magnetic field respectively which depend on the time $t \geq 0$ and the space-variable $x \in \Omega$. On the perfectly conducting boundary $\partial\Omega$ the tangential-component of \mathbf{E} must vanish, which is expressed by boundary condition 1.3. \mathbf{j}_0 is a prescribed external current, which is assumed to be divergence-free. The charge current $\sigma(t, x, \mathbf{E}_\varepsilon)$ may depend nonlinearly on the electric field. σ is a monotone function (with respect to \mathbf{E}_ε) with the property that

$$\sigma(t, x, \mathbf{y}) = 0 \text{ for all } x \in \Omega_0 \stackrel{\text{def}}{=} \Omega \setminus G$$

with some bounded set $G \subset \Omega$. The dielectric susceptibility $\varepsilon > 0$ is considered as a parameter in the above system. $\mu \in L^\infty(\mathbb{R}^3)$ is the magnetic susceptibility, which is assumed to be uniformly positive. The initial-data must be compatible with Maxwell's equations, i.e.

$$\operatorname{div} \mathbf{E}_0 = 0 \text{ on } \Omega_0 = \Omega \setminus G, \operatorname{div} \mathbf{h}_0 = 0 \text{ on } \Omega \quad (1.5)$$

$$\text{and } \int_{\mathcal{C}_k} \vec{n} \mathbf{E}_0(x) dS = Q_k \text{ for all } k \in \{1, \dots, N\}. \quad (1.6)$$

If these compatibility conditions are fulfilled then every solution of the system 1.1-1.4 satisfies

$$\operatorname{div} \mathbf{E}_\varepsilon = 0 \text{ on } [0, \infty) \times \Omega_0 \text{ and } \operatorname{div} \mathbf{h}_\varepsilon = 0 \text{ on } [0, \infty) \times \Omega.$$

$$\text{and } \int_{\mathcal{C}_k} \vec{n} \mathbf{E}_\varepsilon(t, x) dS = Q_k \text{ for all } t \in [0, \infty), k \in \{1, \dots, N\}. \quad (1.7)$$

Here $\mathcal{C}_k, k = 1, \dots, N$ denote the connected components of $\mathbb{R}^3 \setminus \overline{\Omega}_0$. The physical meaning of 1.7 is that the total electric charge Q_k on each \mathcal{C}_k is invariant under the time evolution of $\mathbf{E}_\varepsilon, \mathbf{h}_\varepsilon$ governed by 1.1-1.4. This is a consequence of the assumptions $\sigma(t, x, \mathbf{y}) = 0$ for $\mathbf{y} \in \Omega_0$ and $\operatorname{div} \mathbf{j}_0 = 0$ on \mathbb{R}^3 . In many situations the displacement current $\varepsilon \partial_t \mathbf{E}_\varepsilon$ is small in comparison with the charge currents and is often neglected. Therefore, it is the aim of this paper to investigate the singular limit $\varepsilon \rightarrow 0$. By letting $\varepsilon \rightarrow 0$ one obtains formally from 1.1-1.4 and 1.7 the quasi-stationary Maxwell-System

$$\operatorname{curl} \mathbf{h} = \sigma(t, x, \mathbf{E}) + \mathbf{j}_0 \quad (1.8)$$

$$\operatorname{curl} \mathbf{E} = -\mu \partial_t \mathbf{h} \quad (1.9)$$

$$\operatorname{div} \mathbf{E} = 0 \text{ on } [0, \infty) \times \Omega_0 \quad (1.10)$$

supplemented by the initial-boundary-conditions

$$\vec{n} \wedge \mathbf{E} = 0 \text{ on } [0, \infty) \times \partial\Omega, \quad (1.11)$$

$$\int_{\mathcal{C}_k} \vec{n} \mathbf{E}(t, x) dS = Q_k \text{ for all } t \in [0, \infty), k \in \{1, \dots, N\}. \quad (1.12)$$

$$\text{and } \mathbf{h}(0) = \mathbf{h}_0 \text{ on } \Omega \quad (1.13)$$

This problem has been investigated in [8] in the case that the domain Ω is bounded and σ is linear with respect to \mathbf{E} and uniformly positive. See also [3] and [4], where a temperature dependent electrical conductivity is considered. However, in this paper Ω is

an exterior domain and the electrical conductivity vanishes on a subset $\Omega_0 = \Omega \setminus \overline{G}$. It is assumed that

$$\mathbf{y}\sigma(t, x, \mathbf{y}) \geq \gamma(x)|\mathbf{y}|^2 \text{ for all } x \in G, \mathbf{y} \in \mathbb{R}^3 \quad (1.14)$$

with some $\gamma \in L^\infty(G)$, $\gamma > 0$. σ may have mild degeneracies on G , i.e. it is not assumed that γ is uniformly positive. More precisely,

$$\gamma^{-1} \in L^{r_0/(2-r_0)}(G) \text{ with some } 6/5 < r_0 \leq 2. \quad (1.15)$$

Note that 1.8-1.13 can be considered as a problem of mixed type. In the case that μ is constant on Ω and σ is linear and independent of $x \in G$ it is elliptic on Ω_0 (with respect to the space variable x for fixed time) and parabolic on $(0, \infty) \times G$. It is shown in this paper that 1.8-1.13 admits for given $\mathbf{h}_0, \mathbf{j}_0$ and Q_k a unique global weak solution (\mathbf{E}, \mathbf{h}) with

$$\begin{aligned} \mathbf{E} &\in W_{loc}^{-1}([0, \infty), L^{r_0}(\Omega \cap B_{R_0})) \cap W_{loc}^{-1}([0, \infty), L^6(\mathbb{R}^3 \setminus B_{R_0})) \cap L_{loc}^2([0, \infty), L_\gamma^2(G)) \\ &\subset \mathcal{D}'((0, \infty) \times \Omega) \text{ and } \mathbf{h} \in C([0, \infty), L^2(\Omega)), \end{aligned}$$

where $\gamma, r_0 \in (6/5, 2]$ as in 1.14, 1.15 and $L_\gamma^2(G)$ is the weighted L^2 -space and $R_0 > 0$, such that $G \cup (\mathbb{R}^3 \setminus \Omega) \subset B_{R_0} = \{|x| < R_0\}$. Moreover, it is shown that

$$\begin{aligned} \mathbf{h}_\varepsilon &\xrightarrow{\varepsilon \rightarrow 0} \mathbf{h} \text{ in } L^\infty((0, T), L^2(\Omega)) \text{ weak} - * \text{ for all } T \in (0, \infty) \\ \text{and } \mathbf{E}_\varepsilon &\xrightarrow{\varepsilon \rightarrow 0} \mathbf{E} \text{ in } \mathcal{D}'((0, \infty) \times \Omega). \end{aligned}$$

In the case that σ is independent of t and $(\mathbf{E}_0, \mathbf{h}_0), \mathbf{j}_0$ are sufficiently regular it is shown that 1.8-1.13 admits global strong solution (\mathbf{E}, \mathbf{h}) with

$$\begin{aligned} \mathbf{E} &\in L_{loc}^2([0, \infty), L^{r_0}(\Omega \cap B_{R_0})) \cap L_{loc}^2([0, \infty), L^6(\mathbb{R}^3 \setminus B_{R_0})) \cap L_{loc}^2([0, \infty), L_\gamma^2(G)) \\ \text{and } \mathbf{h} &\in W_{loc}^{1,\infty}([0, \infty), L^2(\Omega)). \end{aligned}$$

Moreover, one has in this case

$$\begin{aligned} \mathbf{h}_\varepsilon &\xrightarrow{\varepsilon \rightarrow 0} \mathbf{h} \text{ in } W^{1,\infty}((0, T), L^2(\Omega)) \text{ weak} - * \\ \text{curl } \mathbf{h}_\varepsilon &\xrightarrow{\varepsilon \rightarrow 0} \text{curl } \mathbf{h} \text{ in } L^2((0, T), L^2(\Omega)) \text{ weakly} \\ \text{and } \mathbf{E}_\varepsilon &\xrightarrow{\varepsilon \rightarrow 0} \mathbf{E} \text{ in } L^2((0, T), L^{r_0}(\Omega \cap B_{R_0}) \cap L^6(\mathbb{R}^3 \setminus B_{R_0})) \text{ weakly}. \end{aligned}$$

for all $T \in (0, \infty)$. In the case that also $\partial\Omega$ is sufficiently regular this implies also

$$\mathbf{h}_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \mathbf{h} \text{ in } C([0, T], L^2(\Omega \cap B_R)) \text{ strongly}$$

for all $R \in (0, \infty)$ and $T \in (0, \infty)$.

Finally, the asymptotic behavior of the solution to 1.8-1.13 for $t \rightarrow \infty$ is investigated in the case $\mathbf{j}_0 = 0$.

It is shown that $\|\mathbf{h}(t)\|_{L^2}$ decays exponentially provided that

$$\int_{\Omega} \mu \mathbf{h}_0 \mathbf{g} dx = 0 \text{ for all } \mathbf{g} \in L^2(\Omega) \text{ with } \text{curl } \mathbf{g} = 0.$$

This condition includes $\text{div}(\mu \mathbf{h}_0) = 0$ on Ω and $\vec{n} \mathbf{h} = 0$ on $\partial\Omega$ weakly.

2 Notation, definitions and assumptions

For an arbitrary open set $K \subset \mathbb{R}^3$ the space of all infinitely differentiable functions with compact support contained in K is denoted by $C_0^\infty(K)$. Moreover, $\mathcal{D}'(K)$ is the space of distributions on K .

For $p \in [1, \infty)$ we define $H_{curl}^p(K)$ as the space of all $\mathbf{E} \in L^p(K)$ with $\text{curl } \mathbf{E} \in L^p(K)$. $\overset{0}{H}_{curl}^p(K)$ denotes the set of all $\mathbf{E} \in H_{curl}^p(K)$, such that

$$\int_K \mathbf{E} \text{curl } \mathbf{h} - \mathbf{h} \text{curl } \mathbf{E} dx = 0 \text{ for all } \mathbf{h} \in H_{curl}^{p^*}(K),$$

which includes a weak formulation of the boundary-condition $\vec{n} \wedge \mathbf{E} = 0$ on ∂K . Here $\frac{1}{p} + \frac{1}{p^*} = 1$. Let $W^{1,p}(K)$ is the usual first order Sobolev space consisting of all functions in $L^p(K)$ whose distributional gradient belongs to $L^p(K)$. $\overset{0}{W}^{1,p}(K)$ denotes the closure of $C_0^\infty(K)$ in $W^{1,p}(K)$.

In the sequel $\Omega \subset \mathbb{R}^3$ is an open set with bounded complement, $G \subset \Omega$ and $\Omega_0 \stackrel{\text{def}}{=} \Omega \setminus \overline{G}$. It is assumed that $\mathcal{C} \stackrel{\text{def}}{=} \mathbb{R}^3 \setminus \overline{\Omega_0} = \mathcal{C}_1 \cup \dots \cup \mathcal{C}_N$ is a bounded Lipschitz-domain, where \mathcal{C}_k are its connected components with $\overline{\mathcal{C}_j} \cap \overline{\mathcal{C}_k} = \emptyset$ if $j \neq k$.

$\mu \in L^\infty(\mathbb{R}^3)$ is a uniformly positive function.

The assumptions on $\sigma : [0, \infty) \times \Omega \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ are the following.

$$\sigma(t, x, \mathbf{y}) = 0 \text{ if } x \in \Omega_0 = \Omega \setminus \overline{G}, \quad (2.16)$$

$$\sigma(\cdot, \cdot, \mathbf{y}) \in L_{loc}^\infty([0, \infty), L^\infty(\Omega)) \text{ for fixed } \mathbf{y} \in \mathbb{R}^3, \quad (2.17)$$

and Lipschitz-continuous, i.e. there exists $L \in (0, \infty)$, such that

$$|\sigma(t, x, \mathbf{y}) - \sigma(t, x, \tilde{\mathbf{y}})| \leq L|\mathbf{y} - \tilde{\mathbf{y}}| \text{ for all } \mathbf{y}, \tilde{\mathbf{y}} \in \mathbb{R}^3, t \in [0, \infty) \text{ and } x \in \Omega. \quad (2.18)$$

Moreover,

$$|\sigma(t, x, \mathbf{y})| \leq C_0 \gamma(x) |\mathbf{y}| \text{ and } \mathbf{y} \sigma(t, x, \mathbf{y}) \geq \gamma(x) |\mathbf{y}|^2 \quad (2.19)$$

for all $\mathbf{y} \in \mathbb{R}^3$, $t \in [0, \infty)$ and $x \in G$ with some $\gamma \in L^\infty(G)$, $\gamma > 0$ and $C_0 \in (0, \infty)$.

Finally, σ is assumed to be monotone, i.e.

$$(\mathbf{y} - \tilde{\mathbf{y}})(\sigma(t, x, \mathbf{y}) - \sigma(t, x, \tilde{\mathbf{y}})) > 0 \text{ for all } \mathbf{y} \neq \tilde{\mathbf{y}} \in \mathbb{R}^3, t \in [0, \infty) \text{ and } x \in G. \quad (2.20)$$

Next, let $p_0 \in (2, 6)$, such that for all $p \in [2, p_0]$ the following holds.

$$\varphi \in \overset{0}{W}^{1,2}(\Omega_0 \cap B_{2R_0}) \text{ and } \Delta \varphi \in W^{-1,p}(\Omega_0 \cap B_{2R_0}) \stackrel{\text{def}}{=} \left(W^{1,p^*}(\Omega_0 \cap B_{2R_0}) \right)^* \quad (2.21)$$

$$\implies \nabla \varphi \in L^p(\Omega_0 \cap B_{2R_0}).$$

Since $\Omega_0 \cap B_{2R_0} = B_{2R_0} \setminus \overline{\mathcal{C}}$ is a Lipschitz-domain, it follows from the result in [2] that there exist such a $p_0 > 2$. The following compatibility conditions will be imposed on p_0 and γ .

$$\gamma^{-1} \in L^{r_0/(2-r_0)}(G) \text{ with } r_0 \stackrel{\text{def}}{=} p_0^* \quad (2.22)$$

Let $L_\gamma^2(G)$ be the weighted L^2 -space consisting of all functions $\mathbf{f} : G \rightarrow \mathcal{C}^8$ with

$$\|\mathbf{f}\|_{L_\gamma^2(G)}^2 \stackrel{\text{def}}{=} \int_G |\mathbf{f}|^2 \gamma dx < \infty.$$

It follows from 2.19, 2.22 and Hölder's inequality that

$$L_\gamma^2(G) \subset L^{r_0}(G) \quad (2.23)$$

and for $\mathbf{F} \in L_\gamma^2(G)$ one has

$$\sigma(t, x, \mathbf{F}) \in L_{\gamma^{-1}}^2(G) \subset L^2(G) \quad (2.24)$$

$$\text{with } \|\sigma(t, x, \mathbf{F})\|_{L_{\gamma^{-1}}^2(G)} = \|\sigma(t, x, \mathbf{F})\gamma^{-1/2}\|_{L^2(G)} \leq K \|\mathbf{F}\|_{L_\gamma^2(G)}$$

with some $K \in (0, \infty)$ independent of t, \mathbf{f} .

For $\mathbf{E} \in L_{loc}^1(\Omega)$ with $\text{div } \mathbf{E} = 0$ on Ω_0 define

$$q_j(\mathbf{E}) \stackrel{\text{def}}{=} \int_{\Omega} \mathbf{E} \nabla \chi_j dx = \int_{\Omega_0} \mathbf{E} \nabla \chi_j dx, \quad (2.25)$$

where $\chi_j \in C_0^\infty(\mathbb{R}^3)$ obey $\chi_j|_{\mathcal{C}_k} = \delta_{j,k}$. Due to the condition $\text{div } \mathbf{E} = 0$ this definition is independent of the choice of the functions χ_j .

Now, the following function spaces are defined.

Let for $q \in [1, 2]$ Y_0^q be the space of all measurable functions $\mathbf{E} : \Omega \rightarrow \mathcal{C}^8$,

such that $\mathbf{E} \in L^q(\Omega \cap B_{R_0})$ and $\mathbf{E} \in L^6(\mathbb{R}^3 \setminus B_{R_0})$. Here $B_{R_0} \stackrel{\text{def}}{=} \{|x| < R_0\}$ and $R_0 > 0$, such that $G \subset B_{R_0/2}$ and $\mathbb{R}^3 \setminus \Omega \subset B_{R_0/2}$.

Y_1^q is defined as the space of all $\mathbf{E} \in Y_0^q$ with the property that $\text{div } \mathbf{E} = 0$ on Ω_0 ,

$\text{curl } \mathbf{E} \in L^2(\Omega)$ and $\varphi \mathbf{E} \in H_{curl}^0(\Omega)$ for all $\varphi \in C_0^\infty(\mathbb{R}^3)$.

The following norms are introduced.

$$\|\mathbf{E}\|_{Y_0^q} \stackrel{\text{def}}{=} \|\mathbf{E}\|_{L^6(\mathbb{R}^3 \setminus B_{R_0})} + \|\mathbf{E}\|_{L^q(\Omega \cap B_{R_0})},$$

$$\|\mathbf{E}\|_{Y_1^q} \stackrel{\text{def}}{=} \|\text{curl } \mathbf{E}\|_{L^2(\Omega)} + \|\mathbf{E}\|_{L^q(G)} + \sum_{k=1}^N |q_k(\mathbf{E})|$$

$$\text{and } \|\mathbf{E}\|_{Y_1^q} \stackrel{\text{def}}{=} \|\mathbf{E}\|_{Y_1^q} + \|\mathbf{E}\|_{Y_0^q}.$$

Next, let $\tilde{\mathcal{X}}$ be the space of all $\mathbf{h} \in H_{curl}^2(\Omega)$ with $\text{curl } \mathbf{h} \in L^{r_0^*}(\Omega_0) \cap L_{\gamma-1}^2(G)$, such that the support of $\text{curl } \mathbf{h}$ is bounded.

Moreover, let \mathcal{X} be the space of all $\mathbf{E} \in Y_1^{r_0} \cap L_\gamma^2(G)$, such that

$$\int_{\Omega} \mathbf{E} \text{curl } \mathbf{h} - \mathbf{h} \text{curl } \mathbf{E} dx = 0 \text{ for all } \mathbf{h} \in \tilde{\mathcal{X}}. \quad (2.26)$$

Let \mathcal{W} be the space of all $T \in \mathcal{D}'((0, \infty) \times \mathbb{R}^3) \cap L_{loc}^2([0, \infty), L_\gamma^2(G))$, such that $T = \partial_t \mathbf{A}$ with some $\mathbf{A} \in L_{loc}^\infty([0, \infty), \mathcal{X}) \cap W_{loc}^{1,2}([0, \infty), L_\gamma^2(G))$.

