Ergodic Fluctuations in a Stock Market Model with Interacting Agents - The Mean Field Case

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Abstract

We consider a financial market model with interacting agents and study the long run behaviour of both aggregate behaviour and equilibrium prices. Investors are heterogeneous in their price expectations and they get stochastic signals about the "mood" of the market described by the empirical distributions of the agents' characteristics. We give sufficient conditions for the distribution of equilibrium prices to converge to a unique equilibrium, and we study the asymptotic dynamics of individual expectations. Simulations show that these dynamics may exhibit large and sudden fluctuations which are not due to rational adjustments to new market information but to a distinct herd behaviour.

Key Words: Random systems with complete connections, interacting Markov processes, mean-field models

AMS 1991 subject classification: 60J20, 60K35

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1 Introduction

Standard financial market models usually assume identical investors who share the same rational expectations of a future asset price, and who instantaneously and rationally discount all market information into the present price. From this academic point of view, temporary price overreactions like bubbles and crashes reflect rational changes in the valuation of an asset rather than irrational shifts in sentiment of investors.

Traders, by contrast, often consider markets as being less rational. Many believe that technical trading is possible, that some kind of "Market psychology" exists and, that herd effects unrelated to economic fundamentals can cause bubbles or crashes. Some traders even see the market itself as possessing its own mood or personality; for example they may describe the market as nervous or enthusiastic.

In recent years, financial theory is searching for alternative approaches that could explain these market realities. One approach is to model economies with interacting agents, see, e.g., Kirman (1998), Kirman (1993) or Föllmer (1974). This allows to bring in techniques from the theory of interacting Markov processes or from Markov random field theory; see, e.g., Durlauf (1993), Ioannides (1995) or Brock (1991).

Most of these papers share the same analytical core: A countable infinity of agents is interacting either locally with their "neighbours" or globally via the empirical distribution of individual agents' characteristics. In the context of statistical physics, the emphasis is on models with local interaction. But for applications in economics, it makes immediate sense to introduce an additional dependence on macroeconomic information incorporated in the empirical distribution, i.e., one should combine both local and global interaction.

In this paper we concentrate on the role of global interaction and provide the technical background for more general models. The joint effects of global and local interaction will be investigated in Horst (1999a).

We propose a simple financial market model where agents are heterogenous in their expectation about the future price of some risky asset. These expectations are based on exogenous economic fundamentals, on their own past expectation and on a random signal about the "mood" of the market. This "mood" is described by the empirical distribution of the agents' individual states.

First we will analyse the dynamics of the empirical distribution of individual opinions, using a law of large numbers across the set of agents. Given the external economic fundamental h_t and some random signal e_{t+1} about the "mood" m_t of the market in period t, the evolution of the process $\{m_t\}_{t=0,1,2,\ldots}$ will be deterministic:

$$m_{t+1} = u(m_t, e_{t+1}, h_t).$$

Using some results from the theory of "random systems with complete connections", we can provide sufficient conditions for the "mood" of the market to converge to a unique equilibrium. In a second step we shall show that the microscopic process, i.e., the process describing the states of agents' individual characteristics converges weakly to a unique stationary measure.

Finally, we study the dynamics of the induced equilibrium prices. The equilibrium price process $\{p_t\}_{t\in\mathbb{N}}$ will evolve in a random environment, where the environment is related to a random fluctuation in the behavioural characteristics of agents. More precisely, the process of temporary equilibrium prices $\{p_t\}_{t\in\mathbb{N}}$ will satisfy the recurrence relation

$$p_{t+1} = f(p_t, m_{t+1}).$$

First of all we analyse the dynamics of this process given that it evolves in a stationary environment, i.e., we shall assume that the "mood" of the market is already in equilibrium. In a second step we consider affine linear price dynamics in a non-stationary environment. Furthermore, we obtain a continuous-time limit for the stock price process both in the stationary and in the non-stationary situation.

Numerical simulations show that, even if the distribution of the process describing the empirical distribution individual agents' characteristics converges weakly to a unique stationary measure, this process may exhibit large and sudden fluctuations. These fluctuations do not reflect rational adjustments to new economic fundamentals but are due to a distinct herd behaviour.

The paper is organised as follows. In Section 2 we specify our model. Section 3 is devoted to the study of the process $\{m_t\}_{t\in\mathbb{N}}$. In Section 4 we consider the asymptotic distribution of the microscopic process, i.e., we analyse the long run dynamics of the process describing the individual characteristics of agents. Section 5 is devoted to the analysis of the dynamics of temporary equilibrium prices. In Section 6, we obtain a continuous time limit for the stock price process $\{P_t\}_{t\geq 0}$ if the transformation f takes an affine linear form. In Section 7 we consider some numerical simulations which show that, although the equilibrium price process converges in law to a unique equilibrium, it may exhibit large and sudden fluctuations. Section 8 concludes.

2 The Model

2.1 Asset Prices as Temporary Equilibria

Let us describe the model for the price evolution of a speculative asset. We consider a financial market model with a countable infinite set $\mathcal{A}$ of economic agents who are active on this market. The market contains a risk-free bond bearing interest at a constant rate r and a single risky asset. The price process of the risky asset is denoted by $\{p_t\}_{t=1,2,\ldots}$.

Given a proposed stock price p in period t, each agent $a \in \mathcal{A}$ forms an excess demand $z(p, \hat{p}_t^a)$, where $\hat{p}_t^a$ denotes an individual reference level for agent a in period t. We shall interpret $\hat{p}_t^a$ as a price expectation for the following period. $\hat{p}_t^a$ is allowed to depend on the proposed stock price p.

In this paper we concentrate on the following binary situation. At each time t the **individual state** x_t^a of agent $a \in \mathcal{A}$ takes one of two different values, i.e,

$$x_t^a \in \{-1, +1\},$$

and his price expectation takes the form

$$\hat{p}_t^a(p) = g(x_t^a, p_{t-1}, p)$$

for some measurable function $g : \{-1, +1\} \times \mathbb{R}^2 \to \mathbb{R}$.

Example 2.1 *1. Consider a model with optimistic ($x_t^a = +1$) and pessimistic ($x_t^a = -1$) information traders. Independent of the proposed price p, the price expectation of an information trader takes the log-linear form*

$$\ln \hat{p}_t^a(p) = \ln p_{t-1} + \beta(\ln(F + x_t^a) - \ln p_{t-1}), \quad \beta > 0. \tag{1}$$

Thus, the expectation of an information trader is based on the idea that the next price moves closer to his current subjective perception $F + x_t^a$ of the fundamental value of the asset at time t.

2. Alternatively, we could study a model where fundamentalists ($x_t^a = +1$) and chartists ($x_t^a = -1$) are active on the market and consider the following log-linear expectations of the form

$$\begin{aligned} g(+1, p_{t-1}, p) &= c_F \ln(F - p_{t-1}), \quad c_F > 0, \\ g(-1, p_{t-1}, p) &= c_N \ln(p - p_{t-1}), \quad c_N > 0. \end{aligned} \tag{2}$$

Thus, the dynamics of equilibrium prices will be induced by an underlying microscopic process $\{x_t\}_{t\in\mathbb{N}} = \{(x_t^a)_{a\in\mathcal{A}}\}_{t\in\mathbb{N}}$ on the **configuration space** $S := \{-1, +1\}^{\mathcal{A}}$ which describes the stochastic evolution of all the individual states. Let us first consider a situation, where only finitely many investors are active on the market. To this end we fix a sequence $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ of finite subsets of $\mathcal{A}$ which satisfies $\mathcal{A}_n \uparrow \mathcal{A}$. If only the traders in $\mathcal{A}_n$ are active on the market, the equilibrium stock price p_t at time t is determined by the market clearing condition of zero total excess demand. In this case the sequence $\{p_t\}_{t\in\mathbb{N}}$ of temporary equilibrium prices becomes a process defined by the implicit equation

$$\frac{1}{|\mathcal{A}_n|} \sum_{a\in\mathcal{A}_n} z(p_t, g(x_t^a, p_{t-1}, p_t)) = 0,$$

i.e., p_t solves

$$\frac{m_t^n + 1}{2} z(p_t, g(+1, p_{t-1}, p_t)) + \frac{1 - m_t^n}{2} z(p_t, g(-1, p_{t-1}, p_t)) = 0.$$

Here, $m_t^n := \frac{1}{|\mathcal{A}_n|} \sum_{a\in\mathcal{A}_n} x_t^a$ denotes the "average opinion" (at time t) for all traders belonging to the set $\mathcal{A}_n$.

Let S_0 be the subset of all configurations $x = \{x^a\}_{a\in\mathcal{A}} \in S$ which admit an empirical average along the fixed sequence of finite subsets $\mathcal{A}_n \uparrow \mathcal{A}$, i.e. for each $x \in S_0$ the limit

$$m(x) = \lim_{n\to\infty} \frac{1}{|\mathcal{A}_n|} \sum_{a\in\mathcal{A}_n} x^a \tag{3}$$

exists. The microscopic process will live on the subspace S_0 and we will call $m_t := m(x_t)$ the **mood of the market** at time t.

In the limit of an infinite set of agents the equilibrium price p_t at time t solves the implicit equation

$$\frac{m_t+1}{2} z(p_t, g(+1, p_{t-1}, p_t)) + \frac{1-m_t}{2} z(p_t, g(-1, p_{t-1}, p_t)).$$

Assumption 2.1 *There exists a real valued measurable transformation* $f : [-1,1] \times \mathbb{R} \to \mathbb{R}$ *such that the implicit equation*

$$\frac{m+1}{2} z(p, g(+1, \rho, p)) + \frac{1-m}{2} z(p, g(-1, \rho, p)) = 0.$$

admits a unique solution

$$p = f(m, \rho)$$

for any pair $(m, \rho) \in [-1,1] \times \mathbb{R}$.

In particular, the stock price process $\{p_t\}$ obeys the relation

$$p_t = f(m_t, p_{t-1}). \tag{4}$$

Thus, in our model the microeconomic characteristics represented by the process $\{x_t\}_{t\in\mathbb{N}}$ generate – via the aggregate or macroeconomic quantities $\{m_t\}_{t\in\mathbb{N}}$ – an **endogenous** random environment for the evolution of the stock price. In order to study the asymptotic dynamics of equilibrium prices, we have to analyse the long run dynamics of individual characteristics.

Example 2.2 *Let us illustrate our general setting by choosing a special demand function* z *derived from a standard mean-variance utility maximisation problem.*

Note that for each agent $a \in \mathcal{A}$ *and a for any given price process* $\{p_t\}$*, the dynamics of his wealth process* $\{W_t^a\}_{t=1,2,\ldots}$ *is given by*

$$W_{t+1}^a = (1+r)W_t^a + (p_{t+1} - (1+r)p_t)z_t^a,$$

where z_t^a *is the number of shares purchased at date* t*.*

Suppose that all investors are myopic mean variance maximisers. In this case, the utility function of an agent $a \in \mathcal{A}$ *takes the form*

$$U^a(z_t^a) = \mathbb{E}_t^a(W_{t+1}^a) - \frac{\eta^a}{2} V_t^a(W_{t+1}^a),$$

where $\mathbb{E}_t^a$ *and* V_t^a *denote the conditional expectation and the conditional variance,*

$$V_t^a = \mathbb{E}_t^a[(W_{t+1}^a)^2 - (\mathbb{E}_t^a W_{t+1}^a)^2],$$

based on the information available to this agent at time t*. The parameter* η^a *describes the risk aversion of agent* a*.*

Let us be more specific and assume that the risk aversion and the beliefs about the conditional variance of excess returns per share are the same for all investors.[1]*:*

$$\eta^a = \eta, \quad V_t^a(W_{t+1}^a) = \sigma_t^2$$

for all $a \in \mathcal{A}$*. However, we assume that investors are heterogenous in their expected value about future prices. More precisely, for a proposed stock price* p *at time* t *each agent* a *forms an individual expectation* $\hat{p}_{t+1}^a(p)$ *about the equilibrium price in the next period* $t+1$*. His optimal demand* $\tilde{z}_t^a$ *for shares at time* t *solves*

$$\max_{z_t^a}\{\mathbb{E}_t^a(W_{t+1}^a) - \frac{\eta}{2}V_t^a(W_{t+1}^a)\}.$$

Thus,

$$\tilde{z}_t^a(p) = \frac{\hat{p}_{t+1}^a(p) - (1+r)p}{\eta\sigma_t^2}.$$

Similar dynamics are considered in, e.g., Brock (1991), and Brock and Hommes (1995). Suppose that individual expectations given a proposed stock price p *take the form*

$$\hat{p}_{t+1}^a(p) = p + g(x_t^a, p_{t-1}) \tag{5}$$

for some measurable real valued function $g : \{-1, +1\} \times \mathbb{R} \to \mathbb{R}$*, which is assumed to be linear in the first argument. The total supply of the risky asset is equal to zero.*

Under this assumptions, the dynamics of equilibrium prices is described as follows. If only the traders in $\mathcal{A}_n$ *are active on the market, the market clearing condition yields*

$$p_t = \frac{1}{r}\frac{1}{|\mathcal{A}_n|}\sum_{a\in\mathcal{A}} g(x_t^a, p_{t-1}) = \frac{1}{r}g\left(\frac{1}{|\mathcal{A}_n|}\sum_{a\in\mathcal{A}_n} x_t^a, p_{t-1}\right).$$

In the limit of an infinite set of agents, the equilibrium price process takes the form

$$p_t = \frac{1}{r}g(m_t, p_{t-1}) := f(m_t, p_{t-1}),$$

where $m_t = m(x_t)$ *is the mood of the market at time* t*.*

[1] The assumption of homogenous beliefs on variance is made for analytical tractability. However, Nelson (1992) provides some justification for homogeneity of beliefs on variance.

2.2 The Dynamics of Individual Characteristics

How does an agent a choose his next state x^a_{t+1}? Consider price expectations of the form (1). We assume that the conditional probability that agent $a \in \mathcal{A}$ is optimistic at date $t+1$ is a function of

1. his state x^a_t in the previous period,
2. some (exogenous) economic fundamentals h_t revealed at date t,
3. the "mood" m_t of the market in the previous period.

From an economic point of view it seems reasonable to assume that investors do not know the exact value m_t. Instead, we assume that they get some noisy signal e_{t+1} about the "mood" of the market. For example, e_{t+1} could be a signal about the fraction of chartists or optimistic information traders among all investors who are active in the market.

Let us be more precise. The processes $\{e_t\}_{t\in\mathbb{N}}$ and $\{h_t\}_{t\in\mathbb{N}}$ will evolve on metric state spaces E and H, respectively, equipped with their respective Borel-σ-algebras $\mathcal{E}$ and $\mathcal{H}$. Throughout this paper the state space H is assumed to be compact.

We assume that the external conditions $\{h_t\}_{t\in\mathbb{N}}$ follow a Markov process with transition kernel K. Furthermore, the conditional law $Q(m;\cdot)$ of the signal $e \in E$ given the "mood" $m \in [-1,1]$ of the market is described by a stochastic kernel Q from $([-1,+1], \mathcal{B}([-1,+1]))$ to $(E,\mathcal{E})$. Thus,

$$e_{t+1} \sim Q(m(x_t);\cdot).$$

Since in this paper we concentrate on the role of mean field interaction, we specify the conditional choice probabilities π_a of investor a at time t, given the signal e_{t+1}, the "external field" h_t at date t and given the present configuration $x_t = \{x^a_t\}_{a\in\mathcal{A}}$ as

$$\pi_a(x^a_{t+1} = s | x_t e_{t+1}, h_t) = \pi(x^a_{t+1} = s | x^a_t, e_{t+1}, h_t).$$

In particular, we assume that this decision only depends on the agent's own state x^a_t; the interactive influence of the whole configuration $x_t = \{x^a_t\}_{a\in\mathcal{A}}$ only is felt through the signal e_{t+1} about the empirical average m_t defined in (3). Due to the binary structure of the individual states, these probabilities may be written in the form

$$\pi(x^a_{t+1} = s | x^a_t, e_{t+1}, h_t) = \frac{1}{2}\left(1 + s(\lambda(e_{t+1}, h_t) + \bar{\lambda}(e_{t+1}, h_t) x^a_t)\right), \tag{6}$$

where $s \in \{-1,+1\}$ and λ, $\bar{\lambda} : E \times H \to \mathbb{R}$ are measurable functions.

Example 2.3 *Let us assume a "logit" form for the conditional choice probabilities, i.e., suppose that*

$$\pi(x^a_{t+1} = s | x^a_t, e_{t+1}, h_t) =$$

$$\frac{\exp\{s\beta\,(Je_{t+1}+Tx_t^a+h_t)\}}{\exp\{s\beta\,(Je_{t+1}+Tx_t^a+h_t)\}+\exp\{s\beta\,(Je_{t+1}+Tx_t^a+h_t)\}}, \tag{7}$$

where $J, T, \beta > 0$, as in, e.g., Kirman (1998) or Ioannides (1995). In this case we have we that

$$\begin{aligned}\lambda(e,h) &= \tanh\{\beta(Je+T+Kh)\}+\tanh\{\beta(Je-T+Kh)\},\\ \bar{\lambda}(e,h) &= \tanh\{\beta(Je+T+Kh)\}-\tanh\{\beta(Je-T+Kh)\}.\end{aligned}$$

In this paper we assume that different agents make their transitions to a new state independently of each other, given the common signal e_{t+1} and the external condition h_t. Thus, the dynamics of the microscopic process $\{x_t\}$ is described by a Markov chain

$$\Pi(dx_{t+1}|x_t,e_{t+1},h_t)=\prod_{a\in\mathcal{A}}\pi(dx_{t+1}^a|x_t^a,e_{t+1},h_t). \tag{8}$$

on the product space $S=\{-1,+1\}^{\mathcal{A}}$ in the random environment described by the process $\{(e_{t+1},h_t)\}_{t\in\mathbb{N}}$. Such a process may be viewed as a probabilistic cellular automata (PCA) in a random environment. The special case $h_t\equiv h$ and $e_t=m_t$ is analysed in Föllmer (1994).

