

On Artin's L-functions. II: Dirichlet Coefficients

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Let $a_n, n \geq 1$, be complex numbers. If for a real number θ the summatory function $\sum_{n \leq x} a_n$ is $\mathbf{O}(x^\theta)$ as $x \rightarrow \infty$, then the Dirichlet series $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ converges in the half-plane $\operatorname{Re}(s) > \theta$. For a non-principal Dirichlet character modulo m it holds for every $x \geq 1$

$$\left| \sum_{n \leq x} \chi(n) \right| \leq \varphi(m),$$

so $\sum_{n \leq x} \chi(n)$ is $\mathbf{O}(1)$ and the series $\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$ converges for $\operatorname{Re}(s) > 0$. The main result of this paper is

Theorem. *Let K/\mathbb{Q} be a finite Galois extension, let χ be a character of the Galois group $G = \operatorname{Gal}(K/\mathbb{Q})$ which does not contain the principal character, let $L_{ur}(s, \chi, K/\mathbb{Q})$ be the unramified part of the corresponding Artin L-function, and let*

$$L_{ur}(s, \chi, K/\mathbb{Q})^{\frac{1}{x(1)}} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

for $\operatorname{Re}(s) > 1$. Then:

- (i) *The coefficients a_n are algebraic numbers of the field $\mathbb{Q}(e^{\frac{2\pi i}{|G|}})$ and $|a_n| \leq 1$ for every $n \geq 1$;*
- (ii) *The summatory function $\sum_{n \leq x} a_n$ is $\mathbf{o}(x)$ as $x \rightarrow \infty$.*

The theorem says in the case $K = \mathbb{Q}(e^{\frac{2\pi i}{m}})$ that $\sum_{n \leq x} \chi(n)$ is $\mathbf{o}(x)$ as $x \rightarrow \infty$, which is weaker than: $\sum_{n \leq x} \chi(n)$ is $\mathbf{O}(1)$. It should be seen as an assertion

about a general Artin L -function in the case of a non-abelian extension K/\mathbb{Q} .

An evaluation $\mathbf{O}(x^\theta)$ with $\theta < 1$ of the summatory function of the Dirichlet coefficients of an Artin L -function had as a consequence that its Dirichlet series converges in $\operatorname{Re}(s) > \theta$, so the L -function is holomorphic in that half plane, which would be an important step towards Artin's holomorphy conjecture. This is not reached here. It is the motivation for future research.

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P r o o f of the theorem: Let d_K be the discriminant of the algebraic number field K , and let ([2], P. 297, (7))

$$(1) \quad L_{ur}(s, \chi, K/\mathbb{Q}) := \prod_{(p, d_K)=1} \frac{1}{\det(I - \frac{A_p}{p^s})}$$

be the product of the Euler factors of $L(s, \chi, K/\mathbb{Q})$ corresponding to the unramified primes p . This is absolutely convergent in $\operatorname{Re}(s) > 1$. The dimension of the matrix A_p is $r = \chi(1)$, and it holds

$$\det(I - \frac{A_p}{p^s}) = \prod_{j=1}^r (1 - \frac{\zeta_{j,p}}{p^s}),$$

where $\zeta_{1,p}, \dots, \zeta_{r,p}$ are the eigenvalues of A_p , which are roots of unity of order dividing the order $|G|$ of the Galois group G . By Newton's binomial formula it holds for $\operatorname{Re}(s) > 0$

$$(2) \quad \begin{aligned} \frac{1}{\det(I - \frac{A_p}{p^s})^{\frac{1}{r}}} &= \prod_{j=1}^r \frac{1}{(1 - \frac{\zeta_{j,p}}{p^s})^{\frac{1}{r}}} = \prod_{j=1}^r (1 - \frac{\zeta_{j,p}}{p^s})^{-\frac{1}{r}} = \\ &= \prod_{j=1}^r [1 + \sum_{n=1}^{\infty} \frac{-\frac{1}{r} \cdot (-\frac{1}{r} - 1) \cdot \dots \cdot (-\frac{1}{r} - n + 1)}{1 \cdot 2 \cdot \dots \cdot n} (-\frac{\zeta_{j,p}}{p^s})^n] = \\ &= \prod_{j=1}^r [1 + \sum_{n=1}^{\infty} \frac{1 \cdot (1+r) \cdot \dots \cdot (1+(n-1)r)}{r^n \cdot n!} \frac{\zeta_{j,p}^n}{p^{ns}}] = 1 + \sum_{k=1}^{\infty} \frac{a_p^k}{p^{ks}}, \end{aligned}$$

