Singularities of linear time-varying DAEs

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Abstract

Singular issues arising in linear time-varying differential-algebraic equations are addressed in this paper. We review the use of projector methods based upon the tractability index concept for the analysis of regular problems. A taxonomy of singularities which describes the failing of some assumption in the tractability index definition is then introduced. The analysis of singular problems is focused on situations in which the degeneracy is a minimal one, namely, equations which admit a well-defined extension of the solution set through the singularity. In this framework, so-called weak singularities are shown to display a non-singular local flow despite the singular nature of the problem, extending previous results proved for low-index autonomous systems. Several examples illustrate the scope of the work.

1 Introduction

Singular differential-algebraic equations (DAEs) may be roughly described as implicit system of differential equations $F(x', x, t) = 0$ which fail to satisfy the conditions supporting the definition of an index at a given triple (p_0, x_0, t_0). Singular problems are sometimes defined by a DAE which has a well-defined index except on a lower-dimensional subset of the state space. Other problems are defined by parameterized regular systems which undergo a singularity at a particular value of the parameter. Autonomous singular equations arise, for example, in magnetohydrodynamics [5, 16], nonlinear circuit theory [6, 7], power systems [33, 34] or root-finding problems [28, 29, 30]. Quasilinear singular ODEs [22, 29, 32] and semiexplicit singular index 1 DAEs [5, 6, 7, 16, 27, 31, 33, 34] have received

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special attention. A broader class of quasilinear singular problems has been studied in [23, 24].

Singularities may also be displayed in the non-autonomous context of linear time-varying (LTV) DAEs

$$A(t)x' + B(t)x = q(t), \tag{1}$$

where A and B take values in $\mathbb{R}^{n\times n}$ ($\mathbb{R}^{p\times r}$ standing for the set of real matrices with p rows and r columns), whereas x and q are n-dimensional vectors. A classical source of singular problems in the ODE context comes from the study of systems of the form (1) for which A has maximum rank except at an isolated t^*. Note that these problems comprise, after a straightforward reformulation, higher order scalar LTV equations whose leading coefficient vanishes at a given t^* [13, 35]. The purpose of the present paper is to address some related issues in the context of LTV DAEs. Generally speaking, equation (1) will be a DAE if A is rank-deficient on a whole interval: in this context, singularities will be displayed when there is some rank drop or, more generally, some index change in this interval.

Linear time-varying DAEs have been extensively studied in the last decades [2, 3, 11, 14, 17, 25, 26]. Results in this field are of central importance in the control of descriptor systems and in electrical circuit theory: see references in [8, 15]. Most of these approaches are based on the definition of an *index*, usually through some kind of iterative procedure which assumes a number of non-degeneracy (or *regularity*) conditions. Let us focus, for simplicity, on homogeneous problems

$$A(t)x' + B(t)x = 0. \tag{2}$$

From a geometrical point of view, the above-mentioned procedures usually lead to a lower-dimensional set which smoothly varies with t and where the solutions of the DAE (the so-called *DAE flow* or, sometimes, *reduced flow* of the problem) lie. A smooth basis on this space would theoretically allow for a description of the DAE flow through a LTV ODE of the form

$$y' + C(t)y = 0, \tag{3}$$

where y would denote an m-dimensional ($m < n$) vector of coordinates with respect to such a basis. Equation (3) may be alternatively interpreted, together with $t' = 1$, as defining a flow on a smooth manifold in (t, x)-space.

Most authors explicitly acknowledge the fact that the non-degeneracy assumptions defining the index may fail at some t^*, yielding a singularity at this point. In this case, the equation describing the DAE flow would typically be a singular ODE

$$D(t)y' + E(t)y = 0, \tag{4}$$

rank deficiencies in D characterizing the singularities of the problem. In (t, x)-space, the hyperplane $t = t^*$ would then be an impasse set [22, 23, 24].

Among these approaches, projector methods, based upon the tractability index concept [1, 8, 10, 11, 17, 18, 19, 20], provide a tool for the analytical and numerical study of

LTV DAEs. According to the general discussion above, the tractability index is supported on a set of non-degeneracy assumptions: in the present paper, we will address situations in which some of these assumptions fail at an isolated t^*.

Our main purpose is to discuss, within this family of singular equations, some problems which may in a certain sense be considered the less degenerate ones. Specifically, we will consider situations in which, despite the presence of a singularity, the solution manifold may be smoothly extended through t^*. Equivalently, in these equations there exists a smooth extension of a basis of the manifold where the solutions of the DAE lie. This rules out sudden jumps on this manifold which may be displayed in non-analytic systems, and allows for a theoretical reduction to a singular ODE of the form (4) (or (3), where $C(t)$ may become undefined at t^*), y still denoting coordinates with respect to a *smooth* basis.

Within this type of problems, we will then consider cases in which the DAE flow is in fact a non-singular one, in spite of the singular nature of the equations. In these systems, solutions are well-defined through the singular value t^*. This phenomenon will be analyzed using the *weak singularity* concept, which arises as a natural analog of weak singular points in autonomous equations [27, 28, 29]. A characterization of weak singularities will be performed using desingularization techniques. Special attention will be paid to *critical* problems, for which such behavior is not a straightforward extension of that of the quasilinear system obtained after adding the equation $t' = 1$. The attention will be mainly restricted to singular index 1 equations; however, we will also address some index 2 systems which may be of help in the understanding of general higher index singular DAEs.

The paper is organized as follows. Section 2 reviews the tractability index concept, and presents some background on the use of projector methods for the analysis of regular linear DAEs. This section is self-contained and has been written in a tutorial spirit: we believe that it might be of interest for a reader aimed at getting some intuition on linear DAEs and their analysis via projector techniques. A taxonomy of singularities for linear time-varying DAEs is presented in Section 3, where some geometrical issues concerning the solution manifold are also addressed. Section 4 reviews the weak singularity concept for certain autonomous problems, and extends this notion to LTV DAEs with arbitrary index. Weak singularities describe the existence of a non-singular flow through the singular value: an explicit characterization of these is discussed for ODEs and index 1 DAEs. Some remarks on index 2 cases provide a hint for the analysis of higher index problems. Section 5 illustrates the scope of these results through several examples. Finally, Section 6 compiles some concluding remarks.

2 Regular LTV DAEs: projector methods and the tractability index

We review in this section the tractability index concept, together with the main features of projector methods for index 1 and index 2 linear DAEs: see a thorough discussion in [10, 17]. Recent developments can be found in [1, 8, 19, 20]. Problems with a well-defined tractability index will be called *regular* tractable DAEs: when some of the conditions supporting this definition fail at a given t^*, we will be led to a singularity, as detailed in Section 3.

Although our interest is focused on time-varying problems, a summary of projector techniques for time-invariant DAEs will be included for the sake of clarity. In our discussion, certain special (so-called *canonical*) projectors will play a central role. These projectors provide an explicit description of the dynamics on the solution manifold, and will be particularly useful for the purposes of the present paper.

2.1 The tractability index

Consider a homogeneous linear equation of the form

$$A(t)x' + B(t)x = 0, \tag{5}$$

where $A(t)$ and $B(t)$ are continuous matrix-valued functions. The tractability index concept is based on the construction of the following sequence (see [1, 17, 20] and references therein):

$$\begin{aligned} A_0(t) &= A(t), \\ B_0(t) &= B(t) - A(t)P_0'(t), \\ A_{i+1}(t) &= A_i(t) + B_i(t)Q_i(t), \\ B_{i+1}(t) &= B_i(t)P_i(t) - A_{i+1}(t)(P_0(t)P_1(t)\dots P_{i+1}(t))'P_0(t)P_1(t)\dots P_i(t), \end{aligned} \tag{6}$$

where $Q_i(t)$ denotes a projector onto $N_i(t) = \mathrm{Ker}A_i(t)$, and $P_i(t) = I - Q_i(t)$. In this construction, it is assumed that $Q_i(t)$ may be chosen smooth (continuously differentiable), and such that $Q_j(t)Q_i(t) = 0$ for $j > i$. The existence of a smooth projector $Q_i(t)$ onto $N_i(t)$ is equivalent to the smoothness of $N_i(t)$, that is, to the existence of a continuously differentiable basis $n_1(t), \dots, n_{p_i}(t)$ for $N_i(t)$. Recent results show that this assumption may in fact be relaxed to the requirement that $P_0(t)P_1(t)\dots P_i(t)$ (that is, $P_0(t)$, $P_0(t)P_1(t)$, etc.) be continuously differentiable [20].

Definition 1. *The DAE (5) is* regular with tractability index k *on a given interval* $\mathcal{I}$ *if* k *is the minimum non-negative integer such that*

1. *The matrices* $A_i(t)$ *are singular with constant rank and smooth kernel* $N_i(t)$ *on* $\mathcal{I}$*, for* $0 \leq i < k$.
2. $A_k(t)$ *is non-singular on* $\mathcal{I}$.

Note that, for tractable systems, we are indeed allowed to choose $Q_i(t)$ of class C^1 from the first assumption. The existence of smooth projectors satisfying $Q_j(t)Q_i(t) = 0$ and independence on the choice of the projectors follow from results proved in [11, 12, 20].

It is worth indicating that the constant rank assumption is in fact comprised in the smooth kernel one. Conversely, if $A_i(t)$ is smooth, the smoothness of $\mathrm{Ker}A_i(t)$ follows from the constant rank assumption (see e.g. [25]). Since in many problems $A_i(t)$ will be smooth, the most apparent aspect in this regard is the constant rank condition: for this reason, constant rank is explicitly required in the definition. Remark also that the definition above is directly applicable to time-dependent matrix pencils, to non-homogeneous linear DAEs, as well as to constant coefficient problems, where a simplified chain is obtained after removing all the terms which involve any derivative. The tractability index is, in the latter case, nothing else than the classical Kronecker index of a regular matrix pencil [9].

In particular, linear time varying ODEs, characterized by an invertible $A(t)$ for all t, are regular with index 0. Regular problems with index 1 or higher display one of the main features of DAEs: solutions will only be defined on a lower-dimensional subset of the state space. In the linear setting, the (non-singular) matrix $A_k(t)$ will play a key role in the description of this solution set, and also in the actual computation of solutions.

