Affine Stochastic Differential Equations with Infinite Delay on Abstract Phase Spaces

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Abstract

A stochastic delay differential equation is considered which is of the form $dX(t) = \int_{(-\infty,0]} \nu(ds)X(t+s)dt + dW(t)$, $t \geq 0$, with the initial condition $X(u) = \Upsilon(u)$ for $u \leq 0$. As it is successfully done in the deterministic theory for delay equations, an axiomatic approach describing the set of admissible initial functions is utilized to treat the stochastic equation, which permits the use of semi-group and spectral theory. Moreover, it is obtained a representation of the solution in the abstract setting and for a certain class of measures ν one can give sufficient and necessary conditions for the existence of a stationary solution.

1 Introduction

In a large variety of applications stochastic delay differential equations are used for modeling purposes. In contrast to ordinary stochastic differential equation the past of the process is taken into account. In discrete time there is a well-developed theory for modeling such effects, e.g. ARMA, ARCH and GARCH–processes. However, in many applications continuous time models seem to be more adequate, e.g. in economy or biology, but the theory is much less covered. The time delays arise from effects which are well–known as "time to build","time to maturity", "gestation lag" and others, see e.g. [BeRu91], [Mac89], [MDo78].

Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$. On this probability space let $\{W(t), \mathcal{F}_t, t \geq 0\}$ be a Brownian motion with values in $\mathbb{R}^d$. We consider the affine stochastic differential equation with infinite delay, that is

$$
\begin{aligned}
dX(t) &= \int_{(-\infty,0]} \nu(ds)X(t+s)dt + dW(t), \quad t \geq 0, \qquad (1.1)\\
X(t) &= \Upsilon(t), \quad t \leq 0,
\end{aligned}
$$

where ν is a $d \times d$-matrix with real–valued locally finite measures as entries. The initial condition is given by a stochastic process $\{\Upsilon(t), t \in (-\infty, 0]\}$, which is supposed to be measurable with respect to $\mathcal{F}_0$. The trajectories of the initial process Υ are assumed to be P-a.s. elements of a linear function space $\mathcal{B}$, the

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so–called phase space, which is a subset of the set of all measurable $\mathbb{R}^d$–valued functions on the negative half–line.
Equation (1.1) includes equation with a finite delay, that is the measure ν is concentrated on an interval $[-r, 0]$ for fixed $r > 0$. In this case it is sufficient to consider initial functions with support in $[-r, 0]$. With regard to applications it is quite natural to choose the phase space $\mathcal{B}$ as the set of continuous functions on $[-r, 0]$. But in the case of an infinite delay it is not sufficient to assume the continuity of the initial function. Assume there is a process $\{X(t), t \geq 0\}$ satisfying equation (1.1) and fulfilling the initial condition. Then the integral in (1.1) should be finite which yields in the demand

$$\int_{(-\infty,-t]} |\nu| (ds) \, |\Upsilon(t+s)| < \infty \quad \text{for all } t \geq 0.$$

Obviously that is not necessarily fulfilled by initial functions which are only assumed to be continuous. Even for ν–integrable initial functions Υ the condition is not necessarily satisfied because the shifted function $\Upsilon(t+\cdot)$ may not to be ν–integrable. What is the proper choice for the set of initial functions? There is a large variety of appropriate phase spaces but none of them seems to be a "canonical choice". Hence, building up a theory of the deterministic analogue of equation (1.1) (see the following section), it became desirable to approach the problem purely axiomatically. In the deterministic theory it is successfully done in several contributions which are summarized in [HaKa78], [Ka90] and [HiMuNa91]. The main aim of this article is to establish for the stochastic differential equation (1.1) the axiomatic approach and to answer in the abstract setting some questions which are motivated by the occurence of a noise term as that on the existence of a stationary solution.
For the first time, equations of the form (1.1) and in a more general kind are considered in [ItNi64]. But the above mentioned well–definedness of the integral over the shifted initial function is neglected there.
If the functions $\varphi(s) = \alpha \, \mathbb{1}_{\{0\}}(s)$, $s \leq 0$, for all $\alpha \in \mathbb{R}^d$ are elements of the phase space $\mathcal{B}$, also Volterra integrodifferential equations can be treated by equation (1.1). These are equations of the form

$$dX(t) = \int_{[-t,0]} \nu(ds) X(t+s) dt + dW(t), \quad t \geq 0, \quad X(0) = \Upsilon(0).$$

More general, equation (1.1) is included in that of the form

$$dX(t) = a(t, X_t)dt + b(t, X_t)dW(t), \quad t \geq 0, \quad X(t) = \Upsilon(t), \quad t \leq 0, \quad (1.2)$$

where $X_t := \{X(t+u), u \leq 0\}$ and a and b are functionals from $\mathbb{R}_+ \times \mathcal{B}$ to $\mathbb{R}^d$ and $\mathbb{R}^{d \times d}$, respectively. Equations of the form (1.2) can also be handled by describing the phase space $\mathcal{B}$ in the abstract way. For a concrete phase space $\mathcal{B}$ the existence and uniqueness and some stability aspects of the solution of equation (1.2) are treated in [MiTru84]. The non–autonomous case of equation (1.1),

$$dX(t) = \int_{(-\infty,0]} \nu(t, ds) X(t+s) dt + dW(t), \quad t \geq 0, \quad X(t) = \Upsilon(t), \quad t \leq 0,$$

can also be treated in the proposed axiomatic way, one can even obtain a representation of the solution. By rewriting such equations as Volterra equations a representation of the solution is given in [Di89]. However, the representation is no longer coupled in a direct way with the initial condition.
In section 1 we summarize some known results for the deterministic equation corresponding to the stochastic one, which are necessary for the sequel. This part is mainly based on [HiMuNa91] where one can also find a more comprehensive discussion of the results and many references. Some of the results can be derived by the general theory of semi-groups for linear evolution equation such as it is introduced in [EnNa00]. In addition the deterministic Volterra integro–differential equation is considered and some new properties of its solution are proved.
Without specifying the phase space $\mathcal{B}$ we obtain a representation of the solution of the stochastic equation (1.1) in section 2. In the case of a finite delay the question of the existence of a stationary solution is sophisticatedly answered, see [GuKu00]. In the second part of this section we generalize the results on stationarity to equation (1.1) with infinite delay in the abstract setting, for that we have to impose a condition on the considered measure ν.

