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Abstract

We investigate a stationary model for turbulent flows, in which the Navier-Stokes system is coupled to an equation for the density of turbulent kinetic energy through a bounded coefficient of eddy viscosity. We extend the results of [Lew97b] by proving the existence of weak solutions for this model.

We use the method developed in [Nau05], in order to prove the higher integrability of the gradients of weak solutions. Finally, we show that the model is well posed in the two dimensional case, provided that the external forcing remains sufficiently small.

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1 Introduction

In this paper we would like to study some mathematical aspects of the turbulence model given by the following system of equations

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p + \operatorname{div}\Big((\nu + \nu_t(k))\, Du\Big) + f\,, \tag{1}$$

$$\operatorname{div}\, u = 0\,, \tag{2}$$

$$\frac{\partial k}{\partial t} + u \cdot \nabla k = \operatorname{div}\Big((\nu + \nu_t(k))\, \nabla k\Big) + \nu_t(k)\, D(u,u) - k^{\frac{3}{2}} \qquad \text{in } \Omega\,, \tag{3}$$

where Ω is a bounded domain in $\mathbb{R}^n$ ($n = 2, 3$), and $\nu > 0$ is the dynamical viscosity of the fluid multiplied by two.

The model (1), (2), (3) belongs to the class of so-called RANS models (Reynolds Averaged Navier-Stokes). The basic idea of Reynolds's Ansatz is to understand the turbulence of a flow as the result of random fluctuations around a (time)-averaged mean flow. According to this hypothesis, each quantity that appears in the customary Navier-Stokes equations splits into a mean value part and a fluctuation part, i. e.

$$u = \overline{u} + u', \qquad p = \overline{p} + p', \text{ etc. }\,,$$

with different possible choices for the averaging operator $f \mapsto \overline{f}$ (see [MP94]). Applying such a 'filter' to the Navier-Stokes equations, one derives the new relations

$$\frac{\partial \overline{u}}{\partial t} + (\overline{u} \cdot \nabla)\overline{u} = -\nabla \overline{p} + \operatorname{div}\big(\nu\, D\overline{u}\big) + \operatorname{div}\big(\mathcal{R}\big) + \overline{f}\,, \tag{4}$$

$$\operatorname{div}\, \overline{u} = 0\,.$$

The symbol $\mathcal{R}$ denotes the so-called Reynolds stress tensor. It is given by the formula

$$\mathcal{R}_{i,j} = -\overline{u_i'\, u_j'} \quad (i,j = 1,\ldots,n)\,.$$

Rigorous constitutive relations for determining the new variables $\overline{u_i'\, u_j'}$ are at this time not availaible: this is the problem of the *closure of turbulence models*. The most common closure assumption in current models is to consider that the Reynolds stress tensor is proportional to the sum of the deformation tensor of the mean flow and of some pressure term (see, for instance, [MP01]), i. e.

$$\mathcal{R} = \nu_t\, D\overline{u} + \alpha\, I\,,$$

with a proportionality factor ν_t, which is called, for dimensional reasons, the *eddy viscosity*.

The eddy viscosity must also be modeled. To this purpose, our model (1) introduces the new variable k, the (averaged)-density of turbulent kinetic energy. We set

$$e := \frac{1}{2}\,|u'|^2, \qquad k := \overline{e}\,,$$

and we suppose that the turbulent viscosity depends only on the turbulent kinetic energy k and on the characteristic length l of the turbulence (here assumed to be a given constant), by the relation

$$\nu_t = \nu_t(k) = C\, k^{\frac{1}{2}}\, l\,, \tag{5}$$

with a dimensionless, empirical constant C.

Now, in order to close the model, one derives an equation for the variable k. Subtracting equation (4) to the averaged Navier-Stokes equations, one obtains an evolution equation for the fluctuation part of the velocity u'. Multiplying this equation by u', one gets, after averaging again, an evolution equation for k

$$\frac{\partial k}{\partial t} = \ldots\,.$$

The terms on the right-hand side involve $u', \overline{u}$ and products of these variables and their derivatives. Again, those terms will have to be modeled. This makes the derivation of the equation for k highly heuristical (one finds a list of the assumptions under which the derivation is valid in [MP94], [MP01]). In [Lew97a], one can follow the derivation of

the equation for k in the frame of a concrete model used in oceanography (see also the references therein).

In this paper we focus on the following stationary problem (P):

$$
\begin{aligned}
(u \cdot \nabla)u &= -\nabla p + \operatorname{div}\Big((\nu + \nu_t(k))\, Du\Big) + f\,, \\
\operatorname{div}\, u &= 0\,, \\
u \cdot \nabla k &= \operatorname{div}\Big((\nu + \nu_t(k))\, k\Big) + \nu_t(k)\, D(u,u) - g(k)\, k^{\frac{1}{2}} \qquad \text{in } \Omega\,,
\end{aligned}
$$

with the boundary conditions

$$
u = 0, \qquad k = 0 \ \text{ on } \partial\Omega\,. \tag{6}
$$

Here and troughout the remainder of the paper, we write again u, f instead of $\overline{u}, \overline{f}$.

We will consider a bounded eddy viscosity ν_t, i. e.

$$
0 \le \nu_t(s) \le M \ \text{ for all } s \in \mathbb{R}\,, \tag{7}
$$

with a positive constant M.

We wrote the term $k^{\frac{3}{2}}$ of (1), which represents the dissipation of kinetic energy by the smaller eddies, in the somewhat more general form $g(k)\, k^{\frac{1}{2}}$, with a positive function g.

The requirement (7) has to be understood as a mathematical assumption which will simplify the analysis. As a matter of fact, the hypothesis (7) excludes the real-life case (5). However, the remarks of [Lew97b] show that the problem simplified in this way remains significant for the numerical practice in concrete applications.

The boundary conditions (6) represent also an idealization. They are consistent with the usual no-slip boundary condition, at least in the case of a time averaging filter (observe that $k = \overline{|u'|^2/2} = \overline{|u - \overline{u}|^2/2}$). However, the no-slip assumption usually leads to neglect the turbulence near the boundary. As in [Lew97b] and [Lew97a], we will assume, though, that the approximation (6) is reasonably good.

Note that one of the main mathematical difficulties of RANS-models, which consists in dealing with the L^1-term $D(u,u)$, is present in our model.

In the paper [Lew97b], an existence result was stated for (P). In the first part of the present paper, we are going to give a detailed proof of this result. The difficulties of the existence proof arise from the term $D(u,u)$, which, in the natural $\big[W^{1,2}(\Omega)\big]^n$-context of the Navier-Stokes theory, belongs only to the space $L^1(\Omega)$.

In a second section, we discuss the regularity of the solution. Note that because of the dependence of ν_t on k, regularity is a difficult issue in this context. We will show, though, the higher integrability of ∇u, i. e. $\nabla u \in [L^\sigma(\Omega)]^{n^2}$ for some $\sigma > 2$. This property leads to an appreciable improvement in the two dimensional case, which will permit us to discuss the uniqueness issue in the last section.

2 Weak formulation

We begin by introducing some notations.
We will need the functional spaces,

$$D_0^{1,p}(\Omega) := \left\{ w \in \left[W_0^{1,p}(\Omega)\right]^n \,:\, \text{div } w = 0 \right\}, \qquad \text{for } 1 \le p \le \infty\,.$$

We set

$$Du = D_{i,j}(u) := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial u_i}\right), \qquad D(u,v) := Du \,:\, Dv\,, \qquad \mu(k) := \nu + \nu_t(k)\,,$$

Through the paper, we make the following assumptions on the growth of the data:

$$0 \le \nu_t(z) \le M \qquad \text{for all } z \in \mathbb{R}\,, \tag{8}$$

$$f \in \left[L^s(\Omega)\right]^n \quad \begin{cases} 1 < s < \infty & \text{if } n = 2\,, \\ s = \frac{6}{5} & \text{if } n = 3\,, \end{cases} \tag{9}$$

$$0 \le g(z) \le C_0\,(1 + |z|^\alpha) \quad \begin{cases} 0 \le \alpha < \infty & \text{if } n = 2\,, \\ 0 \le \alpha < \frac{5}{2} & \text{if } n = 3\,, \end{cases} \tag{10}$$

where M, C_0 are given positive constants.