For $\mathbf{A} \in L_{loc}^\infty([0, \infty), \mathcal{X}) \cap W_{loc}^{1,2}([0, \infty), L_\gamma^2(G))$ and $T \stackrel{\text{def}}{=} \partial_t \mathbf{A} \in \mathcal{W}$ we define

$$\tilde{q}_k(T) \in \mathcal{D}'((0, \infty)) \text{ by } \tilde{q}_k(T) \stackrel{\text{def}}{=} \frac{d}{dt} q_k(\mathbf{A}(t)).$$

Next, the notion of weak solutions to 1.8 - 1.13 is given precisely.

Definition 1 Let $\mathbf{j}_0 \in L_{loc}^2([0, \infty), L^2(\Omega))$, $\mathbf{h}_0 \in L^2(\Omega)$ and $Q_k \in \mathbb{R}$.

Then $(\mathbf{E}, \mathbf{h}) \in \mathcal{W} \times C([0, \infty), L^2(\Omega))$ is called a solution to 1.8 - 1.13, if

$$\text{curl } \mathbf{h} = \sigma(t, x, \mathbf{E}) + \mathbf{j}_0, \quad \text{curl } \mathbf{E} = -\mu \partial_t \mathbf{h}, \quad (2.27)$$

$$\tilde{q}_k(\mathbf{E}) = Q_k \text{ for all } k \in \{1, \dots, N\}, \quad (2.28)$$

$$\text{and } \mathbf{h}(0) = \mathbf{h}_0. \quad (2.29)$$

Note that by the definition of \mathcal{W} it follows $\mathbf{E}|_{(0, \infty) \times G} \in L_{loc}^2([0, \infty), L_\gamma^2(G))$. Therefore $\sigma(t, x, \mathbf{E}) \in L_{loc}^2([0, \infty), L_{\gamma-1}^2(G))$.

3 Some auxiliary lemmata

The following lemma will be used frequently.

Lemma 1 Let $q \in (1, 3)$ and $\mathbf{E} \in \mathcal{D}'(\mathbb{R}^3)$, such that

$$\lim_{|x| \rightarrow \infty} (T * \varphi)(x) = 0 \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^3),$$

where $*$ denotes the convolution.

Moreover suppose that $\text{curl } \mathbf{E} \in L^q(\mathbb{R}^3)$ and $\text{div } \mathbf{E} \in L^q(\mathbb{R}^3)$. Then $\mathbf{E} \in L^p(\mathbb{R}^3)$ with $p = \frac{3q}{3-q}$ and

$$\|\mathbf{E}\|_{L^p} \leq C_q (\|\text{curl } \mathbf{E}\|_{L^q(\mathbb{R}^3)} + \|\text{div } \mathbf{E}\|_{L^q(\mathbb{R}^3)})$$

with some konstant $C_q \in (0, \infty)$ independent of \mathbf{E} .

Proof:

Let $S(x) \stackrel{\text{def}}{=} \frac{1}{4\pi|x|}$ be the fundamental solution of $-\Delta$ on \mathbb{R}^3 . Then

$$\nabla S = \frac{-x}{4\pi|x|^3} \in L_w^{3/2}(\mathbb{R}^3) \quad (3.30)$$

in the sense of distributions, where $L_w^p(\mathbb{R}^3)$ denotes the weak L^p -space or Marcinciewicz space, see [10].

It follows from 3.30, $\text{div } \mathbf{E} \in L^q(\mathbb{R}^3)$ and the generalized Young inequality [10] that

$$\mathbf{F}_1 \stackrel{\text{def}}{=} (\nabla S) * \text{div } \mathbf{E} \in L^p(\mathbb{R}^3) \quad (3.31)$$

For all $\mathbf{g} \in C_0^\infty(\mathbb{R}^3, \mathcal{C}^8)$ one has

$$\begin{aligned} \langle \Delta \mathbf{F}_1, \mathbf{g} \rangle_{\mathcal{D}'} &= \int_{\mathbb{R}^3} (\text{div } \mathbf{E})(y) \int_{\mathbb{R}^3} (\nabla S)(x-y) (\Delta \mathbf{g})(x) dx dy \\ &= - \int_{\mathbb{R}^3} (\text{div } \mathbf{E})(y) \int_{\mathbb{R}^3} S(x-y) (\Delta \text{div } \mathbf{g})(x) dx dy \\ &= \int_{\mathbb{R}^3} (\text{div } \mathbf{E})(y) (\text{div } \mathbf{g})(y) dy = - \langle \nabla \text{div } \mathbf{E}, \mathbf{g} \rangle_{\mathcal{D}'}, \end{aligned}$$

i.e.

$$\Delta \mathbf{F}_1 = -\nabla \text{div } \mathbf{E} \text{ in } \mathcal{D}'(\mathbb{R}^3). \quad (3.32)$$

Let

$$\mathbf{F}_2(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} (\text{curl } \mathbf{E})(y) \wedge (\nabla S)(x-y) dy.$$

It follows from similar conclusions as before that $\mathbf{F}_2 \in L^p(\mathbb{R}^3)$ and

$$\Delta \mathbf{F}_2 = \text{curl curl } \mathbf{E} \text{ in } \mathcal{D}'(\mathbb{R}^3). \quad (3.33)$$

3.32 and 3.33 yield

$$\Delta(\mathbf{E} + \mathbf{F}_1 + \mathbf{F}_2) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3). \quad (3.34)$$

Now, let $\varphi \in C_0^\infty(\mathbb{R}^3)$. It follows from 3.34 that $(\mathbf{E} + \mathbf{F}_1 + \mathbf{F}_2) * \varphi$ is harmonic on \mathbb{R}^3 . By the assumption on \mathbf{E} it follows $\lim_{|x| \rightarrow \infty} (\mathbf{E} + \mathbf{F}_1 + \mathbf{F}_2) * \varphi = 0$, since $\mathbf{F}_k \in L^p(\mathbb{R}^3)$. By the maximum-principle (applied to each component) this implies $(\mathbf{E} + \mathbf{F}_1 + \mathbf{F}_2) * \varphi = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^3)$ and hence $-\mathbf{E} = \mathbf{F}_1 + \mathbf{F}_2 \in L^p(\mathbb{R}^3)$.

The norm-estimate follows from 3.30, the generalized Young- inequality and the definitions of $\mathbf{F}_1, \mathbf{F}_2 \in L^p(\mathbb{R}^3)$.

□

For our purposes it is essential that only the L^q -norms of $\operatorname{curl} \mathbf{E}$ and $\operatorname{div} \mathbf{E}$ occur and not any norm of \mathbf{E} itself.

Let $Z \stackrel{\text{def}}{=} \{\varphi \in L^6 \mid \nabla \varphi \in L^2(\mathbb{R}^3)\}$. It is a Hilbert-space endowed with the scalar-product $\langle \varphi, \psi \rangle_Z \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} \nabla \varphi \nabla \overline{\psi} dx$

Let Z_0 be the closed subspace consisting of all $\varphi \in Z$ with $\nabla \varphi = 0$ on \mathcal{C} , i.e. which are constant on each component \mathcal{C}_k .

Lemma 2 *Let $p \in [2, p_0]$, $\mathbf{F} \in L^p(\Omega_0) \cap L^2(\Omega_0)$ and $\psi_0 \in Z_0$ with*

$$\int_{\Omega_0} \nabla \psi_0 \nabla \psi dx = \int_{\Omega_0} \mathbf{F} \nabla \psi dx \text{ for all } \psi \in Z_0.$$

Then $\psi_0 \in W^{1,p}(B_{2R_0})$ and

$$\|\nabla \psi_0\|_{L^p(B_{2R_0})} + \sum_{k=1}^N |\psi_0|_{\mathcal{C}_k} \leq K_3 (\|\mathbf{F}\|_{L^p(\Omega_0)} + \|\mathbf{F}\|_{L^2(\Omega_0)})$$

with some $K_3 \in (0, \infty)$ independent of \mathbf{F} .

Proof:

Let $\alpha_k \stackrel{\text{def}}{=} \psi_0|_{\mathcal{C}_k}$. Then the trace theorem yields

$$\sum_{k=1}^N |\alpha_k| \leq c_1 \|\psi_0\|_Z \leq c_1 \|\mathbf{F}\|_{L^2} \quad (3.35)$$

Let $\varphi_0 \stackrel{\text{def}}{=} \eta \cdot (\psi_0 - \sum_{k=1}^N \alpha_k \chi_k) \in H^1_0(\Omega_0 \cap B_{3R_0})$, where χ_k as in the definition of q_k and $\eta \in C_0^\infty(B_{3R_0})$ with $\eta(x) = 1$ on B_{2R_0} .

Now, for all $\varphi \in H^1_0(\Omega_0 \cap B_{3R_0})$ one has by 3.35 and the embedding $W^{1,6/5}(\Omega_0 \cap B_{3R_0}) \hookrightarrow L^2(\Omega_0 \cap B_{3R_0})$

$$\begin{aligned} & \left| \int_{\Omega_0 \cap B_{3R_0}} \nabla \varphi_0 \nabla \varphi dx \right| \leq \left| \int_{\Omega_0 \cap B_{3R_0}} \nabla (\eta \psi_0) \nabla \varphi dx \right| \\ & \quad + \sum_{k=1}^N |\alpha_k| \left| \int_{\Omega_0 \cap B_{3R_0}} \nabla (\eta \chi_k) \nabla \varphi dx \right| \\ & \leq \left| \int_{\Omega_0 \cap B_{3R_0}} \nabla \psi_0 \nabla (\eta \varphi) dx \right| + c_1 \int_{\Omega_0 \cap B_{3R_0}} (|\psi_0| + \sum_{k=1}^N |\alpha_k|) |\nabla \varphi| + |\nabla \psi_0| |\varphi| dx \\ & \leq \left| \int_{\Omega_0 \cap B_{3R_0}} \mathbf{F} \nabla (\eta \varphi) dx \right| + c_2 (\|\psi_0\|_{L^6} + \|\nabla \psi_0\|_{L^2} + \sum_{k=1}^N |\alpha_k|) \|\varphi\|_{W^{1,6/5}} \\ & \leq c_3 \|\mathbf{F}\|_{L^p} \|\eta \varphi\|_{W^{1,p^*}} + c_3 \|\psi_0\|_Z \|\varphi\|_{W^{1,6/5}} \end{aligned}$$

$$\begin{aligned} &\leq c_4 \|\mathbf{F}\|_{L^p} \|\varphi\|_{W^{1,p^*}} + c_3 \|\mathbf{F}\|_{L^2} \|\varphi\|_{W^{1,6/5}} \\ &\leq c_5 (\|\mathbf{F}\|_{L^p} + \|\mathbf{F}\|_{L^2}) \|\varphi\|_{W^{1,p^*}}, \end{aligned}$$

i.e. $\Delta\varphi_0 \in W^{-1,p}(\Omega_0 \cap B_{3R_0})$. Since $p \leq p_0$, it follows $\varphi_0 \in W^{1,p}(\Omega_0 \cap B_{3R_0})$ by 2.21.

□

In the sequel let $\mathcal{U} \subset L^2(\Omega_0)$ be the space of all $\mathbf{E} \in L^2(\Omega_0)$ with $\int_{\Omega_0} \mathbf{E} \nabla \psi dx = 0$ for all $\psi \in Z_0$. This is the weak formulation of $\operatorname{div} \mathbf{E} = 0$ on Ω_0 and $\int_{\partial \mathcal{C}_k} \bar{\mathbf{n}} \mathbf{E} dS = 0$ for all $k = 1, \dots, N$.

Lemma 3 *i) Let $\mathbf{E} \in L^2(\mathbb{R}^3)$ with $\operatorname{div} \mathbf{E} = 0$ on \mathbb{R}^3 . Then there exist a unique $\mathbf{h} \in L^6(\mathbb{R}^3)$ with $\operatorname{curl} \mathbf{h} = \mathbf{E}$ and $\operatorname{div} \mathbf{h} = 0$ on \mathbb{R}^3 in the sense of distributions.*

ii) There exists a bounded operator $\mathcal{T} : \mathcal{U} \rightarrow L^6(\mathbb{R}^3, \mathcal{C}^8)$ such that $\operatorname{curl}(\mathcal{T}\mathbf{E}) = \mathbf{E}$ on Ω_0 for all $\mathbf{E} \in \mathcal{U}$.

Proof:

i) Let H be the space of all $\mathbf{h} \in L^6(\mathbb{R}^3)$ with $\operatorname{div} \mathbf{h} = 0$ on \mathbb{R}^3 and $\operatorname{curl} \mathbf{h} \in L^2(\mathbb{R}^3)$. It follows easily from lemma 1 that this space is a Hilbert-space endowed with the scalar-product

$$\langle \mathbf{g}, \mathbf{h} \rangle_H \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} \operatorname{curl} \mathbf{g} \operatorname{curl} \bar{\mathbf{h}} dx.$$

Now, let $\mathbf{E} \in L^2(\mathbb{R}^3)$ with $\operatorname{div} \mathbf{E} = 0$ on \mathbb{R}^3 . Then there exists a unique $\mathbf{h} \in H$ with

$$\int_{\mathbb{R}^3} \operatorname{curl} \mathbf{h} \operatorname{curl} \mathbf{g} dx = \langle \mathbf{h}, \bar{\mathbf{g}} \rangle_H = \int_{\mathbb{R}^3} \mathbf{E} \operatorname{curl} \mathbf{g} dx \text{ for all } \mathbf{g} \in H.$$

Then $\mathbf{F} \stackrel{\text{def}}{=} \mathbf{E} - \operatorname{curl} \mathbf{h} \in L^2$ satisfies $\operatorname{curl} \mathbf{F} \stackrel{\text{def}}{=} 0$ and $\operatorname{div} \mathbf{F} = 0$. Hence, lemma 1 yields $\mathbf{F} = 0$, i.e. $\mathbf{E} = \operatorname{curl} \mathbf{h}$. Uniqueness of $\mathbf{h} \in L^6$ follows again from lemma 1.

Proof of ii):

The idea is to extend \mathbf{E} to a divergence-free field on \mathbb{R}^3 and to apply the first part of the lemma. Let $E : H^1(\mathcal{C}) \rightarrow H^1(\mathbb{R}^3)$ be an extension operator, see [12]. Since the \mathcal{C}_k are connected, a norm equivalent to the H^1 -norm can be defined by $\|f\|^2 \stackrel{\text{def}}{=} \int_{\mathcal{C}} |\nabla f|^2 dx$ on the space $X \stackrel{\text{def}}{=} \{f \in H^1(\mathcal{C}) : \int_{\mathcal{C}_k} f dx = 0\}$.

Let $\mathbf{E} \in \mathcal{U}$. Then there exists a unique $f_0 \in X$ with

$$\int_{\mathcal{C}} \nabla f_0 \nabla f dx = \langle f_0, \bar{f} \rangle_X = \int_{\Omega_0} \mathbf{E} \nabla(Ef) dx \text{ for all } f \in X. \quad (3.36)$$

Next, define $F\mathbf{E} \in L^2(\mathbb{R}^3, \mathcal{C}^8)$ by

$$(F\mathbf{E})(x) \stackrel{\text{def}}{=} \mathbf{E}(x) \text{ if } x \in \Omega_0, \text{ and } (F\mathbf{E})(x) \stackrel{\text{def}}{=} -\nabla f_0(x) \text{ if } x \in \bar{\mathcal{C}} = \mathbb{R}^3 \setminus \Omega_0 \quad (3.37)$$

Now, assume $\varphi \in C_0^\infty(\mathbb{R}^3)$ and set $\alpha_k \stackrel{\text{def}}{=} |\mathcal{C}_k|^{-1} \int_{\mathcal{C}_k} \varphi dx$, where $|\mathcal{C}_k|$ denotes the Lebesgue measure of \mathcal{C}_k .

Let $\psi \in X \subset H^1(\mathcal{C})$ be defined by $\psi(x) \stackrel{\text{def}}{=} \varphi(x) - \alpha_k$ for $x \in \mathcal{C}_k$.

Moreover, define $\tilde{\psi} \in Z_0$ by $\tilde{\psi}(x) \stackrel{\text{def}}{=} \varphi(x) - (E\psi)(x)$ if $x \in \Omega_0$ and $\tilde{\psi}(x) \stackrel{\text{def}}{=} \alpha_k$ for $x \in \mathcal{C}_k$. Then 3.36 and 3.37 yield

$$\begin{aligned} \int_{\mathbb{R}^3} (F\mathbf{E})\nabla\varphi dx &= \int_{\Omega_0} \mathbf{E}\nabla\varphi dx - \int_{\mathcal{C}} \nabla f_0 \nabla\psi dx \\ &= \int_{\Omega_0} \mathbf{E}\nabla[\varphi - E\psi] dx = \int_{\Omega_0} \mathbf{E}\nabla\tilde{\psi} dx = 0, \end{aligned}$$

since $\mathbf{E} \in \mathcal{U}$. Therefore,

$$\int_{\mathbb{R}^3} (F\mathbf{E})\nabla\varphi dx = 0 \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^3),$$

i.e. $\text{div}(F\mathbf{E}) = 0$ on \mathbb{R}^3 and by i) there exist a unique $\mathbf{h} \in L^6(\mathbb{R}^3)$ with $\text{curl } \mathbf{h} = F\mathbf{E}$ and $\text{div } \mathbf{h} = 0$ on \mathbb{R}^3 .

Finally, define $\mathcal{T}\mathbf{E} \stackrel{\text{def}}{=} \mathbf{h}$. Then $\text{curl}(\mathcal{T}\mathbf{E}) = F\mathbf{E}$ on \mathbb{R}^3 and in particular $\text{curl}(\mathcal{T}\mathbf{E}) = \mathbf{E}$ on Ω_0 .

□

Now, the main estimate can be proved.