2 The deterministic case

In the deterministic context consider the homogeneous linear equation

$$\begin{aligned} \dot{x}(t) &= \int_{(-\infty,0]} \nu(ds)x(t+s), \quad t \geq 0, \\ x(t) &= \varphi(t), \quad t \leq 0, \end{aligned} \tag{2.3}$$

where ν is a $d \times d$-matrix with $\mathbb{K}$–valued locally finite measures as entries, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. The initial condition $x(t) = \varphi(t)$ for $t \leq 0$ is given by the function φ, which is an element of a phase space $\mathcal{B}$. The phase space $\mathcal{B}$ is a linear space equipped with a semi-norm $\|\cdot\|_{\mathcal{B}}$ consisting of functions mapping $(-\infty, 0]$ into $\mathbb{K}^d$:

$$\varphi \in \mathcal{B} \subseteq \{\eta : (-\infty, 0] \to \mathbb{K}^d : \eta \text{ is measurable}\}.$$

For any function $x : (-\infty, a) \to \mathbb{K}^d$, $a > 0$, denote the segment by $x_t = \{x(t+u), u \leq 0\}$ for fixed $t < a$, which is a function on the negative line. Such a function x is called *admissible with respect to $\mathcal{B}$ on the interval* $[0, a)$, if $x_0 \in \mathcal{B}$ holds and x is continuous on $[0, a)$. We may rewrite equation (2.3) in operator notation by

$$\dot{x}(t) = Lx_t, \quad t \geq 0, \quad x_0 = \varphi \text{ with } \varphi \in \mathcal{B}, \tag{2.4}$$

where the linear operator L maps $\mathcal{B}$ into $\mathbb{K}^d$ and is given by

$$L : \mathcal{B} \to \mathbb{K}^d, \quad L\eta = \int_{(-\infty,0]} \nu(ds)\eta(s). \tag{2.5}$$

Obviously it has to be guaranteed that the linear operator L is bounded on $\mathcal{B}$, which is always assumed in this article. The results mentioned below holds more generally for any bounded linear map on $\mathcal{B}$ into $\mathbb{K}^d$ not only for an integral operator as specified above. Since we are mainly interested in expanding the theory of affine stochastic differential equation with finite delay we are focusing on the case, where L is the integral operator (2.5).
We denote the Euclidean metric on $\mathbb{K}^d$ by $|\cdot|$ and the semi-norm on $\mathcal{B}$ by $\|\cdot\|_{\mathcal{B}}$. The total variation of the matrix–valued measure ν on a set $F \subseteq \mathbb{R}_-$ is denoted by $|\nu|(F)$.
The following definition postulates first conditions on the phase space $\mathcal{B}$. The same notation as in [HiMuNa91] is used.

Definition 2.1

A) The phase space $\mathcal{B}$ satisfies condition (A) if for all functions $x : (-\infty, a) \to \mathbb{K}^d$, $a > 0$, which are admissible with respect to $\mathcal{B}$, it holds:

- *(a) $x_t \in \mathcal{B}$ for every $t \in [0, a)$;*
- *(b) it exists $H > 0$, such that $|x(t)| \leq H \|x_t\|_{\mathcal{B}}$ for every $t \in [0, a)$;*
- *(c) it exists $K : [0, \infty) \to [0, \infty)$, continuous, independent of x it exists $M : [0, \infty) \to [0, \infty)$, locally bounded, independent of x :*
  $\|x_t\|_{\mathcal{B}} \leq K(t) \sup_{0 \leq u \leq t} |x(u)| + M(t) \|x_0\|_{\mathcal{B}}$ *for every $t \in [0, a)$.*

A1) The phase space $\mathcal{B}$ satisfies condition (A1) if for all functions $x : (-\infty, a) \to E$, $a > 0$, which are admissible with respect to $\mathcal{B}$, it holds:
$t \mapsto x_t$ is a $\mathcal{B}$–valued continuous function for $t \in [0, a)$.

It is quite easy to establish that a phase space $\mathcal{B}$ satisfying condition A and A1 includes all continuous functions on the negative line with compact support.

Example 2.2 *Examples for phase spaces $\mathcal{B}$ satisfying conditions A and A1 are the following ones:*
1) Fix a constant $\Upsilon \in \mathbb{R}$ and set

$$\mathcal{B} := C_\gamma := \{\varphi : (-\infty, 0] \to \mathbb{K}^d \text{ continuous } : \lim_{u \to -\infty} e^{\Upsilon u} \varphi(u) \text{ exists } \},$$
$$\|\varphi\|_{C_\gamma} := \sup_{u \leq 0} \left| e^{\Upsilon u} \varphi(u) \right| ;$$

2) Suppose that $1 \leq p < \infty$ and $0 \leq \rho < \infty$. Let $g : (-\infty, -\rho) \to \mathbb{R}_+$ be a locally integrable function satisfying the following condition:
$\exists\, G : (-\infty, 0] \to \mathbb{R}_+$ locally bounded:
$g(u + s) \leq G(u) g(s)$ for every $u \leq 0$ and every $s \in (-\infty, -\rho) \setminus N_u$,
where N_u has Lebesgue measure 0. Define

$$\mathcal{B} := C[-\rho,0] \times L^p(g) := \{\varphi : (-\infty,0] \to \mathbb{K}^d : \varphi \textit{ continuous on } [-\rho,0] \text{ and } \int_{-\infty}^{-\rho} |\varphi(s)|^p g(s)ds < \infty\}$$

$$\|\varphi\|_{\mathcal{B}} := \sup_{-\rho\le u\le 0} |\varphi(u)| + \left(\int_{-\infty}^{-\rho} |\varphi(s)|^p g(s)ds\right)^{1/p}.$$

A solution of equation (2.3) is a function $x : \mathbb{R} \to \mathbb{K}^d$ such that x_t is in $\mathcal{B}$ and that x is locally absolutely continuous on $\mathbb{R}_+$ and satisfies equation (2.3) (Lebesgue) almost everywhere on $\mathbb{R}_+$.

Theorem 2.3
Let the function space $\mathcal{B}$ satisfy condition A and A1. Then for any function $\varphi \in \mathcal{B}$, there exists a unique solution x of equation (2.3) with $x_0 = \varphi$.

Proof: See theorem 4.1.2 in [HiMuNa91]. □

For the use of the condition A and A1 in the proof of this theorem we refer to [HiMuNa91].

Example 2.4 Let us consider the following scalar example at this point:

$$\dot{x}(t) = a\int_{-\infty}^{0} x(t+s)e^{\gamma s}ds, \quad t \ge 0, \quad x(u) = \varphi(u), \quad u \le 0, \quad \varphi \in \mathcal{B}, \quad (2.6)$$

where $a, \gamma \in \mathbb{R}$. An appropriate choice of the phase space $\mathcal{B}$ is given by $\mathcal{B} = \mathbb{C} \times L^1(g)$, the one introduced in Example 2.2.2 with $g(u) = \exp(\gamma u)$, $u \le 0$, and $\rho = 0$. By the so–called chain trick, see e.g. [MDo78], one can derive the solution as

$$x(t) = \begin{cases} c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, & t \ge 0, \\ \varphi(t), & t \le 0, \end{cases} \quad (2.7)$$

where λ_1 and λ_2 are the complex zeros of the polynomial $p(\lambda) := \lambda^2 + \lambda\gamma - a$, assuming that the zeros are distinct. The complex constants c_1 and c_2 depend on a, γ and the values of $\varphi(0)$ and $\int_{\mathbb{R}_-} e^{\gamma s}\varphi(s)ds$.

