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On L^p-stability of numerical schemes for Affine Stochastic Delay Differential Equations: stochastic recurrence relations

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On L^p-stability of numerical schemes for Affine Stochastic Delay Differential Equations: stochastic recurrence relations *

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Abstract

Numerical solutions of SDDE often reflect to only a limited extent the exact solution behaviour. Hence it is necessary to identify those parameters of SDDE and algorithm for which a numerical method in use is reliable. For affine SDDE test equations, there exist estimates of the stability regions of a numerical method. However, these results rely on bounds for covariance terms. In this paper exact but high dimensional stochastic affine (linear) recurrence relations are derived for some $p > 1$. A reduction method presented here allows the representation of the corresponding characteristic polynomial as a determinant of a matrix of polynomial coefficients and lower dimension. This can be used to compute non-zero coefficients of the characteristic polynomial for application to stability questions concerning SDDE. A number of areas where work is continuing is indicated.

Keywords: recurrence relation, stochastic recurrence relation, SDDE, SFDE, stochastic delay equations, numerical algorithms, stability, stability regions

AMS 2000 Subject Classification: 34K50, 34F05, 60Hxx

1 Introduction

1.1 Background on deterministic recurrences

Related with the **affine recurrence relation**

$$\begin{array}{lcll} v_{n+1} & = & B + Av_n, & n \in \mathbb{N}, \\ v_n & \in & \mathbb{R}^d, & n \in \mathbb{N}, \\ v_0 & = & c \in \mathbb{R}^d, & \\ B & \in & \mathbb{R}^d, & \\ A & \in & \mathbb{M}_d(\mathbb{R}), & \end{array} \tag{1.1}$$

are often two questions. The first question is that for the qualitative behaviour of the sequence $v = \{v_n\}_{n\in\mathbb{N}}$ of state values in its time evolution. That is the question, whether v (or some p-th moment $|v|^p = \{\,|v_n|^p\,\}_{n\in\mathbb{N}}$, $p \in [1,\infty)$) is goes or oscillates to $\pm\infty$ (explosion), remains bounded or converges to some limit. A second question is that of stability with respect to changes in the initial values of v. That is, consider the series $\tilde{v} = \{\tilde{v}_n\}_{n\in\mathbb{N}}$ with

$$\begin{array}{lcll} \tilde{v}_{n+1} & = & B + A\tilde{v}_n, & n \in \mathbb{N}, \\ \tilde{v}_n & \in & \mathbb{R}^d, & n \in \mathbb{N}, \\ \tilde{v}_0 & = & \tilde{c} \in \mathbb{R}^d. & \end{array} \tag{1.2}$$

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Then $\tilde{v}$ is generated by the same affine coefficients B and A, but with a start vector $\tilde{c}$ differing from c. Understanding $\tilde{c}$ as perturbed initial vector, it is of interest to ask, how the initial error $\delta_0 = c - \tilde{c}$ propagates with the time evolution. Due to the affine nature of v and $\tilde{v}$ the error propagation can be described by the linear recurrence relation

$$\begin{aligned} \delta_{n+1} &= A\delta_n, \quad n \in \mathbb{N}, \\ \delta_n &= v_n - \tilde{v}_n \in \mathbb{R}^d, \quad n \in \mathbb{N}, \\ \delta_0 &= c - \tilde{c} \in \mathbb{R}^d. \end{aligned} \tag{1.3}$$

It is of interest to know the qualitative behaviour of $\delta = \{\delta_n\}_{n\in\mathbb{N}}$, that is whether δ goes to infinity (explosion), remains bounded or converges to zero. Both questions can be answered.

In case of homogeneity ($B = 0$) of (1.1) we know that $\vec{v}_n = A^n v_0 = PJ^nP^{-1}v_0$, where the column vectors of P form a set of independent (generalized) eigenvectors of A and J is the corresponding Jordan block matrix of A. The behaviour of v depends on the eigenvalue structure of A or J, respectively. Denote by

$$\begin{aligned} \sigma(M) &= \{\, \lambda \in \mathbb{C} \mid |A - \lambda I| = 0 \,\} \text{ the } \textbf{spectrum} \text{ of } M, \\ \mu(\lambda) & \quad \text{the } \textbf{geometric multiplicity} \text{ of } \lambda \in \sigma(M), \\ \gamma(\lambda) & \quad \text{the } \textbf{algebraic multiplicity} \text{ of } \lambda \in \sigma(M), \\ \sigma^s(M) &= \{\, \lambda \in \mathbb{C} \mid \lambda \in \sigma(M), \mu(\lambda) = \gamma(\lambda) \,\}, \text{ the } \textbf{semi-simple eigenvalues} \text{ of } M, \\ \sigma_{<1}(M) &= \{\, \lambda \in \mathbb{C} \mid \lambda \in \sigma(M), |\lambda| < 1 \,\}, \\ \sigma^{s,1}_{\leq 1}(M) &= \{\, \lambda \in \mathbb{C} \mid \lambda \in \sigma(M), |\lambda| \leq 1, |\lambda| = 1 \to \lambda \in \sigma^s(M) \,\}, \\ \bar{\sigma}^{s,1}_{\leq 1}(M), &= \mathbb{C} \setminus \sigma^{s,1}_{\leq 1}(M), \end{aligned} \tag{1.4}$$

where for some $D \in \mathbb{N}\setminus\{0\}$ $M, I \in \mathbb{M}_D(\mathbb{R})$, I is the identity matrix. If $\sigma(A) = \sigma_{<1}(A)$, then $\lim_{n\to\infty} x_n = 0$. If $\sigma(A) = \sigma_{\leq 1}(A)$, then x remains bounded. But if $\sigma(A) \cap \bar{\sigma}_{\leq 1}(A) \neq \emptyset$ then there exist start vectors $c \in \mathbb{R}^d$ for which v explodes. The eigenvalues are roots of the **characteristic** or **stability polynomial** $|A - \lambda I|$.

For the non-homogeneous case $B \neq 0$ of (1.1), the reformulation of (1.1) to

$$\begin{aligned} \bar{v}_{n+1} &= \bar{A}\bar{v}_n, \quad n \in \mathbb{N}, \\ \bar{v}_n &= (v_n^\top, 1)^\top \in \mathbb{R}^d \times \{1\}, \quad n \in \mathbb{N}, \\ \bar{v}_0 &= \bar{c} \in \mathbb{R}^d \times \{1\}, \\ \bar{A} &= \begin{pmatrix} A & B \\ 0 & 1 \end{pmatrix}, \end{aligned} \tag{1.5}$$

allows us to answer the question about the evolution of v. It is easy to verify that $\bar{v}_n = \bar{A}^n \bar{v}_0$ or $v_n = A^n v_0 + (\sum_{i=0}^{n} A^i)B$ and that the behaviour of v depends on the eigenvalue structure of $\bar{A}$. As $\bar{A}$ is an upper block triangular matrix, the eigenvalues of $\bar{A}$ are solutions of the characteristic polynomial $|\bar{A} - \lambda I| = (1 - \lambda)|A - \lambda I|$, hence $\sigma(\bar{A}) = \sigma(A) \cup \{1\}$. If $\sigma(\bar{A}) = \sigma_{<1}(A)$, then $\lim_{n\to\infty} v_n = (E - A)^{-1}b$. If $\sigma(A) = \sigma^{s,1}_{\leq 1}(A)$, then v remains bounded. But if $\sigma(A) \cap \bar{\sigma}^{s,1}_{\leq 1}(A) \neq \emptyset$, then there exist start vectors $\bar{c}$ for which $\bar{v}$ and hence v explodes.

It is remarkable that the structure of the eigensystems of A or $\bar{A}$, respectively, completely characterizes the qualitative behaviour of v in its course of evolution under each $\mathbb{R}^d$-norm $|\ |$. This is due to the facts that the requested properties explosion, boundedness, convergence are topological properties that from the behaviour of v one can determine the behaviour of $|v| = \{\, |v_n| \,\}_{n\in\mathbb{N}}$ and that in $\mathbb{R}^d$ all norms are equivalent.