Definition 1 comprises some geometrical properties which will play an important role in singular problems. The key result in this direction is proved in [10]:

Proposition 1. *Let A, $B \in \mathbb{R}^{n\times n}$. Assume that A is singular and let Q be a projector onto $N = \mathrm{Ker}A$. Define $S = \{x \in \mathbb{R}^n : Bx \in \mathrm{Im}A\}$. Then, the following statements are equivalent:*

1. *$A + BQ$ is non-singular;*
2. *$\mathbb{R}^n = N \oplus S$;*
3. *$\{A, B\}$ is a regular matrix pencil with index 1.*

Let us elaborate on some implications of this result. First, assume that condition 1 in definition 1 is satisfied for a linear time varying DAE such as (5). We may then apply proposition 1 to $A_{k-1}(t)$, $B_{k-1}(t)$, $Q_{k-1}(t)$ for any t in $\mathcal{I}$. Hence, condition 2 in the tractability index definition can be alternatively written, with the notation $S_{k-1}(t) = \{x \in \mathbb{R}^n : B_{k-1}(t)x \in \mathrm{Im}A_{k-1}(t)\}$, as $\mathbb{R}^n = N_{k-1}(t) \oplus S_{k-1}(t)$ or $\mathrm{ind}(A_{k-1}(t), B_{k-1}(t)) = 1$ for all t in $\mathcal{I}$.

In this situation, from the constant rank assumption on $A_{k-1}(t)$, it follows that there exist a continuous mapping $W_{k-1} : \mathcal{I} \to \mathbb{R}^{p\times n}$, with $p = n - r$, $r = \mathrm{rk}A_{k-1}(t)$, such that $\mathrm{rk}\ W_{k-1}(t) = p\ \forall t \in \mathcal{I}$, and $v \in \mathrm{Im}A_{k-1}(t) \Leftrightarrow W_{k-1}(t)v = 0$. Hence, the set $S_{k-1}(t)$ may be described as Ker $W_{k-1}(t)B_{k-1}(t)$. This allows one to show that assumption 2 in definition 1 actually comprises two different properties:

1. $S_{k-1}(t)$ varies continuously with t.
2. $S_{k-1}(t)$ is transversal to $N_{k-1}(t)$, for all t in $\mathcal{I}$.

Whilst the latter is obvious, the former follows from the fact that $S_{k-1}(t)$ must be r-dimensional and, therefore, $W_{k-1}(t)B_{k-1}(t)$ must have constant rank p. In consequence, $S_{k-1}(t) = \mathrm{Ker}\ W_{k-1}(t)B_{k-1}(t)$ will be continuous in t. The key aspect here is that the constant dimension of $S_{k-1}(t)$ does not follow from that of $N_{k-1}(t)$ alone, as it will be illustrated in Section 3.

Some interesting conclusions may also be derived from the regularity of the (index 1) matrix pencil $\{A_{k-1}(t), B_{k-1}(t)\}$ in tractable systems, namely

$$\mathrm{ind}(A_{k-1}(t), B_{k-1}(t)) = 1 \Rightarrow \mathrm{rk}\ (A_{k-1}(t) \quad B_{k-1}(t)) = n \Rightarrow \mathrm{rk}\ W_{k-1}(t)B_{k-1}(t) = p. \tag{7}$$

Continuity of $S_{k-1}(t)$ follows from the latter, as indicated above. The first part of (7) is in fact true for any regular matrix pencil, regardless of the index, since a rank deficiency in $(A_{k-1}(t) \quad B_{k-1}(t))$ would imply the existence of a non-vanishing left null vector p, which would make $p(\lambda A_{k-1}(t) + B_{k-1}(t)) = 0$ for any λ and, in turn, this would imply the singularity of the matrix pencil. The second part easily follows from the identity $W_{k-1}(t)(A_{k-1}(t) \quad B_{k-1}(t)) = (0 \quad W_{k-1}(t)B_{k-1}(t))$. The important consequence is that, in singular problems in which assumption 2 in definition 1 fails, we may still guarantee the existence of a continuously defined manifold $S_{k-1}(t)$ from any of the less restrictive conditions in (7).

The following subsections show how the definition of regular tractable DAEs allows for a direct treatment of linear DAEs with index 1 or 2. The index 2 case provides also some hints to address problems with index greater than 2, which are not explicitly considered in the present paper. These subsections will also illustrate some consequences of the geometrical properties discussed above.

2.2 Projector methods for index 1 problems

2.2.1 Time-invariant case

The behavior of a linear, constant coefficient (time-invariant) homogeneous DAE

$$Ax' + Bx = 0 \tag{8}$$

may be easily described, when $\{A, B\}$ is a regular matrix pencil, in terms of the Kronecker index and the corresponding Kronecker canonical form (see e.g. [2]). This process involves a linear change of coordinates, and can be alternatively formulated in terms of projectors. The fact supporting this equivalence is that the tractability index presented in definition 1 is, in time-invariant problems, exactly the Kronecker index.

Let us focus on the index 1 case. Tractability with index 1 amounts to require invertibility of the matrix $A_1 = A + BQ$, Q being a projector onto $N_0 = \mathrm{Ker} A$ (for notational simplicity, the 0-subscript will be removed in the sequel for these projectors). According to proposition 1, this is equivalent to the transversality of N_0 and $S_0 = \{x \in \mathbb{R}^n : Bx \in \mathrm{Im} A\}$.

In index 1 problems, the set S_0 is in fact the solution manifold of the DAE. This means that all solutions lie on S_0 and they completely fill it. To get an explicit representation of these solutions, let us consider the projector $P = I - Q$ along N_0 , and remark that

$$\begin{aligned} A_1^{-1}BQ &= Q \\ A_1^{-1}A &= P \end{aligned}$$

since $AQ = 0$ and, therefore, $A_1Q = (A + BQ)Q = BQQ = BQ$. It follows that $Q = A_1^{-1}BQ$ and, finally, $A_1^{-1}A = A_1^{-1}(A_1 - BQ) = I - A_1^{-1}BQ = I - Q = P$.

We may then premultiply the DAE by A_1^{-1} and decompose $x = Px + Qx$ to get

$$A_1^{-1}Ax' + A_1^{-1}Bx = A_1^{-1}Ax' + A_1^{-1}BPx + A_1^{-1}BQx = 0$$

or, equivalently,

$$Px' + A_1^{-1}BPx + Qx = 0,$$

which in turn can be decoupled into the system

$$\begin{aligned} Px' + PA_1^{-1}BPx &= 0 && \text{(9a)} \\ QA_1^{-1}BPx + Qx &= 0. && \text{(9b)} \end{aligned}$$

With the notation $u = Px,\ v = Qx$, (9) becomes

$$\begin{aligned} u' + PA_1^{-1}Bu &= 0 && \text{(10a)} \\ QA_1^{-1}Bu + v &= 0. && \text{(10b)} \end{aligned}$$

We may consider (10a) as an *inherent ODE* defined for $u \in \mathbb{R}^n$. The subspace ImP is then invariant for this ODE, and the solutions of the DAE can be recovered from the differential variable u using the algebraic relation (10b).

Notice that ImP will generally not be the true solution manifold S_0, and that the solution manifold is not necessarily invariant for the inherent equation. However, the identification between ImP and S_0 is possible for a special choice of the projectors P, Q. Let us define the *canonical projector* Q_c as the unique projector onto $N_0 = \text{Ker}A$ along S_0 [18]. Therefore, $P_c = I - Q_c$ is a projector onto the solution manifold S_0. It may be shown that $QA_1^{-1}B = Q_c$ for any initial choice of the projector Q. This makes it possible to restate (9) as

$$\begin{aligned} P_cx' + P_cA_1^{-1}BP_cx &= 0 && \text{(11a)} \\ Q_cx &= 0, && \text{(11b)} \end{aligned}$$

where now A_1 is understood to be defined from Q_c. In this case, (10) reads

$$\begin{aligned} u' + P_cA_1^{-1}Bu &= 0 && \text{(12a)} \\ v &= 0. && \text{(12b)} \end{aligned}$$

Inherent flow and DAE flow. It is worth remarking the distinction between the related concepts of *inherent* or *inner* flow and *DAE* (occasionally called *reduced*) flow. The choice of any projector P along N_0 leads to equation (10a), which defines an *inherent* flow in $\mathbb{R}^n$. The linear subspace ImP is invariant for this flow, and the solutions of the DAE can be recovered, from the restriction of the inherent flow to this space, through the algebraic relation (10b). For the particular case defined by the choice of canonical projectors, the restriction of the corresponding inherent flow to the solution manifold $S_0 = \text{Im}P_c$ yields the *DAE* flow, which comprises the true solutions of the DAE. In this case, the inherent flow may also be called an *underlying* flow.

An explicit coordinate description of the equations defining the DAE flow can be easily obtained from the choice of linear coordinates in $S_0 = \text{Im}P_c$, that is, from the choice of a basis $(s_1, \ldots, s_r)$ in this space, where $r = \text{dim}S_0 = \text{rk}A$. If the linear operator $P_cA_1^{-1}B$ is understood to be restricted to S_0 and expressed in the aforementioned basis, we may see equation (12a) as an r-dimensional ODE, u being now a vector of coordinates with respect to this basis. The DAE flow may then be expressed from the solutions of this ODE simply as $u_1(t)s_1 + \ldots + u_r(t)s_r$. The choice of any other basis leads to another ODE which is linearly conjugate to the previous one. For this reason, with abuse in the terminology, equation (12a) is sometimes referred to, in the LTI context, simply as "the" reduced ODE.

2.2.2 Time-varying case

From definition 1, it follows that a LTV DAE (5) has tractability index 1 on $\mathcal{I}$ if on this interval $A(t)$ and $B(t)$ are continuous, $A(t)$ is singular but has a smooth kernel $N_0(t)$, and $A_1(t) = A(t) + B_0(t)Q(t)$ is non-singular, where $B_0(t) = B(t) - A(t)P'(t)$, $Q(t)$ is a smooth projector onto $N_0(t)$, and $P(t) = I - Q(t)$.

As it was discussed in 2.1, tractability with index 1 on $\mathcal{I}$ is equivalent to the condition $\mathbb{R}^n = N_0(t) \oplus S_0(t)\ \forall t \in \mathcal{I}$, with $S_0(t) = \{x \in \mathbb{R}^n : B_0(t)x \in \mathrm{Im}A(t)\}$. Note that the solutions of the DAE must now lie on the manifold $\{x \in \mathbb{R}^n : B(t)x \in \mathrm{Im}A(t)\}$, but this set may be proved to be in fact $S_0(t)$ since $B_0(t) = B(t) - A(t)P'(t)$ and, therefore, $B(t)x \in \mathrm{Im}A(t) \Leftrightarrow B_0(t)x \in \mathrm{Im}A(t)$. It follows that all the geometrical remarks discussed in 2.1 for $S_{k-1}(t)$ are directly applicable to the solution manifold of a LTV index 1 DAE. Also, it is straightforward to show that, in this situation, tractability with index 1 is equivalent to the smoothness of $\mathrm{Ker}A(t)$ together with the index 1 assumption on the local matrix pencil $\{A(t), B(t)\}$ for all t. Basically, this fact expresses that the tractability index and the local (Kronecker) index are the same for index 1 LTV problems. In turn, this makes it possible to identify the tractability index with other index concepts in the index 1 setting: see [2, Theorem 2.4.2] and [25, Theorem 7.1].