Theorem 1 *For $r \in [r_0, 2]$, i. e. $r^* \leq p_0$ there exists a constant $K_r \in \mathbb{R}^+$, such that for all $\mathbf{E} \in Y_1^r$ the estimate*

$$\begin{aligned} \|\mathbf{E}\|_{Y_0^r} &= \|\mathbf{E}\|_{L^6(\mathbb{R}^3 \setminus B_{R_0})} + \|\mathbf{E}\|_{L^r(\Omega \cap B_{R_0})}, \\ &\leq K_r \|\mathbf{E}\|_{Y_1^r} = K_r (\|\text{curl } \mathbf{E}\|_{L^2(\Omega)} + \|\mathbf{E}\|_{L^r(G)} + \sum_{k=1}^N |q_k(\mathbf{E})|) \end{aligned}$$

holds. In particular $\|\cdot\|_{Y_1^r}$ and $\|\cdot\|_{Y_1^r}$ are equivalent norms on Y_1^r .

Proof:

Suppose $\mathbf{E} \in Y_1^r$. Let $\tilde{\mathbf{E}} \in L^r(\mathbb{R}^3) + L^6(\mathbb{R}^3)$ be defined by $\tilde{\mathbf{E}}(x) \stackrel{\text{def}}{=} \mathbf{E}(x)$ if $x \in \Omega$ and $\tilde{\mathbf{E}}(x) \stackrel{\text{def}}{=} 0$ if $x \in \mathbb{R}^3 \setminus \Omega$. Since $\text{curl } \mathbf{E} \in L^2(\Omega)$ and $\varphi\mathbf{E} \in \overset{0}{H}{}^r_{\text{curl}}(\Omega)$ for all $\varphi \in C_0^\infty(\mathbb{R}^3)$, one has $\text{curl } \tilde{\mathbf{E}} \in L^2(\mathbb{R}^3)$. By lemma 3 there exists a unique $\mathbf{E}_1 \in L^6(\mathbb{R}^3)$ with

$$\text{curl } \mathbf{E}_1 = \text{curl } \tilde{\mathbf{E}}, \text{div } \mathbf{E}_1 = 0 \text{ on } \mathbb{R}^3 \tag{3.38}$$

and $\|\mathbf{E}_1\|_{L^6} \leq K_1 \|\text{curl } \mathbf{E}\|_{L^2} \leq K_1 \|\mathbf{E}\|_{Y_1^r}$.

Now, $\mathbf{E}_2 \stackrel{\text{def}}{=} \tilde{\mathbf{E}} - \mathbf{E}_1 \in L^r(\mathbb{R}^3) + L^6(\mathbb{R}^3)$ satisfies

$$\text{curl } \mathbf{E}_2 = 0 \text{ on } \mathbb{R}^3 \text{ and } \text{div } \mathbf{E}_2 = 0 \text{ on } \Omega_0. \tag{3.39}$$

Let $\chi \in C^\infty(\mathbb{R}^3)$ with

$$\chi(x) = 0 \text{ on } B_{R_0} \text{ and } \chi(x) = 1 \text{ outside } B_{2R_0}. \quad (3.40)$$

Since $\text{supp } \chi \subset \Omega_0$ and $r \geq 6/5$, 3.39 yields $\text{div}(\chi \mathbf{E}_2) = \mathbf{E}_2 \nabla \chi \in L^{6/5}(\mathbb{R}^3)$ and $\text{curl}(\chi \mathbf{E}_2) = -\mathbf{E}_2 \wedge \nabla \chi \in L^{6/5}(\mathbb{R}^3)$. Thus, lemma 1 yield $\chi \mathbf{E}_2 \in L^2(\mathbb{R}^3)$ and therefore

$$\mathbf{E}_2 \in L^2(\mathbb{R}^3 \setminus B_{R_0}). \quad (3.41)$$

Next, define $\mathbf{F} \in L^{r^*}(\Omega_0) \cap L^2(\Omega_0)$ by $\mathbf{F}(x) \stackrel{\text{def}}{=} |\mathbf{E}_2(x)|^{r-2} \mathbf{E}_2(x)$ if $x \in \Omega_0 \cap B_{2R_0}$ and $\mathbf{F}(x) \stackrel{\text{def}}{=} 0$ if $x \in \mathbb{R}^3 \setminus B_{2R_0}$. Then there exists some $\psi_0 \in Z_0$, such that

$$\int_{\Omega_0} \nabla \psi_0 \nabla \psi dx = \langle \psi_0, \bar{\psi} \rangle_{Z_0} = \int_{\Omega_0} \mathbf{F} \nabla \psi dx \quad (3.42)$$

for all $\psi \in Z_0$, since $\mathbf{F} \in L^2(\Omega_0)$.

Recall that Z_0 be the space of all $\varphi \in L^6$ with $\nabla \varphi \in L^2$ and $\nabla \varphi = 0$ on \mathcal{C} , i.e. φ is constant on each component \mathcal{C}_k . It follows from lemma 2 that $\nabla \psi_0 \in W^{1,r^*}(B_{2R_0})$ with

$$\|\nabla \psi_0\|_{L^{r^*}(B_{2R_0})} + \sum_{k=1}^N |\psi_0|_{\mathcal{C}_k}| \leq c_3 \|\mathbf{F}\|_{L^{r^*}(\Omega_0)} = c_3 \|\mathbf{E}_2\|_{L^r(\Omega_0 \cap B_{2R_0})}^{r-1}. \quad (3.43)$$

Let $\varphi(x) \stackrel{\text{def}}{=} \psi_0(x) - \sum_{k=1}^N \alpha_k \chi_k(x)$, where $\alpha_k \stackrel{\text{def}}{=} \psi_0|_{\mathcal{C}_k}$ and χ_k as in the definition of q_k . Moreover, let $h_n(x) \stackrel{\text{def}}{=} h(x/n)$, $x \in \mathbb{R}^3$, where $h \in C_0^\infty(\mathbb{R}^3)$, such that $h(x) = 1$ on a neighbourhood of B_1 . By 3.40 and 3.43 one has $(1 - \chi)h_n \varphi \in W^{1,r^*}_0(\Omega_0 \cap B_{2R_0})$. Hence, 3.39 yields

$$\int_{\Omega_0} \mathbf{E}_2 \nabla [(1 - \chi)h_n \varphi] dx = 0 \quad (3.44)$$

Moreover, by 3.40 one has $\chi h_n \varphi \in W^{1,2}_0(B_{2n} \setminus B_{R_0})$.

Therefore, it follows from 3.39 and $\mathbf{E}_2 \in L^2(B_{2n} \setminus B_{R_0})$ that $\int_{\Omega_0} \mathbf{E}_2 \nabla [\chi h_n \varphi] dx = 0$ and hence according to 3.44

$$\int_{\Omega_0} \mathbf{E}_2 \nabla [h_n \varphi] dx = 0 \text{ for all } n \in \mathbb{N} \quad (3.45)$$

By 3.41 $\mathbf{E}_2 \nabla \psi_0$ is integrable on Ω_0 , and hence

$$\int_{\Omega_0} \mathbf{E}_2 \nabla \psi_0 dx = \lim_{n \rightarrow \infty} \left(\int_{\Omega_0} \mathbf{E}_2 \nabla [h_n \psi_0] dx + \beta_n \right),$$

where

$$|\beta_n| \leq \left| \int_{\Omega} \psi_0 \mathbf{E}_2 \nabla h_n dx \right| \leq K/n \int_{\{|x|>n\}} |\mathbf{E}_2| |\psi_0| dx$$

$$\begin{aligned} &\leq K/n \|\mathbf{E}_2\|_{L^2(\{|x|>n\})} \|\psi_0\|_{L^6(\{|x|>n\})} |B_{2n}|^{1/3} \leq C \|\mathbf{E}_2\|_{L^2(\{|x|>n\})} \|\psi_0\|_{L^6(\{|x|>n\})} \\ &\xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

by Hölder's inequality, since $\mathbf{E}_2 \in L^2(\{|x| > R_0\})$ and $\psi_0 \in L^6(\{|x| > R_0\})$. By 3.45 it follows

$$\begin{aligned} \int_{\Omega_0} \mathbf{E}_2 \nabla \psi_0 dx &= \lim_{n \rightarrow \infty} \int_{\Omega_0} \mathbf{E}_2 \nabla [h_n \psi_0] dx \\ &= \lim_{n \rightarrow \infty} \int_{\Omega_0} \mathbf{E}_2 \nabla [h_n (\varphi + \sum_{k=1}^N \alpha_k \chi_k)] dx = \sum_{k=1}^N \alpha_k q_k(\mathbf{E}_2). \end{aligned} \quad (3.46)$$

Now, 3.38, 3.43 and 3.46 yield

$$\begin{aligned} \left| \int_{\Omega_0} \mathbf{E}_2 \nabla \psi_0 dx \right| &\leq \sum_{k=1}^N |\psi_0|_{c_k} q_k(\mathbf{E}_2) \\ &\leq c_3 \|\mathbf{E}_2\|_{L^r(\Omega_0 \cap B_{2R_0})}^{r-1} (\|\mathbf{E}_1\|_{L^6} + \sum_{k=1}^N |q_k(\mathbf{E})|) \leq c_4 \|\mathbf{E}_2\|_{L^r(\Omega_0 \cap B_{2R_0})}^{r-1} |\mathbf{E}|_{Y_1^r}. \end{aligned} \quad (3.47)$$

By 3.42 one has $\mathbf{F} - \nabla \psi_0 \in \mathcal{U}$ and lemma 3 yields

$$\mathbf{F} - \nabla \psi_0 = \operatorname{curl} \mathbf{h} \text{ where } \mathbf{h} \stackrel{\text{def}}{=} \mathcal{T}(\mathbf{F} - \nabla \psi_0) \in L^6(\mathbb{R}^3) \quad (3.48)$$

with

$$\|\mathbf{h}\|_{L^6} \leq c_4 \|\mathbf{E}_2\|_{L^r(\Omega_0 \cap B_{2R_0})}^{r-1}.$$

Let $A : H_{\operatorname{curl}}^r(\mathcal{C}) \rightarrow H_{\operatorname{curl}}^r(\mathbb{R}^3)$ be an extension operator constructed in the appendix with the properties

$$(A\mathbf{w})(x) = \mathbf{w}(x) \text{ for all } \mathbf{w} \in H_{\operatorname{curl}}^r(\mathcal{C}), x \in \mathcal{C}, \quad (3.49)$$

and $\|A\mathbf{w}\|_{H_{\operatorname{curl}}^r(\mathbb{R}^3)} \leq C_r \|\mathbf{w}\|_{H_{\operatorname{curl}}^r(\mathcal{C})}$ and $\operatorname{supp} (A\mathbf{w}) \subset B_{R_0}$ for all $\mathbf{w} \in H_{\operatorname{curl}}^r(\mathcal{C})$. Let

$$\mathbf{E}_3 \stackrel{\text{def}}{=} A(\mathbf{E}_2|_{\mathcal{C}}) \in H_{\operatorname{curl}}^r(\mathbb{R}^3).$$

Then $\operatorname{supp} \mathbf{E}_3 \subset B_{R_0}$ and 3.39 yields the estimate

$$\begin{aligned} \|\mathbf{E}_3\|_{H_{\operatorname{curl}}^r} &\leq C_r \|\mathbf{E}_2\|_{H_{\operatorname{curl}}^r(\mathcal{C})} = C_r \|\mathbf{E}_2\|_{L^r(\mathcal{C})} \\ &\leq c_5 \left(\|\tilde{\mathbf{E}}\|_{L^r(\mathcal{C})} + \|\mathbf{E}_1\|_{L^6(B_{R_0})} \right) \\ &\leq c_6 \left(\|\mathbf{E}\|_{L^r(G)} + \|\operatorname{curl} \mathbf{E}\|_{L^2(\Omega)} \right) \leq c_7 |\mathbf{E}|_{Y_1^r}. \end{aligned} \quad (3.50)$$

By 3.39, 3.40 and 3.49 one has $(1 - \chi)h_n(\mathbf{E}_2 - \mathbf{E}_3) \in H_{curl}^0(\Omega_0 \cap B_{2R_0})$.
 Moreover, $\mathbf{h} \in H_{curl}^{r^*}(\Omega_0 \cap B_{2R_0})$ by 3.48, since $\mathbf{F} - \nabla\psi_0 \in L^{r^*}(\Omega_0 \cap B_{2R_0})$ and $r^* \leq 6/5$.
 Hence,

$$\int_{\Omega_0} (1 - \chi)h_n[\mathbf{E}_2 - \mathbf{E}_3] \operatorname{curl} \mathbf{h} dx = \int_{\Omega_0} \mathbf{h} \operatorname{curl} ((1 - \chi)h_n[\mathbf{E}_2 - \mathbf{E}_3]) dx \quad (3.51)$$

By 3.40 one has $\chi h_n[\mathbf{E}_2 - \mathbf{E}_3] \in H_{curl}^2(B_{2n} \setminus B_{R_0})$.

Therefore

$$\int_{\Omega_0} \chi h_n[\mathbf{E}_2 - \mathbf{E}_3] \operatorname{curl} \mathbf{h} dx = \int_{\Omega_0} \mathbf{h} \operatorname{curl} (\chi h_n[\mathbf{E}_2 - \mathbf{E}_3]) dx$$

and hence according to 3.39, 3.48 and 3.51

$$\begin{aligned} \int_{\Omega_0} h_n[\mathbf{E}_2 - \mathbf{E}_3](\mathbf{F} - \nabla\psi_0) dx &= \int_{\Omega_0} h_n[\mathbf{E}_2 - \mathbf{E}_3] \operatorname{curl} \mathbf{h} dx \\ &= \int_{\Omega_0} \mathbf{h} \operatorname{curl} (h_n[\mathbf{E}_2 - \mathbf{E}_3]) dx = \gamma_n - \int_{\Omega_0} h_n \mathbf{h} \operatorname{curl} \mathbf{E}_3 dx. \end{aligned} \quad (3.52)$$

Here

$$\begin{aligned} |\gamma_n| &\leq \int_{\Omega} |\mathbf{h}| |\mathbf{E}_2 - \mathbf{E}_3| |\nabla h_n| dx = \int_{\Omega} |\mathbf{h}| |\mathbf{E}_2| |\nabla h_n| dx \leq K/n \int_{\{|x|>n\}} |\mathbf{E}_2| |\mathbf{h}| dx \\ &\leq K/n \|\mathbf{E}_2\|_{L^2(\{|x|>n\})} \|\mathbf{h}\|_{L^6(\{|x|>n\})} |B_{2n}|^{1/3} \leq C \|\mathbf{E}_2\|_{L^2(\{|x|>n\})} \|\mathbf{h}\|_{L^6(\{|x|>n\})} \\ &\xrightarrow{n \rightarrow \infty} 0, \end{aligned}$$

since $\mathbf{E}_2 \in L^2(\{|x| > R_0\})$ by 3.41 and $\mathbf{h} \in L^6(\{|x| > R_0\})$. It follows now from 3.43, 3.50, the estimate in 3.48 and 3.52 by letting $n \rightarrow \infty$ that

$$\begin{aligned} \left| \int_{\Omega_0} \mathbf{E}_2(\mathbf{F} - \nabla\psi_0) dx \right| &= \left| \int_{\Omega_0} \mathbf{E}_3(\mathbf{F} - \nabla\psi_0) dx - \int_{\Omega_0} \mathbf{h} \operatorname{curl} \mathbf{E}_3 dx \right| \\ &\leq \|\mathbf{E}_3\|_{H_{curl}^r} (\|\mathbf{F} - \nabla\psi_0\|_{L^{r^*}(\Omega_0 \cap B_{2R_0})} + \|\mathbf{h}\|_{L^{r^*}(B_{2R_0})}) \\ &\leq c_8 |\mathbf{E}|_{Y_1^r} \|\mathbf{E}_2\|_{L^r(\Omega_0 \cap B_{2R_0})}^{r-1}. \end{aligned} \quad (3.53)$$

Next, 3.47 and 3.53 yield

$$\|\mathbf{E}_2\|_{L^r(\Omega_0 \cap B_{2R_0})}^r = \int_{\Omega_0} \mathbf{E}_2 \mathbf{F} dx \leq c_9 \|\mathbf{E}_2\|_{L^r(\Omega_0 \cap B_{2R_0})}^{r-1} |\mathbf{E}|_{Y_1^r}.$$

and hence

$$\|\mathbf{E}_2\|_{L^r(\Omega_0 \cap B_{2R_0})} \leq c_9 |\mathbf{E}|_{Y_1^r}.$$

Since $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$ on Ω , this yields together with 3.38

$$\|\mathbf{E}\|_{L^r(\Omega_0 \cap B_{2R_0})} \leq C|\mathbf{E}|_{Y_1^r}. \quad (3.54)$$

with some constant C independent of \mathbf{E} .

Now, choose $\chi_a \in C_0^\infty(\Omega_0 \cap B_{2R_0})$ and $\chi_b \in C^\infty(\mathbb{R}^3)$ with $\chi_a(x) = 1$ on $\text{supp } \nabla \chi_b$ and $\chi_b(x) = 1$ for $|x| > R_0$. Then $\chi_a \mathbf{E} \in L^r(\mathbb{R}^3)$ (extended by zero outside Ω) satisfies $\text{curl}(\chi_a \mathbf{E}) \in L^r(\mathbb{R}^3)$ and $\text{div}(\chi_a \mathbf{E}) \in L^r(\mathbb{R}^3)$, since $r \leq 2$, $\text{curl } \mathbf{E} \in L^2(\Omega)$ and $\text{div } \mathbf{E} = 0$ on Ω_0 . By 3.54 it follows

$$\|\text{curl}(\chi_a \mathbf{E})\|_{L^r} + \|\text{div}(\chi_a \mathbf{E})\|_{L^r} \leq C_1(\|\text{curl } \mathbf{E}\|_{L^2} + |\mathbf{E}|_{Y_1^r}) \leq (C_1 + 1)|\mathbf{E}|_{Y_1^r}.$$

Since $r \geq 6/5$ and χ_a has bounded support, lemma 1 yields $\chi_a \mathbf{E} \in L^2(\mathbb{R}^3)$ and

$$\|\chi_a \mathbf{E}\|_{L^2} \leq C_2|\mathbf{E}|_{Y_1^r}. \quad (3.55)$$

Since $\chi_a(x) = 1$ on $\text{supp } \nabla \chi_b$, $\text{curl } \mathbf{E} \in L^2(\Omega)$ and $\text{div } \mathbf{E} = 0$ on Ω_0 . it follows from 3.54 that $\text{curl}(\chi_b \mathbf{E}) \in L^2(\mathbb{R}^3)$ and $\text{div}(\chi_b \mathbf{E}) \in L^2(\mathbb{R}^3)$,

$$\|\text{curl}(\chi_b \mathbf{E})\|_{L^2} + \|\text{div}(\chi_b \mathbf{E})\|_{L^2} \leq C_3(\|\text{curl } \mathbf{E}\|_{L^2} + |\mathbf{E}|_{Y_1^r}) \leq (C_3 + 1)|\mathbf{E}|_{Y_1^r}.$$

and thus $\chi_b \mathbf{E} \in L^6(\mathbb{R}^3)$ and

$$\|\chi_b \mathbf{E}\|_{L^6} \leq C_4|\mathbf{E}|_{Y_1^r} \quad (3.56)$$

again from lemma 1. Finally, the desired estimate follows immediately from 3.54 and 3.56.