The full dynamics is described by the Markov process $\{(x_t,e_{t+1},h_t)\}_{t\in\mathbb{N}}$ on the state space $S_0\times E\times H$ with transition kernel

$$\mathbb{P}(d\tilde{x},d\tilde{e},d\tilde{h}|x,e,h)=\Pi(d\tilde{x}|x,e,h)\otimes Q(m(x);d\tilde{e})\otimes K(h;d\tilde{h}).$$

Note that in our case the random variables $\{x_t^a\}_{a\in\mathcal{A}}$ are conditionally independent for a given pair (e_{t+1},h_t). Using the strong law of large numbers, the dynamics of the empirical mean on the interval $[-1,1]$ is easily calculated.

Proposition 2.1 *We have*

$$m_{t+1}=u(m_t,(e_{t+1},h_t)),$$

where

$$u(m,e,h)=\lambda(e,h)+\bar{\lambda}(e,h)m. \tag{9}$$

Proof: Let $e\in E$ and $h\in H$ be given. We put $\mathcal{A}_t^+ := \{a\in\mathcal{A} : x_t^a=+1\}$ and $\mathcal{A}_t^- := \{a\in\mathcal{A} : x_t^a=-1\}$. Observe that the random variables $\{x_{t+1}^a\}_{a\in\mathcal{A}^+}$ and the random variables $\{x_{t+1}^a\}_{a\in\mathcal{A}^-}$ are conditionally independent and identically distributed. Thus, the assertion follows immediately by applying the strong law of large numbers for *i.i.d.* random variables across agents. □

In particular, we conclude from the above proposition that the configuration x_{t+1} admits almost surely the empirical average defined in (3) if x_t does. With other words, the microscopic process $\{x_t\}_{t\in\mathbb{N}}$ stays almost surely in S_0 as soon as the initial configuration x_0 belongs to S_0.

Definition 2.1 *The process* $\{(m_t, h_t)\}_{t\in\mathbb{N}}$ *will be called the* ***macroscopic process****.*

In section 3 we shall see that the macroscopic process is a Markov process, while the process $\{(e_{t+1}, h_t)\}$ which generates all **observable** "macroeconomic" information will not have the Markov property.

Suppose that the Markov process $\{(m_t, h_t)\}_{t\in\mathbb{N}}$ admits a unique stationary measure μ^* and that the initial distribution is given by this invariant distribution. In this case the macroscopic process is stationary and ergodic. Thus, since the sequence $\{p_t\}_{t\in\mathbb{N}}$ of temporary equilibrium prices satisfies the recurrence relation

$$p_{t+1} = f(m_t, p_t),$$

the stock price process evolves in a stationary environment. Based on analytical properties of the transformation f we will be able to formulate sufficient conditions which guarantee ergodicity of the induced equilibrium price process $\{p_t\}_{t\in\mathbb{N}}$. If the transformation f takes a linear form, then the "mood" drives the price process into a probabilistic equilibrium as soon as the sequence $\{m_t\}_{t\in\mathbb{N}}$ settles down to a unique equilibrium in the long run. This will be done in Section 5.

3 Dynamics of the Macroscopic Process

3.1 Motivation

As outlined at the end of Section 2, one of the main purposes of this paper is to analyse the asymptotic behaviour of the process $\{m_t\}_{t\in\mathbb{N}}$ describing the dynamics of the "mood" of the market. Observe that, due to the strong law of large numbers, the conditional distribution of m_{t+1} given the microscopic configuration x_t and the external condition h_t depends on x_t only through the empirical average $m(x_t) = m_t$. In particular, given an initial configuration $x_0 \in S_0$ and an initial external condition $h_0 \in H$, it suffices to study a stochastic process $\{(m_t, e_{t+1}, h_t)\}_{t\in\mathbb{N}}$ on the state space $[-1, +1] \times E \times H$ given by

$$\begin{aligned} m_{t+1} &= u(m_t, e_{t+1}, h_t) \\ e_{t+1} &\sim Q(m_t; \cdot) \\ h_{t+1} &\sim K(h_t; \cdot), \end{aligned} \tag{10}$$

in order to analyse the asymptotic distribution of the sequence $\{m_t\}_{t\in\mathbb{N}}$. Note that by the theorem of Ionescu Toulcea there exists for each pair $(m, h) \in [-1, +1] \times H$ of initial values a probability measure $\mathbb{P}_{m,h}$ on the canonical pace space, such that $(m_0, h_0) = (m, h)$ $\mathbb{P}_{m,h} - a.s.$ and such that

$$\begin{aligned} m_{t+1} &= u(m_t, e_{t+1}, h_t), \qquad (11) \\ \mathbb{P}_{m,h}[e_{t+1} \in A | e_t, h_t, m_t, e_{t-1}, \ldots] &= \mathbb{P}_{m,h}[e_{t+1} \in A | m_t] = Q(m_t, A), \;\; A \in \mathcal{E}, \\ \mathbb{P}_{m,h}[h_{t+1} \in B | e_t, h_t, m_t, e_{t-1}, \ldots] &= \mathbb{P}_{m,h}[h_{t+1} \in B | h_t] = K(h_t, B), \;\; B \in \mathcal{H}, \end{aligned}$$

i.e., the process $\{(m_t, e_{t+1}, h_t)\}_{t\in\mathbb{N}}$ "exists". Let us first verify that the macroscopic process has the Markov property. To this end, we denote by $\mathbb{E}_{m,h}$ the expectation with respect to the probability measure $\mathbb{P}_{m,h}$.

Lemma 3.1 *The macroscopic process $\{(m_t, h_t)\}_{t\in\mathbb{N}}$ is a homogeneous Markov chain with transition operator U defined on $B([-1,+1]\times H)$, the set of all bounded measurable real valued functions f on $[-1,+1]\times H$, by the equation*

$$Uf(m,h) = \int_{E\times H} f(u(m,e,h),\tilde{h})Q(m,de)\otimes K(h,d\tilde{h}).$$

Proof: Let $f \in B(X \times H)$. For $t \in \mathbb{N}$ we put $m^{(t)} = (m_1, \ldots, m_t)$; $e^{(t)}$ and $h^{(t)}$ are defined analogously. Using the law of iterated expectations we obtain

$$\begin{aligned}
& \mathbb{E}_{m,h}[f(m_{t+1}, h_{t+1})|m^{(t)}, h^{(t)}] \\
= \; & \mathbb{E}_{m,h}\left[\mathbb{E}_{m,h}[f(u(m_t, e_{t+1}, h_t), h_{t+1})|m^{(t)}, h^{(t)}, e^{(t+1)}]|m^{(t)}, h^{(t)}\right] \\
= \; & \mathbb{E}_{m,h}\left[\int_H f(u(m_t, e_{t+1}, h_t), h_{t+1})K(h_t, dh_{t+1})|m^{(t)}, h^{(t)}\right] \\
= \; & \int_{E\times H} f(u(m_t, e_{t+1}, h_t), h_{t+1})Q(m_t, de_{t+1})\otimes K(h_t, dh_{t+1}) \quad \mathbb{P}_{m,h} - a.s.
\end{aligned}$$

□

The rest of this section is devoted to the analysis of the macroscopic process. To this end, we propose a "random system with complete connections" as a mathematical framework to describe and analyse the sequence $\{(m_t, h_t)\}_{t\in\mathbb{N}}$. We will use some general results about random systems with complete connections provided in Iosefescu and Theodorescu (1968), Iosefescu and Gregorescu (1993) and Norman (1963).

3.2 Random System with Complete Connections and Compact Markov Processes

In this subsection we introduce the concept of a "random system with complete connections" and consider a special system, whose associated Markov process turns out to be our macroscopic process. Using general ergodicity criteria provided in Iosefescu and Theodorescu (1968) and Norman (1963), we will formulate sufficient conditions which guarantee the existence of a invariant measure for the Markov chain $\{(m_t, h_t)\}_{t\in\mathbb{N}}$ Imposing an additional mixing condition will then ensure convergence in law of the macroscopic process to a unique equilibrium μ^* as $t \to \infty$.

Let us first introduce some notation. $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a given probability space. $(M, \mathcal{M})$ and $(X, \mathcal{X})$ are measurable spaces. Z denotes a stochastic kernel defined on $M \times \mathcal{X}$ and $v : M \times X \to M$ is a measurable mapping. For each $l \in \mathbb{N}$ we put $X^{(l)} := \prod_{i=1}^{l} X$, $\mathcal{X}^{(l)} := \bigotimes_{i=1}^{l} \mathcal{X}$, and $\zeta^{(l)} := (\zeta_1, \zeta_2, \ldots, \zeta_l) \in X^{(l)}$.

We define a sequence $\{v_t\}_{t\in\mathbb{N}}$ of mappings $v_t : M \times X^{(t)} \to M$, by the relation

$$\begin{aligned} v_1(\xi, \zeta_1) &= v(\xi, \zeta_1) \\ v_t(\xi, \zeta^{(t)}) &= v(v_{t-1}(\xi, \zeta^{(t-1)}), \zeta_t). \end{aligned}$$

Abusing our notation we put

$$v(\xi, \zeta^{(t)}) := v_t(\xi, \zeta^{(t)}) = v(\cdot, \zeta_t) \circ \ldots \circ v(\xi, \zeta_1), \quad \xi \in M. \tag{12}$$

Definition 3.1 • *Following Iosefescu and Theodorescu (1968) we will call the quadruple*

$$\Sigma = ((M, \mathcal{M}), (X, \mathcal{X}), Z, v)$$

a homogeneous random system with complete connections *(RSCC).*

- *Given an initial value ξ, a RSCC induces two stochastic processes (on the canonical path space) $\{\xi_t\}_{t=0,1,2,..}$ and $\{\zeta_t\}_{t=1,2,...}$ taking values in $(M, \mathcal{M})$ and $(X, \mathcal{X})$, respectively, by the relation*

$$\xi_{t+1} = v(\xi_t, \zeta_{t+1}), \tag{13}$$

and

$$\mathbb{P}_\xi(\zeta_{t+1} \in A | \xi_t, \zeta_t, \xi_{t-1}, \zeta_{t-1}, \ldots) = Z(\xi_t, A), \quad A \in \mathcal{X}.$$

These processes are called the ***associated Markov process*** *and* ***signal process****, respectively.*

- *Let d denote a metric on M. A RSCC is called a* ***distance diminishing model****, if the transformation v satisfies the following* ***mean contraction property****: There exists a constant $\theta < 1$ such that*

$$\left| \int d(v(\xi, \zeta), v(\xi', \zeta)) Z(\xi, d\zeta) \right| \leq \theta d(\xi, \xi'),$$

uniformly in $\zeta \in X$ $(\xi, \xi' \in M)$.

- *For any $l, n \in \mathbb{N}$ and for each $\xi \in M$ we define functions $Z_l^n(\xi, \cdot)$ on $X^{(l)}$ by the relation (see Iosefescu and Theodorescu (1968), Chapter 2.1.1.1.3):*

$$Z_l(\xi, A^{(l)}) := \begin{cases} Z(\xi, A^{(l)}) & l = 1 \\ \int Z(\xi, d\zeta_1) \int Z(v(\xi, \zeta_1), d\zeta_2) & \\ \cdots Z(v(\xi, \zeta^{(l-1)}), d\zeta_l) \mathbf{1}_{A^{(l)}}(\zeta^{(l)}) & l \geq 2 \end{cases}, \tag{14}$$

$$Z_l^n(\xi, A^{(l)}) := \begin{cases} Z_l(\xi, A^{(l)}) & n = 1 \\ \int Z_l(\xi, d\zeta_1) Z_l^{n-1}(v(\xi, \zeta_1), A^{(l)}) & n \geq 2 \end{cases}.$$

With other words: $Z_l(\xi, \cdot)$, respectively, $Z_l^n(\xi, \cdot)$ denotes the distribution of the vectors $\zeta^{(l)}$ and $(\zeta_{n+1}, \ldots, \zeta_{n+l})$, respectively. The system Σ is called ***uniformly***

ergodic *(in the strong sense), if for every* $l \in \mathbb{N}$ *there exists a probability measure* Z_l^∞ *on* $(X^{(l)}, \mathcal{X}^{(l)})$ *such that*

$$\lim_{n\to\infty} Z_l^n(\xi, A^{(l)}) = Z_l^\infty(A^{(l)})$$

uniformly with respect to $l \in \mathbb{N}$, $\xi \in M$ *and* $A^{(l)} \in X^{(l)}$. *Thus* Σ *is uniformly ergodic if* $Z_l^n(\xi;\cdot)$ *converges uniformly in* $\xi \in M$ *to* $Z_l^\infty(\cdot)$ *in the norm of total variation.*

Observe that by the recursive structure (13), the Markov process $\xi = \{\xi_t\}_{t\in\mathbb{N}}$ associated to the Random system Σ satisfies almost surely the relation

$$\xi_{t+1} = v(\xi_0, \zeta^{(t+1)}).$$

Let us denote by $\widetilde{Z}$ the transition kernel of the process ξ, i.e., $\widetilde{Z}$ is a stochastic kernel on $M \times \mathcal{X}$ defined by

$$\widetilde{Z}(\xi, B) := Z(\xi, B_m) \quad \text{with} \quad B_m := \{\zeta \in X : v(\xi,\zeta) \in B\}, \quad B \in \mathcal{M}.$$

By $Supp_n(\xi)$ we denote the support of the measure $\widetilde{Z}^n(\xi,\cdot)$.

For distance diminishing models with compact state space M the following two conditions are sufficient for weak convergence of the distributions μ_t of ξ_t to a unique probability measure μ^* on $(M, \mathcal{M})$ as $t \to \infty$; see, e.g., Lemma 2.1.55, Lemma 2.1.58, Lemma 2.1.59 and Theorem 2.1.40 in Iosefescu and Theodorescu (1968).

Condition (C1): ("Lipschitz Condition for the transition kernel Z") There exists a constant $C > 0$ such that

$$\sup_{A\in\mathcal{X}} \sup_{\xi\neq\xi'} \frac{|Z(\xi, A) - Z(\xi', A)|}{d(\xi,\xi')} \le C < \infty. \tag{15}$$

Condition (C2):

$$\lim_{n\to\infty} d(Supp_n(\xi), Supp_n(\xi')) = 0$$

for all $\xi, \xi' \in M$.

Usually, Condition (C1) can easily be checked. Below, we shall provide a sufficient condition for (43) to hold true. In many cases, however, Condition (C2) is hard to verify. Imposing an additional "mixing condition" on the transition kernel Z, we shall obtain another criteria for regularity of the associated Markov process (see Theorem 3.2 below).

Definition 3.2 *A Markov process* $\{\xi_t\}_{t\in\mathbb{N}}$ *with state space* $(M, \mathcal{M})$ *and transition operator* U *is called* ***aperiodic*** *on a Banach space* $(L, ||\cdot||_L)$ *if there exists a bounded linear operator* U^∞ *on* L *such that*

$$\lim_{n\to\infty} ||U^n - U^\infty||_L. \tag{16}$$

The process is called ***regular****, if in addition* $U^\infty(L)$ *is one dimensional.*

Remark 3.1 *Let $\xi = \{\xi_t\}_{t\in\mathbb{N}}$ be a Markov chain taking values in a compact metric space (M, d). Suppose that its transition operator U is regular on a dense subset L of $C(M)$. From the above definition, and by the Riesz representation theorem it is easily seen, that in this case the process ξ admits a unique stationary probability μ^* on $(M, \mathcal{M})$. Furthermore, the distributions μ_t of ξ_t converge weakly to μ^* for any initial distribution μ_0 as $t \to \infty$.*

Suppose that the measures $Z(\xi;\cdot)$, $\xi \in M$ have a "lower bound" which does not depend on ξ. For distance diminishing models the following condition ensures regularity of the associated Markov process; see Norman (1963), Theorem 4.1.1.

Condition (C3): There exists a measure ν on $(X, \mathcal{X})$, a constant $c > 0$ and a set $A^* \in \mathcal{X}$ such that $\nu(A^*) = 1$,

$$Z(\xi; A) \geq c\nu(A^* \cap A), \quad A \in \mathcal{X}$$

and

$$\int d(v(\xi,\zeta), v(\xi',\zeta))\nu(d\xi) \leq \theta d(\xi,\xi'). \tag{17}$$

Note that (17) is satisfied if, for example, the mapping v satisfies the contraction condition

$$d(v(\xi,\zeta), v(\xi',\zeta)) \leq \theta d(\xi,\xi')$$

uniformly in $\zeta \in X$.

Let us now consider the dynamics of the macroscopic process. In order to analyse the asymptotic behaviour of the Markov chain $\{(m_t, h_t)\}_{t\in\mathbb{N}}$ we now introduce a random system Σ^* with the following ingredients:

$$\begin{aligned} M &:= [-1,+1] \times H, \quad \xi_t = (m_t, h_t), \\ X &:= E \times H, \quad \zeta_{t+1} = (e_{t+1}, h_{t+1}), \\ v(\xi_t, \zeta_{t+1}) &:= (u(m_t, e_{t+1}, h_t), h_{t+1}), \\ Z(\xi_t;\cdot) &:= Q(m_t;\cdot) \otimes K(h_t;\cdot). \end{aligned} \tag{18}$$

Here, u is the mapping defined in (9). For the rest of this paper H is assumed to be a compact subset of $\mathbb{R}$ and $||\cdot||$ denotes the Euclidian distance on $\mathbb{R}^2$. Furthermore, we assume that the function u defined in (9) has the following **mean contraction property**.

