LINEAR BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL ALGEBRAIC EQUATIONS

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Abstract. By the use of the corresponding shift matrix, the paper gives a criterion for the unique solvability of linear boundary value problems posed for linear differential algebraic equations up to index 2 with well-matched leading coefficients. The solution is constructed by a proper Green function. Another characterization of the solutions is based upon the description of arbitrary affine linear subspaces of solutions to linear differential algebraic equations in terms of solutions to the adjoint equation. When applied to boundary value problems, the result provides a constructive criterion for unique solvability and allows reducing the problem to initial value problems and linear algebraic equations.

Introduction

For the linear differential algebraic equations (DAEs) of the form

$$A(t)\left(D(t)x(t)\right)' + B(t)x(t) = q(t) \tag{1}$$

with continuous, quadratic matrix-valued functions A, D and B with complex entries, the index-1 and index-2 notion was introduced in [3]. A theorem on unique solvability of the properly formulated initial value problems (IVPs) for (1) equipped with these indices was proven. It was shown that under the same conditions, the adjoint equation

$$-D^*(t)\left(A^*(t)y(t)\right)' + B^*(t)y(t) = p(t) \tag{2}$$

is of the same index and the proper IVP for (2) is solvable simultaneously. Meanwhile, some properties of the inherent ordinary differential equation (ODE) of (1) were investigated. The fundamental matrices for (1) and a specific one called normalized fundamental matrix were introduced, too.

The main goal of this paper is studying boundary value problems (BVPs) for (1) up to index 2. The assertion on simultaneous solvability of (1) and (2) turns to be the keystone in the analysis of BVPs.

The paper is organized as follows. In Section 1 we recall the basic definitions and some propositions concerning equation (1). For completeness, we define the index-0 equations, too. The results of [3] can be extended to index-0 equations in a trivial way. The solvability theorem for IVPs posed for the pair (1) and (2) is cited in this section. Existence results for two-point BVPs in terms of the Green

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function and shift matrix will be stated in Section 2. In Section 3 we describe affine linear subspaces of solutions to (1) by the help of solutions to (2). The transfer of boundary conditions (BCs) for BVPs both with separated and non-separated BCs and the related constructive existence theorem will be the topic of Section 4. The paper is accomplished with some remarks on numerical implementation in the final Section 5.

1. Preliminaries

We consider equations of the form (1) where A, D and B are continuous $m \times m$ matrix functions with complex entries on closed interval $\mathcal{I} = [a, b]$, q is a continuous vector-valued function with complex components on $\mathcal{I}$. In parallel to (1), equation (2) is involved into our study, p also is a continuous vector-valued function with complex components on $\mathcal{I}$. The pair of leading terms in (1) is assumed to be *well matched* in the following sense:

Condition C1 [3]: For each $t \in \mathcal{I}$,

$$\ker A(t) \oplus \operatorname{im} D(t) = \mathbb{C}^m\,, \tag{3}$$

and there exist continuously differentiable functions $a_1, \dots, a_{m-r}$ and $d_1, \dots, d_r$ such that

$$\ker A(t) = \operatorname{span}\,\{a_1(t), \dots, a_{m-r}(t)\}, \quad \operatorname{im} D(t) = \{d_1(t), \dots, d_r(t)\}, \quad t \in \mathcal{I}. \tag{4}$$

We proved

Lemma 1.1. (Lemma 2.1 [3]) *Equation (1) has well matched leading coefficients A and D if and only if the leading coefficients A^* and D^* of equation (2) are so.*

If R is the continuously differentiable projector function realizing the decomposition (3), i.e. $\ker R(t) = \ker A(t)$ and $\operatorname{im} R(t) = \operatorname{im} D(t)$, $t \in \mathcal{I}$, then R^* is the projector function corresponding to the decomposition induced by A^* and D^*.

Remark 1.1. Our main interest is in considering singular well matched leading pairs $A(t)$ and $D(t)$. Assumption (3), however, includes the case when both matrices $A(t)$ and $D(t)$ are nonsingular over the whole interval $\mathcal{I}$. Then, $r = m$ and $R(t) \equiv I$ where I is the $m \times m$ identity matrix. Considerations in [3] trivially can be extended to nonsingular well matched leading terms. Equation (1) turns into a standard explicit ODE if $A(t) \equiv D(t) \equiv I$.

Definition 1.1. (Definition 2.1 [3]) *A vector function $x : \mathcal{I} \to \mathbb{C}^m$ is called solution of (1) if $x \in \mathcal{C}^1_D(\mathcal{I}) := \{x \in \mathcal{C}(\mathcal{I}) : \ Dx \in \mathcal{C}^1(\mathcal{I})\}$ and (1) is satisfied pointwise.*

A solution of (2) is defined similarly. A kind of Lagrange identity was stated.

Lemma 1.2. *Let the matrix functions A and D be well matched. Then, for each pair of solutions $x \in C^1_D$, $y \in C^1_{A^*}$ of the homogeneous equations (1) and (2), respectively, the identity*

$$y^*(t)A(t)D(t)x(t) = \ const \quad t \in \mathcal{I}. \tag{5}$$

holds.

A key tool in the investigations of problems in [3] was a chain of matrix– and subspace–valued functions associated with (1):

$$
\begin{array}{ll}
G_0 := AD, & B_0 := B; \\
\text{for } i = 0, 1, & Q_i, P_i, W_i \text{ are projector functions: } Q_i^2 = Q_i, \quad W_i^2 = W_i, \\
& N_i := \ker G_i = \operatorname{im} Q_i, \quad P_i = I - Q_i, \\
& \ker W_i = \operatorname{im} G_i, \\
& G_{i+1} := G_i + B_i Q_i, \quad B_{i+1} = B_i P_i, \\
& S_i := \{z \in \mathbb{C}^m : B_i z \in \operatorname{im} G_i\} = \ker W_i B_i.
\end{array} \tag{6}
$$

Further on, D^- denotes the reflexive generalized inverse (RGI) function of D such that $DD^- = R$ and $D^-D = P_0$; A^- is an RGI function of A such that $A^-A = R$ and $AA^- = I - W_0$; G_1^- stands for the RGI function of G_1 such that $G_1G_1^- = I - W_1$ and $G_1^-G_1 = P_1$. We recall [5]] that a matrix $T^- \in \mathrm{L}(\mathbb{C}^k, \mathbb{C}^l)$ is an RGI of a matrix $T \in \mathrm{L}(\mathbb{C}^l, \mathbb{C}^k)$ if it satisfies the equalities $T^-TT^- = T^-$ and $TT^-T = T$. The products $\mathrm{P}_{RGI1} := TT^-$ and $\mathrm{P}_{RGI2} := T^-T$ are projectors. If $\mathrm{P}_{RGI1}, \mathrm{P}_{RGI2}$ are given projectors such that $\operatorname{im} \mathrm{P}_{RGI1} = \operatorname{im} T$ and $\ker \mathrm{P}_{RGI2} = \ker T$, then they define an RGI T^- uniquely.

