Differential Algebraic Systems with Properly Stated Leading Term and MNA Equations

R. März
Humboldt-University Berlin
Institute of Mathematics
Unter den Linden 6
D-10099 Berlin, Germany

Abstract

Differential algebraic equations with properly stated leading term are equations of the form $A(x(t),t)(d(x(t),t))' + b(x(t),t) = 0$ with in some sense well-matched coefficients. Systems resulting from the modified nodal analysis (MNA) in circuit simulation promptly fit into this form.
Recent results concerning solvability and numerical treatment of those equations are discussed. An index notion that works via linearization is given. This allows for index criteria just in terms of the coefficients A, d, b and their first partial derivatives, no further derivative arrays are used.

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1 Introduction

Differential algebraic equations (DAEs) with properly stated leading term are equations of the form

$$A(x(t),t)(d(x(t),t))' + b(x(t),t) = 0, \tag{1.1}$$

with in some sense well-matched coefficients A and d. Roughly speaking, there is no gap but also no overlap between A and d, and the seam in between is solution independent. By means of the DAE (1.1), more information on the process to be modelled is preserved than by a standard form DAE

$$E(x(t),t), x'(t) + f(x(t),t) = 0 \tag{1.2}$$

Namely, it is precisely figured out which derivatives of the unknown function are involved and in what way. In contrast to this, equation (1.2) with singular $E(x,t)$ leaves this question open and suggests that all components of the solution are involved together with their derivatives.

Stronger solvability results, and a better performance of numerical methods can be realized as benefits of the more precise model. For instance, as it is well known, the famous index two DAE (cf. [BrCaPe], page 46)

$$\begin{pmatrix} 0 & 0 \\ 1 & \eta t \end{pmatrix} x'(t) + \begin{pmatrix} 1 & \eta t \\ 0 & 1+\eta \end{pmatrix} x(t) = \begin{pmatrix} g(t) \\ 0 \end{pmatrix}, \tag{1.3}$$

which has the solution $x_1(t) = g(t) + \eta t g'(t)$, $x_2(t) = -g'(t)$, cannot be solved by the implicit Euler method if $\eta < \frac{1}{2}$. However, rewriting (1.3) with properly stated leading term as

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} (x_1(t) + \eta t x_2(t))' + \begin{pmatrix} 1 & \eta t \\ 0 & 1 \end{pmatrix} x(t) = \begin{pmatrix} g(t) \\ 0 \end{pmatrix}, \tag{1.4}$$

and then applying the implicit Euler method yields $x_{1n} = g(t_n) + \eta t_n \frac{1}{h_n}(g(t_n) - g(t_{n-1}))$, $x_{2n} = -\frac{1}{h_n}(g(t_n) - g(t_{n-1}))$, i.e., in this version the implicit Euler method provides good approximations.

The idea to consequently formulate the leading term in a DAE by means of two well-matched coefficients was born in [BaMä]. First, the reason was only the desire for more symmetry in DAE theory, in particular, a uniform treatment of linear DAEs and their adjoints. While a linear standard form DAE $E(t)x'(t) + F(t)x(t) = q(t)$ and its adjoint $-(E(t)^* y(t))' + F(t)^* y(t) = r(t)$ are equations of completely different type, a DAE with proper leading term

$$A(t)(D(t)x(t))' + B(t) = q(t) \tag{1.5}$$

has the adjoint equation

$$-D(t)^*(A(t)^* y(t))' + B(t)^* y(t) = r(t), \tag{1.6}$$

which, in turn, has a properly stated leading term. The solutions $x(.)$ and $y(.)$ satisfy a generalized Lagrange identity, and there are interesting relations among the characteristic subspaces corresponding to (1.5) and (1.6), respectively ([BaMä]). Positive consequences for the optimal control can be expected here.
Numerical integration methods for lower index DAEs with properly stated leading term are considered in [HiMä], [HiMäTi], [HiMäTi2], [Te], [Fl].

A completely different aspect, which emphasizes the importance of equations of the form (1.1), is the fact that the modified nodal analysis (MNA) used in circuit simulation provides DAEs promptly fitting into the form (1.1) provided that the structured description is used [EsTi], cf. Section 2 below). From this point of view the results on the DAE (1.1) and their numerical treatment give the theoretical background for many things that are already common practice in circuit simulation due to practical experience and intuition.

At present, one of the challenging tasks in circuit simulation is monitoring the DAE index before or during the simulation. In this context let us remark that, unfortunately, the so-called structural index of linear constant coefficients, which can be calculated relatively easily by means of the Pantelides algorithm, is not relevant here (cf. [ReMaBa]).

The development of a derivative array ([BrCaPe]) is impossible because of the missing smoothness on the one hand as well as because of the huge dimensions in circuit simulation on the other hand.
Basing on the tractability index, an index monitor that can detect whether the given DAE has index one, two or greater than two has been developed for the simulation package TITAN (cf. [Es et all]). This index monitor uses topological methods as well as numerical criteria. Now this gives rise to the explicit wish for exactly determining higher indices, too. This paper aims at developing appropriate index criteria for that. We hope that these criteria can later be applied by using structural properties by means of topological methods.
The paper is organized as follows. In Section 2, structured MNA equations due to [EsTi] are described. Section 3 contains the basics on DAEs with properly stated leading term. Results on index one DAEs are collected in Section 4. Section 5 concerns linearizations, and the tractability index is introduced for linear DAEs. Finally, in Section 6, the tractability index of nonlinear DAEs is give in such a way that the corresponding notion for linear DAEs appears to be a special case, and, furthermore, all admissible linearizations of a nonlinear index μ DAE have index μ, too. Let us emphasize here that just the first partial derivatives of A, d, b are used, but no higher derivatives and derivative arrays. We finish with proving further necessary index μ criteria concerning the so-called local matrix pencil.

2 MNA equations

The modified nodal analysis (MNA), which is one of the most applied modelling techniques in circuit simulation packages, provides equation systems of the form (cf. [EsTi])

$$\begin{aligned}
A_C(q(A_C^Te(t),t))' \; &+A_Rr(A_R^Te(t),t)+A_Lj_L(t)+A_Vj_V(t) && \\
&+A_Ii(\tilde{A}^Te(t),j_L(t),j_V(t),t) &&= 0, && (2.1)\\
(\phi(j_L(t),t))' \; &-A_L^Te(t) &&= 0, && (2.2)\\
&A_V^Te(t)-v(\tilde{A}^Te(t),j_L(t),j_V(t)) &&= 0. && (2.3)
\end{aligned}$$