Solutions are then defined only for initial points $x(t_0) \in S_0(t_0)$ and remain on $S_0(t)$ for all t. This allows one to define, as in the LTI case, the canonical projector $P_c(t)$ onto $S_0(t)$ along $N_0(t)$. The canonical projectors can be computed as $Q_c = QA_1^{-1}B_0$, $P_c = I - Q_c$: see details in [1, 8, 17]. The inherent ODE would read, in this case,

$$u' - P_c'u + P_cA_1^{-1}B_0u \;=\; 0, \tag{13}$$

B_0 and A_1 being now defined from the canonical projectors. The restriction of the corresponding flow to $\mathrm{Im}P_c(t) = S_0(t)$ yields the DAE flow.

If $S_0(t)$ is not only continuous but smooth (condition which follows immediately from smoothness assumptions on the operators A, B), we may choose coordinates with respect to a smooth basis in $S_0(t)$, and a LTV ODE describing the DAE flow in these coordinates can be derived from (13). Remark that the DAE flow can also be seen as an autonomous one lying on the manifold $S_0 = \cup_{t\in\mathcal{I}} S_0(t)$ (more precisely, $\cup_{t\in\mathcal{I}}\{t\} \times S_0(t)$) in (t, x)-space.

2.3 Projector methods for index 2 problems

2.3.1 Time-invariant case

Consider again a LTI DAE $Ax' + Bx = 0$ and let us now focus on the index 2 case. Tractability with index 2 for LTI problems amounts to the following. Let Q be any projector onto $N_0 = \mathrm{Ker}A$, and $P = I - Q$. Assume that $A_1 = A + BQ$ is a singular matrix. Define $B_1 = BP$, and let Q_1 be any projector onto $N_1 = \mathrm{Ker}A_1$ such that $Q_1Q = 0$. Then, tractability with index 2 is equivalent to the regularity of $A_2 = A_1 + B_1Q_1$, or to the condition $\mathbb{R}^n = N_1 \oplus S_1$, where $S_1 = \{x \in \mathbb{R}^n : B_1x \in \mathrm{Im}A_1\}$. This is also equivalent to the Kronecker index 2 condition on the matrix pencil.

Again, a special role is played by the canonical projector Q_{1c} onto N_1 along S_1, which can be obtained from any other projector Q_1 onto N_1 through the relation $Q_{1c} = Q_1A_2^{-1}B_1$.

An extension of the reasoning presented for the index 1 case yields

$$\begin{aligned} A_2^{-1}A &= P_{1c}P \\ A_2^{-1}BQ &= Q \\ A_2^{-1}BPQ_{1c} &= Q_{1c}, \end{aligned}$$

where $P_{1c} = I - Q_{1c}$ and A_2 is now understood to be defined from Q_{1c}. Premultiplying the DAE by A_2^{-1} and then by PP_{1c}, PQ_{1c} and QP_{1c}, respectively, leads to the decoupling

$$\begin{aligned} PP_{1c}x' + PP_{1c}A_2^{-1}BPP_{1c}x &= 0 && \text{(14a)} \\ PQ_{1c}x &= 0 && \text{(14b)} \\ QP_{1c}A_2^{-1}BPP_{1c}x + Qx &= 0. && \text{(14c)} \end{aligned}$$

Denoting $u = PP_{1c}x$, $v = PQ_{1c}x$, $w = Qx$, (14) becomes

$$\begin{aligned} u' + PP_{1c}A_2^{-1}Bu &= 0 && \text{(15a)} \\ v &= 0 && \text{(15b)} \\ QP_{1c}A_2^{-1}Bu + w &= 0. && \text{(15c)} \end{aligned}$$

Again, we may consider (15a) as an *inherent ODE* defined for $u \in \mathbb{R}^n$. The subspace $\mathrm{Im}PP_{1c}$ is then invariant for this ODE, and the solutions of the DAE can be recovered from the differential variable u using (15b) and (15c).

The projector Q can be chosen in a way such that the space $\mathrm{Im}PP_{1c}$ be the true solution manifold for the index 2 LTI DAE [18]. Namely, if after the previous development we define $Q_{0c} = QP_{1c}A_2^{-1}B$, $P_{0c} = I - Q_{0c}$, and redefine the matrices A_i, B_i accordingly, system (14) would read

$$\begin{aligned} P_{0c}P_{1c}x' + P_{0c}P_{1c}A_2^{-1}BP_{0c}P_{1c}x &= 0 \\ P_{0c}Q_{1c}x &= 0 \\ Q_{0c}x &= 0, \end{aligned}$$

and the inherent ODE becomes

$$u' + P_{0c}P_{1c}A_2^{-1}Bu = 0. \tag{16}$$

In the sense discussed in 2.2.1, (16) may also be understood as the reduced ODE of the system.

2.3.2 Time-varying case

Following again definition 1, a LTV DAE has tractability index 2 on $\mathcal{I}$ if on this interval $A(t)$ and $B(t)$ are continuous, $A(t)$ and $A_1(t)$ are singular with constant rank and smooth kernel, the projectors $Q(t)$ and $Q_1(t)$ satisfying $Q_1(t)Q(t)$ are smooth, and $A_2(t) = A_1(t)+B_1(t)Q_1(t)$ is non-singular, with $B_1(t) = B_0(t)P(t) - A_1(t)(P(t)P_1(t))'(t)P(t)$.

It is worth indicating that regularity of the local matrix pencil $\{A(t), B(t)\}$ does not follow from the tractability index 2 assumption. This represents a substantial difference with LTI problems, and makes LTV DAEs with index ≥ 2 more intricate than index 1 equations. The key aspect is that regularity of the local matrix pencil cannot assure (neither is necessary for) unique solvability of the DAE. It is true, however, that tractability with index 2 implies that the local matrix pencil $\{A_1(t), B_1(t)\}$ is regular with Kronecker index 1, as it was discussed in 2.1. Moreover, it may be shown that the modified matrix pencil $\{A(t), B_0(t)\} \equiv \{A(t), B(t) - A(t)P'(t)\}$ is regular with Kronecker index 2.

On the other hand, the identification between the solution manifold and any of the spaces $S_0(t)$ or $S_1(t)$ does no longer hold in problems with index higher than 1. The solution manifold will now be a lower dimensional subset of $\{x \in \mathbb{R}^n : B(t)x \in \mathrm{Im}A(t)\} \equiv S_0(t)$. Note that smoothness of the solution set may be derived from appropriate smoothness assumptions on the operators characterizing the DAE.

The solution manifold may also be described using canonical projectors. Besides the projector $Q_{1c}(t)$ onto $N_1(t)$ along $S_1(t)$, which may be computed as $Q_{1c} = Q_1 A_2^{-1} B_1$ from any other projector Q_1 onto N_1, we may introduce (see e. g. [1]) the additional canonical projector

$$Q_{0c}(t) = Q(t)P_{1c}(t)A_2^{-1}(t)B_0(t) + Q(t)Q_{1c}(t)(P(t)Q_{1c}(t))'(t)P(t).$$

With this construction, the solution manifold may indeed be shown to be $\mathrm{Im}P_{0c}P_{1c}(t)$. Solutions are defined only for initial points $x(t_0) \in \mathrm{Im}P_{0c}P_{1c}(t_0)$ and remain on $\mathrm{Im}P_{0c}P_{1c}(t)$ for all t. The inherent ODE would read in this case

$$u' - (P_{0c}P_{1c})'u + P_{0c}P_{1c}A_2^{-1}B_0 u \;=\; 0, \tag{17}$$

all operators being dependent on t and defined from Q_{0c}.

2.4 Examples

We present in this section two index 1 examples with a two-fold goal. First, we attempt to illustrate the use of projector methods for regular LTV problems. Our second purpose is to motivate the kind of different behaviors which can be displayed at singular points. Index 2 examples may be found in Sections 3 and 5.

2.4.1 Example 1 (I)

Let us consider the LTV homogeneous DAE

$$\begin{pmatrix} 1 & -t \\ 1 & -t \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.$$

With the above-introduced notation, we have

$$\begin{aligned} N_0(t) &= \mathrm{Ker}A(t) = \mathrm{span}\{(t,1)\} \\ S_0(t) &= \{x \in \mathbb{R}^2 : \; B(t)x \in \mathrm{Im}A(t)\} = \mathrm{span}\{(1,1)\}. \end{aligned}$$

Therefore, the DAE has tractability index 1 except at $t^* = 1$, where there is a singularity, which can be understood in terms of the the non-transversality of N_0 and S_0. In this simple example, it is easy to obtain by inspection the algebraic constraint $x_1 = x_2$, the flow being determined by the equation $(1-t)x_1' + x_1 = 0$. A systematic approach to derive this conclusion follows from the choice of the projectors

$$Q(t) = \begin{pmatrix} 0 & t \\ 0 & 1 \end{pmatrix}, \; P(t) = \begin{pmatrix} 1 & -t \\ 0 & 0 \end{pmatrix},$$

from which we obtain the canonical ones

$$Q_c(t) = \frac{1}{1-t} \begin{pmatrix} -t & t \\ -1 & 1 \end{pmatrix}, \; P_c(t) = \frac{1}{1-t} \begin{pmatrix} 1 & -t \\ 1 & -t \end{pmatrix},$$

and

$$P_c'(t) = \frac{1}{(1-t)^2} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

After some computations, we get

$$\begin{aligned} A_1^{-1}(t) &= \frac{1}{(1-t)^2} \begin{pmatrix} t^2 - t + 2 & -t^2 - 1 \\ t + 1 & -2t \end{pmatrix}, \\ B_0(t) &= \frac{1}{1-t} \begin{pmatrix} -t & 1 \\ -1 & 2 - t \end{pmatrix}, \end{aligned}$$

and, finally,

$$(-P_c' + P_c A_1^{-1} B_0)(t) = \frac{1}{1-t} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

This means that the inherent ODE is

$$u_1' + \frac{1}{1-t} u_2 = 0, \; u_2' + \frac{1}{1-t} u_2 = 0.$$

Solutions have the form $u_1 = C_1(1-t) + C_2, \; u_2 = C_1(1-t)$. In particular, given an initial point $u_1(t_0) = u_2(t_0) = C_0$ in the solution manifold for $t_0 \neq 1$, the corresponding solution of the DAE is (written in x-coordinates)

$$x_1 = x_2 = \frac{C_0}{1-t_0}(1-t). \tag{18}$$

This flow displays a singularity at $t_0 = 1$, which yields an infinite number of analytic solutions satisfying $x_1(1) = 0$ (namely, $x_1 = C(1-t)$ for any $C \in \mathbb{R}$), and no solution for $x_1(1) \neq 0$.