Definition 2.1. A *weak solution* of (P) is a pair

$$\{u,k\} \in D_0^{1,2}(\Omega) \times \bigcap_{p_0 \le p < p_1} W_0^{1,p}(\Omega), \quad \begin{cases} p_0 = 1 \text{ and } p_1 = 2 & \text{for } n = 2\,, \\ p_0 = \frac{6}{5} \text{ and } p_1 = \frac{3}{2} & \text{for } n = 3\,, \end{cases}$$

such that for all pairs $\{v,\phi\} \in D_0^{1,2}(\Omega) \times W_0^{1,q}(\Omega)\,(q > n)$, the three following relations are satisfied:

$$\int_\Omega u_j \frac{\partial u_i}{\partial u_j}\, v_i + \int_\Omega \mu(k)\, D(u,v) = \int_\Omega f \cdot v\,, \tag{11}$$

$$\int_\Omega u_j \frac{\partial k}{\partial x_j}\, \phi + \int_\Omega \mu(k)\, \nabla k \cdot \nabla \phi = \int_\Omega \nu_t(k)\, D(u,u)\, \phi - \int_\Omega g(k)\, k^{\frac{1}{2}}\, \phi\,, \tag{12}$$

$$k \ge 0 \text{ a. e. in } \Omega\,. \tag{13}$$

One easily verifies that under the assumptions (8), (9) and (10), this definition is well-posed.

3 An existence result.

In this section we prove the following theorem:

Theorem 3.1. If the assumptions (8), (9) and (10) are satisfied, then (P) has at least one weak solution.

First, we will prove the existence of suitable approximate solutions.
To this purpose, we can somewhat relax the hypothesis (10). We assume in this section that

$$0 \le g(z) \le C_0\,(1+|z|^\alpha) \qquad \begin{cases} 0 \le \alpha < \infty & \text{for } n=2\,, \\ 0 \le \alpha < \frac{9}{2} & \text{for } n=3\,, \end{cases} \tag{14}$$

where C_0 is a given constant.
We introduce the space

$$H := D_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)\,.$$

Because of Korn's inequality, we see that H is a Hilbert space with respect to the scalar-product

$$\Big(\{u,\psi\},\{v,\phi\}\Big) := \int_\Omega D(u,v) + \nabla\psi\cdot\nabla\phi\,.$$

We have the natural identity

$$H^* = \Big(D_0^{1,2}(\Omega)\Big)^* \times \Big(W_0^{1,2}(\Omega)\Big)^*\,.$$

For $\{v^*,\phi^*\} \in H^*$, we define a duality product

$$\big\langle \{v^*,\phi^*\},\, \{v,\phi\}\big\rangle_H \;:=\; \big\langle v^*,\, v\big\rangle_{D_0^{1,2}(\Omega)} + \big\langle \phi^*, \phi\big\rangle_{W_0^{1,2}(\Omega)}\,. \tag{15}$$

Proposition 3.2. Under the hypothesis (8), (9) and (14), there exists for all $\epsilon > 0$ a pair $\{u^\epsilon, k^\epsilon\} \in H$ such that the relations

$$\int_\Omega u_j^\epsilon \frac{\partial u_i^\epsilon}{\partial x_j}\, v_i + \int_\Omega \mu(k^\epsilon)\, D(u^\epsilon, v) = \int_\Omega f\cdot v\,, \tag{16}$$

$$\int_\Omega u_j^\epsilon \frac{\partial k^\epsilon}{\partial x_j}\, \phi + \int_\Omega \mu(k^\epsilon)\, \nabla k^\epsilon\cdot\nabla\phi = \int_\Omega \nu_t(k^\epsilon)\, \frac{D(u^\epsilon,u^\epsilon)}{1+\epsilon\, D(u^\epsilon,u^\epsilon)}\,\phi - \int_\Omega g(k^\epsilon)\,\sqrt{k^\epsilon}\,\phi\,, \tag{17}$$

$$k^\epsilon \ge 0 \qquad \text{a. e. in } \Omega\,, \tag{18}$$

are satisfied for all $\{v,\phi\} \in H$.

Proof. Consider the operator

$$\big\langle A\big(\{u,k\}\big),\{v,\phi\}\big\rangle := \int_\Omega u_j\,\frac{\partial u_i}{\partial x_j}\,v_i + \int_\Omega \mu(k)\,D(u,v) + \int_\Omega u_j\,\frac{\partial k}{\partial x_j}\,\phi$$
$$+\int_\Omega \mu(k)\,\nabla k\cdot\nabla\phi - \int_\Omega \nu_t(k)\,\frac{D(u,u)}{1+\epsilon\,D(u,u)}\,\phi + \int_\Omega g(k)\,\sqrt{k^+}\,\phi\,.$$

Then, by representation (15), we easily see that A is continuous and bounded from H into H^*.

We now prove that A is coercive and pseudomonotone.
Thanks to the bound (8) and Young's inequality, we have the estimate

$$\left|\int_\Omega \nu_t(k)\,\frac{D(u,u)}{1+\epsilon\,D(u,u)}\,k\right| \le M\,\frac{1}{\epsilon}\,\parallel k\parallel_{L^1(\Omega)} \le \frac{\nu}{2}\,\parallel k\parallel^2_{W^{1,2}_0(\Omega)} + C\,.$$

On the other hand, since u is divergence free and vanishes on the boundary, we can write that

$$\int_\Omega u_j\,\frac{\partial u_i}{\partial x_j}\,u_i = \int_\Omega u_j\,\frac{1}{2}\,\frac{\partial u_i^2}{\partial x_j} = 0, \qquad \int_\Omega u_j\,\frac{\partial k}{\partial x_j}\,k = \int_\Omega u_j\,\frac{1}{2}\,\frac{\partial k^2}{\partial x_j} = 0\,.$$

Finally, we have

$$\int_\Omega g(k)\,\sqrt{k^+}\,k = \int_\Omega g(k)\,\sqrt{k^+}^3 \ge 0\,,$$

such that one directly obtains the bound

$$\big\langle A\big(\{u,k\}\big),\{u,k\}\big\rangle \;\ge\; \frac{\nu}{2}\,\parallel\{u,k\}\parallel^2_H - C\,,$$

with a fixed positive constant C. This proves the coercivity of A.

In order to prove that A is pseudomonotone, we consider an arbitrary sequence $\{u_m,k_m\}\subset H$ such that

$$\{u_m,k_m\}\rightharpoonup\{u,k\} \quad \text{in } H\,, \qquad \limsup_{m\to\infty}\big\langle A\big(\{u_m,k_m\}\big),\{u_m-u,k_m-k\}\big\rangle \le 0\,.$$

We solve the problem of the lower semicontinuity of the term

$$\nu_t(k_m)\,\frac{D(u_m,u_m)}{1+\epsilon\,D(u_m,u_m)}\,,$$

by proving the existence of a subsequence $\{u_m,\,k_m\}$ that converges strongly in H.

