A class of Heath-Jarrow-Morton models in which the unbiased expectations hypothesis holds

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Abstract

The unbiased expectations hypothesis states that forward rates are unbiased estimates for future short rates. Cox, Ingersoll and Ross [1] conjectured that this hypothesis should be inconsistent with the absence of arbitrage possibilities. Using the framework of Heath, Jarrow and Morton [4] we show that this is not always the case. The unbiased expectations hypothesis together with the existence of an equivalent martingale measure is equivalent to a certain condition on the volatilities of the forward rates.

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1 Introduction

Expectations hypotheses about the term structure of interest rates traditionally play a major role both in theoretical and in empirical work. The qualitative requirement that expectations about yields should be correct can be interpreted in several ways. The absence of arbitrage possibilities for example implies that the *local* expectation hypotheses (LE henceforth) holds under the equivalent martingale measure $P^\star$:

$$B(t,T) = E^\star[e^{-\int_t^T r_u du}|\mathcal{F}_t] \, .$$

Here, $B(t,T)$ denotes the price of a zero-coupon bond with maturity T and r_t is the short rate at time t. (LE) is equivalent to the fact that the expected local rate of return of every zero-coupon-bond equals the actual short rate r_t , thus explaining the epithet *"local"*.
An alternative version of the expectations hypothesis has often been used in empirical research, the so-called *unbiased* expectations hypothesis (UE). It states that actual forward rates (f_t^T) are unbiased estimates of future short rates (r_T):

$$E[r_T|\mathcal{F}_t] \quad = \quad f_t^T \tag{1}$$

where the forward rates (f_t^T) are given by

$$B(t,T) = e^{-\int_t^T f_t^u du} \, .$$

Using Fubini's theorem one obtains for the yields $(Y_t^T) = -\frac{1}{T-t}\log B(t,T)$

$$Y_t^T \quad = \quad E\left[\frac{1}{T-t}\int_t^T r_u du|\mathcal{F}_t\right] \tag{2}$$

which is the *yield to maturity* expectation hypothesis.
It follows that bond prices ($B(t,T)$) satisfy the relation

$$B(t,T) \quad = \quad e^{-E\left[\int_t^T r_u du|\mathcal{F}_t\right]} \tag{3}$$

and Jensen's inequality implies immediately that (UE) cannot hold under the equivalent martingale measure $P^\star$.

Using the forward measure technique one obtains that the (UE) is always true in the following sense:

Weak unbiased expectation hypothesis For every maturity T there exists a measure P^T given by

$$D_t = \frac{dP^T}{dP^\star}|_{\mathcal{F}_t} = \frac{B(t,T)}{B(0,T)\beta_t}$$

such that for all times $t \leq T$

$$E^T[r_T|\mathcal{F}_t] = f_t^T .$$

$\beta_t = \exp(\int_0^t r_u du)$ denotes the money account.

PROOF : By definition we have

$$\begin{aligned} f_t^T &= \frac{-\frac{\partial}{\partial T} B_t^T}{B_t^T} \\ &= -\frac{\partial}{\partial T} E^\star \left[e^{-\int_t^T r_u du} \right] (B_t^T)^{-1} \end{aligned}$$

Assuming $E^\star \left[\int_0^T r_u du \right] < \infty$ and a positive short rate $r_t > 0$ we may interchange derivative and expectation:

$$\begin{aligned} f_t^T &= E^\star \left[r_T e^{-\int_t^T r_u du} |\mathcal{F}_t \right] (B_t^T)^{-1} \\ &= E^\star \left[r_T \frac{D_T}{D_t} |\mathcal{F}_t \right] \\ &= E^T[r_T|\mathcal{F}_t] \end{aligned}$$

where the last equality follows by Bayes' rule.[1] □

Note that the measures P^T are all different because of

$$\left.\frac{dP^T}{dP^{T'}}\right|_{\mathcal{F}_T} = \frac{B(0,T')}{B(0,T)\, B(T,T')}$$

which is not equal to 1 if there is any randomness at all.

[1]The notion of forward measure has been introduced by Geman [3]. It is implicit in the earlier work of Jamshidian [5]. The power of this tool is developed in El Karoui et al. [2].

Thus, the weak (UE) does not tell us whether the forward rates are good predictors for future short rates under the "objective" measure P.
This leads us to consider the following stronger version of the (UE):

Strong unbiased expectation hypothesis

There exists a measure P such that for all maturities T and all times $t \leq T$ the relation

$$E[r_T|\mathcal{F}_t] = f_t^T$$

holds.
It is important to note that the measure P which appears in the strong (UE) is not a global forward measure.
Cox, Ingersoll and Ross ([1]) prove that in a single factor economy the strong (UE) does not hold in equilibrium and they claim that it should indeed always allow for arbitrage. This last statement was refuted by McCulloch ([6]) who gave an example of an exchange economy where the strong (UE) holds by constructing an appropriate process of aggregate consumption. He then conjectures that the strong (UE) is consistent with no-arbitrage conditions if there are infinitely many factors determining the term structure.
In this paper we show that the strong (UE) holds in a rather general class of Heath-Jarrow-Morton models when the source of randomness has at least dimension 2.
We also show that in single factor models the strong (UE) is inconsistent with the absence of arbitrage.