Due to condition C1, $\dim \operatorname{im} G_0(t) \equiv r$. Let $\dim \operatorname{im} G_1(t) = r_1(t)$. Based on the properties of terms in the chain, an index may be assigned to some equations of the form (1) if, in addition to condition C1, another requirement also is fulfilled.

<u>Condition C2</u> [3]:

$$\dim D(t)S_1(t) \equiv \text{ const} =: \rho \quad \text{and} \quad \dim D(t)N_1(t) \equiv \text{ const} =: \nu, \tag{7}$$

and there exist continuously differentiable functions $s_1^D, \dots, s_\rho^D$ and $n_1^D, \dots, n_\nu^D$ such that for all $t \in \mathcal{I}$,

$$D(t)S_1(t) = \operatorname{span}\{s_1^D(t), \dots, s_\rho^D(t)\} \quad \text{and} \quad D(t)N_1(t) = \operatorname{span}\{n_1^D(t), \dots, n_\nu^D(t)\}.$$

Here we extend the Definition 2.2 from [3] as follows:

Definition 1.2. *Let conditions C1 and C2 be valid. Equation (1) is said to be*
(0) *an* index-0 tractable *DAE if*

$$N_0(t) = \{0\}, \qquad t \in \mathcal{I}, \tag{8}$$

(1) *an* index-1 tractable *DAE if*

$$N_0(t) \neq \{0\}, \tag{9}$$

$$N_0(t) \cap S_0(t) = \{0\}, \qquad t \in \mathcal{I}, \tag{10}$$

(2) *an* index-2 tractable *DAE if*

$$\dim N_0(t) \cap S_0(t) = \mathit{const} > 0, \tag{11}$$

$$N_1(t) \cap S_1(t) = \{0\}, \qquad t \in \mathcal{I}. \tag{12}$$

When $r < m$, the chain associated with an equation is not uniquely defined due to the freedom in the choices of the projectors. The index, however, does not depend on these choices. Thus, the index value, if exists, is an inherent property of the equation. One may choose a specific projector $\hat{Q}_1$ so that $\ker \hat{Q}_1(t) = S_1(t)$. The related terms in the chain will be marked by "^" (hat), too. For equations equipped with an index the assumptions ensure $r_1(t) \equiv \text{ const} =: r_1$ and $\rho = r + r_1 - m$, $\nu = m - r_1$. In the index-0 and index-1 cases, $r_1 = m$. Function $D\hat{P}_1D^-$ is a continuously differentiable projector function: for each t it projects onto DS_1 along $DN_1 \oplus \ker A$.

Setting $A_* = -D^*,\ D_* = A^*,\ B_* = B^*$ one may form the chain similar to (6) beginning with $A_*,\ D_*,\ B_*$, i.e. for the equation (2). The terms derived in this chain will be marked by an additional first subscript " $_*$ " (star).

With the inclusion of the index-0 equations, Theorem 5.1 of [3] reads as follows:

Theorem 1.1. *Equation (1) is of index μ, $\mu = 0, 1, 2$, if and only if equation (2) is so.*

The main point in the proof was showing that

(13)
$$DS_1 = R(A^*N_{*1})^\perp = (A^*N_{*1} \oplus \ker D^*)^\perp, \quad A^*S_{*1} = R^*(DN_1)^\perp = (DN_1 \oplus \ker A)^\perp$$

The so called inherent regular ODE for DAE (1) is

$$u' + DG_0^{-1}BD^{-1}u = A^{-1}q \tag{14}$$

in the index-0 case while

$$u' - R'u + DG_1^{-1}BD^{-}u = DG_1^{-1}q \tag{15}$$

in the index-1 case. If the DAE is of index 2, then the inherent ODE is

$$u' - (D\hat{P}_1D^-)'u + D\hat{P}_1\hat{G}_2^{-1}BD^-u = \mathcal{N}_0 q. \tag{16}$$

where

$$\mathcal{N}_0 q := D\hat{P}_1\hat{G}_2^{-1}q + (D\hat{P}_1D^-)'D\hat{Q}_1\hat{G}_2^{-1}q, \quad \mathcal{N}_0 q = D\hat{P}_1D^-\mathcal{N}_0 q.$$

For the inherent ODEs derived from an index-1 DAE it was shown that if $u(\tilde{t}) \in \operatorname{im} D(\tilde{t})$ for some $\tilde{t} \in \mathcal{I}$, then $u(t) \in \operatorname{im} D(t)$ for all $t \in \mathcal{I}$. Similarly, in the index-2 case $u(\tilde{t}) \in \operatorname{im} D(\tilde{t})\hat{P}_1(\tilde{t})$ involves $u(t) \in \operatorname{im} D(t)\hat{P}_1(t)$. Equations (15) and (16) are independent of the choice of P_0 and P_0, P_1, respectively.

Let x be a solution of the equation (1). If (1) is a DAE of index 0, then Dx is a solution of (14). In the index-1 case, Dx is a solution of (15). In the index-2 case, function $D\hat{P}_1x$ is a solution of (16).

Finally, we recall the solvability statement for IVPs.

Theorem 1.2. (Theorem 3.1, 3.2 [3]) *Let $t_0 \in \mathcal{I}$. When*

1. *(1) is an index-0 or index-1 DAE, $q \in \mathcal{C}(\mathcal{I})$ holds and the initial condition is*

$$D(t_0)x(t_0) = d_0 \quad \textit{with} \quad d_0 \in \operatorname{im} D(t_0),$$

or, when

2. *(1) is an index-2 DAE, $q \in \mathcal{C}^1_{DQ_1G_2^{-1}}(\mathcal{I})$ holds and the initial condition is*

$$D(t_0)\hat{P}_1(t_0)x(t_0) = d_0 \quad \textit{with} \quad d_0 \in \operatorname{im} D(t_0)\hat{P}_1(t_0), \tag{17}$$

then there exists a unique solution x of the IVP.

Now, the assertion on simultaneous solvability of DAEs (1) and (2) with proper right hand sides and initial conditions appears to be a direct consequence of Theorems 1.1 and 1.2.