In the sequel we shall always assume that the phase space $\mathcal{B}$ satisfies condition A and $A1$. One can define the solution operators for $t \ge 0$ by

$$T(t) : \mathcal{B} \to \mathcal{B}, \quad T(t)\varphi = x_t(\cdot,\varphi),$$

where $x(\cdot,\varphi) : \mathbb{R} \to \mathbb{K}^d$ is the solution of equation (2.3) with $x_0(\cdot,\varphi) = \varphi$. The uniqueness of the solution and the condition A1 on $\mathcal{B}$ imply that $\{T(t)\}_{t\ge 0}$ is a strongly continuous semi-group of bounded linear operators on $\mathcal{B}$. Further let $\{S(t)\}_{t\ge 0}$ be the strongly continuous semi-group of solution operators defined by the trivial equation $\dot{x} = 0$. To benefit from the theory of semi-groups one requires that the quotient space $\mathcal{B} / \|\cdot\|_{\mathcal{B}}$ is a Banach space which motivates the first of the following conditions on the phase space $\mathcal{B}$.

Definition 2.5

B) The phase space $\mathcal{B}$ satisfies condition (B) if $\mathcal{B}$ is complete.

C) The phase space $\mathcal{B}$ satisfies condition (C) if for every Cauchy sequence $\{\varphi_n\} \subseteq \mathcal{B}$ with respect to the semi-norm $\|\cdot\|_{\mathcal{B}}$, the following implication holds:

$$\forall\, \varphi : \mathbb{R}_- \to \mathbb{K}^d : \varphi_n \to \varphi \text{ uniformly on every compact subsets of } \mathbb{R}_- \\ \Rightarrow \varphi \in \mathcal{B} \text{ and } \|\varphi_n - \varphi\|_{\mathcal{B}} \to 0, \quad n \to \infty.$$

The examples C_γ and $C[-\rho, 0] \times L^p(g)$ of phase spaces $\mathcal{B}$ introduced in example 2.2, which fulfill the condition A and A1, satisfy also the conditions B and C. For the following discussion suppose that the phase space $\mathcal{B}$ satisfies conditions A,A1, B and C. For φ in $\mathcal{B}$, the symbol $\hat{\varphi}$ denotes the equivalence class $\{\psi : \|\psi - \varphi\|_{\mathcal{B}} = 0\}$. Hence, the quotient space $\hat{B} := \mathcal{B} / \|\cdot\|_{\mathcal{B}}$ is a Banach space with the norm $\|\hat{\varphi}\|_{\hat{\mathcal{B}}} = \|\varphi\|_{\mathcal{B}}$. Let $\hat{T}(t)$ and $\hat{S}(t)$ be the operators on $\hat{\mathcal{B}}$ induced by $T(t)$ and $S(t)$, respectively.
Call the generator A and its induced operator $\hat{A}$. Obviously $\hat{A}$ is the generator of the strongly continuous semi-group $\{\hat{T}(t)\}_{t\geq 0}$ on the Banach space $\hat{\mathcal{B}}$.
In [Ha74] the essential spectra of the operators $S(t)$ and $T(t)$ are investigated for the first time. Let $T : X \supseteq \mathcal{D}(T) \to X$ be a linear operator on a Banach space X. We denote by $\rho(T)$, $\sigma(T)$ and $\sigma_P(T)$ the resolvent set, spectrum and point spectrum of the operator T, respectively. The essential spectrum is defined by

$$\sigma_e(T) := \{\lambda \in \sigma(T) : \operatorname{rank}(\lambda \operatorname{Id} - T) \text{ is not closed or } \lambda \text{ is a limit point of } \sigma(T) \\ \text{or } \cup_{k\geq 1} \operatorname{kern}((\lambda \operatorname{Id} - T)^k) \text{ is infinite dimensional }\}$$

Furthermore, let $\alpha(F)$ be the Kuratowski measure of non–compactness of a bounded set F of the Banach space X, that is

$$\alpha(F) = \inf\{d > 0 : F \text{ has a finite cover of diameter } < d\}.$$

Now, let the operator $T : X \to X$ be bounded. Then the Kuratowski measure of the operator T is defined by

$$\alpha(T) := \inf\{s \geq 0 : \alpha(TF) \leq s\alpha(F) \quad \text{for all bounded sets } F \subseteq X\}.$$

By Nussbaum's theorem, see [Ha74], the essential spectral radius of the bounded linear operator T is given by

$$r_e(T) : = \sup\{|\lambda| : \lambda \in \sigma_e(T)\} = \lim_{n\to\infty} (\alpha(T^n))^{1/n}.$$

Now, define a parameter $\beta \in [-\infty, \infty]$ by the following relation

$$\beta := \lim_{t\to\infty} \frac{\log \alpha(\hat{S}(t))}{t} = \inf_{t\geq 0} \frac{\log \alpha(\hat{S}(t))}{t}, \tag{2.8}$$

where the second identity is a consequence of the sub-additivity of the function $t \mapsto \alpha(\hat{S}(t))$.
The constant β depends on the operator L only by the condition that L has to be bounded on $\mathcal{B}$. Apart from that the constant β is only defined by the solution semi-group $\{S(t)\}_{t\geq 0}$ of the trivial equation on $\mathcal{B}$ which is independent of the operator L. In this sense we refer to the constant β as *the constant of the phase space* $\mathcal{B}$.
In the case of a finite delay with the phase space $\mathcal{B}$ as the set of continuous functions on the interval $[-r, 0]$, the semi–group $\{S(t)\}_{t\geq 0}$ is eventually compact. Hence, because the Kuratowski measure of a compact operator equals zero, the constant is derived as $\beta = -\infty$.

Example 2.4 cont. The parameter β of the phase space $\mathcal{B} = \mathbb{C} \times L(g)$, chosen in example 2.4, can be derived as $\beta = -\Upsilon$.

Because of a decomposition of $T(t)$ into a sum of a compact operator and the operator $S(t)$, see [HaKa78], one obtains for the essential spectral radius $r_e(\hat{T}(t)) = e^{\beta t}$.
For fixed $\lambda \in \mathbb{C}$ define the function

$$e(\lambda) : \mathbb{C}^d \to \{\psi : \mathbb{R}_- \to \mathbb{C}^d \text{ continuous }\}, \quad (e(\lambda)b)(u) := e^{\lambda u} b, \quad u \leq 0.$$

In the case that the phase space $\mathcal{B}$ is a complex linear space, that means $\mathbb{K} = \mathbb{C}$, it is shown in [HiMuNa91] that for $\mathrm{Re}\lambda > \beta$ the functions $e(\lambda)b$ are elements of $\mathcal{B}$ for every $b \in \mathbb{C}^d$ and $\hat{e}(\lambda)b$ is a $\hat{\mathcal{B}}$–valued analytic function. By defining $\mathcal{B}_{\mathbb{C}} := \mathcal{B} \oplus i\,\mathcal{B}$ one obtains a complex phase space, if the original one is real, and note that $\mathcal{B}_{\mathbb{C}}$ fulfills all the conditions which the original space $\mathcal{B}$ does.
The point spectrum of the generator $\hat{A}$ is given by the following theorem:

Theorem 2.6
Let the phase space $\mathcal{B}$ be a complex space satisfying the conditions A, A1, C and D and let $\hat{A}$ be the generator of the semi-group $\hat{T}(t)$. Then:

$$\sigma_P(\hat{A}) = \{\lambda \in \mathbb{C} : \exists b \in \mathbb{C}^d \setminus \{0\} : e(\lambda)b \in \mathcal{B} \text{ and } \lambda b - L(e(\lambda)b) = 0\}$$