The unknown functions are $e(t), j_L(t)$ and $j_V(t)$, where $e(t)$ consists of the node potentials (excepting the datum node), $j_L(t)$ and $j_V(t)$ combine the currents of inductances and voltage sources, respectively. A_C, A_L, A_R, A_V and A_I are the element-related incidence matrices, they describe the branch-current relations for the capacitive branches, inductive branches, resistive branches, branches of voltage sources, and branches of current sources.
$\tilde{A} := (A_C, A_L, A_R, A_V, A_I)$ is the (reduced) incidence matrix of the whole system, its entries are just from $\{-1, 0, 1\}$. The functions $q(u,t)$ and $\phi(j,t)$ describe charges and fluxes, and the partial Jacobians

$$C(u,t) := q_u(u,t), \quad L(j,t) := \phi_j(j,t)$$

are positive-definite.
Choosing $x := \begin{pmatrix} e \\ j_L \\ j_V \end{pmatrix}$, $A := \begin{pmatrix} A_C & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix}$, $d(x,t) := \begin{pmatrix} A_C^+A_Cq(A_C^Te,t) \\ \phi(j_L,t) \end{pmatrix}$, with the

Moore-Penrose Inverse A_C^+ of A_C,

$$b(x,t) := \begin{pmatrix} A_R r(A_R^T e,t) + A_L j_L + A_V j_V + A_I i(\tilde{A}^T e, j_L, j_V, t) \\ -A_L^T e \\ A_V^T e - v(\tilde{A}^T e, j_L, j_V, t) \end{pmatrix},$$

we rewrite the MNA system (2.1)-(2.3) as

$$A(d(x(t),t))' + b(x(t),t) = 0. \tag{2.4}$$

The partial Jacobian $D(x,t) := d_x(x,t)$ is now of the special form

$$D(x,t) = \begin{pmatrix} A_C^+ AC(A_C^T e,t)A_C^T & 0 \\ 0 & L(j_L,t) \end{pmatrix} = \begin{pmatrix} A_C^+ AC(A_C^T e,t) & \\ & L(j_L,t) \end{pmatrix} A^T. \tag{2.5}$$

We observe that

$$AD(x,t) = A \begin{pmatrix} C(A_C^T e,t) & 0 \\ 0 & L(j_L,t) \end{pmatrix} A^T,$$

where the inner factor on the right-hand side is a positive-definite $n \times n$ matrix. Recall that A is a large rectangular matrix. It follows that (cf. [EsTi], Lemma 2.4)

$$ker AD(x,t) = ker A^T, \quad im AD(x,t) = im A \tag{2.6}$$

is valid independently of x and t. Then we also have the relations

$$ker A \cap im D(x,t) = 0, \;\; im D(x,t) = im A^+ A = (ker A)^\perp, \tag{2.7}$$

since $Az = 0, z = D(x,t)w$ yield $AD(x,t)w = 0$, thus $D(x,t)w = 0$, i.e., $z = 0$. It comes out that we are given the decomposition

$$\mathbb{R}^n = ker A \oplus im D(x,t). \tag{2.8}$$

Obviously, the constant projector $R := A^+ A = \begin{pmatrix} A_C^+ A_C & 0 \\ 0 & I \end{pmatrix}$ realizes the decomposition (2.8).

At this place we emphasize that (2.4) is, formally, slightly different from (2.1)-(2.3). Namely, while the first term in (2.1) is $A_C(q(A_C^T e(t),t)' = A_C A_C^+ A_C (q(A_C^T e(t),t))'$, the corresponding one in (2.4) is $A_C(A_C^+ A_C q(A_C^T e(t),t))'$. But, since we are allowed to move the constant matrix factor $A_C^+ A_C$ from outside into the inner of the derivative, we know these terms to coincide in fact.

Notice that the factor $A_C^+ A_C$ is introduced to obtain a constant subspace $im D(x,t)$. As we shall see below, there is no need for using this factor $A_C^+ A_C$ in practical computations.

Theoretically, the orthoprojector $A_C^+ A_C$ may be replaced by any projector $P_{A_C} \in L(\mathbb{R}^n)$ with $ker P_{A_C} = ker A_C$ if the subspace $im\, C(A_C^T e,t)A_C^T$ is independent of x and t a priori, the factor $A_C^+ A_C$ in the definition of $d(x,t)$ can be dropped completely.

Since the MNA system (2.1)-(2.3) is not in the standard form $f(x'(t),x(t),t) = 0$, which is mostly used in DAE theory as well as for numerical methods, one turns to the so-called conventional MNA or to the so-called charge-oriented MNA (cf. [EsTi]). Both of them are in standard DAE form. In contrast to this we shall treat the equations (2.1)-)(2.3) resp. (2.4) as they are.

3 DAEs with properly stated leading term

We consider equations

$$A(x(t),t)(d(x(t),t))' + b(x(t),t) = 0 \tag{3.1}$$

with coefficients $A(x,t) \in L(\mathbb{R}^n, \mathbb{R}^m)$, $d(x,t) \in \mathbb{R}^n$, $b(x,t) \in \mathbb{R}^m$ given for $x \in \mathcal{D}$, $t \in \mathcal{I}$. $\mathcal{D} \subseteq \mathbb{R}^m$ is open and connected, $\mathcal{I} \subseteq \mathbb{R}$ is an interval. A, b, d are assumed to be continuous with continuous partial derivatives $A_x, b_x, d_x, d_t, d_{xt}, d_{xx}$. Denote

$$D(x,t) := d_x(x,t).$$

A continuous function $x : \mathcal{I}_x \subseteq \mathcal{I} \to \mathbb{R}^m$ is said to be a solution of equation (3.1) if $x(t) \in \mathcal{D}$, for $t \in \mathcal{I}_x$, $d(x(t),t)$ is continuously differentiable with respect to t, and (3.1) is satisfied for $t \in \mathcal{I}_x$.
In this context, the coefficients are not supposed to have higher derivatives and the solution is not expected to belong to C^1, as it is usually the case in standard DAE theory [RaRh]. Of course, if a solution is C^1, then

$$A(x(t),t)D(x(t),t)x'(t) + b(x(t),t) + A(x(t),t)d_t(x(t),t) = 0 \tag{3.2}$$

is satisfied. Turning from (3.1) to (3.2) corresponds to the solution of the MNA system (2.1)-(2.3) by means of the conventional MNA.

Definition 3.1 *Equation (3.1) is a DAE with properly stated leading term if the decomposition*

$$ker A(x,t) \oplus im D(x,t) = \mathbb{R}^n, \quad x \in \mathcal{D},\ t \in \mathcal{I}, \tag{3.3}$$

is valid, $d(x,t) \in im D(x,t), x \in \mathcal{D}, t \in \mathcal{I}$, *is satisfied , and if the subspaces* $ker A(x,t)$ *and* $im D(x,t)$ *are independent of* x *but have bases that are continuously differentiable in* t.

In the following, the leading term in (3.1) is supposed to be stated properly. In particular, $A(x,t)$ and $D(x,t)$ have common constant rank then. Denote by $R(t) \in L(\mathbb{R}^n)$ the projector matrix that realizes the decomposition (3.3), i.e., $R(t)^2 = R(t)$, $im D(x,t) = im R(t)$, $ker A(x,t) = ker R(t)$, $t \in \mathcal{I}$. Since the subspaces $im R(t)$, $ker R(t)$ have continuously differentiable bases, the projector function R is continuously differentiable. Further it holds that $d(x,t) = R(t)d(x,t)$. Obviously, Definition 3.1 coincides with [HiMä], Definition 5.1.