2.4.2 Example 2 (I)

Let us now consider the system

$$\begin{pmatrix} 1 & -t \\ 1 & -t \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} + \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.$$

In this case, it is

$$\begin{aligned} N_0(t) &= \mathrm{Ker} A(t) = \mathrm{span}\{(t,1)\} \\ S_0(t) &= \{x \in \mathbb{R}^2 :\ B(t)x \in \mathrm{Im} A(t)\} = \mathrm{span}\{(1,t)\}. \end{aligned}$$

Therefore, the DAE has tractability index 1 except at $t^* = \pm 1$, where N_0 and S_0 are non-transversal. The (non-canonical) projectors introduced for example 1 are still valid, since they are defined only from the matrix $A(t)$:

$$Q(t) = \begin{pmatrix} 0 & t \\ 0 & 1 \end{pmatrix},\ P(t) = \begin{pmatrix} 1 & -t \\ 0 & 0 \end{pmatrix}.$$

However, the canonical ones now read

$$Q_c(t) = \frac{1}{1-t^2}\begin{pmatrix} -t^2 & t \\ -t & 1 \end{pmatrix},\ P_c(t) = \frac{1}{1-t^2}\begin{pmatrix} 1 & -t \\ t & -t^2 \end{pmatrix},$$

and

$$P_c'(t) = \frac{1}{(1-t^2)^2}\begin{pmatrix} 2t & -1-t^2 \\ 1+t^2 & -2t \end{pmatrix},$$

from which we obtain

$$\begin{aligned} A_1^{-1}(t) &= \frac{1}{(1-t^2)^2}\begin{pmatrix} t^3-t+2 & -t^3-t^2+t-1 \\ t^2+2t-1 & -t^3-t^2-t+1 \end{pmatrix}, \\ B_0(t) &= \frac{1}{1-t^2}\begin{pmatrix} -t^3 & 1 \\ -t & 2-t^2 \end{pmatrix}. \end{aligned}$$

It then follows that the operator in the inherent ODE (13) reads

$$(-P_c' + P_c A_1^{-1} B_0)(t) = \frac{1}{(1-t^2)^2}\begin{pmatrix} -t & 1 \\ -1 & 2t-t^3 \end{pmatrix}.$$

The solution manifold is defined in this case by the condition $x_1 t - x_2 = 0$. If we restrict the solutions of the inherent ODE to this manifold, with a given initial condition $u_1(t_0) = C_0,\ u_2(t_0) = u_1(t_0)t_0 = C_0 t_0$, we get

$$x_1 = C_0,\ x_2 = C_0 t. \tag{19}$$

This is a well defined flow even through the singular value. The reason for this nice behavior is that the singularity at $t^* = 1$ is a weak one, as illustrated in Sections 4 and 5.

3 Singularities of LTV DAEs

As it was discussed in Section 1, most definitions of an index are based upon an iterative procedure which assumes a number of non-degeneracy conditions. This is the case of the tractability index, as presented in 2.1. The fail of one non-degeneracy assumption in the sequence defining regular tractable DAEs leads naturally to the notion of a singularity presented below.

Definition 2. *The DAE (5) is said to have a* j-singularity *or an* order-j singularity *at* t^*, *j being a non-negative integer, if there exists a neighborhood* $\mathcal{I}^{t^*}$ *of* t^* *for which*

1. *There exist integers* $k_1,\ k_2 \geq j$ *such that the DAE is regular with tractability index* k_1 *on* $\{t \in \mathcal{I}^{t^*} : t < t^*\}$ *and tractability index* k_2 *on* $\{t \in \mathcal{I}^{t^*} : t > t^*\}$.
2. j *is the minimum non-negative integer such that* $A_j(t)$ *in (6) fails to have either constant rank or smooth kernel on* $\mathcal{I}^{t^*}$.

Most of the singular cases considered in this paper will be defined by a non-constant rank in $A_j(t)$. For the sake of consistency it is necessary, however, to consider situations in which it is the smoothness of $\mathrm{Ker} A_j(t)$ what fails at the singularity, since in this case it may be not possible to continue the sequence up to $A_{j+1}(t)$ due to the non-smoothness of the corresponding projector. Note also that definition 2 may be proved independent of the choice of the projectors. The reason supporting this is the fact that the constant rank condition is not dependent on the projectors, which bases the definition of tractable systems: see [11, 12, 20].

Regardless of the order j of the singularity, if $k_1 = k_2 \equiv k$ we will say that the DAE has *singular index* k around t^*, meaning that it behaves as a regular index k problem except at t^*, where some pathological behavior may be displayed. On the other hand, if $k_1 \neq k_2$, we will say that the system undergoes an *index jump* at t^*, as in

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \alpha(t) & 0 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0, \tag{20}$$

where $\alpha(t)$ is a C^l function (for any positive l or $l = \infty$) such that $\alpha(t) = 0 \Leftrightarrow t \leq 0$. A 0-singularity is displayed at the origin, since there is a change in the rank of $A(t)$ there, and the system may be easily shown to jump from index 1 at $t < 0$ to index 2 at $t > 0$. It is worth indicating that, despite the index jump at the singularity, this DAE displays a nice behavior for all t, as it is discussed in 5.2. Note also that it is easy to construct DAEs with a similar shape but without index jumps, e.g.,

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha(t) & 0 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0,$$

which has tractability index 2 for every $t \neq 0$.

The above-mentioned jump behavior cannot be displayed in analytic systems. This follows from the fact that rank drops in analytic matrices occur only at isolated values [25]. Moreover, despite the rank change in $A_j(t)$, there exist a projector $Q_j(t)$ which admits an analytic extension to the singularity. The sequence may therefore be continued this way, yielding a chain which must lead to the same tractability index at both sides of the singular value. In the sequel, we will restrict the attention to singular index k problems, that is, singular DAEs without index jumps.

According to definition 2, 0-singularities, such as the one occurring in (20), are those for which $A(t)$ has non-constant rank (or non-smooth kernel). Notice that order 0 in the singularity of (20) is entirely independent of the non-analytic character of the system or the index jump displayed at the origin. These are the only singularities which may happen in the ODE setting [13, 35]. The family of 0-singularities also comprises problems of the form $(t-t^*)\tilde{A}(t)x' + B(t)x = 0$, where $\tilde{A}(t)$ is typically a matrix with constant (not necessarily maximum) rank: see e.g. [21].

Intermediate singularities are those for which the order j satisfies $0 < j < k$. The following system, with singular index 2, displays a 1-singularity at the origin:

$$\begin{pmatrix} 1 & 0 & 0 \\ t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & t \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0. \tag{21}$$

Note that $\mathrm{rk}A(t) = 1, \forall t \in \mathbb{R}$. It is straightforward to check that

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a projector onto $\mathrm{Ker}A$, which yields, after some simple computations

$$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ t & 0 & t \\ 0 & 0 & 0 \end{pmatrix}.$$

A rank drop in $A_1(t)$ occurs at the origin. It is easy to prove that the DAE has tractability index 2 on $\mathbb{R} - \{0\}$, the problem hence having singular index 2 with a 1-singularity at the origin.

Remark that, at j-singularities of singular index k systems with $0 \leq j < k$, the matrix sequence may well lead to a non-singular extension of $A_k(t)$ through the singular value. Some examples of this situation, which provides a particularly simple setting, are presented in 5.2. Of course, this cannot happen in k-singularities, for which by definition A_k becomes singular at t^*. The fact that all the matrices $A_j(t)$ for $j < k$ have constant rank reduces somehow the number of different phenomena which may be displayed at k-singularities. Additionally, it is worth noting that, as it happens in some autonomous problems [27], the rank deficiency in $A_k(t)$ does not necessarily imply a singularity on the solution manifold: this issue is further analyzed below for the case of singular index 1 systems.

Geometrical remarks. Consider a 1-singularity of a singular index 1 LTV, that is, a problem in which $A(t)$ has constant rank and smooth kernel on $\mathcal{I}$, $A_1(t)$ being invertible on $\mathcal{I} - \{t^*\}$ and singular at t^*. According to the discussion in 2.1, the set $S_0(t)$ (which is in fact the solution manifold, since we are dealing with an index 1 problem) must display any of the following degeneracies:

1. $S_0(t)$ is not continuous at t^*.
2. $S_0(t)$ being continuous, is not transversal to $N_0(t)$ at t^*.

Obviously, from a geometrical point of view, the former is more degenerate than the latter. In the first case, we may further distinguish problems in which there exists a continuous extension of $S_0(t)$ to t^*, that is, there exists a continuous space $S_0^*(t)$ such that $S_0^*(t) = S_0(t)\ \forall t \in \mathcal{I} - \{t^*\}$, from those which do not admit such an extension. An example of the former is defined by

$$\begin{pmatrix} 1 & -t \\ 1 & -t \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} + \begin{pmatrix} 1-t & 0 \\ 0 & 1-t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0. \tag{22}$$

It is easy to prove that $S_0(t) = \mathrm{span}\{(1,1)\}$ for $t \neq 0$, whereas $S_0(0) = \mathbb{R}^2$. Obviously, $S_0(t)$ admits the smooth extension $S_0^*(t) = \mathrm{span}\{(1,1)\}$, $\forall t$.

However, this is not always the case, as the following example shows:

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \alpha(t) & \beta(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0, \tag{23}$$

where $\alpha(t)$ (resp. $\beta(t)$) is a C^∞ function which vanishes exactly on $t \leq 0$ (resp. $t \geq 0$). Again, it is immediate to prove that $S_0(t) = \mathrm{span}\{(1,0)\}$ for $t < 0$ and $S_0(t) = \mathrm{span}\{(0,1)\}$ for $t > 0$. No continuous extension may exist for this singularity.

Accordingly, the appropriate taxonomy regarding the existence of well-defined solutions through the singularity should distinguish between problems in which

1'. $S_0(t)$ does not admit a continuous extension at t^*.

2'. $S_0(t)$ admits a continuous extension which is not transversal to $N_0(t)$ at t^*.

The constant rank assumption on $A(t)$ leads to a description of $S_0(t)$ as Ker $W(t)B(t)$, with the notation introduced in 2.1 ($W(t)$ standing for $W_{k-1}(t) \equiv W_0(t)$ since $k = 1$). From the results in [25], it follows that analyticity on the operators $A(t)$, $B(t)$ rules out case 1' above. Analytic 1-singularities always satisfy 2', where "continuous" may be replaced by "analytic". Nevertheless, in non-analytic C^l ($l \geq 0$) problems, this is not always the case, as illustrated by system (23). It is therefore of interest to formulate conditions which guarantee the existence of a continuous or smooth extension of the solution manifold through the singularity or, in particular, conditions which assure that this manifold is itself continuous or smooth.