□

By 2.23 and the previous theorem the space \mathcal{X} is complete with respect to the norm

$$\|\mathbf{E}\|_{\mathcal{X}} \stackrel{\text{def}}{=} \|\text{curl } \mathbf{E}\|_{L^2(\Omega)} + \|\mathbf{E}\|_G + \sum_{k=1}^N |q_k(\mathbf{E})| \text{ for } \mathbf{E} \in \mathcal{X}.$$

The estimate

$$\|\mathbf{E}\|_{L^6(\{|x| > R_0\})} + \|\mathbf{E}\|_{L^{r_0}(\Omega \cap B_{R_0})} \leq K\|\mathbf{E}\|_{\mathcal{X}} \text{ for all } \mathbf{E} \in \mathcal{X} \quad (3.57)$$

holds.

4 Uniqueness for the quasistationary ME

In this section uniqueness for problem 2.27 - 2.29 is shown. Let $\mathbf{h}_0 \in H_{curl}^2(\Omega)$ and $Q_k \in \mathbb{R}$. Suppose $(\mathbf{E}^{(1)}, \mathbf{h}^{(1)})$ and $(\mathbf{E}^{(2)}, \mathbf{h}^{(2)})$ are solutions to 2.27 - 2.29 in the sense of definition 1 with $\mathbf{E}^{(k)} \in \mathcal{W}$ and $\mathbf{h}^{(k)} \in C([0, \infty), L^2(\Omega))$ and $\mathbf{E}^{(k)} = \partial_t \mathbf{A}^{(k)}$, where $\mathbf{A}^{(k)} \in L_{loc}^\infty([0, \infty), \mathcal{X}) \cap W_{loc}^{1,2}([0, \infty), L_\gamma^2(G))$.

Let $\mathbf{A} \stackrel{\text{def}}{=} \mathbf{A}^{(1)} - \mathbf{A}^{(2)}$, $\mathbf{E} \stackrel{\text{def}}{=} \mathbf{E}^{(1)} - \mathbf{E}^{(2)}$, $\mathbf{h} \stackrel{\text{def}}{=} \mathbf{h}^{(1)} - \mathbf{h}^{(2)}$, and $\mathbf{J}(t) \stackrel{\text{def}}{=} [\sigma(t, x, \mathbf{E}^{(1)}(t)) - \sigma(t, x, \mathbf{E}^{(2)}(t))] \in L_{\gamma^{-1}}^2(G)$ by 2.24.

A regularizing procedure with respect to the time-variable is used.

Let $\omega_n \in C_0^\infty((-\infty, 0), n \in \mathbb{N})$ be an usual mollifier-sequence and define

$$\begin{aligned} \mathbf{A}_n(t) &\stackrel{\text{def}}{=} \int_{\mathbb{R}} \omega_n(t-s) \mathbf{A}(s) ds \in \mathcal{X}, \\ \text{and } \mathbf{h}_n(t) &\stackrel{\text{def}}{=} \int_0^\infty \omega_n(t-s) \mathbf{h}(s) ds \in L^2(\Omega). \end{aligned}$$

Since $\mathbf{h} \in C([0, \infty), L^2(\Omega)) = 0$ and $\mathbf{h}(0) = 0$ one has

$$\|\mathbf{h}_n(0)\|_{L^2} \xrightarrow{n \rightarrow \infty} 0. \quad (4.58)$$

For $t \geq 0$ one has $\text{supp } \omega_n(t - \cdot) \subset (0, \infty)$. Since $\partial_t \mathbf{A} = \mathbf{E}$ and $\partial_t(\mu \mathbf{h}) = -\text{curl } \mathbf{E}$ in $\mathcal{D}'((0, \infty) \times \Omega)$ it follows easily that

$$\mu \partial_t \mathbf{h}_n(t) = -\text{curl } \partial_t \mathbf{A}_n(t). \quad (4.59)$$

By 2.27 and 2.24 it follows that

$$\text{curl } \mathbf{h}_n(t) = \int_0^\infty \omega_n(t-s) \text{curl } \mathbf{h}(s) ds = \mathbf{J}_n(s) \stackrel{\text{def}}{=} \int_0^\infty \omega_n(t-s) \mathbf{J}(s) ds \quad (4.60)$$

$\in L_{\gamma^{-1}}^2(G)$ and $\text{curl } \mathbf{h}_n(t, x) = 0$ if $x \in \Omega_0 = \Omega \setminus \overline{G}$. In particular

$$\mathbf{h}_n(t) ds \in \tilde{\mathbf{X}}. \quad (4.61)$$

Now, 2.26 and 4.59 - 4.61 yield

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mu^{1/2} \mathbf{h}_n(t)\|_{L^2}^2 &= - \int_{\Omega} \mathbf{h}_n(t) \text{curl } \partial_t \mathbf{A}_n(t) dx \\ &= - \int_G \partial_t \mathbf{A}_n(t) \text{curl } \mathbf{h}_n(t) dx = - \int_G \partial_t \mathbf{A}_n(t) \mathbf{J}_n(t) dx. \end{aligned} \quad (4.62)$$

Since $\partial_t \mathbf{A}|_{[0, \infty) \times G} = \mathbf{E} \in L_{loc}^2([0, \infty), L_\gamma^2(G))$ and $\text{supp } \omega_n(t - \cdot) \subset (0, \infty)$ one has

$$\partial_t \mathbf{A}_n(t)|_G = \int_{\mathbb{R}} \omega_n(t-s) \mathbf{E}(s) ds \in L_\gamma^2(G) \quad (4.63)$$

and $\partial_t \mathbf{A}_n|_{(0,T) \times G} \xrightarrow{n \rightarrow \infty} \mathbf{E}$ in $L^2((0, T), L^2_\gamma(G))$

strongly for all $T > 0$. Moreover, it follows from 4.60 that

$$\mathbf{J}_n = \text{curl } \mathbf{h}_n \xrightarrow{n \rightarrow \infty} \mathbf{J} \text{ in } L^2((0, T), L^2_{\gamma^{-1}}(G)) \text{ strongly.} \quad (4.64)$$

Now, 4.58 and 4.62-4.63 yield

$$\frac{1}{2} \|\mu^{1/2} \mathbf{h}_n(T)\|_{L^2}^2 = - \int_0^T \int_G \partial_t \mathbf{A}_n(t) \text{curl } \mathbf{h}_n(t) dx dt \quad (4.65)$$

$$\begin{aligned} & \xrightarrow{n \rightarrow \infty} - \int_0^T \int_G \mathbf{E}(t) \mathbf{J}(t) dx dt \\ & = \int_0^T \int_G (\mathbf{E}^{(2)}(t) - \mathbf{E}^{(1)}(t)) [\sigma(t, x, \mathbf{E}^{(1)}(t)) - \sigma(t, x, \mathbf{E}^{(2)}(t))] dx dt \end{aligned}$$

for all $T > 0$. By the monotonicity of σ , 2.20, this implies

$$\mathbf{h}(t) = \lim_{n \rightarrow \infty} \|\mathbf{h}_n(t)\|_{L^2} = 0 \text{ for all } t \in (0, \infty) \quad (4.66)$$

and $\partial_t \mathbf{A}|_{[0, \infty) \times G} = \mathbf{E}|_{[0, \infty) \times G} = 0$ on $(0, \infty) \times G$.

Since $\mathbf{E} = \partial_t \mathbf{A}$ it follows

$$\partial_t \text{curl } \mathbf{A} = \text{curl } \mathbf{E} = \partial_t(\mu \mathbf{h}) = 0. \quad (4.67)$$

$$\text{Moreover, } \frac{d}{dt} q_k(\mathbf{A}(t)) = \tilde{q}(\mathbf{E}(t)) = \tilde{q}(\mathbf{E}^{(1)}(t)) - \tilde{q}(\mathbf{E}^{(2)}(t)) = 0. \quad (4.68)$$

Since $\mathbf{A} \in L^\infty_{loc}(\mathbb{R}, Y_1^{r_0})$, it follows from 4.66 - 4.68 and lemma 1 that \mathbf{A} is constant with respect to t and hence $\mathbf{E} = \partial_t \mathbf{A} = 0$. This completes the proof.

□

5 The semistatic limit for ME

In the sequel let $\mathbf{j}_0 \in L^2_{loc}(\mathbb{R}, L^2(\Omega))$, such that there exists some $\mathbf{g}_0 \in L^2_{loc}(\mathbb{R}, H^2_{curl}(\Omega)) \cap W^{1,1}_{loc}(\mathbb{R}, L^2(\Omega))$, with

$$\mathbf{j}_0(t) = \text{curl } \mathbf{g}_0(t), \quad (5.69)$$

in particular $\mathbf{j}_0(t)$ is assumed to be divergence-free .

Moreover let $(\mathbf{E}_0, \mathbf{h}_0) \in X_0 = L^2(\Omega)$ with

$$\mathbf{g}_0(0) - \mathbf{h}_0 \in \overline{\mathcal{X}_0}, \quad (5.70)$$

where the closure is taken in the $L^2(\Omega)$ -topology. Here $\tilde{\mathcal{X}}_0$ denotes the space of all $\mathbf{h} \in H_{curl}^2(\Omega)$ with $\text{curl } \mathbf{h} = 0$ on Ω_0 and $\text{curl } \mathbf{h} \in L_{\gamma^{-1}}^2(G)$. It is assumed that \mathbf{E}_0 satisfies

$$\text{div } \mathbf{E}_0 = 0 \text{ on } \Omega_0, \quad Q_k \stackrel{\text{def}}{=} q_k(\mathbf{E}_0) \quad (5.71)$$

Now, Maxwell's equations

$$\varepsilon \partial_t \mathbf{E}_\varepsilon = \text{curl } \mathbf{h}_\varepsilon - \mathbf{j}_0 - \sigma(t, x, \mathbf{E}_\varepsilon), \quad (5.72)$$

$$\mu \partial_t \mathbf{h}_\varepsilon = - \text{curl } \mathbf{E}_\varepsilon, \quad (5.73)$$

are considered, where $\varepsilon > 0$ is a parameter. 5.72, 5.73 is supplemented by the initial-boundary-conditions

$$\vec{n} \wedge \mathbf{E}_\varepsilon = 0 \text{ on } (0, \infty) \times \partial\Omega, \quad (5.74)$$

$$\mathbf{E}_\varepsilon(0, x) = \mathbf{E}_0(x), \mathbf{h}_\varepsilon(0, x) = \mathbf{h}_0(x). \quad (5.75)$$

For the weak formulation of 5.72 - 5.75 the operator B defined by

$$B(\mathbf{E}, \mathbf{h}) \stackrel{\text{def}}{=} (\text{curl } \mathbf{h}, -\mu^{-1} \text{curl } \mathbf{E})$$

for $(\mathbf{E}, \mathbf{h}) \in D(B) \stackrel{\text{def}}{=} H_{curl}^2(\Omega) \times H_{curl}^2(\Omega)$ is introduced. It turns out that B is a densely defined skew self-adjoint operator in the Hilbert-space $X_0 \stackrel{\text{def}}{=} L^2(\Omega, \mathcal{C}^6)$ endowed with the scalar-product

$$\langle (\mathbf{E}, \mathbf{h}), (\mathbf{F}, \mathbf{g}) \rangle_{X_0} \stackrel{\text{def}}{=} \int_\Omega (\mathbf{E}\bar{\mathbf{F}} + \mu \mathbf{h}\bar{\mathbf{g}}) dx .$$

Setting $\mathbf{w}_\varepsilon \stackrel{\text{def}}{=} (\varepsilon^{1/2} \mathbf{E}_\varepsilon, \mathbf{h}_\varepsilon - \mathbf{g}_0)$ 5.72-5.75 reads as

$$\partial_t \mathbf{w}_\varepsilon = \varepsilon^{-1/2} B \mathbf{w}_\varepsilon + \varepsilon^{-1/2} F_\sigma(t, \mathbf{E}_\varepsilon) + \mathbf{f}_0, \quad \mathbf{w}_\varepsilon(0) = \mathbf{w}_{\varepsilon,0} \stackrel{\text{def}}{=} (\varepsilon^{1/2} \mathbf{E}_0, \mathbf{h}_0 - \mathbf{g}_0(0)) \quad (5.76)$$

where $\mathbf{f}_0 \stackrel{\text{def}}{=}} -(0, \partial_t \mathbf{g}_0)$ and $F_\sigma : [0, \infty) \times L^2(\Omega, \mathbb{R}^3) \rightarrow X_0 = L^2(\Omega, \mathcal{C}^6)$ is defined by

$$(F_\sigma(t, \mathbf{E}))(t, x) \stackrel{\text{def}}{=}} -(\sigma(t, x, \mathbf{E}(x)), 0) \text{ for } \mathbf{E} \in L^2(\Omega).$$

Now, $\mathbf{w}_\varepsilon \in C([0, \infty), X_0)$ is called a weak solution to 5.76, if $\mathbf{w}_\varepsilon \stackrel{\text{def}}{=} (\varepsilon^{1/2} \mathbf{E}_\varepsilon, \mathbf{h}_\varepsilon)$ satisfies for all $\mathbf{u} \in D(B)$

$$\begin{aligned} \frac{d}{dt} \langle \mathbf{w}_\varepsilon(t), \mathbf{u} \rangle_{X_0} &= -\varepsilon^{-1/2} \langle \mathbf{w}_\varepsilon(t), B \mathbf{u} \rangle_{X_0} \\ &+ \langle \varepsilon^{-1/2} F_\sigma(t, \mathbf{E}_\varepsilon(t)) + \mathbf{f}_0(t), \mathbf{u} \rangle_{X_0} \end{aligned} \quad (5.77)$$

This is equivalent to the variation of constant formula

$$\begin{aligned} \mathbf{w}_\varepsilon(t) &= \exp(\varepsilon^{-1/2}tB)\mathbf{w}_{\varepsilon,0} \\ &+ \int_0^t \exp(\varepsilon^{-1/2}(t-s)B)[\varepsilon^{-1/2}F_\sigma(s, \mathbf{E}(s)) + \mathbf{f}_0(s)]ds, \end{aligned} \quad (5.78)$$

where $\exp(tB), t \in \mathbb{R}$ is the unitary group generated by B . Since F_σ is Lipschitz-continuous with respect to $\mathbf{E} \in L^2(\Omega)$, it follows from a standard result that this integral equation has a unique solution $\mathbf{w}_\varepsilon = (\varepsilon^{1/2}\mathbf{E}_\varepsilon, \mathbf{h}_\varepsilon) \in C([0, \infty), X_0)$, see [7], chapter 6.

This yields the energy balance

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{w}_\varepsilon(t)\|_{X_0}^2 &= \langle \varepsilon^{-1/2}F_\sigma(t, \mathbf{E}_\varepsilon(t)) + \mathbf{f}_0(t), \mathbf{w}_\varepsilon(t) \rangle_{X_0} \\ &= - \int_G \mathbf{E}_\varepsilon(t) \sigma(t, x, \mathbf{E}_\varepsilon(t)) dx + \int_\Omega \mu(\mathbf{g}_0(t) - \mathbf{h}_\varepsilon(t)) \partial_t \mathbf{g}_0(t) dx. \end{aligned} \quad (5.79)$$

The aim of the following considerations is the investigation of the limit $\varepsilon \rightarrow 0$. Let $\mathbf{A}_\varepsilon \in C(\mathbb{R}, L^2(\Omega))$ be defined by

$$\mathbf{A}_\varepsilon(t) \stackrel{\text{def}}{=} \int_0^t \mathbf{E}_\varepsilon(s) ds \text{ for } t \geq 0. \quad (5.80)$$

Lemma 4 *The family*

$$(\mathbf{w}_\varepsilon = (\varepsilon^{1/2}\mathbf{E}_\varepsilon, \mathbf{h}_\varepsilon - \mathbf{g}_0))_{\varepsilon>0} \text{ is bounded in } L^\infty((0, T), X_0), \quad (5.81)$$

$$(\mathbf{E}_\varepsilon)_{\varepsilon>0} = (\partial_t \mathbf{A}_\varepsilon)_{\varepsilon>0} \text{ is bounded in } L^2((0, T), L_\gamma^2(G)) \quad (5.82)$$

$$\text{and } (\mathbf{A}_\varepsilon)_{\varepsilon>0} \text{ is bounded in } L^\infty((0, T), \mathcal{X}) \subset L^\infty((0, T), Y_1^{r_0} \cap L_\gamma^2(G)), \quad (5.83)$$

for every $T > 0$. Moreover,

$$\text{div } \mathbf{E}_\varepsilon(t) = 0 \text{ on } \Omega_0, \quad q_k(\mathbf{E}_\varepsilon(t)) = Q_k. \quad (5.84)$$

$$\text{and } \text{curl } \mathbf{A}_\varepsilon(t) = \mu[\mathbf{h}_0 - \mathbf{h}_\varepsilon(t)] \in L^2(\Omega). \quad (5.85)$$

Proof: The energy balance 5.79 and assumption 2.19 on σ yield

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{w}_\varepsilon(t)\|_{X_0}^2 \leq - \int_G \gamma |\mathbf{E}_\varepsilon(t)|^2 dx + \|\mathbf{w}_\varepsilon(t)\|_{X_0}^2 + \|\partial_t \mathbf{g}_0(t)\|_{L^2}^2.$$

This implies 5.81 and 5.82.