The MNA system (2.1)-(2.3) forms a particular class of DAEs with properly stated leading term. Moreover, so-called semi-explicit DAEs

$$\begin{aligned} x_1'(t) + b_1(x_1(t), x_2(t), t) &= 0, \\ b_2(x_1(t), x_2(t), t) &= 0 \end{aligned}$$

have properly stated leading terms by simply choosing

$$A = \begin{pmatrix} I \\ 0 \end{pmatrix},\ d(x,t) = x_1,\ D(x,t) = (I, 0).$$

A properly formulated leading term contains more precise information on which part of the derivative of the unknown function is actually involved. This information gets lost if one turns to standard form DAEs like (3.2).

Sometimes, the augmented system

$$A(x(t),t)(R(t)y(t))' + b(x(t),t) = 0 \quad (3.4)$$

$$y(t) - d(x(t),t) = 0 \quad (3.5)$$

is easier to handle. It has a properly stated leading term, too.
A solution of (3.4), (3.5) is a pair of continuous functions $x : \mathcal{I}_x \to I\!R^m, y : \mathcal{I}_x \to I\!R^n, x(t) \in \mathcal{D}, t \in \mathcal{I}_x$, such that $R(t)y(t)$ is continuously differentiable in t, and the equations (3.4), (3.5) are satisfied. Then, also $y(t)$ is continuously differentiable in t, since $y(t) = d(x((t),t) = R(t)d(x(t),t) = R(t)y(t)$. Consequently, the DAE (3.1) and its augmented form (3.4), (3.5) are equivalent via $y(.) = d(x(.),.)$.
Notice that the augmented form (3.4), (3.5) of the MNA system (2.1)-(2.3) corresponds to the charge-oriented MNA.
Obviously, if $x : \mathcal{I}_x \to I\!R^m$ is a solution of the DAE (3.1), then $x(t) \in \mathcal{M}_0(t),\ t \in \mathcal{I}_x$, must hold, where

$$\mathcal{M}_0(t) := \{x \in \mathcal{D} : b(x,t) \in imA(x,t)\}, \quad t \in \mathcal{I}, \quad (3.6)$$

is the so-called constraint set. The flow of the DAE is restricted to $\mathcal{M}_0(t)$.
For given $t_0 \in \mathcal{I}, x_0 \in \mathcal{M}_0(t)$, there is a uniquely determined $y_0 \in I\!R^n$ such that

$$A(x_0,t_0)y_0 + b(x_0,t_0) = 0, \quad y_0 = R(t_0)y_0. \quad (3.7)$$

Definition 3.2 *The value $x_0 \in \mathcal{M}_0(t)$ is said to be consistent if there is a solution passing through (x_0,t_0). Then, with y_0 from (3.7), the pair (y_0,x_0) is a consistent initialization at t_0.*

In case of DAEs (3.1) with index 1 (cf. Section 4 below), the flow generated by the DAEs is restricted to $\mathcal{M}_0(t)$, but $\mathcal{M}_0(t)$ is completely filled by this flow, thus each $x_0 \in \mathcal{M}_0(t)$ is consistent. The flow of higher index DAEs is additionally restricted by so-called hidden constraints, and the set of consistent values at t_0 is a proper subset of $\mathcal{M}_0(t)$ only.

Introduce the matrix functions

$$G_0(x,t) := A(x,t)D(x,t), \quad (3.8)$$

$$B_0(y,x,t) := b_x(x,t) + A(x,t)y)_x \quad (3.9)$$

and the subspaces

$$N_0(x,t) := kerG_0(x,t) = kerD(x,t), \quad (3.10)$$

$$S_0(y,x,t) := \{z \in I\!R^m : B_0(y,x,t)z \in imG_0(x,t)\}, \quad (3.11)$$

for $x \in \mathcal{D}, t \in \mathcal{I}, y \in I\!R^n$.
Because of the identity $A(x,t) \equiv A(x,t)R(t)$ it holds that $B_0(y,x,t) \equiv B_0(R(t)y,x,t)$ and $S_0(y,x,t) \equiv S_0(R(t)y,x,t)$.
If the tangent space $T_{x_0}\mathcal{M}_0(t)$ is well-defined for $x_0 \in \mathcal{M}_0(t)$, then it coincides with $S_0(y_0,x_0,t_0)$, y_0 from (3.7).

4 Index-1 DAEs

In this section we collect certain relevant results obtained recently.

Definition 4.1 *([HiMä]) The DAE (3.1) has tractability index* $\mu = 1$ *if*

$$N_0(x,t) \cap S_0(y,x,t) = 0, \quad x \in \mathcal{D},\ t \in \mathcal{I},\ y \in \mathbb{R}^n, \tag{4.1}$$

is satisfied, i.e., these subspaces intersect transversally.

It is well known that (4.1) is equivalent to

$$N_0(x,t) \oplus S_0(y,x,t) = \mathbb{R}^m, \quad x \in \mathcal{D},\ t \in \mathcal{I},\ y \in \mathbb{R}^n.$$

In particular, if $T_x\mathcal{M}_0(t)$ is the tangent space to $\mathcal{M}_0(t)$ at $x \in \mathcal{M}_0(t)$, then $N_0(x,t) \oplus T_x\mathcal{M}_0(t) = \mathbb{R}^m$ must be true for $x \in \mathcal{M}_0(t)$.

Theorem 4.2 *([HiMä]) Let the DAE (3.1) have tractability index* $\mu = 1$.

(i) Then, for each $t_0 \in \mathcal{I}, x_0 \in \mathcal{M}_0(t_0)$*, there is exactly one solution passing through* (x_0, t_0).

(ii) If $x_* : \mathcal{I}_* \to \mathbb{R}^m$ *is a solution of (3.1),* $\mathcal{I}_* \subseteq \mathcal{I}$ *compact, then the perturbed initial value problems*

$$A(x(t),t)(d(x(t),t))' + b(x(t),t) = q(t), \tag{4.2}$$

$$d(x(t_0),t_0) = y^0, \tag{4.3}$$

with arbitrary $q \in C(\mathcal{I}_*, \mathbb{R}^m)$, $y^0 \in imR(t_0)$, $\|q\|_\infty$ *and* $|d(x_*(t_0),t_0) - y^0|$ *sufficiently small, are uniquely solvable on* $\mathcal{I}_*$*, and the solution* $x : \mathcal{I}_* \to \mathbb{R}^m$ *satisfies the inequality*

$$\|x - x_*\|_\infty \le K(\|q\|_\infty + |d(x_*(t_0),t_0) - y^0|). \tag{4.4}$$

Theorem 4.3 *Let the DAE (3.1) have tractability index* $\mu = 1, t_0 \in \mathcal{I}, y^0 \in imR(t_0)$*. Then, supposed the system*

$$\begin{aligned} A(x_0,t_0)y_0 + b(x_0,t_0) &= 0, & (4.5)\\ (I - R(t_0))y_0 + d(x_0,t_0) - y^0 &= 0, & (4.6) \end{aligned}$$

is solvable for x_0, y_0*, it is locally uniquely solvable and provides a consistent initialization.*