The complete characterization of the qualitative behaviour of solutions v of (1.1) allows us also to characterize thoroughly **deterministic affine recurrences with memory**

$$\begin{aligned} x_{n+1} &= b + \sum_{i=0}^{k} a_i x_{n-i}, \quad n \in \mathbb{N}, \\ x_n &\in \mathbb{R}, \quad n \in (-k + \mathbb{N}), \\ x_n &= c_n \in \mathbb{R}, \quad n \in (-k + \mathbb{N}_k), \\ b, a_i &\in \mathbb{R}, \quad i \in \mathbb{N}_k. \end{aligned} \tag{1.6}$$

Such recurrences with memory are an appropriate model for a wide range of deterministic phenomena. As for the affine recurrence relation (1.1) the two questions for the evolution of the state variables $x = \{x_n\}_{n\in\mathbb{N}}$ and the error propagation in time are of interest. Concerning the error propagation, consider time the series $\tilde{x} = \{\tilde{x}_i\}_{i\in\mathbb{N}}$ with

$$\begin{aligned} \tilde{x}_{n+1} &= b + \sum_{i=0}^{k} a_i \tilde{x}_{n-i}, \qquad n \in \mathbb{N}, \\ \tilde{x}_n &\in \mathbb{R}, \qquad n \in (-k + \mathbb{N}) \\ \tilde{x}_n &= c_n \in \mathbb{R}, \qquad n \in (-k + \mathbb{N}_k) \end{aligned} \tag{1.7}$$

Then the error $\delta = \{\delta_n\}_{n\in\mathbb{N}}$, where $\delta_n = \tilde{x}_n - x_n$, $n \in (-k + \mathbb{N})$ is given by a linear recurrence with memory

$$\begin{aligned} \delta_{n+1} &= \sum_{i=0}^{k} a_i \delta_{n-i}, \qquad n \in \mathbb{N}, \\ \delta_n &\in \mathbb{R}, \qquad n \in (-k + \mathbb{N}), \\ \delta_n &= c_n - \tilde{c}_n \in \mathbb{R}, \qquad n \in (-k + \mathbb{N}_k). \end{aligned} \tag{1.8}$$

Both our questions can be answered by reformulating (1.6) and (1.8) to provide appropriate affine and recurrence relations (1.1), respectively. As before we treat the homogeneous case of (1.6) (linear recurrence with memory) and the inhomogeneous case of (1.6) (proper affine recurrence with memory) separately.

In the case of homogeneity ($b = 0$) the reformulation of equation (1.1) to the **augmented linear recurrence** relation is

$$\begin{aligned} \vec{x}_{n+1} &= A\,\vec{x}, \qquad n \in \mathbb{N}, \\ \vec{x}_n &= (x_n, \dots, x_{n-k})^\top \in \mathbb{R}^{k+1}, \qquad n \in \mathbb{N} \\ \vec{x}_0 &= (c_0, \dots, c_k)^\top \in \mathbb{R}^{k+1} \\ A &= \begin{pmatrix} a_0 & a_1 & \dots & a_{k-1} & a_k \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & & \dots & & \dots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \end{aligned} \tag{1.9}$$

The characterization of the solution behaviour can now be performed with the so-called **amplification matrix** A. Its spectrum are the roots of the characteristic polynomial $|A - \lambda I| = (-1)^{k+1}(\lambda^{k+1} - \sum_{i=0}^{k} a_i \lambda^{k-i})$ If $\sigma(A) = \sigma_{<1}(A)$, then $\lim_{n\to\infty} x_n = 0$. If $\sigma(A) = \sigma_{\leq 1}(A)$, then x remains bounded. But if $\sigma(\lambda) \cap \bar{\sigma}_{\leq 1}(\lambda) \neq \emptyset$ then there exist start vectors $c_h = (c_0, \dots, c_k)^\top$ for which x explodes.

For the non-homogeneous case ($b \neq 0$), the reformulation of equation 1.1 to the **augmented affine recurrence** relation is

$$\begin{aligned} \vec{x}_{n+1} &= \bar{A}\,\vec{x}_n, \qquad n \in \mathbb{N}, \\ \vec{x}_n &= (x_n, \dots, x_{n-k}, y_n)^\top \in \mathbb{R}^{k+2}, \qquad n \in \mathbb{N} \\ \vec{x}_0 &= (c_0, \dots, c_k, 1)^\top \in \mathbb{R}^{k+1} \times \{1\} \\ \bar{A} &= \begin{pmatrix} A & B \\ 0 & 1 \end{pmatrix} \text{ where } B = (b, 0, \dots, 0)^\top \in \mathbb{R}^{k+1}, \end{aligned} \tag{1.10}$$

The spectrum of the amplification matrix are the roots of the characteristic polynomial $|\bar{A} - \lambda I| = (-1)^{k+2}(\lambda - 1)(\lambda^{k+1} - \sum_{i=0}^{k} a_i \lambda^{k-i})$. If $\sigma(A) = \sigma_{<1}(A)$, then $\lim_{n\to\infty} x_n = (1 - \sum_{i=0}^{k} a_i)^{-1} b$. If $\sigma(\bar{A}) = \sigma^{s,1}_{\leq 1}(\bar{A})$, then x remains bounded. But if $\sigma(\bar{A}) \cap \bar{\sigma}_{\leq 1}(\bar{A}) \neq \emptyset$, then there exist start vectors $c = (c_0, \dots, c_k, 1)^\top$ for which x explodes.

So in the qualitative behaviour of affine recurrences with memory x in can be completely characterized by the structure of eigenvalues and eigenspaces of the corresponding amplification matrix.

1.2 Stochastic recurrences

Now let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Then a quite natural question arises: what happens, if one changes the deterministic affine recurrence with memory (1.6) into a stochastic affine recurrence with memory

$$\begin{array}{lcl} X_{n+1} & = & \beta_{n+1} + \sum\limits_{i=0}^{k} \alpha_{i,n+1} X_{n-i}, \quad n \in \mathbb{N}, \\ X_i & \in & \mathbb{R}, \quad i \in (-k + \mathbb{N}), \\ X_i & = & c_i \in \mathbb{R}, \quad i \in (-k + \mathbb{N}_k), \\ \beta_n, \alpha_{0,n}, \ldots, \alpha_{k,n} & \in & (\Omega, \mathbb{R}), \quad n \in \mathbb{N}, \end{array} \tag{1.11}$$

with

$$\forall n \in \mathbb{N} : (\beta_n, \alpha_{0,n}, \ldots, \alpha_{k,n+1}) \overset{iid}{\sim} (\beta, \alpha_0, \ldots, \alpha_k),$$

where (1.12)

$$\beta, \alpha_0, \ldots, \alpha_k \in (\Omega, \mathbb{R}) \text{ and „sufficiently“ integrable ?}$$

Can one answer the two initial questions for the characterization of the evolution of the state variables and appropriate powers of it and the error propagation ? Of course, as in the deterministic case, one can immediately derive an equivalent amplified system

$$\begin{array}{lcl} \vec{X}_{n+1} & = & \bar{A}_h \, \vec{x}, \qquad n \in \mathbb{N}, \\ \vec{X}_n & = & (X_n, \ldots, X_{n-k})^\top \in \mathbb{R}^{k+1}, \qquad n \in \mathbb{N}, \\ \vec{X}_0 & = & (c_0, \ldots, c_k)^\top \in \mathbb{R}^{k+1}, \\ \bar{A}_n & = & \begin{pmatrix} \alpha_{0,n} & \alpha_{1,n} & \ldots & \alpha_{k-1,n} & \alpha_{k,n} & \beta_n \\ 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 & 0 \\ \ldots & & \ldots & & \ldots & \\ 0 & 0 & \ldots & 1 & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 1 \end{pmatrix}. \end{array} \tag{1.13}$$

But the difference with the deterministic case is that (1.13) generates a recurrence with memory $X = \{X_n\}_{n \in \mathbb{N}}$ for each $\omega \in \Omega$. As it is usually too difficult to investigate $X = \{X_n\}_{n \in \mathbb{N}}$ ω-wise or as this is not required, a typical approach is to limit oneself to the investigation of the evolution of corresponding moments of X by trying to find appropriate affine recurrence relations (1.1) for them. As expectations are involved in moments, an averaged behaviour of X is considered and a ω-wise consideration is avoided. This creates several difficulties, which we explore now.