where for $k \geq 1$

$$a_p^k = \sum_{k_1 \geq 0, \dots, k_r \geq 0, k_1 + \dots + k_r = k} \prod_{j=1}^r c_{k_j} \zeta_{j,p}^{k_j} = \sum_{k_1 \geq 0, \dots, k_r \geq 0, k_1 + \dots + k_r = k} \prod_{j=1}^r c_{k_j} \cdot \prod_{j=1}^r \zeta_{j,p}^{k_j},$$

with

$$(3) \quad c_0 = 1, \quad c_l = \frac{\prod_{u=1}^l (1 + (u-1)r)}{r^l \cdot l!}, \quad l \geq 1,$$

hence

$$(4) \quad |a_{p^k}| \leq \sum_{k_1 \geq 0, \dots, k_r \geq 0, k_1 + \dots + k_r = k} \left| \prod_{j=1}^r c_{k_j} \cdot \prod_{j=1}^r \zeta_{j,p}^{k_j} \right| = \sum_{k_1 \geq 0, \dots, k_r \geq 0, k_1 + \dots + k_r = k} \prod_{j=1}^r c_{k_j}.$$

Lemma. For $r \geq 1$ and $k \geq 1$ it holds

$$(5) \quad \sum_{k_1 \geq 0, \dots, k_r \geq 0, k_1 + \dots + k_r = k} \prod_{j=1}^r c_{k_j} = 1.$$

P r o o f of the lemma: For $l \geq 0$ let $d_l = r^l \cdot l! \cdot c_l$. By (3) it holds

$$(6) \quad d_0 = 1, \quad d_l = \prod_{u=1}^l (1 + (u-1)r), \quad l \geq 1.$$

For $k_1 \geq 0, \dots, k_r \geq 0$ with $k_1 + \dots + k_r = k$ it holds

$$\prod_{j=1}^r c_{k_j} = \frac{\prod_{j=1}^r d_{k_j}}{r^k \cdot k_1! \cdot \dots \cdot k_r!} = \frac{1}{r^k \cdot k!} \binom{k}{k_1 \dots k_r} \prod_{j=1}^r d_{k_j},$$

where

$$\binom{k}{k_1 \dots k_r} = \frac{k!}{k_1! \cdot \dots \cdot k_r!}$$

is the multinomial coefficient, so (5) is equivalent to

$$(7) \quad \sum_{k_1 \geq 0, \dots, k_r \geq 0, k_1 + \dots + k_r = k} \binom{k}{k_1 \dots k_r} \prod_{j=1}^r d_{k_j} = r^k \cdot k!.$$

For $k = 1$ it holds by (6)

$$d_1 + \dots + d_1 = r \cdot d_1 = r \cdot 1,$$

so (7) is true. Suppose that (7) is true for a number $k \geq 1$. Then

$$(8) \quad \begin{aligned} r^{k+1} \cdot (k+1)! &= r(k+1)r^k \cdot k! = \\ &= r(k+1) \sum_{k_1 \geq 0, \dots, k_r \geq 0, k_1 + \dots + k_r = k} \binom{k}{k_1 \dots k_r} \prod_{j=1}^r d_{k_j} = \\ &= \sum_{k_1 \geq 0, \dots, k_r \geq 0, k_1 + \dots + k_r = k} (rk+r) \binom{k}{k_1 \dots k_r} \prod_{j=1}^r d_{k_j} = \end{aligned}$$

$$= \sum_{k_1 \geq 0, \dots, k_r \geq 0, k_1 + \dots + k_r = k} \binom{k}{k_1 \dots k_r} [(rk_1 + 1) + \dots + (rk_r + 1)] \prod_{j=1}^r d_{k_j}.$$