This type of conditions may be easily obtained from any of the assumptions on (7). Remarkably, the condition

$$\mathrm{rk}\ (A(t) \quad B(t)) = n, \tag{24}$$

which may be obviously satisfied at 1-singularities, guarantees (together with the constant rank condition on $A(t)$) that $W(t)B(t)$ has a continuous kernel $S_0(t)$. It is worth noting that (24) is an intermediate requirement in the index definition presented in [25]. Furthermore, it is shown in this reference that tractability with index 1 is equivalent to the index 1 definition there. This means that singularities will be displayed exactly at the same points with both definitions of the index. Nevertheless, if at a singular point we still assume constant rank on $A(t)$ together with (24), continuity (or smoothness) of the solution manifold in C^l problems follows.

Note that the discussion above does by no means imply that only at 1-singularities a continuous extension of the solution set may exist. Indeed, the condition rk $W(t)B(t) = p$, which also leads to a continuous $S_0(t)$, may be required even without assuming constant rank on $A(t)$ ($W(t)$ hence being non-continuous), that is, at 0-singularities of index 1 DAEs. Nevertheless, constant rank on $W(t)B(t)$ but not on $W(t)$ seems to require a special matching between A and B. For instance, (24) would in this case imply rk $W(t)B(t) =$ rk $W(t)$, yielding a non-constant rank in $W(t)B(t)$. Furthermore, in this situation $N_0(t)$ is non-smooth: these facts suggest that this kind of problems may be more intricate from a geometrical point of view.

Finally, remark that in problems with index $k \geq 2$, continuity conditions for the solution manifold may not be formulated in such a straightforward manner. This is due to the fact that the analog of the discussion above would lead to the continuity or smoothness of $S_{k-1}(t)$ at k-singularities, but this set is no longer the solution manifold in DAEs with index $k \geq 2$. The corresponding study for these higher index problems would require a more detailed geometrical discussion, which is beyond the purposes of this work.

4 Weak singularities

Despite the singular nature of some differential or differential-algebraic equations, there exist cases in which solutions are well-defined through the singular set. This behavior has been characterized in certain autonomous, low-index nonlinear problems through the concept of a *weak singularity* [27, 28, 29] under a simplifying non-critical assumption [22]. The purpose of the present section is to address this phenomenon in the context of linear time-varying DAEs. Note that, in critical problems, this behavior does not follow from a straightforward extension of that of the quasilinear system obtained after adding the equation $t' = 1$.

To motivate this analysis, we review the weak singularity concept for autonomous problems in 4.1. We then present a definition of weak singularities for linear time-varying problems of arbitrary index in 4.2. Notice that a previous requirement for the existence of a well-defined flow through a singular point is the existence of a continuous extension of the solution manifold, in the terms discussed in Section 3: this is acknowledged in the weak singularity definition. We then provide some criteria for a singular point to be weak using desingularization tools. This is performed for singular index 0 and index 1 problems in 4.2.1 and 4.2.2, respectively. This framework will explain (see 5.1) the different behavior depicted by examples 1 and 2 presented in 2.4. Finally, in 4.2.3 we provide some hints for the analysis of higher index LTV problems inspired in the behavior of index 2 equations.

4.1 Autonomous low-index DAEs and weak singularities

As indicated above, around weak singularities of certain low-index autonomous DAEs there exists an n-dimensional flow which is well-defined even through the singular manifold [27, 28, 29, 30]. This is a complementary behavior to the one described by the Singular Flow Theorem [27, 33, 34]. Although this is a non-generic phenomenon, it comprises certain key singular features whereas, on the other hand, its analysis may be naturally carried out using classical tools. More general cases may sometimes be addressed as extensions, in a certain sense, of this weak behavior. This approach has been successfully applied to the stability study of a class of singular equilibria in quasilinear index 0 equations [29], and also to the synthesis of quadratically convergent discretizations of continuous-time methods for singular root-finding problems [30].

4.1.1 Quasilinear index 0 and semiexplicit index 1 autonomous problems

Generally speaking, an autonomous DAE is called singular at a given point x^* if the (differential, tractability or geometrical) index is well-defined on an open dense subset of a neighborhood of x^*, but not at the point x^* itself. In some simple cases (namely, quasilinear index 0 and semiexplicit index 1 equations under a non-critical hypothesis), the singularity is simply described by a rank drop at x^*. These cases have received considerable recent attention, and some of the main results are summarized below.

Let us first consider *quasilinear* differential equations [22, 23, 24, 29, 32]

$$A(x)x' = f(x), \tag{25}$$

where $A \in C^l(\mathbb{R}^n, \mathbb{R}^{n\times n})$, and $f \in C^l(\mathbb{R}^n, \mathbb{R}^n)$, with $l \geq 1$. System (25) may be trivially reduced to the explicit ODE $x' = A(x)^{-1}f(x) \equiv h(x)$ around points where $A(x)$ is regular. If, on the other hand, $A(x)$ has constant rank $r < n$ on a neighborhood of a singular point x^*, the equation often has a regular finite index on this neighborhood (see [2]).

We are nevertheless interested in problems in which $A(x)$ is singular on a hypersurface $\mathcal{S}$, with $x^* \in \mathcal{S}$. This occurs if x^* is a *non-critical singular point* [22], that is, if the condition $\nabla \mathrm{det} A(x^*) = (\mathrm{det} A)'(x^*) \neq 0$ is satisfied, (25) being in this case a singular index 0 DAE [29]. The main taxonomy of singular points in non-critical quasilinear equations (25) classifies them into *algebraic singularities*, where $f(x^*) \notin \mathrm{Im} A(x^*)$, and *geometric singularities*, satisfying $f(x^*) \in \mathrm{Im} A(x^*)$ [23, 24]. Algebraic singularities typically behave as *impasse points* [6, 7, 22, 23, 24], where trajectories collapse in finite time with infinite speed. On the other hand, different phenomena such as singularity crossing or bifurcation of solutions may happen at geometric singular points [29, 32]. It is worth mentioning that, under the non-critical assumption, geometric singularities of quasilinear problems are characterized by the condition $\mathrm{Adj} A(x^*)f(x^*) = 0$ [22, 27, 29], where Adj denotes the adjoint matrix (transpose of the matrix of cofactors). Equivalently, geometric singularities are equilibria of the *desingularized field* $\mathrm{Adj} A(x)f(x)$. Weak singularities fall within the class of geometric singular points, and are discussed below.

Similar ideas apply to semiexplicit index 1 DAEs

$$u' = f(u, v) \tag{26a}$$

$$0 = g(u, v), \tag{26b}$$

with $f \in C^l(\mathbb{R}^n \times \mathbb{R}^m,\ \mathbb{R}^n)$, $g \in C^{l+1}(\mathbb{R}^n \times \mathbb{R}^m,\ \mathbb{R}^m)$, $l \geq 1$. Equation (26b) represents a *solution manifold* $\mathcal{M}$ where the solutions of the DAE lie. From a local point of view, if we consider a point (u^*, v^*) in the solution manifold, the assumption that $g_v(u^*, v^*)$ is an invertible matrix defines (26) as an *index 1* problem. In this situation, (26b) defines locally a smooth manifold, and there exists a function $\tilde{g}$ verifying $g(u, v) = 0 \Leftrightarrow v = \tilde{g}(u)$ on a neighborhood of (u^*, v^*). Dynamics on this manifold may then be described, using u-coordinates, by the so-called *reduced ODE* $u' = f(u, \tilde{g}(u))$.

Singularities of these equations occur at points (u^*, v^*) of the solution manifold such that $g_v(u^*, v^*)$ is singular but where there exist arbitrarily close points with invertible g_v. These singularities have received considerable recent attention [5, 6, 7, 16, 27, 31, 33, 34], coming from problems within fields such as magnetohydrodynamics, nonlinear electrical circuits and power systems.

The above-mentioned non-critical condition has a natural extension to semiexplicit index 1 problems, which reads $\nabla \mathrm{det} g_v(u^*, v^*) \neq 0$. This condition may be easily shown to make the *underlying ODE* [2, 27]

$$\begin{aligned} u' &= f(u,v) && \text{(27a)} \\ g_v(u,v)v' &= -g_u(u,v)f(u,v) && \text{(27b)} \end{aligned}$$

also non-critical. The non-critical condition yields an index 1 DAE on the solution manifold $\mathcal{M}$ except on the intersection with the underlying singular hypersurface, defined in this case as $\mathcal{S} = \{(u,v) \in \mathbb{R}^{n+m} : \ \mathrm{det} g_v(u,v) = 0\}$. We are therefore led to a singular index 1 problem whenever the solution manifold is not entirely included in the singular hypersurface, which is the case under generic transversality assumptions [27].

The analog of a geometric singular point in this setting is a *pseudoequilibrium point* [33, 34], defined by $g(u^*, v^*) = 0$, $\mathrm{det} g_v(u^*, v^*) = 0$ and $\mathrm{Adj} g_v(u^*, v^*) g_u(u^*, v^*) f(u^*, v^*) = 0$. The analogy follows from the characterization of geometric singularities in non-critical quasilinear problems by the condition $\mathrm{Adj} A(x^*) f(x^*) = 0$. As it happens in the singular quasilinear index 0 setting, weak singularities of semiexplicit index 1 DAEs will be framed in the group of pseudoequilibria.

4.1.2 Weak singularities of non-critical autonomous equations

Let us briefly examine two specific examples in order to introduce the weak singularity concept. Consider the quasilinear system which results from applying the continuous Newton method $-J(x)x' = f(x)$ (J standing for the Jacobian matrix of f) to the vector function $f(x_1, x_2) = (x_1^2, x_2)$, that is,

$$\begin{pmatrix} -2x_1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_1^2 \\ x_2 \end{pmatrix}.$$

Obviously, this system is singular and non-critical on the manifold $x_1 = 0$. Nevertheless, $h(x_1, x_2) = (-x_1/2, -x_2)$ is well defined on the whole $\mathbb{R}^2$, yielding a non-singular flow and a description of the dynamics in terms of a vector field. A weak singular behavior occurs in this case: see a more detailed discussion in [28, 29, 30].

