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Common Cycles: A frequency domain approach*

by

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Abstract

In this paper we decompose the Serial Correlation Common Feature (SCCF) of Engle and Kozicki (1993) in the frequency domain. A collection of time series is said to share a common cycle if there exists a linear combination of the predicted series with a zero spectral density at some frequency. Estimation and inference can be performed using an Instrumental Variables (IV) approach or a Canonical Correlation Analysis (CCA). The asymptotic and finite sample properties are studied and an analysis of the comovement between Germany, Austria and the United Kingdom is presented.

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1 Introduction

Economic analysis is often concerned with the analysis of comovement between time series. The cointegration framework suggested by Granger (1981) and Engle and Granger (1987) focuses on long-term comovement (that is comovement at the zero frequency) between a set of non-stationary variables. Yet, the analysis of short-term comovement (for example at business cycle frequencies) has not gained the same popularity. An important contribution was the concept of "codependence" (Gourieroux and Peaucelle (1993)) or "common features" (Engle and Kozicki (1993)), where it is assumed that a set of stationary time series shares common short-term movements (called common cycles), that is, there exists a linear combination of the series that is unpredictable. To test such "*serial correlation common feature*" (SCCF) two different approaches have been proposed. Engle and Kozicki (1993) construct a test using instrument variables (IV) and Vahid and Engle (1993) employ a canonical correlation analysis (CCA).

This framework has been applied to investigate problems like the relative purchasing power parity (Gourieroux and Peaucelle (1993)), the permanent income hypothesis (Vahid and Engle (1993)), cross-country real interest rate differentials (Kugler and Neusser (1991)), convergence of European economies (Beine and Hecq (1998)) and Okun's law (Candelon and Hecq (2000)). However, as pointed out by Cubadda (1999a), even a perfect comovement at business cycle frequencies does not imply the presence of a SCCF. In other words, there is only a weak correspondence between the concept of SCCF and comovement at particular frequencies. Therefore, it is difficult to disentangle high frequency comovement (for example due to seasonal behavior) and common business cycle dynamics that are related to different ranges of frequencies. Furthermore, as the concept of SCCF requires that the spectral density matrix is strictly positive at all frequencies, the use of seasonal adjusted data is problematical (cf. Cubadda (1999a, 1999b)).

In this paper we focus on comovement at particular frequencies by translating the SCCF concept in the frequency domain. Such common cycle tests can be used to analyze comovement at different frequencies. In Section 2 we introduce the basic definitions of our concept. Empirical procedures to estimate and test common cycles at a particular frequency are considered in Section 3. The power of the tests are analyzed in Section 4 and in Section 5 the methodology is applied to study the degree of synchronization between three european countries (Germany, Austria and the United Kingdom). Finally, Section 6 concludes.

2 Basic Concepts

To illustrate the main ideas, it is useful to consider a simple example. Let y_{1t} and y_{2t} be two stationary time series that can be decomposed as

$$y_{1t} = c_t + u_t \quad (1)$$

$$y_{2t} = \gamma c_t + v_t \ , \quad (2)$$

where c_t is the common cycle and u_t and v_t are uncorrelated white noise series. According to the definition of Engle and Kozicki (1993) the series have a "serial correlation common feature" (SCCF) with the property:

$$E(\gamma y_{1t} - y_{2t} | I_{t-1}) = 0 \ , \quad (3)$$

where I_{t-1} is the information set given by $I_{t-1} = \{y_{1,t-1}, y_{2,t-1}, y_{1,t-2}, y_{2,t-2}, \ldots\}$. In other words, there exists a linear combination $z_t = \gamma' y_t$ that is not predictable conditional on I_{t-1}.

For business cycle analysis this concept of a SCCF seems to be overly restrictive. For example, assume that u_t (or v_t) has a high frequency component that is predictable using I_{t-1}. An example is a joint seasonal pattern in the series. Using the concept of Engle and Kozicki (1993) we conclude that there is no "common cycle" although the predictability is irrelevant for business cycle analysis. On the other hand the series may share a common (stochastic) trend so that their long run relationship is predictable. However, such long-run properties are analyzed by theories of economic growth while business cycle theories focus on the cyclical properties of the time series.

Hence, for our analysis we modify the original concept of "common cycles" (or SCCF) in order to investigate the comovement at particular business cycle frequencies. To do so we consider the forecast ability of the series in a frequency domain. Under some well known conditions the optimal prediction of z_t is the conditional expectation $\xi_t = (z_t|I_{t-1})$. If z_t and the variables in I_t are stationary, then the series ξ_t is also stationary and possess a spectral density $f_\xi(\omega)$. In the following definition, we define predictability at some frequency ω^* by using the spectral density of ξ_t.

Definition 1: *Assume that z_t is a stationary time series with $E(z_t) = 0$ and spectral density $f_z(\omega) > 0$ for all $\omega \in [0, \pi]$. Furthermore, the information set is defined as $I_{t-1} = \{y_{t-1}, y_{t-2}, \ldots\}$ and $f_\xi(\omega)$ is the spectral density of the forecast $\xi_t = E(z_t|I_{t-1}) = \psi(L)y_{t-1}$. The series z_t is not predictable at frequency ω^*, if $f_\xi(\omega^*) = 0$.*

Our definition is similar to the definition of noncausality at some given frequency as suggested by Geweke (1982) and Hosoya (1991). The latter measure is also used by Granger and Lin (1996).