Condition (C4):

$$\int_E |u(m_1, e, h_1) - u(m_2, e, h_2)| Q(m_1; de) \leq \theta ||(m_1, h_1) - (m_2, h_2)||, \quad \theta < 1. \tag{19}$$

Remark 3.2 *Some straightforward calculations show that the function u satisfies the mean contraction condition (19) as soon as the individual transition probabilities defined in (6) satisfy the following condition: There exists a constant $\theta < 1$ such that*

$$\sup_{e,h} |\bar{\lambda}(e,h)| < \theta, \quad |\bar{\lambda}(e,h) - \bar{\lambda}(e,h')| + |\lambda(e,h) - \lambda(e,h')| < \theta|h - h'|.$$

Lemma 3.2 *1. If the transformation u defined in (9) satisfies the mean contraction property (19) then the system Σ^* is distance diminishing.*

2. If the transition kernels Q and K satisfy Condition (C1) with constants C_Q and C_K respectively, then the kernel Z defined by (18) satisfies Condition (C1) with constant $C = C_Q + C_K$.

3. Suppose that the measures $Q(m;\cdot)$, $m \in [-1,+1]$, have a density $g(m,\cdot)$ which is uniformly Lipschitz continuous in the first argument, i.e.,

$$\sup_e |g(m_1, e) - g(m_2, e)| \leq C_0 |m_1 - m_2|,$$

for some constant C_0. In this case the stochastic kernel Q satisfies Condition (C1) with constant $C = C_0$.

Proof:

1. This assertion follows from

$$\begin{aligned} & \int ||(u(m,e,h),\tilde{h}) - (u(\bar{m},e,\bar{h}),\tilde{h})|| Z(m,h;de,d\tilde{h}) \\ = & \int |u(m,e,h) - u(\bar{m},e,\bar{h})| Q(m,de) \\ \leq & \quad \theta ||(m,h) - (\bar{m},\bar{h})||. \end{aligned}$$

2. Let $A \in \mathcal{E} \otimes \mathcal{H}$ and $(m_1,h_1),(m_2,h_2) \in [-1,+1] \times H$ be given. For each $h \in H$ and any $e \in E$ we put $A(h) := \{e \in E : (e,h) \in A\}$ and $A(e) := \{h \in H : (e,h) \in A\}$ respectively. Using Fubini's theorem we obtain

$$\begin{aligned} & (||(m_1,h_1) - (m_2,h_2)||)^{-1} \left| \int_A (Z(m_1,h_1;de,d\tilde{h}) - Z(m_2,h_2;de,d\tilde{h})) \right| \\ \leq & (||(m_1,h_1) - (m_2,h_2)||)^{-1} \Big(\left| \int_A (Q(m_1;de) - Q(m_2;de)) \otimes K(h_1,d\tilde{h}) \right| \\ & + \left| \int_A (K(h_1;d\tilde{h}) - K(h_2;d\tilde{h})) \otimes Q(m_2,de) \right| \Big) \\ \leq & \left| \int_H \frac{Q(m_1;A(\tilde{h})) - Q(m_2;A(\tilde{h}))}{|m_1 - m_2|} K(h_1,d\tilde{h}) \right| \\ & + \left| \frac{\int_E K(h_1;A(e)) - K(h_2;A(e))}{|h_1 - h_2|} Q(m_2,de) \right| \\ \leq & \quad C_Q + C_K. \end{aligned}$$

Taking suprema yields the desired result.

3. This assertion is obvious.

□

3.2.1 Compact Markov Processes

In order to analyse the asymptotic distribution of the process $\{m_t\}_{t\in\mathbb{N}}$ we will now use results about compact Markov processes provided in Norman (1963), Chapter 3. Our aim is to show that the transition operator of the macroscopic process is regular on the set of all bounded Lipschitz continuous functions on $[-1,1]\times H$. According to Remark 3.1 this guarantees the existence of a unique stationary measure μ^* and, moreover, convergence in law of the process $\{(m_t,h_t)\}_{t\in\mathbb{N}}$ to μ^*.

To this end, we introduce some additional notation. If (M,d) if a metric space and $f\in B(M)$ is a bounded measurable function we put

$$m(f) \ := \ \sup_{x\neq y}\frac{|f(x)-f(y)|}{d(x,y)}, \quad \|f\|_{Lip} := m(f)+|f|_\infty.$$

A function f on M will be called **Lipschitz** if $m(f)<\infty$. By $L(M)$ we denote the set of all bounded Lipschitz functions on M, i.e.,

$$L(M) := \{f : \|f\|_{Lip}<\infty\}.$$

Note that $L(M)$ is a normed linear space with respect to both $|\cdot|_\infty$ and $||\cdot||_{Lip}$. We denote by $\Delta(f)$ the oscillation of f, i.e.,

$$\Delta(f) := \sup_{x,y}|f(x)-f(y)|.$$

After this preliminaries we can now show that the transition operator U of the macroscopic process satisfies the so-called **Doeblin-Fortet-Inequality** (inequality (20) below), provided that the transformation u defined in (9) satisfies the mean contraction property (19).

Lemma 3.3 *Assume that the transition kernel Z defined in (18) satisfies Condition (C1). If the transformation u defined in (9) satisfies the mean contraction property (19), then there exists a positive constant R such that the transition operator U of the process $\{(m_t,h_t)\}_{t\in\mathbb{N}}$ satisfies the following inequality on $L([-1,+1]\times H)$:*

$$m(Uf)\le\theta m(f)+R|f|_\infty\le(R+1)||f||_{Lip}. \tag{20}$$

Proof: The proof of this lemma is merely a modification of the proofs of Lemma 1.3 and Theorem 1.2 in Norman (1963). For all $f\in L([-1,+1]\times H)$ we have

$$Uf(m,h)=\int f(u(m,e,h),\tilde{h})Q(m,de)\otimes K(h,d\tilde{h}).$$

Therefore

$$\begin{aligned} & |Uf(m_1,h_1)-Uf(m_2,h_2)| \\ \le & \int |f(u(m_1,e,h_1),\tilde{h})-f(u(m_2,e,h_2),\tilde{h})|Q(m_1,de)\otimes K(h_1,d\tilde{h}) \qquad (21) \\ + & \left|\int f(u(m_2,e,h_2),\tilde{h})(Q(m_1,de)\otimes K(h_1,d\tilde{h})-Q(m_2,de)\otimes K(h_2,d\tilde{h}))\right|. \end{aligned}$$

Condition (C1) yields the following estimate for the second term:

$$\begin{aligned}
&\left| \int f(u(m_2, e, h_2), \tilde{h})(Q(m_1, de) \otimes K(h_1, d\tilde{h}) - Q(m_2, de) \otimes K(h_2, d\tilde{h})) \right| \\
\leq\;& 2C|f|_\infty ||(m_1, h_1) - (m_2, h_2)||.
\end{aligned}$$

Since f is Lipschitz continuous we have that

$$\begin{aligned}
&\int_{E\times H} \left| \left(f(u(m_1, e, h_1), \tilde{h}) - f(u(m_2, e, h_2), \tilde{h}) \right) \right| Z(m_1, h_1; d(e, \tilde{h})) \\
=\;& \int \frac{|f(u(m, e, h), \tilde{h}) - f(u(\bar{m}, e, \bar{h}), \tilde{h})|}{||(u(m, e, h), \tilde{h}), (u(\bar{m}, e, \bar{h}), \tilde{h})||} \Delta u(\cdot, e, \cdot) Z(m_1, h_1; de, d\tilde{h}) \\
\leq\;& m(f) \int_E |u(m_1, e, h_1) - u(m_2, e, h_2)| Q(m_1; de) \\
\leq\;& \theta m(f) ||(m_1, h_1) - (m_2, h_2)||.
\end{aligned}$$

The last inequality follows from the mean contraction condition (19). Hence we can estimate term (21) by

$$\theta m(f) ||(m_1, h_1) - (m_2, h_2)||$$

and therefore

$$\frac{|Uf(m_1, h_1) - Uf(m_2, h_2)|}{||(m_1, h_1) - (m_2, h_2)||} \leq 2C|f|_\infty + \theta m(f).$$

Taking the supremum over all $(m, h) \neq (m', h')$ on both sides yields the desired result. □

Definition 3.3 *A Markov process with compact state space M, whose transition operator U satisfies the Doeblin-Fortet inequality (20) and maps $L(M)$ into $L(M)$ boundedly with respect to the norm $\|\cdot\|_{Lip}$ is called a **compact Markov process** in the sense of Norman (1963).*

Corollary 3.1 *The Markov process $\{(m_t, h_t)\}_{t\in\mathbb{N}}$ is a compact Markov process if the function u satisfies the mean contraction property.*

Proof: The proof follows immediately from Lemma 3.3 because

$$||Uf||_{Lip} \leq (R+1)||f||_{Lip} + |Uf|_\infty \leq (R+2)||f||_{Lip}$$

for any Lipschitz function f. □

3.2.2 Existence and Uniqueness of Invariant Measures

The proof of the following theorem can be found in, e.g, Iosefescu and Gregorescu (1993), page 94.

Theorem 3.1 *Let u satisfy the mean contraction property. Under Condition (C1) the macroscopic process admits at least one stationary probability.*

The next theorem, which follows from general results about RSCCs in Iosefescu and Theodorescu (1968) and Norman (1963), provides sufficient conditions for the macroscopic process to converge to a unique equilibrium.

Theorem 3.2 *Suppose that the function u satisfies the mean contraction property.*

1. *Under Conditions (C1) and (C2) the process $\{(m_t, h_t)\}_{t \in \mathbb{N}}$ is regular on the Banach space $(L([-1, +1] \times H), || \cdot ||_{Lip})$.*

2. *Under Condition (C1) and (C3) the macroscopic process is regular on the Banach space $(L([-1, +1] \times H), || \cdot ||_{Lip})$.*

In particular, in both cases the Markov chain $\{(m_t, h_t)\}_{t \in \mathbb{N}}$ admits a unique stationary measure μ^* and the macroscopic process converges in distribution to μ^*.

Proof:

1. Due to Corollary 3.1 the process $\{(m_t, h_t)\}_{t \in \mathbb{N}}$ is a compact Markov process. Recall that the theorem of Ionescu-Tulcea and Marinescu is applicable to the transition operator U of a compact Markov process; see, e.g., Chapter 3 in Norman (1963). Thus, one can show that U is regular as soon as $\iota = +1$ is the only eigenvalue of modulus one of U and if the only continuous solutions of $Uf = f$ are constants; see Theorem 2.1.57 in Iosefescu and Theodorescu (1968).

   However, due to condition (C2) $\iota = +1$ is indeed the only eigenvalue of modulus 1 of U by Lemma 2.1.55 in Iosefescu and Theodorescu (1968). Lemma 2.1.56 in Iosefescu and Theodorescu (1968) states that the only eigenfunctions corresponding to the eigenvalue $\iota = 1$ are in fact constants.

2. This assertion follows immediately from Theorem 4.1.1 in Norman (1963).

□

Definition 3.4 *In the sequel we shall call the random system Σ^* regular, if the associated Markov process $\{(m_t, h_t)\}_{t \in \mathbb{N}}$ is regular on $(L([-1, +1] \times H), || \cdot ||_{Lip})$.*

4 Dynamics of the Microscopic Process

This section is devoted to the study of the long run dynamics of the microscopic process $\{x_t\}_{t\in\mathbb{N}}$, $x_t = \{x_t^a\}_{a\in\mathcal{A}}$, describing the evolution of all the individual states. We shall assume that the individual transition probability π in (6) is bounded away from zero uniformly in (e,h) by some constant c'. If, for example, π takes the "logit" form (7), this assumption is obviously satisfied, if E is some bounded subset of $\mathbb{R}$.

Our aim is to show that the regularity of the macroscopic process $\{(m_t, h_t)\}_{t\in\mathbb{N}}$ induced by Conditions (C1) and (C2) or by Conditions (C1) and (C3), implies regularity of the microscopic process $\{x_t\}_{t\in\mathbb{N}}$.

In Subsection 4.1 we will show that for any fixed $a \in \mathcal{A}$ the Markov chain $M_{\{a\}} := \{(x_t^a, m_t, h_t)\}_{t\in\mathbb{N}}$ is regular on the set of all bounded Lipschitz continuous functions on $\{-1,+1\} \times [-1,+1] \times H$. From this it is easy to show that for any finite set $A \subset \mathcal{A}$ the Markov process $M_A := \{(\{x_t^a\}_{a\in A}, m_t, h_t)\}_{t\in\mathbb{N}}$ is also regular. As the random variables $\{x_t^a\}_{a\in\mathcal{A}}$ are conditionally independent given the pair (e,h) it will turn out that the family of the invariant measures corresponding to the processes M_A $(A \in \mathcal{A})$ is consistent. From this we will deduce that the distribution of the process $\{x_t\}_{t\in\mathbb{N}}$ converge weakly to a unique equilibrium as $t \to \infty$.

Observe that the microscopic process $\{x_t\}_{t\in\mathbb{N}}$ may be regarded as a Markov chain evolving in a radom environment. In Subsection 4.2 we shall consider the dynamics of this process given that the macroscopic process is already in equilibrium, i.e., we study the situation where the process $\{x_t\}_{t\in\mathbb{N}}$ evolves in a stationary environment.

4.1 Regularity of the Microscopic Process

We define a metric d on $\{-1,+1\} \times [-1,+1] \times H$ by

$$d((x,m,h),(x',m',h')) = |x-x'| + |m-m'| + |h-h'|.$$

Let us first verify the following simple result.

Remark 4.1 *Suppose that Condition (C1) is satisfied. Then the following Lipschitz condition holds true for all $a \in \mathcal{A}$ and for all $(e,h) \in E \times H$:*

$$\sup_{(m,x^a)\neq(\bar{m},y^a),(A,B)} \frac{\left|\int_B \{\pi(A;x^a,e,h)Q(m;de) - \pi(A;y^a,e,h)Q(\bar{m};de)\}\right|}{|m-\bar{m}| + |x^a - y^a|} \leq C < \infty. \tag{22}$$

Proof: For $x^a = y^a$ the assertion follows from Condition (C1). For $x^a \neq y^a$ the assertion follows from $|m-\bar{m}| + |x^a - y^a| \geq 2$. □

Theorem 4.1 *Assume that the kernel Z defined by (18) satisfies Conditions (C1) and (C2) or Conditions (C1) and (C3). Suppose further that the transformation u in (9) satisfies the mean contraction condition (19). Then the Markov process $M_{\{a\}} := \{(x_t^a, m_t, h_t)\}_{t\in\mathbb{N}}$ with state space $\{-1,+1\} \times [-1,1] \times H$ is regular on $L(\{-1,+1\} \times [-1,+1] \times H)$ for any $a \in \mathcal{A}$.*

Proof: Let us fix $a \in \mathcal{A}$. Without loss of generality we may assume that the external field $\{h_t\}_{t\in\mathbb{N}}$ is almost surely constant and neglect the dependence on the parameter h in our notation. First, we shall show that the process $M_{\{a\}}$ is a compact and regular Markov process. Its transition operator is defined on $B(\{-1,+1\}\times[-1,+1])$ by the relation

$$Uf(x^a,m) = \int f(y^a,u(m,e))\pi(dy^a|x^a,e)Q(m,de).$$

Since u satisfies the mean contraction condition (19), we can deduce that this operator satisfies the Doeblin-Fortet inequality (20). Indeed, for each $f \in L(\{-1,+1\}\times[-1,+1])$ we have

$$\begin{aligned}
&|Uf(x^a,m) - Uf(y^a,\bar{m})| && (23)\\
&\leq \left|\int f(y,u(m,e))(\pi(dy|x^a,e)Q(m;de) - \pi(dy|y^a,e)Q(\bar{m};de))\right| \\
&\quad + \left|\int (f(y,u(m,e)) - f(y,u(\bar{m},e)))\,\pi(dy|y^a,e)Q(\bar{m};de)\right| && (24)\\
&\leq C|f|_\infty(|m-\bar{m}| + |x^a - y^a|) + \theta m(f)|m-\bar{m}| \\
&\leq \{C|f|_\infty + \theta m(f)\}\,(|m-\bar{m}| + |x^a - y^a|)
\end{aligned}$$

by Remark 4.1. This shows that the Markov process $\{(x_t^a, m_t)\}_{t\in\mathbb{N}}$ is compact in the sense of Norman (1963).

As the transition probability π in (6) is assumed to be uniformly bounded away from zero, the distribution of the process $M_{\{a\}}$ after n steps, given $m_0 = m$, has support

$$\{-1,+1\}\times Supp_n(m).$$

Here $Supp_n(m)$ denotes the support of the distribution μ_n on m_n (given $m_0 = m$). Hence, regularity follows from Condition (C2) and Theorem 3.2 (1.) above.

If Conditions (C1) and (C3) are satisfied we can use techniques provided in Norman (1963), Chapter 4, and proceed as follows: For each $l \in \mathbb{N}$, for any given pair $(x,m) \in \{-1,+1\}\times[-1,+1]$ and for each $A \in \mathcal{B}((\{-1,+1\}\times E)^l)$ we put

$$\begin{aligned}
R_l(x,m;A) &:= \int Q(m,de_1)\pi(dx_1|x,e_1)Q(u(m,e_1);de_2)\pi(dx_2|x_1,e_2)\cdots \\
&\qquad \cdots Q(u(m;e^{(l-1)});de_l)\pi(dx_l|x_{l-1},e_l)\mathbf{1}_A \circ (e^{(l)},x^{(l)}),\\
r_l &:= \sup_A \sup_{(x,m)\neq(y,\bar{m})} \frac{|R_l(x,m;A) - R_l(y,\bar{m};A)|}{|m-\bar{m}| + |x-y|},\\
r_l^* &:= \sup_A \sup_{(x,m)\neq(y,\bar{m})} |R_l(x,m;A) - R_l(y,\bar{m};A)|.
\end{aligned}$$

Using (24) and the mean contraction property (19) it is easy to see that

$$\Delta(Uf) \leq \Delta(f)r_1^* + 2\theta m(f)$$

for any Lipschitz continuous function f on $\{-1,+1\} \times [-1,+1]$.