As a second example, consider the system [4, 27]

$$u_1' = 1, \ u_2' = v, \ u_1 v - u_2 = 0.$$

This is a semiexplicit DAE , for which the solution manifold $g(u_1, u_2, v) = u_1 v - u_2 = 0$ has no singularities, but the DAE itself is (non-critically) singular at points where $g_v(u_1, u_2, v) = u_1 = 0$. Nevertheless, given any initial point $(u_1(0), u_2(0), v(0))$ in the solution manifold, there exists a well-defined solution $u_1(t) = u_1(0) + t$, $u_2(t) = u_2(0) + v(0)t$, $v = v(0)$, even if $u_1(0) = 0$. The reason for this is that there exists a vector field which describes the underlying dynamics, and which may be smoothly extended even through the singular manifold. See details in [27], where it is also shown that even in cases in which the vector field may be extended to the singular set only along the solution manifold (that is, underlying dynamics not being weak) there exists a regular flow through the singularity.

The same phenomenon is responsible for the behavior displayed by both examples. In the quasilinear index 0 setting, a non-critical geometric singular point x^* is said to be a *weak singularity* if there exists a singular neighborhood $U^{x^*} \cap \mathcal{S}$ of x^* entirely formed by geometric singularities. With the notation $g(x) = \mathrm{Adj}A(x)f(x)$, $\omega(x) = \det A(x)$, weak singular points may be equivalently defined as singularities around which there exists a neighborhood U^{x^*} where $\omega(x) = 0 \Rightarrow g(x) = 0$ [28, 29].

Weak singular points verify that $h(x) = A(x)^{-1}f(x) = g(x)/\omega(x)$ may be extended as a C^{l-1} vector field on a whole neighborhood of x^* (including singular points) if this is a non-critical weak singularity [28]. This is based on the fact that the quotient $g(x)/\omega(x)$ may be shown to be well-defined by an argument similar to the one supporting Hadamard lemma [28]. It is easy to check that this is satisfied in the example concerning the continuous Newton method presented above.

A similar behavior may happen in semiexplicit index 1 problems. Note that the underlying equation (27) is described on $\mathbb{R}^{n+m} - \mathcal{S}$ by the C^l vector field

$$\tilde{h}(u, v) = \left(\begin{array}{c} f(u, v) \\ -g_v^{-1}(u, v)g_u(u, v)f(u, v) \end{array} \right),$$

which can be rewritten as $\tilde{h}(u, v) = \tilde{h}_d(u, v)/\det g_v(u, v)$, $\tilde{h}_d$ being the *transformed* (or *desingularized*) C^l field [33, 34]

$$\tilde{h}_d(u, v) = \left(\begin{array}{c} \det g_v(u, v)f(u, v) \\ -\mathrm{Adj}g_v(u, v)g_u(u, v)f(u, v) \end{array} \right).$$

The restriction of $\tilde{h}$ (resp. $\tilde{h}_d$) to $\mathcal{M} - \mathcal{S}$ yields a C^l vector field $\hat{h}$ (resp. $\hat{h}_d$) tangent to $\mathcal{M}$, since the solution manifold is invariant for the flow defined by the underlying ODE.

Regarding the singular set, pseudoequilibria are the only singular candidates for the field $\hat{h}$ to be continuously defined along $\mathcal{M}$, since at other singularities the condition $\mathrm{Adj}g_v(u^*, v^*)g_u(u^*, v^*)f(u^*, v^*) \neq 0$ forces $\lim_{(u,v)\to(u^*,v^*)} \hat{h}(u, v) = \infty$ (in the one-point compactification of $\mathbb{R}^{n+m}$). In this direction, let us define a non-critical pseudoequilibrium as *weak* if $\mathrm{Adj}g_v(u, v)g_u(u, v)f(u, v) = 0$ for all $(u, v) \in \mathcal{M}\cap\mathcal{S}$ in a neighborhood of (u^*, v^*) [27]. This notion expresses that weak singularities are pseudoequilibrium points around which all singularities *on the solution manifold* are themselves pseudoequilibria.

As it happens in the quasilinear index 0 case, it may be shown that around non-critical weak singularities there actually exists a non-singular n-dimensional flow [27]. More precisely, if $(u^*, v^*) \in \mathcal{M}$ is a non-critical weak pseudoequilibrium, and $\mathcal{M}_1$ is a smooth n-dimensional subset of $\mathcal{M}$ through (u^*, v^*), transversal to $\mathcal{S}$, then $\hat{h} = \hat{h}_d/\det g_v$ may be defined as a C^{l-1} vector field on a neighborhood $U \cap \mathcal{M}_1$ of (u^*, v^*).

In the semiexplicit example presented above, the underlying ODE is described by the vector field $\tilde{h}(u_1, u_2, v) = (1, v, 0)$, which admits a smooth extension through the singular set and yields a well-defined vector field $\hat{h}$ when restricted to the whole solution manifold (including singular points). See [27] for a discussion of other examples in which the vector field $\hat{h}$ is well-defined at weak singularities where the underlying vector field $\tilde{h}$ cannot be continuously defined on the (underlying) singular set.

4.2 Weak singularities of linear time-varying problems

Let us now drive our attention back to linear time-varying equations

$$A(t)x' + B(t)x = 0, \tag{28}$$

with a singularity at an isolated t^*, in the terms defined in Section 3. Regardless of the order of the singularity, and even in cases displaying an index jump, a well-defined flow may exist through this singular value:

Definition 3. *A singularity of (28) is called* weak *if:*

1. *There exists a neighborhood $\mathcal{I}^{t^*}$ and a smoothly time-dependent set $\mathcal{M}^*(t)$ defined on $\mathcal{I}^{t^*}$ such that $\mathcal{M}^*(t)$ is the solution manifold of (28) for $t \neq t^*$.*
2. *For every $x^* \in \mathcal{M}^*(t^*)$, there exists a solution of (28) with $x(t^*) = x^*$.*

The first requirement expresses that, even in problems with an index jump at the singularity, the solution manifold must have the same dimension at both sides of the singular value, and there must exist a smooth extension of this manifold through t^*. The second item requires the existence of a solution through *any* point of this extended manifold. Note that the trivial solution is always defined through the origin, regardless of the regular or singular nature of the problem: in weak cases, a trajectory must exist for *every* point in the extension of the solution set.

The following subsections provide some criteria for a singularity to be weak, based on a desingularization approach. Note that the definition above does not require additional smoothness properties on the operators defining the equation: in the rest of this section, however, for the sake of simplicity we will restrict the discussion to analytic problems. This assumption notably simplifies the desingularization analysis, avoiding the need for somewhat cumbersome assumptions on non-analytic functions, and also rules out jumps either in the solution manifold or in the index of the problem. Nevertheless, note that weak phenomena may of course happen in non-analytic problems: an instance of this situation, displaying an index jump at the singular value, will be discussed in Section 5.

Regardless of the (singular) index k of the problem, a general remark concerning the order j of the singularity must be made. In cases in which $j = k$, that is, in problems in which the order of the singularity equals the singular index of the problem, the matrix $A_k(t)$ must necessarily undergo a singularity at t^*, and this means that the desingularization analysis is strictly necessary. This is always the case in singular ODEs, since only 0-singularities can occur in this setting. However, j-singularities in singular index k problems, with $j < k$, display a rank change on $A_j(t)$, but an analytic extension of $A_k(t)$ might be non-singular even at t^*, and analytic canonical projectors might be defined through the singular value. This would yield a trivial desingularization, since the quotient describing the linear operator which characterizes the problem would have a non-vanishing denominator at the singularity. Examples of this particular instance of weak behavior will be presented in 5.2. The present section is mainly oriented to situations in which $A_k(t)$ indeed loses invertibility at the singular value, which is in particular the case for k-singularities.

4.2.1 Briefly on singular index 0 cases

Given an open interval $\mathcal{I} \subseteq \mathbb{R}$, a singular homogeneous LTV index 0 DAE or, simply, a singular homogeneous LTV ODE

$$A(t)x' + B(t)x = 0 \tag{29}$$

is characterized by a matrix function $A(t)$ which is singular at a given $t^* \in \mathcal{I}$, and non-singular for all t in $\mathcal{I} - \{t^*\}$ [13, 35]. For values of t different from t^*, the solution manifold of the problem is obviously $\mathbb{R}^n$. This means that the first requirement in definition 3 is trivially satisfied in this setting. In the analysis of weak singularities we must then focus on the second requirement.

In general, since the adjoint matrix $\mathrm{Adj}A(t)$ is non-singular if and only if $A(t)$ is non-singular, equation (29) is equivalent, on $\mathcal{I} - \{t^*\}$, to

$$\mathrm{det}A(t)x' + \mathrm{Adj}A(t)B(t)x = 0.$$

Hence, the behavior as t approaches t^* can be analyzed in terms of the operator

$$L(t) \equiv \frac{1}{\mathrm{det}A(t)}\mathrm{Adj}A(t)B(t) = \frac{G(t)}{\omega(t)},$$

with $G(t) = \mathrm{Adj}A(t)B(t)$, $\omega(t) = \mathrm{det}A(t)$, and obvious notational abuse in the quotient G/ω, which stands for $(1/\omega)G$. Note that, in the LTV setting, the problem is remarkably simpler than in autonomous quasilinear index 0 equations, since the singular function $\omega(t) = \mathrm{det}A(t)$ is now dependent only on a scalar parameter t. Indeed, the characterization of weak singularities presented below for analytic singular ODEs is immediate. Note that the boundedness notion there is not dependent on the choice of a specific norm in $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ due to the finite dimensionality of this space (of course, in the understanding that the norm does not change with t).

Theorem 1. *A singularity t^* of the analytic ODE (29) is weak if and only if there exists a neighborhood $\mathcal{I}^{t^*}$ of t^* such that $L(t)$ is uniformly bounded on $\mathcal{I}^{t^*} - \{t^*\}$.*

Proof: Since all the functions involved are analytic, and $\omega(t)$ undergoes an isolated zero at t^*, there must exist an integer $d \geq 1$ such that $\omega(t) = a_d(t-t^*)^d + O((t-t^*)^{d+1})$, with $a_d \neq 0$. We may then write

$$L(t) = \frac{G(t)}{\omega(t)} = \frac{G(t)}{a_d(t-t^*)^d + O((t-t^*)^{d+1})} = \frac{G(t)}{a_d(t-t^*)^d}(1 + O(t-t^*)). \tag{30}$$

The boundedness of $L(t)$ amounts to the requirement that $G(t)$ meet a zero at t^* of order $O((t-t^*)^z)$ with $z \geq d$. In this case, the Taylor expansion of $G(t)$ trivially yields an analytic quotient in (30). The operator $L(t)$ then provides an analytic extension of the flow through the singularity.