We can write

$$\int_\Omega \mu(k_m)\left[D(u_m-u,u_m-u)+|\nabla(k_m-k)|^2\right]$$

$$=\big\langle A\big(\{u_m,k_m\}\big),\{u_m-u,k_m-k\}\big\rangle-\int_\Omega \mu(k_m)\left[D(u,u_m-u)+\nabla k_m\cdot\nabla(k_m-k)\right]$$

$$-\int_\Omega u_{m,j}\,\frac{\partial u_{m,i}}{\partial x_j}\,(u_{m,i}-u_i)-\int_\Omega u_{m,j}\,\frac{\partial k_m}{\partial x_j}\,(k_m-k)+\int_\Omega \nu_t(k_m)\,\frac{D(u_m,u_m)}{1+\epsilon\,D(u_m,u_m)}\,(k_m-k)$$

$$-\int_\Omega g(k_m)\,\sqrt{k_m^+}\,(k_m-k)\,. \tag{19}$$

According to (14), if p is such that $p\,(\alpha+\frac{1}{2})=6$, then we see that $p':=p/(p-1)<6$. Thus, by well-known compactness results, we will be able to find a subsequence $\{u_m,k_m\}$, that we not relabel, such that

$$u_m\to u\ \text{ in } [L^4(\Omega)]^n\,,\quad k_m\to k\ \text{ in } L^{p'}(\Omega)\,,\quad k_m(x)\to k(x)\ \text{ for almost all } x\in\Omega\,. \tag{20}$$

We then have for $i,j=1,\dots,n$ that

$$\mu(k_m)\,D_{i,j}(u)\to\mu(k)\,D_{i,j}(u)\ \text{ in } L^2(\Omega),\qquad \mu(k_m)\,\nabla k\to\mu(k)\,\nabla k\ \text{ in } [L^2(\Omega)]^n\,.$$

Considering that

$$\left\|\nu_t(k_m)\,\frac{D(u_m,u_m)}{1+\epsilon\,D(u_m,u_m)}\right\|_{L^\infty(\Omega)}\le\frac{M}{\epsilon},\quad \|\,g(k_m)\,\sqrt{k_m^+}\,\|_{L^p(\Omega)}\le C\,,$$

we can pass to the limit in (19) in order to obtain

$$\nu\,\limsup_{m\to\infty}\int_\Omega\left[D(u_m-u,u_m-u)+|\nabla(k_m-k)|^2\right]\le 0\,.$$

This implies for a subsequence that

$$\{u_m,k_m\}\to\{u,k\}\ \text{ in } H,\quad D(u_m,u_m)\to D(u,u)\ \text{ a. e. in } \Omega\,.$$

Now, for this subsequence, it clearly follows that

$$\mu(k_m)\,D_{i,j}(u_m)\to\mu(k)\,D_{i,j}(u)\ \text{ in } L^2(\Omega)\,,\quad \mu(k_m)\,\nabla k_m\to\mu(k)\,\nabla k\ \text{ in } [L^2(\Omega)]^n\,,$$

$$\nu_t(k_m)\,\frac{D(u_m,u_m)}{1+\epsilon\,D(u_m,u_m)}\to\nu_t(k)\,\frac{D(u,u)}{1+\epsilon\,D(u,u)}\ \text{ in } L^q(\Omega)\quad (q<\infty\ \text{arbitrary})\,. \tag{21}$$

With the help of (21), we easily can show that for each pair $\{v,\phi\}\in H$, we now have

$$\liminf_{m\to\infty}\big\langle A\big(\{u_m,k_m\}\big),\{u_m-v,k_m-\phi\}\big\rangle=\big\langle A\big(\{u,k\}\big),\{u-v,k-\phi\}\big\rangle\,,$$

which proves the pseudomonotonicity of A.

By the theorem 2.7 of [Lio69], we find that A is surjective, and we obtain the existence of a pair $\{u^\epsilon, k^\epsilon\} \in H$ such that (16) and (17) are satisfied. It remains to prove that k^ϵ is positive.

Testing in (17) , with $k^{\epsilon-}$, we obtain the relation

$$\int_\Omega u_j^\epsilon \frac{\partial k^\epsilon}{\partial x_j} k^{\epsilon-} + \int_\Omega \mu(k^\epsilon) \left|\nabla k^{\epsilon-}\right|^2 = \int_\Omega \left(\nu_t(k^\epsilon) \frac{D(u^\epsilon, u^\epsilon)}{1 + \epsilon\, D(u^\epsilon, u^\epsilon)} - g(k^\epsilon) \sqrt{k^{\epsilon+}} \right) k^{\epsilon-} .$$

Considering that

$$\int_\Omega u_j^\epsilon \frac{\partial k^\epsilon}{\partial x_j} k^{\epsilon-} = \frac{1}{2} \int_\Omega u_j^\epsilon \frac{\partial}{\partial x_j} (k^\epsilon)^{-^2} = 0 , \quad \int_\Omega g(k^\epsilon) \sqrt{k^{\epsilon+}}\, k^{\epsilon-} = 0 ,$$

we get the inequality

$$\int_\Omega \mu(k^\epsilon) \left|\nabla k^{\epsilon-}\right|^2 \leq 0 ,$$

which leads to (18) and finishes the proof of the proposition. □

Now the second step consists in finding some *a priori* estimates for the sequence of aproximate solutions.

Proposition 3.3. For the sequence $\{u^\epsilon, k^\epsilon\}$ constructed in Proposition 3.2, there exists positive constants C_1, C_2, that do not depend on ϵ, such that

$$\| u^\epsilon \|_{D_0^{1,2}(\Omega)} \leq C_1 \ \| f \|_{[L^s(\Omega)]^n} , \tag{22}$$

$$\| k^\epsilon \|_{W_0^{1,p}(\Omega)} \leq C_2 \left(\| f \|_{[L^s(\Omega)]^n} + \| f \|_{[L^s(\Omega)]^n}^{\frac{2}{1-\delta}} \right) , \tag{23}$$

Here, we choose s according to (9), whereas p and δ satisfy

$$\begin{cases} p \in]1, 2[, \ \delta \in]0, 1[& \text{for } n = 2 , \\ p \in \left]1, \frac{3}{2}\right[, \ \delta = \frac{3-2p}{3-p} & \text{for } n = 3 . \end{cases} \tag{24}$$

Proof. We test the equation (16) with u^ϵ, and observing as usual that the convective term vanishes, we get the estimate

$$\nu \int_\Omega D(u^\epsilon, u^\epsilon) \leq \int_\Omega |f|\, |u^\epsilon| .$$

Using Korn's inequality and standard embedding arguments, we prove the estimate (22).

In order to prove the second estimate, we use the test function

$$\phi = 1 - \frac{1}{(1+k^\epsilon)^\delta}\,,$$

where δ is given by (24). If we set $\Psi(s) := \int_0^s \left(1 - \frac{1}{(1+\tau)^\delta}\right) d\tau$, then we can write

$$\frac{\partial \phi}{\partial x_i} = \frac{\delta}{(1+k^\epsilon)^{1+\delta}} \frac{\partial k^\epsilon}{\partial x_i}, \qquad \frac{\partial k^\epsilon}{\partial x_i}\,\phi = \frac{\partial}{\partial x_i}\Psi(k^\epsilon)\,,$$

$$\int_\Omega g(k^\epsilon)\,\sqrt{k^\epsilon}\,\phi \geq 0\,.$$

Considering that the convective term in relation (17) will again vanish, we can write

$$\delta \int_\Omega \mu(k^\epsilon)\,\frac{|\nabla k^\epsilon|^2}{(1+k^\epsilon)^{1+\delta}} \;\leq\; \int_\Omega \nu_t(k^\epsilon)\,\frac{D(u^\epsilon,u^\epsilon)}{1+\epsilon\,D(u^\epsilon,u^\epsilon)}\,\phi \;\leq\; M \int_\Omega D(u^\epsilon,u^\epsilon)\,.$$