The Jacobian of the system (4.5), (4.6) is of the form

$$J = \begin{bmatrix} B_0 & A \\ D & I - R \end{bmatrix}.$$

$J\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0$ implies $z_2 = Rz_2,\ z_1 \in kerD,\ B_0 z_1 + Az_2 = 0$, hence $z_1 \in kerD = N_0$, $B_0 z_1 = -Az_2 = -ADD^- z_2$, thus $z_1 \in N_0 \cap S_0$, i.e., $z_1 = 0$, further $z_2 = Rz_2,\ Az_2 = 0$, which leads to $z_2 = 0$. Consequently, J is nonsingular. ◇

Naturally, a variable stepsize variable order (up to k) BDF applied to (3.1) reads

$$A(x_n, t_n) \frac{1}{h_n} \sum_{j=0}^{k} \alpha_{nj}\ d(x_{n-j}, t_{n-j}) + b(x_n, t_n) = 0. \tag{4.7}$$

A stiffly accurate s-stage Runge-Kutta method applied to (3.1) is of the form

$$x_n = X_{ns},$$

$$A(X_{ni}, t_{ni}) \frac{1}{h_n} \sum_{j=1}^{s} \alpha_{ij} (d(X_{nj}, t_{nj}) - d(x_{n-1}, t_{n-1})) + b(X_{ni}, t_{ni}) = 0, \tag{4.8}$$
$$i = 1, \dots, s,$$

where X_{ni} are the stage approximations, $t_{ni} = t_{n-1} + c_i h_n$ are the stages and $\alpha_{ij} = (\mathcal{A}^{-1})_{ij}$ are the entries of the inverse of the Runge Kutta matrix $\mathcal{A}$.
These methods are discussed in [HiMä], [HiMäTi]. For index-1 DAEs (3.1), stability, consistency and convergence is proved. Roughly speaking, for sufficiently small stepsizes these integration methods work well as expected. However, caused by subspaces $imR(t) = imD(x,t)$ varying with t, certain extra stepsize restrictions arise. Conversely, if the matrix $D(x,t)$ has a constant image-space, then, by means of (4.7) resp. (4.8), the so-called inherent regular ODE of (3.1) is automatically integrated by the same method. No conversion, e.g. of the implicit Euler method into the explicit one, may happen. Contractivity and dissipativity properties are preserved (cf. [HiMäTi]).

Definition 4.4 *([HiMäTi] The DAE (3.1) with tractability index $\mu = 1$ is* numerically qualified *if the subspace $imD(x,t)$ is constant.*

The implementations of the BDF in [Fl] and of Radau IIA in [Te] confirm the discussed theoretical aspects. It is evident that one should try for numerically qualified models from the very beginning. If the nullspace $kerA(x,t)$ is constant, but $imD(x,t)$ is not, we may put $\tilde{R} := A(x,t)^+ A(x,t)$ and reformulate (3.1) as

$$A(x(t), t)(\tilde{R} d(x(t), t))' + b(x(t), t) = 0 \tag{4.9}$$

which is now numerically qualified. Due to $A(x,t) \equiv A(x,t)\tilde{R}$ the integration methods (4.7) resp. (4.8) are invariant under this procedure. Therefore, there is no need for computing $\tilde{R}$ in practice. In this sense also DAEs (3.1) with constant $kerA(x,t)$ are numerically qualified.

Remark 4.5 Fortunately, as it was realized in Section 2, the MNA system (2.1)-(2.3) generates just a constant subspace $imD(x,t)$, namely (cf. (2.7)) $imD(x,t) = (kerA)^{\perp}$. Hence, the MNA system is, if it has tractability index 1, just numerically qualified. Furthermore, there is no need for any practical computation of $A_C^+ A_C$.

Notice that the conventional MNA system resulting from (2.1)-(2.3) (cf. (3.2))

$$AD(x(t),t)x'(t) + b(x(t),t) + Ad_t(x(t),t) = 0$$

may be rewritten in its turn as (cf. (2.6))

$$AD(x(t),t)(AA^+x(t))' + b(x(t),t) + Ad_t(x(t),t) = 0, \tag{4.10}$$

which has a properly stated leading term and which is numerically qualified, too. Again, when realizing numerical integration methods for (4.10) there is no need to calculate AA^+ in fact. However, as it has been well known in circuit simulation for a long time, when applying numerical integration methods to (4.10) one is confronted with the need of practical computations of the second derivatives d_{xt}, d_{xx}, whereas, in (4.7) and (4.8) one can do with $D = d_x$.

Remark 4.6 In essence, the investigations of DAEs with properly stated leading terms in the index 1 case confirm long, practical experience in circuit simulation. However, what is of interest less for practical simulation but for its mathematical backing up are the now simpler proofs and investigations. Namely, in [Gü] a loop way via an artificially enlarged standard form index-two DAE was still expedient. Having a closer look at the assumptions in [Gü] reveals that we just have numerically qualified DAEs (3.1) with tractability index 1 there.

5 Linearization

Observe the MNA equation (2.4) to be just a special case of the general DAE (3.1). Namely, the matrix function $A(x,t)$ and the nullspace $N_0(x,t)$ ($N_0 = kerA^T$, cf. (2.6)) are invariant at all of x and t. In this case we have

$$A(x,t) \equiv A, \quad N_0(x,t) \equiv N_0, \quad G_0(x,t) \equiv AD(x,t),$$
$$B_0(y,x,t) \equiv b_x(x,t) =: B_0(x,t).$$

Further, if $Q_0 \in L(\mathbb{R}^m))$ denotes a projector onto N_0 (i.e., $Q_0^2 = Q_0,\ imQ_0 = N_0$), $P_0 := I - Q_0$, we derive

$$d(x,t) - d(P_0x,t) = \int_0^1 D(sx + (1-s)P_0x, t)Q_0x ds = 0,$$

for all $x \in \mathcal{D}, t \in \mathcal{I}, [x, P_0x] \subset \mathcal{D}$. Recall that $\mathcal{D} \times \mathcal{I} \subseteq \mathbb{R}^m \times \mathbb{R}$ was introduced in §3 to denote the definition domain of the functions d and b.

In accordance with this, and to be more transparent, we are now going to consider more special DAEs of the form

$$A(d(x)t,t))' + b(x(t),t) = 0, \tag{5.1}$$

with properly stated leading term, constant subspace $N_0 = kerAD(x,t)$ and a domain $\mathcal{D}$ that includes $[x, P_0x]$ together with x. Let us stress that all semi-explicit systems

$$x_1'(t) + b_1(x_1(t), x_2(t), t) = 0, \tag{5.2}$$
$$b_2(x_1(t), x_2(t), t) = 0, \tag{5.3}$$

and, in particular, all Hessenberg form DAEs, have this simpler form, where $A = \begin{pmatrix} I \\ 0 \end{pmatrix}$, $d(x,t) \equiv x_1$, $D(x,t) = (I,0)$, $N_0 = O \times \mathbb{R}^{m_2}$.