On the other hand, if $G(t)$ does not have such an order-z zero at t^*, the resulting pole in (30) shows that there is no way to extend a (non-trivial) solution through the singularity. $\square$

For later use, let us explicitly state the computations needed to check the weak singular nature of the equation:

Corollary 1. *Write $\omega(t) = a_d(t-t^*)^d + O((t-t^*)^{d+1})$ with $a_d \neq 0$ and $d \geq 1$. Then, t^* is a weak singularity of (29) if and only if there exists $z \geq d$ such that $G(t) = O((t-t^*)^z)$.*

In particular, a non-critical singular ODE (that is, a singular problem for which $(\det A)'(t^*) \neq 0$) will be weak if and only if $\mathrm{Adj}A(t^*)B(t^*) = 0$, which in turn amounts to the condition $\mathrm{Im}B(t^*) \subseteq \mathrm{Im}A(t^*)$, since $(\det A)'(t^*) \neq 0$ implies that $\mathrm{rk}A(t^*) = n-1$: in this situation, $\mathrm{Adj}A(t^*)$ does not vanish and $v \in \mathrm{Im}A(t^*) \Leftrightarrow \mathrm{Adj}A(t^*)v = 0$ (see e.g. [29]).

4.2.2 Singular index 1 cases

Consider now a singular index 1 analytic DAE

$$A(t)x' + B(t)x = 0, \tag{31}$$

defined on $\mathcal{I} \subseteq \mathbb{R}$, displaying a singularity at t^*. This point may be either a 0-singularity or a 1-singularity, in the terms discussed in Section 3. Note that, here or in higher index cases, a j-singularity in a singular index k problem, with $j < k$, might admit a non-singular analytic extension of $A_k(t)$ even at the singular value. This would provide a straightforward simplification of the desingularization analysis: instances of this situation are presented in 5.2. The study here performed is mainly oriented to situations leading to a non-invertible $A_k(t^*)$.

As it was indicated in Section 3, an analytic extension of the solution manifold may be guaranteed in the present setting. Remark that, in index-1 problems, the solution set is the space $S_0(t) = \{x \in \mathbb{R}^n : \ B(t)x \in \mathrm{Im}A(t)\}$. This means that there exists a set $S_0^*(t)$ with an analytic basis $(s_1^*(t), \ldots, s_r^*(t))$ on $\mathcal{I}$ such that $S_0^*(t) = S_0(t)$ for all $t \in \mathcal{I} - \{t^*\}$. Note that it might be $S_0^*(t^*) \neq S_0(t^*)$, as it was the case for system (22). Conditions guaranteeing that the set $S_0(t)$ is itself analytic (or smooth, in non-analytic systems) are discussed in Section 3.

Following the notation introduced in Section 2, let $P_c(t)$ be the canonical projector onto $S_0(t)$ along $N_0(t)$ for all $t \in \mathcal{I} - \{t^*\}$. Define $\mathcal{I}^- = \{t \in \mathcal{I} : \ t < t^*\}$, $\mathcal{I}^+ = \{t \in \mathcal{I} : \ t > t^*\}$. The inherent ODE reads on $\mathcal{I} - \{t^*\}$

$$u' + (-P_c' + P_c A_1^{-1} B_0)u = 0. \tag{32}$$

Solutions starting on $\mathrm{Im}P_c(t_0) = S_0(t_0)$ with $t_0 \in \mathcal{I}^-$ (resp. $\mathcal{I}^+$) remain on $\mathrm{Im}P_c(t) = S_0(t)$ for all $t \in \mathcal{I}^-$ (resp. $\mathcal{I}^+$), and define the DAE flow. Equivalently, if we add the equation $t' = 1$ to (31), thus obtaining a quasilinear autonomous system, the manifolds $S_0^- = \cup_{t<t^*} S_0(t)$ and $S_0^+ = \cup_{t>t^*} S_0(t)$ in (t, x)-space are locally invariant for the corresponding flow, and are foliated by the solutions of the DAE. Note that S_0^- and S_0^+ can be glued through a well-defined linear space $S_0^*(t^*)$ yielding a manifold $S_0^* = S_0^- \cup S_0^*(t^*) \cup S_0^+$.

The DAE (31) then defines a flow on this manifold, except maybe on $S_0^*(t^*)$. Nevertheless, in some cases there may exist weak singularities leading to a regular DAE flow on all S_0^*. In this regard, remark that the singular nature of (32) may be due to the presence of the inverse matrix A_1^{-1}, and also to the fact that the canonical projectors may become unbounded as $t \to t^*$. The latter would yield singularities not only in P_c, P_c' but also in A_1 and B_0.

These issues make the desingularization of (32) slightly more difficult than in the setting of singular LTV ODEs. Let us first recall that the canonical projectors may be constructed from analytic projectors $Q(t)$, $P(t)$ through the relations $B_0(t) = B(t) - A(t)P'(t)$, $A_1(t) = A(t) + B_0(t)Q(t)$, $Q_c(t) = Q(t)A_1^{-1}(t)B_0(t)$, $P_c(t) = I - Q_c(t)$. The matrix $A_1(t)$ must then be redefined from $Q_c(t)$. As indicated above, some 0-singularities might admit a non-singular analytic extension of $A_1(t)$ through the singularity, notably simplifying the study (see some examples in the index-2 context in 5.2). Excluding these cases, let us assume that $\det A_1(t)$ has a zero of order d_1 at t^*, so that

$$\det A_1(t) = a_{d_1}(t-t^*)^{d_1} + O((t-t^*)^{d_1+1}) = (t-t^*)^{d_1}(a_{d_1} + O(t-t^*)),$$

with $a_{d_1} \neq 0$. This leads to the desingularization

$$P_c(t) = \frac{\overline{P}(t)}{(t-t^*)^{d_1}},$$

where $\overline{P}(t)$ is analytic on $\mathcal{I}$. In a similar way, we may derive the existence of operators $\overline{R}(t), \overline{C}(t), \overline{B_0}(t)$ which are analytic on $\mathcal{I}$ and such that

$$P_c'(t) = \frac{\overline{R}(t)}{(t-t^*)^{d_1+1}}, \quad A_1^{-1}(t) = \frac{\overline{C}(t)}{(t-t^*)^{d_0}}, \quad B_0(t) = \frac{\overline{B_0}(t)}{(t-t^*)^{d_1}},$$

for some $d_0 \in \mathbb{N}$. The order d_1 in the latter expression follows from the fact that $B_0(t) = B(t) - A(t)P_c'(t)$, and $A(t)P_c'(t) = A'(t)Q_c(t)$ if $t \neq t^*$. We may then rewrite the linear operator in (32) as

$$L(t) = (-P_c' + P_c A_1^{-1} B_0)(t) = \frac{G(t)}{(t-t^*)^d}, \tag{33}$$

where $d = d_0 + 2d_1$, and $G(t) = -(t-t^*)^{d_0+d_1-1}\overline{R}(t) + \overline{P}(t)\overline{C}(t)\overline{B_0}(t)$ is analytic on $\mathcal{I}$.

In a way similar to the one followed for singular LTV ODEs, conditions guaranteeing that t^* is a weak singularity may be stated in terms of the operators appearing in (33). Nevertheless, there is a substantial difference between this case and the previous one: whereas in index 0 problems $L(t)$ should remain uniformly bounded while approaching the singularity, now it suffices to require this boundedness only *along the solution manifold* in (t, x)-space, in a sense which is detailed below. This is similar to the discussion of semiexplicit index 1 autonomous problems in 4.1.2. The case in which $L(t)$ is itself uniformly bounded is a particular one, meaning that *the inherent ODE* is weak. Nevertheless, the DAE flow may be regular under milder conditions.

Formulating these conditions requires some preliminary discussion. The key aspect now is that it is *the restriction of* $L(t)$ *to* $S_0(t)$ what must be uniformly bounded. Nevertheless, since the domain $S_0(t)$ of this restriction varies with t, it is not straightforward to work with an operator norm independent of t. To illustrate how this ambiguity may be overcome, fix any norm in $\mathbb{R}^n$ and define, for $t \neq t^*$, the norm $||l(t)|| =$

$\max_{\|v\|=1,\ v \in S_0(t)} \|l(t)v\|$ in the space of linear operators $\mathcal{L}(S_0(t), \mathbb{R}^n)$. This is equivalent to the definition of the seminorm $\|L(t)\| = \max_{\|v\|=1,\ v \in S_0(t)} \|L(t)v\|$ in $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$. We may then state the following analog of theorem 1:

Theorem 2. *A singularity t^* of the analytic singular index 1 DAE (31) is weak if there exists a neighborhood $\mathcal{I}^{t^*}$ of t^* such that $L(t)$ is uniformly bounded with respect to the above-defined seminorm. In particular, this is the case if the operator $L(t)$ is uniformly bounded with respect to any norm in $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$.*

Proof: let us consider the enlargement defined by the inherent equation (32) together with $t' = 1$. Both equations describe an ODE in this enlarged space, which may be written as $u' = -L(t)u,\ t' = 1$. The manifold S_0^* is invariant for the corresponding flow, which may become undefined on $S_0^*(t^*)$. Nevertheless, if we show that the vector field $(-L(t)u, 1)$ admits an analytic extension to $S_0^*(t^*)$ along the solution manifold, it would follow that the flow crosses smoothly the singular set, hence defining t^* as a weak singularity.

To achieve this, let us remark that a parameterization $(t, y_1, \ldots, y_r)$ of this manifold is easily obtained after choosing an analytic basis $(s_1^*(t), \ldots, s_r^*(t))$. This means that a vector $u \in S_0^*(t)$ can be written as $u = y_1 s_1^*(t) + \ldots + y_r s_r^*(t)$. With the compact notation $s(t) = (s_1^*(t) \ldots s_r^*(t))$, $y = (y_1 \ldots y_r)^T$, u would read $s(t)y$.

Now, writing $L(t)u = L(t)s(t)y$, it becomes apparent that the boundedness of the restriction $L|_{S_0(t)}(t)$ implies that $L(t)s(t)$ must itself be uniformly bounded. Since $L(t) = G(t)/(t - t^*)^d$ and therefore $L(t)s(t) = G(t)s(t)/(t - t^*)^d$, it follows that $G(t)s(t)$ must display a zero at t^* with order $\geq d$. This fact trivially yields an analytic quotient and means that $L(t)s(t)$ admits an analytic extension at $t = t^*$. In turn, this expresses that the vector field itself can be smoothly extended along the solution manifold, yielding a well-defined flow through the singularity.