With the help of estimate (22), we obtain that

$$\int_\Omega \mu(k^\epsilon)\,\frac{|\nabla k^\epsilon|^2}{(1+k^\epsilon)^{1+\delta}} \leq \frac{C\,M}{\delta\,\nu}\;\|\,f\,\|^2_{[L^s(\Omega)]^n}\;.$$

Arguing as in [Nau05], we can further estimate

$$\begin{aligned}
\int_\Omega |\nabla k^\epsilon|^p &= \int_\Omega \frac{|\nabla k^\epsilon|^p}{(1+k^\epsilon)^{(1+\delta)\,p/2}}\,(1+k^\epsilon)^{(1+\delta)\,p/2} \\
&\leq \left(\int_\Omega \frac{|\nabla k^\epsilon|^2}{(1+k^\epsilon)^{1+\delta}}\right)^{p/2} \left(\int_\Omega (1+k^\epsilon)^{\frac{(1+\delta)\,p}{(2-p)}}\right)^{(2-p)/2} \\
&\leq \left(\frac{C\,M}{\delta\,\nu^2}\;\|\,f\,\|^2_{[L^s(\Omega)]^n}\right)^{p/2} \left(|\Omega|^{\frac{2-p}{2}} + \|\,k^\epsilon\,\|^{p\,(1+\delta)/2}_{L^{\frac{(1+\delta)\,p}{(2-p)}}(\Omega)}\right).
\end{aligned}$$

With the help of Young's inequality, we now obtain for an arbitrary $\gamma > 0$ that

$$\int_\Omega |\nabla k^\epsilon|^p \leq \tilde{C}_\delta \left(\,\|\,f\,\|^p_{[L^s(\Omega)]^n} + c_\gamma\;\|\,f\,\|^{\frac{2p}{1-\delta}}_{[L^s(\Omega)]^n} + \gamma\;\|\,k^\epsilon\,\|^p_{L^{\frac{(1+\delta)\,p}{(2-p)}}(\Omega)}\right),$$

with a positive constant c_γ that depends on γ.
Now, considering our choice of δ according to (24), we have $(1+\delta)\,p/(2-p) \leq p^*$, where p^* denotes the number

$$p^* := \frac{n\,p}{n-p}\,.$$

In view of Sobolev's embedding theorems, we can find a positive constant c^* independent of ϵ such that

$$\| k^\epsilon \|^p_{L^{\frac{(1+\delta)p}{(2-p)}}(\Omega)} \leq c^* \int_\Omega |\nabla k^\epsilon|^p \,.$$

At this point, we just have to choose γ sufficiently small in order to finish the proof of the proposition. □

Remark 3.4. We introduce the function

$$K_0(t) := (C_1 + C_2)\, t + C_2\, t^{\frac{2}{1-\delta}} \quad \text{for } t \in \mathbb{R}^+ \,, \tag{25}$$

where C_1, C_2 and δ are given by Proposition 3.3. The result of Proposition 3.3 allows us to write that

$$\| u^\epsilon \|_{D_0^{1,2}(\Omega)} + \| k^\epsilon \|_{W_0^{1,p}(\Omega)} \leq K_0(\| f \|_{[L^s(\Omega)]^n}) \,.$$

We now give the proof of the main result.

Proof of theorem 3.1. Taking (10) into account, we consider a number p in the range fixed by Proposition 3.3, such that $\alpha + \frac{1}{2} < p^*$. As above, p^* denotes the number $n\,p/(n-p)$. Using the estimates of Proposition 3.3, and standard compactness results, we obtain the existence of a subsequence $\{u^\epsilon, k^\epsilon\} \subset H$ such that

$$\begin{aligned} &u^\epsilon \rightharpoonup u \ \text{ in } D_0^{1,2}(\Omega), \quad k^\epsilon \rightharpoonup k \ \text{ in } W_0^{1,p}(\Omega)\,, \\ &u^\epsilon \to u \ \text{ in } [L^4(\Omega)]^n, \quad k^\epsilon \to k \ \text{ in } L^p(\Omega)\,, \\ &k^\epsilon(x) \to k(x) \ \text{ for almost all } x \in \Omega\,. \end{aligned} \tag{26}$$

We test in (16) with $v = u^\epsilon - u$, we rearrange the terms, and we write

$$\int_\Omega \mu(k^\epsilon)\, D(u^\epsilon - u, u^\epsilon - u) = \int_\Omega f \cdot (u^\epsilon - u) - \int_\Omega u_j^\epsilon \frac{\partial u_i^\epsilon}{\partial x_j}\,(u_i^\epsilon - u_i) - \int_\Omega \mu(k^\epsilon)\, D(u, u^\epsilon - u)\,.$$

Since

$$\mu(k^\epsilon)\, D_{i,j}(u) \to \mu(k)\, D_{i,j}(u), \qquad u_j^\epsilon\, u_i^\epsilon \to u_j\, u_i\,, \qquad \text{in } L^2(\Omega)\,,$$

we see that

$$\nu \lim \int_\Omega D(u^\epsilon - u, u^\epsilon - u) \leq 0\,.$$

This implies that $u^\epsilon \to u$ in $D_0^{1,2}(\Omega)$. We find a subsequence such that

$$D(u^\epsilon, u^\epsilon) \to D(u, u) \ \text{ in } L^1(\Omega)\,. \tag{27}$$

Now, the properties (26) imply, first, that for any $v \in D_0^{1,2}(\Omega)$ we have

$$\mu(k^\epsilon)\, D_{i,j}(v) \to \mu(k)\, D_{i,j}(v), \qquad u_j^\epsilon\, v_i \to u_j\, v_i\,, \qquad \text{in } L^2(\Omega)\,.$$

Thus, passing to the limit $\epsilon \to 0$ in (16), we find the relation

$$\int_\Omega u_j\, \frac{\partial u_i}{\partial x_j}\, v_i + \int_\Omega \mu(k)\, D(u,v) = \int_\Omega f \cdot v\,.$$

In order to pass to the limit in (17), we first note that for any $\phi \in W_0^{1,q}(\Omega)\,(q > p')$, it holds that

$$u^\epsilon\, \phi \to u\, \phi, \qquad \mu(k^\epsilon)\, \nabla\phi \to \mu(k)\, \nabla\phi\,, \quad \text{in } [L^{p'}(\Omega)]^n\,. \tag{28}$$

Setting $r := \frac{p^*}{\alpha+1/2} > 1$, we can again pass to subsequences, and obtain that

$$g(k^\epsilon)\, \sqrt{k^\epsilon} \rightharpoonup g(k)\, \sqrt{k} \;\text{ in } L^r(\Omega)\,. \tag{29}$$

Finally we observe that

$$\begin{aligned}
\Big| D(u^\epsilon, u^\epsilon) - D(u,u) \Big| &= \Big| D_{i,j}(u^\epsilon - u)\, D_{i,j}(u^\epsilon) + D_{i,j}(u)\, D_{i,j}(u^\epsilon - u) \Big| \\
&\le \Big| D_{i,j}(u^\epsilon - u) \Big| \Big| D_{i,j}(u^\epsilon) \Big| + \Big| D_{i,j}(u) \Big| \Big| D_{i,j}(u^\epsilon - u) \Big| \\
&= \Big| D_{i,j}(u^\epsilon - u) \Big| \, \Big(\Big| D_{i,j}(u^\epsilon) \Big| + \Big| D_{i,j}(u) \Big| \Big)\,.
\end{aligned} \tag{30}$$