1) Taking expectations, usually one has two options: to investigate moments or absolute moments. This distinction is more than formal, as the evolution of moments and absolute moments can differ substantially. Let $x_0 = 0$, $\forall n \in \mathbb{N}$: $\alpha_n = (1, 0, \ldots, 0)$, $\forall z \in \mathbb{R}_+ : \mathbb{P}[\beta_n < -z] = \mathbb{P}[\beta_n > z]$ and $\mathbb{E}[\beta_n] = 1$. Then $\forall n \in \mathbb{N}$: $X_n = \sum_{i=0}^{n} \beta_n$, $\mid \mathbb{E}[X_n] \mid = 0$ and $\mathbb{E}[\mid X_n \mid] = 2n$. So in general: while moments can converge, the corresponding absolute moments can explode, but on the other hand, if the absolute moments converge, the moments converge too (inequality of Jensen).
2) From the ω-wise affine recurrence relation (1.12) and the independence assumption (1.13) one can derive an affine recurrence relation

$$\mathbb{E}\,[\vec{X}_{n+1}] = \mathbb{E}\,[\,\bar{A}_n\,]\,\mathbb{E}\,[\vec{X}_n] \tag{1.14}$$

for the expected values $\{\,\mathbb{E}\,[\vec{X}_n\,]\,\}_{n \in \mathbb{N}}$. From this one can conclude as in the deterministic case to the evolutionary behaviour of $\{\mid \mathbb{E}\,[X_n] \mid\}_{n \in \mathbb{N}}$.
On the other hand, as $\mid . \mid$ is only subadditive and from (1.12) and (1.13) one can only derive the componentwise inequalities

$$\mathbb{E}\,[\mid \vec{X}_{n+1} \mid] \;\le\; \mathbb{E}\,[\mid \bar{A}_n \mid]\;\mathbb{E}\,[\mid \vec{X}_n \mid]. \tag{1.15}$$

The inequality (1.15) no longer describes the exact evolution of $\{\mathbb{E}[\mid \vec{X}_n \mid]\}_{n \in \mathbb{N}}$. But these inequalities allow at least a verification of convergence or boundedness of $\{\,\mathbb{E}[\mid \vec{X}_n \mid]\,\}_{n \in \mathbb{N}}$, if the eigenvalues of $\mathbb{E}[\mid \bar{A}_n \mid]$ are appropriate. This gives an estimate for stability regions.

3) Let $q(>0), p(>q) \in \mathbb{N}$. Although for any random variable $X \in \mathbb{L}^p(\Omega, \mathbb{P})$ we know that $|\, X\,|_q \leq |\, X\,|_p$, the $\mathbb{L}^q$- and $\mathbb{L}^p$-norm are not equivalent. So from the convergence of $X_q := \{|\, X_n\,|_q\}_{n\in\mathbb{N}}$ one cannot make conclusions about the boundedness or unboundedness of $X_p := \{|\, X_n\,|_q\}_{n\in\mathbb{N}}$ and conversely, from the unboundedness of X_p one cannot draw conclusions about the convergence or non-convergence of X_q.

4) Consider (1.11) with $k = 1$, $\beta_n = 0$, $n \in \mathbb{N}$ and assume that we are interested in $\{\, \mathbb{E}[\,|\, X_n\,|^2\,]\,\}_{n\in\mathbb{N}}$. Then one easily derives

$$\mathbb{E}[\,|\, X_{n+1}\,|^2\,] = \mathbb{E}[\alpha_0^2]\,\mathbb{E}[\,|\, X_n^2\,|\,] + 2\mathbb{E}[\alpha_0\alpha_1]\,\mathbb{E}[X_nX_{n-1}] + \mathbb{E}[\alpha_1^2]\,\mathbb{E}[\,|\, X_{n-1}^2\,|\,]. \tag{1.16}$$

We observe the occurrence of the covariance term $\mathbb{E}[X_nX_{n-1}]$. In general $\mathbb{E}[X_nX_{n-1}]$ is not a linear functional of $\mathbb{E}[|\, X_n^2\,|]$ and $\mathbb{E}[|\, X_{n-1}^2\,|]$. So in order to create an exact linear recurrence relation from (1.16) one not only augments (1.16) by $\mathbb{E}[X_{n-1}^2]$, but has also to include this covariance term. This leads to the system of equations

$$\begin{aligned}
\mathbb{E}[\,|\, X_{n+1}\,|^2\,] &= \mathbb{E}[\alpha_0^2]\,\mathbb{E}[\,|\, X_n^2\,|\,] + 2\mathbb{E}[\alpha_0\alpha_1]\,\mathbb{E}[\,X_nX_{n-1}\,] + \mathbb{E}[\alpha_1^2]\,\mathbb{E}[\,|\, X_{n-1}\,|^2\,] \\
\mathbb{E}[X_{n+1}X_{n+1-1}] &= \mathbb{E}[\alpha_0]\,\mathbb{E}[\,|\, X_n^2\,|\,] + \mathbb{E}[\alpha_1]\,\mathbb{E}[\,X_nX_{n-1}\,]
\end{aligned} \tag{1.17}$$

$$\mathbb{E}[\,|\, X_{n+1-1}^2\,|\,] = \mathbb{E}[\,|\, X_n^2\,|\,]$$

Although there are cases, where exact affine recurrence relations can be derived, the inclusion of covariance terms can increase the dimension of the exact recurrence relation considerably.

The observations 1) $-$ 4) can be summarized as follows. The difference between the deterministic affine recurrence relation (1.1) and its stochastic parallel (1.11) is that (1.11) is a ω-wise relation. The usual way to deal with it is by considering the evolution of moments and absolute moments of X. The difficulties arising are: *i*) the affine recurrence relation (1.11) does not necessarily carries through the moment case and requires estimates, *ii*) p-th order absolute moments usually have to be investigated separately, *iii*) exact affine augmented systems can suffer scaling problems (large dimension).

The encouraging side of the the observation 4) is the fact that, for moments (and with this even for absolute moments) exact affine recurrence relations in the moments can be derived by considering augmented systems.

An example, where systems (1.11) naturally arise, is the numerical solution of stochastic delay differential equations (SDDEs). SDDEs play a large rôle in describing dynamics and explaining phenomena in economics, e.g. population dynamics, oscillations and chaotic behaviour in control systems with time-delay feedback, price fluctuations in presence of time lags. Since many SDDE are too difficult to be solved analytically, the simulation of solutions of SDDE on computers is important to gain insight into complex models and for a parametric tuning of models.

Numerical solutions of SDDE are mostly time-discrete approximations of the solutions of the SDDE considered and depend on the underlying numerical methods and their parameters. From stiff deterministic ODEs it is already known that numerical solutions not always reflect appropriately the behaviour of the exact solutions of the ODE. This can also happen with numerical solutions of SDDE. So a question of vital interest is, for which parameters of the numerical method (this often includes the step width) the numerical solution of SDDE can be accepted as a „valuable" approximation of the exact solution.

This question is too complex to be answered in general for all equations, all properties of solutions and all numerical methods. That is why one often limits consideration to certain test classes of equations, properties of solutions that are of interest and particular classes of numerical methods. Let us assume here the following class of real scalar affine SDDEs

$$\begin{aligned}
dX(t) &= (a + \sum_{i=0}^{n_d} a_i X(t-\tau_i^d))\,dt + (b + \sum_{j=0}^{n_n} b_j X(t-\tau_j^n))\,dW(t), \qquad t \in [0,\infty), \\
X(t) &= \xi(t), \qquad t \in [-\tau, 0],
\end{aligned} \tag{1.18}$$

where

$$\begin{aligned}
\tau &= \max\{\, \tau_i^d, \tau_j^n \mid i \in \mathbb{N}_{n^d}, j \in \mathbb{N}_{n^n}\,\} > 0, \\
\tau_i^d, \tau_j^n &\in [0,\infty), \qquad i \in \mathbb{N}_{n^d}, j \in \mathbb{N}_{n^n}, \\
a_i, b_j &\in \mathbb{R}, \qquad i \in \mathbb{N}_{n^d}, j \in \mathbb{N}_{n^n},
\end{aligned} \tag{1.19}$$

$$\begin{array}{lcl} \xi & \in & \mathbb{C}([-\tau,0],\mathbb{R}), \\ (W(t),\mathcal{F}_t,t\in[0,\infty)) & & \text{real, scalar standard Wiener process.} \end{array}$$

We study (1.18) within the common stochastic setting of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the adapted right continuous increasing filtration $(\mathcal{F}_t, t \in [0,\infty))$. Furthermore we assume strong numerical methods leading to numerical schemes

$$\tilde{X}_{n+1} = \beta_n + \sum_{i=0}^{m} \alpha_{i,n} \tilde{X}_{n-k_i} \qquad (1.20)$$

where

$$\begin{array}{lcl} m & = & \#\left(\{\tau_i^d \mid i \in \mathbb{N}_{n^d}, \tau_i^d > 0\} \cup \{\tau_j^n \mid j \in \mathbb{N}_{n^n}, \tau_j^n > 0\}\right), \\ k_i & \in & \mathbb{N}, \quad i \in \mathbb{N}_m, \quad 0 = k_0, \quad k_{j-1} < k_j \quad \forall j \in \mathbb{N}_m \setminus \{0\}, \\ \beta_n, \alpha_{i,n} & \in & \mathbb{R}, \quad i \in \mathbb{N}_m, \\ \beta_n, \alpha_{i,n} & & \text{independent of } \{\tilde{X}_l\}_{l \in \mathbb{N}_{n-1}}, \quad i \in \mathbb{N}_m, \\ \tilde{X}_{-n} & = & \xi(-\frac{n}{k}\tau), \quad n \in \mathbb{N}_k. \end{array} \qquad (1.21)$$

An example of a numerical method, which is covered by (1.20), is the Euler-Maruyama explicit method.