2 A necessary and sufficient condition for the strong (UE)

The essential content of the strong (UE) is:
For every maturity T the process of forward rates (f_t^T) follows a martingale.
Thus we model the forward rates as

$$f_t^T = f_0^T + \sum_{j=1}^{d} \int_0^t \sigma_j(T,u) dW_u^j \tag{4}$$

where W is a d-dimensional Wiener process and $\sigma(T,t,\omega) : \Re^+ \times (\Re^+ \times \Omega) \rightarrow \Re^d$ is $\mathcal{B} \otimes \mathcal{P}$-measurable where $\mathcal{B}$ is the Borel σ-algebra and $\mathcal{P}$ the predictable

σ -algebra. Additionally, we need the condition

$$E \int_0^T \int_0^u \|\sigma(u,s)\|^2 \, ds du < \infty$$

for all T. This ensures that the stochastic integrals are martingales and that the stochastic version of Fubini's theorem holds.
Denote

$$\sigma^\star(T,t) \;=\; \int_t^T \sigma(u,t) du \,. \tag{5}$$

The question is whether the strong (UE) (4) is consistent with the existence of an equivalent martingale measure. The following theorem provides the answer.

Theorem 2.1 *A necessary and sufficient condition for the strong (UE) to hold under no-arbitrage conditions is*

$$\frac{1}{2} \|\sigma^\star(T,t)\|^2 \;=\; \lambda_t \sigma^\star(T,t) \tag{6}$$

for an appropriate $\Re^d$-valued market price of risk[2] λ.

PROOF :
Let $P^\star$ be given by

$$\frac{dP^\star}{dP}|_{\mathcal{F}_t} = \exp\left(- \int_0^t \lambda_u dW_u - \frac{1}{2} \int_0^t \|\lambda_u\|^2 du \right)$$

$P^\star$ is an equivalent martingale measure if and only if the discounted bond prices

$$B(t,T) \exp\left(- \int_0^t r_u du \right)$$

[2]We call λ a market price of risk if it is predictable and the stochastic exponential

$$\exp\left(\int_0^t \lambda_u dW_u - \frac{1}{2} \int_0^t \|\lambda_u\|^2 du \right)$$

is a martingale.

form $P^\star$-martingales. The definition (4) implies

$$\log\left(B(t,T)\exp\left(-\int_0^t r_u du\right)\right) \quad = \quad -\int_t^T f_t^u\, du - \int_0^t r_u\, du$$

$$= -\int_t^T \left(f_0^u + \int_0^t \sigma(u,s)\, dW_s\right) du \quad - \quad \int_0^t \left(f_0^u + \int_0^u \sigma(u,s)\, dW_s\right) du$$

Now apply Fubini's theorem to obtain

$$\begin{aligned} &= -\int_0^T f_0^u\, du - \int_0^t \int_s^T \sigma(u,s)\, du\, dW_s \\ &= -\int_0^T f_0^u du - \int_0^t \sigma^\star(T,s) dW_s \end{aligned}$$

Under $P^\star$ the process Z with

$$dZ^j = dW^j + \lambda^j dt$$

is a standard Wiener process by Girsanov's theorem.
Plugging in Z for W we have

$$\begin{aligned} &\log\left(B(t,T)\exp\left(-\int_0^t r_u du\right)\right) \\ &= -\int_0^T f_0^u du - \int_0^t \sigma^\star(T,s) dZ_s - \int_0^t \sigma^\star(T,s)\lambda_s ds\,, \end{aligned}$$

This means that discounted bond prices are $P^\star$-martingales if and only if

$$\sum_{j=1}^d \sigma_j^\star(T,s)\lambda_s^j = \frac{1}{2}\left\|\sigma^\star\right\|^2$$

□

We obtain as a corollary the result of Cox, Ingersoll and Ross [1] :

Corollary 1 *In single factor models the strong (UE) does not hold if interest rates are stochastic.*

Proof : Differentiating (6) by T one obtains

$$\sigma(T,t) = 0\,.$$

□

In higher dimensions, however, it is always possible to determine volatilities $\sigma_j \neq 0$ such that (6) holds for a given market price of risk $\lambda > 0$.

Example 2.1 The case of two dimensions
Let the market price of risk $\lambda = (\lambda_1, \lambda_2) > 0$ be given. We show that there exist functions $\sigma^\star(T,t)$ with the property (6) and which are not constant (this ensures stochastic interest rates). The volatility of bond prices tends to zero at maturity (confer (5)), so we also need

$$\sigma^\star(T,T) \;=\; 0\,. \tag{7}$$

Let $\sigma^\star(T,t) = (\phi(T-t), \eta(T-t))$. Then (6) is equivalent to

$$\phi^2 + \eta^2 = 2\lambda_1\phi + 2\lambda_2\eta$$

or

$$\eta = \lambda_2 \pm \sqrt{(\lambda_1)^2 + (\lambda_2)^2 - (\phi - \lambda_1)^2}\,.$$

Because of (7)

$$\phi(0) = \eta(0) = 0\,,$$

thus

$$\eta = \lambda_2 - \sqrt{(\lambda_1)^2 + (\lambda_2)^2 - (\phi - \lambda_1)^2}\,.$$

A solution η exists if

$$\lambda_1 - \|\lambda\| \leq \phi \leq \lambda_1 + \|\lambda\|$$

Examples for such functions are

$$\begin{aligned}\phi(\tau) &= \frac{\lambda_1\tau}{1+\tau}\\ \phi(\tau) &= \lambda_1\left(1 - e^{-\tau}\right)\end{aligned}$$

3 Conclusion

In the framework of Heath, Jarrow, and Morton we have proved that under suitable conditions on the volatilities of forward rates the strong unbiased expectations hypotheses is consistent with the absence of arbitrage.
Tha analysis of the problem in terms of the forward rates is particularly convenient and makes the proof very simple. Using equilibrium theory (cf. McCulloch [6]) or the usual "short-rate"-approach the argument becomes more complicated.

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