Using the definition of predictability at some given frequency, we are able to define a common cycle at frequency ω^*.

Definition 2 (Common Cycle): *Let y_t be a stationary $n \times 1$ vector with a positive definite spectral density matrix at all frequencies. Then, y_t is said to possess a common cycle with frequency $0 < \omega^* < \pi$, if (i) the components of y_t are predictable at frequency ω^* and (ii) there exists a linear combination $z_t = \gamma' y_t$ that is not predictable at frequency ω^*.*

This definition relaxes the original concept of Engle and Kozicki (1993) by assuming that only the part of the series related to the business cycle frequencies possesses the SCCF. Obviously, if y_t has the SCCF feature according to

Engle and Kozicki (1993), then there are common cycles at all frequencies. Before considering empirical test procedures we make some remarks that are related to alternative ways to generalize the SCCF definition introduced by Engle and Kozicki (1993).

Remark A: The measure can also be applied to cointegrated time series. Consider the vector error correction model (VECM):

$$\Delta y_t = \alpha\beta' y_{t-1} + \Gamma_1 \Delta y_{t-1} + \cdots + \Gamma_p \Delta y_{t-p} + u_t \ , \tag{4}$$

where y_t is a $n \times 1$ vector of integrated time series and α, β are $n \times r$ matrices with $0 < r < n$. The one-step ahead prediction is $\xi_t = \alpha\beta' y_{t-1} + \Gamma_1 \Delta y_{t-1} + \cdots + \Gamma_p \Delta y_{t-p}$ is stationary with positive spectral density, in general. However, if $\Gamma_1, \ldots, \Gamma_p$ are singular with $\alpha_j^* \beta'$ and some arbitrary $n \times r$ matrices α_j^* for $j = 1, \ldots, p$, then the prediction of $z_t = \alpha_\perp' \Delta y_t$ with $\alpha_\perp' \alpha = 0$ can be expressed by using the lagged differences of $v_t = \beta' y_t$ alone. In this case z_t is unpredictable at frequency zero because ξ_t is a function of $\Delta v_{t-1}, \ldots, \Delta v_{t-p}$ and, therefore, $f_\xi(0) = 0$ (see also Gonzalo and Granger (1995)). Hecq et al. (1998) suggest the concept of "weak form SCCF". Accordingly, their framework requires that there exists a matrix γ such that $\gamma' \Gamma_j = 0$ for $j = 1, \ldots, p$ but not necessarily $\gamma' \alpha = 0$ as in Vahid and Engle (1993). Thus, this definition allows for a linear combination that is predictable in the *long-run* (at frequency zero).

Remark B: Gourieroux and Peaucelle (1993) and Vahid and Engle (1997) propose a generalization in a different direction. According to their definition, a vector of time series, y_t, with a codependent cycle has the property

$$E(\gamma' y_t | I_{t-q}) = 0 \tag{5}$$

for some $q \geq 1$. Thus, the linear combination $\gamma' y_t$ is predictable at forecast horizons less than q. This generalization is similar in spirit to our concept as it allows for predictability at short forecast horizons. Our framework, however, operates in a frequency domain so that predictability is allowed at high frequencies.

Remark C: In a cautious note Cubadda (1999 a) points out that the SCCF concept of Engle and Kozicki (1993) is not (or only weakly) related to the co-movement at business cycle frequencies measured by the coherence between the series. Indeed the original SCCF concept and the frequency domain version considered here are based on a measure of comovement between the *predicted* series. If there is a common cycle as defined in Definition 2, then the predicted series show a perfect comovement at that frequency, while the *observed* series (the predictable and the unpredictable part) may exhibit a weak comovement only. Usually, economic theory make statements about the expected response to economic shocks. For example, the heterogeneous consumer hypothesis of Campbell and Mankiw (1989) implies that the impulse responses of income and consumption with respect to the transitory shock are proportional (see also Vahid and Engle (1993)). If the permanent component is a pure random walk, this implies that predicted income and consumption are perfectly correlated although the observed series may only be weakly correlated.

Remark D: Cubadda (1999a) notes that the SCCF cannot be used for seasonally adjusted data because the usual seasonal adjustment procedures introduce a zero spectrum at the seasonal frequencies. This, however, is in conflict with the SCCF feature that requires a flat spectrum. By focusing on the frequencies of interest, our frequency based concept is able to cope with this problem. A zero spectral density at other frequencies does not affect the analysis, at least in large samples.

3 Empirical test procedures

In order to test for a common cycle at some given frequency ω^*, we first have to test, whether the involved time series are predictable at ω^*. If they are not predictable at ω^* it does not make sense to speak of a common cycle (see also Engle and Kozicki 1993). In a second step, we test whether there exists a linear combination that is unpredictable at the frequency of interest. For both steps of the analysis, similar tests can be performed.