One typical criterion of having appropriate numerical method is whether the numerical method generates a numerical solution, which has „approximately“ the same asymptotic behaviour as the exact solution of the SDDE (1.18). A partial task is to answer this question is to determine for the parameters given in (1.21), whether (1.20) converges, is bounded or explodes. A particular point of interest is how the stability behaviour changes if the step width $h = \frac{\tau}{k}$ is changed. A decreasing step width h leads to an increasing k and a family of numerical schemes (1.20), where the fractions k_i/k of index shifts of memory coefficients and the memory length remain nearly constant ($k_i/k \sim \tau_i$).

An approach to compute exact regions of mean-convergence of the numerical solutions (1.20) can be found in [2]. In this paper the authors can exploit the special structure of an affine test equation to characterize the expected growth of solutions and provide stability regions in the parameter space.

A first approach to approximate the region of mean-square-convergence of the numerical solutions (1.20) can be found in [1]. In this paper the authors use bounds for the occurring covariance terms of higher order and use Halany-type inequalities to estimate the growth of solutions and to provide stability regions in the parameter space.

In this paper we derive and investigate exact recurrence relations of moments of recurrence relations (1.11) as sketched in observation 4). A method is suggested to reduce the dimension of the exact recurrences. This can be used to compute the nonzero coefficients of the corresponding stability polynomial. This generalizes also the approach in [1], as further criteria can then be applied to decide numerically the question of stability of the underlying recurrence relation.

In section 2 an exact linear recurrence relation is derived for the homogeneous case $\beta = 0$. In section 3 the dimension of the resulting linear recurrence relation is quantified. Furthermore, properties of a large submatrix of the underlying amplification matrix of this linear recurrence scheme are shown. These properties allow one to find a representation of the stability polynomial of the linear recurrence relation by computing a determinant of a matrix with polynomial coefficients and a dimension that is lower than that of the amplification matrix. An abstract algorithm is provided to compute this matrix for the general case. For some special cases the matrix of reduced dimension can be computed directly. Although we shall here restrict attention to the homogeneous case $\beta = 0$, the inhomogeneous case $\beta \neq 0$ can also be treated. Corresponding exact affine linear recurrence relations can be established and their treatment can be reduced to the consideration of the homogeneous type case. Elsewhere, we will provide remarks on how to implement the computation of the determinant of the matrix representation in order to compute the non-zero coefficients of the stability polynomial corresponding to the exact affine recurrence scheme. The preceding comments indicate some areas in which the author will present results later.

2 A linear recurrence relation involving p-th order moments

In this section we describe, how, starting from the stochastic linear recurrence relation (1.11) ($\beta = 0$ $\mathbb{P}$-a.s.), one can construct an amplified linear recurrence relation. In the first subsection we consider examples and introduce definitions. In the second subsection we describe the amplified linear recurrence relation for the special case $m = 1$. In the third subsection we describe the amplified linear recurrence relation for the general case. The special case was introduced first, as it is easier to understand and as the proof of the main lemma of this section can be reduced to this case.

2.1 Examples and definitions

Consider the linear recurrence relation

$$\tilde{X}_{n+1} = \sum_{j=0}^{m} \alpha_{j,n} \tilde{X}_{n-k_j} \tag{2.1}$$

with stochastic coefficients $\alpha_n := (\alpha_{0,n}, \dots, \alpha_{m,n})$, $n \in \mathbb{N}$, which satisfy

$\forall n \in \mathbb{N} : \alpha_n$ is independent of $\{\tilde{X}_l\}_{l \in \mathbb{N}_{n-1}}$

$\forall n \in \mathbb{N} : \alpha_n \sim \alpha := (\alpha_0, \dots, \alpha_m)$.

Let $p \in \mathbb{N} \setminus \{0\}$. Define $I(m,p) := \{ \pi \in \mathbb{N}_p^{m+1} \mid \langle \pi, 1_{m+1} \rangle = p \}$ and $\binom{p}{\pi} := \frac{p!}{\pi_0! \dots \pi_m!}$
Then

$$\tilde{X}_{n+1}^p = \sum_{\pi \in I(m,p)} \binom{p}{\pi} \prod_{j=0}^{m} \alpha_{j,n}^{\pi_j} X_{n-k_j}^{\pi_j}, \tag{2.2}$$

and due to the assumed independence and distributional properties of the stochastic coefficients $\{\alpha_n\}_{n \in \mathbb{N}}$ the p-th moment of $\tilde{X}_{n+1}^p$ is

$$\mathbb{E}[\, \tilde{X}_{n+1}^p \,] = \sum_{\pi \in I(m,p)} \binom{p}{\pi} \mathbb{E}[\, \alpha_n^{\circ\pi} \,] \, \mathbb{E}[\, \tilde{X}_n^{\circ\pi} \,], \qquad \text{with} \quad \alpha_n^{\circ\pi} := \prod_{j=0}^{m} \alpha_{j,n}^{\pi_j}, \quad \tilde{X}_n^{\circ\pi} := \prod_{j=0}^{m} \tilde{X}_{n-k_j}^{\pi_j}. \tag{2.3}$$

The formula (2.3) obviously relates the p-th moment $\mathbb{E}[\tilde{X}_{n+1}^{\circ\pi}]$ to p-th order mixed covariance moments $\mathbb{E}[\, \tilde{X}_n^{\circ\pi} \,]$, $\pi \in I(m,p)$. Since $\mathbb{E}[\, \tilde{X}_n^{\circ\pi} \,]$ cannot in general be represented as a function of $\{\mathbb{E}[\, \tilde{X}_l^p \,]\}_{l=n-k,\dots,n}$, one could use the Hölder inequality to find upper bounds for the mixed covariances $\mathbb{E}[\, \tilde{X}_n^{\circ\pi} \,]$, $\pi \in I(m,p)$, and to transform the above equality (2.3) into a inequality. This approach was used in [1] for $m = 1$, $p = 2$ and to the characterization of regions contained in the exact stability region.

Here we do not want to eliminate the p-th order mixed covariance moments. Instead we understand the p-th order mixed covariance moments $\mathbb{E}[\tilde{X}_n^{\circ\pi}]$, $\pi \in I(m,p)$, to be part of a more general linear recurrence relation in a set of p-th order mixed covariance moments. The foundations for this linear recurrence relationship are laid by the equations

$$\tilde{X}_{n+1}^q = \sum_{\pi \in I(m,q)} \binom{q}{\pi} \mathbb{E}[\, \alpha_n^{\circ\pi} \,] \, \mathbb{E}[\, \tilde{X}_n^{\circ\pi} \,], \qquad q \in \mathbb{N}_p, \tag{2.4}$$

and the expectations of the q-potentials suitably complemented by mixed covariances of order p.

EXAMPLE 2.1.1
Consider $p = 3$, $k = 2$ and $m = 1$. Then the $p + 1$ identities are well known:

$$X_{n+1}^3 = \alpha_{0,n}^3 X_n^3 + 3\alpha_{0,n}^2 \alpha_{1,n} X_n^2 X_{n-2} + 3\alpha_{0,n}\alpha_{1,n}^2 X_n X_{n-2}^2 + \alpha_{0,n}^3 X_{n-2}^3 \tag{2.5}$$

$$X_{n+1}^2 = \alpha_{0,n}^2 X_n^2 + 2\alpha_{0,n}\alpha_{1,n} X_n X_{n-2} + \alpha_{1,n}^2 X_{n-2}^2 \tag{2.6}$$

$$X_{n+1} = \alpha_{0,n} X_n + \alpha_{1,n} X_{n-2} \tag{2.7}$$

$$X_{n+1}^0 = 1 \tag{2.8}$$

These identities lead to the following system of linear equations

$$\mathbb{E}[\, X_{n+1}^3 \,] \underset{(2.5)}{=} \mathbb{E}[\alpha_0^3]\, \mathbb{E}[X_n^3] + 3\mathbb{E}[\alpha_0^2\alpha_1]\, \mathbb{E}[X_n^2 X_{n-2}] + 3\mathbb{E}[\alpha_0\alpha_1^2]\, \mathbb{E}[X_n X_{n-2}^2] + \mathbb{E}[\alpha_0^3]\, \mathbb{E}[X_{n-2}^3],$$

$$\mathbb{E}[X_{n+1}^2 X_{n+1-1}] \underset{(2.6)*X_n}{=} \mathbb{E}[\alpha_0^2]\, \mathbb{E}[X_n^3] + 2\mathbb{E}[\alpha_0\alpha_1]\, \mathbb{E}[X_n^2 X_{n-2}] + \mathbb{E}[\alpha_1^2]\, \mathbb{E}[X_n X_{n-2}^2],$$