Consider a continuous function $x_*: \mathcal{I}_* \to \mathbb{R}^m$, $\mathcal{I}_* \subseteq \mathcal{I}$, with values in $\mathcal{D}$, i.e., $x_*(t) \in \mathcal{D}$, $t \in \mathcal{I}_*$. Introduce the accompanying function $y_*: \mathcal{I}_* \to \mathbb{R}^n$, $y_*(t) := d(x_*(t),t), t \in \mathcal{I}_*$. A priori, y_* is continuous. However, is it also continuously differentiable?

Lemma 5.1 *If the component $P_0 x_*$ is continuously differentiable, then y_* is so, and vice versa.*

Proof: Supposed $P_0 x_* \in C^1(\mathcal{I}, \mathbb{R}^m)$, the function $y_* = d(x_*(.),.) = d(P_0 x_*(.),.)$ is continuously differentiable as a superposition of C^1 functions. Conversely, assume $y_* \in C^1(\mathcal{I}, \mathbb{R}^n)$, and consider the equation $F(y,x,t) = 0$, where the map $F: \mathbb{R}^n \times (\mathcal{D} \cap imP_0) \times \mathcal{I} \to imA$ is given by $F(y,x,t) := Ay - Ad(x,t)$. The derivative $F_x(y,x,t) = -AD(x,t) = -AD(x,t)P_0$ acts bijectively between imP_0 and imA. Due to $F(y_*(t), P_0 x_*(t), t) = 0$, $t \in \mathcal{I}_*$, the Implicit Function Theorem provides a continuously differentiable function $x = \xi(y,t)$, which solves the equation $F(y,x,t) = 0$. In particular, $P_0 x_*(t) = \xi(y_*(t),t), t \in \mathcal{I}_*$, holds true. Thus, $P_0 x_*$ is continuously differentiable. $\diamond$

The function space

$$C^1_{N_0}(\mathcal{I}_*, \mathbb{R}^m) := \{x \in C(\mathcal{I}_*, \mathbb{R}^m) : P_0 x \in C^1(\mathcal{I}_*, \mathbb{R}^m)\}$$

is independent of the special choice of the projector P_0. Namely, if $P_0, \tilde{P}_0$ are two projectors, $kerP_0 = ker\tilde{P}_0 = N_0$, then it holds that $P_0 = P_0 \tilde{P}_0$, $\tilde{P}_0 = \tilde{P}_0 P_0$.

Now we are well prepared to define the map $\mathcal{F}: \mathcal{D}_\mathcal{F} \to C(\mathcal{I}_*, \mathbb{R}^m)$,

$$(\mathcal{F}x)(t) := A(d(x(t),t))' + b(x(t),t), \quad t \in \mathcal{I}_*, \ x \in \mathcal{D}_\mathcal{F}, \tag{5.4}$$

with the definition domain

$$\mathcal{D}_\mathcal{F} := \{x \in C^1_{N_0}(\mathcal{I}_*, \mathbb{R}^m) : x(t) \in \mathcal{D}, \ \ t \in \mathcal{I}_*\}.$$

The operator equation $\mathcal{F}x = 0$ represents the DAE (5.1) on the interval $\mathcal{I}_* \subseteq \mathcal{I}$.
For fixed $x_* \in \mathcal{D}_\mathcal{F}$ being not necessarily a solution of the DAE we form the coefficients

$$D_*(t) := D(x_*(t),t), \quad B_*(t) := b_x(x_*(t),t), \quad t \in \mathcal{I}_*. \tag{5.5}$$

If $\mathcal{I}_*$ is compact, the map $\mathcal{F}$ is Frechet differentiable, and we obtain

$$\mathcal{F}'(x_*)x = A(D_* x)' + B_* x, \quad x \in C^1_{N_0}(\mathcal{I}_*, \mathbb{R}^m).$$

Hence, the operator equation linearized at x_*, $\mathcal{F}'(x_*)x = q$, is just the abstract form of the so-called *DAE linearized along the function x_**

$$A(D_* x)' + B_* x = q. \tag{5.6}$$

Coming from the linear DAE theory (e.g. [Mä1, Mä2]) one would consider the function space

$$C^1_{D_*}(\mathcal{I}_*, I\!R^m) := \{x \in C(\mathcal{I}_*, I\!R^m) : D_* x \in C^1(\mathcal{I}_*, I\!R^n)\}$$

as the solution space for (5.6). Does it depend on D_*? And is the leading term in (5.6) also properly stated? These questions are answered by the next lemma.

Lemma 5.2

(i) The linearized DAE (5.6) has a properly stated leading term with the same projector function $R \in C^1(\mathcal{I}_*, L(I\!R^n))$ as given for the nonlinear DAE (5.1).

(ii) The function spaces $C^1_{D_*}(\mathcal{I}_*, I\!R^m)$ and $C^1_{N_0}(\mathcal{I}_*, I\!R^m)$ coincide.

Proof: (i) The properties of the leading term (cf. Definition 3.1) apply simply to the linearized DAE (5.6) via $D_*(t) = D(x_*(t), t)$.
(ii) Let the generalized inverse D_*^- be determined (pointwise for $t \in \mathcal{I}_*$) by the four conditions

$$D_*^- D_* D_*^- = D_*^-, \quad D_* D_*^- D_* = D_*, \quad D_* D_*^- = R, \quad D_*^- D_* = P_0. \tag{5.7}$$

Since $D_* = D(P_0 x_*(.), .)$ belongs to the class C^1, so does D_*^-. Therefore, because of $P_0 x = D_*^- D_* x,\ D_* x = D_* P_0 x$, the above function spaces coincide in fact. ◇

To apply the index notion proposed for DAEs in [Mä1, Mä2] we construct the following sequence of matrix functions and possibly time-dependent subspaces. We begin with

$$\begin{aligned} G_{*0} &:= AD_*, \quad B_{*0} := B_*, \\ N_{*0} &:= kerG_{*0} = N_0, \quad Q_{*0} := Q_0, \quad P_{*0} := I - Q_{*0}, \end{aligned}$$

and introduce D_*^- as in Lemma 5.2 by (5.7).
Then, for $i \geq 0$, we define

$$\begin{aligned} G_{*i+1} &:= G_{*i} + B_{*i} Q_{*i}, \qquad (5.8)\\ N_{*i+1} &:= kerG_{*i+1}, \\ Q_{*i+1} &= Q^2_{*i+1}, \ imQ_{*i+1} = N_{*i+1}, \ P_{*i+1} := I - Q_{*i+1}, \\ B_{*i+1} &:= B_{*i} P_{*i} - G_{*i+1} D_*^- (D_* P_{*0} \cdots P_{*i+1} D_*^-)' D_* P_{*0} \cdots P_{*i}. \qquad (5.9) \end{aligned}$$

All expressions are meant pointwise for $t \in \mathcal{I}_*$. When realizing this sequence, one has to take care for the existence of the derivative in B_{*i+1}. As discussed in [Mä1, Mä2], this derivative corresponds to a consecutive decomposition of the *smooth* subspace imR into further such subspaces. This is closely related to solvability properties of the linear DAE (5.6).