Our approach is based on a vector autoregressive (VAR) framework. Let y_t be generated by a VAR(p) model

$$\begin{aligned} y_t &= A_1 y_{t-1} + \cdots + A_p y_{t-p} + \varepsilon_t \\ &= A(L) y_{t-1} + \varepsilon_t \ , \end{aligned} \tag{6}$$

where ε_t is a $n \times 1$ time series vector and $A(L) = A_1 + A_2 L + \cdots + A_p L^{p-1}$ and L is the lag operator. According to Definition 1, if the matrix $A(e^{i\omega^*})$ is singular, then there exists a linear combination $z_t = \gamma' y_t$ that is unpredictable at frequency ω^*. To simplify the exposition we confine ourselves to a bivariate system ($n = 2$). In this case we have

$$z_t = a_1 y_{1,t-1} + \cdots + a_p y_{1,t-p} + b_1 y_{2,t-1} + \cdots + b_p y_{2,t-p} + u_t \tag{7}$$

$$ = a(L) y_{1,t-1} + b(L) y_{2,t-1} + u_t \ , \tag{8}$$

where u_t is white noise with $E(u_t^2) = \sigma_u^2$. Notice that the coefficients a_j and b_j depend on the vector γ. However, to keep the notation clear this dependency is suppressed in our notation.

The one-step ahead prediction is equal to the conditional expectation

$$\xi_t = E(z_t | I_{t-1}) = a(L) y_{1,t-1} + b(L) y_{2,t-1} \ .$$

If we want to test whether y_{1t} is unpredictable at frequency ω^* we let $\gamma = [1, 0]'$, whereas y_{2t} is tested by setting $\gamma = [0, 1]'$.

The prediction ξ_t has a zero spectral density at frequency ω^* if (i) y_{1t} and y_{2t} have a zero spectral density at ω^* or/and (ii) the gain functions of the filters $a(L)$ and $b(L)$ have a zero at frequency ω^*. Since we assume that the series are predictable at ω (see Definition 2) condition (i) does not hold. Condition (ii) is satisfied if

$$|a(e^{i\omega^*})| = 0 \quad \text{and} \quad |b(e^{i\omega^*})| = 0 \ , \tag{9}$$

which is equivalent to

$$\begin{array}{ll} \sum\limits_{s=1}^{p} a_s \cos(\omega^* s) = 0 & \sum\limits_{s=1}^{p} a_s \sin(\omega^* s) = 0 \\ \sum\limits_{s=1}^{p} b_s \cos(\omega^* s) = 0 & \sum\limits_{s=1}^{p} b_s \sin(\omega^* s) = 0 \ . \end{array} \tag{10}$$

Let $\beta = [a_1, \ldots, a_p, b_1, \ldots, b_p]'$. Then, we can test the hypothesis by using the usual Wald, LM or LR statistics for the null hypothesis

$$H_0 : \quad R(\omega^*)\beta = 0 \ , \tag{11}$$

where

$$R(\omega^*) = \begin{bmatrix} \cos(\omega^*) & \cdots & \cos(\omega^* p) & 0 & \cdots & 0 \\ \sin(\omega^*) & \cdots & \sin(\omega^* p) & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cos(\omega^*) & \cdots & \cos(\omega^* p) \\ 0 & \cdots & 0 & \sin(\omega^*) & \cdots & \sin(\omega^* p) \end{bmatrix} .$$

Notice that $\sin(\omega^*) = 0$ for $\omega^* = 0$ and $\omega^* = \pi$ so that the respective rows can be dropped in these cases.

To test the hypothesis (11), an instrumental variable approach can be used that is similar to the one suggested by Engle and Kozicki (1993). Let $\gamma = [1, -\gamma_2]'$, $x_t = [y_{1,t-1}, \ldots, y_{1,t-p}, y_{2,t-1}, \ldots, y_{2,t-p}]$ and $Q = [R'_\perp, R']$, where $R_\perp$ is an orthogonal complement of R. Then, the regression (7) can be rewritten as

$$\begin{aligned} y_{1t} &= \gamma_2 y_{2t} + \beta^{*\prime} x_t^* + u_t \\ &= \gamma_2 y_{2t} + \beta_1^{*\prime} x_{1t}^* + \beta_2^{*\prime} x_{2t}^* + u_t \ , \end{aligned}$$

where

$$\beta^* = Q'\beta = \begin{bmatrix} \beta_1^* \\ \beta_2^* \end{bmatrix} \quad \text{and} \quad x_t^* = Q^{-1} x_t = \begin{bmatrix} x_{1t}^* \\ x_{2t}^* \end{bmatrix} .$$

Under the hypothesis that $\gamma = [1, -\gamma_2]'$ is a cofeature vector, the model reduces to

$$y_{1t} = \gamma_2 y_{2t} + \beta_1^{*\prime} x_{1t}^* + u_t \ . \tag{12}$$

This model can be estimated by using x_t (or x_t^*) as the vector of instrumental variables. The common cycle hypothesis implies an over-identifying restriction that can be tested by using Sargan's test statistic:

$$\lambda_S = \widehat{\sigma}_u^{-2} \left(\sum_{t=p+1}^{T} \widehat{u}_t x_t' \right) \left(\sum_{t=p+1}^{T} x_t x_t{}' \right)^{-1} \left(\sum_{t=p+1}^{T} x_t \widehat{u}_t \right) \ , \tag{13}$$

where $\widehat{u}_t$ denotes the residual of (12) and $\widehat{\sigma}_u^2 = T^{-1} \sum \widehat{u}_t^2$ is the respective variance estimate. Under the null hypothesis of a common cycle, the test statistic is asymptotically χ^2 distributed with 3 degrees of freedom.