$$\mathbb{E}[X_{n+1}^2 X_{n+1-2}] \underset{(2.6)*X_{n-1}}{=} \mathbb{E}[\alpha_0^2]\, \mathbb{E}[X_n^2 X_{n-1}] + 2\mathbb{E}[\alpha_0\alpha_1]\, \mathbb{E}[X_n^1 X_{n-1} X_{n-2}] + \mathbb{E}[\alpha_1^2]\, \mathbb{E}[X_{n-1} X_{n-2}^2],$$

$$\mathbb{E}[X_{n+1} X_{n+1-1}^2] \underset{(2.7)*X_n^2}{=} \mathbb{E}[\alpha_0]\, \mathbb{E}[X_n^3] + \mathbb{E}[\alpha_1]\, \mathbb{E}[X_n^2 X_{n-2}],$$

$$\mathbb{E}[X_{n+1} X_{n+1-1} X_{n+1-2}] \underset{(2.7)*X_n X_{n-1}}{=} \mathbb{E}[\alpha_0]\, \mathbb{E}[X_n^2 X_{n-1}] + \mathbb{E}[\alpha_1]\, \mathbb{E}[X_n X_{n-1} X_{n-2}]$$

$$\begin{aligned}
\mathbb{E}[X_{n+1}X^2_{n+1-2}] &\underset{(2.7)*X^2_{n-1}}{=} \mathbb{E}[\alpha_0]\,\mathbb{E}[X_nX^2_{n-1}] + \mathbb{E}[\alpha_1]\,\mathbb{E}[X^2_{n-1}X_{n-2}],\\
\mathbb{E}[X^3_{n+1-1}] &\underset{(2.8)*X^3_n}{=} \mathbb{E}[X^3_n],\\
\mathbb{E}[X^2_{n+1-1}X_{n+1-2}] &\underset{(2.8)*X^2_nX_{n-1}}{=} \mathbb{E}[X^2_nX_{n-1}],\\
\mathbb{E}[X_{n+1-1}X^2_{n+1-2}] &\underset{(2.8)*X_nX^2_{n-1}}{=} \mathbb{E}[X_nX^2_{n-1}],\\
\mathbb{E}[X^3_{n+1-2}] &\underset{(2.8)*X^3_{n-1}}{=} \mathbb{E}[X^3_{n-1}].
\end{aligned}$$

Let us now introduce the set of triples $\mathbb{X}(2,3) = \{\,(i_1,i_2,i_3) \mid 0 \le i_1 \le i_2 \le i_3 \le 2\,\}$. Then on $\mathbb{X}(2,3)$ a complete order relation $\prec_{2,3}$ can be introduced by the definition

$$\begin{aligned}
\forall\,(i_1,i_2,i_3),(j_1,j_2,j_3) \in \mathbb{X}(2,3)\colon (i_1,i_2,i_3) \prec_{2,3} (j_1,j_2,j_3) \quad &:\Longleftrightarrow\\
\text{(i)}\quad & i_1 < j_1\\
\text{or}\quad \text{(ii)}\quad & i_1 = j_1,\ i_2 < j_2\\
\text{or}\quad \text{(iii)}\quad & i_1 = j_1,\ i_2 = j_2,\ i_3 < j_3.
\end{aligned}$$

$\prec_{2,3}$ provides a linear ordering and allows the iteration through $\mathbb{X}(2,3)$ from the largest to the smallest element of $\mathbb{X}(2,3)$. Defining $\forall n \in \mathbb{N}, \pi \in \mathbb{X}(2,3)$, $Y_n(\pi) := \mathbb{E}[\prod_{i=1}^{3} X_{n-\pi(i)}]$, and defining $\forall n \in \mathbb{N}$, $\{\pi_i\}_{i=1,\ldots,10}$ with $\pi_i \prec_{2,3} \pi_{i+1}$, $i = 1,\ldots,9$, $Z_n(\prec_{2,3}) := (\,Y_n(\pi_1),\ldots,Y_n(\pi_{10})\,)^\top$, we can now derive the linear recurrence relation

$$Z_{n+1}(\prec_{2,3}) = A(\alpha,\prec_{2,3})Z_n(\prec_{2,3}) \tag{2.9}$$

where

$$A(\alpha,\prec_{2,3}) = \begin{pmatrix}
\mathbb{E}[\alpha_0^3] & 0 & 3\mathbb{E}[\alpha_0^2\alpha_1] & 0 & 0 & 3\mathbb{E}[\alpha_0\alpha_1^2] & 0 & 0 & 0 & \mathbb{E}[\alpha_1^3]\\
\mathbb{E}[\alpha_0^2] & 0 & 2\mathbb{E}[\alpha_0\alpha_1] & 0 & 0 & \mathbb{E}[\alpha_1^2] & 0 & 0 & 0 & 0\\
0 & \mathbb{E}[\alpha_0^2] & 0 & 0 & 2\mathbb{E}[\alpha_0\alpha_1] & 0 & 0 & \mathbb{E}[\alpha_1^2] & 0 & 0\\
\mathbb{E}[\alpha_0] & 0 & \mathbb{E}[\alpha_1] & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \mathbb{E}[\alpha_0] & 0 & 0 & \mathbb{E}[\alpha_1] & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \mathbb{E}[\alpha_0] & 0 & 0 & 0 & \mathbb{E}[\alpha_1] & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}. \tag{2.10}$$

It can be observed that the components of $Z(\prec_{2,3})$ consist of the 3rd order mixed covariances $\mathbb{E}[\,X_{n-i_1}X_{n-i_2}X_{n-i_3}\,]$, where $0 \le i_1 \le i_2 \le i_3 \le 2$ and $n \in \mathbb{N}$. Furthermore, although p and k are quite small, the amplification matrix $A(\alpha,\prec_{2,3})$ are high dimensional compared with p, k and are n-independent due to the distributional assumptions concerning the stochastic coefficients $\{\,\alpha_n\,\}_{n\in\mathbb{N}}$ of the numerical scheme considered in this example.

Example 2.1.2
Consider $p = 2$, $k = 3$, $(k_1,k_2,k_3) = (0,1,3)$, and $m = 2$. Then the $p+1$ identities are well known:

$$\begin{aligned}
X^2_{n+1} &= \alpha^2_{0,n}X^2_n + \alpha^2_{1,n}X^2_{n-1} + \alpha^2_{2,n}X^2_{n-3} \qquad (2.11)\\
&\quad +2\alpha_{0,n}\alpha_{1,n}X_nX_{n-1} + 2\alpha_{0,n}\alpha_{2,n}X_nX_{n-3} + 2\alpha_{1,n}\alpha_{2,n}X_{n-1}X_{n-3},\\
X_{n+1} &= \alpha_{0,n}X_n + \alpha_{1,n}X_{n-1} + \alpha_{2,n}X_{n-3}, \qquad (2.12)\\
X^0_{n+1} &= 1. \qquad (2.13)
\end{aligned}$$

These identities lead to the following system of linear equations:

$$
\begin{aligned}
\mathbb{E}[\, X_{n+1}^2 \,] &\underset{(2.11)}{=} \mathbb{E}[\alpha_0^2]\,\mathbb{E}[X_n^2] + \mathbb{E}[\alpha_1^2]\,\mathbb{E}[X_{n-1}^2] + \mathbb{E}[\alpha_2^2]\,\mathbb{E}[X_{n-3}^2] \\
&\quad +2\mathbb{E}[\alpha_0\alpha_1]\,\mathbb{E}[X_n X_{n-1}] + 2\mathbb{E}[\alpha_0\alpha_2]\,\mathbb{E}[X_n X_{n-3}] + 2\mathbb{E}[\alpha_1\alpha_2]\,\mathbb{E}[X_{n-1}X_{n-3}], \\
\mathbb{E}[X_{n+1}X_{n+1-1}] &\underset{(2.12)*X_n}{=} \mathbb{E}[\alpha_0]\,\mathbb{E}[X_n^2] + \mathbb{E}[\alpha_1]\,\mathbb{E}[X_n X_{n-1}] + \mathbb{E}[\alpha_2]\,\mathbb{E}[X_n X_{n-3}], \\
\mathbb{E}[X_{n+1}X_{n+1-2}] &\underset{(2.12)*X_{n-1}}{=} \mathbb{E}[\alpha_0]\,\mathbb{E}[X_n X_{n-1}] + \mathbb{E}[\alpha_1]\,\mathbb{E}[X_{n-1}^2] + \mathbb{E}[\alpha_2]\,\mathbb{E}[X_{n-1}X_{n-3}], \\
\mathbb{E}[X_{n+1}X_{n+1-3}] &\underset{(2.12)*X_{n-2}}{=} \mathbb{E}[\alpha_0]\,\mathbb{E}[X_n X_{n-2}] + \mathbb{E}[\alpha_1]\,\mathbb{E}[X_{n-1}X_{n-2}] + \mathbb{E}[\alpha_2]\,\mathbb{E}[X_{n-2}X_{n-3}], \\
\mathbb{E}[X_{n+1-1}X_{n+1-1}] &\underset{(2.13)*X_n X_n}{=} \mathbb{E}[X_n X_n], \\
\mathbb{E}[X_{n+1-1}X_{n+1-2}] &\underset{(2.13)*X_n X_{n-1}}{=} \mathbb{E}[X_n X_{n-1}], \\
\mathbb{E}[X_{n+1-1}X_{n+1-3}] &\underset{(2.13)*X_n X_{n-2}}{=} \mathbb{E}[X_n X_{n-2}], \\
\mathbb{E}[X_{n+1-2}X_{n+1-2}] &\underset{(2.13)*X_{n-1} X_{n-1}}{=} \mathbb{E}[X_{n-1} X_{n-1}], \\
\mathbb{E}[X_{n+1-2}X_{n+1-3}] &\underset{(2.13)*X_{n-1} X_{n-2}}{=} \mathbb{E}[X_{n-1} X_{n-2}], \\
\mathbb{E}[X_{n+1-3}X_{n+1-3}] &\underset{(2.13)*X_{n-2} X_{n-2}}{=} \mathbb{E}[X_{n-2} X_{n-2}].
\end{aligned}
$$

Let us now introduce the set of pairs $\mathbb{X}(3,2) = \{\, (i_1, i_2) \mid 0 \le i_1 \le i_2 \le 3 \,\}$. Then on $\mathbb{X}(3,2)$ a complete order relation $\prec_{3,2}$ can be introduced by

$$
\forall\, (i_1,i_2), (j_1,j_2) \in \mathbb{X}(3,2)\colon (i_1,i_2) \prec_{3,2} (j_1,j_2) \quad :\Longleftrightarrow \quad \begin{array}{ll} & \text{(i)} \quad i_1 < j_1 \\ \text{or} & \text{(ii)} \quad i_1 = j_1,\ i_2 < j_2 \end{array}
$$

$\prec_{3,2}$ provides also a linear ordering and allows the iteration through $\mathbb{X}(3,2)$ from the largest to the smallest element of $\mathbb{X}(3,2)$. Defining $\forall n \in \mathbb{N}, \pi \in \mathbb{X}(3,2)\ Y_n(\pi) := \mathbb{E}[\, \prod_{i=1}^{2} X_{n-\pi(i)} \,]$ and defining $\forall n \in \mathbb{N}, \{\pi_i\}_{i=1,\ldots,10}$ with $\pi_i \prec_{3,2} \pi_{i+1},\ i = 1,\ldots,9,\ Z_n(\prec_{3,2}) := (\, Y_n(\pi_1), \ldots, Y_n(\pi_{10}) \,)^\top$, we can derive the linear recurrence relation

$$
Z_{n+1}(\prec_{3,2}) = A(\alpha, \prec_{3,2})\, Z_n(\prec_{3,2}) \tag{2.14}
$$

$$
A(\alpha,\prec_{3,2}) = \begin{pmatrix}
\mathbb{E}[\alpha_0^2] & 2\mathbb{E}[\alpha_0\alpha_1] & 0 & 2\mathbb{E}[\alpha_0\alpha_2] & \mathbb{E}[\alpha_1^2] & 0 & 2\mathbb{E}[\alpha_1\alpha_2] & 0 & 0 & \mathbb{E}[\alpha_2^2] \\
\mathbb{E}[\alpha_0] & \mathbb{E}[\alpha_1] & 0 & \mathbb{E}[\alpha_2] & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbb{E}[\alpha_0] & 0 & 0 & \mathbb{E}[\alpha_1] & 0 & \mathbb{E}[\alpha_2] & 0 & 0 & 0 \\
0 & 0 & \mathbb{E}[\alpha_0] & 0 & 0 & \mathbb{E}[\alpha_1] & 0 & 0 & \mathbb{E}[\alpha_2] & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix} \tag{2.15}
$$

It can be seen that the components of $Z(\prec_{3,2})$ consist of the 2nd order mixed covariances $\mathbb{E}[X_{n-i_1}X_{n-i_2}]$, where $0 \le i_1 \le i_2 \le 3$ and $n \in \mathbb{N}$. As in the previous example, although p and k are quite small, the amplification matrix $A(\alpha, \prec_{3,2})$ is in terms of p, k high dimensional. Finally, one easily verifies that, for $\mathbb{P}[\alpha_1 = 0] = 1$, the amplification matrix $A(\alpha, \prec_{3,2})$ is the same as the amplification matrix for the numerical method

$$
\tilde{X}_{n+1} = \alpha_{0,n}\tilde{X}_n + \alpha_{2,n}\tilde{X}_{n-3} \quad ,
$$

where $k = 3$, $p = 2$, $m = 1$.

The previous examples exhibit the main tools we will use to derive a linear recurrence relation of a selected set of covariances, including the p-th order moments. In order to extend this example, we will work with (k,p)-indices and operations on them, with (k,p)-products and (k,p)-covariances. These concepts will be introduced by the following definitions.

DEFINITION 2.1.3
Let

$$k \in \mathbb{N},\ p \in \mathbb{N} \setminus \{0\}.$$

Then

$$\mathbb{X}(k,p) := \{\, x = (x_1, \ldots, x_p) \in \mathbb{N}_k^p \ \mid\ 0 \le x_1 \le \ldots \le x_p \,\}.$$

An element of $\mathbb{X}(k,p)$ *is called* $(\mathbf{k},\mathbf{p})$**-index**.

We also define for later purposes

$$\begin{array}{lcl} \mathbb{X}_l(k,p) & := & \{\, x \in \mathbb{X}(k,p) \mid l \le x_1 \,\}, \quad l \in \mathbb{N}_k, \\ \mathbb{X}_{l_1,l_2}(k,p) & := & \mathbb{X}_{l_1} \setminus \mathbb{X}_{l_2}, \quad l_1, l_2 (\ge l_1) \in \mathbb{N}_k, \\ \mathbb{X}_d(k,p) & := & \{\, x \in \mathbb{X}(k,p) \mid x_1 = x_p = i, i \in \mathbb{N}_k \,\}. \end{array}$$

DEFINITION 2.1.4
Let x *be a* (k,p)*-index. Define the following operations and functions:*

$$\begin{array}{lcl} +(x) & := & ((x_1+1) \wedge k, \ldots, (x_p+1) \wedge k), \\ -(x) & := & ((x_1-1) \vee 0, \ldots, (x_p-1) \vee 0), \\ x^{+r} & := & \begin{cases} x & r = 0, \\ +(x^{+(r-1)}) & r \in \mathbb{N}_k \setminus \{0\}, \end{cases} \\ x^{-r} & := & \begin{cases} x & r = 0, \\ -(x^{-(r-1)}) & r \in \mathbb{N}_k \setminus \{0\}, \end{cases} \\ s(i,j,x) & := & (x_{i+1}, \ldots, x_j), \quad \text{where } 1 \le i \le j \le p, \\ r(i,j,x) & := & (\{0\}^i, x_{j+1}, \ldots, x_p, \{k\}^{j-i}), \quad 0 \le i \le j \le p, \\ -r(i,j,x) & := & (\{0\}^i, (x_{j+1}-1) \vee 0, \ldots, (x_p - 1) \vee 0, \{k\}^{j-i}), \quad 0 \le i \le j \le p, \\ n(j,x) & := & \begin{cases} max\{\, i = 1, \ldots, p \mid x_i = j \,\} + 1 - min\{\, i = 1, \ldots, p \mid x_i = j \,\} & \exists i \in \mathbb{N}_p \setminus \{0\} : x_i = j, \\ 0 & \text{otherwise.} \end{cases} \end{array}$$

DEFINITION 2.1.5
Let $x \in \mathbb{X}(k,p)$, $y \in \mathbb{X}(k,q)$. *We define the function:*

$$c(x,y) := z \in \mathbb{X}(k, p+q) \ \text{with}\ \forall\, l \in \mathbb{N}_{p+q} \setminus \{0\} : n(z_l, z) = n(z_l, x) + n(z_l, y)$$

DEFINITION 2.1.6
Let $x \in \mathbb{X}(k,p)$.