Consider

$$\begin{aligned}
&\int_\Omega \left| \nu_t(k^\epsilon)\, \frac{D(u^\epsilon, u^\epsilon)}{1 + \epsilon\, D(u^\epsilon, u^\epsilon)} - \nu_t(k)\, D(u,u) \right| \\
&\le \int_\Omega \nu_t(k^\epsilon)\, \left| \frac{D(u^\epsilon, u^\epsilon) - D(u,u) - \epsilon\, D(u^\epsilon, u^\epsilon)\, D(u,u)}{1 + \epsilon\, D(u^\epsilon, u^\epsilon)} \right| + \int_\Omega |\nu_t(k^\epsilon) - \nu_t(k)|\, D(u,u) \\
&\le \int_\Omega \nu_t(k^\epsilon)\, |D(u^\epsilon, u^\epsilon) - D(u,u)| + \int_\Omega M\, \epsilon\, \frac{D(u^\epsilon, u^\epsilon)\, D(u,u)}{1 + \epsilon\, D(u^\epsilon, u^\epsilon)} + \int_\Omega |\nu_t(k^\epsilon) - \nu_t(k)|\, D(u,u)\,.
\end{aligned}$$

The two last integrals converge to zero by the dominated convergence theorem. In view of (30), we can estimate the first term by

$$M \;\| u^\epsilon - u \|_{D_0^{1,2}(\Omega)}\; (\| u^\epsilon \|_{D_0^{1,2}(\Omega)} + \| u \|_{D_0^{1,2}(\Omega)}) \to 0\,.$$

We obtain that

$$\nu_t(k^\epsilon)\, \frac{D(u^\epsilon, u^\epsilon)}{1 + \epsilon\, D(u^\epsilon, u^\epsilon)} \to \nu_t(k)\, D(u,u) \qquad \text{in } L^1(\Omega)\,.$$

This last result combined with (28), (29) allow us to pass to the limit in relation (17), finishing the proof of the theorem. □

4 Higher integrability of the gradient of the mean flow.

In this section we make the stronger hypothesis that

$$f \in [L^2(\Omega)]^n \,. \tag{31}$$

Identifying f with an element of $\left([W_0^{1,\frac{2n}{n+2}}(\Omega)]^n\right)^*$, we can find some

$$g \in [L^s(\Omega)]^{n^2} \quad \begin{cases} s < \infty & \text{for } n = 2\,, \\ s = 6 & \text{for } n = 3\,, \end{cases}$$

such that for all $v \in [W_0^{1,\frac{2n}{n+2}}(\Omega)]^n$, the representation

$$\int_\Omega f \cdot v = \int_\Omega g : \nabla v \,, \tag{32}$$

is valid.

Proposition 4.1. Let $\{u, k\}$ be a weak solution of (P), and let the regularity (31) for f be satisfied.
Then there exists $\sigma > 2$ such that

$$\nabla u \in [L^\sigma_{\text{loc}}(\Omega)]^{n^2} \,.$$

In order to prove this theorem, we first note two lemmas.

Lemma 4.2. If $\{u, k\}$ is a weak solution of (P), there exists $p \in L^2(\Omega)/\mathbb{R}$ such that

$$\int_\Omega u_j \frac{\partial u_i}{\partial x_j} v_i + \int_\Omega \mu(k)\, D(u, v) = \int_\Omega f \cdot v + \int_\Omega \tilde{p} \operatorname{div} v \,, \tag{33}$$

for all $v \in [W_0^{1,2}(\Omega)]^n$ and $\tilde{p} \in p$.

Proof. This is a standard result. See for example [Gal94]. □

In the following we use the notation

$$h_R := \frac{1}{B_R(x_0)} \int_{B_R(x_0)} h \,,$$

whenever $B_R(x_0) \subseteq \Omega$ and $h : \Omega \longrightarrow \mathbb{R}$.

Lemma 4.3. Let $x_0 \in \Omega$ and $R > 0$ be such that $B_{2R}(x_0) \subset \Omega$. Then there exits a constant $c > 0$, which does not depend on $B_{2R}(x_0)$, such that

$$\int_{B_{2R}(x_0)} |p - p_{2R}|^2 \le c \int_{B_{2R}(x_0)} \left\{ |\nabla u|^2 + |u - u_{2R}|^2 \, |u|^2 + |g|^2 \right\} . \tag{34}$$

Proof. Consider the coordinate transformation:

$$\begin{aligned} \Phi : B_1(0) &\longrightarrow B_{2R}(x_0) \\ z &\longmapsto x_0 + 2\,R\,z\,. \end{aligned}$$

We have

$$\det\big(\nabla\Phi\big) \equiv (2R)^n \qquad \text{in } B_1(0)\,.$$

With the transformation formula for the Lebesgue integral, we easily verify that

$$p_{2R} = \frac{1}{B_1(0)} \int_{B_1(0)} p(\Phi(z))\,dz =: [p(\Phi)]_{B_1}\,.$$

Therefore, still using the transformation formula, we can write that

$$\int_{B_{2R}(x_0)} |p - p_{2R}|^2\,dx = (2R)^n \int_{B_1(0)} \big|p(\Phi(z)) - [p(\Phi)]_{B_1}\big|^2\,dz\,.$$

Now exploiting the results of part III.3 in [Gal94], we can choose a $w \in [W_0^{1,2}(B_1(0))]^n$ such that

$$\begin{aligned} &\operatorname{div} w = p(\Phi(z)) - [p(\Phi)]_{B_1} \qquad \text{in } B_1(0)\,, \\ &\| w \|_{[W_0^{1,2}(B_1(0))]^n} \le c^* \ \| p(\Phi(z)) - [p(\Phi)]_{B_1} \|_{L^2(B_1(0))}\,, \end{aligned} \tag{35}$$

with a constant $c^* > 0$ that does not depend on p, Φ. Define

$$\tilde{w}(x) \ := \ 2\,R\,w(\Phi^{-1}(x)) \qquad \text{for } x \in B_{2R}(x_0)\,.$$

We see that

$$\frac{\partial \tilde{w}_i}{\partial x_j} = 2\,R \sum_{l=1}^{n} \frac{\partial w_i}{\partial z_l}\big(\Phi^{-1}(x)\big)\,\frac{\partial \Phi_l^{-1}}{\partial x_j}(x) = \frac{\partial w_i}{\partial z_j}\big(\Phi^{-1}(x)\big)\,.$$

It follows that

$$\big(\operatorname{div} \tilde{w}\big)(x) = \big(\operatorname{div} w\big)(\Phi^{-1}(x)) = p(x) - p_{2R}\,.$$

By a straightforward computation, where we use only the transformation formula, we find that

$$\int_{B_{2R}(x_0)} |\nabla \tilde{w}|^2 \le c^{*2} \int_{B_{2R}(x_0)} |p - p_{2R}|^2\,dx\,.$$

with the constant c^* from (35).
The last estimate ensures that $\tilde{w} \in [W_0^{1,2}(B_{2R}(x_0))]^n$. Extending $\tilde{w}$ by zero into $\Omega \setminus B_{2R}(x_0)$, we see that we can use it as a test function in (33).