This test can easily be generalized for a vector of n time series with the common feature vector $\gamma = [1, -\gamma_2, \dots, -\gamma_n]'$. In this case, we let $x_t = [y_{1,t-1}, \dots, y_{1,t-p}, y_{2,t-1}, \dots, y_{2,t-p}, \dots, y_{n,t-1}, \dots, y_{n,t-p}]'$ and the restriction matrix is $R = \text{diag}[\Phi(\omega^*), \dots, \Phi(\omega^*)]$, where

$$\Phi(\omega^*) = \begin{bmatrix} \cos(\omega^*) & \cdots & \cos(\omega^* p) \\ \sin(\omega^*) & \cdots & \sin(\omega^* p) \end{bmatrix} \ .$$

The resulting Sargan statistic is asymptotically χ^2 distributed with $n+1$ degrees of freedom.

To avoid the normalization that one element of γ is set to unity, a limited information maximum likelihood (LIML) approach or a canonical correlation approach (CCA) can be employed. The LIML estimator is obtained from maximizing the variance ratio:

$$\lambda_{Liml} = \sup_{\gamma} \left\{ \frac{\gamma' Y' X_2^* (X_2^{*\prime} X_2^*)^{-1} X_2^{*\prime} Y \gamma}{\gamma' Y' X (X'X)^{-1} X' Y \gamma} \right\} \ , \tag{14}$$

where $Y = [y_{p+1}, \dots, y_T]'$, $X_2^* = [x_{2,p+1}^*, \dots, x_{2T}^*]'$ and $X = [x_{p+1}, \dots, x_T]'$. The resulting LIML estimate is the eigenvector corresponding to the largest eigenvalue of the matrix

$$C = Y' X_2^* (X_2^{*\prime} X_2^*)^{-1} X_2^{*\prime} Y (Y' X (X'X)^{-1} X' Y)^{-1} \ .$$

Following Vahid and Engle (1993) we may also use a Canonical Correlation approach. Let $P_1 = I - X_1^*(X_1^{*\prime}X_1^*)^{-1}X_1^{*\prime}$, where $X_1^* = [x_{1,p+1}^*, \ldots, x_{1T}^*]'$. Furthermore, we define $\widetilde{Y} = P_1 Y$. Then, the CCA is based on the eigenvalue problem:

$$|\lambda \widetilde{Y}'\widetilde{Y} - \widetilde{Y}'X_2^*(X_2^{*\prime}X_2^*)^{-1}X_2^{*\prime}\widetilde{Y}| = 0 \ . \tag{15}$$

Estimates for the vector γ are obtained as the eigenvectors corresponding to the *smallest* eigenvalues. The existence of a common cycle according to Definition 2 can be tested by comparing the smallest eigenvalue to the critical value of the χ^2 distribution with $n+1$ degrees of freedom. Using

$$Y'X(X'X)^{-1}X'Y = Y'X_1^*(X_1^{*\prime}X_1^*)^{-1}X_1^{*\prime}Y + Y'X_2^*(X_2^{*\prime}X_2^*)^{-1}X_2^{*\prime}Y$$

it is easy to show that the LIML and CCA approaches yield identical test statistics.

4 Power

To study the local power of the test, we consider the simple model

$$y_{2,t} = b_0(L)y_{1,t-1} + u_t \ , \tag{16}$$

where $b_0(L) = \alpha[1 - 2\cos(\omega_0)L + L^2]$ and $\{y_{1,t}\}$, $\{u_t\}$ are white noise. Despite of the simplicity of this model we are able to derive some important features of the test for common cycles. A more general model implies additional parameters and more complicated formulae without gaining additional insight. Since, the gain function of the filter $b_0(L)$ is zero at ω_0, the process is unpredictable at frequency ω_0.

