Asymptotic equivalence of discretely observed geometric Brownian motion to a Gaussian shift

Cristina Butucea, Michael Nussbaum
SFB 373, Humboldt-Universität zu Berlin,
Weierstrass Institute Berlin, Germany

Abstract

Financial models consider often stochastic processes satisfying certain differential equations. We show that the solution of a particular geometric Brownian motion observed in discrete time is asymptotically equivalent with a Gaussian white noise model.

1 Introduction

In finance, stock prices are modelled by geometric Brownian motion since the well known Black-Scholes model, classically written as follows

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where $\{W_t, t \in [0,1]\}$ is a Brownian motion, with appreciation rate $\mu \in I\!R$ and volatility $\sigma > 0$. The process S_t describes the evolution of the stock price at continuous time t.

This model is further used in computing the price of derivative securities (e.g. option valuation). Thus, a good knowledge of the volatility structure, in particular, is needed in the analysis of financial analysis and forecasting. For further details of financial calculations we refer to Musiela and Rutkowski [4].

We also put an accent here on the study of the time-dependent volatility. The following geometric Brownian motion is considered

$$dS_t = \sigma(t) S_t dW_t, \tag{1}$$

where the volatility function σ is time-dependent, $t \in [0,1]$, belongs to some set H and is bounded away from 0 ($\sigma(t) \geq \varepsilon > 0$ for all t in $[0,1]$). Without loss of generality we assume that the initial value is $S_0 = 1$.

Consider $Y_t = \log S_t$ and substitute in equation (1). The volatility is a continuous function and then a solution to this last equation is known to satisfy

$$dY_t = -\frac{\sigma^2(t)}{2} dt + \sigma(t) dW_t. \tag{2}$$

Remark that, from a practical point of vue, the continuous time model (2) is obviously unrealistic. We have at our disposal discrete time observations. Therefore, we consider discrete observations of the process Y_t in (2) at times $t_i = i/n$, for $i = 1, \ldots, n$ and the following normalized independent increments

$$Z_i = \sqrt{n}\left(Y_{t_i} - Y_{t_{i-1}}\right) = -\frac{\sqrt{n}}{2}\int_{t_{i-1}}^{t_i} \sigma^2(u)\,du + \sqrt{n}\int_{t_{i-1}}^{t_i} \sigma(u)\,dW_u. \qquad (3)$$

We show that the statistical experiment (as defined later on) induced by observations in (3) has the same asymptotic behavior as the following white noise shift model:

$$dX_t = \frac{1}{\sqrt{2}}\log\sigma(t)\,dt + \frac{1}{\sqrt{n}}dW_t, \qquad (4)$$

with $t \in [0,1]$.

The approach is in the sense of Le Cam theory of asymptotic equivalence of statistical experiments, in a nonparametric setup. We assume that the volatility belongs to a Hölder class of functions, $H(\beta, L)$, with $\beta \in (1/2, 1]$ of functions that are bounded away from 0:

$$H(\beta, L) = \left\{\sigma : [0,1] \to [\varepsilon, \infty) \;\middle|\; \varepsilon > 0,\ |\sigma(x) - \sigma(y)| \leq L\,|x-y|^{\beta}\right\}.$$

Note that we prove the asymptotic equivalence without indicating constructive procedures (Markov kernels) which allow to pass from one model to the other (see Nussbaum and Klemelä [6] for such results).

In this paper, we obtain the result that the geometric Brownian motion (1), observed over a regular grid is asymptotically equivalent with the white noise model drifted observations in (4), in the sense of Le Cam's deficiency measure tending to 0 with $n \to \infty$. In Section 2 we introduce briefly the asymptotic equivalence notion and the main tools in our proofs. Results are stated and proven in several consecutive steps in Section 3.

2 Notation, definitions

Let $E_{i,n} = \{\Omega, \mathcal{A}, (P_{i,n} = P_{i,n}(\sigma) : \sigma \in H)\}$, $i = 1, 2$ be two statistical experiments defined on the same probability space $(\Omega, \mathcal{A})$ and indexed by the same parameter space.