Note that for the index-0 equations the initial condition is equivalent simply to condition $x_0 \in \mathbb{C}^m$ and the equation may be considered formally as a particular case of index–1 equations with $Q_0 = W_0 = 0$, $r = m$. In turn, an index–1 DAE may be considered formally as a particular case of the index–2 equations with $Q_1 = W_1 = 0$;

then $\rho = r$, $D\hat{P}_1 D^- = R$, $D(t)N_1(t) \equiv \{0\}$, $G_2 = G_1$. Thus, in the next sections it is sufficient to prove the statements only for the index-2 DAEs.

2. Green function approach

Let equation (1) be tractable with index μ, $\mu \in \{0,1,2\}$. Denote the maximal fundamental solution matrix normalized at $t_0 \in \mathcal{I}$ by $X(t,t_0)$, i.e. $X(t,t_0) \in L(\mathbb{C}^m)$ and $X(.,t_0)$ is the matrix–valued solution of the IVP

$$A(DX)' + BX = 0, \quad D(t_0)\hat{P}_1(t_0)(X(t_0) - I) = 0. \tag{18}$$

We recall from [3] the properties of the maximal fundamental solutions:

$$\text{im } X(t,t_0) = \text{im } \Pi_{can\,\mu}(t) \quad \ker X(t,t_0) = \ker \Pi_{can\,\mu}(t_0), \qquad t \in \mathcal{I},$$

where $\Pi_{can\,\mu}$ is a projector function onto the geometric solution space of the homogeneous DAE (1.1) ($q = 0$), $S_{ind\,\mu}(t) = \text{im } \Pi_{can\,\mu}(t)$,

$$\Pi_{can\,\mu} := KP_0\hat{P}_1, \quad K := I - Q_0\hat{P}_1\hat{G}_2^{-1}BP_0 - Q_0\hat{Q}_1D^-(D\hat{Q}_1D^-)'D, \tag{19}$$

U is nonsingular.

Let the RGI $X(t,t_0)^- \in L(\mathbb{C}^m)$ of $X(t,t_0)$ be defined by

$$\begin{array}{rcl} X(t,t_0)X(t,t_0)^- & = & \Pi_{can\,\mu}(t), \\ X(t,t_0)^-X(t,t_0) & = & \Pi_{can\,\mu}(t_0). \end{array}$$

(See Section 1 for the definition of RGI). The usual group properties

$$X(t_1,t_2)X(t_2,t_3) = X(t_1,t_3), \qquad X(t_1,t_2)^- = X(t_2,t_1).$$

hold. From Theorem 1.2 it follows that for each $q \in C_{DQ_1G_2^{-1}}(\mathcal{I})$ and $x^0 \in \mathbb{C}^m$, the IVP

$$A(Dx)' + Bx = q, \;\; D(t_0)\hat{P}_1(t_0)(x(t_0) - x^0) = 0, \tag{20}$$

is uniquely solvable. Due to linearity, the solution can be split into two terms

$$x(t) = X(t,t_0)x^0 + \tilde{x}(t), \;\; t \in \mathcal{I}. \tag{21}$$

where $\tilde{x}$ denotes the solution of the IVP

$$A(Dx)' + Bx = q, \qquad D(t_0)\hat{P}_1(t_0)x(t_0) = 0. \tag{22}$$

In [3] it was stated that each solution of (1) can be represented in the form $x = \Pi_{can\,\mu}x + \mathcal{N}_1 q$ where

$$\mathcal{N}_1 q := (P_0\hat{Q}_1 + Q_0\hat{P}_1)\hat{G}_2^{-1}q + Q_0\hat{Q}_1D^-(DQ_1G_2^{-1}q)'.$$

Thus, we can derive

$$\tilde{x}(t) = \int_{t_0}^{t} X(t,s)(\mathcal{N}_0 q)(s)ds + (\mathcal{N}_1 q)(t), \;\; t \in \mathcal{I}. \tag{23}$$

Now we turn to the BVP for (1) with the boundary condition

$$K_a x(a) + K_b x(b) = d, \tag{24}$$

$d \in L_{BC}$, where $L_{BC} := \text{im } (K_a \;|K_b\;) \subseteq \mathbb{C}^m$ is the linear subspace associated with the boundary condition. The values $x^0 \in \mathbb{C}^m$ in (21) that yield solutions of the BVP (1), (24) must satisfy the linear system

$$Mx^0 = d - K_a\tilde{x}(a) - K_b\tilde{x}(b) \tag{25}$$

with the "shift matrix" M

$$M := K_a X(a,t_0) + K_b X(b,t_0). \tag{26}$$

Theorem 2.1. *Let the DAE (1.1) be tractable with index $\mu, \mu \in \{0,1,2\}$. Then, for each $d \in L_{BC}$ and $q \in C^1_{DQ_1G_2^{-1}}(\mathcal{I})$, the BVP (1), (24) is uniquely solvable if and only if the shift matrix M satisfies the conditions*

$$\ker M = \ker \Pi_{can\ \mu}(t_0), \tag{27}$$

$$\operatorname{im} M = L_{BC}. \tag{28}$$

Proof. Due to the construction, the relations $\ker \Pi_{can\ \mu}(t_0) \subseteq \ker M$, $\operatorname{im} M \subseteq L_{BC}$ are valid.

Let the BVP (1), (24) be uniquely solvable for all $d \in L_{BC}$ and $q \in C^1_{DQ_1G_2^{-1}}(\mathcal{I})$. Put $q = 0$. For each $d \in L_{BC}$ there is an $x^0 \in \mathbb{C}^m$ such that $Mx^0 = d$. Hence $L_{BC} \subseteq \operatorname{im} M$, i.e., (28) holds.

Moreover, since the homogeneous BVP (1), (24) with $d = 0$ and $q = 0$ has only the trivial solution, the IVP

$$A(Dx)' + Bx = 0, \quad D(t_0)\hat{P}_1(t_0)(x(t_0) - x^0) = 0, \quad x^0 \in \ker M$$

may have only the identically vanishing solution. This means that

$$\ker M \subseteq \ker D(t_0)\hat{P}_1(t_0) = \ker \Pi_{can\ \mu}(t_0)$$

must be true, and consequently, (27) holds.

Conversely, let (27) and (28) be valid. Then for each $d \in L_{BC}$ and $q \in C^1_{DQ_1G_2^{-1}}(\mathcal{I})$ a solution of the BVP is determined by (21) and (25). $d = 0$ and $q = 0$ imply $\tilde{x} = 0$ and $Mx^0 = 0$. Thus $x^0 \in \ker M = \ker X(t,t_0)$. Now (21) leads to an identically vanishing solution x. □

Remark 2.1. The conditions (27), (28) ensure that rank $M = \varrho = r + r_1 - m$.