Assume, however, that instead of ω_0 we test against the frequency $\omega^* = \omega_0 + c/\sqrt{T}$ and, thus, we study the local power of the test when the frequency under test converges to the true frequency with a suitable rate. Using a Taylor expansion around ω_0, the process can be represented as

$$y_{1,t} \ \simeq \ b^*(L)y_{2,t-1} - T^{-1/2}2c\alpha\sin(\omega^*)y_{2,t-2} + u_t \ . \tag{17}$$

Table 1: Actual and asymptotic power

	$\omega_0 = \pi/4$		$\omega_0 = \pi/2$		$\omega_0 = 3\pi/4$	
c	actual	asympt.	actual	asympt.	actual	asympt.
0.5	0.068	0.069	0.128	0.133	0.066	0.069
1.0	0.131	0.133	0.409	0.416	0.123	0.133
1.5	0.278	0.250	0.768	0.771	0.225	0.250
2.0	0.494	0.416	0.953	0.957	0.355	0.416
2.5	0.716	0.603	0.995	0.996	0.500	0.603
3.0	0.878	0.771	1.000	1.000	0.633	0.771

Note: Entries of this tables report the rejection frequencies of 5,000 simulations generated according to model (16) with $\alpha = 1$. The sample size is $T = 500$ and the 0.05 significance level is used.

where $b^*(L) = \alpha[1 - 2\cos(\omega^*)L + L^2]$ It follows that the power of the test depends on c and the frequency ω^*. The following proposition gives the distribution of the IV test statistic under such local alternatives.

Theorem 1: *Let $y_{1,t}$ be generated as in (16), where $\{y_{2,t}\}$ and $\{u_t\}$ are independent white noise processes with finite variances. Under the local alternative $\omega^* = \omega_0 + c/\sqrt{T}$ the test statistic λ_S is asymptotically distributed as a noncentral χ^2 distributed random variable with noncentrality parameter*

$$\eta^2 = \frac{[2c\alpha\sin(\omega_0)]^2}{1 + 2\cos(\omega_0)^2} \; .$$

From this result two important conclusions can be drawn. First, the test suffers from a "leakage problem", that is, the power of the test deteriorates if the frequency under test tends towards the common cycle frequency. Hence, the test statistic has problems to detect deviations from the null hypothesis, if there exists a common cycle with a similar frequency.

Second, the power of the test depends sensitively on the frequency under test. The maximal power against local alternative is at $\omega_0 = \pi/2$ and for

$\omega_0 \to 0$ and $\omega_0 \to \pi$, the local power against common cycles with frequencies close to the hypothesized frequency tends to zero. This should be taken into account when comparing the test results at different frequencies. Moreover, this result suggests that the sampling frequency of the test is important for testing the common cycle hypothesis. For example, using a monthly frequency instead of quarterly frequency increases the number of observations by a factor of 3 but on the other hand, the frequency under test also shifts by the same factor towards zero. Hence, increasing the sampling frequency does not need to improve the power of the test.

To investigate the reliability of our asymptotic results we simulate time series according to (16) with $\alpha = 1$ and $T = 500$. The frequency under test is $\omega^* = \omega_0 + c/\sqrt{T}$, that is, we test at frequencies close to the frequency of the common cycle. The results are shown in Table 1. For a common cycle at $\omega_0 = \pi/2$ the empirical power is very close to the asymptotic powers. For $\omega_0 = \pi/4$ and $\omega_0 = 3\pi/4$ we found that for small values of c the actual power is well approximated by the asymptotic power but for increasing values of c the asymptotic theory provides a less accurate approximation to the actual power. Furthermore, the actual power tends to be asymmetric for substantial values of c.

Next, we study the power of a VAR(3) process of the form

$$y_t = \begin{bmatrix} \theta & 0.4 \\ 0.4 & \theta \end{bmatrix} y_{t-3} + \varepsilon_t \quad \text{with} \quad E(\varepsilon_t \varepsilon_t') = \begin{bmatrix} 0.3 & 0.2 \\ 0.2 & 0.3 \end{bmatrix} . \tag{18}$$

If $\theta = 0.4$, then $\gamma = [1, -1]'$ is a co-feature vector and $\gamma' y_t$ is white noise. In this case we can use the SCCF-tests suggested by Engle and Kozicki (1993) and Vahid and Engle (1993). These tests can be seen as a joint test for the hypothesis that there is a common cycle at all frequencies. In contrast, the tests suggested in Section 3 are based on the hypothesis that there is a common cycle at some pre-specified frequency. For $\theta \neq 0.4$ the process has no common common cycle and, thus, we are able to compare the power of the different test statistics. Since the common cycle hypothesis is violated for all frequencies, it is natural that the joint test is more powerful against

Table 2: Size and Power

Common cycle Tests								
	$\theta = 0.1$		$\theta = 0.2$		$\theta = 0.3$		$\theta = 0.4$	
	IV	CCA	IV	CCA	IV	CCA	IV	CCA
$\pi/2$	0.715	0.688	0.368	0.359	0.127	0.127	0.048	0.051
$\pi/3$	0.835	0.819	0.477	0.473	0.151	0.153	0.051	0.052
$\pi/4$	0.866	0.883	0.513	0.515	0.157	0.161	0.047	0.054
$\pi/10$	0.901	0.897	0.552	0.556	0.172	0.174	0.051	0.055
$\pi/15$	0.901	0.901	0.556	0.559	0.167	0.170	0.051	0.050
SCCF tests								
all	0.935	0.926	0.599	0.570	0.177	0.166	0.055	0.056

Note: Entries of this tables are the rejection frequencies for 10,000 realizations of model (18). The sample size is T=200 and a 0.05 significance level is used.

such type of alternatives. We will therefore use the original SCCF tests as a benchmark for assessing the loss of power when focusing on a particular frequency only.