For $l \in \mathbb{N}$ we denote by U^l the $n-$fold iteration of the operator U. Using the mean contraction property (19) again, we obtain by induction that

$$\Delta(U^l f) \leq \Delta(f) r_l^* + 2\theta^l m(f) \leq r^* \Delta(f) + 2\theta^l m(f), \tag{25}$$

where $r^* := \sup_l r_l^*$.

Suppose that $r^* < 1$. In this case it follows immediately from the proof of Theorem 4.1.1 in Norman (1963) that U is regular on $L(\{-1,+1\} \times [-1,+1])$. Indeed, we can adopt this proof word by word. Thus, it remains to show that $r^* < 1$.

To this end, we will first show that $r_\infty := \sup_l r_l < \infty$. For that it is obviously enough to show that

$$\tilde{r}_l := \sup_l \sup_A \sup_{x, m \neq \bar{m}} \frac{|R_l(x,m;A) - R_l(x,\bar{m};A)|}{|m - \bar{m}|} < \infty.$$

For $l \in \mathbb{N}$ we put $\pi(dx^{(l)}|x, e^{(l)}) = \pi(dx_1|x, e_1) \cdots \pi(dx_l|x_{l-1}, e_l)$. Thus, for all $i, j \in \mathbb{N}$ and for any $A \in \mathcal{B}((\{-1,+1\} \times [-1,+1])^{i+j})$ we have that

$$\begin{aligned}
&|R_{i+j}(x,m;A) - R_{i+j}(x,\bar{m};A)| \\
\leq\ & \left| \int \left\{ Q_i(m; de^{(i)}) Q_j(u(m; e^{(i)}); de^{(j)}) - Q_i(\bar{m}; de^{(i)}) Q_j(u(\bar{m}; e^{(i)}); de^{(j)}) \right\} \right. \\
& \left. \pi(dx^{(i+j)}|x, e^{(i+j)}) \mathbf{1}_A \circ (e^{(i+j)}, x^{(i+j)}) \right| \\
\leq\ & \left| \int Q_i(m; de^{(i)}) \left\{ Q_j(u(m; e^{(i)}); de^{(j)}) - Q_j(u(\bar{m}; e^{(i)}); de^{(j)}) \right\} \right. \\
& \left. \pi(dx^{(i+j)}|x, e^{(i+j)}) \mathbf{1}_A \circ (e^{(i+j)}, x^{(i+j)}) \right| \\
& + \left| \int \left\{ Q_i(m; de^{(i)}) - Q_i(\bar{m}; de^{(i)}) \right\} Q_j(u(\bar{m}; e^{(i)}); de^{(j)}) \right. \\
& \left. \pi(dx^{(i+j)}|x, e^{(i+j)}) \mathbf{1}_A \circ (e^{(i+j)}, x^{(i+j)}) \right|.
\end{aligned}$$

Using Condition (C1) and the contraction property of the transformation u we obtain

$$\begin{aligned}
&|R_{i+j}(x,m;A) - R_{i+j}(x,\bar{m};A)| \\
\leq\ & \tilde{r}_j \int |u(m, e^{(i)}) - u(\bar{m}, e^{(i)})| Q_i(m, de^{(i)}) + \tilde{r}_i |m - \bar{m}| \leq (\tilde{r}_j \theta^i + \tilde{r}_i)|m - \bar{m}|.
\end{aligned}$$

Dividing by $|m - \bar{m}|$ and taking the supremum on both sides, we get

$$\tilde{r}_{i+j} \leq \tilde{r}_j \theta^i + \tilde{r}_i.$$

Since $\tilde{r}_{j+1} \geq \tilde{r}_j$ we can conclude that

$$\sup_l \tilde{r}_l \leq \frac{\tilde{r}_1}{1 - \theta} < \infty,$$

because

$$\begin{aligned}\tilde{r}_1 &= \sup_{A\in\mathcal{B}(\{-1,+1\}\times E)} \sup_{x,m\neq\bar{m}} \left|\int_A \frac{Q(m;de)-Q(\bar{m},de)}{|m-\bar{m}|}\pi(dy|x,e,h)\right| \\ &\leq \sup_{A\in\mathcal{E}} \sup_{m\neq\bar{m}} \frac{|Q(m;A)-Q(\bar{m};A)|}{|m-\bar{m}|} < \infty\end{aligned}$$

by Condition (C1). Now recall that the individual transition probability π in (6) is assumed to be bounded away from zero by some constant $c' > 0$. Without loss of generality we may assume that $2c' = c$, where $c > 0$ is the constant given by Condition (C3).

We define a measure ν_* on $(\{-1,+1\}\times E; \mathcal{B}(\{-1,+1\}\times E))$ by the relation

$$\nu_*(A) = \left(\frac{1}{2}\delta_{+1}(A_1) + \frac{1}{2}\delta_{-1}(A_1)\right)\nu(A^*\cap A_2), \quad A = A_1\times A_2. \tag{26}$$

Using this measure, we can define a family $\{\nu_{r,l}\}_{r,l\in\mathbb{N}}$ of stochastic kernels on $[-1,+1]\times \mathcal{B}((\{-1,+1\}\times E)^l)$ by the following equation:

$$\nu_{r,l}(m;A) = \begin{cases} \nu_*^l(A) & l\leq r \\ \int \nu_*^r(dx^{(r)},de^{(r)})R_{l-r}(x_r,u(m,e^{(r)});d(x^{(l-r)},e^{(l-r)}))\mathbf{1}_A(x^{(l)},e^{(l)}) & l>r. \end{cases}$$

Here, ν_*^l denotes the product measure $\prod_{i=1}^l \nu_*(\cdot)$ on $\mathcal{B}((\{-1,+1\}\times E)^l)$. Observe that $|\nu_{r,l}(m;A) - \nu(\bar{m};A)| = 0$ for $l\leq r$. For $l>r$ we have that

$$\begin{aligned}&|\nu_{r,l}(m;A) - \nu_{r,l}(\bar{m};A)| \\ &= \left|\int \nu_*^r(dx^{(r)},de^{(r)})\left[R_{l-r}(x_r,u(m,e^{(r)});de_{r+1},dx_{r+1},\cdots,de_l,dx_l)\right.\right. \\ &\qquad \left.\left. -R_{l-r}(x_r,u(\bar{m},e^{(r)});de_{r+1},dx_{r+1},\cdots,de_l,dx_l)\right]\mathbf{1}_A(e^{(l)},x^{(l)})\right| \\ &\leq r_{l-r}\int \nu_*^r(dx^{(r)},de^{(r)})|u(m,e^{(r)}) - u(\bar{m},e^{(r)})| \\ &\leq r_{l-r}\theta^r|m-\bar{m}| \leq r_\infty\theta^r|m-\bar{m}|,\end{aligned}$$

where the last but one inequality follows from Condition (C3). In particular, we can deduce that

$$\Delta(\nu_{r,l}(\cdot;A)) \leq 2r_\infty\theta^r. \tag{27}$$

Note furthermore that

$$R_l((x,m);A) \geq c^{2\min\{r,l\}}\nu_{r,l}(m;A).$$

Thus, we can define another family $\{\mu_{r,l}\}_{r,l\in\mathbb{N}}$ of stochastic kernels on $\{-1,+1\}\times[-1,+1]\times\mathcal{B}((\{-1,+1\}\times E)^l)$ via the equation

$$R_l((x,m);A) = c^{2\min\{r,l\}}\nu_{r,l}(m;A) + \left(1-c^{2\min\{r,l\}}\right)\mu_{r,l}((x,m);A).$$

From (27) we deduce that

$$\sup_A \Delta R_l(\cdot; A) \le 2c^{2\min\{r,l\}}\theta^r r_\infty + \left(1 - c^{2\min\{r,l\}}\right),$$

and, as $r_\infty < \infty$, it is easily seen that for r sufficiently large

$$\Delta R_l(\cdot; A) < 1$$

uniformly in $l \in \mathbb{N}$. Thus $r^* < 1$ which proves our assertion. □

Theorem 4.1 allows us to prove local convergence of the microscopic process $\{x_t\}_{t\in\mathbb{N}}$ towards a unique equilibrium. To this end, we denote by $\mathcal{A}_{Loc}$ the system of all finite subsets of $\mathcal{A}$. For any $A \in \mathcal{A}_{Loc}$ we put

$$\begin{aligned} S_A &:= \{-1,+1\}^A \times X \times H, \\ M_A &:= \{(\{x_t^a\}_{a\in A}, m_t, h_t)\}_{t\in\mathbb{N}}. \end{aligned}$$

Obviously, M_A is a Markov chain with compact state space S_A. On S_A we define a metric d_A by the relation

$$d_A((x,m,h),(x',m',h')) := \sum_{a\in A} 2^{-w(a)}|x^a - y^a| + ||(m,h) - (m',h')||,$$

where $w : \mathcal{A} \to \mathbb{N}$ is a weight function which satisfies $\sum_{a\in\mathcal{A}} 2^{-w(a)} < \infty$.

Note that for any $A \in \mathcal{A}_{Loc}$ the conditional transition probability

$$\Pi_A(dx|x,e,h) = \prod_{a\in A} \pi(dy^a|x^a,e,h)$$

of the process $\{\{x^a\}_{a\in A}\}_{t\in\mathbb{N}}$ is bounded away from zero uniformly in (e,h). Thus, Remark 4.1 remains valid if we replace $\pi(\cdot|x,e,h)$ by $\Pi_A(\cdot|x,e,h)$ $(A \in \mathcal{A}_{Loc})$ and $|m - \bar{m}| + |x - y|$ by $d_A((x,m,h),(x',m',h'))$ in (22). In particular, we can deduce that the transition operator U_A of the process M_A is a Doeblin-Fortet operator.

An inspection of the proof of Theorem 4.1 shows that all arguments remain valid, if we consider the process M_A instead of the process $M_{\{a\}}$. We just have to modify the measure ν_* in (26). Thus, we can conclude that the Markov chain M_A is regular on the set $L(S_A)$ for any $A \in \mathcal{A}_{Loc}$.

Since the processes M_A, $A \in \mathcal{A}_{Loc}$, are regular, we can study the family $\{\mu_A\}_{A\in\mathcal{A}_{Loc}}$ of the corresponding stationary measures.

Lemma 4.1 *The family $\{\mu_A\}_{A\in\mathcal{A}_{Loc}}$ is consistent.*

Proof: Let $A \subset A' \in \mathcal{A}_{Loc}$ and $B \in \mathcal{B}(S_A)$. Without loss of generality we may assume $|A| = |A'| + 1$. Let $f \in C(S_A)$ be Lipschitz continuous. The regularity of the processes M_A on $L(S_A)$ and $M_{A'}$ on $L(S_{A'})$ yields

$$\begin{aligned} 0 &= \lim_{n\to\infty} |U_{A'}^n f - \mu_{A'}(f)|_\infty \\ &= \lim_{n\to\infty} |U_A^n f - \mu_{A'}(f)|_\infty \\ &= \lim_{n\to\infty} |U_A^n f - \mu_A(f)|_\infty. \end{aligned}$$

Thus, $\mu_A(f) = \mu_{A'}(f)$ for any $f \in C(S_A)$ and therefore

$$\mu_{A'}(B \times \{-1, +1\}) = \mu_A(B) \quad (B \in \mathcal{B}(S_A)).$$

This proves consistency. □

In order to show that the microscopic process $\{x_t\}_{t \in \mathbb{N}}$ converges in law to a unique stationary measure as $t \to \infty$, we will now define an operator U on the set $B(\{-1, +1\}^{\mathcal{A}} \times [-1, 1] \times H)$ by the equation

$$Uf(x, m, h) = \int f(y, u(m, e, h), \bar{h}) \Pi(dy|x, e, h) \otimes Q(m, de) \otimes K(h, d\bar{h}),$$

where Π is defined in (8). Note that for any $A \in \mathcal{A}_{Loc}$ the operator U_A is the restriction of U to the set $B(S_A)$, the set of all bounded measurable and real valued functions f on $S_{\mathcal{A}} := \{-1, +1\}^{\mathcal{A}} \times [-1, 1] \times H$, which depend only on the coordinates in S_A. Thus, for any $\mathcal{B}(S_A)$ measurable function f we have

$$U_A f = Uf.$$

Let us denote by μ the unique measure with marginal distributions $\{\mu_A\}_{A \in \mathcal{A}_{Loc}}$. If $f : S_{\mathcal{A}} \to \mathbb{R}$ is a bounded continuous **local** function, i.e, if f depends only on finitely many coordinates, there exists a local set $A \in \mathcal{A}_{Loc}$ such that

$$\lim_{n \to \infty} |U^n f(x) - \mu(f)|_\infty = \lim_{n \to \infty} |U_A^n f(x) - \mu_A(f)|_\infty = 0. \tag{28}$$

If f is **quasilocal**, i.e., if f can be uniformly approximated by local functions, equation (28) follows by uniform approximation. (Recall that any continuous real valued function f on $S_{\mathcal{A}}$ is quasilocal.) Hence,

$$\lim_{n \to \infty} |U^n f - \mu(f)| = 0$$

for any bounded continuous function f on $S_{\mathcal{A}}$. This proves local, and therefore weak convergence of the distributions of the Markov chain $\{(x_t, m_t, h_t)\}_{t \in \mathbb{N}}$ to the unique equilibrium μ. Thus, we have shown:

Theorem 4.2 *If the macroscopic process $\{(m_t, h_t)\}_{t \in \mathbb{N}}$ is regular, and if the individual transition probability π in (6) is bounded away from zero, then the microscopic process $\{x_t\}_{t \in \mathbb{N}}$ converges in law to a unique stationary measure.*

4.2 The Microscopic Process in a Stationary Environment

Next, we shall analyse the microscopic dynamics given that the macroscopic process is already in equilibrium, i.e., we assume that the initial distribution μ_0 of the process $\{(m_t, h_t)\}_{t \in \mathbb{N}}$ is given by its unique probabilistic equilibrium μ^*. Without loss of generality we will restrict ourselves to almost surely constant external fields.

The proof of the following proposition can be found in Iosefescu and Theodorescu (1968), p. 135.

Proposition 4.1 *Suppose that $\mu_0 = \mu^*$. Then the signal sequence $\{e_{t+1}\}_{t\in\mathbb{N}}$ is stationary and ergodic.*

We shall now show that the microscopic process $\{x_t\}_{t\in\mathbb{N}}$ becomes stationary in the long run, provided it evolves in a stationary environment. To this end, we shall use some general results about recursive chains and about stochastically recursive sequences provided in Borovkov (1998).

Definition 4.1 1. *A sequence $\{X(t)\}_{t\in\mathbb{N}}$ of random variables taking values in a measurable space $(X, \mathcal{X})$ is called a* ***recursive chain*** *(RC) governed by the sequence $\{\eta_t\}_{t\in\mathbb{N}}$ if for all $t \geq 0$ and for all $A \in \mathcal{X}$ the following holds true:*

$$\mathbb{P}[X(t+1) \in A|\mathcal{F}_t] = \mathbb{P}[X(t+1) \in A|\eta_{t+1}, X(t)].$$

Here, $\mathcal{F}_t = \sigma(\eta_1, \ldots, \eta_{t+1}, X(0), \ldots, X(t))$.

2. *A random sequence $\{X(t)\}_{t\in\mathbb{N}}$ taking values in $(X, \mathcal{X})$ is called a* ***stochastically recursive sequence*** *(SRS) controlled by the* ***driving sequence*** *$\{\eta_t\}_{t\in\mathbb{N}}$ taking values in a measurable space $(E, \mathcal{E})$ if the random variables $X(t)$ $(t \in \mathbb{N})$ obey the equation*

$$X(t+1) = f(X(t), \eta_{t+1})$$

for some measurable function $f : X \times E \to \mathbb{R}$.

Remark 4.2 1. *Observe that the microscopic process $\{x_t\}_{t\in\mathbb{N}}$ is a RC with governing sequence $\{e_{t+1}\}_{t\in\mathbb{N}}$.*

2. *Recall that with no loss of generality, the stationary sequence $\{e_t\}_{t\in\mathbb{N}}$ can always be extended to a sequence defined for all integers $-\infty < t < \infty$.*

Let us introduce the shift transformation T, generated by the stationary sequence $\{e_t\}_{t\in\mathbb{Z}}$, which acts on the canonical path space Ω according to the formula

$$T((\omega_t, \omega_{t+1}, \ldots)) = (\omega_{t+1}, \omega_{t+2}, \ldots),$$

where $\omega = (\omega_t)_{t\in\mathbb{N}} \in \Omega$.

We can now use general ergodicity criteria for RCs provided in Borovkov (1998), to show that the microscopic process converges in an appropriate sense to a stationary process.

Theorem 4.3 *Suppose that the macroscopic process is in equilibrium and that the individual transition probability π is uniformly bounded away from zero. Then there exists a stationary sequence of random variables $Y = \{Y(t)\}_{t\in\mathbb{N}}$, $Y(t) = \{Y_a(t)\}_{a\in\mathcal{A}}$, defined on a common probability space with $\{(x_t, e_{t+1})\}_{t\in\mathbb{N}}$, such that the microscopic process converges locally to Y in the sense of a strong coupling. More precisely, for any local set $A \in \mathcal{A}_{Loc}$ the following holds true:*

$$\lim_{t\to\infty} \mathbb{P}_{\mu^*}[Y_a(k) = x_k^a \quad \text{for all } k \geq t \text{ and for all } a \in A] = 1.$$

The distribution μ_0 of $Y(0)$ is invariant for the process $\{x_t\}_{t\in\mathbb{N}}$.