In order to evaluate how close the two experiments are, we consider Le Cam deficiency measure or the Δ-pseudodistance. The experiments $E_{i,n}$ are said to be asymptotically equivalent if this distance, $\Delta(E_{1,n}, E_{2,n})$, tends to 0 with n. For more details about the asymptotic equivalence theory we refer to Le Cam and Yang [3], Nussbaum [5]. In the following, we describe upper bounds of this distance by means of Hellinger distance.

Assume that there exists σ_0 in H such that the associated probability measure dominates all other probability measures of the same experiment. If the likelihood processes $\Lambda_{i,n}(\sigma) = dP_{i,n}(\sigma)/dP_{i,n}(\sigma_0)$ are defined on the same probability space (or have such versions), then the Δ-distance satisfies

$$\Delta(E_{1,n}, E_{2,n}) \leq \frac{1}{2}\sup_{\sigma\in H} E\left|\Lambda_{1,n}(\sigma) - \Lambda_{2,n}(\sigma)\right|.$$

This upper bound can be expressed by means of Hellinger distance as follows. The Hellinger distance between two likelihood processes is by definition

$$H^2(\Lambda_{1,n}, \Lambda_{2,n}) = \frac{1}{2}E\left(\sqrt{\Lambda_{1,n}} - \sqrt{\Lambda_{2,n}}\right)^2.$$

Then we deduce easily that

$$\Delta(E_{1,n}, E_{2,n}) \leq \sup_{\sigma\in H} H(\Lambda_{1,n}, \Lambda_{2,n}). \tag{5}$$

In particular, we use the following formulas, where $N(\mu, \sigma^2)$ denotes the Gaussian law with mean μ and variance σ^2,

$$H^2\left(N\left(\mu_1, \sigma^2\right), N\left(\mu_2, \sigma^2\right)\right) \leq \frac{(\mu_1 - \mu_2)^2}{4\sigma^2}, \tag{6}$$

$$H^2\left(N\left(\mu, \sigma_1^2\right), N\left(\mu, \sigma_2^2\right)\right) \leq 2\frac{(\sigma_1 - \sigma_2)^2}{\sigma_1^2 + \sigma_2^2}. \tag{7}$$

3 Equivalence results

Let P_n^Y and P_n^X denote the distribution of $\{Y_i = Y_{t_i}\}_{i=1,\ldots,n}$ in (2) and $\{X_t(n)\}_{t\in[0,1]}$ in (4), respectively, and $E_n^Y = \left\{[0,1]^n, \mathcal{B}^n_{[0,1]}, \left(P_n^Y(\sigma), \sigma \in H(\beta, L)\right)\right\}$, $E_n^X = \left\{C_{[0,1]}, \mathcal{B}_{C_{[0,1]}}, \left(P_n^X(\sigma), \sigma \in H(\beta, L)\right)\right\}$ the associated experiments. We prove that these experiments are equivalent in a few steps.

Consider the following random variables

$$Z_i^* = -\frac{\sigma^2(t_i)}{2\sqrt{n}} + \sigma(t_i)\,\xi_i, \tag{8}$$

where ξ_i are independent, having standard Gaussian law.

Lemma 1 *If $E_{1,n}$ is the experiment generated by $\{Z_i\}_{i=1,\ldots,n}$ in (3) and $E_{1,n}^*$ by $\{Z_i^*\}_{i=1,\ldots,n}$ in (8), then these experiments are asymptotically equivalent, i.e.*

$$\lim_{n\to\infty} \Delta\left(E_{1,n}, E_{1,n}^*\right) = 0.$$

Proof. Let us remark that observations Z_i and Z_i^* in (3) and (8) are independent and have Gaussian law,

$$P_i : N\left(-\frac{h_i(\sigma^2)}{2\sqrt{n}}, h_i(\sigma^2)\right), \text{ where } h_i(\sigma^2) = n\int_{t_{i-1}}^{t_i} \sigma^2(u)\, du,$$