Definition 5.3 *([Mä2]) The linear DAE (5.6) has tractability index μ if there is a sequence (5.8), (5.9) such that, for $i \geq 1$,*

(i) $G_{*i}(t)$ has constant rank r_{*i} on $\mathcal{I}_*$,

(ii) $N_{*i} \oplus \cdots \oplus N_{*i-1} \subseteq kerQ_{*i}$,

(iii) $Q_{*i} \in C(\mathcal{I}_*, L(\mathbb{R}^m)),\ D_* P_{*0} \cdots P_{*i} D_*^- \in C^1(\mathcal{I}_*, L(\mathbb{R}^n))$
hold true, and further $r_{*\mu-1} < r_{*\mu} = m$.

Remark 5.4 The flexible part within the sequence (5.8), (5.9) is given by the projector functions Q_{*i}. The index μ as well as the characteristic values $r_{*0}, \dots, r_{*\mu-1}$ are independent of the choice of these projector functions in Definition 5.3 ([Mä2]). The index is also invariant under regular transformations of the unknown function and refactorizations of the leading term ([Mä1]).

It is a common practice to define properties of nonlinear differentiable maps as $\mathcal{F}$ is one via linearizations. In accordance with this, in the next section, we try to construct a pointwise matrix function sequence for the nonlinear DAE (5.1) and to give a constructive index definition of tractability index μ in such a way that all admissible linearizations have tractability index μ, too. With this we intend to propose rather simple index criteria, which can be checked by linear algebra tools.

If, reversely, the desired properties of the pointwise nonlinear matrix function sequence are always given in case all admissible linearizations have tractability index μ, is not clear yet. Clarifying this here would go far beyond the scope of this paper. Thus, we do with the following conjecture, which will be proved for the case of $\mu = 1$ in Theorem 5.6.

Conjecture 5.5 *It is a mere supposition that the nonlinear DAE (5.1) has tractability index μ (in the sense of Definition 6.1 below) if all linearizations at $x_* \in \mathcal{D}_{\mathcal{F}} \cap C^{(\mu-1)}(\mathcal{I}_*, \mathbb{R}^m)$ with $\mathcal{I}_* \subseteq \mathcal{I}$ an interval, have tractability index μ.*

Clearly, the index notion below shall cover Definition 4.1 in the index-1 case.

Theorem 5.6 *If the DAE (5.1) has tractability index $\mu = 1$, then, for each $x_* \in \mathcal{D}_{\mathcal{F}}, \mathcal{I}_* \subseteq \mathcal{I}$, the linearized DAE (5.6) has tractability index $\mu = 1$. Conversely, if all these linearizations have tractability index $\mu = 1$, then the nonlinear DAE (5.1) has tractability index $\mu = 1$.*

Proof: Let the DAE (5.1) have tractability index $\mu = 1$. Then, due to (4.1), the matrix function $G_1(x,t) := G_0(x,t) + B_0(x,t)Q_0$ remains nonsingular for all $x \in \mathcal{D}, t \in \mathcal{I}$. Consider the linearization (5.6) for fixed $x_* \in \mathcal{D}_{\mathcal{F}}$. Choosing $Q_{*0} = Q_0$ we find $G_{*1}(t) = AD_*(t) + B_*(t)Q_{*0} = G_1(x_*(t), t),\ t \in \mathcal{I}_*$. Thus, the linear DAE (5.6) has tractability index $\mu = 1$. Conversely, let all linearizations have index $\mu = 1$. Consider the function $G_1(x,t)$ given above and fix an arbitrary $(\bar{x}, \bar{t}) \in \mathcal{D} \times \mathcal{I}$. Take any function $x_* \in \mathcal{D}_{\mathcal{F}}$ passing through $(\bar{x}, \bar{t})$. Since the linearization along x_* has index $\mu = 1$, $AD_*(\bar{t}) + B_*(\bar{t}) = G_1(\bar{x}, \bar{t})$ must be nonsingular. ◇

6 Index criteria for nonlinear DAEs

We continue to investigate the nonlinear DAE (5.1) as we agreed upon in Section 5. For $x \in \mathcal{D}, t \in \mathcal{I}$, we apply once more the matrices

$$G_0(x,t) := AD(x,t),\ \ B_0(x,t) := b_x(x,t),$$

as well as the subspaces $N_0 := kerG_0(x,t)$. $Q_0 \in L(I\!R^m)$ is a projector onto N_0, $P_0 := I - Q_0$.
Let $D(x,t)^-$ denote the generalized inverse of $D(x,t)$, which satisfies the four conditions $DD^-D = D,\ D^-DD^- = D^-,\ \ DD^- = R,\ \ D^-D = P_0$ pointwise on $\mathcal{D}\times\mathcal{I}$. Since $D(x,t)$ depends continuously differentiably on x,t and has constant rank, $D(x,t)^-$ does so, too.

Introduce further, for $x \in \mathcal{D}, t \in \mathcal{I}$,

$$\begin{aligned} G_1(x,t) &:= G_0(x,t) + B_0(x,t)Q_0, \\ N_1(x,t) &:= kerG_1(x,t), \\ Q_1(x,t) &\in L(I\!R^m) \quad \text{a projector onto } N_1(x,t), \\ P_1(x,t) &:= I - P_1(x,t). \end{aligned} \tag{6.1}$$

Then we construct, for $x \in \mathcal{D}, t \in \mathcal{I}, x^1 \in I\!R^m$,

$$Diff_1(x^1,x,t) := (DP_0P_1D^-)_x(x,t)x^1 + (DP_0P_1D^-)_t(x,t),$$

$$B_1(x^1,x,t) := B_0(x,t)P_0 - G_1(x,t)D(x,t)^-Diff_1(x^1,x,t)D(x,t). \tag{6.2}$$

Of course, when using these expression we have to take care of the existence of the involved partial derivatives. For $i \geq 1$, we form the sequence

$$\begin{aligned} G_{i+1} &:= G_i + B_iQ_i, & (6.3)\\ N_{i+1} &:= kerG_{i+1}, \\ Q_{i+1} &= Q_{i+1}^2,\ imQ_{i+1} = N_{i+1},\ P_{i+1} := I - Q_{i+1}, \\ B_{i+1} &:= B_iP_i - G_{i+1}D^-Diff_{i+1}DP_0\cdots P_i & (6.4) \end{aligned}$$

pointwise for $x \in \mathcal{D}, t \in \mathcal{I}, x^1,\ldots,x^{i+1} \in I\!R^m$. Thereby, $G_{i+1}, N_{i+1}, Q_{i+1}, P_{i+1}$, $DP_0\cdots\ \cdots P_{i+1}D^-$ depend on the arguments $x^i,\ldots,x^1,x,t$ while $Diff_{i+1}$ and B_{i+1} have the arguments $x^{i+1},\ldots,x^1,x,t$. The definition

$$\begin{aligned} Diff_{i+1}(x^{i+1},\ldots,x^1x,t) &:= \sum_{j=1}^{i}(DP_0\cdots P_{i+1}D^-)_{x^j}(x^i,\ldots x^1x,t)x^{j+1} \\ +(DP_0\cdots P_{i+1}D^-)_x(x^i,\ldots x^1x,t)x^1 &+ (DP_0\cdots P_{i+1}D^-)_t(x^i,\ldots x^1x,t) \end{aligned} \tag{6.5}$$

shows how the new variable x^{i+1} comes into this expression at this stage.