Proof: Let us first fix a finite set $A \in \mathcal{A}_{Loc}$ and put $x_t^A = \{x_t^a\}_{a\in A}$. Using Theorem 13.3 in Borovkov (1998) and the conditional independence of the random variables x_t^a ($a \in A$), we can define a stochastically recursive sequence $z^A = \{z^A(t)\}_{t\in\mathbb{N}}$, $z^A(t) = \{z_a(t)\}_{a\in A}$, on a common probability space with the sequence $\{(x_t, e_{t+1})\}_{t\in\mathbb{N}}$ such that

$$z_t^A \stackrel{Law}{=} x_t^A \quad \text{for all} \quad t \in \mathbb{N},$$

and such that the sequence z^A satisfies the relation

$$z^A(t+1) = \{f(z_t^a, e_{t+1}, \alpha_t^a)\}_{a\in A}, \quad i.e., \quad z_a(t+1) = f(z_t^a, e_{t+1}, \alpha_t^a), \tag{29}$$

for some measurable transformation $f : \{-1,+1\}\times E\times[0,1] \to \{-1,+1\}$ independent of a. Here, $\{\alpha_t^a\}_{t\in\mathbb{N}}$ ($a \in \mathcal{A}$) is an *i.i.d.* sequence of random variables taking values in $[0,1]$, and the sequences $\{e_{t+1}\}_{t\in\mathbb{N}}$, $\{\alpha_t^a\}_{t\in\mathbb{N}}$ ($a \in A$) are independent.

For a fixed $s \geq 1$ we shall define a sequence of random variables $\{Y_s^A(t)\}_{t\geq s}$ by the relation

$$Y_s^A(t) = T^{-s} z^A(s+t),$$

i.e., $Y_s^A(-s) = z^A(0) = x_0^A$, $Y^A(-s+1) = \{f(z_a(0), e_{-s+1}, \alpha_{-s}^a)\}_{a\in A}$, etc. Obviously we have that $Y_s^A(t) = z^A(t)$ implies $Y_s^A(t+n) = z^A(t+n)$ for all $n \in \mathbb{N}$.

As the individual transition probability π is bounded away from zero, it is easy to see that

$$\lim_{n\to\infty} \mathbb{P}_{\mu^*}\left(\bigcup_{t=1}^{n} \{Y_s^A(t) = z^A(t)\}\right) = 1.$$

Therefore we can deduce from the proof of Theorem 11.3 in Borovkov (1998) that there exists a stationary sequence $Y^A = \{Y^A(t)\}_{t\in\mathbb{N}}$, $Y^A(t) = \{y_a(t)\}_{a\in A}$ which satisfies the relation (29) and such that

$$\lim_{n\to\infty} \mathbb{P}_{\mu^*}[y_a(t) = z_a(t) \quad \text{for all } t \geq n \quad \text{and for all } a \in A] = 1. \tag{30}$$

In particular, the sequences $\{y_a(t)\}_{t\in\mathbb{N}}$ ($a \in A$) are conditionally independent given the process $\{e_{t+1}\}_{t\in\mathbb{N}}$.

Now we can define the desired process Y by the relation $Y(t) = \{y_a(t)\}_{a\in\mathcal{A}}$. Obviously, this process is stationary and the sequences $\{y_a(t)\}_{t\in\mathbb{N}}$, $a \in \mathcal{A}$, are conditionally independent given the signal sequence $\{e_{t+1}\}_{t\in\mathbb{N}}$.

From (30) and from stationarity of the process Y it is now obvious that the distribution μ of $Y(0)$ is invariant for the microscopic process $\{x_t\}_{t\in\mathbb{N}}$ and that x_t converges locally to μ as $t \to \infty$. □

At the end of this section we would like the draw the readers attention to the following point. At the first glance it might seem that the conditional invariant probability ν^* of the microscopic process is given by a product measure, i.e., $\nu^*(\cdot) = \prod_{a\in\mathcal{A}} \nu(\cdot)$. This, however, does not necessarily hold true as the following example shows.

Example 4.1 *Suppose that $\{e_t\}_{t\in\mathbb{N}}$ is a stationary Markov process taking values in the space $E = \{+,-\}$. The transition matrix D^e is given by*

$$D^e := \left\{ \begin{array}{cc} 3/4 & 1/4 \\ 1/4 & 3/4 \end{array} \right\}.$$

Furthermore, we fix $a, a' \in \mathcal{A}$, and $\{x_t^a\}_{t\in\mathbb{N}}$ and $\{a_t^{a'}\}_{t\in\mathbb{N}}$ are Markov chains with state space $\{-1,+1\}$ evolving in the random environment generated by the process $\{e_t\}_{t\in\mathbb{N}}$. The respective conditional transition probabilities D^+ and D^- are given by

$$D^+ := \left\{ \begin{array}{cc} 2/3 & 1/3 \\ 1/3 & 2/3 \end{array} \right\} \quad \text{and} \quad D^- = \left\{ \begin{array}{cc} 1/3 & 2/3 \\ 2/3 & 1/3 \end{array} \right\}.$$

Some straightforward calculations show that $\mathbb{P}[x_{t+1}^a = 1 | x_t^a = 1, x_t^{a'} = 1] = \mathbb{P}[x_{t+1}^a = 1 | x_t^a = 1] = \frac{1}{2}$. Observe, however, that

$$\mathbb{P}[e_t = + | x_t^a = x_{t+1}^a = 1] = \frac{3}{4}$$

and therefore

$$\mathbb{P}[x_t^{a'} = 1 | x_t^a = x_{t+1}^a = 1] = \frac{3}{4}\frac{2}{3} + \frac{1}{4}\frac{1}{3} = \frac{7}{12}. \tag{31}$$

Now suppose that the invariant measure ν^* of the microscopic process is a product measure. In this case we would have that $\nu^*(\cdot) = \int \mathbb{P}_{\mu^*}(\cdot; x^a)v(dx^a)$ ($a \in \mathcal{A}$) and we would get the following equality:

$$\begin{aligned} \nu^*(x_{t+1}^{a'} = x_{t+1}^a = 1) &= \nu(x_t^a = 1)\nu(x_t^{a'} = 1) \\ &= \int \mathbb{P}[x_{t+1}^{a'} = 1 | x_t^{a'}]\nu(dx_t^{a'}) \int \mathbb{P}[x_{t+1}^a = 1 | x_t^a]\nu(dx_t^a). \end{aligned}$$

As ν^* is invariant it would follow that

$$\begin{aligned} \mathbb{P}[x_{t+1}^a = 1 | x_t^a]\mathbb{P}[x_{t+1}^{a'} = 1 | x_t^{a'}] &= \mathbb{P}[x_{t+1}^{a'} = 1 = x_{t+1}^a | x_t^{a'}, x_t^a] \\ &= \mathbb{P}[x_{t+1}^{a'} = 1 | x_{t+1}^a = 1, x_t^{a'}, x_t^a]\mathbb{P}[x_{t+1}^a | x_t^a]. \end{aligned}$$

In general, however, this equation is wrong as (31) shows.

5 Dynamics of Equilibrium Prices

This section is devoted to the study of equilibrium price dynamics in a random environment. The random environment is related to a random fluctuation in the behavioural characteristics of agents, for example, in the proportion between different types of traders who are active on the financial market. More precisely, we will analyse price processes $\{p_t\}_{t\in\mathbb{N}}$ which take the form

$$p_{t+1} = f(p_t, m_{t+1}), \tag{32}$$

as in (4). Here $f : \mathbb{R}^2 \to \mathbb{R}$ denotes a measurable function and $\{m_t\}_{t\in\mathbb{N}}$ is the stochastic process which describes the mood of the market analysed in Section 3.

In the first subsection we will analyse the stock price process corresponding to (32) in a situation, where the mood of the market forms a stationary environment. Thus, we study the dynamics of the process $\{p_t\}_{t\in\mathbb{N}}$ given that $\{m_t\}_{t\in\mathbb{N}}$ is a stationary and ergodic sequence of random variables, i.e., we assume that the mood of the market is already in equilibrium.

Using some general results provided in Borovkov (1998), we can prove that the distribution of p_t converges weakly to a unique equilibrium as $t \to \infty$, provided that the environment is on average not too destabilising.

In the second subsection we study affine linear price dynamics in a non-stationary environment. More precisely, we consider the case

$$p_{t+1} = f(m_{t+1})p_t + g(m_{t+1}), \tag{33}$$

where $f, g : [-1, 1] \to \mathbb{R}$ are Lipschitz continuous functions, and where the macroscopic process is a non-stationary Markov chain. Here again, our aim is to analyse the asymptotic distribution of p_t as $t \to \infty$.

We know from Section 3 that there exists a unique stationary probability measure μ^* for the Markov process $\{(m_t, h_t)\}_{t\in\mathbb{N}}$, provided that the transition kernel Z defined in (18) and the function u in (9) satisfy certain technical assumptions. Throughout this section we will assume that the random system Σ^* analysed in Section 3 is distance diminishing and regular. By $\mathbb{P}_{\mu^*}$ we denote the induced probability measure on the canonical path space if the initial distribution of the macroscopic process is given by μ^*.

5.1 Asymptotic Dynamics of Equilibrium Prices in a Stationary Environment

In this subsection we assume that this macroscopic process already starts in its probabilistic equilibrium, i.e., its initial distribution μ_0 is given by μ^*. In particular, the sequence $\{m_t\}_{t\in\mathbb{N}}$ is stationary and ergodic. Therefore, the price process $\{p_t\}_{t\in\mathbb{N}}$ evolves in a stationary environment.

Several ergodicity conditions of SRSs are discussed in Borovkov (1998). Some of the most general ergodicity and stability theorems for stochastically recursive sequences are based on the concept of so-called renovation events. Other ergodicity conditions are stated in terms of certain analytical properties of the transformation f. For our purposes an approach which uses a *mean contraction property* of the transformation f seems to be appropriate.

Let us consider the following conditions concerning the iterates

$$f(p, m^{(n)}) := f(\cdot, m_n) \circ \cdots \circ f(\cdot, m_2) \circ f(p, m_1) \quad (n \in \mathbb{N}),$$

where $p_0 = p$ is the initial price at time $t = 0$; see also Borovkov (1998), Chapter 2, Section 8.

Assumption 5.1 *1. ("Boundedness in Probability") For some* $p_0 \in \mathbb{R}$ *and for each* $\delta > 0$ *there exists* $N = N(\delta)$ *such that for all* $n \geq 1$ *we have that*

$$\mathbb{P}_{\mu^*}\{|p_0 - f(p_0, m^{(n)})| > N\} < \delta.$$

2. *("Contraction in the mean") The function* $f = f(p,m)$ *is continuous in* p *and there exists an integer* $r \geq 1$*, a number* $\beta > 0$*, and a measurable function* $c : \mathbb{R}^r \to \mathbb{R}_+$ *such that for all* $p_1, p_2 \in \mathbb{R}$ *the following inequalities hold true:*

$$\begin{aligned} |f(p_1, m^{(r)}) - f(p_2, m^{(r)})| &\leq c(m_1, \ldots, m_r)|p_1 - p_2|, \\ \frac{1}{r}\mathbb{E}_{\mu^*} \ln c(m_1, \ldots, m_r) &\leq -\beta. \end{aligned} \tag{34}$$

3. *("Strong law of large numbers") The sequence* $\{\ln c(m_{jr}, \ldots, m_{jr+r-1})\}_{j\in\mathbb{N}}$ *satisfies the strong law of large numbers.*

Recall that with no loss of generality, the sequence $\{m_t\}_{t\in\mathbb{N}}$ can always be extended to a sequence defined for all integers $-\infty < t < \infty$. For stationary sequences, the necessary extension to $t < 0$ can always be realised using a theorem of Kolmogorov.

The following result appears as Theorem 12.2 in Borovkov (1998).

Theorem 5.1 *Suppose that Assumption 5.1 is satisfied. Then there exists a stationary sequence of random variables* $\{X_t\}_{t\in\mathbb{N}}$*,* $X_t(\omega) = \omega_t$*, on the canonical path space, which obeys the equation* $X_{t+1} = TX_t = f(X_t, m_t)$*, and such that for each fixed* p *we have*

$$T^{-t} f(p, m^{(t+s)}) \to X_s \quad \text{as } t \to \infty \quad \text{with probability one.} \tag{35}$$

The distribution μ *of* X_0 *is invariant for the sequence* $\{p_t\}_t$*. In particular, the process* $\{p_t\}_{t\in\mathbb{N}}$ *converges in distribution to* μ*.*

Remark 5.1 *1. For our financial market model the above theorem provides a bound for the aggregate effect of interaction between different types of traders, which ensures that the induced price fluctuations are stationary in the long run. Beyond this bound the price process may become highly unstable. In fact, in an affine linear model, c.f. Example 5.2 below, the trajectories converge to zero or to infinity and their growth or decay exceeds an exponential rate if the term* $\frac{1}{r}\mathbb{E}_{\mu^*} \ln c(m_1, \ldots, m_r)$ *in (34) is positive.*

2. *If the transformation* f *takes an affine linear form, Theorem 5.1 is a discrete time version of Theorem 4.2 in Föllmer and Schweizer (1993)*

Example 5.1 *(Generalised auto-regression) Suppose that the equilibrium price dynamics obey the recurrence relation*

$$p_{t+1} = G(f(m_t)F(p_t) + g(m_t)), \tag{36}$$

where $F, G : \mathbb{R} \to \mathbb{R}$ *are Lipschitz continuous:*

$$|F(p_1) - F(p_2)| \leq c_F|p_1 - p_2|, \quad |G(p_1) - G(p_2)| \leq c_G|p_1 - p_2|.$$

The following proposition is taken from Borovkov (1998).

Proposition 5.1 *Under the above restriction the sequence (36) specified in Example 5.1 satisfies parts 1 and 2 of Assumption 5.1 if*

$$\ln |c_F c_G| + \mathbb{E}_{\mu^*} \ln |f(m_0)| < 0 \quad \textit{and} \quad \mathbb{E}_{\mu^*} (\ln |g(m_0)|)^+ < \infty. \tag{37}$$

Example 5.2 *Let us consider the price dynamics induced by the interaction of noise traders and fundamentalists as in Föllmer and Schweizer (1993). In this case, the evolution of the logarithmic stock price process $\{\ln p_t\}_{t \in \mathbb{N}}$ is governed by (36) with*

$$G(p) = F(p) = p, \quad f(m) = \frac{1 + rc_1 + (1-r)c_2}{1 + (1-r)c_2}, \quad g(m) = -\frac{rc_1 F^*}{1 + (1-r)c_2},$$

for some constants $c_1, c_2 < 0$. Here, $F^* \in \mathbb{R}$ denotes some fixed, say long run, fundamental value of the asset, and $r := \frac{2m-1}{2}$ denotes the fraction of fundamentalists among all traders being active in the market.

Suppose that the inequalities (37) are satisfied. By Proposition 5.1, in order to prove the a.s. convergence of the sequence $\{\ln p_t\}_{t \in \mathbb{N}}$ to a stationary process $\{X_t\}_{t \in \mathbb{N}}$ in the sense of (35), it remains to verify part 3 of Assumption 5.1: We have to show that the sequence $\{\ln |f(m_t)|\}_{t \in \mathbb{N}}$ satisfies a strong law of large numbers. This, however, follows immediately from Proposition 4.1 and Proposition 4.3 in Krengel (1988) as $\{m_t\}_{t \in \mathbb{N}}$ is strictly stationary and ergodic.

On the other hand, we can show that the trajectories of the price process are in no sense stable if (37) in not satisfied. Suppose that

$$\mathbb{E}_{\mu^*} \ln |f(m_0)| = c > 0.$$

In this case, the paths of the equilibrium price process exhibits super-exponential growth or decay. In fact, our stock price process takes the form

$$p_t = \exp(Y_t + \ln F),$$

where the process $\{Y_t\}_{t \in \mathbb{N}}$ satisfies the recurrence relation

$$Y_{t+1} - Y_t = f(m_t) Y_t;$$

see Föllmer and Schweizer (1993). Thus

$$|\ln p_t| = |Y_0| \prod_{i=1}^{t} |f(m_i)|, \quad Y_0 = \ln p_0 - \ln F.$$

Therefore

$$\frac{1}{t} \ln |\ln p_t| = \frac{1}{t} \left(\ln |Y_0| + \sum_{i=0}^{t} \ln |f(m_i)| \right).$$

and, due to the strong law of large numbers, we obtain

$$\lim_{t \to \infty} \frac{1}{t} \ln |\ln p_t| = c \quad \mathbb{P}_{\mu^*} - a.s.$$

This is a discrete-time version of Theorem 4.1 in Föllmer and Schweizer (1993).

5.2 Affine Linear Price Dynamics in a Non-Stationary Environment

Let us now consider an affine linear dynamics for the equilibrium price process. More precisely, we assume that the process $\{\ln p_t\}_{t\in\mathbb{N}}$ obeys the recurrence relation

$$\ln p_{t+1} = f(m_t)\ln p_t + g(m_t),$$

where f and g are Lipschitz continuous functions. In this subsection we do not assume that the random environment for the evolution of the stock price process is generated by a stationary and ergodic process. As far as we know very little is known about stochastically recursive sequences evolving in a non-stationary environment. If the governing sequence $\{\eta_t\}_{t\in\mathbb{N}}$ of a SRS $\{X_t\}_{t\in\mathbb{N}}$ converges in the sense of a strong coupling to a stationary process $\{\eta'_t\}_{t\in\mathbb{N}}$, then the results of the preceeding subsection remain valid under some strong contraction conditions; see Borovkov (1998), Chapter 3. However, this assumption seems to be too restrictive for our purposes. Furthermore, the author believes that such a condition is, if at all, hard to verify. However, using the special structure of the process $\{m_t\}_{t\in\mathbb{N}}$ one can prove the following theorem, see Horst (1999b), Theorem 2.4.