respectively,

$$P_i^* : N\left(-\frac{\sigma^2(t_i)}{2\sqrt{n}}, \sigma^2(t_i)\right).$$

In order to prove that $\Delta\left(E_{1,n}, E_{1,n}^*\right) \to 0$, when $n \to \infty$, it suffices by (5) to prove that $H^2\left(P_{1,n}, P_{1,n}^*\right) \to 0$, as $n \to \infty$, where $P_{1,n} = \bigotimes_{i=1}^n P_i$ and $P_{1,n}^* = \bigotimes_{i=1}^n P_i^*$. We have

$$\begin{aligned} H^2\left(P_{1,n}, P_{1,n}^*\right) &\leq 2\sum_{i=1}^n H^2\left(P_i, P_i^*\right) \\ &\leq 2\sum_{i=1}^n \left[H^2\left(P_i, \widetilde{P}_i\right) + H^2\left(\widetilde{P}_i, P_i^*\right)\right], \end{aligned}$$

where $\widetilde{P}_i$ denotes the law of the following independent random variables

$$\widetilde{Z}_i = -\frac{h_i(\sigma^2)}{2\sqrt{n}} + \sigma(t_i)\,\xi_i.$$

Using formulas (6) and (7), we get

$$\begin{aligned} H^2\left(P_{1,n}, P_{1,n}^*\right) &\leq 2\sum_{i=1}^n \left[2\frac{\left(\sqrt{h_i(\sigma^2)} - \sigma(t_i)\right)^2}{h_i(\sigma^2) + \sigma^2(t_i)} + \frac{1}{4\sigma^2(t_i)}\frac{\left(h_i(\sigma^2) - \sigma^2(t_i)\right)^2}{4n}\right] \\ &\leq \sum_{i=1}^n \left(h_i(\sigma^2) - \sigma^2(t_i)\right)^2 \left[\frac{1}{\varepsilon^2} + \frac{1}{8n\varepsilon^2}\right]. \end{aligned}$$

Remark that for the continuous function σ there exists a value $u_i \in [t_{i-1}, t_i]$ such that $h_i(\sigma^2) = \sigma^2(u_i)$ and then

$$\begin{aligned} \left|h_i(\sigma^2) - \sigma^2(t_i)\right| &= \left|\sigma^2(u_i) - \sigma^2(t_i)\right| \\ &\leq 2B(\beta, L)\left|u_i - t_i\right|^\beta = O(1)\, n^{-\beta}, \end{aligned}$$

where $B(\beta, L)$ is an upper bound of functions σ in $H(\beta, L)$. Finally,

$$H^2\left(P_{1,n}, P_{1,n}^*\right) \leq n^{1-2\beta}(1 + o(1)),$$

which tends to 0 for all $\beta > 1/2$. ■

Remark 1 *In addition to the main result we notice the following facts. Similarly, let us consider the process* M_t *satisfying*

$$dM_t = \sigma(t)\, dW_t$$

and discrete observations of this process at times $t_i = i/n$, *for* $i = 1, \ldots, n$, *together with*

$$U_i = \sqrt{n}\left(M_{t_i} - M_{t_{i-1}}\right) = \sqrt{n}\int_{t_{i-1}}^{t_i} \sigma(u)\, dW_u. \tag{9}$$

Analogously to Lemma 1, *it is stated in the following Lemma that the Gaussian random variables*

$$U_i^* = \sigma(t_i)\, \xi_i \tag{10}$$

introduce an equivalent experiment to (9). *Via the same steps as for the observations* Z_i^* *we obtain in our Theorem the equivalence of* (10) *to observations* X_i *in* (4). *This proves the fact that the drift part of the discrete observations* Y_i *in* (2) *is asymptotically non significant.*

Lemma 2 *If* $E_{2,n}$ *is the experiment associated to* $\{U_i\}_{i=1,\ldots,n}$ *in* (9) *and* $E_{2,n}^*$ *to* $\{U_i^*\}_{i=1,\ldots,n}$ *in* (10) *then*

$$\lim_{n\to\infty} \Delta\left(E_{2,n}, E_{2,n}^*\right) = 0.$$

Proof. The proof goes similarly to that of Lemma 1, by writing the laws of $\{U_i\}_{i=1,\ldots,n}$ and $\{U_i^*\}_{i=1,\ldots,n}$, $P_{2,n} : \bigotimes_{i=1}^n N(0, h_i(\sigma^2))$ and $P_{2,n}^* : \bigotimes_{i=1}^n N(0, \sigma^2(t_i))$, respectively. By inequalities (6) and (7) on the Hellinger distance and using the Hölder property of the volatility function σ, we conclude immediately. ■

The following result states that the solution Y_t of the geometric Brownian motion, discretely observed and the Gaussian drifted model X_t are equivalent. Moreover, the non random expectation term in (8) is asymptotically non informative and all the information can be 'extracted' from the variance term, in model (10).