Definition 6.1 *The DAE (5.1) has the tractability index μ if there is a sequence (6.1)-(6.4) such that, for $i \geq 1$,*

(i) $G_i(x^{i-1},\ldots,x^1,x,t)$ *has constant rank* r_i *for* $x \in \mathcal{D}, t \in \mathcal{I}, x^1,\ldots,x^{i-1} \in I\!R^m$,

(ii) $N_0 \oplus \cdots \oplus N_{i-1} \subseteq kerQ_i$ *pointwise,*

(iii) Q_i *is continuous, but* $DP_0\cdots P_iD^-$ *is continuously differentiable,*

and $r_{\mu-1} < r_\mu = m$.

Remark 6.2 The notion of the tractability index allows for an index monitoring via linear algebra tools (rank detection, projector calculations). A special numerical algorithm realizing the condition $N_0 \oplus \cdots \oplus N_{i-1} \subseteq kerQ_i$ is proposed in [La]. There are several possibilities to reduce the computational amount.
In particular, G_μ is nonsingular if and only if

$$N_{\mu-1} \cap ker((I - G_{\mu-1}G^-_{\mu-1})B_0) = 0, \tag{6.6}$$

hence, one can avoid to compute $B_{\mu-1}$ and G_μ. Moreover, this proves the equivalence of Definition 4.1 for the case of $\mu - 1$ in Definition 6.1.

For more details and computational experiments we refer to [La].

Remark 6.3 An index monitor to be used in TITAN for circuit simulation is reported in [Es et all]. Its mathematical background is given by the tractability index (former version for standard DAEs, $\mu \leq 2$) and the careful and detailed structural analysis of electrical circuits [EsTi] providing the characteristic subspaces and projectors. Due to the combination of topological and numerical tools the index can be calculated (up to $\mu = 2$) even for very large circuits. It is hoped that it will become possible to update this index monitor according to the new higher index criteria given here to realize at least index three precisely, too.

Obviously, if the DAE (5.1) itself is just a linear one, then Definition 6.1 coincides with the index notion given for linear DAEs in [Mä2] (cf. Definition 5.3). In this case, the sequence (6.1)-(6.4) is simply independent of the values x and x^j, and (6.5) reads $Diff_{i+1}(t) = (DP_0 \cdots P_{i+1}D^-)_t(t)$.

Theorem 6.4 *If the DAE (5.1) has tractability index μ, then, for each $x_* \in \mathcal{D}_\mathcal{F} \cap C^{\mu-1}(\mathcal{I}_*, I\!R^m)$, $\mathcal{I}_* \subseteq \mathcal{I}$, the linearized DAE (5.6) has tractability index μ, too. All these linearizations have common characteristic values $r_{*i} = r_i$, $i = 0, \ldots, \mu$.*

Proof: Fix an $x_* \in \mathcal{D}_\mathcal{F} \cap C^{\mu-1}(\mathcal{I}_*, I\!R^m)$, $\mathcal{I}_* \subseteq \mathcal{I}$ and construct an appropriate sequence for the linearized DAE (5.6) (cf. Definition 5.3). Choose $P_{*0} := P_0, Q_{*0} := Q_0$. It results that $G_{*1}(t) := AD_*(t) + B_*(t)Q_0 = G_1(x_*(t), t)$, $t \in \mathcal{I}_*$, and $r_{*1} = r_1$. Because of $N_0 \subseteq kerQ_1(x,t), x \in \mathcal{D}, t \in \mathcal{I}$, we may choose $Q_{*1}(t) := Q_1(x_*(t), t), t \in \mathcal{I}_*$. This implies $N_0 \subseteq kerQ_{*1}(t)$. Further, Q_{*1} is obviously continuous, and $(D_*P_{*0}P_{*1}D_*^-)(t) = (DP_0P_1D^-)(x_*(t), t)$ depends continuously differentiably on t as a superposition of C^1 functions.
From $Diff_1(x_*'(t), x_*(t), t) = \frac{d}{dt}(DP_0P_1D^-)(x_*(t), t) = (D_*P_{*0}P_{*1}D_*^-)'(t)$ we derive

$$B_{*1}(t) = B_1(x_*'(t), x_*(t), t), \;\; G_{*2}(t) = G_2(x_*'(t), x_*(t), t), \; r_{*2} = r_2.$$

Next we choose $Q_{*2}(t) := Q_2(x_*'(t), x_*(t), t)$ and so on. $\diamond$

Example 6.5 Consider the Hessenberg form DAE of size three (cf. [BrCaPe])

$$\begin{aligned} x_1'(t) + x_1(t) + x_4(t) &= q_1(t), \\ x_2'(t) + \alpha(x_3(t), t)x_4(t) &= q_2(t), \\ x_3'(t) + x_1(t) + x_2(t) + x_3(t) &= q_3(t), \\ x_3(t) &= q_4(t), \end{aligned} \tag{6.7}$$

α is continuously differentiable, $1+\alpha(x_3,t)\neq 0$ for all x and t. With
$A = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{pmatrix}$, $D = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}$, $D^- = A$, $G_0 = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}$,
$P_0 = G_0$, $B_0 = \begin{pmatrix}1&0&0&1\\0&0&\gamma&\alpha\\1&1&1&0\\0&0&1&0\end{pmatrix}$, $\gamma(x_3,x_4,t) := \alpha_{x_3}(x_3,t)x_4$ we put the DAE(6.7) in the form (5.1). Derive further
$G_1 = \begin{pmatrix}1&0&0&1\\0&1&0&\alpha\\0&0&1&0\\0&0&0&0\end{pmatrix}$, $Q_1 = \begin{pmatrix}1&0&0&0\\\alpha&0&0&0\\0&0&0&0\\-1&0&0&0\end{pmatrix}$, $DP_0P_1D^- = \begin{pmatrix}0&0&0\\-\alpha&1&0\\0&0&1\end{pmatrix}$, $\mathit{Diff}_1 =$
$\begin{pmatrix}0&0&0\\\delta&0&0\\0&0&0\end{pmatrix}$, $\delta(x_3^1,x_3,t) := \alpha_{x_3}(x_3,t)x_3^1+\alpha_t(x_3,t)$, $G_2 = \begin{pmatrix}2&0&0&1\\\delta&1&0&\alpha\\1+\alpha&0&1&0\\0&0&0&0\end{pmatrix}$, $Q_2 =$
$\begin{pmatrix}0&0&-\frac{1}{1+\alpha}&0\\0&0&\frac{\delta-2\alpha}{1+\alpha}&0\\0&0&1&0\\0&0&\frac{2}{1+\alpha}&0\end{pmatrix}$, $DP_0P_1P_2D^- = \begin{pmatrix}0&0&0\\-\alpha&1&\frac{\alpha-\delta}{1+\alpha}\\0&0&0\end{pmatrix}$, and check that the intersection (cf. 6.6) $N_2\cap ker((I-G_2G_2^-)B_0) = \{z\in I\!R^4 : z_3=0,\ (1+\alpha)z_1=0,\ \delta z_1+z_2+\alpha z_4=0,\ 2z_1+z_4=0\}$ is trivial. Hence, system (6.7) has tractability index $\mu=3$ as expected. Observe that $DP_0P_1P_2D^-$ actually depend, via δ, on x^1.