When (27) and (28) are valid we can introduce an RGI $M^- \in L(\mathbb{C}^m)$ of M such that $M^-M = \Pi_{can\ \mu}(t_0)$ holds.

Theorem 2.2. *Let the DAE (1) have tractability index $\mu, \mu \in \{0,1,2\}$ and let the conditions (27) and (28) be valid. Then the solution of the BVP (1), (24) with $d \in L_{BC}$, $q \in C^1_{DQ_1G_2^{-1}}(\mathcal{I})$ is*

$$\begin{aligned} x(t) &= X(t,t_0)M^-d + \int_a^b \mathcal{G}(t,s)(\mathcal{N}_0 q)(s)ds \\ &+ (\mathcal{N}_1 q)(t) - X(t,t_0)M^-\{K_a(\mathcal{N}_1 q)(a) + K_b(\mathcal{N}_1 q)(b)\}, \end{aligned} \tag{29}$$

where the Green function is defined as

$$\mathcal{G}(t,s) = \begin{cases} X(t,t_0)M^-K_aX(a,t_0)X(s,t_0)^-, & s \le t, \\ -X(t,t_0)M^-K_bX(b,t_0)X(s,t_0)^-, & s > t. \end{cases}$$

Proof. ¿From (25), $\Pi_{can\ \mu}(t_0)x^0 = M^-(d - K_a\tilde{x}(a) - K_b\tilde{x}(b))$ and from (21) one obtains (29) by standard calculations. Note that (29) is defined uniquely while there is a freedom in the choice of M^-. □

Remark 2.2. The map $\mathcal{L} : C^1_D(\mathcal{I}) \to C(\mathcal{I}) \times L_{BC}$ defined by

$$\mathcal{L}x := (A(Dx)' + Bx, K_a x(a) + K_b x(b)), \ x \in C^1_D(\mathcal{I})$$

is linear and bounded. It acts bijectively between $C^1_D(\mathcal{I})$ and $C^1_{DQ_1G_2^1}(\mathcal{I}) \times L_{BC}$. Recall that, in case of $\mu = 2$, the set $C^1_{DQ_1G_2^{-1}}(\mathcal{I})$ is a proper, non-closed dense subset in $C(\mathcal{I})$. Hence, when $\mu = 2$, $\mathcal{L}$ has a densely defined unbounded inverse $\mathcal{L}^{-1}$. However, if we equip $C^1_{DQ_1G_2^{-1}}(\mathcal{I})$ with a natural norm and consider $\mathcal{L} : C^1_D(\mathcal{I}) \to C^1_{DQ_1G_2^{-1}}(\mathcal{I}) \times L_{BC}$, in this setting $\mathcal{L}$ has a bounded inverse.

3. Affine linear subspaces of solutions

In [1] for a subclass of homogeneous index-1 DAEs (1) with $D = P_0$ and continuously differentiable coefficients A, P_0 and B we described the linear subspaces of solutions in terms of the adjoint equation. In this section we formulate and prove an analogous theorem for the affine linear subspaces of the solutions for DAE (1) with arbitrary function D well matched with A. The DAE is assumed to be of index–μ, $\mu \in \{0, 1, 2\}$ and it is not necessarily homogeneous. The smoothness conditions on the coefficients A, D and B are exactly as in Section 1, i.e. they must allow for assignment of an index, only. The function q is assumed to be of the class required by Theorem 1.2; in the index–0 and index–1 case q is only continuous, in the index–2 case $q \in \mathcal{C}^1_{DQ_1G_2^{-1}}(\mathcal{I})$.

A set of functions $\mathcal{M} \subset C^1_D(\mathcal{I})$ is called an affine linear subspace of functions $x \in C^1_D(\mathcal{I})$ if $\mathcal{M} = \tilde{x} + \mathcal{L}_{\mathcal{M}}$, where $\tilde{x} \in C^1_D(\mathcal{I})$ and $\mathcal{L}_{\mathcal{M}} \subset C^1_D(\mathcal{I})$ is a linear subspace. Denote $M(t) = \{v \in \mathbb{C}^m : \ v = x(t), \ x \in \mathcal{M}\}$ and $L_M(t) = \{w \in \mathbb{C}^m : w = z(t), \ z \in \mathcal{L}_{\mathcal{M}}\}$. If $\dim L_M(t) \equiv \text{ const} =: l$ then $\dim \mathcal{M} := \dim \mathcal{L} := l$.

The set $\mathcal{M}_{ind\,\mu}$ of all solutions of the DAE (1) is an affine linear subspace of dimension $\varrho = r + r_1 - m$ in $C^1_D(\mathcal{I})$. This follows immediately from the representation (21). The linear subspace $L_{M_{ind\,\mu}}(t) \in \mathbb{C}^m$ corresponding to the affine linear subspace $M_{ind\,\mu}(t)$ describes the geometric constraint to which each solution of the homogeneous equation is subjected to. It reads

$$L_{M_{ind\,\mu}}(t) = S_{ind\,\mu}(t) = \operatorname{im} \Pi_{can\,\mu}(t).$$

Lemma 3.1. *The set $\mathcal{M}_{ind\,\mu}$ has an equivalent description*

$$\{x \in C^1_D(\mathcal{I}) : W_0Bx = W_0q, \ Hx = \mathcal{H}(q)\} \tag{30}$$

where the matrix function H is defined by

$$H = D\hat{Q}_1D^-[A^-B - (D\hat{Q}_1D^-)'D] \tag{31}$$

and the linear map $\mathcal{H} : \ C^1_{DQ_1G_2^{-1}}(\mathcal{I}) \to C(\mathcal{I})$ is

$$\mathcal{H}(q) = D\hat{Q}_1D^-[A^-q - (DQ_1G_2^{-1}q)']. \tag{32}$$

Proof. Denote the set (30) by $\tilde{\mathcal{M}}$. If equation (1) is multiplied by W_0 then we get the first relation while the second one, $Hx = \mathcal{H}(q)$ is the so-called hidden constraint derived in [3]. Thus, $\mathcal{M}_{ind\,\mu} \subset \tilde{\mathcal{M}}$. Now it is enough to check the kernels of W_0B and H (argument t is omitted). Instead of showing that $\dim(\ker W_0B \cap \ker H) = \rho$, we check the intersection of kernels of W_0BK and HK with invertible matrix function K from (19), noting that the identities $W_0BK = W_0\hat{G}_2$ and $HK = D\hat{Q}_1D^-A^-\hat{G}_2(I - \hat{P}_1P_0)$ can be checked by direct computations (for brevity we omit the details). If $W_0\hat{G}_2x = 0$, then there exist $y : \ y = P_0y, \ x = \hat{G}_2^{-1}ADy = \hat{P}_1P_0y = \hat{P}_1y$, i.e. $\hat{Q}_1x = 0$. If, additionaly, $0 = HKx = D\hat{Q}_1D^-A^-\hat{G}_2(I - \hat{P}_1P_0)x$,