Theorem 1 *The experiments* E_n^Y *and* E_n^X *associated to* $\{Y_i\}_{i=1,\ldots,n}$ *in* (3) *and* $\{X_t(n)\}_{t\in[0,1]}$ *in* (4), *respectively, are asymptotically equivalent, i.e.*

$$\lim_{n\to\infty} \Delta\left(E_n^Y, E_n^X\right) = 0.$$

Moreover, the drift part of observations $\{Y_i\}_{i=1,\ldots,n}$ *in* (3) *is asymptotically non significant in this model.*

Proof. We use several times the transitivity property of the asymptotic equivalence property. It is sufficient to prove that $E_{1,n}^{*}$ (see Lemma 1), respectively $E_{2,n}^{*}$ (see Lemma 2) are equivalent with the same global experiment E_n^X.

Let us consider first $E_{2,n}^{*} = \left\{[0,1]^n, \mathcal{B}_{[0,1]}^n, \left(P_{2,n}^{*}(\sigma), \sigma \in H(\beta, L)\right)\right\}$ and remark that $P_{2,n}^{*} = \bigotimes_{i=1}^n P_{\sigma(t_i)}^2$ can be written as a product measure of distributions from an exponential family having parameters $\sigma(t_i)$. By following the same steps as in Grama and Nussbaum [2] (see Example 4.2, p 189), we deduce that this model is equivalent to

$$Z_i^{**} = \frac{\log \sigma(t_i)}{\sqrt{2}} + \varepsilon_i, \tag{11}$$

where $\sigma \in H(\beta, L)$ and ε_i are i.i.d. standard normal distributed.

The proof that $E_{1,n}^{*} = \left\{[0,1]^n, \mathcal{B}_{[0,1]}^n, \left(P_{1,n}^{*}(\sigma), \sigma \in H(\beta, L)\right)\right\}$ with $P_{1,n}^{*}(\sigma) = \bigotimes_{i=1}^n P_{\sigma(t_i)}^1$ is asymptotically equivalent to (11) follows the same arguments slightly differently, as given below.

The final step is immediate and consists of applying Brown and Low [1] in order to deduce that the experiment induced by observations Z_i^{**} is is asymptoticaly equivalent to E_n^X in (4).These facts finish the proof of our theorem.

The global experiment $E_{1,n}^{*}$ has to be reduced to the local experiment of the same observations with $\sigma \in H_{\sigma_0,\gamma_n}(\beta, L)$, a shrinking neighborhood of σ_0. For a fixed function σ_0 in $H(\beta, L)$ define

$$H_{\sigma_0,\gamma_n}(\beta, L) = \left\{\sigma \in H(\beta, L) : \|\sigma - \sigma_0\|_\infty \le \gamma_n\right\},$$

where $\gamma_n = (n/\log n)^{-\beta/(2\beta+1)}$.

Observations Z_i^{*}, (8), have a normal density written as an exponential model as follows

$$\begin{aligned} p_i(x) &= \frac{1}{\sigma(t_i)\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2(t_i)}\left(x + \frac{\sigma^2(t_i)}{2\sqrt{n}}\right)^2\right\} \\ &= \exp\left\{\lambda_i \frac{x^2}{2} - \frac{x}{2\sqrt{n}} + \frac{1}{8\lambda_i n} + \frac{1}{2}\log\left(-\frac{\lambda_i}{2\pi}\right)\right\}, \end{aligned}$$