Proof: See theorem 5.2.1 in [HiMuNa91]. □

Define the $d \times d$ matrix $\Delta(\lambda)$, the characteristic matrix of equation (2.3), by

$$\Delta(\lambda) := \lambda I - L(e(\lambda)I) \quad \text{for } \lambda \in \mathbb{C} \text{ with } \mathrm{Re}\lambda > \hat{\beta},$$

where $\hat{\beta} := \inf\{x \in \mathbb{R} : \int_{\mathbb{R}_-} e^{xs} |\nu| (ds) < \infty\}$ and I denotes the $d \times d$–identity matrix. Recall that $\hat{\beta} \leq \beta$. All the entries of $\Delta(\lambda)$ are analytic functions for $\mathrm{Re}\lambda > \beta$ and by the previous theorem

$$\{\lambda \in \sigma_P(\hat{A}) : \mathrm{Re}\lambda > \beta\} = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > \beta \text{ and } \det[\Delta(\lambda)] = 0\}.$$

Because the operator L is assumed to be the integral operator (2.5), the function $\lambda \mapsto L(e(\lambda)I)$ is the Laplace transform of the measure ν. Hence, the entries of the matrix $\Delta(\cdot)$ are analytic functions even in $\mathrm{Re}\lambda > \hat{\beta}$. Define the set $\Lambda(c) = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \geq c \text{ and } \det[\Delta(\lambda)] = 0\}$ for a constant $c > \hat{\beta}$. For all $\lambda \in \Lambda(c)$ it holds

$$|\lambda| \leq \int_{\mathbb{R}_-} e^{u\mathrm{Re}\lambda} |\nu| (du) \leq \int_{\mathbb{R}_-} e^{uc} |\nu| (du) \leq K$$

for a constant $K > 0$. Hence, the set $\Lambda(c)$ is bounded and because the function $\lambda \mapsto \det[\Delta(\lambda)]$ is analytic for $\mathrm{Re}\lambda \geq \hat{\beta}$, the set $\Lambda(c)$ is finite for $c > \hat{\beta}$ and it contains for $c > \beta$ the eigenvalues of the generator $\hat{A}$.
In the case of a finite delay, the function $\lambda \mapsto \Delta(\lambda)$ is an entire function, because it holds $\hat{\beta} = -\infty$. In general for an infinite delay the function $\Delta(\lambda)$ is not defined on the half plane $\mathrm{Re}\lambda < \hat{\beta}$.

Theorem 2.7
Let $\mathcal{B}$ be a complex phase space with parameter β satisfying the conditions A, A1, C and D and let $c > \beta$ be a constant. Then $\hat{\mathcal{B}}$ is decomposed by $\Lambda(c)$:

$$\hat{\mathcal{B}} = \hat{P}_\Lambda \oplus \hat{Q}_\Lambda,$$

where $\hat{P}_\Lambda$ and $\hat{Q}_\Lambda$ are $\hat{T}(t)$–invariant subspaces of $\hat{B}$ and $\hat{P}_\Lambda$ is of finite dimension. Furthermore, for sufficiently small $\epsilon > 0$ there exists a constant $k := k(\epsilon) > 0$ such that

$$\left\|\hat{T}(t)\hat{\varphi}\right\|_{\hat{\mathcal{B}}} \leq k e^{(c-\epsilon)t} \|\hat{\varphi}\|_{\hat{\mathcal{B}}} \quad \text{for every } t \leq 0,\ \hat{\varphi} \in \hat{P}_\Lambda,$$
$$\left\|\hat{T}(t)\hat{\varphi}\right\|_{\hat{\mathcal{B}}} \leq k e^{(c-\epsilon)t} \|\hat{\varphi}\|_{\hat{\mathcal{B}}} \quad \text{for every } t \geq 0,\ \hat{\varphi} \in \hat{Q}_\Lambda,$$

where the semigroup $\{T(\cdot)\}$ can be extended to the whole line $\mathbb{R}$ on $\hat{P}_\Lambda$ as a solution of a differential equation.

Proof: See theorem 5.3.1 in [HiMuNa91]. □

Remark 2.8 *In the special case of $\beta < 0$ and $\det[\Delta(\lambda)] \neq 0$ for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda \geq 0$ the estimate can be strengthened:*

$$\left\|\hat{T}(t)\hat{\varphi}\right\|_{\hat{\mathcal{B}}} \leq k e^{-vt} \|\hat{\varphi}\|_{\hat{\mathcal{B}}}, \quad t \geq 0,\ \varphi \in \mathcal{B},$$

for some constants $k > 0$ and $v > 0$.

The representation of the solution of the stochastic equation (1.1) strongly bases on the so-called *differential resolvent of the locally finite measure ν*. That is a function $r : \mathbb{R}_+ \to \mathbb{K}^{d\times d}$ solving the deterministic matrix equation

$$\begin{aligned} \dot{r}(t) &= \int_{[-t,0]} \nu(ds) r(t+s) = \int_{[-t,0]} r(t+s)\nu(ds), \quad t \geq 0, \quad \text{a.e.}, \\ r(0) &= I, \end{aligned} \tag{2.9}$$

where I is the $d \times d$–identity matrix. There exists a unique locally absolutely continuous solution of equation (2.9) for almost all $t \in \mathbb{R}_+$, see theorem 3.3.1 in [GrLoSt90].
In the case of a finite delay the differential resolvent is nothing else than the so–called fundamental solution, that is the solution of the equation

$$\dot{x}(t) = \int_{[-r,0]} \nu(du)x(t+s), \quad t \geq 0, \quad x(u) = \mathbb{1}_{\{0\}}(u)\,\mathrm{I}, \quad u \in [-r, 0].$$

Example 2.4 cont. Because the function $\varphi(u) = \mathbb{1}_{\{0\}}(u)$, $u \leq 0$, is an element of the phase space $\mathcal{B} = \mathbb{C} \times L(g)$ in example 2.4, the differential–resolvent of the measure $\nu(ds) = a\exp(\gamma s)ds$ in equation (2.6) is also of the form (2.7) with special chosen parameters c_1 and c_2.

Theorem 2.9
Let the measure ν be finite and let $\Lambda(c) = \{\lambda \in \mathbb{C} : Re\lambda \geq c$ and $det[\Delta(\lambda)] = 0\}$ be given by $\Lambda(c) = \{\lambda_1, \ldots, \lambda_n\}$ for a constant $c > \hat{\beta}$. Then it holds:

$$r(t) = \sum_{i=1}^{n} p_i(t)e^{\lambda_i t} + F(t) \quad \textit{for } t > 0,$$

where p_i are some polynomials over $\mathbb{K}^{d \times d}$ for $i = 1, \ldots n$ and $F : \mathbb{R}_+ \to \mathbb{K}^{\mathrm{d}\times\mathrm{d}}$ is a continuous function with $F(t) = o(e^{ct})$ for $t \to \infty$.