Theorem 5.2 *Let f and g be Lipschitz continuous and suppose that the macroscopic process is regular on $L([-1,+1]\times H)$. If, moreover, $\mathbb{E}_{\mu^*}\ln|f(m_0)| < 0$ and if $\mathbb{E}_{\mu^*}\max\{\ln|g(m_0),0\} < \infty$, then there exists a unique probability measure ν^* on $\mathbb{R}$ such that the the stock price process converges in law to ν^* as $t\to\infty$.*

Remark 5.2 *Observe that in our present setting the mood of the market drives to price process $\{p_t\}_{t\in\mathbb{N}}$ into equilibrium as soon as the macroscopic process settles down to a stationary measure in the long run.*

6 Convergence to a Diffusion Model

Large part of financial economic theory is based on models with continuous-time security trading. These models are relevant insofar as they may characterise the behaviour of models in which trading occurs discretely in time. Thus, it seems natural to check whether the limit of our discrete-time security market model, as the length of periods between trades shrink to zero, produces the effect of continuous-time trading and to identify a "canonical candidate" for the resulting continuous-time asset price process. This approach has been taken by Duffie and Protter (1992) and by Föllmer and Schweizer (1993) among others.

In this section we shall obtain a continuous-time stock price process $S = \{S_t\}_{t\geq 0}$ by passage to the limit from the discrete-time equilibrium price process analysed in the previous subsection. The convergence concept we use is weak convergence on the Skorohod space $\mathbb{D}^d$, endowed with the Skorohod topology.

We denote by p_t the logarithmic stock price at time t and consider an affine linear dynamics, i.e.,

$$p_{t+1} - p_t = f(m_{t+1})p_t + g(m_{t+1}).$$

The functions f and g are assumed to be Lipschitz continuous and $\{m_t\}_{t\in\mathbb{N}}$ is again the stochastic process which describes the random fluctuations in the behavioural characteristics of agents.

For each $n \in \mathbb{N}$ we shall consider a suitably defined sequence of random variables $\psi^n = \{(f_t^n, g_t^n)\}_{t\in\mathbb{R}}$ obtained from the process $\psi = \{(f(m_t), g(m_t))\}_{t\in\mathbb{N}}$. Applying an invariance principle to this process and assuming a certain "goodness" property, we obtain a convergence result for the logarithmic security price processes $P^n = \{P_t^n\}_{t\in\mathbb{N}}$ ($n \in \mathbb{N}$) defined by

$$P_{t+1}^n - P_t^n = f_t^n P_t^n + g_t^n. \tag{38}$$

In the following subsection we follow the approach taken by Föllmer and Schweizer (1993) who analysed an Ornstein-Uhlenbeck process evolving in an *exogenously* produced random environment. However, in contrast to their model, we consider equilibrium price dynamics generated by *endogenously* produced random fluctuations in the proportion between different types of traders who are active on the market.

Throughout this section we will assume that the RSCC Σ^* analysed in Section 3 is regular. In Subsection 6.1 we consider price dynamics evolving in a *stationary* environment, i.e., we will assume that the macroscopic process already starts in its probabilistic equilibrium μ^*. Using a functional central limit theorem we show that the process $\{\psi^n\}_{n\in\mathbb{N}}$ converges in distribution to a continuous diffusion process Z with deterministic drift vector B and volatility matrix C. Therefore, in the continuous-time limit the evolution of the random environment is described by a diffusion process.

In Subsection 6.2 we study the continuous-time limit in a *non-stationary* situation. We establish a convergence result for the equilibrium price process evolving in a non-stationary environment. It is possible to apply an invariance principle to functionals of non-stationary Markov processes if, for example, the distributions converge strongly, i.e., in the norm of total variation, to some invariant measure μ^*. As we can only verify weak convergence of the macroscopic process to μ^*, we will approximate the process $\psi = \{(f(m_t), g(m_t))\}_{t\in\mathbb{N}}$ and consider a slight perturbation of our original stock price process. In this case we shall see that in the continuous-time limit the dynamics of the random environment can be described by a diffusion process with deterministic drift vector B and volatility matrix $\tilde{C}$ describing the random environment. The quadratic characteristic $\tilde{C}$ lies in an ϵ-neighbourhood of C for a given $\epsilon > 0$. Furthermore, we shall demonstrate that the induced sequence of semimartingales describing the evolution of the random environment is indeed "good". This allows us to derive a continuous-time limit for the equilibrium price process.

From our point of view these results may justify the assumption of a stationary and ergodic random environment for the asset price process made in Subsection 6.1.

The following basic property of our regular and distance-diminishing RSCC Σ^* analysed in Section 3 which appears as Theorem 2.1.57 in Iosefescu and Theodorescu

(1968), turns out to be essential for the proof of Theorem 6.1 below.

Proposition 6.1 *There exists for each Lipschitz continuous function f on $[-1,+1]\times H$ constants $L(f)>0$ and $\alpha<1$ such that*

$$|U^t f - \mu^*(f)|_\infty \le L(f)\alpha^t. \tag{39}$$

The constant $L(f)$ depends on f only through its Lipschitz constant $m(f)$ and through its global maximum $|f|_\infty$.

6.1 Continuous-Time Dynamics in a Stationary Environment

Let us introduce the following notation. By $Law(X, \mathbb{P})$ we denote the distribution of a random variable X under the measure $\mathbb{P}$. Furthermore, we define

$$\mu_f := \mathbb{E}_{\mu^*} f(m_0) = \mu^*(f), \quad \mu_g := \mathbb{E}_{\mu^*} g(m_0) = \mu^*(g),$$

$$f_t^n := \frac{1}{\sqrt{n}} \sum_{k=0}^{[nt]} (f(m_k) - \mu_f), \quad g_t^n := \frac{1}{\sqrt{n}} \sum_{k=0}^{[nt]} (g(m_k) - \mu_g) \quad (t \in \mathbb{R}),$$

and put

$$\mathcal{F}_t := \sigma(\{m_s, h_s\}; s \le t) \quad (t \ge 0).$$

In this subsection we will apply an invariance principle to the stochastic processes $\psi^n = \{(f_t^n, g_t^n)\}_{t\in\mathbb{R}}$, $(n \in \mathbb{N})$ and show that this sequence of semimartingales converges in law to the continuous Gaussian Martingale CW, where $W = (W_f, W_g)$ is a 2-dimensional Brownian motion and

$$C := \begin{pmatrix} \sigma_f & \sigma_{fg} \\ \sigma_{fg} & \sigma_g \end{pmatrix}$$

with

$$\sigma_f^2 := \mathbb{E}_{\mu^*}[(f(m_0) - \mu_f)^2] + 2\sum_{t\ge1} \mathbb{E}_{\mu^*}[(f(m_0) - \mu_f)(f(m_t) - \mu_f)], \tag{40}$$

$$\sigma_g^2 := \mathbb{E}_{\mu^*}[(g(m_0) - \mu_g)^2] + 2\sum_{t\ge1} \mathbb{E}_{\mu^*}[(g(m_0) - \mu_g)(g(m_t) - \mu_g)], \tag{41}$$

$$\sigma_{fg}^2 := \mathbb{E}_{\mu^*}[(f(m_0) - \mu_f)(g(m_0) - \mu_g)] + \sum_{t\ge1} \mathbb{E}_{\mu^*}[(f(m_t) - \mu_f)(g(m_0) - \mu_g) + (g(m_t) - \mu_g)(f(m_0) - \mu_f)]. \tag{42}$$

Theorem 6.1 *Suppose that the initial distribution μ_0 of the macroscopic process is given by the unique stationary measure μ^* and let f and g be Lipschitz continuous functions. Then the sequence of processes $\psi = \{\psi^n\}_{n\in\mathbb{N}} = \{(f_t^n, g_t^n)_{t\in\mathbb{R}}\}_{n\in\mathbb{N}}$ converges in distribution to the Gaussian martingale CW:*

$$Law(\psi^n, \mathbb{P}_{\mu^*}) \xrightarrow{w} Law(CW, \mathbb{P}_{\mu^*}) \quad (n \to \infty).$$

Proof: Without loss of generality we may assume that $\mu^*(f) = \mu^*(g) = 0$. First, recall that the sequence $\{(f(m_t), g(m_t))\}_{t\in\mathbb{N}}$ is stationary and ergodic. Next, observe that due to the Lipschitz continuity of the function f there exists constants $L(f) > 0$ and $\alpha < 1$ such that

$$|\mathbb{E}_{\mu^*}[f(m_{t+1})|\mathcal{F}_0]| = |\mathbb{E}_{m_0,h_0}[f(m_{t+1})]| = |U^t f(m_0) - \mu^*(f)| \leq L(f)\alpha^t \quad \mathbb{P}_{\mu^*} - a.s. \tag{43}$$

by (39). In particular,

$$\sum_{t\geq 0} ||\mathbb{E}_{\mu^*}[f(m_t)|\mathcal{F}_0]||_{L_2} < \infty, \tag{44}$$

which implies that the series on the right hand side in (40) converges absolutely; see, e.g., Jacod and Shiryayev (1987), Theorem VIII 3.97, and note that the results about invariance principles for stationary sequences in Jacod and Shiryayev (1987) remain valid, if one considers a one-sided (stationary) sequence $\{m_t\}_{t\in\mathbb{N}}$ instead of the doubly infinite one $\{m_t\}_{t\in\mathbb{Z}}$.

As the function $f+g$ is also Lipschitz continuous, (44) is satisfied with f replaced by $f+g$, i.e.,

$$\sum_{t\geq 0} ||\mathbb{E}_{\mu^*}[f(m_t) + g(m_t)|\mathcal{F}_0]||_{L_2} < \infty, \tag{45}$$

and we can conclude that the series in (42) is also absolutely convergent.

Following the proof of Theorem VIII 3.79 and of Theorem VIII 3.97 in Jacod and Shiryayev (1987), the process ψ^n admits a representation of the form

$$\psi_t^n = \frac{1}{\sqrt{n}}[Y_t^n - Y_0^n + M_t^n],$$

where $Y_t^n = Y_{[nt]}$, $M_t^n = (M_{[nt]}^f, M_{[nt]}^g) = M_{[nt]}$. Here, $Y = \{Y_t\}_{t\in\mathbb{N}} = \{(Y_t^f, Y_t^g)\}_{t\in\mathbb{N}}$ is a vector of stationary processes and $M = \{M_t\}_{t\in\mathbb{N}} = \{(M_t^f, M_t^g)\}_{t\in\mathbb{N}}$ is a two dimensional vector of square integrable $\{\mathcal{F}_t\}$-martingales which satisfy

$$\mathbb{E}_{\mu^*}[(M_t^f)^2] = t\sigma_f^2, \quad \mathbb{E}_{\mu^*}[(M_t^g)^2] = t\sigma_g^2.$$

More precisely, M_t^f takes the form

$$M_t^f = \sum_{s\geq 0} \{\mathbb{E}_{\mu^*}[f(m_s)|\mathcal{F}_t] - \mathbb{E}_{\mu^*}[f(m_s)|\mathcal{F}_0]\}. \tag{46}$$

M^g is defined analogously. It is well known, see, e.g., Jacod and Shiryayev (1987), Chapter VIII that

$$Law\left(\frac{1}{\sqrt{n}}M^{f,n}, \mathbb{P}_{\mu^*}\right) \xrightarrow{w} Law(\sigma_f W_f, \mathbb{P}_{\mu^*}) \quad (n\to\infty)$$

and

$$Law\left(\frac{1}{\sqrt{n}}M^{g,n}, \mathbb{P}_{\mu^*}\right) \xrightarrow{w} Law(\sigma_g W_g, \mathbb{P}_{\mu^*}) \quad (n\to\infty).$$

Here, W_f and W_g are standard Wiener processes. Furthermore it follows from the proof of Theorem VIII 3.79 in Jacod and Shiryayev (1987) that

$$\sup_{t\leq T}\left|\left|\psi_t^n - \frac{1}{\sqrt{n}}M_t^n\right|\right|^2 \leq \sup_{t\leq T}\left|f_t^n - \frac{1}{\sqrt{n}}M_t^{f,n}\right|^2 + \sup_{t\leq T}\left|g_t^n - \frac{1}{\sqrt{n}}M_t^{g,n}\right|^2 \xrightarrow{\mathcal{P}} 0 \quad (n\to\infty).$$

Here, $||\cdot||$ denotes the Euclidian distance on $\mathbb{R}^2$ and $\xrightarrow{\mathcal{P}}$ means convergence in probability.

Using Theorem 4.1 in Billingsley (1968) and Corollary VIII 3.26 in Jacod and Shiryayev (1987) it suffices to show that

1. $\frac{1}{n}[M^f, M^g]_{[nt]} \xrightarrow{\mathcal{P}} t\sigma_{fg}$, where $[M^f, M^g]$ denotes the quadratic co-variation of the martingales M^f and M^g,
2. $\nu^n([0,t], \{||x|| \geq \epsilon\}) \xrightarrow{\mathcal{P}} 0$ for all $\epsilon > 0$, where $\nu^n = \nu^n(dt, dx)$ is the jump measure of the martingale M^n .

To show (1.) we can proceed as follows. Using (45) and Theorem VIII 3.97 in Jacod and Shiryayev (1987) again, we deduce that the processes $\phi^n = \{\phi_t^n\}_{t\in\mathbb{R}}$ $(n\in\mathbb{N})$, defined by

$$\phi_t^n = \frac{1}{\sqrt{n}}\sum_{s=0}^{[nt]}(f(m_s) + g(m_s))$$

admit the representation

$$\phi_t^n = \frac{1}{\sqrt{n}}(Y_{[nt]}^{f+g} - Y_0^{f+g} + M_{[nt]}^{f+g})$$

for a square integrable martingale M^{f+g} defined as in (46). Observe that

$$M^{f+g} = M^f + M^g.$$

Thus, by the polarisation identity

$$[M^f, M^g] = \frac{1}{2}([M^f + M^g, M^f + M^g] - [M^f, M^g] - [M^g - M^g])$$

we obtain

$$\frac{1}{n}[M^f, M^g]_{[nt]} = \frac{1}{2n}([M^f + M^g, M^f + M^g]_{[nt]} - [M^f, M^f]_{[nt]} - [M^g, M^g]_{[nt]}). \quad (47)$$

Using Birkhoff's ergodic theorem, Lemma 9.4.1 in Liptser and Shiryayev (1986), and ergodicity of the process $\{f(m_t) + g(m_t)\}_{t\in\mathbb{N}}$, we conclude from our previous considerations that the right hand side in (47) converges almost surely to

$$
\begin{aligned}
&\frac{t}{2}\left\{\mathbb{E}_{\mu^*}[(M_1^f+M_1^g)^2]-\mathbb{E}_{\mu^*}[(M_1^f)^2]-\mathbb{E}_{\mu^*}[(M_1^g)^2]\right\}\\
&= \frac{t}{2}\left\{\mathbb{E}_{\mu^*}[(M_1^{f+g})^2]-\mathbb{E}_{\mu^*}[(M_1^f)^2]-\mathbb{E}_{\mu^*}[(M_1^g)^2]\right\}\\
&= \frac{t}{2}\Bigg\{\mathbb{E}_{\mu^*}[(f(m_0)+g(m_0))^2]+\sum_{s\geq 1}\mathbb{E}_{\mu^*}[(f(m_s)+g(m_s))(f(m_0)+g(m_0))]\\
&\quad -\mathbb{E}_{\mu^*}[(f(m_0))^2]-\sum_{s\geq 1}\mathbb{E}_{\mu^*}[f(m_s)f(m_0)]\\
&\quad -\mathbb{E}_{\mu^*}[(g(m_0))^2]-\sum_{s\geq 1}\mathbb{E}_{\mu^*}[g(m_s)g(m_0)]\Bigg\}\\
&= t\left\{\mathbb{E}_{\mu^*}[f(m_0)g(m_0)]+\sum_{s\geq 1}\mathbb{E}_{\mu^*}[f(m_s)g(m_0)+g(m_s)f(m_0)]\right\}\\
&= t\sigma_{f,g}.
\end{aligned}
$$

Thus,

$$\frac{1}{n}[M,M]_{[nt]} \xrightarrow{\mathcal{P}} tC \quad (n\to\infty).$$

It remains to show (2.). To this end, we denote by $\nu^{n,f}$ and $\nu^{n,g}$, respectively, the jump measure of the martingale $M^{n,f}$ and $M^{n,g}$ respectively.

Since $\nu^{n,f}((0,t]\times\{|x_f|>\epsilon\})$ and $\nu^{n,g}((0,t]\times\{|x_g|>\epsilon\})$ tend to zero in probability as $n\to\infty$ ($\epsilon>0$), our assertion follows immediately from the inequality

$$\nu^n([0,t]\times\{||(x_f,x_g)||>\epsilon\})\leq \nu^{n,f}([0,t]\times\{|x_f|>\epsilon/2\})+\nu^{n,g}([0,t]\times\{|x_g|>\epsilon/2\}).$$

□

Now we can easily introduce a drift vector B and show how our discrete-time model converges to a diffusion model with drift B and volatility matrix C. The canonical drift vector is $B:=(\mu_f,\mu_g)$. To this end, we define for each $n\in\mathbb{N}$ continuous-time processes $X^n=\{X_t^n\}_{t\geq 0}$ and $Y^n=\{Y_t^n\}_{t\geq 0}$ by the relation

$$X_t^n := \frac{1}{\sqrt{n}}\sum_{k=0}^{[nt]}\left\{f(m_k)-\mu_f+\frac{\mu_f}{\sqrt{n}}\right\},\quad Y_t^n := \frac{1}{\sqrt{n}}\sum_{k=0}^{[nt]}\left\{g(m_k)-\mu_g+\frac{\mu_g}{\sqrt{n}}\right\}$$

and show that the sequence $\{Z^n\}_{n\in\mathbb{N}}=\{(X^n,Y^n)\}_{n\in\mathbb{N}}$ converges in distribution to a diffusion process with drift B and dispersion matric C.