where $\lambda_i = -1/\sigma^2(t_i)$. We write this exponential model as

$$p_i(x) = \exp\left\{\lambda_i U(x) - V_n(\lambda_i)\right\} \exp\left\{-\frac{x}{2\sqrt{n}}\right\},$$

where $U(x) = x^2/2$ and

$$\begin{aligned} V_n(\lambda_i) &= -\frac{1}{2}\log\left(-\frac{\lambda_i}{2\pi}\right) - \frac{1}{8\lambda_i n}, \\ V_n'(\lambda_i) &= -\frac{1}{2\lambda_i} + \frac{1}{8\lambda_i^2 n}, \\ V_n''(\lambda_i) &= \frac{1}{2\lambda_i^2}\left(1 - \frac{1}{2\lambda_i n}\right) = I_n(\lambda_i). \end{aligned}$$

Via Theorem 3.2 in Grama and Nussbaum [2], the local $E_{1,n}^{*}$ is equivalent to the discrete Gaussian experiment:

$$Z_i^{(1)} = \sigma(t_i) + I_n(\sigma_0(t_i))^{-1/2}\varepsilon_i,\ \sigma \in H_{\sigma_0,\gamma_n}(\beta, L). \tag{12}$$

Denote $I(\lambda_i) = 1/(2\lambda_i^2)$ and consider the variables

$$Z_i^{(2)} = \sigma(t_i) + I(\sigma_0(t_i))^{-1/2}\varepsilon_i,\ \sigma \in H_{\sigma_0,\gamma_n}(\beta, L). \tag{13}$$

We prove that the experiments generated by $Z^{(1)} = \left\{Z_i^{(1)}\right\}_{i=1,\dots,n}$ and $Z^{(2)} = \left\{Z_i^{(2)}\right\}_{i=1,\dots,n}$ are asymptotically equivalent, in order to bring our problem back to the case of Grama and Nussbaum [2] and apply the variance-stabilizing transformation .

We show that the Hellinger distance tends to 0 with n, as follows

$$\begin{aligned}
H^2\left(Z^{(1)}, Z^{(2)}\right) &\leq 2\sum_{i=1}^{n} H^2\left(N\left(\sigma(t_i), I_n(\sigma_0(t_i))^{-1}\right), N\left(\sigma(t_i), I(\sigma_0(t_i))^{-1}\right)\right) \\
&\leq 4\sum_{i=1}^{n} \frac{\left(I_n(\sigma_0(t_i))^{-1/2} - I(\sigma_0(t_i))^{-1/2}\right)^2}{I_n(\sigma_0(t_i))^{-1} + I(\sigma_0(t_i))^{-1}} \\
&\leq 4\sum_{i=1}^{n} \frac{\left(\left(1 - \frac{1}{2n\sigma_0(t_i)}\right)^{-1/2} - 1\right)^2}{\left(1 - \frac{1}{2n\sigma_0(t_i)}\right)^{-1} + 1}.
\end{aligned}$$

If we denote $a_i = 1/(2n\sigma_0(t_i))$ then

$$\frac{\left((1-a_i)^{-1/2} - 1\right)^2}{(1-a_i)^{-1} + 1} = \frac{\left(1 - (1-a_i)^{-1/2}\right)^2}{2 - a_i} \leq a_i^2$$

and

$$H^2\left(Z^{(1)}, Z^{(2)}\right) \leq \frac{1}{n^2}\sum_{i=1}^{n}\frac{1}{\sigma_0^2(t_i)} \leq \frac{1}{n\varepsilon^2} \underset{n\to\infty}{\to} 0.$$

The last step is to apply a variance-stabilizing transformation Γ, solution of the differential equation $\Gamma'(\lambda) = \sqrt{I(\lambda)}$, which gives $\Gamma(\lambda) = \log\lambda/\sqrt{2}$. We obtain the equivalent local Gaussian experiment

$$Z_i^{(3)} = \frac{\log\sigma(t_i)}{\sqrt{2}} + \varepsilon_i,\ \sigma \in H_{\sigma_0,\gamma_n}(\beta, L).$$

By globalization arguments of Grama and Nussbaum [2] (Theorem 3.8) we obtain the global experiment in (11) (where σ in the class $H(\beta, L)$). ■

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