Choosing in particular $\tilde{p} = p - p_{2R}$, we see that

$$\int_{B_{2R}(x_0)} |p - p_{2R}|^2 = \int_{B_{2R}(x_0)} (p - p_{2R}) \operatorname{div} \tilde{w}$$
$$= \int_{B_{2R}(x_0)} u_j \frac{\partial u_i}{\partial x_j} \tilde{w}_i + \int_\Omega \mu(k)\, D(u, \tilde{w}) - \int_\Omega g : \nabla \tilde{w}\,.$$

Using the identity

$$\int_{B_{2R}(x_0)} u_j \frac{\partial u_i}{\partial x_j} \tilde{w}_i = - \int_\Omega u_j \,(u_i - u_{i,2R}) \frac{\partial \tilde{w}_i}{\partial x_j}\,,$$

and the bound (35), we can now deduce the assertion with the help of Hölder's inequality. □

Proof of Propostion 4.1. We prove the claim by reverse Hölder inequality (see [Gia83]). For $B_R(x_0) \subset B_{2R}(x_0) \subseteq \Omega$ consider a function $\zeta \in C_c^\infty(B_{2R}(x_0))$ such that

$$\begin{cases} \zeta \equiv 1 & \text{in } B_R(x_0)\,, \\ 0 \le \zeta \le 1 & \text{in } B_{2R}(x_0) \setminus B_R(x_0)\,, \\ |\nabla \zeta| \le \frac{c}{R} & \text{in } \Omega\,. \end{cases}$$

We define $v := (u - u_{2R})\,\zeta^2$. From straightforward computations, we get

$$\operatorname{div} v = 2\,(u_i - u_{i,2R})\,\zeta \frac{\partial \zeta}{\partial x_i}\,,$$
$$D_{i,j}(u)\,\zeta^2 = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - 2\,(u_i - u_{i,2R})\,\zeta \frac{\partial \zeta}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - 2\,(u_j - u_{j,2R})\,\zeta \frac{\partial \zeta}{\partial x_i} \right)\,.$$

We can therefore write that

$$\begin{aligned} &\int_\Omega \mu(k)\, D(u,u)\,\zeta^2 \\ &= \int_\Omega \mu(k)\, D_{i,j}(u)\, \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} - 2\,(u_i - u_{i,2R})\,\zeta \frac{\partial \zeta}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - 2\,(u_j - u_{j,2R})\,\zeta \frac{\partial \zeta}{\partial x_i} \right) \\ &= - \int_\Omega \mu(k)\, D_{i,j}(u) \left((u_i - u_{i,2R})\,\zeta \frac{\partial \zeta}{\partial x_j} + (u_j - u_{j,2R})\,\zeta \frac{\partial \zeta}{\partial x_i} \right) + \int_\Omega \mu(k)\, D(u,v)\,. \end{aligned} \tag{36}$$

Our goal is now to prove the existence of a positive constant c, and of numbers $R_0 > 0$ and $\tilde{\theta} \in]0, \frac{1}{2^n}[$ such that for all $R < R_0$ we have

$$\int_{B_R(x_0)} |\nabla u|^2 \le c \left(\frac{1}{R^2} \int_{B_{2R}(x_0)} |u - u_{2R}|^2 + \int_{B_{2R}(x_0)} |f|^{\frac{6}{5}} + |g|^2 \right) + \tilde{\theta} \int_{B_{2R}(x_0)} |\nabla u|^2\,, \tag{37}$$

provided that $B_{2R}(x_0) \subset \Omega$. If (37) is satisfied, then we can multiply this relation by $|B_R(x_0)|^{-1}$, and the claim follows from Proposition 1.1 in part V of [Gia83].

In order to prove (37), we consider in turn each term on the right-hand side of (36). Using Young's inequality, we first obtain

$$\begin{aligned}
&\left|\int_\Omega \mu(k)\, D_{i,j}(u)\, \left((u_i - u_{i,2R})\,\zeta\,\frac{\partial\zeta}{\partial x_j} + (u_j - u_{j,2R})\,\zeta\,\frac{\partial\zeta}{\partial x_i}\right)\right| \\
&\qquad \le 2\sqrt{M+\nu}\int_\Omega \sqrt{\mu(k)\, D(u,u)}\,|u - u_{2R}|\,\zeta\,|\nabla\zeta| \\
&\qquad \le \delta\int_\Omega \mu(k)\, D(u,u)\,\zeta^2 + \frac{c_\delta}{R^2}\int_\Omega |u - u_{2R}|^2\,, \qquad (38)
\end{aligned}$$

where $\delta > 0$ is arbitrary and c_δ is a positive constant depending on δ.
On the other hand, we have from (33) that

$$\begin{aligned}
\int_\Omega \mu(k)\, D(u,v) &= \int_\Omega f\cdot(u - u_{2R})\,\zeta^2 + 2\int_\Omega (p - p_{2R})\,(u - u_{2R})\cdot\nabla\zeta\,\zeta \\
&\quad - \int_\Omega (u\cdot\nabla)u\cdot(u - u_{2R})\,\zeta^2\,. \qquad (39)
\end{aligned}$$

Again, we want to estimate each term of the right-hand side in order to prove the assertion (37). We apply Young's inequality and first find that

$$\begin{aligned}
\left|\int_\Omega f\cdot(u - u_{2R})\,\zeta^2\right| &\le \delta\int_{B_{2R}(x_0)} |u - u_{2R}|^6 + c_\delta\int_{B_{2R}(x_0)} |f|^{\frac{6}{5}} \\
&\le c\,\delta\, R^{6\left[n(\frac{1}{6}-\frac{1}{2})+1\right]}\left(\int_{B_{2R}(x_0)} |\nabla u|^2\right)^3 + c_\delta\int_{B_{2R}(x_0)} |f|^{\frac{6}{5}}\,,
\end{aligned}$$

with an arbitrary parameter $\delta > 0$. We observe that

$$\beta := 6\left[n\left(\frac{1}{6} - \frac{1}{2}\right) + 1\right] \ge 0\,.$$

For all $R \le R_0$ we find the estimate

$$\left|\int_\Omega f\cdot(u - u_{2R})\,\zeta^2\right| \le c\, R_0^\beta\, K_0^2\,\delta\int_{B_{2R}(x_0)} |\nabla u|^2 + c_\delta\int_{B_{2R}(x_0)} |f|^{\frac{6}{5}}\,, \qquad (40)$$

where K_0 is given by (25).
Now, we consider the second term on the right-hand side of (39). Using again Young's inequality and the estimate of lemma 4.3, we can write that

$$\begin{aligned}
&\left|\int_\Omega (p - p_{2R})\,(u - u_{2R})\cdot\nabla\zeta\,\zeta\right| \le \delta\int_{B_{2R}(x_0)} |p - p_{2R}|^2 + \frac{c_\delta}{R^2}\int_{B_{2R}(x_0)} |u - u_{2R}|^2 \\
&\quad \le \delta\, c\int_{B_{2R}(x_0)} \left\{|\nabla u|^2 + |u - u_{2R}|^2\,|u|^2 + |g|^2\right\} + \frac{c_\delta}{R^2}\int_{B_{2R}(x_0)} |u - u_{2R}|^2\,.
\end{aligned}$$

Now, using the continuity of the embedding $W_0^{1,2}(\Omega) \hookrightarrow L^4(\Omega)$, we can write that

$$\int_{B_{2R}(x_0)} |u-u_{2R}|^2\,|u|^2 \le \left(\int_{B_{2R}(x_0)} |u-u_{2R}|^4\right)^{\frac12} \left(\int_{B_{2R}(x_0)} |u|^4\right)^{\frac12}$$
$$\le c\,K_0^2\,R^{2\left[n(\frac14-\frac12)+1\right]} \int_{B_{2R}(x_0)} |\nabla u|^2\,. \tag{41}$$