Corollary 6.1 *Let f and g be Lipschitz continuous functions. Then*

$$Law(Z^n,\mathbb{P}_{\mu^*}) \longrightarrow Law(Z,\mathbb{P}_{\mu^*})$$

where Z is a continuous diffusion with deterministic characteristics $(\mu,C,0)$, i.e., $dZ_t=\mu dt+CdW_t$. Here, $\mu=(\mu_f,\mu_g)$ and C and W are as in Theorem 6.1.

Proof: Following the proof of Theorem 6.1, the process Z^n admits the representation

$$Z_t^n = Y_t^n - Y_0^n + M_t^n + \frac{[nt]}{n}\mu,$$

where Y^n and M^n are as in the proof of Theorem 6.1. Since $\frac{[nt]}{n}\mu \to t\mu$ as $n \to \infty$, the semimartingale $\left\{M_t^n + \frac{[nt]}{n}\mu\right\}_{t\in n}$ converges to Z by Theorem VIII 3.8 in Jacod and Shiryayev (1987). □

Remark 6.1 *The convergence results remain valid for arbitrary measurable functions f and g as long as the series in (44) and (45) are convergent.*

Now we can proceed as in Föllmer and Schweizer (1993). We consider for each $n \in \mathbb{N}$ a continuous-time process $P^n := \{P_t^n\}_{t\in\mathbb{N}}$ given by

$$dP_t^n = P_{t-}^n dX_t^n + dY_t^n.$$

We assume that the sequence $\{(X^n, Y^n)\}_{n\in\mathbb{N}}$ converges in distribution to the semimartingale $Z = (X, Y)$ and that it is "good" in the following sense.

Definition 6.1 *(Duffie and Protter (1992)) A sequence $\{Z^n\}_{n\in\mathbb{N}}$ of semimartingales defined on probability spaces $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$ is **good**, if for any sequence $\{H^n\}_{n\in\mathbb{N}}$ of càdlàg adapted processes the convergence $Law((Z^n, H^n), \mathbb{P}^n) \stackrel{w}{\longrightarrow} Law((Z, H), \mathbb{P})$ implies the convergence $Law((Z^n, H^n, \int H_-^n dZ^n, \mathbb{P}^n)) \stackrel{w}{\longrightarrow} Law((Z, H, \int H_- dZ), \mathbb{P})$ as $n \to \infty$.*

The following result appears as Theorem 3.1 in Föllmer and Schweizer (1993).

Theorem 6.2 *The sequence $\{(X^n, Y^n, P^n)\}_{n\in\mathbb{N}}$ converges in law to (X, Y, P). Here P satisfies the stochastic differential equation*

$$dP_t = P_{t-} dX_t + dY_t. \tag{48}$$

In particular, the prices process $\{S_t\}_{t\geq 0}$ takes the form

$$S_t = S_0 \exp(P_t - P - 0).$$

Remark 6.2 *Note that in our model the "volatility coefficients" σ_f and σ_g reflect the aggregate behaviour of traders. For example, the higher the volatility of aggregate behaviour at time $t = 0$ reflected by the terms $\mathbb{E}_{\mu^*}[(f(m_0) - \mu_f)^2]$ and $\mathbb{E}_{\mu^*}[(g(m_0) - \mu_g)^2]$, the higher the volatility of the resulting stock price process. Thus, in our model, volatility is generated also by the interaction of agents and not only by random fluctuations of certain fundamental values.*

6.2 Continuous-Time Dynamics in a Non-stationary Environment

In the previous subsection we derived a continuous-time model for the evolution of the stock price using a functional central limit theorem for strictly stationary and ergodic processes. It turned out that, given that the mood of the market is already in equilibrium, the random environment for the evolution of the security price process can be approximated by a diffusion process with deterministic volatility matrix C.

However, this approach, although common in literature, seems to be inconsistent. The logarithmic security price process $\{p_t\}_{t\in\mathbb{N}}$ evolves in an environment generated by random fluctuations in the behavioural characteristics of agents. These fluctuations will typically not be in equilibrium, and therefore it does not seem to be appropriate to assume that the process $\{p_t\}_{t\in\mathbb{N}}$ evolves in a stationary environment. If the "mood" of the market admits a unique invariant measure, the assumption that the asset price process evolves in a stationary environment may be regarded as merely a simplification to circumvent deeper problems related to a non-stationary random environment.

In this subsection we shall consider a slight perturbation of our original model. We show how to obtain for any $\epsilon > 0$ a suitable "ϵ-approximation" of the process Z describing the dynamics of the environment under the assumption that $\{m_t\}_{t\in\mathbb{N}}$ is a *non-stationary* sequence of random variables. Furthermore, we shall verify that the induced sequence $\{Z^{n,\epsilon}\}_{n\in\mathbb{N}}$ of semimartingales is indeed "good". This results may be viewed as a justification for the approach taken in the previous subsection.

To motivate our considerations recall that an invariance principle holds true for a strictly stationary and φ-mixing sequence $\{\xi_t\}_{t\in\mathbb{N}}$ of real valued random variables if, for example, the mixing coefficients satisfy the relation $\varphi(t) = L\alpha^t$ for some $\alpha < 1$; see, e.g. Billingsley (1968), Chapter 20. Furthermore, the functional central limit theorem remains valid if the original probability measure $\mathbb{P}$ governing the sequence $\{\xi_t\}_{t\in\mathbb{N}}$ is replaced by an absolutely continuous probability $\mathbb{P}_0$. However, under $\mathbb{P}_0$ the process $\{\xi_t\}_{t\in\mathbb{N}}$ need no longer be stationary. Techniques related to those used in the proof of Theorem 20.2 in Billingsley (1968) can be applied to establish Theorem 6.3 below.

Let us now consider a slight perturbation of our original model. For notational simplicity we restrict ourselves to almost surely constant external fields. Let $\epsilon > 0$ be given and let $\{\eta_t^f, \eta_t^g\}_{t\in\mathbb{N}}$ be independent and $N(0, \epsilon^2)$ distributed random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P}_m^\epsilon)$ with the sequence $\{m_t\}_{t\in\mathbb{N}}$ and independent of this process. As usual, we we put $\mathbb{P}_{\mu^*}^\epsilon = \int \mathbb{P}_m^\epsilon \mu^*(dm)$, where μ^* denotes the unique invariant probability of the macroscopic process. We replace $f(m_t)$ and $g(m_t)$ by

$$f_t^\epsilon := f(m_t) + \eta_t^f \quad \text{and} \quad g_t^\epsilon := g(m_t) + \eta_t^g \quad (t \in \mathbb{N})$$

respectively and set $\psi_t^\epsilon := (f_t^\epsilon, g_t^\epsilon)$.

Let us start with some preliminaries about the sequence $\psi^\epsilon = \{\psi^\epsilon_t\}_{t\in\mathbb{N}}$. For any pair $(l,n) \in \mathbb{N}^2$ we denote by $K^n_l(m;\cdot)$ the common distribution of the random variables $(\psi^\epsilon_{n+1},\cdots,\psi^\epsilon_{n+l})$ under the measure $\mathbb{P}^\epsilon_m$.

Using the fact that the function f and g are Lipschitz continuous and that the random variables η^f_t, η^g_t $(t \in \mathbb{N})$ are normally distributed, one can easily verify the existence of a constant $C^\epsilon < \infty$ such that

$$\sup_{m\neq m'} \frac{|K^n_l(m;A) - K^n_l(m';A)|}{|m-m'|} \leq C^\epsilon \tag{49}$$

uniformly in $l,n \in \mathbb{N}$ and in $A \in \mathcal{B}(\mathbb{R}^{2l})$; see, e.g., Horst (1999b), Lemma 5.1. Using the Lipschitz continuity of the mappings $m \mapsto K^n_l(m,A)$ $(n,l \in \mathbb{N}, A \in \mathcal{B}(\mathbb{R}^{2l}))$, (39) and a monotone class argument as in Horst (1999b), it is easy to show that there exists a constant $L^\epsilon < \infty$ and a constant $\alpha < 1$ such that

$$|\mathbb{P}^\epsilon_m(A) - \mathbb{P}^\epsilon_{\mu^*}(A)| \leq L^\epsilon \alpha^t \tag{50}$$

for all $A \in \sigma(\{\psi^\epsilon_i : i \geq t\})$. Now we are able to establish some basic mixing properties of the sequence ψ^ϵ.

Definition 6.2 *Let $(X,\mathcal{X})$ be a measurable space and $\zeta = \{\zeta_t\}_{t\in\mathbb{Z}}$ is a doubly infinite sequence of X-valued random variables defined on some probability space $(\Omega,\mathcal{F},\mathbb{P})$. (The usual case of an infinite sequence $\{\zeta_t\}_{t\in\mathbb{N}}$ is obtained by taking $\zeta_k = 0$ for $k < 0$.) The sequence ζ is called **φ-mixing**, if $\varphi(n) \to 0$ as $n \to \infty$, where the mixing coefficients $\varphi(n)$ are defined by*

$$\varphi(n) = \sup_{l\in\mathbb{Z}} \sup |\mathbb{P}(B|A) - \mathbb{P}(B)|.$$

Here the second supremum is taken over all $B \in \sigma(\{\zeta_i : i \leq l\})$ and over all $A \in \sigma(\{\zeta_i : i \geq l+n\})$.

Proposition 6.2 *The sequence ψ^ϵ is φ-mixing both under $\mathbb{P}^\epsilon_m$ and under $\mathbb{P}^\epsilon_{\mu^*}$. The mixing coefficients satisfy the relation*

$$\varphi(n) \leq 2L^\epsilon\alpha^n.$$

Proof: Let us first show that the sequence ψ^ϵ is $\varphi-$mixing in the non-stationary situation, i.e., under the measure $\mathbb{P}^\epsilon_m$. Let $r,l,n \in \mathbb{N}$ and $\psi^{\epsilon,(l)} := (\psi^\epsilon_0,\ldots,\psi^\epsilon_l)$. We have that

$$\begin{aligned}
& \mathbb{P}^\epsilon_m[(\psi^\epsilon_{l+n+1},\ldots,\psi^\epsilon_{l+m+r}) \in A|\psi^{\epsilon,(l)}] \\
= \; & \mathbb{P}^\epsilon_m\left[\mathbb{P}^\epsilon_m[(\psi^\epsilon_{l+n+1},\ldots,\psi^\epsilon_{l+m+r}) \in A|\psi^{\epsilon,(l)}, e^{(l)}]|\psi^{\epsilon,(l)}\right] \\
= \; & \mathbb{P}^\epsilon_m[K^n_r(u(m,e^{(l)});A)|\psi^{\epsilon,(l)}] \\
= \; & K^n_r(u(m,e^{(l)});A)
\end{aligned}$$

by the law of iterated conditional expectations. Therefore we get

$$\begin{aligned}
&\left|\mathbb{P}^\epsilon_m[(\psi^\epsilon_{l+n+1},\ldots,\psi^\epsilon_{l+n+r})\in A]-\mathbb{P}^\epsilon_m[(\psi^\epsilon_{l+n+1},\ldots,\psi^\epsilon_{l+n+r})\in A|\psi^{\epsilon,(l)}]\right|\\
\le\;& |U^nK^0_r(\cdot;A)-\mu^*(K^0_r(\cdot;A))|+|U^{n+l}K^0_r(\cdot;A)-\mu^*(K^0_r(\cdot;A))|\\
\le\;& 2L^\epsilon\alpha^n \quad \mathbb{P}^\epsilon_m-a.s.
\end{aligned}$$

for some constant $L^\epsilon<\infty$. Here, the last but one inequality follows from (49) and (50). From this it is easily seen that

$$|\mathbb{P}^\epsilon_m(B|A)-\mathbb{P}^\epsilon_m(B)|\le 2L^\epsilon\alpha^n \tag{51}$$

for all $B\in\sigma(\{\psi^\epsilon_i: i\ge l+n\})$ and all $A\in\sigma(\{\psi^\epsilon_i: i\le l\})$ satisfying $\mathbb{P}^\epsilon_m(A)>0$. This establishes $\varphi(n)\le 2L^\epsilon\alpha^n$ in the non-stationary situation. The mixing property under the measure $\mathbb{P}_{\mu^*}$ follows immediately from (51) and from the definition of $\mathbb{P}^\epsilon_{\mu^*}$. □

Corollary 6.2 *The sequences $\{f^\epsilon_t\}_{t\in\mathbb{N}}$, $\{g^\epsilon_t\}_{t\in\mathbb{N}}$, and $\{f^\epsilon_t+g^\epsilon_t\}_{t\in\mathbb{N}}$ are $\varphi-$mixing under both $\mathbb{P}^\epsilon_m$ and $\mathbb{P}^\epsilon_{\mu^*}$. The respective $n-$th mixing coefficient $\varphi^f(n)$, $\varphi^g(n)$ and $\varphi^{f+g}(n)$ are bounded above by $2L^\epsilon\alpha^n$.*

Proof: The assertion follows immediately from Proposition 6.1 and from the definition of the mixing coefficients. □

Before we state and prove the main result of this subsection we introduce some notation. We let $X^{n,\epsilon}=\{X^{n,\epsilon}_t\}_{t\ge0}$ and $Y^{n,\epsilon}=\{Y^{n,\epsilon}_t\}_{t\ge0}$ be defined by

$$X^{n,\epsilon}_t:=\frac{1}{\sqrt{n}}\sum_{i=0}^{[nt]}\left\{f^\epsilon_i-\mu_f+\frac{\mu_f}{\sqrt{n}}\right\},\quad Y^{n,\epsilon}_t:=\frac{1}{\sqrt{n}}\sum_{i=0}^{[nt]}\left\{g^\epsilon_i-\mu_g+\frac{\mu_g}{\sqrt{n}}\right\}$$

and put

$$\mu_f:=\mathbb{E}_{\mu^*}f(m_0)=\mu^*(f),\quad \mu_g:=\mathbb{E}_{\mu^*}g(m_0)=\mu^*(g),$$
$$f^n_t:=\frac{1}{\sqrt{n}}\sum_{k=0}^{[nt]}(f(m_k)-\mu_f),\quad g^n_t:=\frac{1}{\sqrt{n}}\sum_{k=0}^{[nt]}(g(m_k)-\mu_g).$$

The proof of the following theorem is based on the mixing properties of the process ψ^ϵ, which allows us to apply an invariance principle for φ-mixing sequences.

Theorem 6.3 *Suppose that the macroscopic process is regular and that the functions f and g are Lipschitz continuous. For each $\epsilon>0$ the sequence of processes $\{Z^{n,\epsilon}\}_{n\in\mathbb{N}}=\{(X^{n,\epsilon},Y^{n,\epsilon})\}_{n\in\mathbb{N}}$ converges in distribution to a continuous diffusion process with deterministic drift vector $B=(\mu_f,\mu_g)$ and with volatility matrix C^ϵ, where*

$$C^\epsilon:=\begin{pmatrix}\sqrt{\sigma_f^2+\epsilon^2} & \sigma_{fg}\\ \sigma_{fg} & \sqrt{\sigma_g^2+\epsilon^2}\end{pmatrix}.$$

Here, σ_f,σ_g and σ_{fg} are defined as in the previous subsection.

Proof: Let $\epsilon > 0$ be given. First, we consider the case $\mu_f = \mu_g = 0$. According to Corollary 6.2 we have that

$$\sum_{n\in\mathbb{N}} \sqrt{\varphi^f(n)} < \infty, \quad \sum_{n\in\mathbb{N}} \sqrt{\varphi^g(n)} < \infty, \quad \sum_{n\in\mathbb{N}} \sqrt{\varphi^{f+g}(n)} < \infty.$$

Step 1: Let us first analyse the stationary situation. Due to the above mixing properties we can use a functional limit theorem for strictly stationary and φ-mixing sequences. By Corollary VIII 3.106 in Jacod and Shiryayev (1987), the respective sequences in (38), (39) and (40) with $f(m_t)$ and $g(m_t)$ replaced by $f(m_t) + \eta_t^f$ and $g(m_t) + \eta_t^g$ respectively, are absolutely convergent. Therefore

$$Law(X^{n,\epsilon}, \mathbb{P}^\epsilon_{\mu^*}) \xrightarrow{w} Law(\sigma_f^\epsilon W_f, \mathbb{P}^\epsilon_{\mu^*}) \text{ and } \quad Law(Y^{n,\epsilon}, \mathbb{P}^\epsilon_{\mu^*}) \xrightarrow{w} Law(\sigma_g^\epsilon W_g, \mathbb{P}^\epsilon_{\mu^*})$$

where W_f, W_g are standard Brownian motions and

$$\begin{aligned}(\sigma_f^\epsilon)^2 &= \mathbb{E}^\epsilon_{\mu^*}[f(m_0) + \epsilon_0] + 2\sum_{t>0} \mathbb{E}^\epsilon_{\mu^*}[(f(m_t) + \epsilon_t)(f(m_0) + \epsilon_0)] = \sigma_f^2 + \epsilon^2, \\ (\sigma_g^\epsilon)^2 &= \sigma_g^2 + \epsilon^2.\end{aligned}$$

Using the same arguments as in the proof of Theorem 6.1 it is easily seen that $Law(Z^{n,\epsilon}, \mathbb{P}^\epsilon_{\mu^*}) \xrightarrow{w} Law(Z^\epsilon, \mathbb{P}^\epsilon_{\mu^*})$ as $n \to \infty$. Here $Z^\epsilon := C^\epsilon W$ and $Z^{n,\epsilon} = (X^{n,\epsilon}, Y^{n,\epsilon})$.