then $0 = D\hat{Q}_1 P_0 y$, i.e. $\hat{G}_2 y = ADy$. Therefore, $\hat{G}_2 y = \hat{G}_2 x$, i.e. $x = y$. Finally, $x = y = P_0 y = P_0 x = P_0 \hat{P}_1 x$. It yields $\ker H \cap \ker W_0 B = \operatorname{im} K P_0 \hat{P}_1 = \operatorname{im} \Pi_{can\,\mu}$. □

Remark 3.1. Observe that $M_{ind\,0}(t)$ coincides with $\mathbb{C}^m$ since $W_0 = 0$ and $\hat{Q}_1 = 0$. For $\mu = 1$, W_0 is non-trivial while $\hat{Q}_1$ is vanishing. For $\mu = 2$, both W_0 and $\hat{Q}_1$ are non-trivial, and the hidden constraint $H(t)z = \mathcal{H}(q)(t)$ is active.

For the purposes of the next assertions, we decompose equation (1) using the identity

$$I = Q_{*0}^* + \hat{Q}_{*1}^* P_{*0}^* + \hat{P}_{*1}^* P_{*0}^*. \tag{33}$$

We obtain

$$Q_{*0}^* Bx = Q_{*0}^* q, \tag{34}$$

$$\hat{Q}_{*1}^* A(Dx)' + \hat{Q}_{*1}^* P_{*0}^* Bx = \hat{Q}_{*1}^* P_{*0}^* q, \tag{35}$$

$$\hat{P}_{*1}^* A(Dx)' + \hat{P}_{*1}^* P_{*0}^* Bx = \hat{P}_{*1}^* P_{*0}^* q. \tag{36}$$

Equations (35) and (36) are equivalent to

$$A^{*-*}\hat{Q}_{*1}^* A(Dx)' + A^{*-*}\hat{Q}_{*1}^* P_{*0}^* Bx = A^{*-*}\hat{Q}_{*1}^* P_{*0}^* q, \tag{37}$$

$$A^{*-*}\hat{P}_{*1}^* A(Dx)' + A^{*-*}\hat{P}_{*1}^* P_{*0}^* Bx = A^{*-*}\hat{P}_{*1}^* P_{*0}^* q. \tag{38}$$

Since $\hat{Q}_{*1}^* Q_{*0}^* B = \hat{Q}_{*1}^* AD$, a consequence of (34) is

$$A^{*-*}\hat{Q}_{*1}^* ADx = A^{*-*}\hat{Q}_{*1}^* Q_{*0}^* q. \tag{39}$$

Condition C2 and Theorem 1.1 yield that the projectors $A^*\hat{Q}_{*1}A^{*-}$ and $A^*\hat{P}_{*1}A^{*-}$ are differentiable functions, therefore so are functions $A^{*-*}\hat{Q}_{*1}^* A$ and $A^{*-*}\hat{P}_{*1}^* A$. It immediately gives that $A^{*-*}\hat{Q}_{*1}^* Q_{*0}^* q \in \mathcal{C}^1$ is a necessary condition for a function x to be a solution. One can check, however, that

$$D\hat{Q}_1 \hat{G}_2^{-1} = A^{*-*}\hat{Q}_{*1}^* Q_{*0}^*. \tag{40}$$

Further, due to $AD = \hat{G}_2 P_1 P_0$ and $AD = -P_{*0}^* P_{*1}^* \hat{G}_{*2}^*$, the identities $D\hat{P}_1 = D\hat{G}_2^{-1} AD = -D\hat{G}_{*2}^{-*} AD = A^{*-*}\hat{P}_{*1}^* AD$ hold. Thus, in Theorem 1.2 we could use matrix functions connected with the equation (2), i.e. we could suppose

$$A^{*-*}\hat{Q}_{*1}^* Q_{*0}^* q \in \mathcal{C}^1,$$

and replace (17) by

$$P_{*1}^*(t_0)A(t_0)D(t_0)x(t_0) = d, \quad d \in \operatorname{im} P_{*1}^*(t_0)A(t_0)D(t_0).$$

If $x \in \mathcal{C}_D^1$, then the first term in (37) rewrites as

$$A^{*-*}\hat{Q}_{*1}^* A[(A^{*-*}\hat{Q}_{*1}^* ADx)' - (A^{*-*}\hat{Q}_{*1}^* A)' Dx].$$

Thus, combined with (39), i.e. with (34), equation (37) rewrites as

$$A^{*-*}\hat{Q}_{*1}^* A[(A^{*-*}\hat{Q}_{*1}^* Q_{*0}^* q)' - (A^{*-*}\hat{Q}_{*1}^* A)' Dx] = A^{*-*}\hat{Q}_{*1}^* P_{*0}^*(q - Bx). \tag{41}$$

This is exactly the so-called hidden constraint $Hx = \mathcal{H}(q)$ in a different form. Indeed, since $\ker W_0 = \ker Q_{*0}^*$ and (34) hold, the multiplier $I - W_0$ may be inserted before the term $q - Bx$ in (41). On the other hand, the relation $\ker W_0 = \ker Q_{*0}^*$ involves that (34) and $W_0 Bx = W_0 q$ are equivalent. In fact, we checked the following statement:

Lemma 3.2. *A function $x \in \mathcal{C}_D^1$ satisfies (34) and (35) if and only if $x \in \mathcal{M}_{ind\,\mu}$.*

Exploiting (34) and (41), we can rewrite (38) as a regular ODE

$$\begin{aligned} u' &- (A^{*-*}\hat{P}^*_{*1}A)'u - (A^{*-*}\hat{P}^*_{*1}A)A^{*-*}BG^{-*}_{*2}Au \\ &= [(A^{*-*}\hat{P}^*_{*1}A)'D - A^{*-*}\hat{P}^*_{*1}P^*_{*0}B]G^{-*}_{*2}Q^*_{*0}q + A^{*-*}\hat{P}^*_{*1}P^*_{*0}q. \end{aligned} \tag{42}$$

for $u := A^{*-*}\hat{P}^*_{*1}A(Dx)$. Equation (42) is nothing else but the inherent ODE (16) in terms connected with (2)! Indeed, the term by term coincidence can be verified by direct computation. The forms (41) and (42) show that both the hidden constraint and the inherent ODE are independent of the chosen projectors P_0, P_1 since so are P_{*0}, P_{*1}; this assertion was proven in [3] in a different way.