This is the case if, in particular, $L(t)$ itself is uniformly bounded (as in the last case considered in the statement of the theorem), and hence admits an analytic extension. This assumption expresses that the inherent ODE is weak and, therefore, the restriction of the flow to the solution manifold is obviously well-defined. $\square$

Again, computations which allow one to check whether a given singular point is weak can be immediately formulated from the reasoning above:

Corollary 2. *Let $(s_1^*(t), \ldots, s_r^*(t))$ be an analytic basis of $S_0^*(t)$. The singularity t^* is weak if there exist integers $z_1, \ldots, z_r \geq d$ such that $G(t)$ satisfies $G(t)s_i^*(t) = O((t - t^*)^{z_i})$ for all $i = 1, \ldots, r$. This is the case if, in particular, $G(t) = O((t - t^*)^z)$ with $z \geq d$.*

4.2.3 Remarks on index 2 and higher index cases

The analysis of weak singularities discussed above may be extended to the index 2 case using the framework presented in 2.3.2. In turn, there is an extension to problems with index greater than 2 which proceeds along the same ideas. However, a detailed exposition would be somewhat cumbersome and we will simply summarize how a weak singular behavior may be displayed in index 2 cases.

Let us then assume that an analytic DAE

$$A(t)x' + B(t)x = 0, \tag{34}$$

has singular index 2 on $\mathcal{I} \subseteq \mathbb{R}$, with a singularity at an isolated $t = t^*$. In this situation, the matrix $A_2(t)$ will usually become singular at t^* (it might remain non-singular for certain 0- or 1-singularities, and this would lead to a direct simplification of the analysis: see examples in 5.2). The appearing of $A_2^{-1}(t)$ in the canonical projectors

$$Q_{0c}(t) = Q(t)P_{1c}(t)A_2^{-1}(t)B_0(t) + Q(t)Q_{1c}(t)(P(t)Q_{1c}(t))'(t)P(t),$$

and

$$Q_{1c}(t) = Q_1(t)A_2^{-1}(t)B_1(t),$$

would typically make these operators unbounded as $t \to t^*$. The inherent ODE

$$u' - (P_{0c}P_{1c})'u + P_{0c}P_{1c}A_2^{-1}B_0u \;\; = \;\; 0, \tag{35}$$

for which the restriction to $\mathrm{Im}P_{0c}P_{1c}$ at non-singular points yields the DAE flow, would then become singular as $t \to t^*$, not only because of the explicit appearing of A_2^{-1} but also due to the unbounded character of the canonical projectors P_{0c} and P_{1c}. Nevertheless, if the restriction of the operator $-(P_{0c}P_{1c})' + P_{0c}P_{1c}A_2^{-1}B_0$ to $\mathrm{Im}P_{0c}P_{1c}$ (or, in particular, the operator itself) is bounded as $t \to t^*$, a weak behavior will be displayed, as in the index 1 case. The DAE flow would in this case be a non-singular one and analytic dynamics would be displayed on the whole $\mathcal{I}$. Explicit conditions characterizing this phenomenon can be naturally obtained from the desingularization of the operators P_{0c}, P_{1c}, A_2^{-1}, etc., in a way identical to the one discussed for index 1 problems. Some instances of weak phenomena in singular index 2 problems are presented in the following section.

5 Examples

We discuss in this section some examples which illustrate the above-presented results. The index 1 examples introduced in 2.4 are revisited in 5.1: the different behavior at the singular point $t^* = 1$ is explained in the light of theorem 2. Some index 2 examples are then discussed in 5.2.

5.1 Index 1 problems

5.1.1 Example 1 (II)

The linear time-varying homogeneous DAE

$$\begin{pmatrix} 1 & -t \\ 1 & -t \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

displaying a 1-singularity in $t^* = 1$, was shown in section 2.4 to yield the linear operators

$$P_c(t) = \frac{1}{1-t} \begin{pmatrix} 1 & -t \\ 1 & -t \end{pmatrix}, \; A_1^{-1}(t) = \frac{1}{(1-t)^2} \begin{pmatrix} t^2 - t + 2 & -t^2 - 1 \\ t + 1 & -2t \end{pmatrix},$$

that is, $d_1 = 1,\ d_0 = 2$ and, therefore, $d = 2d_1 + d_0 = 4$. On the other hand, $G(t)$ reads, after some computations,

$$G(t) \ = \ (1-t)^3 \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},$$

showing that $z = 3$, with the notation of Corollary 2. Since $z < d$, the inherent ODE is not weak and this explains why the inherent flow becomes undefined at the singular value. The linear operator characterizing inherent dynamics is

$$L(t) = (-P_c' + P_c A_1^{-1} B_0)(t) = \frac{1}{1-t} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},$$

where the exponent 1 in the denominator $(1-t)$ is exactly $d-z$. Let us fix $s_1^*(t) = (1,1)$ to check whether the DAE flow might be well-defined through the singularity. We have

$$G(t)s_1^*(t) \ = \ (1-t)^3 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Hence, $z_1 = 3 < d$ and dynamics on the solution manifold is not weak, either. This fact supports the singular expression (18) for the DAE flow, that is, $x_1 = x_2 = C_0(1-t)/(1-t_0)$.

5.1.2 Example 2 (II)

The second example presented in section 2 was a defined by the system

$$\begin{pmatrix} 1 & -t \\ 1 & -t \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} + \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0,$$

which has singular index 1 and displays 1-singularities at $t^* = \pm 1$. Now we have

$$P_c(t) = \frac{1}{1-t^2} \begin{pmatrix} 1 & -t \\ t & -t^2 \end{pmatrix}, \ A_1^{-1}(t) = \frac{1}{(1-t^2)^2} \begin{pmatrix} t^3 - t + 2 & -t^3 - t^2 + t - 1 \\ t^2 + 2t - 1 & -t^3 - t^2 - t + 1 \end{pmatrix},$$

Again, $d_1 = 1,\ d_0 = 2$ and $d = 2d_1 + d_0 = 4$. Note that we are desingularizing with respect to both $t^* = 1$ and $t^* = -1$. Now we have

$$G(t) \ = \ (1-t^2)^2 \begin{pmatrix} -t & 1 \\ -1 & 2t - t^3 \end{pmatrix},$$

with $z = 2 < d$ and, as in example 1, the inherent ODE is not weak. The linear operator characterizing this ODE is

$$L(t) = (-P_c' + P_c A_1^{-1} B_0)(t) = \frac{1}{(1-t^2)^2} \begin{pmatrix} -t & 1 \\ -1 & 2t - t^3 \end{pmatrix}.$$

Again, the exponent 2 in the denominator $(1-t^2)^2$ is exactly $d-z$. Let us now fix $s_1^*(t) = (1,t)$. Then

$$G(t)s_1^*(t) \ = \ (1-t^2)^4 \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

and therefore $z_1 = 4 = d$, leading to a weak singularity at ±1 and a non-singular DAE flow. This is the reason supporting the analytic expression for the flow depicted in (19), namely, $x_1 = C_0,\ x_2 = C_0 t$.

5.2 Index 2 problems

Let us now discuss some index 2 examples, aimed at illustrating specific phenomena. Example 3, in subsection 5.2.1, displays a 0-singularity which nevertheless leads to an invertible $A_2(t)$, well-defined analytic projectors, and a weak inherent ODE. Example 4, in 5.2.2, is a straightforward modification of example 3 which illustrates that weak phenomena may also occur in non-analytic systems undergoing an index jump at the singularity. These examples admit a natural extension to problems in standard canonical form [2, 3]. The simplicity of both examples avoids the need for long computations: remark, however, that situations similar to the ones depicted by the index 1 examples in 5.1 may of course happen at 2-singularities of singular index 2 problems. In particular, it may well occur that a singular index 2 problem have a weak singularity and therefore a non-singular DAE flow without requiring the inherent ODE to be weak.

5.2.1 Example 3

Consider

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & t & 0 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} + \begin{pmatrix} a(t) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0,$$

where $a(t)$ is continuous on $\mathbb{R}$. This system obviously displays a 0-singularity at the origin. It is not difficult to check that, for $t \neq 0$,

$$Q_{0c} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_1(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & t & 1 \end{pmatrix},$$

$$Q_{1c} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -t & 0 \end{pmatrix}, \quad A_2(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{pmatrix}.$$

Despite the singularity, $A_2(t)$ admits an invertible extension through the origin, and the canonical projectors remain analytic. This is a specific phenomenon of some j-singularities in which the order j is strictly lower than the (singular) index of the problem, as discussed in Sections 3 and 4. This avoids the need for a desingularization analysis (or makes it trivial, since the denominators arising there would remain non-null), and yields a weak behavior in the inherent ODE, for which the operator $L(t) = -(P_{0c}P_{1c})' + P_{0c}P_{1c}A_2^{-1}B_0$ simply reads

$$L(t) = \begin{pmatrix} a(t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The restriction of the corresponding flow to the solution manifold (defined by the restriction $x_2 = x_3 = 0$) yields the non-singular DAE flow

$$x_1 = C_0 e^{-\int_{t_0}^{t} a(\tau)d\tau}, \; x_2 = x_3 = 0.$$

5.2.2 Example 4

Example 3 may be modified in a straightforward manner to provide a system displaying a weak behavior even with an index jump at the singularity. Specifically, consider

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \alpha(t) & 0 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} + \begin{pmatrix} a(t) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0,$$

where $\alpha(t)$ is a C^∞ function such that $\alpha(t) = 0 \Leftrightarrow t \leq 0$. The case $a(t) \equiv 1$ yields, in particular, system (20). This system has index 1 on $\mathbb{R}^-$ and index 2 on $\mathbb{R}^+$. It is immediate to show that the solution manifold of this DAE is defined by $x_2 = x_3 = 0$ at both sides of the singularity, and that the system behaves exactly as the one in example 3, hence displaying a weak singularity at the origin.

6 Concluding remarks

The present paper has addressed some singular aspects of linear time-varying DAEs. The tractability index concept and the use of projector methods for index 1 and index 2 regular problems have been reviewed. The failing of any non-degeneracy assumption in the sequence supporting the tractability index concept leads naturally to the definition of *singular* DAEs. These problems comprise situations in which the solution manifold may not admit a continuous extension through the singular value, as well as cases in which the index undergoes a jump at the singularity.

Singular problems with minimal degeneracies have then been considered. In this regard, a wide class of singular equations admit a continuous or smooth extension of the solution manifold through the singular value. Within this context, there are (so-called weak) singular systems for which there exist a well-defined DAE flow through the singularity. This is the case if, in particular, the inherent equation displays a non-singular behavior. Such phenomena have been characterized in terms of projectors using desingularization techniques. Several examples illustrate this behavior.

The results might be relevant in actual applications arising in the control of descriptor systems or in circuit problems modeled using differential-algebraic equations, as it has been the case in other contexts involving autonomous equations. Finally, it is worth mentioning that the work seems to have natural extensions concerning semilinear problems of the form $A(t)x' + f(x,t) = 0$, quasilinear equations $A(x,t)x' + f(x,t) = 0$, and also recently-proposed formulations with a properly stated leading term $A(x,t)(D(x,t))' + f(x,t) = 0$ [1, 19, 20].

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