The data is generated using normally distributed errors generated with the Gauss 3.2 package. The sample size is $T = 200$ and 10,000 replications are used. Table 2 presents the rejection frequencies of the tests when the IV version of the test is used. Letting $\theta = 0.4$ we obtain empirical sizes that are close to the nominal size of 0.05. As expected we find a lower power for the single-frequency test. Nevertheless, the tests have substantial power for all frequencies and the power increases when the frequency tends to zero. Furthermore, the powers of the IV and CCA statistics are very similar.

5 European Business Cycles

The common cycle framework can be used to investigate the synchronisation of European business cycles. The fact that a set of economies share common dynamics constitutes an important condition for the characterization of an

"Optimal Currency Area" (see Mundell (1961), for example). A number of empirical studies coped with this problem for the European countries (see for example Artis and Zhang (1995), Bayoumi and Eichengreen (1993)). Among them, Rubin and Thygesen (1997) and Beine, Candelon and Hecq (2000) test for the existence of an European common cycle by using the SCCF framework of Engle and Kozicki (1993) and Vahid and Engle (1993).

For our application we focus on three European countries: Germany, Austria and the United Kingdom. The first two countries are expected to show a strong comovement, whereas Germany and the United Kingdom are supposed to be loosely linked and should therefore present weaker comovements. We focus on seasonally unadjusted industrial production (IP) indices for the period ranging from 1975m1-1997m4 extracted from the datastream database. These indices are also analyzed by Rubin and Thygesen (1997) and Beine, et al. (2000).

First the series are tested for possible unit roots. In Table 3 the results of the tests suggested by Hylleberg et al. (1990) are presented. The results suggest that the series have unit roots at frequency zero and some seasonal frequencies. It is therefore necessary to remove the unit roots by applying the annual difference filter $\Delta_{12} = (1 - L^{12})$.[1] The resulting series are compared in Figure 1 a) for Germany and Austria and in Figure 1 b) for Germany and the United Kingdom.

The Akaike information criterion (AIC) suggests that the first system (German and Austria IP indices) can be represented by a VAR (11) and the second one (German and United Kingdom industrial IP indices) by a VAR (9). All equations include a constant term. Table 5 presents the SCCF statistics for both systems. The test statistic based on a 2SLS approach suggested by Engle and Kozicki (1992) is denoted by "IV" and the canonical correlation statistic due to Vahid and Engle (1993) is denoted by "CCA". It turns out that both tests reject the presence of a common cycle for both system.

[1] It is important to note that a possible "overdifferencing" at some seasonal frequencies does not affect the inference on other frequencies (e.g. business cycle frequencies).

Table 3: HEGY Seasonal Unit Roots Tests

	Ger		*Au*		*UK*	
Lags	1 *to* 11		1 *to* 6		1 *to* 10	
Model	c, sd	c, sd, t	c, sd	c, sd, t	c, sd	c, sd, t
π_1	-1.02	-1.83	-0.38	-3.14	-0.54	-2.09
π_2	-3.16*	-3.13*	-1.66	-1.70	-1.16	-0.99
$\pi_3 \cap \pi_4$	6.57*	6.48*	4.95	4.68	2.60	3.21
$\pi_5 \cap \pi_6$	11.68*	11.45*	6.83*	6.51*	4.33	2.56
$\pi_7 \cap \pi_8$	3.61	3.56	15.37*	12.54*	2.88	3.17
$\pi_9 \cap \pi_{10}$	12.86*	12.64*	14.61*	15.06*	12.35*	9.08*
$\pi_{11} \cap \pi_{12}$	8.17*	8.24*	16.57*	16.62*	1.69	2.31

Note: This table presents the results of unit root tests at zero and seasonal frequencies (see Franses (1990)) for models with a constant and seasonal dummies (c,sd) and with a constant, seasonal dummies and a deterministic trend (c,sd,t). * indicates a rejection with respect to the 5% critical value.

This confirms the results of Beine et al. (2000), stressing the surprisingly low degree of synchronisation between European economies.