Next, 5.84 is proved. For this purpose assume that $\varphi \in C_0^\infty(\mathbb{R}^3)$ with $\nabla \varphi = 0$ on $\mathcal{C} = \mathbb{R}^3 \setminus \overline{\Omega_0}$. Then one has for all $\mathbf{g} \in C_0^\infty(\mathbb{R}^3) \subset H_{\text{curl}}^2(\Omega)$

$$\int_\Omega (\nabla \varphi) \text{curl } \mathbf{g} dx = \int_{\mathbb{R}^3} (\nabla \varphi) \text{curl } \mathbf{g} dx = 0.$$

Hence, $\nabla\varphi \in H^2_{curl}(\Omega)$, i. e. $(\nabla\varphi, 0) \in \ker B$. Therefore 5.69 and 5.78 yield

$$\begin{aligned} \int_{\Omega} \mathbf{E}_{\varepsilon}(t) \nabla\varphi dx &= \varepsilon^{-1/2} \langle \mathbf{w}_n(t), (\nabla\varphi, 0) \rangle_{X_0} \\ &= \varepsilon^{-1/2} \left[\langle \mathbf{w}_{\varepsilon,0}, (\nabla\varphi, 0) \rangle_{X_0} + \int_0^t \langle (\varepsilon^{-1/2} F_{\sigma}(t, \mathbf{E}_{\varepsilon}(s)) + \mathbf{f}_0(s)), (\nabla\varphi, 0) \rangle_{X_0} ds \right] \\ &= \int_{\Omega} \mathbf{E}_0 \nabla\varphi dx, \end{aligned}$$

since $\nabla\varphi = 0$ on G whereas $\sigma(t, x, \mathbf{E}_{\varepsilon}(t, x)) = 0$ on $\Omega_0 = \Omega \setminus \overline{G}$. By the definition of q_k 5.84 is shown.

In order to prove 5.83 note that 5.84 implies

$$\operatorname{div} \mathbf{A}_{\varepsilon}(t) = 0 \text{ on } \Omega_0 \text{ and } q_k(\mathbf{A}_{\varepsilon}(t)) = tQ_k. \quad (5.86)$$

Next, it is shown that $\mathbf{A}_{\varepsilon}(t) \in H^2_{curl}(\Omega)$ and that 5.85 holds.

For this purpose let $\mathbf{g} \in C_0^{\infty}(\mathbb{R}^3) \subset H^2_{curl}(\Omega)$. Then one has $(0, \mathbf{g}) \in D(B)$ and hence

$$\begin{aligned} \int_{\Omega} \mathbf{A}_{\varepsilon}(t) \operatorname{curl} \mathbf{g} dx &= \int_0^t \int_{\Omega} \mathbf{E}_{\varepsilon}(s) \operatorname{curl} \mathbf{g} dx ds \\ &= \varepsilon^{-1/2} \int_0^t \langle \mathbf{w}_{\varepsilon}(s), B(0, \mathbf{g}) \rangle_{X_0} ds \\ &= \int_0^t \left[\langle \varepsilon^{-1/2} F_{\sigma}(s, \mathbf{E}_{\varepsilon}(s)) + \mathbf{f}_0(s), (0, \mathbf{g}) \rangle_{X_0} - \frac{d}{ds} \langle \mathbf{w}_{\varepsilon}(s), (0, \mathbf{g}) \rangle_{X_0} \right] ds \\ &= - \int_0^t \frac{d}{ds} \int_{\Omega} \mu \mathbf{h}_{\varepsilon}(s) \mathbf{g} dx ds = \int_{\Omega} \mu [\mathbf{h}_0 - \mathbf{h}_{\varepsilon}(t)] \mathbf{g} dx. \end{aligned}$$

Hence 5.85 is proved.

Next, let $\chi \in C_0^{\infty}(\mathbb{R}^3)$ with $\chi(x) = 1$ outside some bounded set and $\chi(x) = 0$ on \mathcal{C} .

Since $\operatorname{supp} \chi \subset \Omega_0$, 5.86 and 5.85 yield

$$\operatorname{div} (\chi \mathbf{A}_{\varepsilon}(t)) = \mathbf{A}_{\varepsilon}(t) \nabla \chi \in L^2(\mathbb{R}^3)$$

and

$$\operatorname{curl} (\chi \mathbf{A}_{\varepsilon}(t)) = -\mathbf{A}_{\varepsilon}(t) \wedge \nabla \chi + \chi \mu [\mathbf{h}_0 - \mathbf{h}_{\varepsilon}(t)] \in L^2(\mathbb{R}^3)$$

This implies $\chi \mathbf{A}_{\varepsilon}(t) \in H^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$. Hence $\mathbf{A}_{\varepsilon}(t) \in H^2_{curl}(\Omega) \cap L^6(\{|x| > R_0\})$. By 5.86 this yields $\mathbf{A}_{\varepsilon}(t) \in \mathcal{X}$. Finally, 5.83 follows from estimate 3.57 (as a consequence

of lemma 1), 5.82, 5.85, 5.86 and the boundedness of $(\mathbf{h}_\varepsilon)_{\varepsilon>0}$ in $L^\infty((0, T), X_0)$ which follows from 5.81.

□

By lemma 4 there exist $\mathbf{A} \in L^\infty_{loc}(\mathbb{R}, \mathcal{X}) \cap W^{1,2}_{loc}([0, \infty), L^2_\gamma(G))$ and a sequence $\varepsilon_n, n \in \mathbb{N}$ with $\lim_{n \rightarrow \infty} \varepsilon_n = 0$, such that

$$\mathbf{A}_{\varepsilon_n} \xrightarrow{n \rightarrow \infty} \mathbf{A} \text{ in } L^\infty((0, T), L^6(\{|x| > R_0\})) \text{ weak } - *, \quad (5.87)$$

$$L^2((0, T), L^{r_0}(\Omega_0 \cap B_{R_0})) \text{ weak } - * \text{ and in } W^{1,2}((0, T), L^2_\gamma(G)) \text{ weakly}$$

$$\text{with } \text{curl } \mathbf{A}_{\varepsilon_n} \xrightarrow{n \rightarrow \infty} \text{curl } \mathbf{A} \text{ in } L^\infty((0, T), L^2(\Omega)) \text{ weak } - *,$$

Since $\sigma(\cdot, \mathbf{E}_{\varepsilon_n})_{n \in \mathbb{N}}$ is bounded in $L^2((0, T), L^2_{\gamma-1}(G))$ by 2.24 and 5.82, it can be also assumed that

$$\sigma(\cdot, \mathbf{E}_{\varepsilon_n}) \xrightarrow{n \rightarrow \infty} \mathbf{J} \text{ in } L^2((0, T), L^2_{\gamma-1}(G)) \text{ weakly } . \quad (5.88)$$

From 5.80, 5.84 and 5.87 it follows that

$$q_k(\mathbf{A}(t)) = tQ_k \text{ for all } k \in \{1, \dots, N\} \text{ and} \quad (5.89)$$

$$\mathbf{E}_{\varepsilon_n}|_{(0, \infty) \times G} = \partial_t \mathbf{A}_{\varepsilon_n}|_{(0, \infty) \times G} \xrightarrow{n \rightarrow \infty} \mathbf{e} \stackrel{\text{def}}{=} \partial_t \mathbf{A}|_{(0, \infty) \times G} \quad (5.90)$$

in $L^2((0, T), L^2_\gamma(G))$ weakly, in particular $\mathbf{A}(t)|_G = \int_0^t \mathbf{e}(s) ds \in L^2_\gamma(G)$.

Moreover, it follows from 5.85 and 5.87 that

$$\mathbf{h}_{\varepsilon_n} = \mathbf{h}_0 - \mu^{-1} \text{curl } \mathbf{A}_{\varepsilon_n} \xrightarrow{n \rightarrow \infty} \mathbf{h} \stackrel{\text{def}}{=} \mathbf{h}_0 - \mu^{-1} \text{curl } \mathbf{A} \quad (5.91)$$

in $L^\infty((0, T), L^2(\Omega))$ weak $-*$.

Since $\|\varepsilon_n \mathbf{E}_{\varepsilon_n}\|_{L^\infty((0, T), L^2(\Omega))} \xrightarrow{n \rightarrow \infty} 0$ by 5.81, it follows easily from 5.72 that

$$\text{curl } \mathbf{h} = \mathbf{J} + \mathbf{j}_0 \in L^2_{loc}([0, \infty), L^2(\Omega)), \quad (5.92)$$

in particular $\mathbf{h}(t) - \mathbf{g}_0(t) \in \tilde{\mathcal{X}}$ with

$$\text{curl } (\mathbf{h} - \mathbf{g}_0) = \mathbf{J} \in L^2_{loc}([0, \infty), L^2_{\gamma-1}(G)).$$

Lemma 5 i)

$$\int_\Omega \mu \mathbf{h}(t) \mathbf{G} dx = \int_\Omega \mu \mathbf{h}_0 \mathbf{G} dx - \int_0^t \int_G \mathbf{e}(r) \text{curl } \mathbf{G} dx dr$$

for all $\mathbf{G} \in \tilde{\mathcal{X}}_0$, that means $\mathbf{G} \in H^2_{curl}(\Omega)$ with $\text{curl } \mathbf{G} = 0$ on Ω_0 and $\text{curl } \mathbf{G} \in L_{\gamma-1}(G)$.

$$ii) \int_\Omega \mu |\mathbf{h}(t) - \mathbf{g}_0(t)|^2 dx = \int_\Omega \mu |\mathbf{h}_0 - \mathbf{g}_0(0)|^2 dx$$

$$- 2 \int_0^t \left(\int_G \mathbf{e}(r) \mathbf{J}(r) dx + \int_\Omega \mu [\mathbf{h}(r) - \mathbf{g}_0(r)] \partial_t \mathbf{g}_0(r) dx \right) dr$$

Proof:

i) By 2.26, 5.90 and 5.91 one has for all $\mathbf{G} \in \tilde{\mathcal{X}}_0$

$$\begin{aligned} \int_{\Omega} \mu \mathbf{h}(t) \mathbf{G} dx &= \int_{\Omega} [\mu \mathbf{h}_0 - \operatorname{curl} \mathbf{A}(t)] \mathbf{G} dx = \int_{\Omega} [\mu \mathbf{h}_0 \mathbf{G} - \mathbf{A}(t) \operatorname{curl} \mathbf{G}] dx \\ &= \int_{\Omega} \mu \mathbf{h}_0 \mathbf{G} dx - \int_0^t \int_G \mathbf{e}(r) \operatorname{curl} \mathbf{G} dx dr. \end{aligned}$$

Proof of ii)

Let $\omega_n \in C_0^\infty(0, \infty)$, $n \in \mathbb{N}$ be a mollifier-sequence and define

$$\begin{aligned} \mathbf{F}_n(t) &\stackrel{\text{def}}{=} \int_0^\infty \omega_n(t-s) \mathbf{A}(s) ds \in \mathcal{X} \text{ and} \\ \mathbf{H}_n(t) &\stackrel{\text{def}}{=} \int_0^\infty \omega_n(t-s) [\mathbf{h}(s) - \mathbf{g}_0(s)] ds \end{aligned}$$

Then 5.71 and 5.92 yield

$$\operatorname{curl} \mathbf{H}_n(t) = \mathbf{J}_n(t) \stackrel{\text{def}}{=} \int_0^\infty \omega_n(t-s) \mathbf{J}(s) ds \in L_{\gamma-1}^2(G). \quad (5.93)$$

In particular $\mathbf{H}_n(t) \in \tilde{\mathcal{X}}_0$.

Since $\omega_n(-s) = 0$ for $s \geq 0$, one has

$$\mathbf{H}_n(0) = 0. \quad (5.94)$$

Next, 5.90 yields

$$\partial_t \mathbf{F}_n(t)|_G = \mathbf{e}_n(t) \stackrel{\text{def}}{=} \int_0^\infty \omega_n(t-s) \mathbf{e}(s) ds. \quad (5.95)$$

Moreover, it follows from 5.85 that

$$\begin{aligned} \partial_t \mathbf{H}_n &= \int_0^\infty \omega_n'(t-s) [\mathbf{h}_0 - \mu^{-1} \operatorname{curl} \mathbf{A}(s) - \mathbf{g}_0(s)] dx \\ &= \omega_n(t) [\mathbf{h}_0 - \mathbf{g}_0(0)] - \mu^{-1} \operatorname{curl} \partial_t \mathbf{F}_n(t) - \omega_n * \partial_t \mathbf{g}_0(t) \end{aligned} \quad (5.96)$$

Since $\partial_t \mathbf{F}_n(t) \in \mathcal{X}$, 5.93, 5.95 5.96 and 2.26 yield

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\sqrt{\mu} \mathbf{H}_n(t)\|_{L^2}^2 \\ &= \omega_n(t) \int_{\Omega} \mu \mathbf{H}_n(t) [\mathbf{h}_0 - \mathbf{g}_0(0)] dx \\ &\quad - \int_{\Omega} \mathbf{H}_n(t) \operatorname{curl} \partial_t \mathbf{F}_n(t) dx - \int_{\Omega} \mu \mathbf{H}_n(t) (\omega_n * \partial_t \mathbf{g}_0)(t) dx \end{aligned} \quad (5.97)$$

$$= \omega_n(t) \int_0^\infty \omega_n(t-s) f(s) ds - \int_G \mathbf{J}_n(t) \mathbf{e}_n(t) dx - \int_\Omega \mu \mathbf{H}_n(t) (\omega_n * \partial_t \mathbf{g}_0)(t) dx,$$

where $f(t) \stackrel{\text{def}}{=} \int_\Omega \mu (\mathbf{h}(t) - \mathbf{g}_0(t)) [\mathbf{h}_0 - \mathbf{g}_0(0)] dx$. Here the fact is used that the support of $\text{curl } \mathbf{H}_n(t) = \mathbf{J}_n(t)$ is contained in \overline{G} by 5.93. By the definition of \mathbf{F}_n , \mathbf{H}_n , \mathbf{e}_n and \mathbf{J}_n it follows that

$$\mathbf{H}_n \xrightarrow{n \rightarrow \infty} \mathbf{h} - \mathbf{g}_0 \text{ in } L^\infty((0, T), L^2(\Omega)) \text{ weak } *, \quad (5.98)$$

$$\mathbf{e}_n \xrightarrow{n \rightarrow \infty} \mathbf{e} \text{ in } L^2((0, T), L^2_\gamma(G)) \text{ strongly}, \quad (5.99)$$

$$\mathbf{J}_n|_{(0, T) \times G} \xrightarrow{n \rightarrow \infty} \mathbf{J} \text{ in } L^2((0, T), L^2_{\gamma^{-1}}(G)) \text{ strongly}. \quad (5.100)$$

$$\text{and } \omega_n * \partial_t \mathbf{g}_0 \xrightarrow{n \rightarrow \infty} \partial_t \mathbf{g}_0 \text{ in } L^1((0, T), L^2(\Omega)) \text{ strongly}. \quad (5.101)$$

Now, 5.97-5.101 and 5.94 yield

$$\begin{aligned} & \frac{1}{2} \|\sqrt{\mu} \mathbf{H}_n(\tau)\|_{L^2}^2 - \int_0^\tau \omega_n(t) \int_0^\infty \omega_n(t-s) f(s) ds dt \\ & \xrightarrow{n \rightarrow \infty} - \int_0^\tau \int_G \mathbf{e}(t) \mathbf{J}(t) dx dt - \int_0^\tau \int_\Omega \mu [\mathbf{h}(t) - \mathbf{g}_0(t)] \partial_t \mathbf{g}_0(t) dx dt. \end{aligned} \quad (5.102)$$

By 5.70 there exists a sequence $(\mathbf{G}_m)_{m \in \mathbb{N}}$ in $\tilde{\mathcal{X}}_0$ with

$$\|\mathbf{h}_0 - \mathbf{g}_0(0) - \mathbf{G}_m\|_{L^2(\Omega)} \xrightarrow{m \rightarrow \infty} 0.$$

Let $f_m(t) \stackrel{\text{def}}{=} \int_\Omega \mu (\mathbf{h}(t) - \mathbf{g}_0(t)) \mathbf{G}_m dx$. Then

$$\|f_m - f\|_{L^\infty(0, T)} \xrightarrow{m \rightarrow \infty} 0.$$

$$\text{and } f_m(t) = \int_\Omega \mu (\mathbf{h}_0 - \mathbf{g}_0(t)) \mathbf{G}_m dx - \int_0^t \int_G \mathbf{e}(r) \text{curl } \mathbf{G}_m dx dr$$

by lemma 5.

This implies that f is continuous and

$$f(0) = \lim_{m \rightarrow \infty} f_m(0) = \lim_{m \rightarrow \infty} \int_\Omega \mu (\mathbf{h}_0 - \mathbf{g}_0(0)) \mathbf{G}_m dx = \int_\Omega \mu |\mathbf{h}_0 - \mathbf{g}_0(0)|^2 dx.$$

Hence

$$\begin{aligned} & \left| \int_0^\tau \omega_n(t) \int_0^\infty \omega_n(t-s) f(s) ds dt - 1/2 \int_\Omega \mu |\mathbf{h}_0 - \mathbf{g}_0(0)|^2 dx \right| \\ & = \left| \int_0^\tau \omega_n(t) \int_0^t \omega_n(u) f(t-u) du dt - 1/2 f(0) \right| \end{aligned}$$

$$= \left| \int_0^\tau \omega_n(t) \int_0^t \omega_n(u) [f(t-u) - f(0)] du dt \right| \leq \sup_{0 \leq t \leq 1/n} |f(t-u) - f(0)| du dt$$

$$\xrightarrow{m \rightarrow \infty} 0.$$

Finally, assertion ii) follows from 5.102.

□

Next, let $\mathbf{E} \stackrel{\text{def}}{=} \partial_t \mathbf{A} \in \mathcal{W} \subset \mathcal{D}'((0, \infty) \times \Omega)$.

It follows from 5.89 and 5.91 that $\partial_t(\mu \mathbf{h}) = -\text{curl } \mathbf{E}$ and $\tilde{q}_t(\mathbf{E}) = Q_k$.