By (6) it holds for $l \geq 0$

$$d_{l+1} = (1 + lr)d_l,$$

so (8) is equivalent to

$$(9) \quad r^{k+1} \cdot (k+1)! = \sum_{k_1 \geq 0, \dots, k_r \geq 0, k_1 + \dots + k_r = k} \binom{k}{k_1 \dots k_r} \sum_{j=1}^r d_{k_1} \dots d_{k_{j+1}} \dots d_{k_r}.$$

It holds

$$\begin{aligned} (10) \quad & \sum_{k_1 \geq 0, \dots, k_r \geq 0, k_1 + \dots + k_r = k} \binom{k}{k_1 \dots k_r} \sum_{j=1}^r d_{k_1} \dots d_{k_{j+1}} \dots d_{k_r} = \\ & = \sum_{l_1 \geq 0, \dots, l_r \geq 0, l_1 + \dots + l_r = k+1} \left[\sum_{j=1, l_j \geq 1}^r \binom{k}{l_1 \dots l_j - 1 \dots l_r} \right] d_{l_1} \dots d_{l_r} = \\ & = \sum_{l_1 \geq 0, \dots, l_r \geq 0, l_1 + \dots + l_r = k+1} \binom{k+1}{l_1 \dots l_r} d_{l_1} \dots d_{l_r}. \end{aligned}$$

From (8), (9) and (10) it follows that (7) is true for $k+1$. \square

By (4) and the preceding lemma it holds for every $k \geq 1$

$$(11) \quad |a_p^k| \leq 1.$$

Let $a_1 := 1$, $a_n := 0$ for $(n, d_K) \neq 1$, $a_n := a_{p_1}^{k_1} \dots a_{p_m}^{k_m}$ for $n = p_1^{k_1} \dots p_m^{k_m} > 1$, $(n, d_K) = 1$. Then it holds by (1) and (2)

$$L_{ur}(s, \chi, K/\mathbb{Q})^{\frac{1}{x(1)}} = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

with

$$(12) \quad a_n \in \mathbb{Q}(e^{\frac{2\pi i}{|G|}}), \quad |a_n| \leq 1,$$

for every $n \geq 1$. The first part of the theorem is proved. For the second part apply

The Tauberian Theorem of Wiener-Ikehara. *Let $G(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ be a Dirichlet series. Suppose there exists a Dirichlet series $H(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$ with positive real coefficients such that*

a) $|a_n| \leq b_n$ for all $n \geq 1$;

- b) the series $H(s)$ converges for $\operatorname{Re}(s) > 1$;
 c) the function $H(s)$ (respectively $G(s)$) can be extended to a meromorphic function in the region $\operatorname{Re}(s) \geq 1$ having no poles except (respectively except possibly) for a simple pole at $s = 1$ with residue $C \geq 0$ (respectively c).

Then

$$\sum_{n \leq x} a_n = cx + \mathbf{o}(x)$$

as $x \rightarrow \infty$. In particular, if $G(s)$ is holomorphic at $s = 1$, then $c = 0$ and $\sum_{n \leq x} a_n = \mathbf{o}(x)$ as $x \rightarrow \infty$. ([3], Theorem 1.1, P. 7)

Take

$$G(s) = L_{ur}(s, \chi, K/\mathbb{Q})^{\frac{1}{x(1)}} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

and

$$H(s) = \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The assumptions of the tauberian theorem of Wiener-Ikehara are satisfied:

- a) It was proved above that $|a_n| \leq 1$ for every $n \geq 1$;
 b) the series $H(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ converges for $\operatorname{Re}(s) > 1$;
 c) the Riemann zeta function $H(s) = \zeta(s)$ is meromorphic in the whole complex plane with exactly one pole at $s = 1$, which is simple with residue $C = 1 \geq 0$. The Artin L -function $L_{ur}(s, \chi, K/\mathbb{Q})$ is holomorphic and has no zeroes for $\operatorname{Re}(s) \geq 1$ ([1], Satz 3, P. 105), so the function $L_{ur}(s, \chi, K/\mathbb{Q})^{\frac{1}{x(1)}}$ is defined and holomorphic for $\operatorname{Re}(s) \geq 1$. It has no pole at $s = 1$: $c = 0$.
 By the theorem of Wiener-Ikehara it holds

$$\sum_{n \leq x} a_n = \mathbf{o}(x).$$

□

References

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