Again observing that

$$\hat\beta := 2\left[n\left(\frac14-\frac12\right)+1\right] \ge \frac12\,,$$

we find that

$$\left|\int_\Omega (p-p_{2R})\,(u-u_{2R})\cdot\nabla\zeta\,\zeta\right| \le c\,\delta\left(1+K_0^2\,R_0^{\hat\beta}\right)\int_{B_{2R}(x_0)} |\nabla u|^2$$
$$+\frac{c_\delta}{R^2}\int_{B_{2R}(x_0)} |u-u_{2R}|^2 + c\,\delta\int_{B_{2R}(x_0)} |g|^2\,, \tag{42}$$

where we can choose $\delta>0$ arbitrary small.
We now consider the third term in (39). Using Young's inequality and (41), we find that

$$\left|\int_\Omega (u\cdot\nabla)u\cdot(u-u_{2R})\,\zeta^2\right| \le \delta\int_{B_{2R}(x_0)} |\nabla u|^2 + c_\delta\int_{B_{2R}(x_0)} |u-u_{2R}|^2\,|u|^2$$
$$\le \delta\int_{B_{2R}(x_0)} |\nabla u|^2 + c_\delta R^{\hat\beta}\int_{B_{2R}(x_0)} |\nabla u|^2\,. \tag{43}$$

By estimates (36), (38), (39), (40), (42), (43), we get

$$\frac{\nu}{2}\int_\Omega D(u,u)\,\zeta^2$$
$$\le \frac{c_\delta}{R^2}\int_{B_{2R}(x_0)} |u-u_{2R}|^2 + c\int_{B_{2R}(x_0)} |f|^{\frac65}+|g|^2 + C\left(\delta+\frac{1}{\delta}\,R^{\hat\beta}\right)\int_{B_{2R}(x_0)} |\nabla u|^2\,.$$

The constants $c, C>0$ do not depend on $B_R(x_0)$.
Now, for each given $\tilde\theta \in]0,\frac{1}{2^n}[$, we can achieve by suitable choices of δ and R_0 that for all $R\le R_0$

$$\int_{B_R(x_0)} |\nabla u|^2 \le c_K\int_{B_R(x_0)} D(u,u) \le c_K\int_\Omega D(u,u)\,\zeta^2$$
$$\le \frac{C}{R^2}\int_{B_{2R}(x_0)} |u-u_{2R}|^2 + C\int_{B_{2R}(x_0)} |f|^{\frac65}+|g|^2 + \tilde\theta\int_{B_{2R}(x_0)} |\nabla u|^2\,.$$

Note that by using the same coordinate transformation as in lemma 4.3, we can show easily that the constant $c_K > 0$ that appears in the first inequality can be chosen independently of $B_R(x_0)$. Thus, the constant C depends only on the data. This proves (37) and the theorem. □

The proof of Proposition 4.1 motivates the following more general result.

Theorem 4.4. If $\{u, k\}$ is a weak solution of (P), then there exists $\sigma > 2$ and $\tau > \frac{n}{n-1}$ such that

$$\nabla u \in [L^\sigma(\Omega)]^{n^2}\,, \quad \nabla k \in [L^\tau(\Omega)]^n\,.$$

In addition, there exists a positive constant $\bar{c}$ such that

$$\|\nabla u\|_{[L^\sigma(\Omega)]^{n^2}} + \|\nabla k\|_{[L^\tau(\Omega)]^3} \leq \bar{c}\, K_0(\|f\|_{[L^s(\Omega)]^n})\,.$$

Proof. By the arguments of [Nau05] and a reasoning similar to the proof of Propostion 4.1, one will get the global higher integrability of ∇u. The higher integrability of ∇k follows also as in [Nau05]. □

5 Uniqueness in the two dimensional case under a smallness assumptions on the data

Through this section, we assume that $n = 2$.
We recall the notation (25). We make the assumptions

$$\exists\, L_\nu > 0 \,:\, \forall\, t_1,\, t_2 \in \mathbb{R}^+ \,:\, |\nu_t(t_1) - \nu_t(t_2)| \leq L_\nu\, |t_1 - t_2|\,, \tag{44}$$

$$\exists\, c_g > 0\,,\, \beta \geq 0 \,:\, \forall\, t_1,\, t_2 \in \mathbb{R}^+ \,:\, |g(t_1)\sqrt{t_1} - g(t_2)\sqrt{t_2}| \leq c_g\, |t_1 + t_2|^\beta\, |t_1 - t_2|\,. \tag{45}$$

Then we have the following qualitative result:

Theorem 5.1. Let $n = 2$. If the number $\|f\|_{[L^2(\Omega)]^3}$ is sufficiently small, then there exists at most one weak solution of (P).

Proof. In this proof, we denote by $c_{p,q} > 0$ the embedding constant of

$$W^{1,q}(\Omega) \hookrightarrow L^p(\Omega)\,,$$

whenever this embedding is continuous. Recalling the notation (25), we write $\hat{K}_0$ instead of $K_0(\|f\|_{[L^2(\Omega)]^3})$.
We suppose that $\{u, k\}$ and $\{\tilde{u}, \tilde{k}\}$ are two weak solutions of (P). We write (11) for both u and $\tilde{u}$, and we use the test function $v = u - \tilde{u}$. Substracting the respective integral identities, we obtain after straightforward rearrangements of terms that

$$\begin{aligned}\int_\Omega \mu(k)\, D(u - \tilde{u}, u - \tilde{u}) = &- \int_\Omega \Big((u \cdot \nabla)u - (\tilde{u} \cdot \nabla)\tilde{u}\Big) \cdot (u - \tilde{u}) \\ &- \int_\Omega [\mu(k) - \mu(\tilde{k})]\, D(\tilde{u}, u - \tilde{u})\,. \end{aligned} \tag{46}$$

Since as usual

$$\int_\Omega (u\cdot\nabla)(u-\tilde{u})\cdot(u-\tilde{u}) = 0\,,$$

we obtain the estimate

$$\begin{aligned}\left|\int_\Omega \Big((u\cdot\nabla)u-(\tilde{u}\cdot\nabla)\tilde{u}\Big)\cdot(u-\tilde{u})\right| &\le \|\nabla\tilde{u}\|_{[L^2(\Omega)]^4}\,\|u-\tilde{u}\|^2_{[L^4(\Omega)]^2}\\ &\le \hat{K}_0\,c_{4,2}^2\,\|\nabla(u-\tilde{u})\|^2_{[L^2(\Omega)]^4}\,.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\left|\int_\Omega [\mu(k)-\mu(\tilde{k})]\,D(\tilde{u},u-\tilde{u})\right| &\le L_\nu\int_\Omega |k-\tilde{k}|\,|\nabla\tilde{u}|\,|\nabla(u-\tilde{u})|\\ &\le L_\nu\,\|\nabla\tilde{u}\|_{[L^\sigma(\Omega)]^4}\,\|\nabla(u-\tilde{u})\|_{[L^2(\Omega)]^4}\,\|k-\tilde{k}\|_{L^{\frac{2\sigma}{\sigma-2}}(\Omega)}\\ &\le L_\nu\,\bar{c}\,\hat{K}_0\,c_{\frac{2\sigma}{\sigma-2},2}\,\|\nabla(u-\tilde{u})\|_{[L^2(\Omega)]^4}\,\|\nabla(k-\tilde{k})\|_{[L^2(\Omega)]^3}\,,\end{aligned}$$

where we made use of Theorem 4.4.
Using Korn's inequality, we can therefore write

$$\nu\,c_{\text{Korn}}^{-2}\,\|\nabla(u-\tilde{u})\|_{[L^2(\Omega)]^4} \le \hat{K}_0\,c_{4,2}^2\,\|\nabla(u-\tilde{u})\|_{[L^2(\Omega)]^4} + L_\nu\,\bar{c}\,\hat{K}_0\,c_{\frac{2\sigma}{\sigma-2},2}\,\|\nabla(k-\tilde{k})\|_{[L^2(\Omega)]^3}\,.$$