Denote by $X_{n+1}^x := \prod_{j=1}^{p} X_{n+1-x_j}$ *the* $(\mathbf{k},\mathbf{p})$**-product of** X_{n+1}.
Denote by $Y_{n+1}^x := \mathbb{E}[\, X_{n+1}^x \,]$ *the* $(\mathbf{k},\mathbf{p})$**-covariance of** X_{n+1}.

In the following we assume $p \in \mathbb{N} \setminus \{0\}$ and denote $k = max\{\, k_i \mid i \in \mathbb{N}_m \,\}$, $\vec{k} = (k_0, \ldots, k_m)$ if not explicitly defined in a different way.

2.2 The recurrence relation for $\mathbf{m = 1}$

In this subsection the special case of $m = 1$ will be considered. It is convenient to use the following notations.

DEFINITION 2.2.1
For $j, q\ (\ge j) \in \mathbb{N}$ *define* $b(j,q,\alpha) := \binom{q}{j} \alpha_0^j \alpha_1^{q-j}$ *and* $c(j,q,\alpha) := \mathbb{E}[\, b(j,q,\alpha) \,]$.

The next Lemma describes, how (k,p)-covariances of (k,p)-products X_{n+1}^{x}, $x \in \mathbb{X}(k,p)$, containing a factor X_{n+1} can be represented as a linear combination of (k,p)-covariances of (k,p)-products X_n^y, $y \in \mathbb{X}(k,p)$ as well as how (k,p)-covariances of (k,p)-products without any factor X_{n+1} can be restated as (k,p)-covariances of (k,p)-products X_n^y, $y \in \mathbb{X}(k,p)$. This lemma is essential for the recurrence relation in the (k,p)-covariances to be found.

LEMMA 2.2.2
Let $x \in \mathbb{X}(k,p)$. Then:

$$n(0,x) > 0: \qquad Y_{n+1}^{x} = \sum_{j=0}^{n(0,x)} c(j,n(0,x),\alpha) Y_n^{r(j,n(0,x),-(x))}, \tag{2.16}$$

$$n(0,x) = 0: \qquad Y_{n+1}^{x} = Y_n^{-(x)}. \tag{2.17}$$

Proof: Deferred to section 4. □

If we understand (2.16) and (2.17) as explicit linear recurrence relations, then they relate the (k,p)-covariances of the left side of the equation to the corresponding sets of (k,p)-covariances of the right sides of the equations. Hence they provide information of the (k,p)-covariances that is necessary to form one complete linear recurrence relation. As (k,p)-covariances are uniquely represented by (k,p)-indices, a recurrence relation based on (2.16) and (2.17) can be derived from any set of (k,p)-indices that contains $\{0\}^p$ and which is invariant under the (k,p)-index transform

$$\begin{aligned} T_1: \quad \mathcal{P}(\mathbb{X}(k,p)) &\to \mathcal{P}(\mathbb{X}(k,p)) \\ X &\to T_1(X) := \bigcup_{\substack{x\in X \\ 0<n(0,x)}} \{\, r(j,n(0,x),-(x)) \mid j=1,\ldots,n(0,x)\,\} \cup \bigcup_{\substack{x\in X \\ 0=n(0,x)}} \{\, -(x)\,\}. \end{aligned} \tag{2.18}$$

This is made rigorous by the following definition:

DEFINITION 2.2.3
Define $\mathcal{X}(k,p) \subset \mathbb{X}(k,p)$ by

$$(i) \qquad \{0\}^p \in \mathcal{X}(k,p), \tag{2.19}$$

$$(ii) \qquad \forall\, x \in \mathcal{X}(k,p): 0 < n(0,x) \longrightarrow \forall j=1,\ldots,n(0,x): r(j,n(0,x),-(x)) \in \mathcal{X}(k,p), \tag{2.20}$$

$$(iii) \qquad \forall\, x \in \mathcal{X}(k,p): 0 = n(0,x) \longrightarrow -(x) \in \mathcal{X}(k,p). \tag{2.21}$$

The next lemma characterizes $\mathcal{X}(k,p)$ and shows its uniqueness.

LEMMA 2.2.4
$\mathcal{X}(k,p) = \mathbb{X}(k,p)$

Proof: Deferred to section 4. □

The preceding lemma shows that the set of (k,p)-covariances determined by (2.19)-(2.21) is actually identically to the complete set of (k,p)-covariances. So the recurrence scheme includes all (k,p)-covariances. In order to describe the corresponding linear recurrence relation formally, it is convenient to iterate through $\mathbb{X}(k,p)$ and to form a corresponding vector of (k,p)-covariances. This is done using the following two definitions.

DEFINITION 2.2.5
Let $x,y \in \mathbb{X}(k,p)$. Then we define

$$\begin{aligned} x \prec_{k,p} y \quad &:\Longleftrightarrow \quad && (i) \quad x_1 < y_1 \\ & && \text{or } (ii) \quad \exists i \in \mathbb{N}_{p-1}\setminus\{0\}: \; x_{i+1} < y_{i+1} \text{ and } \forall j \in \mathbb{N}_i \setminus \{0\}\; x_j = y_j; \\ x \preccurlyeq_{k,p} y \quad &:\Longleftrightarrow \quad && (i) \quad x \prec_{k,p} y \\ & && \text{or } (ii) \quad x = y \end{aligned}$$

REMARK 2.2.6

1) Obviously, $\prec_{k,p}$ introduces a complete ordering on $\mathbb{X}(k,p)$.
2) $\prec_{k,p}$ is k-independent. k is introduced as a parameter for notational convenience to indicate the order relation on $\mathbb{X}(k,p)$.

DEFINITION 2.2.7
Define $n(k,p) := \#(\mathbb{X}(k,p))$
$Z_n(\prec_{k,p}) := (Y_n^{x_1}, \ldots, Y_n^{x_{n(k,p)}})^\top$ *where* $\forall i = 1, \ldots, n(k,p)-1 : x_i \prec_{k,p} x_{i+1},\ n \in \mathbb{N}$

DEFINITION 2.2.8
The matrix

$$A(\alpha, \prec_{k,p}) := (a(\alpha, \prec_{k,p})_{x,y})_{x,y \in \mathbb{X}(k,p)} \tag{2.22}$$

is called **(k,p) amplification matrix**.
Here is

$$a(\alpha, \prec_{k,p})_{x,y} := \begin{cases} c(j, n(0,x), \alpha) & 0 < n(0,x),\ \ y = r(j, n(0,x), -(x)),\ \ j \in \mathbb{N}_{n(0,x)} \\ 1 & 0 = n(0,x),\ \ y = -(x) \\ 0 & \textit{else} \end{cases} \tag{2.23}$$

Having introduced the recurrence vector and amplification matrix, one can now describe the linear relationship between consecutive recurrence vectors by

$$Z_{n+1}(\prec_{k,p}) = A(\alpha, \prec_{k,p}) Z_n(\prec_{k,p}). \tag{2.24}$$

2.3 The recurrence relation for m > 1

In this subsection the more general case of $m > 1$ will be considered. The following notation will be used.

DEFINITION 2.3.1
For $q \in \mathbb{N} \setminus \{0\}$, $\pi \in I(m,q)$, *define* $B(\pi, q, \alpha) := \binom{q}{\pi} \prod_{j=0}^{m} \alpha_j^{\pi_j}$ *and* $C(\pi, q, \alpha) := \mathbb{E}[\, B(\pi, q, \alpha)\,]$.

LEMMA 2.3.2
Let $x \in \mathbb{X}(k,p)$. *Then:*

$$n(0,x) > 0 : \qquad Y_{n+1}^{x} = \sum_{\pi \in I(m, n(0,x))} C(\pi, n(0,x), \alpha) Y_n^{c(s(n(0,x),p,-(x)),\pi)}, \tag{2.25}$$

$$n(0,x) = 0 : \qquad Y_{n+1}^{x} = Y_n^{-(x)}. \tag{2.26}$$

Proof: Deferred to section 4. □

As in the previous section we can understand (2.25) and (2.26) as explicit linear recurrence relations, which relate the (k,p)-covariances of the left-hand side of the equation to the corresponding sets of (k,p)-covariances of the right-hand side of the equations. We can equally express the recurrence relation formally as set of (k,p)-indices which is invariant under the (k,p)-index transform

$$\begin{aligned} T_m : \quad & \mathcal{P}(\mathbb{X}(k,p)) \to \mathcal{P}(\mathbb{X}(k,p)) \\ & X \mapsto T_m(X) := \bigcup_{\substack{x \in X \\ 0 < n(0,x)}} \{\, c(s(n(0,x),p,-(x)),\pi) \mid \pi \in I(m, n(0,x)) \,\} \cup \bigcup_{\substack{x \in X \\ 0 = n(0,x)}} \{\, -(x) \,\}. \end{aligned} \tag{2.27}$$

DEFINITION 2.3.3
Define $\mathcal{X}(\vec{k},p) \subset \mathbb{X}(k,p)$ *by*

$$(i) \quad \{0\}^p \in \mathcal{X}(\vec{k},p), \tag{2.28}$$

$$(ii) \quad \forall\, x \in \mathcal{X}(\vec{k},p) : 0 < n(0,x) \longrightarrow \forall \pi \in I(m, n(0,x)) : c(s(n(0,x),p,-(x)),\pi) \in \mathcal{X}(\vec{k},p), \tag{2.29}$$

$$(iii) \quad \forall\, x \in \mathcal{X}(\vec{k},p) : 0 = n(0,x) \longrightarrow -(x) \in \mathcal{X}(\vec{k},p). \tag{2.30}$$

The next lemma characterizes $\mathcal{X}(\vec{k},p)$ and shows its uniqueness.