Remark 6.6 The index notion proposed in [Mä3] means in essence Definition 6.1 applied to a somewhat restricted class of DAEs. Namely, in [Mä3], the projector $DP_0P_1P_2D^-$ is supposed to depend just on x,t, but to be independent of x^1. In the consequence, Diff_2 depends just on x^1,x,t, and so do B_2 and G_3. At the next stage, $DP_0P_1P_2P_3D^-$ is supposed to depend only on x,t, and so on. These additional structural properties lead to a sequence of matrix functions $G_i(x^1,x,t),\ i\geq 2$, depending on the variables x^1,x,t only. Then, Theorem 6.4 applies to each $x_*\in\mathcal{D}_{\mathcal{F}}\cap C^1(\mathcal{I}_*,I\!R^m)$. However, now we restrict the admissible linearizations to a smoother class, which remains huge enough, and we avoid additional structural restrictions. As we have seen in Example 6.5, the projector $DP_0P_1P_2D^-$ may actually depend on x^1.

In practice it ist hard to carry out formula (6.5) or to approximate the needed derivatives. Sometimes, necessary conditions, which are much easier to handle, will do.

Corollary 6.7 *Let the DAE (5.1) have tractability index μ.*

(i) Then, for each $c\in\mathcal{D}$, the linear DAE

$$A(D(c,t)x(t))'+b_x(c,t)x(t)=q(t) \tag{6.8}$$

is tractable with index μ and it has the characteristic values $r_{ci}=r_i,\ i=0,\dots\mu$.

(ii) *If the DAE (5.1) is autonomous, then, for each* $c \in \mathcal{D}$*, the local matrix pencil*

$$\lambda AD(c) + b_x(c)$$

is regular with Kronecker index μ*, and* $r_i - r_{i-1}$ *is the number of nilpotent Jordan blocks of size* i *in the Kronecker canonical normal form,* $i = 1, \dots, \mu$*.*

(iii) *If the subspaces* $imD(c,t)$*,* $im(DP_0 \cdots P_{i-1}Q_i)(0, \dots, 0, c, t), i = 1, \dots, \mu - 1$*, do not vary at all, then the local pencil*

$$\lambda AD(c,t) + b_x(c,t)$$

is regular with Kronecker index μ *uniformly for* $t \in \mathcal{I}$*.*

Proof: We linearize along constant functions $x_*(t) \equiv c, t \in \mathcal{I}$. The second part is a consequence of [GrMae], Theorem 3, whereas the third one follows from [Mä1], Theorem 4.4. ◇

We finish by stressing that there is some hope for realizing and exploiting constant relevant subspaces in circuit simulation (cf. [HiMäTi2], [EsTi]). This is not only important for the index monitoring, but also for a good performance of numerical integration methods.

References

[BaMä] K. Balla, R. März: A unified approach to linear differential algebraic equations and their adjoints. Zeitschrift f. Analysis und ihre Anwendungen 21(2002)3, 783-802.

[BrCaPe] K. E. Brenan, S. L. Campbell, L. R. Petzold: Numerical solution of initial value problems in differential-algebraic equations. North Holland, Amsterdam, 1989.

[Es et all] D. Estévez Schwarz, U. Feldmann, R. März, S. Sturtzel, C. Tischendorf: Finding beneficial DAE structures in circuit simulation. W. Jäger, H.-J. Krebs (eds.) Mathematics - Key Technology for the future. Springer 2003.

[EsTi] D. Estévez Schwarz, C. Tischendorf: Structural analysis of electric circuits and consequences for MNA. Int. J. Circuit Theory and Applications 28, 131- 162, 2000.

[Fl] J. Flügel: Lösung von Algebro-Differentialgleichungen mit properem Hauptterm durch die BDF.Diplomarbeit, Humboldt-Universität zu Berlin, Institut für Mathematik, 2002

[GrMae] E. Griepentrog, R. März: Basic properties of some differential-algebraic equations. Zeitschrift für Analysis udn ihre Anwendungen 8, 25-40, 1998.

[Gü] M. Günther: Simulating digital circuits numerically - a charge-oriented ROW approach. Numerische Mathematik 79, 203-212, 1998.

[HiMä] I. Higueras, R. Maerz: Differential algebraic equations with properly stated leading terms. Computers Math. Applic. (to appear)

[HiMäTi] I. Higueras, R. Maerz, C. Tischendorf: Numerically well qualified index-1 DAEs. Applied Numerical Mathematics, to appear.

[HiMäTi2] I. Higueras, R. Maerz, C. Tischendorf: Numerically qualified index-2 DAEs. Applied Numerical Mathematics, to appear.

[Mä1] R. März: The index of linear differential algebraic equations with properly stated leading terms. Results in Mathematics, to appear.

[Mä2] R. März: Solvability of linear differential algebraic equations with properly stated leading terms. Humboldt-Universität zu Berlin, Institut f—'ur Mathematik, Preprint 02-12.

[Mä3] R. März:Characterizing differential algebraic equations without the use of derivative arrays. Proc. NMCM 2002, University of Miskolc, Hungary, July 15-19, 2002

[La] R. Lamour: Index determination and calculation of consistent initial values for DAEs. Proc. NMCM 2002, University of Miskolc, Hungary, July 15-19, 2002

[ReMaBa] G. Reißig, W.S. Martinson, P.J. Barton: Differential-algebraic equations of index 1 may have an arbitrarily high structural index. SIAM J. Scie. Comp. 21(6), 1997-1990, 2000.

[RaRh] P.J. Rabier, W.C. Rheinboldt: Theoretical and Numerical Analysis of Differential-Algebraic Equations. Handbook of Numerical Analysis Vol. VIII, Elsevier Publ. Amsterdam 2002.

[Te] S. Tegtmeier: Numerische Lösung von Algebro-Differentialgleichungen mit proper formuliertem Hauptterm durch Runge-Kutta-Verfahren. Diplomarbeit, Humboldt-Universität zu Berlin, Institut für Mathematik, 2002