Remark 3.2. Since $Q^*_{*0}G^*_{*2} = Q^*_{*0}B$, (34) defines the projection of the function x onto im $G^{-*}_{*2}Q^*_{*0}G^*_{*2}$:

$$G^{-*}_{*2}Q^*_{*0}G^*_{*2}x = G^{-*}_{*2}Q^*_{*0}q. \tag{43}$$

This is an equivalent of the first equation in formula (30) of Lemma 3.1. A combination of (43) with the second equation $Hx = \mathcal{H}(q)$ in (30) defines another projection of the function x. Namely, we can derive

$$G^{-*}_{*2}V^*G^*_{*2}x = G^{-*}_{*2}[\hat{Q}^*_{*1}(P^*_{*0} - Q^*_{*0})q - \hat{Q}^*_{*1}A(A^{*-*}\hat{Q}^*_{*1}Q^*_{*0}q)'], \tag{44}$$

$$V = [P_{*0} + A^{*-}(A^*\hat{P}_{*1}A^{*-})'A^*]\hat{Q}_{*1}, \quad V^2 = V, \quad VQ_{*0} = Q_{*0}V = 0,$$

and, the system of equations in (30) becomes equivalent to system (43)-(44) that defines two projections of function x. This observation will be used in Section 4.

Theorem 3.1. *Let (1.1) be tractable with index μ, $\mu \in \{0,1,2\}$ and $q \in C^1_{DQ_1G_2^{-1}}(\mathcal{I})$. Then, a set $\mathcal{K} \subset C^1_D(\mathcal{I})$ is a k-dimensional affine linear subspace of solutions of the DAE (1) if and only if, for all $t \in \mathcal{I}$,*

$$K(t) = \{w \in \mathbb{C}^m : y^*(t)A(t)D(t)w + v^*(t) = 0, \ \ w \in M_{ind\,\mu}(t)\} \tag{45}$$

where $y : \mathcal{I} \to L(\mathbb{C}^s, \mathbb{C}^m)$, $\dim \operatorname{im} y(t) \equiv s$, $v : \mathcal{I} \to \mathbb{C}^s$, $s = \rho - k$, and

$$-D^*(A^*y)' + B^*y = 0, \tag{46}$$

$$v' + q^*y = 0. \tag{47}$$

Theorem 3.1 states that any affine linear subspace within the whole solution set can be segregated by the help of functions that are solutions of the homogeneous adjoint DAE and solutions of an explicit ODE.

Proof. We denote the set on the right hand side of (45) by $\tilde{K}(t)$ and provide the proof for $\mu = 2$.

Let $\mathcal{K} \in C^1_D(\mathcal{I})$ be a k–dimensional affine linear subspace of solutions of the DAE (1) of index 2 and choose an arbitrary $x_a \in \mathcal{K}$. Let $\mathcal{L}_\mathcal{K} := \{\ell \in C^1_D(\mathcal{I}) : \ell = x - x_a, \ x \in \mathcal{K}\}$ be the linear subspace of functions corresponding to $\mathcal{K}$ and $L_K(t) \in \mathbb{C}^m$, $t \in \mathcal{I}$, $t \in \mathcal{I}$, be the corresponding subspaces. Note that for each t, $D(t)\hat{Q}_1(t)L_K(t) \equiv \{0\}$ and $\dim D(t)L_K(t) = \dim L_K(t)$. Fix $t_0 \in \mathcal{I}$. Let

$$L^c_K(t_0) := (D(t_0)L_\mathcal{K}(t_0) \oplus DN_1(t_0) \oplus \ker A(t_0))^\perp.$$

One has $\dim L^c_\mathcal{K}(t_0) = m - [k + \nu + (m - r)] = r - k - \nu = s$. Thus, there exist s linearly independent vectors $z^0_1, \dots, z^0_s$ spanning $L^c_\mathcal{K}(t_0)$. Since $L^c_\mathcal{K}(t_0) \subset (DN_1(t_0) \oplus \ker A(t_0))^\perp = A^*(t_0)S_{*1}(t_0)$, the IVPs for the homogeneous equation (2) (p=0) with initial conditions $A^*(t_0)\hat{P}_{*1}(t_0)y(t_0) = z_i$, $i = 1, \dots, s$ have uniquely defined solutions y_i.

The above solutions $y_1, \dots, y_s$ of the homogeneous equation (2) are linearly independent. Indeed, assume the contrary, that is for the solution $\xi(t) = \sum_{i=1}^{s} c_i y_i(t)$ with at least one non-zero c_i, $\xi(\tilde{t}) = 0$ holds for some $\tilde{t}$. The IVP for homogeneous equation (2) with initial condition $A^*(\tilde{t})\hat{P}_{*1}(\tilde{t})y(\tilde{t}) = 0$ has the unique solution $y = 0$. This is in contradiction with $A^*(t_0)\hat{P}_{*1}(t_0)\xi(t_0) = \sum_{i=1}^{s} c_i z_i \neq 0$.

Set $v_i^0 = -z_i^{0*} D(t_0) x_a(t_0)$ and let $v_i : \mathcal{I} \to \mathbb{C}$ be the solution of the IVP for ODE $v_i' + q^* y_i = 0$ satisfying $v_i(t_0) = v_i^0$.

Let $x \in \mathcal{K}$. Then

$$\begin{aligned}&(y_i^*(t)A(t)D(t)x(t) + v_i^*(t))' = \\ &\quad [(y_i^*(t)A(t))'D(t)]\,x(t) + y_i^*(t)\,[A(t)(D(t)x(t))'] - y_i^*(t)q(t) = \\ &\quad y_i^*(t)B(t)x(t) + y_i^*(t)[q(t) - B(t)x(t)] - y_i^*(t)q(t) = 0.\end{aligned}$$

Thus,

$$y_i^*(t)A(t)D(t)x(t) + v_i^*(t) \equiv y_i^*(t_0)A(t_0)D(t_0)x(t_0) + v_i^*(t_0). \tag{48}$$

Note that $A^*(t_0)\hat{Q}_{*1}(t_0)y_i(t_0) = 0$. Thus, the right hand side expression in (48) equals to

$$y_i^*(t_0)\hat{P}_{*1}^*(t_0)A(t_0)D(t_0)x(t_0) - z_i^{0*}D(t_0)x_a(t_0) = z_i^* D(t_0)[x(t_0) - x_a(t_0)].$$

Since $x(t_0) - x_a(t_0) \in L_{\mathcal{K}}(t_0)$, the latter expression vanishes by construction.