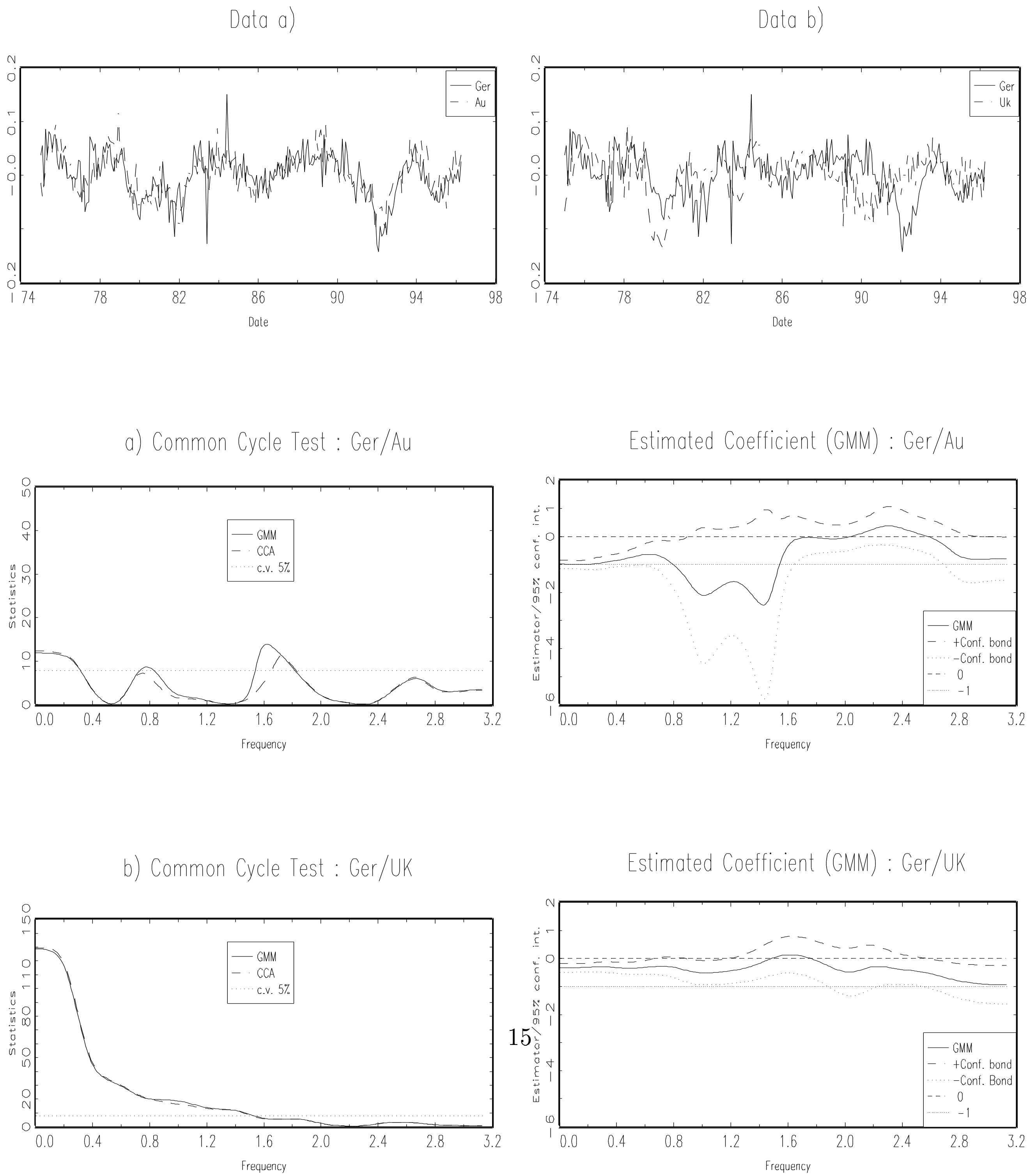
The results of the common cycle analysis are shown in Figure 1. The second row show plots of the test statistics for frequencies in the range $\omega \in (0, \pi)$ and the third row presents the respective IV estimates of the parameter c for the common feature vector $\gamma = [1, c]'$, where both bivariate systems are normalized such that the coefficient for the German IP series is unity.

Table 4: Test of SCCF

	Ger/UK	Ger/Au
CCA	188.627*	59.480*
IV	148.800*	42.926*

Note: Entries of this tables are the rejection frequencies for 10,000 realizations of model (18). The sample size is T=200 and a 0.05 significance level is used.

Figure 1: Common Cycle Analysis



For Germany and Austria a common cycle is accepted in a range $\omega \in [0.35, 0.7]$ corresponding to a wave length between 8 and 22 months. These frequencies are higher than the usual range of business cycle frequencies suggesting that the comovement is to a lesser extend due to a synchronous business cycle. For typical business cycle frequencies the test statistic is roughly equal to the 0.01 critical value. For the interval $\omega \in [0, 0.7]$ the estimated cofeature vector is close to $\gamma = [1, -1]'$ and the value $c = -1$ always lies inside the 95%-confidence intervals. For higher frequencies the confidence interval covers the value $c = 0$ suggesting that the German IP series is not predictable for frequencies higher than 0.8.

Interestingly, the results for Germany/U.K. are quite different. In this application the common cycle hypothesis is clearly rejected for frequencies up to $\omega = \pi/2$ (4 months). For higher frequencies, the estimated cofeature vector is close to $\gamma = [1, 0]'$, suggesting that the acceptance of the null hypothesis at very high frequencies is due to the fact that the German IP is not predictable at high frequencies.

6 Conclusion

By decomposing the SCCF in the frequency domain we propose tests and estimators for common cycles at some pre-specified frequency. Such a concept can be used to analyze the comovement of a collection of time series in the frequency domain. The advantage of this approach is that the analysis focuses on particular frequencies (e.g. business cycle frequencies) and, thus, does not restrict the dynamic relationship of the series at all frequencies. Consequently, the analysis is robust to problems at other frequencies (e.g. resulting from a seasonal adjustment of the data).