Proof: The proof is given in theorem 7.2.1 in [GrLoSt90] beside the estimation of the remaining term F. The proof of the asymptotic behavior of F requires the estimation of an integral given in the proof in [GrLoSt90]. That can be easily done by using the Neumann series expansion and some calculus establishes the result. □

In the case of a measure with compact support, the equivalence of the integrability of x and x^2 is well–known, where x is the fundamental solution. We will show, that, if $\hat{\beta} < 0$, this and other properties of the fundamental solution carry over to the differential resolvent of an arbitrary measure with support $\mathbb{R}_-$. Observe that the space C_{-b} of example 2.4.1 is a proper choice for the phase space for every $b > \hat{\beta}$, if $\hat{\beta} < \infty$. One has to show only the boundness of the integral operator L on C_{-b}:

$$\int_{(-\infty,0]} |\varphi(s)|\,|\nu|\,(ds) \leq \|\varphi\|_{C_{-b}} \int_{(-\infty,0]} e^{bs}\,|\nu|\,ds \leq M\,\|\varphi\|_{C_{-b}}$$

for every $\varphi \in C_{-b}$ and a constant $M > 0$.

Theorem 2.10
Let r be the differential resolvent of the measure ν and suppose $r \in L^p(\mathbb{R}_+)$, $p \in [1, \infty)$, and $\hat{\beta} < 0$. Let x be the solution of equation (2.3) with $x_0 = \varphi$, where $\varphi \in C_{-b}$ for a fixed $b \in (\hat{\beta}, 0)$. Then it holds: $x \in L^p(\mathbb{R}_+)$.

Proof: Rewriting equation (2.3) to a inhomogeneous Volterra equation yields in

$$\begin{aligned}\dot{x}(t) &= \int_{(-\infty,0]} \nu(ds)x(t+s) \\ &= \int_{[-t,0]} \nu(ds)x(t+s) + \int_{(-\infty,-t)} \nu(ds)\varphi(t+s) \\ &=: \int_{[-t,0]} \nu(ds)x(t+s) + f(t), \quad \text{for } t \geq 0, \quad x(0) = \varphi(0). \end{aligned} \tag{2.10}$$

By the variation of constants formula, see e.g. theorem 3.3.5 in [GrLoSt90], the solution of equation (2.10) is given by

$$x(t) = r(t)\varphi(0) + (r * f)(t) \quad \text{for } t \geq 0, \tag{2.11}$$

where the convolution is defined as $(r * f)(t) := \int_0^t r(t-s)f(s)ds$. We have to show, that $f \in L^1(\mathbb{R}_+)$:

$$\begin{aligned}\int_0^\infty |f(s)|\, ds &\leq \int_0^\infty \int_{(-\infty,-s)} |\varphi(s+u)|\, e^{-b(s+u)} e^{b(s+u)}\, |\nu|\, (du) ds \\ &\leq \|\varphi\|_{C_{-b}} \int_0^\infty e^{bs} \int_{(-\infty,0]} e^{bu}\, |\nu|\, (du) ds \\ &< \infty\end{aligned}$$

Hence it holds $r * f \in L^p(\mathbb{R}_+)$, see e.g. theorem 2.2.2 in [GrLoSt90], and in account of (2.11), the assertion is proved. □

Theorem 2.11
Let r be the differential–resolvent of the measure ν and let $\hat{\beta} < 0$. Then it is equivalent:

1. *$det[\Delta(\lambda)] \neq 0$ for all $\lambda \in \mathbb{C}$ with $Re\lambda \geq 0$;*
2. *$r(t) = O(e^{ct})$ for $t \to \infty$ and some $c < 0$;*
3. *$r \in L^1(\mathbb{R}_+)$;*
4. *$r \in L^2(\mathbb{R}_+)$.*

Proof: The equivalence of 1 and 3 is given in theorem 3.3.5 in [GrLoSt90]. Obviously, 2 implies 3 and 4 and theorem 2.9 establishes the implication from 1 to 2. It remains to prove that 4 implies 1. Suppose there are some $\lambda_i \in \mathbb{C}$, $i = 1, \cdots, n$, with $\text{Re}\lambda_i \geq 0$ and $\det[\Delta(\lambda_i)] = 0$. Then there exists $c_i \in \mathbb{C}^d \setminus \{0\}$ for every $i = 1, \ldots, n$, such that $x_i(t) := e^{\lambda_i t} c_i$ is a solution of

$$\dot{x}_i(t) = \int_{(-\infty,0]} \nu(ds)x_i(t+s), \quad t \geq 0, \quad x_i(u) = \varphi_i(u) := e^{\lambda_i u} c_i, \quad u \leq 0.$$

Since for a fixed $b \in (\hat{\beta}, 0)$, the functions φ_i are elements of the phase space C_{-b}, theorem 2.10 implies $x_i \in L^2(\mathbb{R}_+)$. But this does not hold for every λ_i with $\mathrm{Re}\lambda_i \geq 0$. □

3 The Stochastic Case

By means of the results of the deterministic case we establish existence and uniqueness of a solution of equation (1.1). We assume always that the operator L defined in (2.5) is bounded on $\mathcal{B}$. Let the differential resolvent r of the locally finite measure ν be continued on the negative line by $r(t) = 0$ for $t < 0$. Further on, the proof requires an integral equation which is already used in the case of a measure with compact support in [MoSch90].

Lemma 3.1
Let $\Phi : G_0 \to \mathbb{R}^2$, $G_0 \subseteq \mathbb{R}^2$, be given by $\Phi(s,u) := (s+u,u)$ with $G_1 := \Phi(G_0)$ and let $f : G_1 \to \mathbb{R}^d$ be a function with $\int_{G_0} |f(s+u,s)|\,|\nu|\,(du)ds < \infty$. Then it holds

$$\int_{G_0} f(s+u,u)\mu_0(ds,du) = \int_{G_1} f(x,y)\mu_0(dx,dy),$$

where $\mu_0(ds,du) := ds \times \nu(du)$ for each locally finite measure ν.

Proof: Define the measure

$$\mu_1(H) := \mu_0(\Phi^{-1}(H)) \quad \text{for all } H \in \sigma(\mathbb{R}^2) \cap G_1,$$

and let $R := [a,b] \times [c,d] \subseteq G_1$ be a rectangle in G_1. For that we get

$$\mu_1(R) = \mu_0(\Phi^{-1}(R)) = \int_{[c,d]} \int_{a-u}^{b-u} ds\nu(du) = (b-a)\nu([c,d]) = \mu_0(R).$$

This implies

$$\begin{aligned} \int \mathbb{1}_{G_0}(s,u) f(s+u,u)\nu(du)ds &= \int \mathbb{1}_{G_1}(x,y) f(x,y)(\Phi\mu_0)(dx,dy) \\ &= \int \mathbb{1}_{G_1}(x,y) f(x,y)\mu_0(dx,dy), \end{aligned}$$

which completes the proof. □

In contrast to the case of a measure with compact support the assumption $\int_{G_0} |f(s+u,s)|\,|\nu|\,(du)ds < \infty$ in the Lemma above is essential, if the measure ν is not finite and the set G_0 is unbounded.
For the definition of a unique strong solution of the equation (1.1), we refer to definition 5.2.1 in [Mao97], with the obvious completion for the case of an unbounded delay.