Theorem 2 (\mathbf{E}, \mathbf{h}) *with $\mathbf{E} \in \mathcal{W} \subset \mathcal{D}'((0, \infty) \times \Omega)$ is the unique solution of problem 2.27 - 2.29 in the sense of definition 1. Moreover,*

$$\mathbf{A}_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \mathbf{A} \text{ in } L^\infty((0, T), Y_1^{r_0}) \text{ weak } * \text{ and in } W^{1,2}((0, T), L_\gamma^2(G)) \text{ weakly}, \quad (5.103)$$

in particular

$$\mathbf{E}_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \mathbf{E} = \partial_t \mathbf{A} \text{ in } \mathcal{D}'((0, \infty) \times \Omega) \text{ and in } L^2((0, T), L_\gamma^2(G)) \text{ weakly} \quad (5.104)$$

$$\text{and } \mathbf{h}_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \mathbf{h} = \mathbf{h}_0 - \mu^{-1} \text{curl } \mathbf{A} \text{ in } L^\infty((0, T), L^2(\Omega)) \text{ weak } - *. \quad (5.105)$$

Finally,

$$\sigma(t, x, \mathbf{E}_\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} \sigma(t, x, \mathbf{E}) \text{ in } L^2((0, T), L_{\gamma-1}^2(G)) \text{ weakly} \quad (5.106)$$

Proof:

First it is shown that

$$\mathbf{h} \in C([0, \infty), L^2(\Omega)) \quad (5.107)$$

$$\text{and } \mathbf{h} = (0). \quad (5.108)$$

Let $T > 0$ and $\omega_n \in C_0^\infty(-1/n, 0)$, $n \in \mathbb{N}$ be a mollifier-sequence and define

$$\mathbf{h}_n(t) \stackrel{\text{def}}{=} (\mathbf{h} * \omega_n)(t) = \int_{-\infty}^0 \omega_n(u) \mathbf{h}(t-u) du.$$

$$\text{Then } \|\mathbf{h}_n - \mathbf{h}\|_{L^p((0, T), L^2(\Omega))} \xrightarrow{n \rightarrow \infty} 0.$$

Hence there exist a set $\mathcal{N} \subset [0, T]$ of measure zero and a subsequence still labelled by $(\mathbf{h}_n)_{n \in \mathbb{N}}$, such that

$$\|\mathbf{h}_n(t) - \mathbf{h}(t)\|_{L^2(\Omega)} \xrightarrow{n \rightarrow \infty} 0 \text{ for all } t \in (0, T) \setminus \mathcal{N}. \quad (5.109)$$

In particular, assertion i) of lemma 5 holds for all $t \in (0, T) \setminus \mathcal{N}$ and all $\mathbf{G} \in \tilde{\mathcal{X}}_0$. Moreover, it follows from 5.92 that \mathcal{N} can be chosen such that

$$\mathbf{h}(t) - \mathbf{g}_0(t) \in \tilde{\mathcal{X}} \text{ with } \operatorname{curl}(\mathbf{h}(t) - \mathbf{g}_0(t)) = \mathbf{J}(t) \in L^2_{\gamma^{-1}}(G) \quad (5.110)$$

and assertion ii) of lemma 5 holds for all $t \in (0, T) \setminus \mathcal{N}$. Now, it follows from 5.109, 5.110 and lemma 5 that for all $s, t \in (0, T) \setminus \mathcal{N}$

$$\begin{aligned} & \|\mu^{1/2}(\mathbf{h}(t) - \mathbf{g}_0(t)) - \mu^{1/2}(\mathbf{h}(s) - \mathbf{g}_0(s))\|_{L^2}^2 \\ &= \|\mu^{1/2}(\mathbf{h}(t) - \mathbf{g}_0(t))\|_{L^2}^2 - \|\mu^{1/2}(\mathbf{h}(s) - \mathbf{g}_0(s))\|_{L^2}^2 \\ &+ 2 \int_{\Omega} \mu(\mathbf{h}(s) - \mathbf{g}_0(s)) [\mathbf{h}(s) - \mathbf{h}(t) - \mathbf{g}_0(s) + \mathbf{g}_0(t)] dx \\ &= -2 \int_s^t \int_G \mathbf{e}(r) \mathbf{J}(r) dx dr - 2 \int_s^t \int_{\Omega} \mu [\mathbf{h}(r) - \mathbf{g}_0(r)] \partial_t \mathbf{g}_0(r) dx dr \\ &+ 2 \int_s^t \int_G \mathbf{e}(r) \operatorname{curl}(\mathbf{h}(s) - \mathbf{g}_0(s)) dx dr - 2 \int_{\Omega} \mu(\mathbf{h}(s) - \mathbf{g}_0(s)) [\mathbf{g}_0(s) - \mathbf{g}_0(t)] dx \\ &\leq 2 \int_s^t \|\mathbf{e}(r)\|_{L^2_{\gamma}(G)} \|\mathbf{J}(r)\|_{L^2_{\gamma^{-1}}(G)} dr + 2M \int_s^t \|\partial_t \mathbf{g}_0(r)\|_{L^2} dr \\ &+ 2 \int_s^t \int_G \mathbf{e}(r) \mathbf{J}(s) dx dr + 2M \|\mathbf{g}_0(t) - \mathbf{g}_0(s)\|_{L^2}, \end{aligned}$$

where $M \stackrel{\text{def}}{=} \|\mathbf{h} - \mathbf{g}_0\|_{L^\infty((0,T), L^2(\Omega))}$.

Let $\alpha_n \stackrel{\text{def}}{=} \sup_{t \in (0, T)} \left(\int_t^{t+1/n} \|\mathbf{e}(r)\|_{L^2_{\gamma}(G)}^2 dr \right)^{1/2}$ and $\beta_n \stackrel{\text{def}}{=} \sup_{t \in (0, T)} \left(\int_t^{t+1/n} \|\partial_t \mathbf{g}_0(r)\|_{L^2}^2 dr \right)$.

Then the previous estimate yields

$$\begin{aligned} & \|\mu^{1/2}(\mathbf{h}(t) - \mathbf{g}_0(t)) - \mu^{1/2}(\mathbf{h}(s) - \mathbf{g}_0(s))\|_{L^2}^2 \\ &\leq 2\alpha_n \|\mathbf{J}\|_{L^2((0,T), L^2_{\gamma^{-1}}(G))} + 4M\beta_n + 2\alpha_n |t - s|^{1/2} \|\mathbf{J}(s)\|_{L^2_{\gamma^{-1}}(G)} \\ &\leq C_1(\alpha_n + \beta_n) + 2\alpha_n |t - s|^{1/2} \|\mathbf{J}(s)\|_{L^2_{\gamma^{-1}}(G)} \end{aligned}$$

with $C_1 \in (0, \infty)$ independent of s, t, n . Now,

$$\begin{aligned} & \|\mu^{1/2}(\mathbf{h}(t) - \mathbf{g}_0(t)) - \mu^{1/2}(\mathbf{h}_n(t) - (\omega_n * \mathbf{g}_0)(t))\|_{L^2}^2 \\ &= \|\mu^{1/2} \int_{-\infty}^0 \omega_n(u) [(\mathbf{h}(t) - \mathbf{g}_0(t)) - (\mathbf{h}(t-u) - \mathbf{g}_0(t-u))] du\|^2 \\ &\leq \int_{-\infty}^0 \omega_n(u) \|\mu^{1/2}(\mathbf{h}(t) - \mathbf{g}_0(t)) - \mu^{1/2}(\mathbf{h}(t-u) - \mathbf{g}_0(t-u))\|^2 du. \end{aligned}$$

$$\begin{aligned}
&\leq \int_{-\infty}^0 \omega_n(u) \left(C_1(\alpha_n + \beta_n) + 2\alpha_n |u|^{1/2} \|\mathbf{J}(t-u)\|_{L^2_{\gamma^{-1}}(G)} \right) du. \\
&= C_1(\alpha_n + \beta_n) + \alpha_n \left(\int_{-\infty}^0 \omega_n(u)^2 |u| du \right)^{1/2} \left(\int_0^T \|\mathbf{J}(r)\|_{L^2_{\gamma^{-1}}(G)}^2 dr \right)^{1/2} \\
&\xrightarrow{n \rightarrow \infty} 0 \text{ uniformly with respect to } t \in (0, T).
\end{aligned}$$

This proves 5.107.

In order to show 5.108 let $(\mathbf{G}_m)_{m \in \mathbb{N}}$ a sequence in $\tilde{\mathcal{X}}_0$ with

$$\|\mathbf{h}_0 - \mathbf{g}_0(0) - \mathbf{G}_m\|_{L^2(\Omega)} \xrightarrow{m \rightarrow \infty} 0,$$

see 5.70. Let $f(t) \stackrel{\text{def}}{=} \int_{\Omega} \mu(\mathbf{h}(t) - \mathbf{g}_0(t)) [\mathbf{h}_0 - \mathbf{g}_0(0)] dx$.

and $f_m(t) \stackrel{\text{def}}{=} \int_{\Omega} \mu(\mathbf{h}(t) - \mathbf{g}_0(t)) \mathbf{G}_m dx$. Then

$$\|f_m - f\|_{L^\infty(0, T)} \xrightarrow{m \rightarrow \infty} 0.$$

$$\text{and } f_m(t) = \int_{\Omega} \mu(\mathbf{h}_0 - \mathbf{g}_0(t)) \mathbf{G}_m dx - \int_0^t \int_G \mathbf{e}(r) \operatorname{curl} \mathbf{G}_m dx dr$$

by lemma 5.

This implies that f is continuous and

$$f(0) = \lim_{m \rightarrow \infty} f_m(0) = \lim_{m \rightarrow \infty} \int_{\Omega} \mu(\mathbf{h}_0 - \mathbf{g}_0(0)) \mathbf{G}_m dx = \int_{\Omega} \mu |\mathbf{h}_0 - \mathbf{g}_0(0)|^2 dx.$$

$$\text{Hence } \|(\mathbf{h}(t) - \mathbf{g}_0(t)) - (\mathbf{h}_0 - \mathbf{g}_0(0))\|_{L^2}^2$$

$$= \|(\mathbf{h}(t) - \mathbf{g}_0(t))\|_{L^2}^2 + \|(\mathbf{h}_0 - \mathbf{g}_0(0))\|_{L^2}^2 - 2f(t) \xrightarrow{t \rightarrow 0} 0.$$

Since $\mathbf{e} = \mathbf{E}|_{(0, \infty) \times G} \in L^2_{loc}([0, \infty), L^2_{\gamma}(G))$ by 5.90, it remains to show

$$\mathbf{J} = \sigma(x, \mathbf{e}). \tag{5.111}$$

This is will be done using a monotonicity-argument. For this purpose let $\mathcal{T} : X \rightarrow X^*$ with $X \stackrel{\text{def}}{=} L^2((0, T), L^2_{\gamma}(G))$ and $X^* \stackrel{\text{def}}{=} L^2((0, T), L^2_{\gamma^{-1}}(G))$ be defined by

$$(\mathcal{T}\mathbf{f})(t, x) \stackrel{\text{def}}{=} \chi(t) \sigma(t, x, \mathbf{f}(t, x)), \mathbf{f} \in X,$$

where $\chi \in C^\infty(\mathbb{R})$ with $\chi \geq 0$, $\chi' \leq 0$, $\chi(0) = 1$ and $\chi(T) = 0$. The aim of the following considerations is to show

$$\chi \mathbf{J} = \mathcal{T}(\mathbf{e}) \tag{5.112}$$

which implies 5.111, since $T > 0$ can be chosen arbitrary large. Let

$$\mathbf{e}_n \stackrel{\text{def}}{=} \mathbf{E}_{\varepsilon_n}|_{(0,T) \times G} \in L^2((0, T), L^2_\gamma(G)).$$

Then 5.88 and 5.90 yield

$$\mathbf{e}_n \xrightarrow{n \rightarrow \infty} \mathbf{e} \text{ in } X \text{ weakly and } \mathcal{T}(\mathbf{e}_n) \xrightarrow{n \rightarrow \infty} \chi \mathbf{J} \text{ in } X^* \text{ weakly .}$$

Since \mathcal{T} is monotone and Lipschitz-continuous by the assumptions on σ , it suffices to show

$$\overline{\lim}_{n \rightarrow \infty} \langle \mathcal{T}(\mathbf{e}_n), \mathbf{e}_n \rangle_X \leq \langle \chi \mathbf{J}, \mathbf{e} \rangle_X \quad (5.113)$$

The energy balance 5.79 yields

$$\begin{aligned} \langle \mathcal{T}(\mathbf{e}_n), \mathbf{e}_n \rangle_X &= \int_0^T \chi(t) \int_G \mathbf{E}_{\varepsilon_n}(t) \sigma(t, x, \mathbf{E}_{\varepsilon_n}(t)) dx dt \\ &= \int_0^T \chi(t) \left[\int_\Omega \mu(\mathbf{g}_0(t) - \mathbf{h}_{\varepsilon_n}(t)) \partial_t \mathbf{g}_0(t) dx - \frac{1}{2} \frac{d}{dt} \|\mathbf{w}_{\varepsilon_n}(t)\|_{X_0}^2 \right] dt \\ &\leq \int_0^T \chi(t) \int_\Omega \mu(\mathbf{g}_0(t) - \mathbf{h}_{\varepsilon_n}(t)) \partial_t \mathbf{g}_0(t) dx dt + \frac{1}{2} \|\mathbf{w}_{\varepsilon_n}(0)\|_{X_0}^2 \\ &\quad + \frac{1}{2} \int_0^T \chi'(t) \|\sqrt{\mu}(\mathbf{h}_{\varepsilon_n}(t) - \mathbf{g}_0(t))\|_{L^2}^2 dt \end{aligned}$$

Since $\chi' \leq 0$, it follows from 5.91 that

$$\begin{aligned} \overline{\lim}_{n \rightarrow \infty} \langle \mathcal{T}(\mathbf{e}_n), \mathbf{e}_n \rangle_X &\leq \int_0^T \chi(t) \int_\Omega \mu(\mathbf{g}_0(t) - \mathbf{h}(t)) \partial_t \mathbf{g}_0(t) dx dt + \frac{1}{2} \|\sqrt{\mu}(\mathbf{h}_0 - \mathbf{g}_0(0))\|_{L^2}^2 \\ &\quad + \frac{1}{2} \int_0^T \chi'(t) \|\sqrt{\mu}(\mathbf{h}(t) - \mathbf{g}_0(t))\|_{L^2}^2 dt \end{aligned}$$

In order to prove 5.113 it suffices now to show that

$$\begin{aligned} \langle \chi \mathbf{J}, \mathbf{e} \rangle_X &= \int_0^T \chi(t) \int_\Omega \mu(\mathbf{g}_0(t) - \mathbf{h}(t)) \partial_t \mathbf{g}_0(t) dx dt \\ &\quad + \frac{1}{2} \|\sqrt{\mu}(\mathbf{h}_0 - \mathbf{g}_0(0))\|_{L^2}^2 + \frac{1}{2} \int_0^T \chi'(t) \|\sqrt{\mu}(\mathbf{h}(t) - \mathbf{g}_0(t))\|_{L^2}^2 dt. \end{aligned} \quad (5.114)$$

Lemma 5 ii) yields

$$\frac{1}{2} \frac{d}{dt} \|\sqrt{\mu}(\mathbf{h}(t) - \mathbf{g}_0(t))\|_{L^2}^2 = - \int_G \mathbf{e}(t) \mathbf{J}(t) dx$$

$$- \int_{\Omega} \mu[\mathbf{h}(t) - \mathbf{g}_0(t)] \partial_t \mathbf{g}_0(t) dx.$$

Hence

$$\begin{aligned} < \chi \mathbf{J}, \mathbf{e} >_X &= \int_0^T \chi(t) \int_G \mathbf{e}(t) \mathbf{J}(t) dx dt \\ &= - \int_0^T \chi(t) \left[\frac{1}{2} \frac{d}{dt} \|\sqrt{\mu}(\mathbf{h}(t) - \mathbf{g}_0(t))\|_{L^2}^2 + \int_{\Omega} \mu[\mathbf{h}(t) - \mathbf{g}_0(t)] \partial_t \mathbf{g}_0(t) dx \right] dt \\ &= \frac{1}{2} \|\mu^{1/2}(\mathbf{h}_0 - \mathbf{g}_0(0))\|_{L^2}^2 + \frac{1}{2} \int_0^T \chi'(t) \|\sqrt{\mu}(\mathbf{h}(t) - \mathbf{g}_0(t))\|_{L^2}^2 dt \\ &\quad - \int_0^T \chi(t) \int_{\Omega} \mu[\mathbf{h}(t) - \mathbf{g}_0(t)] \partial_t \mathbf{g}_0(t) dx dt \end{aligned}$$

whence 5.114.

Hence, is proved that $(\mathbf{E}, \mathbf{h}) \in \mathcal{W} \times C([0, \infty), L^2(\Omega))$ solve 2.27 - 2.29. Thus, the proof of existence and uniqueness of solutions to 2.27-2.29 is complete.

The above considerations show that each accumulation point of $(\mathbf{E}_\varepsilon, \mathbf{h}_\varepsilon)$, $\varepsilon > 0$ in the weak topologies described in 5.103 - 5.106 provides a solution to 2.27-2.29. Since the solution of this problem is unique, one obtains also assertions 5.103 - 5.106.

□

6 Asymptotic behavior: Decay of the magnetic field

In this section the asymptotic behavior for $t \rightarrow \infty$ of solutions to 2.27-2.29 in the case

$$\mathbf{j}_0 = 0 \tag{6.115}$$

is investigated. In the sequel let $H_{curl,0}(\Omega)$ be the space of all $\mathbf{h} \in H_{curl}^2(\Omega)$ with $\text{curl } \mathbf{h} = 0$ on Ω . It is a closed subspace of $L^2(\Omega)$. Its orthogonal complement with respect to the scalar-product

$\langle \mathbf{h}, \mathbf{g} \rangle_{\mu} \stackrel{\text{def}}{=} \int_{\Omega} \mu \mathbf{h} \bar{\mathbf{g}} dx$ is denoted by X_h . Let P be the orthogonal projection on $X_h \subset L^2(\Omega)$ with respect to $\langle \cdot, \cdot \rangle$.

Since $\nabla \psi \in H_{curl,0}(\Omega)$ for all $\psi \in H^1(\Omega) \stackrel{\text{def}}{=} W^{1,2}(\Omega)$, it follows that each $\mathbf{h} \in X_h$ obeys

$$\text{div}(\mu \mathbf{h}) = 0 \text{ on } \Omega \text{ and } \vec{n} \mathbf{h} = 0 \text{ on } \partial\Omega \text{ weakly.} \tag{6.116}$$

in the sense that

$$- \int_{\Omega} \mu \mathbf{h} \nabla \psi dx = 0 \text{ for all } \psi \in H^1(\Omega).$$

Next, let $(\mathbf{E}, \mathbf{h}) \in \mathcal{W} \times C([0, \infty), L^2(\Omega))$

be the solution to 2.27-2.29. In order to apply a compactness theorem, it is assumed that $\mathbb{R}^3 \setminus \bar{\Omega}$ is also a Lipschitz-domain.