We study the asymptotic and finite sample properties of the tests and illustrate the analysis by studying the comovements between three European countries (Germany, Austria and the United Kingdom). We hope that our concept of common cycles is also useful to explain short-run comovements of other time series.

Appendix: Proof of Theorem 1:

PROOF: The null hypothesis that y_{2t} is unpredictable at frequency ω is equivalent to $\widetilde{R}(\omega^*)\beta = 0$ in the model $y_{2,t} = \beta' x_t + u_t$, where $x_t = [y_{1,t-1}, y_{1,t-2}, y_{1,t-3}]'$, $\beta = [\alpha, -\alpha 2\cos(\omega_0), \alpha]'$ and

$$\begin{aligned}
\widetilde{R}(\omega^*) &= \begin{bmatrix} \cos(-\omega^*) & \cos(0) & \cos(\omega) \\ \sin(-\omega^*) & \sin(0) & \sin(\omega) \end{bmatrix} \\
&= \begin{bmatrix} \cos(\omega^*) & 1 & \cos(\omega) \\ -\sin(\omega^*) & 0 & \sin(\omega) \end{bmatrix}
\end{aligned}$$

Furthermore $\widetilde{R}_\perp(\omega^*) = [1, -2\cos(\omega^*), 1]$. From the inverse of $Q(\omega^*)' = [\widetilde{R}_\perp(\omega^*)', \widetilde{R}(\omega^*)']'$ we obtain under the null hypothesis

$$y_{2,t} = \beta_1^* x_{1,t}^* + \widetilde{u}_t$$

where $\beta_1^* = \alpha$ and $x_{1,t} = y_{1,t-1} - 2\cos(\omega^*) y_{1,t-2} + y_{1,t-3}$.

The IV estimator of β_1^* is given by

$$\widehat{\beta}_1^* = [X_1^{*\prime} X (X'X)^{-1} X' X_1^*]^{-1} X_1^{*\prime} X (X'X)^{-1} X' y_2 \ ,$$

where the matrices are defined as in Section 3. From the model assumptions it follows that

$$\begin{aligned}
&T^{-1} X'X \xrightarrow{p} \sigma_1^2 I_3 \quad \sigma_1^2 = E(y_{1,t}^2) \\
&T^{-1} X' X_1^* = \sigma_1^2 \begin{bmatrix} 1 \\ -2\cos(\omega^*) \\ 1 \end{bmatrix} \ .
\end{aligned}$$

Since the residual vector $y_2 - \widetilde{\beta}_1^* x_{1,t}^*$ is orthogonal to the vector $Z(Z'Z)^{-1} Z' X_1^*$ it follows that

$$\begin{aligned}
& (y_2 - \widetilde{\beta}_1^* X_1^*)' X (X'X)^{-1} X' X_1^* \\
= \ & (y_2 - \widetilde{\beta}_1^* X_1^*)' X \begin{bmatrix} 1 \\ -2\cos(\omega^*) \\ 1 \end{bmatrix} \\
= \ & (y_2 - \widetilde{\beta}_1^* X_1^*)' X_1^* = 0.
\end{aligned}$$

Let

$$X^* = [X_1^*, X_2^*, X_3^*] = X \begin{bmatrix} 1 & \cos(\omega^*) & -\sin(\omega^*) \\ -2\cos(\omega^*) & 1 & 0 \\ 1 & \cos(\omega^*) & \sin(\omega^*) \end{bmatrix} .$$

Since the test statistic remains the same if the instruments X are replaced by X^*, the test statistic can be written as

$$\begin{aligned} \lambda_S &= \widehat{\sigma}_u^{-2}(y_2 - \widetilde{\beta}_1^* X_1^*)' X (X'X)^{-1} X' (y_2 - \widetilde{\beta}_1^* X_1^*) \\ &= \widehat{\sigma}_u^{-2}(y_2 - \widetilde{\beta}_1^* X_1^*)' X^* (X^{*\prime} X^*)^{-1} X^{*\prime} (y_2 - \widetilde{\beta}_1^* X_1^*) \end{aligned}$$

It is easy to verify that

$$T^{-1/2} E[(y_2 - \widetilde{\beta}_1^* X_1^*)' X_2^*] = -2c\alpha \sin(\omega^*) + o(1)$$

and

$$T^{-1/2} E[(y_2 - \widetilde{\beta}_1^* X_1^*)' X_3^*] = 0.$$

Furthermore,

$$T^{-1} X^{*\prime} X^* \xrightarrow{p} \sigma_1^2 \begin{bmatrix} 2 + 4\cos(\omega^*)^2 & 0 & 0 \\ 0 & 1 + 2\cos(\omega^*)^2 & 0 \\ 0 & 0 & 2\sin(\omega^*)^2 \end{bmatrix}$$

and, thus, the limiting distribution of λ_S is a noncentral χ^2 distributed with 2 degrees of freedom and noncentrality parameter

$$\eta^2 = \frac{[2c\alpha \sin(\omega^*)]^2}{1 + 2\cos(\omega^*)^2} .$$

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