Theorem 3.2

Assume that the phase space $\mathcal{B}$ satisfies the conditions A and A1. Then for any stochastic process Υ with $P(\Upsilon \in \mathcal{B}) = 1$ there exists a unique strong solution $\{X(t), t \in \mathbb{R}\}$ of the differential equation (1.1) and it holds for $t \geq 0$:

$$X_t(u) = (T(t)\,\Upsilon)(u) + \int_0^t r(t-s+u)dW(s), \quad u \leq 0,$$
$$X_t \in \mathcal{B} \;\; P\text{-}a.s.$$

Proof: For $u \leq 0$ and $t \geq 0$ the process

$$\begin{aligned} Z_t(u) &:= \int_0^t r(t-s+u)dW(s) \\ &= \begin{cases} 0 & : \; u \leq -t, \\ \int_0^{t+u} r(t+u-s)dW(s) & : \; -t \leq u \leq 0, \end{cases} \end{aligned}$$

has P-a.s. continuous paths with compact support. Hence Z_t is in $\mathcal{B}$ P-a.s. for all $t \geq 0$ and so it holds also $X_t \in \mathcal{B}$. The family $\{X_t\}_{t \geq 0}$ corresponds to a function on $\mathbb{R}$, since for $t \geq 0$ and $u \leq 0$:

$$X_t(u) = X(t+u) = \begin{cases} \Upsilon(t+u) & : \; t+u \leq 0, \\ X_{t+u}(0) & : \; t+u \geq 0. \end{cases}$$

Since the process $\{X(t), t \in \mathbb{R}\}$ has P-a.s. continuous paths for $t \geq 0$ and fulfills $X_0 = \Upsilon$, the process X is an admissible function with respect to $\mathcal{B}$ for P-a.e. $\omega \in \Omega$. Condition A2 implies that $t \mapsto X_t$ is continuous for $t \geq 0$ and as the functional L is also continuous, it holds:

$$\int_0^t \left| \int_{(-\infty,0]} \nu(du)X(s+u) \right| ds = \int_0^t |L(X_s)|\, ds < \infty.$$

To show uniqueness let $X^{(1)}$ and $X^{(2)}$ be two solutions of equation (1.1).Then $\tilde{X} = X^{(1)} - X^{(2)}$ satisfies for $t \geq 0$:

$$\tilde{X}(t) = \int_0^t L(X_s^{(1)})ds - \int_0^t L(X_s^{(2)})ds = \int_0^t L(\tilde{X}_s)ds,$$
$$\tilde{X}_0 = 0.$$

Since equation (2.3) has a unique solution, it follows $\tilde{X} \equiv 0$. By using Lemma 3.1 we show that $X_t(0)$ is a solution of equation (1.1):

$$
\begin{aligned}
&X(t) - \Upsilon(0) - W(t) - \int_0^t LX_s ds \\
&\quad = (T(t)\Upsilon)(0) - \Upsilon(0) + \int_0^t \dot{r}(t-s)W(s)ds - \int_0^t L(T(s)\Upsilon)ds \\
&\qquad - \int_0^t LW_s ds - \int_0^t L\left(\int_0^s \dot{r}(s-v+\cdot)W(v)dv\right) ds \\
&\quad = \int_0^t \dot{r}(t-s)W(s)ds - \int_0^t LW_s ds - \int_0^t L\left(\int_0^s \dot{r}(s-v+\cdot)W(v)dv\right) ds \\
&\quad = \int_0^t \dot{r}(t-s)W(s)ds - \int_0^t \nu([s-t,0])W(s)ds \\
&\qquad - \int_0^t \left(\int_{[v-t,0]} \nu(du) \int_{u+v}^t \dot{r}(s-v+u)ds\right) W(v)dv \\
&\quad = \int_0^t \dot{r}(t-s)W(s)ds - \int_0^t \nu([s-t,0])W(s)ds \\
&\qquad - \int_0^t \left(\int_{[v-t,0]} \nu(du)(r(t-v+u) - \mathrm{I})\right) W(v)dv = 0,
\end{aligned}
$$

which completes the proof. □

Now we establish the existence and representation of a stationary solution of equation (1.1) under the assumption $\hat{\beta} < 0$, that is

$$
\exists\, b < 0 : \int_{(-\infty,0]} e^{bs}\, |\nu|\, (ds) < \infty. \tag{3.12}
$$

The constant $\hat{\beta}$ indicates the half plane $\mathrm{Re}z > \hat{\beta}$, where the Laplace transform $\hat{\nu}(z) = \int_{\mathbb{R}_-} \nu(ds) I e^{zs}$ exists, and depends only on the measure ν. In contrast the constant β is determined by the phase space $\mathcal{B}$ under consideration. In account of the relation $\hat{\beta} \le \beta$, the condition $\beta < 0$ guarantees that condition (3.12) is satisfied. The assumption (3.12) allows to prove the existence of a stationary solution in a similar way as in the case of a finite delay. The assumption (3.12) and some more weak hypothesis admit the formulation of equivalent conditions for the existence for a stationary solution, see remark 3.8.1, similar to the case of a finite delay, see [GuKu00].
Let $\{W(t), t \in \mathbb{R}\}$ be a $\mathbb{R}^d$–valued Brownian motion on the whole real line, defined by $W(t) = \mathbb{1}_{(-\infty,0]}(t) W_1(-t) + \mathbb{1}_{[0,\infty)}(t) W(t)$, where W and W_1 are independent Brownian motions on the positive line. Assume that the differential resolvent r of the measure ν satisfies $|r(t)| = O(e^{ct})$ for $t \to \infty$ and a constant $c < 0$. On account of theorem 2.11 this is equivalent to $r \in L^2(\mathbb{R}_+)$. In this case one can define the following integral as in [GuKu00]:

$$
\int_{-\infty}^t r(t-s)dW(s) \quad \text{for every } t \in \mathbb{R}\,.
$$

Because of (3.12), $|r(t)| = O(e^{ct})$ for $t \to \infty$ implies for a.e. $t \geq 0$

$$|\dot{r}(t)| \leq C_1 \int_{[-t,0]} e^{c(t+s)} |\nu| (du) \leq C_1 e^{(b \vee c)t} \int_{\mathbb{R}_-} e^{bs} |\nu| (du) \leq C e^{(b \vee c)t}$$

for some constants $C, C_1 > 0$. Hence, the formula of partial integration holds for the integral

$$\int_{-\infty}^{t} r(t-s) dW(s) = r(t) + \int_{-\infty}^{t} \dot{r}(t-s) W(s) ds \quad \text{for all } t \in \mathbb{R}_+ .$$

Using the formula we obtain a representation of the stationary solution:

Theorem 3.3
Let condition (3.12) be satisfied by the measure ν and assume that the differential resolvent r of the measure ν fulfills $r \in L^2(\mathbb{R}_+)$.
Then the process

$$\{X(t) := \int_{-\infty}^{t} r(t-s) dW(s), t \in \mathbb{R}\}$$

is a stationary solution of equation (1.1) with the initial condition

$$\Upsilon(u) := \int_{-\infty}^{u} r(u-s) dW(s), \quad \text{for } u \leq 0.$$

Proof: One has to show

$$X(t) - \Upsilon(0) - \int_0^t L(X_s) ds - W(t) = 0 \quad \text{for all } t \geq 0.$$

By partial integration this is equivalent to

$$\begin{aligned} &\int_{-\infty}^{t} \dot{r}(t-s) W(s) ds - \int_{-\infty}^{0} \dot{r}(-s) W(s) ds - \int_0^t \int_{(-\infty,0]} \nu(du) W(s+u) ds \\ &- \int_0^t \left(\int_{(-\infty,0]} \nu(du) \int_{-\infty}^{s+u} \dot{r}(s+u-v) W(v) dv \right) ds = 0 \quad \text{for all } t \geq 0. \end{aligned} \tag{3.13}$$

In contrast to the case of a finite delay the finiteness of the integrals requires a justification. Since $|\dot{r}(t)| = O(e^{c_1 t})$ holds for $t \to \infty$ with $c_1 = c \vee b$, the first two integrals in (3.13) are P-a.s. finite. For the third one choose $-b > \epsilon > 0$ then it holds P-a.s. for every $s \geq 0$ due to the law of the iterated logarithm and the condition (3.12)

$$\int_{\mathbb{R}_-} |W(s+u)| |\nu| (du) \leq C e^{-\epsilon s} \int_{\mathbb{R}_-} e^{bu} |\nu| (du) < \infty \tag{3.14}$$

For $-c_1 > \epsilon > 0$ one obtains

$$\begin{aligned}
&\int_{\mathbb{R}_-} \int_{-\infty}^{s+u} |\dot{r}(s+u-v)W(v)|\, dv\, |\nu|\,(du) \\
&\quad \leq C \int_{\mathbb{R}_-} e^{c_1(s+u)} \int_{-\infty}^{s+u} e^{-(c_1+\epsilon)v} dv\, |\nu|\,(du) \\
&\quad = Ce^{-\epsilon s} \int_{\mathbb{R}_-} e^{-\epsilon u}\, |\nu|\,(du) \\
&\quad < \infty \quad \text{for every } s \geq 0 \text{ P-a.s.}
\end{aligned}$$

Due to (3.14) the assumptions of Lemma 3.1 are satisfied such that the third integral in (3.13) yields

$$\int_0^t \int_{\mathbb{R}_-} \nu(du) W(s+u) ds = \int_{-\infty}^t \nu([v-t, v \vee 0]) W(v) dv, \tag{3.15}$$

where the assumptions are satisfied, because of condition (3.12) the inner integral on the right hand side in (3.15) exists and is continuous in s.
Since the last iterated integral in (3.13) exists for the absolute value of the integrand we get by Fubini's theorem

$$\begin{aligned}
&\int_0^t \left(\int_{\mathbb{R}_-} \nu(du) \int_{-\infty}^{s+u} \dot{r}(s+u-v) W(v) dv \right) ds \\
&= \int_{\mathbb{R}_-} \nu(du) \int_{-\infty}^{u} \int_0^t \dot{r}(s+u-v) ds W(v) dv \\
&\qquad\qquad + \int_{\mathbb{R}_-} \nu(du) \int_u^{u+t} \int_{v-u}^t \dot{r}(s+u-v) ds W(v) dv \\
&= \int_{\mathbb{R}_-} \nu(du) \int_{-\infty}^{t+u} r(t+u-v) W(v) dv - \int_{\mathbb{R}_-} \nu(du) \int_{-\infty}^{u} r(u-v) W(v) dv \\
&\qquad\qquad - \int_{\mathbb{R}_-} \nu(du) \int_u^{t+u} W(v) dv.
\end{aligned} \tag{3.16}$$

In a similar way as above all the integrals in (3.16) exist even for the absolute value of the integrands. By using Fubini's theorem again we obtain

$$\begin{aligned}
&\int_0^t \left(\int_{\mathbb{R}_-} \nu(du) \int_{-\infty}^{s+u} \dot{r}(s+u-v) W(v) dv \right) ds \\
&= \int_{-\infty}^t \dot{r}(t-v) W(v) dv + \int_{-\infty}^0 \dot{r}(-v) W(v) dv + \int_{-\infty}^t \nu([v-t, v \wedge 0]) W(v) dv.
\end{aligned}$$

By the last equality and (3.15) the equation (3.13) is established. □

Remark 3.4 *Under the conditions of the previous theorem the stationary solution* $X(t) = \int_{-\infty}^t r(t-s) dW(s)$ *is a Gaussian process with*

$$EX(t) = 0, \quad Cov(X(t), X(t+h)) = \int_0^\infty r(s) r(s+h) ds, \quad \textit{for all } t, h \geq 0.$$

The spectral density of the process is given by

$$f(s) = \frac{1}{2\pi}(\Delta(-is))^{-1}\left((\Delta(is))^{-1}\right)^T, \quad s \in \mathbb{R}.$$

To prove the last assertion, observe, that the Laplace transform $\hat{r}$ of the differential resolvent can be derived as $\hat{r}(z) = (\Delta(z))^{-1}$ for all $z \in \mathbb{C}$ with $Rez \geq 0$. An application of the inverse Laplace transform and of Parseval's equality finishes the proof.

In theorem 3.3 we do not specify a phase space $\mathcal{B}$ but ensure the existence of the integrals by the assumptions (3.12) and $|r(t)| = O(e^{ct})$, $c < 0$, for $t \to \infty$ instead of requiring $\Upsilon(\cdot) = \int_{-\infty}^{\cdot} r(\cdot - s)dW(s) \in \mathcal{B}$ P-a.s. and using the condition A to conclude the existence of the integrals. On the other hand the phase space C_{-b} is an appropriate choice for every $b > \hat{\beta}$ under the assumption (3.12), as it is established before theorem 2.10. By assuming in addition $|r(t)| = O(e^{ct})$, $c < 0$, for $t \to \infty$, one obtains, that the initial condition $\Upsilon(\cdot) = \int_{-\infty}^{\cdot} r(\cdot - s)dW(s)$ is P-a.s. an element of C_{-b}.

Theorem 3.5
Let $\mathcal{B}$ be a phase space satisfying conditions A, A1, C and D with parameter $\beta < 0$. Further on assume $det[\Delta(\lambda)] \neq 0$ for all $\lambda \in \mathbb{C}$ with $Re\lambda \geq 0$. Then for each solution $\{X(t) := (X_1(t), \ldots, X_d(t))^T, t \in \mathbb{R}\}$ of equation (1.1) with initial condition $X_0 = \Upsilon \in \mathcal{B}$ P-a.s. it holds for arbitrary $0 \leq t_1 < t_2 < \cdots < t_n$ and $n \in \mathbb{N}$:

$$(X_k(t+t_1), \ldots, X_k(t+t_n))^T \overset{\mathcal{D}}{\to} U, \quad t \to \infty, \quad k = 1, \ldots d,$$

$$\text{where } U \overset{\mathcal{D}}{=} N(0, \Sigma) \text{ and } \Sigma = \left(\int_0^\infty < r_k(s), r_k(|t_i - t_j| + s) > ds\right)_{i,j=1,\ldots n},$$

where $\mathcal{D}$ denotes in distribution and r_k denotes the k-th row of r for $k = 1, \ldots, d$.