Lemma 6 *There exists some $\delta > 0$ (depending only on σ), such that*

$$\frac{d}{dt} \frac{1}{2} \|\sqrt{\mu} \mathbf{h}(t)\|_{L^2}^2 \leq -\delta \| \operatorname{curl} \mathbf{h}(t) \|_{L^2(\Omega)}^2.$$

Proof: By 6.115 one has

$$\operatorname{curl} \mathbf{h}(t) = \mathbf{J}(t) \in L_{\gamma^{-1}}(G), \quad (6.117)$$

and $\operatorname{curl} \mathbf{h}(t, x) = 0$ for $x \in \Omega_0 = \Omega \setminus \overline{G}$, in particular $\mathbf{h}(t) \in \tilde{\mathcal{X}}_0$. Therefore it follows from the same arguments as in the proof of 4.62 that

$$\frac{1}{2} \frac{d}{dt} \|\sqrt{\mu} \mathbf{h}(t)\|_{L^2}^2 = - \int_G \sigma(t, x, \mathbf{E}(t)) \mathbf{E}(t) dx.$$

By the assumptions on σ one has

$$\begin{aligned} -\frac{1}{2} \frac{d}{dt} \|\sqrt{\mu} \mathbf{h}(t)\|_{L^2}^2 &= \int_G \mathbf{E}(t, x) \sigma(t, x, \mathbf{E}(t, x)) dx \geq \int_G \gamma(x) |\mathbf{E}(t, x)|^2 dx \\ &\geq \frac{1}{C^2} \int_G \gamma(x)^{-1} |\sigma(t, x, \mathbf{E}(t, x))|^2 dx \geq \delta \int_G |\sigma(t, x, \mathbf{E}(t, x))|^2 dx \\ &= \delta \int_G |\operatorname{curl} \mathbf{h}(t, x)|^2 dx = \delta \int_\Omega |\operatorname{curl} \mathbf{h}(t, x)|^2 dx. \end{aligned}$$

with $\delta \stackrel{\text{def}}{=} (C^2 \|\gamma\|_{L^\infty})^{-1}$. This completes the proof.

□

The aim of the following considerations is to estimate $\|\mathbf{h}\|_{L^2}$ by $\|\operatorname{curl} \mathbf{h}\|_{L^2}$ for $\mathbf{h} \in X_h$.

Lemma 7 *Let $\mathbf{h} \in L^2(\mathbb{R}^3)$ with $\operatorname{curl} \mathbf{h} \in L^{6/5}(\mathbb{R}^3)$ and $\operatorname{div}(\mu \mathbf{h}) \in L^{6/5}(\mathbb{R}^3)$. Then*

$$\|\mathbf{h}\|_{L^2(\mathbb{R}^3)} \leq K (\|\operatorname{curl} \mathbf{h}\|_{L^{6/5}(\mathbb{R}^3)} + \|\operatorname{div}(\mu \mathbf{h})\|_{L^{6/5}(\mathbb{R}^3)})$$

with some $K \in (0, \infty)$ independent of \mathbf{h} .

Proof:

Recall that $Z \stackrel{\text{def}}{=} \{\varphi \in L^6 \mid \nabla \varphi \in L^2(\mathbb{R}^3)\}$ is a Hilbert-space endowed with the scalar-product

$\langle \varphi, \psi \rangle_Z \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} \mu \nabla \varphi \nabla \overline{\psi} dx$. Thus, there exists a unique $\psi_0 \in Z$, such that

$$\int_{\mathbb{R}^3} \mu \nabla \psi_0 \nabla \varphi dx = \langle \psi_0, \overline{\varphi} \rangle_{Z_0} = \int_{\mathbb{R}^3} \varphi \operatorname{div}(\mu \mathbf{h}) dx$$

for all $\varphi \in Z$, since $\operatorname{div}(\mu \mathbf{h}) \in L^{6/5}(\mathbb{R}^3)$, such that the last term defines a continuous linear functional on $Z \subset L^6$.

$$\|\mu^{1/2} \nabla \psi_0\|_{L^2(\mathbb{R}^3)} = \|\psi_0\|_Z \leq K_1 \|\operatorname{div}(\mu \mathbf{h})\|_{L^{6/5}(\mathbb{R}^3)} \quad (6.118)$$

Let $\mathbf{h}_1 \stackrel{\text{def}}{=} \mathbf{h} + \nabla\psi_0 \in L^2(\mathbb{R}^3)$. Then $\text{curl } \mathbf{h}_1 = \text{curl } \mathbf{h} \in L^{6/5}(\mathbb{R}^3)$ and $\text{div}(\mu\mathbf{h}_1) = 0$ on \mathbb{R}^3 . By lemma 3 i) there exists a unique $\mathbf{F} \in L^6(\mathbb{R}^3)$ with $\text{curl } \mathbf{F} = \mu\mathbf{h}_1 \in L^2(\mathbb{R}^3)$ and $\text{div } \mathbf{F} = 0$ on \mathbb{R}^3 . The estimate

$$\|\mathbf{F}\|_{L^6(\mathbb{R}^3)} \leq K_2 \|\mathbf{h}_1\|_{L^2(\mathbb{R}^3)} \quad (6.119)$$

holds with constants $K_1, K_2 \in (0, \infty)$ independent of \mathbf{h} . Let $\psi_1 \in C_0^\infty(B_2)$ with $\psi_1 = 1$ on B_1 and $\psi_n \stackrel{\text{def}}{=} \psi_1(x/n)$. Then

$$\begin{aligned} \|\mu^{1/2}\mathbf{h}_1\|_{L^2(\mathbb{R}^3)}^2 &= \int_{\mathbb{R}^3} \mathbf{h}_1 \text{curl } \mathbf{F} dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} (\mathbf{h}_1 \text{curl } \mathbf{F}) \psi_n dx \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} [(\mathbf{F} \wedge \mathbf{h}_1) \nabla \psi_n + \psi_n \mathbf{F} \text{curl } \mathbf{h}_1] dx \end{aligned}$$

Since $\text{supp}(\nabla\psi_n) \subset \{n < |x| < 2n\}$, $\mathbf{h}_0 \in L^2(\mathbb{R}^3)$ and $\mathbf{F} \in L^6(\mathbb{R}^3)$, Hölder's inequality yields

$$\begin{aligned} \left| \int_{\mathbb{R}^3} (\mathbf{F} \wedge \mathbf{h}_1) \nabla \psi_n dx \right| &\leq \|\mathbf{F}\|_{L^6(\{|x|>n\})} \|\mathbf{h}_1\|_{L^2(\{|x|>n\})} \|\nabla \psi_n\|_{L^\infty(\mathbb{R}^3)} |B_{2n}|^{1/3} \\ &\leq C \|\mathbf{F}\|_{L^6(\{|x|>n\})} \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Therefore 6.119 yields

$$\begin{aligned} \|\mu^{1/2}\mathbf{h}_1\|_{L^2(\mathbb{R}^3)}^2 &= \int_{\mathbb{R}^3} \mathbf{F} \text{curl } \mathbf{h}_1 dx \leq \|\mathbf{F}\|_{L^6(\mathbb{R}^3)} \|\text{curl } \mathbf{h}_1\|_{L^{6/5}(\mathbb{R}^3)} \\ &\leq K_2 \|\mathbf{h}_1\|_{L^2(\mathbb{R}^3)} \|\text{curl } \mathbf{h}_1\|_{L^{6/5}(\mathbb{R}^3)}. \end{aligned} \quad (6.120)$$

By the definition of \mathbf{h}_1 the assertion follows from 6.118 and 6.120.

□

Lemma 8 *Let $(\mathbf{h}_n)_{n \in \mathbb{N}}$ be a sequence in X_h , such that $(\text{curl } \mathbf{h}_n)_{n \in \mathbb{N}}$ is bounded in $L^2(\Omega)$, $\text{supp}(\text{curl } \mathbf{h}_n) \subset B_R$ for some $R \in (R_0, \infty)$ independent of n . Then $(\mathbf{h}_n)_{n \in \mathbb{N}}$ is precompact in $L^2(\Omega)$.*

Proof:

Let $\tilde{\Omega} \stackrel{\text{def}}{=} B_{3R} \cap \Omega$ and choose $\chi_b \in C^\infty(\mathbb{R}^3)$ with $\chi_b(x) = 0$ on B_R , $\chi_b(x) = 1$ for $|x| > 2R$ and let $\chi_a \in C_0^\infty(B_{3R})$ with $\chi_a(x) = 1$ on B_{2R} , in particular

$$\chi_a(x) = 1 \text{ on } \text{supp}(\nabla\chi_b) \quad (6.121)$$

It follows from the assumptions that

$$(\chi_a \mathbf{h}_n)_{n \in \mathbb{N}} \text{ is bounded in } H_{\text{curl}}^2(B_{3R} \cap \Omega) = H_{\text{curl}}^2(\tilde{\Omega}) \quad (6.122)$$

Moreover, it follows from 6.116 that

$$(\operatorname{div} [\mu \chi_a \mathbf{h}_n])_{n \in \mathbb{N}} \text{ is bounded in } L^2(\tilde{\Omega}) \quad (6.123)$$

and $\chi_a \vec{n} \mathbf{h} = 0$ on $\partial \tilde{\Omega}$, in the sense that

$$-\int_{\tilde{\Omega}} \mu \chi_a \mathbf{h}_n \nabla \psi dx = \int_{\tilde{\Omega}} (\operatorname{div} [\mu \chi_a \mathbf{h}_n]) \psi dx = \int_{\tilde{\Omega}} \mu (\mathbf{h}_n \nabla \chi_a) \psi dx$$

for all $\varphi \in H^1(\tilde{\Omega})$.

Since $\tilde{\Omega} = B_{3R} \cap \Omega$ is a Lipschitz-domain, it follows from 6.122, 6.123 and the result in [9], [11] or [5] that the sequence

$$(\chi_a \mathbf{h}_n)_{n \in \mathbb{N}} \text{ is precompact in } L^2(B_{3R} \cap \Omega) = L^2(\tilde{\Omega}). \quad (6.124)$$

Let $\mathbf{g}_n(x) \stackrel{\text{def}}{=} \chi_b \mathbf{h}_n(x)$ if $x \in \mathbb{R}^3 \setminus B_R = \Omega \setminus B_R$ and $\mathbf{g}_n(x) \stackrel{\text{def}}{=} 0$ if $x \in B_R$.

Since $\operatorname{supp} \operatorname{curl} \mathbf{h}_n \subset B_R$ and $\chi_b(x) = 0$ on B_R , it follows from 6.116, 6.122 and 6.124 that

$$(\operatorname{curl} \mathbf{g}_n)_{n \in \mathbb{N}} = ((\nabla \chi_b) \wedge \mathbf{h}_n)_{n \in \mathbb{N}} \text{ is precompact in } L^{6/5}(\mathbb{R}^3) \quad (6.125)$$

and

$$(\operatorname{div} (\mu \mathbf{g}_n))_{n \in \mathbb{N}} = (\mu \mathbf{h}_n \nabla \chi_b)_{n \in \mathbb{N}} \text{ is precompact in } L^{6/5}(\mathbb{R}^3). \quad (6.126)$$

By 6.125 and 6.126 it follows from lemma 7 that the sequence $(\mathbf{g}_n)_{n \in \mathbb{N}}$ is precompact in $L^2(\mathbb{R}^3)$ and hence

$$(\mathbf{h}_n)_{n \in \mathbb{N}} \text{ is precompact in } L^2(\Omega \setminus B_{2R}). \quad (6.127)$$

since $\chi_b(x) = 1$ for $|x| > 2R$. Finally, the precompactness of $(\mathbf{h}_n)_{n \in \mathbb{N}}$ in $L^2(\Omega)$ follows from 6.124 and 6.127.

□

Now, the following estimate can be proved.

Lemma 9 *Let $R \in (0, \infty)$. Then there exists a constant $K_R \in (0, \infty)$, such that for all $\mathbf{h} \in X_h \cap H_{\operatorname{curl}}^2(\Omega)$ with $\operatorname{curl} \mathbf{h}(x) = 0$ if $|x| > R$ the estimate*

$$\|\mathbf{h}\|_{L^2(\Omega)} \leq K_R \|\operatorname{curl} \mathbf{h}\|_{L^2(\Omega \cap B_R)}$$

holds.

Proof:

Suppose that the estimate were not correct, i.e. would exist a sequence $\mathbf{h}_n \in X_h \cap H_{curl}^2(\Omega)$, $n \in \mathbb{N}$ with

$$1 = \|\mathbf{h}_n\|_{L^2} \geq n \|\operatorname{curl} \mathbf{h}_n\|_{L^2} \text{ for all } n \in \mathbb{N} \quad (6.128)$$

By theorem 8 there exist $\mathbf{h} \in X_h$ and a subsequence \mathbf{h}_{n_k} , $k \in \mathbb{N}$ with

$$\|\mathbf{h}_{n_k} - \mathbf{h}\|_{L^2} \xrightarrow{k \rightarrow \infty} 0 \text{ in particular } \|\mathbf{E}\|_{L^2} = 1. \quad (6.129)$$

From 6.128 and 6.129 it follows that $(\mathbf{E}_{n_k})_{k \in \mathbb{N}}$ is convergent in $X_h \cap H_{curl}^2(\Omega)$.

Hence, $\mathbf{E} \in X_h \cap H_{curl}^2(\Omega)$,

Moreover, $\|\operatorname{curl} \mathbf{E}\|_{L^2} = \lim_{k \rightarrow \infty} \|\mathbf{E}_{n_k}\|_{L^2} = 0$, i.e. $\mathbf{E} \in H_{curl,0}(\Omega) \cap X_h = \{0\}$. This contradicts 6.129.

□

Now the asymptotic behavior of the solution (\mathbf{E}, \mathbf{h}) is investigated.

Let $\mathbf{h}^{(1)}(t) \stackrel{\text{def}}{=} P\mathbf{h}(t) \in X_h$.

Theorem 3 *There exists some $C, d > 0$, such that*

$$\|\mathbf{h}^{(1)}(t)\|_{L^2(\Omega)} \leq C \exp(-dt).$$

Proof: Suppose that $\mathbf{E} = \partial_t \mathbf{A}$ with some $\mathbf{A} \in L_{loc}^\infty([0, \infty), \mathcal{X})$.

Then $\mu\mathbf{h} + \operatorname{curl} \mathbf{A} \in L_{loc}^\infty([0, \infty), L^2(\Omega))$ satisfies

$$\partial_t [\mu\mathbf{h} + \operatorname{curl} \mathbf{A}] = 0 \text{ and hence}$$

$$\mathbf{h}(t) + \mu^{-1}\mathbf{A}(t) = \mathbf{h}_2 \text{ for all } t \in (0, \infty) \quad (6.130)$$

with some $\mathbf{h}_2 \in L^2(\Omega)$.

Let P be the orthogonal projection on $X_h \subset L^2(\Omega)$ with respect to the scalar-product $\langle \mathbf{h}, \mathbf{g} \rangle_\mu = \int_\Omega \mu \mathbf{h} \bar{\mathbf{g}} dx$.

Since $H_{curl,0}(\Omega) \subset \mathcal{X}$, it follows from 2.26 that $\mu^{-1} \operatorname{curl} \mathbf{A}(t) \in X_h$. Hence 6.130 yields

$$\mathbf{h}(t) = \mathbf{h}^{(1)}(t) + (1 - P) (\mathbf{h}_2 - \mu^{-1} \mathbf{A}(t)) = \mathbf{h}^{(1)}(t) + (1 - P)\mathbf{h}_2$$

and therefore

$$\|\sqrt{\mu}\mathbf{h}(t)\|_{L^2}^2 = \|\sqrt{\mu}\mathbf{h}^{(1)}(t)\|_{L^2}^2 + \|\sqrt{\mu}(1 - P)\mathbf{h}_2\|_{L^2}^2 \quad (6.131)$$

Since $\operatorname{curl} \mathbf{h}^{(1)}(t) = \operatorname{curl} \mathbf{h}(t)$, it follows from 6.131, lemma 6 and 9 that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\sqrt{\mu}\mathbf{h}^{(1)}(t)\|_{L^2}^2 &= \frac{1}{2} \frac{d}{dt} \|\sqrt{\mu}\mathbf{h}(t)\|_{L^2}^2 \\ &\leq -\delta \|\operatorname{curl} \mathbf{h}(t)\|_{L^2(G)}^2 = -\delta \|\operatorname{curl} \mathbf{h}^{(1)}(t)\|_{L^2(G)}^2 \leq -K_{R_0}^{-2} \delta \|\mathbf{h}^{(1)}(t)\|_{L^2(G)}^2. \end{aligned}$$

This completes the proof

□

7 Strong solutions

In this section a sufficient condition for the existence of strong solutions (\mathbf{E}, \mathbf{h}) with

$$\mathbf{E} \in L^2_{loc}([0, \infty), \mathcal{X}) \subset L^2_{loc}([0, \infty), Y_1^{r_0} \cap L^2_\gamma(G))$$

and

$$\mathbf{h} \in W^{1,2}_{loc}([0, \infty), L^2(\Omega))$$

is given. For this purpose it is assumed that σ is independent of t , i.e. $\sigma(t, x, \mathbf{y}) = \sigma_0(x, \mathbf{y})$. Moreover, additional regularity properties of the prescribed current \mathbf{j}_0 and the initial-data $\mathbf{E}_0, \mathbf{h}_0$ is imposed, namely

$$\mathbf{g}_0 \in L^2_{loc}(\mathbb{R}, H^2_{curl}(\Omega)) \cap W^{2,2}_{loc}(\mathbb{R}, L^2(\Omega)), \quad (7.132)$$

$$\mathbf{E}_0 \in H^2_{curl}(\Omega), \quad \mathbf{h}_0 \in H^2_{curl}(\Omega) \text{ with} \quad (7.133)$$

$$\text{curl } \mathbf{h}_0 = \mathbf{j}_0(0) + \sigma_0(x, \mathbf{E}_0) \text{ on } \Omega. \quad (7.134)$$

Lemma 10 *The family*

$$(\mathbf{w}_\varepsilon = (\varepsilon^{1/2} \mathbf{E}_\varepsilon, \mathbf{h}_\varepsilon - \mathbf{g}_0))_{\varepsilon > 0} \text{ is bounded in } W^{1,\infty}((0, T), X_0), \quad (7.135)$$

$$(\mathbf{E}_\varepsilon)_{\varepsilon > 0} \text{ is bounded in } L^2((0, T), \mathcal{X}) \subset L^2((0, T), Y_1^{r_0} \cap L^2_\gamma(G)), \quad (7.136)$$

$$\text{and } (\text{curl } \mathbf{h}_\varepsilon)_{\varepsilon > 0} \text{ is bounded in } L^2((0, T), L^2(\Omega)), \quad (7.137)$$

for every $T > 0$.