Now, by hypothesis, we can choose $\hat{K}_0$ such that

$$\nu\,c_{\text{Korn}}^{-2} - \hat{K}_0\,c_{4,2}^2 > 0\,.$$

We can conclude that

$$\|\nabla(u-\tilde{u})\|_{[L^2(\Omega)]^4} \le \frac{L_\nu\,\bar{c}\,\hat{K}_0\,c_{\frac{2\sigma}{\sigma-2},2}}{\nu\,c_{\text{Korn}}^{-2}-\hat{K}_0\,c_{4,2}^2}\,\|\nabla(k-\tilde{k})\|_{[L^2(\Omega)]^3}\,. \tag{47}$$

We turn our attention to relation (12), which we write for both k and $\tilde{k}$. In view of Theorem 4.4, we can use the test function $\phi = k-\tilde{k}$. Rearranging the terms, we get

$$\begin{aligned}\int_\Omega \mu(k)\,|\nabla(k-\tilde{k})|^2 &= -\int_\Omega [\mu(k)-\mu(\tilde{k})]\,\nabla\tilde{k}\cdot\nabla(k-\tilde{k}) - \int_\Omega \Big(u\cdot\nabla k-\tilde{u}\cdot\nabla\tilde{k}\Big)(k-\tilde{k})\\ &+\int_\Omega \Big(\nu_t(k)\,D(u,u)-\nu_t(\tilde{k}\,D(\tilde{u},\tilde{u})\Big)\,(k-\tilde{k}) - \int_\Omega \Big(g(k)\,\sqrt{k}-g(\tilde{k})\,\sqrt{\tilde{k}}\Big)\,(k-\tilde{k})\,.\end{aligned} \tag{48}$$

By Theorem 4.4, we have the estimate

$$\begin{aligned}\left|\int_\Omega [\mu(k)-\mu(\tilde{k})]\,\nabla k\cdot\nabla(k-\tilde{k})\right| &\le L_\nu\,\|\nabla\tilde{k}\|_{[L^\tau(\Omega)]^3}\,\|\nabla(k-\tilde{k})\|_{[L^2(\Omega)]^3}\,\|k-\tilde{k}\|_{L^{\frac{2\tau}{2-\tau}}(\Omega)}\\ &\le L_\nu\,\bar{c}\,\hat{K}_0\,c_{\frac{2\tau}{2-\tau},2}\,\|\nabla(k-\tilde{k})\|^2_{[L^2(\Omega)]^3}\,.\end{aligned}$$

On the other hand, since

$$\int_\Omega u \cdot \nabla(k-\tilde{k})\,(k-\tilde{k}) = 0\,,$$

we have

$$\begin{aligned}\left|\int_\Omega \Big(u\cdot\nabla k - \tilde{u}\cdot\nabla\tilde{k}\Big)(k-\tilde{k})\right| &\le \|\nabla\tilde{k}\|_{[L^2(\Omega)]^3}\,\|u-\tilde{u}\|_{[L^4(\Omega)]^3}\,\|k-\tilde{k}\|_{L^4(\Omega)} \\ &\le \bar{c}\,\hat{K}_0\,c_{4,2}^2\,\frac{L_\nu\,\bar{c}\,\hat{K}_0\,c_{\frac{2\sigma}{\sigma-2},2}}{\nu\,c_{\mathrm{Korn}}^{-2}-\hat{K}_0\,c_{4,2}^2}\,\|\nabla(k-\tilde{k})\|^2_{[L^2(\Omega)]^3}\,,\end{aligned}$$

where we made use also of estimate (47).

We can write

$$\begin{aligned}\int_\Omega \Big(\nu_t(k)\,D(u,u)-\nu_t(\tilde{k}\,D(\tilde{u},\tilde{u})\Big)\,(k-\tilde{k}) &= \int_\Omega [\nu_t(k)-\nu_t(\tilde{k})]\,D(u,u)\,(k-\tilde{k}) \\ &\quad + \int_\Omega \nu_t(\tilde{k})\,\Big(D(u,u)-D(\tilde{u},\tilde{u})\Big)\,(k-\tilde{k})\,.\end{aligned}$$

We have, on the one hand

$$\begin{aligned}\left|\int_\Omega [\nu_t(k)-\nu_t(\tilde{k})]\,D(u,u)\,(k-\tilde{k})\right| &\le L_\nu\,\|\nabla u\|^2_{[L^\sigma(\Omega)]^4}\,\|k-\tilde{k}\|^2_{L^{\frac{2\sigma}{\sigma-2}}(\Omega)} \\ &\le L_\nu\,\bar{c}^2\,\hat{K}_0^2\,c^2_{\frac{2\sigma}{\sigma-2},2}\,\|\nabla(k-\tilde{k})\|^2_{[L^2(\Omega)]^3}\,.\end{aligned}$$

On the other hand, we can use again the estimate (47) in order to obtain

$$\begin{aligned}&\left|\int_\Omega \nu_t(\tilde{k})\,\Big(D(u,u)-D(\tilde{u},\tilde{u})\Big)\,(k-\tilde{k})\right| \\ &\quad\le M\,\|\nabla(u+\tilde{u})\|_{[L^\sigma(\Omega)]^4}\,\|\nabla(u-\tilde{u})\|_{[L^2(\Omega)]^4}\,\|k-\tilde{k}\|_{L^{\frac{2\sigma}{\sigma-2}}(\Omega)} \\ &\quad\le M\,\bar{c}\,\hat{K}_0\,\frac{L_\nu\,\bar{c}\,\hat{K}_0\,c^2_{\frac{2\sigma}{\sigma-2},2}}{\nu\,c_{\mathrm{Korn}}^{-2}-\hat{K}_0\,c_{4,2}^2}\,\|\nabla(k-\tilde{k})\|^2_{[L^2(\Omega)]^3}\,.\end{aligned}$$

Finally, using (45), we can write

$$\begin{aligned}\left|\int_\Omega \Big(g(k)\,\sqrt{k}-g(\tilde{k})\,\sqrt{\tilde{k}}\Big)\,(k-\tilde{k})\right| &\le c_g\,\int_\Omega |k+\tilde{k}|^\beta\,|k-\tilde{k}|^2 \le \|k+\tilde{k}\|^\beta_{L^{2\beta}(\Omega)}\,\|k-\tilde{k}\|^2_{L^4(\Omega)} \\ &\le c_g\,c^\beta_{2\beta,2}\,\bar{c}^\beta\,\hat{K}_0^\beta\,c_{4,2}^2\,\|\nabla(k-\tilde{k})\|^2_{[L^2(\Omega)]^3}\,.\end{aligned}$$

With this last result, we have estimated all terms on the right-hand side of (48). We see that under the condition

$$\nu - L_\nu\, \bar{c}\, \hat{K}_0\, c_{\frac{2\tau}{2-\tau},2} + \bar{c}\, \hat{K}_0\, c_{4,2}^2\, \frac{L_\nu\, \bar{c}\, \hat{K}_0\, c_{\frac{2\sigma}{\sigma-2},2}}{\nu\, c_{\mathrm{Korn}}^{-2} - \hat{K}_0\, c_{4,2}^2} + L_\nu\, \bar{c}^2\, \hat{K}_0^2\, c_{\frac{2\sigma}{\sigma-2},2}^2$$

$$+ M\, \bar{c}\, \hat{K}_0\, \frac{L_\nu\, \bar{c}\, \hat{K}_0\, c_{\frac{2\sigma}{\sigma-2},2}^2}{\nu\, c_{\mathrm{Korn}}^{-2} - \hat{K}_0\, c_{4,2}^2} + c_g\, c_{2\beta,2}^{\beta}\, \bar{c}^{\beta}\, \hat{K}_0^{\beta}\, c_{4,2}^2 > 0\,,$$

the claim of the theorem holds true. □

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