Step 2: In a second step we will now show that the sequence $\{Z^{n,\epsilon}\}_{n\in\mathbb{N}}$ satisfies an invariance principle under the original measure $\mathbb{P}^\epsilon_m$ and that

$$Law(Z^{n,\epsilon}, \mathbb{P}^\epsilon_m) \xrightarrow{w} Law(Z^\epsilon, \mathbb{P}^\epsilon_{\mu^*}).$$

To this end, let $\{\beta_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $\beta_n \uparrow \infty$ and $\beta_n/\sqrt{n} \to 0$ as $n \to \infty$. For a given "time horizon" T and for each $n \in \mathbb{N}$ we introduce the two-dimensional process

$$\widetilde{Z}_t^{n,\epsilon} := \begin{cases} \frac{1}{\sqrt{n}} \sum_{i=\beta_n}^{[nt]} (f_i^\epsilon, g_i^\epsilon) & \text{if } \frac{\beta_n}{\sqrt{n}} \le t \le T \\ 0 & \text{otherwise.} \end{cases}$$

We denote by d_0 the Skorohod metric[2] on the space $\mathbb{D}_\mathbb{R}[0,T]$. Note that

$$d_0(Z^{n,\epsilon}, \widetilde{Z}^{n,\epsilon}) \le \frac{\beta_n}{\sqrt{n}} \left\| \left(\frac{1}{\beta_n} \sum_{i=0}^{\beta_n} |f(m_i) + \eta_i^f|, \frac{1}{\beta_n} \sum_{i=0}^{\beta_n} |g(m_i) + \eta_i^g| \right) \right\|.$$

As f and g are bounded and because η_i^f and η_i^g ($i \in \mathbb{N}$) are independent and normally distributed random variables, the terms

$$\frac{1}{\beta_n} \sum_{i=0}^{\beta_n} |f(m_i) + \eta_i^f| \quad \text{and} \quad \frac{1}{\beta_n} \sum_{i=0}^{\beta_n} |g(m_i) + \eta_i^g|$$

[2]For the definition of d_0 see, e.g.,Billingsley (1968),p. 113.

are $\mathbb{P}^\epsilon_m$- and $\mathbb{P}^\epsilon_{\mu^*}-$ almost surely convergent, due to the strong law of large numbers. Since $\frac{\beta_n}{\sqrt{n}} \to 0$ as $n \to \infty$ we obtain

$$\lim_{n\to\infty} d_0(Z^{n,\epsilon}, \widetilde{Z}^{n,\epsilon}) = 0 \quad \mathbb{P}^\epsilon_m - \text{ and } \mathbb{P}^\epsilon_{\mu^*} - \text{almost surely.} \tag{52}$$

Observe now that the event $\{\widetilde{Z}^{n,\epsilon} \in A\}$, $A \in \mathcal{B}(\mathbb{D}_{\mathbb{R}^2}[0,T])$, belongs to the $\sigma-$algebra $\sigma(\{\psi^\epsilon_i : i \geq \beta_n\})$. Thus from (50) we deduce that

$$|\mathbb{P}^\epsilon_m[\{\widetilde{Z}^{n,\epsilon} \in A\}] - \mathbb{P}^\epsilon_{\mu^*}[\{\widetilde{Z}^{n,\epsilon} \in A\}] \leq L^\epsilon \alpha^{\beta_n}. \tag{53}$$

Let us denote by P^* the law of Z^ϵ under the measure $\mathbb{P}^\epsilon_{\mu^*}$. By Step 1 above we know that

$$\lim_{n\to\infty} \mathbb{P}^\epsilon_{\mu^*}[\{Z^{n,\epsilon} \in A\}] = P^*(A) \text{ for any } P^*\text{-continuous set } A.$$

Thus, by (52) and by Theorem 4.2 in Billingsley (1968) we have that

$$\lim_{n\to\infty} \mathbb{P}^\epsilon_{\mu^*}[\{\widetilde{Z}^{n,\epsilon} \in A\}] = P^*(A) \quad \text{ for any } P^* - \text{continuous set } A.$$

Using (53) and Theorem 4.2 in Billingsley (1968) again, we get

$$\lim_{n\to\infty} \mathbb{P}^\epsilon_m[\{\widetilde{Z}^{n,\epsilon} \in A\}] = P^*(A)$$

for any P^*-continuous set A. Therefore (52) implies that

$$Law(Z^{n,\epsilon}, \mathbb{P}^\epsilon_m) \xrightarrow{w} Law(Z^\epsilon, \mathbb{P}^\epsilon_{\mu^*}) \quad (n \to \infty).$$

The general case $\mu_f, \mu_g \in \mathbb{R}$ follows immediately as in Corollary 6.1. □

In the previous subsection we assumed that the sequence $\{Z^n\}_{n\in\mathbb{N}}$ was good. Due to the mixing properties of the sequence ψ^ϵ we can now *verify* that for any given $\epsilon > 0$ the sequence $\{Z^{n,\epsilon}\}_{n\in\mathbb{N}}$ is in fact "good".

Theorem 6.4 *For any $\epsilon > 0$ the sequence $\{Z^{n,\epsilon}\}_{n\in\mathbb{N}}$ of semimartingales is "good"in the sense of Duffie and Protter (1992).*

Proof: Let us first consider the case $\mu_f = \mu_g = 0$. Recall that the sequence $\psi^\epsilon = \{(f^\epsilon_t, g^\epsilon_t)\}_{t\in\mathbb{R}}$ is φ-mixing under the measure $\mathbb{P}^\epsilon_m$ and that the mixing coefficients satisfy

$$\varphi(n) \leq L^\epsilon \alpha^t \quad \text{ for some } \alpha < 1, L^\epsilon < \infty. \tag{54}$$

We put $\mathcal{F}_t = \sigma(\{f^\epsilon_i, g^\epsilon_i\} : i \leq t)$ In order to show that the sequence $\{Z^{n,\epsilon}\}$ is good, we have to find a suitable semimartingale decomposition of $Z^{n,\epsilon}$. Following Ethier and Kurtz (1986) and Duffie and Protter (1992), we define

$$\overline{M}^{n,\epsilon}_t = \sum_{k=0}^{t} f^\epsilon_i + \sum_{k=0}^{\infty} \mathbb{E}^\epsilon_{\mu^*}[f^\epsilon_{t+k}|\mathcal{F}_t], \tag{55}$$

$$\widetilde{M}^{n,\epsilon}_t = \sum_{k=0}^{t} g^\epsilon_i + \sum_{k=0}^{\infty} \mathbb{E}^\epsilon_{\mu^*}[g^\epsilon_{t+k}|\mathcal{F}_t]. \tag{56}$$

The series on the right hand side in (55) and in (56) are convergent as a result of the mixing property (54), see, e.g., Ethier and Kurtz (1986), p. 351. Furthermore, $M^{n,\epsilon} = (\overline{M}^{n,\epsilon}, \widetilde{M}^{n,\epsilon})$ is a vector of square integrable martingales. With

$$A^{n,\epsilon} = \{A^{\epsilon}_{[nt]}\}_{t\in\mathbb{N}}, \quad A^{n,\epsilon}_t = \frac{1}{\sqrt{n}} \begin{pmatrix} \sum_{k=0}^{\infty} \mathbb{E}^{\epsilon}_{\mu^*}[f^{\epsilon}_{k+t}|\mathcal{F}_t] \\ \sum_{k=0}^{\infty} \mathbb{E}^{\epsilon}_{\mu^*}[g^{\epsilon}_{k+t}|\mathcal{F}_t] \end{pmatrix}$$

we consider the following decomposition:

$$Z^{n,\epsilon}_t = \frac{1}{\sqrt{n}} M^{n,\epsilon}_{[nt]} + \frac{1}{\sqrt{n}} A^{n,\epsilon}_t.$$

According to Theorem 4.3 in Duffie and Protter (1992) the sequence $\{Z^{n,\epsilon}\}_{n\in\mathbb{N}}$ is good as soon as the following condition is satisfied ("Condition B"):

$$\sup_{n\in\mathbb{N}}\{\mathbb{E}^{\epsilon}_{m,h}[\sup_{t\leq T}|\Delta M^n_t|]\} < \infty \quad \text{and} \quad \sup_{n\in\mathbb{N}}\{\mathbb{E}^{\epsilon}_{m,h}[|A^{n,\epsilon}|_T]\} < \infty.$$

Obviously, the martingale M^n has uniformly bounded expected jumps. Using standard estimates provided in Duffie and Protter (1992) it follows that

$$\sup_n \mathbb{E}^{\epsilon}_{\mu^*}[|\bar{A}^{n,\epsilon}|_T] < \infty, \quad \sup_n \mathbb{E}^{\epsilon}_{\mu^*}[|\tilde{A}^{n,\epsilon}|_T] < \infty.$$

Thus "Condition B" is satisfied and the sequence $\{Z^{n,\epsilon}\}$ is good. The general case $\mu_f, \mu_g \in \mathbb{R}$ follows easily from the above calculations. □

Corollary 6.3 *Theorem 6.2 holds true in the non-stationary setting and with our original model replaced by a "perturbed one".*

Remark 6.3 *It is not clear to us which conditions would ensure goodness of the sequence $\{Z^{n,0}\}$. The estimate provided in Duffie and Protter (1992) we used to verify that the process $\{A^{n,\epsilon}\}$ has finite variation on compact time intervals for each fixed $\epsilon > 0$ leads to*

$$\lim_{\epsilon\to 0} \mathbb{E}^{\epsilon}_{m,h}[|A^{n,\epsilon}|_T] = \infty,$$

as the constant L^{ϵ} in the proof of Theorem 6.3 tends to infinity as $\epsilon \to 0$. A condition like

$$\sup_{\epsilon} \mathbb{E}^{\epsilon}_{m,h}[|A^{n,\epsilon}|_T] < \infty$$

would imply goodness.

7 Numerical Simulations

This section is devoted to some numerical simulations of the model developed in Section 2. According to Theorem 3.1 we have sufficient conditions for the distributions of the empirical average to converge weakly to a unique equilibrium μ. Nevertheless, our simulations show that this process may exhibit large and sudden fluctuations ("phase transitions"), and that these fluctuations do not reflect rational adjustments to new economic fundamentals. They are due to a distinct herd behaviour. Small changes in the external conditions lead to large and sudden price overreactions, i.e., to bubbles or crashes. Let us consider the mean variance setting as in Example 2.2.

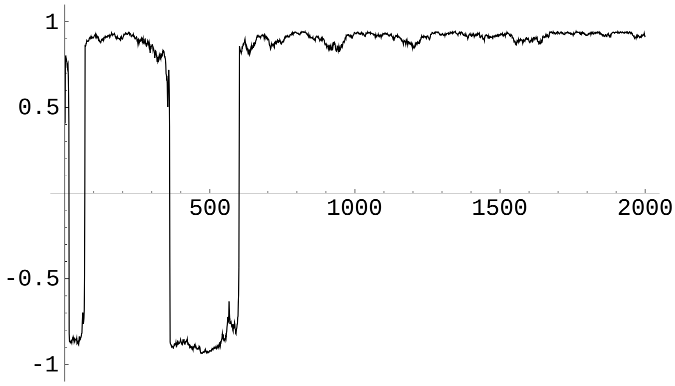

Figure 1: Evolution of the empirical average; less volatile external field

Suppose that only optimistic and pessimistic information trader are active on the market and that for a proposed stock price p the individual expectations are of the form

$$\hat{p}_t^a = p + c(F + x_t^a - p), \quad c > 0.$$

Here F denotes some fixed, say long run, fundamental value of the asset. Thus, the equilibrium price p_t^* at time t is given by

$$p_t^* = \frac{1}{r+c}(F + m_t).$$

The signal sequence $\{e_t\}_{t=1,2,...}$ is defined by

$$e_{t+1} = m_t + \epsilon_t$$

where $\{\epsilon_t\}_{t\in\mathbb{N}}$ is an *i.i.d.* sequence of normally distributed random variables with zero mean and volatility σ_e. We specify the conditional choice probabilities as

$$\pi(x_{t+1}^a = \pm 1 | e_{t+1}, h_t, x_t) = \frac{\exp\{(\pm\beta(Je_{t+1} + Tx_t^a + Kh_t))\}}{2\cosh\{(\pm\beta(Je_{t+1} + Tx_t^a + Kh_t))\}}$$

where $J = 1, T = K = \frac{1}{4}, \beta = \frac{3}{4}$. The (stochastic) external field field $\{h_t\}_{t\in\mathbb{N}}$ is modelled by a random walk:

$$h_{t+1} = h_t + \eta_t.$$

Here $\{\eta_t\}_{t\in\mathbb{N}}$ are independent and normally distributed with volatility σ_h. It is easily seen that the dynamics of the empirical average is given by

$$\begin{aligned} m_{t+1} &= u(m_t, e_{t+1}, h_t) \\ &= \frac{m_t+1}{2}\tanh(\beta(Je_{t+1}+T+Kh_t)) + \frac{1-m_t}{2}\tanh(\beta(Je_{t+1}-T+Kh_t)). \end{aligned}$$

One can easily show that u satisfies a contraction condition for the above parameters. Figure 1 shows the evolution of the empirical average if σ_h is small. Figure 4 shows

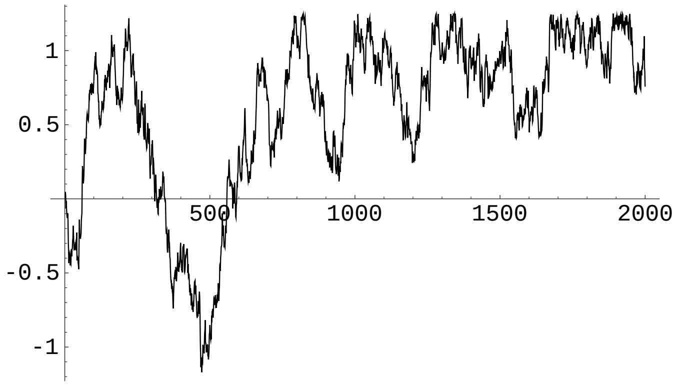

Figure 2: Economic fundamentals

the dynamics for a higher value of σ_h.

These simulations suggest that the equilibrium distribution μ is concentrated around the points ± 0.9. To clarify this point let us assume for the moment that $e_{t+1} = m_t$ and $h_t = h$ a.s. The mapping

$$m \mapsto u(m, m, h)$$

has either one or two stable fixed points, depending on the value h. There exist constants $h^-, h^+ \in \mathbb{R}$ such that for $h \in (h^-, h^+)$ it has two fixed points; otherwise it has just one. Suppose it has two: $m^- < 0 < m^+$. Depending on the initial condition m_t either tends to m^- or to m^+, say to m^+. Suppose now that h_t becomes stochastic. Then m_t fluctuates around m^+ as long as $h_t > h^-$. For $h_t < h^-$ m_t converges to m^- and fluctuates around this point as long as $h_t < h^+$ etc..

The economic interpretation is rather clear. If the market is sufficiently enthusiastic, negative economic fundamentals are neglected. If the external conditions become too negative, a sudden price overreaction occurs.

In case of a stochastic signal process investors may get a "wrong" signal about the aggregate behaviour which forces them to change their individual expectations. As a result, m_t may belong to the attractor of the fixed point m^+ whereas m_{t+1} belongs to the one of m^-. Hence, a crash may occur without any changes in the economic fundamentals.

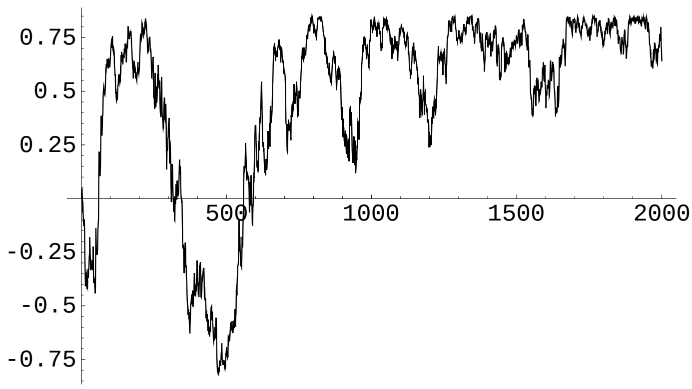

Figure 3: Evolution of the empirical average for J=0

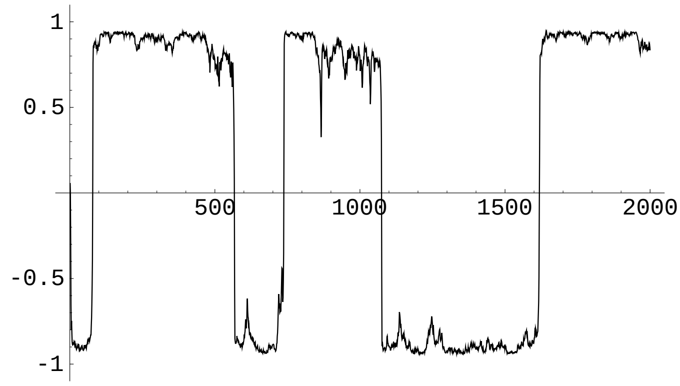

Figure 4: Evolution of the empirical average, highly volatile external field

8 Conclusion

We analysed a simple financial market model where the process of temporary equilibrium prices evolves in a random environment generated by interaction of agents. We have seen how distinct herd behaviour may generate large fluctuations of temporary equilibrium prices. Small changes in economic fundamentals may lead to sudden price overreactions. This may be viewed as the appearance of bubbles and crashes in the context of a financial market model whose overall behaviour is ergodic. In our mean field model, the macroscopic process has a u-shaped rather than a bell-shaped equilibrium distribution. The empirical average would converge to one of possibly several fixed points if the signals are deterministic and external conditions are constant. In case of stochastic signals, from time to time the process is "thrown" into the basin of attraction of different fixed points . Such "phase-transitions" will also appear in more general models, see (Horst 1999).

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