Proof: By lemma 4 we have

$$(\mathbf{w}_\varepsilon = (\varepsilon^{1/2} \mathbf{E}_\varepsilon, \mathbf{h}_\varepsilon - \mathbf{g}_0))_{\varepsilon > 0} \text{ is bounded in } L^\infty((0, T), X_0), \quad (7.138)$$

$$\text{and } (\mathbf{E}_\varepsilon)_{\varepsilon > 0} \text{ is bounded in } L^2(0, T, L^2_\gamma(G)). \quad (7.139)$$

Next, it follows from 5.78 that

$$\mathbf{w}_\varepsilon(t + \tau) - \mathbf{w}_\varepsilon(t) = \exp(\varepsilon^{-1/2} t B) \mathbf{u}_{\varepsilon, \tau, 0} + \int_0^t \exp(\varepsilon^{-1/2}(t - s) B) \mathbf{G}_{\varepsilon, \tau}(s) ds, \quad (7.140)$$

where

$$\begin{aligned} \mathbf{u}_{\varepsilon, \tau, 0} &\stackrel{\text{def}}{=} (\exp(\varepsilon^{-1/2} \tau B) - 1) \mathbf{w}_{\varepsilon, 0} \\ &+ \int_0^\tau \exp(\varepsilon^{-1/2}(\tau - s) B) \left[\varepsilon^{-1/2} F_{\sigma_0}(\mathbf{E}_\varepsilon(s)) + \mathbf{f}_0(s) \right] ds \end{aligned}$$

and

$$\mathbf{G}_{\varepsilon,\tau} \stackrel{\text{def}}{=} \varepsilon^{-1/2} [F_{\sigma_0}(\mathbf{E}_\varepsilon(\tau+s)) - F_{\sigma_0}(\mathbf{E}_\varepsilon(s))] + \mathbf{f}_0(\tau+s) - \mathbf{f}_0(s).$$

Here $F_{\sigma_0} : L^2(\Omega, \mathbb{R}^3) \rightarrow X_0 = L^2(\Omega, \mathcal{C}^6)$ is defined by $F_{\sigma_0}(\mathbf{F})(x) \stackrel{\text{def}}{=} -(\sigma_0(x, \mathbf{F}(x)), 0)$ for $\mathbf{F} \in L^2(\Omega)$.

By 7.134 it follows that $\mathbf{w}_{\varepsilon,0} \in D(B)$ and hence

$$\begin{aligned} \tau^{-1} \mathbf{u}_{\varepsilon,\tau,0} &\xrightarrow{\tau \rightarrow 0} \varepsilon^{-1/2} B \mathbf{w}_{\varepsilon,0} + \varepsilon^{-1/2} F_{\sigma_0}(\mathbf{E}_0) + \mathbf{f}_0(0) \\ &= -(0, \mu^{-1} \text{curl } \mathbf{E}_0 + \partial_t \mathbf{g}_0(0)). \end{aligned} \quad (7.141)$$

7.140 yields the estimate

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{w}_\varepsilon(t+\tau) - \mathbf{w}_\varepsilon(t)\|_{X_0}^2 &= \langle G_{\varepsilon,\tau}(t), \mathbf{w}_\varepsilon(t+\tau) - \mathbf{w}_\varepsilon(t) \rangle_{X_0} \\ &= - \int_G [\sigma_0(x \mathbf{E}_\varepsilon(t+\tau)) - \sigma_0(x \mathbf{E}_\varepsilon(t))] (\mathbf{E}_\varepsilon(t+\tau) - \mathbf{E}_\varepsilon(t)) dx \\ &\quad + \langle \mathbf{f}_0(t+\tau) - \mathbf{f}_0(t), \mathbf{w}_\varepsilon(t+\tau) - \mathbf{w}_\varepsilon(t) \rangle_{X_0} \\ &\leq \|\mathbf{f}_0(t+\tau) - \mathbf{f}_0(t)\|_{X_0}^2 + \|\mathbf{w}_\varepsilon(t+\tau) - \mathbf{w}_\varepsilon(t)\|_{X_0}^2, \end{aligned}$$

since σ_0 is monotone. Now, Gronwall's lemma yields

$$\|\mathbf{w}_\varepsilon(t+\tau) - \mathbf{w}_\varepsilon(t)\|_{X_0}^2 \leq K_T \|\mathbf{f}_0\|_{W^{1,2}((0,T),X_0)}^2 \tau^2 + \|\mathbf{u}_{\varepsilon,\tau,0}\|_{X_0}^2 \quad (7.142)$$

with some $K_T \in (0, \infty)$ independent of ε, τ . By 7.141 this yields $\mathbf{w}_\varepsilon \in W^{1,\infty}((0,T), X_0)$ and

$$\|\partial_t \mathbf{w}_\varepsilon\|_{L^\infty((0,T),X_0)}^2 \leq K_T \|\mathbf{f}_0\|_{W^{1,2}((0,T),X_0)}^2 + \|\mu^{-1} \text{curl } \mathbf{E}_0 + \partial_t \mathbf{g}_0(0)\|_{L^2}^2.$$

Now, 7.135 follows from 7.138 and the previous estimate.

7.135 and 5.77 yield $\mathbf{w}_\varepsilon \in L^2((0,T), D(B))$, in particular

$$\begin{aligned} \mathbf{E}_\varepsilon(t) &\in \overset{0}{H^2}_{\text{curl}}(\Omega), \text{curl } (\mathbf{E}_\varepsilon)_{\varepsilon>0} \text{ is bounded in } L^\infty(0,T, L^2(\Omega)) \\ &\text{and } \text{curl } (\mathbf{h}_\varepsilon)_{\varepsilon>0} \text{ is bounded in } L^2(0,T, L^2(\Omega)). \end{aligned} \quad (7.143)$$

Next, let $\chi \in C_0^\infty(\mathbb{R}^3)$ with $\chi(x) = 1$ outside some bounded set and $\chi(x) = 0$ on \mathcal{C} .

Since $\text{supp } \chi \subset \Omega_0$, 5.84 and 7.143 yield

$$\begin{aligned} \text{div } (\chi \mathbf{E}_\varepsilon(t)) &= \mathbf{E}_\varepsilon(t) \nabla \chi \in L^2(\mathbb{R}^3) \\ \text{and } \text{curl } (\chi \mathbf{E}_\varepsilon(t)) &= -\mathbf{E}_\varepsilon(t) \wedge \nabla \chi + \chi \text{curl } \mathbf{E}_\varepsilon(t) \in L^2(\mathbb{R}^3) \end{aligned}$$

This implies $\chi \mathbf{E}_\varepsilon(t) \in H^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$. Hence $\mathbf{E}_\varepsilon(t) \in \overset{0}{H^2}_{\text{curl}}(\Omega) \cap L^6(\{|x| > R_0\})$. By 5.84 this yields $\mathbf{E}_\varepsilon(t) \in \mathcal{X}$. Finally, 7.136 follows from estimate 3.57, 5.84, 7.139 and 7.143.

□

Theorem 4 *The unique solution (\mathbf{E}, \mathbf{h}) of problem 2.27 - 2.29 obeys $\mathbf{E} \in L^2_{loc}([0, \infty), \mathcal{X})$ and $\mathbf{h} \in W^{1,\infty}_{loc}([0, \infty), L^2(\Omega))$.*

Moreover,

$$\mathbf{E}_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \mathbf{E} \text{ in } L^2((0, T), Y_1^{r_0} \cap L^2_\gamma(G)) \text{ weakly ,} \quad (7.144)$$

$$\sigma_0(x, \mathbf{E}_\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} \sigma_0(x, \mathbf{E}) \text{ in } L^2((0, T), L^2_{\gamma-1}(G)) \text{ weakly} \quad (7.145)$$

and

$$\mathbf{h}_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \mathbf{h} \text{ in } W^{1,\infty}((0, T), L^2(\Omega)) \text{ weak} - * \quad (7.146)$$

and in $L^2((0, T), H^2_{curl}(\Omega))$ weakly .

Proof:

This follows immediately from lemma 10 and theorem 2.

□

Corollary 1 *Suppose that $\mathbb{R}^3 \setminus \overline{\Omega}$ is also a Lipschitz-domain Then*

$$\mathbf{h}_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \mathbf{h} \text{ in } C([0, T], L^2(\Omega \cap B_R)) \text{ strongly}$$

for all $R \in (0, \infty)$ and $T \in (0, \infty)$.

Proof: Choose $\chi \in C^\infty_0(\mathbb{R}^3)$ with $\chi(x) = 1$ on B_R . Then it follows from 6.116 and the result in [11] that for each bounded set $G \subset X_h \cap H^2_{curl}(\Omega)$ the set $\{\chi \cdot \mathbf{g} : \mathbf{g} \in G\}$ is precompact in $L^2(\Omega)$.

By 7.146 the set $(P\mathbf{h}_\varepsilon)_{\varepsilon > 0}$ is bounded in $L^2((0, T), X_h \cap H^2_{curl}(\Omega)) \cap W^{1,\infty}((0, T), L^2(\Omega))$. Hence, it follows for Arzela's theorem that $(\chi P\mathbf{h}_\varepsilon)_{\varepsilon > 0}$ is precompact in $C([0, T], L^2(\Omega))$ for all $T \in [0, \infty)$.

Since $\mu^{-1} \text{curl } \mathbf{E}_\varepsilon(t) \in X_h$, it follows from 5.73 that $(1 - P)\mathbf{h}_\varepsilon(t) = (1 - P)\mathbf{h}_0$. Therefore, $(\chi \mathbf{h}_\varepsilon)_{\varepsilon > 0}$ is precompact in $C([0, T], L^2(\Omega))$ for all $T \in [0, \infty)$. Finally, the assertion follows from 7.146.

□

8 Appendix

In this section the extension theorem for the spaces $H_{curl}^p(\Omega)$ used in the proof of theorem 1 is given.

Theorem 5 *Let $\Omega_1 \subset \mathbb{R}^3$ be a bounded Lipschitz-domain. Then there exists a bounded operator $T : L^1(\Omega_1, \mathcal{C}^8) \rightarrow L^1(\mathbb{R}^3, \mathcal{C}^8)$ with the following properties.*

i) $(T\mathbf{w})(x) = \mathbf{w}(x)$ for all $\mathbf{w} \in L^1(\Omega_1)$, $x \in \Omega_1$.

ii) Let $p \in [1, \infty)$. Then $T\mathbf{w} \in L^p(\mathbb{R}^3, \mathcal{C}^8)$ and $\|T\mathbf{w}\|_{L^p(\mathbb{R}^3)} \leq C_p \|\mathbf{w}\|_{L^p(\Omega_1)}$ for all $\mathbf{w} \in L^p(\Omega_1)$.

iii) Let $p \in [1, \infty)$. Then $T\mathbf{w} \in H_{curl}^p(\mathbb{R}^3)$ and $\|T\mathbf{w}\|_{H_{curl}^p(\mathbb{R}^3)} \leq C_p \|\mathbf{w}\|_{H_{curl}^p(\Omega_1)}$ for all $\mathbf{w} \in H_{curl}^p(\Omega_1)$.

Proof:

By the assumptions on Ω_1 there exist $U_0, U_1, \dots, U_M \subset \mathbb{R}^3$ and Bi-Lipschitz transformations $T_k : (-1, 1)^3 \rightarrow U_k$ with the properties

$$\Omega_1 \subset U_1 \cup \dots \cup U_M, \quad \overline{U_0} \subset \Omega_1,$$

$$U_k \cap \Omega_1 = T_k(V_k) \text{ with } V_k \stackrel{\text{def}}{=} \{x \in (-1, 1)^3 : x_3 < 0\}$$

$$\text{and } U_k \cap \partial\Omega_1 = T_k(\{x \in (-1, 1)^3 : x_3 = 0\}).$$

Let $\chi_k \in C_0^\infty(U_k)$, $k \in \{0, \dots, M\}$ be a partition of unity subordinate to the covering U_0, \dots, U_M of $\overline{\Omega_1}$, i.e. $\sum_{k=0}^M \chi_k = 1$ on $\overline{\Omega_1}$.

Now, suppose $\mathbf{w} \in L^1(\Omega_1, \mathcal{C}^8)$.

Define for $k = 1, \dots, M$ $\mathbf{f}_k \in L^1(V_k, \mathcal{C}^8)$ by

$$\mathbf{f}_k(y) \stackrel{\text{def}}{=} DT_k(y)^* \mathbf{w}(T_k(y)) \text{ for } y \in V_k.$$

Let $\mathbf{F}_k \in L^1((-1, 1)^3, \mathcal{C}^8)$ be an extension defined on $(-1, 1)^3$ by reflection at the plane $y_3 = 0$, i.e.

$$\mathbf{F}_k(y) \stackrel{\text{def}}{=} (\mathbf{f}_{k,1}(y_1, y_2, -y_3), \mathbf{f}_{k,2}(y_1, y_2, -y_3), -\mathbf{f}_{k,3}(y_1, y_2, -y_3)) \text{ if } y_3 \geq 0$$

$$\text{and } \mathbf{F}_k(y) \stackrel{\text{def}}{=} \mathbf{f}_k(y) \text{ if } y_3 \leq 0.$$

Finally, set

$$(T\mathbf{w})(x) \stackrel{\text{def}}{=} \chi_0(x)\mathbf{w}(x) + \sum_{k=1}^M \chi_k(x)D(T_k^{-1})(x)^* \mathbf{F}_k(T_k^{-1}(x)) \quad (8.147)$$

for $x \in U_0 \cup \dots \cup U_M$ and

$$(T\mathbf{w})(x) \stackrel{\text{def}}{=} 0 \text{ for } x \in \mathbb{R}^3 \setminus (U_0 \cup \dots \cup U_M).$$

Now, suppose $x \in \Omega_1$. Then it follows from the definitions of \mathbf{f}_k , \mathbf{F}_k and $T(\mathbf{w}) \in L^1(\mathbb{R}^3, \mathcal{G}^8)$ that

$$\begin{aligned} (T\mathbf{w})(x) &= \chi_0(x)\mathbf{w}(x) + \sum_{k=1}^M \chi_k(x)D(T_k^{-1})(x)^*\mathbf{f}_k(T_k^{-1}(x)) \\ &= \sum_{k=0}^M \chi_k(x)\mathbf{w}(x) = \mathbf{w}(x), \end{aligned}$$

i.e. T obeys i). ii) follows from the substitution formula.

Now, suppose $\mathbf{w} \in H_{curl}^p(\Omega_1)$.

As a consequence of the next lemma one has $\mathbf{f}_k \in H_{curl}^p(V_k)$. Moreover, the extension $\mathbf{F}_k \in L^p((-1, 1)^3)$ of \mathbf{f}_k defined by reflection obeys $\mathbf{F}_k \in H_{curl}^p((-1, 1)^3)$. Finally, it follows from 8.147 and again from the next lemma that $T\mathbf{w} \in H_{curl}^p(\mathbb{R}^3)$.

□

In the previous theorem the following lemma is used.

Lemma 11 *Let $U, V \subset \mathbb{R}^3$ be open sets, $\mathbf{w} \in L_{loc}^p(U)$ with $curl \mathbf{w} \in L_{loc}^p(U)$. Moreover, let $T : V \rightarrow U$ be a Bi-Lipschitz transformation.*

Define

$$\mathbf{f}(y) \stackrel{\text{def}}{=} DT(y)^*\mathbf{w}(T(y)) \text{ for } y \in V.$$

Then $\mathbf{f} \in L_{loc}^p(V)$ with $curl \mathbf{f} \in L_{loc}^p(V)$ and

$$(curl \mathbf{f})(y) = M_T(y)(curl \mathbf{w})(T(y)) \text{ for } y \in V, \quad (8.148)$$

where $M_T \in L_{loc}^\infty(V, \mathbb{R}^{3 \times 3})$ has the line-vectors $\mathbf{a}_1 \stackrel{\text{def}}{=} \partial_2 T \wedge \partial_3 T$, $\mathbf{a}_2 \stackrel{\text{def}}{=} \partial_3 T \wedge \partial_1 T$ and $\mathbf{a}_3 \stackrel{\text{def}}{=} \partial_1 T \wedge \partial_2 T$.

Proof:

By 8.148 and standard density arguments it suffices to consider $\mathbf{w} \in C^\infty(U, \mathcal{G}^8)$. Suppose V_0 is open, bounded with $\overline{V_0} \subset V$. Let $\omega_n \in C_0^\infty(\mathbb{R}^3)$ be a mollifier-sequence and define $T_n \stackrel{\text{def}}{=} \omega_n * T \in C^\infty(V_0)$. Since $T \in W^{1,\infty}(V)$, one has

$$DT_n = \omega_n * DT \xrightarrow{n \rightarrow \infty} DT \text{ in } L^1(V_0) \text{ strongly,} \quad (8.149)$$

$$\|T_n - T\|_{L^\infty(V_0)} \xrightarrow{n \rightarrow \infty} 0 \text{ and } \|DT_n\|_{L^\infty(V_0)} \leq K. \quad (8.150)$$

Define $\mathbf{f}_n \in C^\infty(V_0)$ by

$$\mathbf{f}_n(y) \stackrel{\text{def}}{=} DT_n(y)^*\mathbf{w}(T_n(y)) \text{ for } y \in V_0. \quad (8.151)$$

Then

$$(\operatorname{curl} \mathbf{f}_n)(y) = M_{T_n}(y)(\operatorname{curl} \mathbf{w})(T_n(y)) \text{ for } y \in V_0. \quad (8.152)$$

From 8.149 - 8.152 it follows that

$$\mathbf{f}_n \xrightarrow{n \rightarrow \infty} DT_n^* \cdot (\mathbf{w} \circ T_n) \text{ and } \operatorname{curl} \mathbf{f}_n \xrightarrow{n \rightarrow \infty} M_T \cdot ((\operatorname{curl} \mathbf{w}) \circ T_n) \quad (8.153)$$

in $L^1(V_0)$ strongly. This yields

$$\operatorname{curl} \mathbf{f} = M_T \cdot ((\operatorname{curl} \mathbf{w}) \circ T_n) \in L^p(V_0)$$

in the sense of distributions.

□

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