Proof: Define the vectors

$$u = (u_1, u_2, \ldots, u_n)^T \in \mathbb{R}^d,$$
$$\mathbb{X}(t) = (X_k(t+t_1), \ldots, X_k(t+t_n))^T$$
$$\mathbb{Y}(t) = \left(\int_0^{t+t_1} r_k(t+t_1-s)dW(s), \ldots, \int_0^{t+t_n} r_k(t+t_n-s)dW(s)\right)^T.$$

Because of Remark 2.8 and condition A it holds for $t \to \infty$

$$|(T(t+t_j)\,\Upsilon)(0)| \leq H\left\|\hat{T}(t+t_j)\hat{\Upsilon}\right\|_{\hat{\mathcal{B}}} \leq H\,C\left\|\hat{\Upsilon}\right\|_{\hat{\mathcal{B}}} e^{-v(t+t_j)} \to 0 \quad \text{P-a.s.}$$

Hence one obtains

$$\lim_{t\to\infty}\left|\mathbb{E}\left[e^{i<u,\mathbb{X}(t)>} - e^{i<u,\mathbb{Y}(t)>}\right]\right|$$
$$\leq \lim_{t\to\infty}\left|\mathbb{E}\left[\exp\left(i\sum_{j=1}^n u_j(T(t+t_j))\,\Upsilon)(0)\right) - 1\right]\right| = 0.$$

On account of

$$\mathbb{E}\left[e^{i<u,\mathbb{Y}(t)>}\right] = e^{-\frac{1}{2}u^T\Sigma_t u}$$
$$\text{with} \quad \Sigma_t = \left(\int_0^{t+(t_i\wedge t_j)} < r_k(s), r_k(|t_i - t_j| + s) > ds\right)_{i,j=1,\ldots,n}$$

we obtain

$$\begin{aligned}\lim_{t\to\infty} \mathbb{E}\left[e^{i<u,\mathbb{X}(t)>}\right] &= \lim_{t\to\infty} \mathbb{E}\left[e^{i<u,\mathbb{Y}(t)>}\right] \\ &= \lim_{t\to\infty} \exp\left(-\frac{1}{2}u^T\Sigma_t u\right) \\ &= \exp\left(-\frac{1}{2}u^T\Sigma u\right).\end{aligned}$$

Thus the conclusion follows. □

Remark 3.6 *The assertions $\beta < 0$ and $det[\Delta(\lambda)] \neq 0$ for all $Re\lambda \geq 0$ guarantee that the solution semi–group $\{T(t)\}_{t\geq 0}$ is exponentially stable. It is sufficient to require that the solution semi–group is only stable, that is $\|T(t)\varphi\|_{\mathcal{B}} \to 0$ for $t \to \infty$ for all $\varphi \in \mathcal{B}$.*

Because of theorem 2.11 the condition $\det[\Delta(\lambda)] \neq 0$ for all $\mathrm{Re}\lambda \geq 0$ is equivalent to $r \in L^2(\mathbb{R}_+)$, the condition that guarantees the existence of a stationary solution. The quadratic integrability of the differential resolvent r is even necessary for that:

Theorem 3.7
Let the phase space $\mathcal{B}$ satisfy condition A and A1. For an initial condition $X_0 = \Upsilon \in \mathcal{B}$ P-a.s. assume that there exists a solution $X = \{X(t) := (X_1(t), \ldots, X_d(t))^T : t \in \mathbb{R}\}$ of equation (1.1) converging in distribution. Then for the differential resolvent r of the locally finite measure ν holds: $r \in L^2(\mathbb{R}_+)$.

Proof: Let φ_t^k be the characteristic function of the k-th component $X_k(t)$ of the process X at time t. Then one obtains

$$\exists\, \delta_k \in (0,1)\ \exists\, t_0 \geq 0 : \left|\varphi_t^k(u)\right| \geq \delta_k \quad \forall\, u \in [0,u_0] \text{ and } \forall\, t \geq t_0.$$

Since the initial condition Υ and the Brownian motion $\{W(t), t \geq 0\}$ are independent one gets for $t \geq t_0$ and every $u \in [0,u_0]$

$$\delta_k \leq \left|\mathbb{E}\left[\exp\left(iu\int_0^t r_k(t-s)dW(s)\right)\right]\right| = \exp\left(-\frac{1}{2}u^2\int_0^t |r_k(s)|^2\, ds\right)$$

where r_k denotes the k-th row of r. □

Remark 3.8

1. *Let the phase space $\mathcal{B}$ satisfy the conditions A, A1, C and D with parameter $\beta < 0$ Furthermore, assume that $\int_{-\infty}^{\cdot} r(\cdot - s)dW(s) \in \mathcal{B}$ P-a.s. As in the case of a finite delay (see [GuKu00]), one obtains equivalence between:*

   (a) *there exists a stationary solution $X = \{X(t); t \in \mathbb{R}\}$ of (1.1) with $X_0 \in \mathcal{B}$ P-a.s.;*

   (b) *there is a solution $X = \{X(t); t \in \mathbb{R}\}$ of (1.1) with $X_0 \in \mathcal{B}$ P-a.s., converging in distribution as $t \to \infty$;*

   (c) *any solution $X = \{X(t); t \in \mathbb{R}\}$ of (1.1) with $X_0 \in \mathcal{B}$ P-a.s. converges in distribution for $t \to \infty$;*

   (d) $r \in L^2(\mathbb{R}_+)$.

2. *If the measure ν does not satisfy the condition (3.12) the question concerning the existence of a stationary solution of equation (1.1) is not (yet) answered. A possible way to answer may be viewing equation (1.1) as an equation in the Banach space $\hat{\mathcal{B}}$ and using methods of the theory of infinite dimensional stochastic differential equations in order to proof the existence of an invariant measure on the Banach space $\hat{\mathcal{B}}$.*

Example 2.4 cont. The stochastic equation corresponding to the deterministic one of example 2.4 is given by

$$dX(t) = a \int_{\mathbb{R}_-} X(t+s)e^{\gamma s}ds + dW(t), \quad t \geq 0, \quad X_0 = \Upsilon \in \mathbb{C} \times L(g) \text{ P-a.s.}$$

The differential–resolvent r is square–integrable if and only if the real part of the zeros of the polynomial $p(\lambda) = \lambda^2 + \gamma\lambda - a$ are negative. Hence, the existence of a stationary solution for this equation requires $\gamma > 0$ and $a < 0$. In the case of a finite delay corresponding to this equation, that is $\nu(ds) = a \exp(\gamma s)\, \mathbb{1}_{[-r,0]}(s)ds$, the area of stationarity depends much more on a subtle relation between the parameters a and γ. The area covers only a part of the quadrant $\gamma > 0$ and $a < 0$, but also a part of the quadrant $a < 0$ and $\gamma \leq 0$, see [Rei01].

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