Let $y : \mathcal{I} \to L(\mathbb{C}^s, \mathbb{C}^m)$ be defined "columnwise" by vector-valued functions y_i, $i = 1, \dots, s$, as $y(t) := (y_1(t), \dots, y_s(t))$. Clearly, y is a solution of the DAE (4.3). Similarly, $v : \mathcal{I} \to \mathbb{C}^s$, $v(t) := (v_1(t), \dots, v_s(t))$ satisfies the ODE (4.4).

Thus, we checked that for $x \in \mathcal{K} \subset \mathcal{M}_{ind\ \mu}$ and each fixed $t \in \mathcal{I}$, $w := x(t)$ belongs to the set $\tilde{K}(t)$.

For the second part, for each t let the set $\tilde{K}(t)$ be given. On one hand, $y^*AD = y^*\Pi_{*can\ 2}^* AD = y^*AD\Pi_{can\ 2} = y^*AD\hat{P}_1$. On the other hand, by construction $s = \dim \operatorname{im} y = \dim \operatorname{im} A^*y$ holds. Thus,

$$\dim(\ker y^*A \cap DS_1) = (m - s) - [\nu - (m - r)] = r - s - \nu = r - (\rho - k) - \nu = k,$$

i.e. $\dim \tilde{K}(t) \equiv k$.

Fix a $\tilde{t} \in \mathcal{I}$. Due to above considerations, there exists $w_0 \in \mathbb{C}^m$ such that $w_0 \in \tilde{K}(\tilde{t})$ and there exist k linearly indepent vectors $w_1, \dots, w_k \in \mathbb{C}^m$ such that

$$y^*(\tilde{t})A(\tilde{t})D(\tilde{t})w_i = 0, \quad \text{and } w_i = \Pi_{can\ 2}w_i.$$

Let us consider the solution x_0 of (1) with initial value $x_0(\tilde{t}) = w_0$ and solutions $x_1, \dots, x_k$ of homogeneous equations (1) with initial values $x_i(\tilde{t}) = w_i$, $i = 1, \dots, k$, respectively. For each t, $x_0(t) + \operatorname{span}\{x_1(t), \dots, x_k\} \subset \tilde{K}(t)$. A similar reasoning that we applied when showing the linear independence of the solutions of the homogeneous adjoint equation with linearly independent initial values yields that for each t, the affine linear set $x_0(t) + \operatorname{span}\{x_1(t), \dots, x_k(t)\}$ is of dimension k. Thus, $\tilde{K}(t) = x_0(t) + \operatorname{span}\{x_1(t), \dots, x_k(t)\}$. On the other hand, $x_0 + \operatorname{span}\{x_1, \dots, x_k\}$ is an affine linear solution set as it was claimed.

□

4. Transfer of boundary conditions

4.1. **Separated boundary conditions.** First let us consider the BVP for (1) with separated boundary condition (24), i.e. $K_a^* = (K_{a1}^*|0)$, $K_b^* = (0|K_{b2}^*)$, $d^* = (d_1^*|d_2^*)$,

where $K_{a1} \in L(\mathbb{C}^m, \mathbb{C}^{m_a})$, $K_{b2} \in L(\mathbb{C}^m, \mathbb{C}^{m_b})$, $d_1 \in \mathbb{C}^{m_a}$, $d_2 \in \mathbb{C}^{m_b}$, 0-s stand for zero matrices of proper dimensions.

Both sets of solutions defined by one and the other boundary conditions, i.e.

$$\mathcal{K}_a := \{x \in \mathcal{M}_{ind2} : K_{a1}x(a) = d_1\} \tag{49}$$

and

$$\mathcal{K}_b := \{x \in \mathcal{M}_{ind2} : K_{b2}x(b) = d_2\} \tag{50}$$

are affine linear solution sets and so is $\mathcal{K} := \mathcal{K}_a \bigcap \mathcal{K}_b$, the solution set of the BVP. Due to Remark 3.2, $\mathcal{K}_a$ and $\mathcal{K}_a$ have the equivalent representation

$$\mathcal{K}_a := \{x \in \mathcal{M}_{ind2} : \tilde{K}_{a1}x(a) = \tilde{d}_1\}, \tag{51}$$

$$\mathcal{K}_b := \{x \in \mathcal{M}_{ind2} : \tilde{K}_{b2}x(b) = \tilde{d}_2\}, \tag{52}$$

where

$$\begin{aligned}
\tilde{K}_{a1} &= K_{a1}G_{*2}^{-*}(a)(I - Q_{*0}^*(a) - V^*(a))G_{*2}^*(a),\\
\tilde{K}_{b2} &= K_{b2}G_{*2}^{-*}(b)(I - Q_{*0}^*(b) - V^*(b))G_{*2}^*(b),\\
\tilde{d}_1 &= d_1 - G_{*2}^{-*}(a)[(P_{*1}^*(a)Q_{*0}^*(a) + Q_{*1}^*(a)P_{*0}^*(a))q(a)\\
&\quad -Q_{*1}^*(a)A(a)(A^{*-*}Q_{*1}^*Q_{*0}^*q)'(a)],\\
\tilde{d}_2 &= d_2 - G_{*2}^{-*}(b)[(P_{*1}^*(b)Q_{*0}^*(b) + Q_{*1}^*(b)P_{*0}^*(b))q(b)\\
&\quad -Q_{*1}^*(b)A(b)(A^{*-*}Q_{*1}^*Q_{*0}^*q)'(b)].
\end{aligned}$$

Note that

$$(I - Q_{*0}^*(t) - V^*(t))G_{*2}^*(t) = -[I + A(t)(A^{*-*}Q_{*1}^*A)'(t)A^{*-*}(t)]P_{*1}^*(t)A(t)D(t).$$

We always may assume that the boundary conditions are given in their modified form and the matrices $\tilde{K}_{a1}, \tilde{K}_{a1}$ are of full rank. Set

$$\begin{aligned}
y_{aa} &:= K_*(a)A^{*-}(a)D^{*-}(a)\tilde{K}_{a1}^*,\\
y_{bb} &:= K_*(b)A^{*-}(b)D^{*-}(b)\tilde{K}_{b2}^*,
\end{aligned}$$

where K_* is the counterpart of K from (19), i.e.

$$K_* := I - Q_{*0}\hat{P}_{*1}\hat{G}_{*2}^{-1}B^*P_{*0} - Q_{*0}\hat{Q}_{*1}A^{*-}(A^*\hat{Q}_{*1}A^{*-})'A^*$$

Trivially, conditions $\tilde{K}_{a1}x(a) = \tilde{d}_1$ and $y_{aa}^*A(a)D(a)x(a) = \tilde{d}_1$ are identical and the same is valid for the pair $\tilde{K}_{b2}x(b) = \tilde{d}_2$ and $y_{bb}^*A(a)D(a)x(a)x(b) = \tilde{d}_2$.

Let y_a and y_b be the solutions of (46) with initial values $y_a(a) = y_{aa}$ and $y_b(a) = y_{bb}$. In parallel, let v_a and v_b the solutions of (47) constructed with the corresponding y_a and y_b and initial values $v(a) = \tilde{d}_1$ and $v(b) = \tilde{d}_2$, respectively. Due to Theorem 3.1 a function $x \in C_D^1$ is a solution of the BVP (1) if and only if for each t $x(t)$ satisfies the system

$$y_a^*(t)A(t)D(t)x(t) = -v_a^*(t) \tag{53}$$

$$y_b^*(t)A(t)D(t)x(t) = -v_b^*(t), \tag{54}$$

$$\begin{aligned}
V^*(t)G_{*2}^*(t)x(t) &= \hat{Q}_{*1}^*(t)(P_{*0}^*(t) - Q_{*0}^*(t))q(t)\\
&\quad - \hat{Q}_{*1}^*(t)A(t)(A^{*-*}\hat{Q}_{*1}^*Q_{*0}^*q)'(t), \qquad (55)\\
Q_{*0}^*(t)B(t)x(t) &= Q_{*0}^*(t)q(t). \qquad (56)
\end{aligned}$$

Due to the construction, the first couple of equations is linearly independent of the second couple. Let $\tau := \dim\operatorname{im}\,(y_a|y_b)$. Also by construction,

$$\dim\operatorname{im}\, D^*A^*(y_a|y_b) \equiv \dim\operatorname{im}\,(y_a|y_b) \equiv \dim\operatorname{im}\,(y_{aa}|y_{bb})$$

holds. Formerly we stated $\dim M(t) \equiv r + r_1 - m$. Now, Fredholm alternative for (53)-(56) ensures the following statement:

Theorem 4.1. *For any* $q \in \mathcal{C}^1_{D\hat{Q}_1 G_2^{-1}}$ *and* $\tilde{d}_1 \in \operatorname{im} \tilde{K}_{a1}$, $\tilde{d}_2 \in \operatorname{im} \tilde{K}_{b2}$, *a unique solution* x *exists if and only if* $r + r_1 - m - \tau = 0$.

4.2. **Non-separated boundary conditions.** By the help of Moszyňski's trick [4], we transform the problem into an equivalent one with separated boundary conditions. For $t \in [a, (a+b)/2]$, set

$$\bar{x}(t) := \begin{pmatrix} x(t) \\ x(b+a-t) \end{pmatrix}, \quad \bar{q}(t) := \begin{pmatrix} q(t) \\ q(b+a-t) \end{pmatrix},$$

$$\bar{A}(t) := \operatorname{diag}\,(A(t), A(b+a-t)), \quad \bar{D}(t) := \operatorname{diag}\,(D(t), D(b+a-t)),$$

$$\bar{B}(t) := \operatorname{diag}\,(B(t), -B(b+a-t)),$$

$$\bar{K}_a = \begin{pmatrix} K_a & K_b \\ 0 & 0 \end{pmatrix}, \quad \bar{K}_{\frac{a+b}{2}} = \begin{pmatrix} 0 & 0 \\ I_m & -I_m \end{pmatrix}, \quad \bar{d} = \begin{pmatrix} d \\ 0 \end{pmatrix}.$$

where I_m is the $m \times m$ identity matrix. Trivially, the original boundary value problem is equivalent to the BVP of doubled size on the halfed interval $[a, (a+b)/2]$ with the above data. This latter problem for $\bar{x}$ is a BVP with separated boundary conditions and all considerations of the previous subsection apply.

5. Final remarks

Remark 5.1. The homogeneous IVPs for (2) with our initial data $y(a) = y_{aa}$ and $y(b) = y_{bb}$ in Section 4 are always solvable. Thus, by integration of (2) system from both interval ends up to an arbitrary common point t_0 one obtains $y_a(t_0)$ and $y_b(t_0)$. In parallel, the IVPs for equation (47) are to be solved. One should compute (preserve) the values only at points $\hat{t}$ where the solution x are needed. At these points the other two matrices, $G_{*2}(\hat{t})V(\hat{t})$ and $B^*(\hat{t})Q_{*0}(\hat{t})$ should be calculated, too. If the linear system (53)-(56) is nonsingular at an arbitrary $\hat{t} = t_0$, then so is it for all $\hat{t}$ and one can state solvability and uniqueness of the solution and get the solution at all $\hat{t}$.

Remark 5.2. To go in line with this program, a reliable integrator for (2) is needed and all of the other coefficients occurring in (53)-(56) must be available. It is worth noticing that in this system we need only $A^*(t)y(t) = A^*(t)\hat{P}_{*1}y(t)$, i.e. the solution of the inherent ODE of the adjoint equation. One may prefer solving this homogenous inherent ODE instead of the homogeneous DAE (2). Practically, there is no difference in computational complexity. A reliable integrator for any DAE would use its inherent ODE to keep the numerical solution in the corresponding subspace at least implicitly [2].

Remark 5.3. Theoretically, a properly discretized version of the transfer method would yield an algorithm for numerical solution of the BVPs for (1). However, the resulting procedure may be very sensitive to accumulation of numerical errors. This phenomenon may appear even when the BVP is well conditioned and the relevant subspaces vary slowly. Therefore, a modification of the transfer algorithm seems reasonable. That modification would rely on the orthonormalization of basis vectors of the subspaces in question at the meshpoints or it would build a smoothly varying basis on the whole interval. These issues will be reported in an other publication.

Remark 5.4. There is no gain in the complexity if one avoids using the adjoint equation and the method relies upon any kind of shooting. In that case the computational effort is spent for keeping either a fundamental matrix (not necessarily the maximal normalized fundamental matrix) of the homogeneous DAE (1) in the corresponding subspace or some solutions of the DAE (1) in the corresponding affine subspace. To achieve this aim, one must use projectors not simpler than those in our analysis.

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