LEMMA 2.3.4
$\mathcal{X}(\vec{k},p) = \mathbb{X}(k,p)$.

Proof: Deferred to section 4. □

The preceding lemma confirms that in the case $m > 1$ also the set of (k,p)-covariances that is determined by (2.28)-(2.30) is identical to the complete set of (k,p)-covariances. So the recurrence scheme includes all (k,p)-covariances. Using the definitions 2.2.5 and 2.2.7, from the previous subsection, we can describe the corresponding linear recurrence relation completely by

$$Z_{n+1}(\prec_{k,p}) = A(\alpha, \prec_{k,p}) Z_n(\prec_{k,p}). \tag{2.31}$$

For $m > 1$ the matrix $A(\alpha, \prec_{k,p})$ is defined as follows:

DEFINITION 2.3.5
The matrix

$$A(\alpha, \prec_{k,p}) := (a(\alpha, \prec_{k,p})_{x,y})_{x,y \in \mathbb{X}(k,p)} \tag{2.32}$$

is called the $(\mathbf{k}, \mathbf{p})$ **amplification matrix**.
Here:

$$a(\alpha, \prec_{k,p})_{x,y} := \begin{cases} C(\pi, n(0,x), \alpha) & 0 < n(0,x), \;\; y = c(s(n(0,x), p, -(x)), \pi), \;\; \pi \in I(m, n(0,x)), \\ 1 & 0 = n(0,x), \;\; y = -(x), \\ 0 & \textit{otherwise}. \end{cases} \tag{2.33}$$

REMARK 2.3.6
As for $j, q\,(\geq j) \in \mathbb{N}$: $\binom{q}{j} = \binom{q}{(j, q-j)}$, the recurrence relations (2.24) for $m = 1$ and (2.31) for $m > 1$ are consistent. Furthermore, the definition 2.3.5 of the amplification matrix can be extended consistently to the case $m = 1$.

2.4 The amplified system and the evolution of p-th moments

The evolution and asymptotic behaviour of the linear recurrence relation (2.31) can be completely described by the eigensystem of the amplification matrix. Assume that

$$A(\alpha, \prec_{k,p}) = P(\alpha, \prec_{k,p}) J(\alpha, \prec_{k,p}) P(\alpha, \prec_{k,p})^{-1} \tag{2.34}$$

is the eigen-decomposition of the (k,p)-amplification matrix. That is, $P(\alpha, \prec_{k,p})$ is the matrix of all eigenvectors and $J(\alpha, \prec_{k,p})$ is the Jordan-block matrix of the (k,p)-amplification matrix. Let $Z(\alpha, \prec_{k,p})_0 = P(\alpha, \prec_{k,p}) \tilde{Z}(\alpha, \prec_{k,p})_0$, then

$$Z(\alpha, \prec_{k,p})_0 = P(\alpha, \prec_{k,p}) J(\alpha, \prec_{k,p})^n \tilde{Z}(\alpha, \prec_{k,p})_0. \tag{2.35}$$

A criterion for computing the eigenvalues is to compute them as solutions λ of the equation

$$|A(\alpha, \prec k, p) - \lambda I| = 0 \tag{2.36}$$

where I is the identity matrix. Denote the set of all eigenvalues as the **spectrum** $\sigma(A(\alpha, \prec k, p))$ of $A(\alpha, \prec k, p)$. The polynomial $|A(\alpha, \prec k, p) - \lambda I|$ is a **characteristic** or **stability polynomial** for the recurrence.

We now have to ask what is the relation between the eigenvalues of the (k,p)-amplification matrix and the time evolution of p-th moments and absolute moments of the sequence generated by (2.1) is. An answer is given by the following lemma.

LEMMA 2.4.1

(i) *Let* $q\,(> 0), p\,(\geq q) \in \mathbb{N}, n \in \mathbb{N}, \pi \in I(m,q)$.
Then $\quad |\, \mathbb{E}[X_{n-k_i}^{\circ\pi}] \,| \leq \prod_{i=1}^{q} \mathbb{E}[\,|\, X_{n-k_i} \,|^p\,]^{\frac{\pi_i}{p}}$

(ii) *Let* $p = 1$ *or* $p \in (2 + 2\mathbb{N})$. *Then it holds:*
$\exists\, \lambda \in \sigma(A(\alpha, \prec k, p))$ *with* $|\lambda| > 1$ *or* $1 \in \sigma(A(\alpha, \prec k, p))$ *with Jordan block length* > 1
$\rightarrow \lim_{n\to\infty} \mathbb{E}[\, X_n^p \,] = \infty$
$\forall\, \lambda \in \sigma(A(\alpha, \prec k, p))$ *with* $|\lambda| < 1 \rightarrow \lim_{n\to\infty} \mathbb{E}[\, X_n^p \,] = 0$
$\forall\, \lambda \in \sigma(A(\alpha, \prec k, p))$ *with* $|\lambda| \leq 1$ *or* $1 \in \sigma(A(\alpha, \prec k, p))$ *with Jordan block length* 1
$\rightarrow \lim_{n\to\infty} \mathbb{E}[\, X_n^p \,] = 0$

(*iii*) *Let* $p \in (1+2\mathbb{N})$. *Then it holds:*
$\exists\, \lambda \in \sigma(A(\alpha, \prec k,p))$ *with* $|\lambda| > 1$ *or* $1 \in \sigma(A(\alpha, \prec k,p))$ *with Jordan block length* > 1
$\rightarrow \lim\limits_{n\to\infty} \mathbb{E}[\,|\,X_n\,|^p\,] = \infty$

Proof: Deferred to section 4. □

REMARK 2.4.2
For $p \in (2+2\mathbb{N})$ the p-th order moments and p-th order absolute moments coincide, that is, $\forall n \in \mathbb{N}$: $X_n^p = |X_n|^p$.

The preceding lemma shows that for even p one can characterize the time evolution of the p-th order absolute moments by investigating the eigenvalues of the amplification matrix, that for $p=1$ one can characterize the time evolution of the 1st moment, and that for odd p one has a criterion for the unboundedness of the p-th order absolute moments. Furthermore, for odd $p > 1$ the evolution of p-th order moments depends on the eigenvectors $(P(\alpha, \prec_{k,p}))$ of $A(\alpha, \prec_{k,p})$.

3 Stability polynomials of the amplified system of a linear recurrence relation

The previous section showed that the course of evolution of p-th order moments of a solution $\tilde{X}$ of a linear recurrence relation (2.1) is embedded into the linear recurrence relation (2.31) in vectors of (k,p)-covariances. In the first subsection of this section we show that this amplified linear recurrence relation describes the evolution of p-th order moments exactly. We see that it is computationally prohibitive to compute the eigenvalues of the (k,p)-amplification matrix due to its high dimension. In the second subsection we show some favourable properties of (k,p)-amplification matrices. In the third subsection we show how to reduce by matrix operations the computation of stability polynomial of the (k,p)-amplification matrix to the computation of a determinant of a matrix with polynomial coefficients and a dimension, which is lower than the dimension of the (k,p)-amplification matrix. In subsection 4 this matrix of reduced dimension is computed for some special cases.

3.1 The linear recurrence relation and its dimension

As the amplification matrix $A(\alpha, \prec_{k,p})$ is known, the time evolution of the p-th order moments of the numerical scheme (1.20) can be determined in principle by computing all eigenvalues of $A(\alpha, \prec_{k,p})$ and checking the semi-simplicity of eigenvalues on the complex unit circle. However, a direct evaluation of the spectrum of $A(\alpha, \prec_{k,p})$ is in most cases computationally inefficient or impossible, as the dimension of $A(\alpha, \prec_{k,p})$ can be quite large even for modest k, p. This is illustrated by the following lemma.

LEMMA 3.1.1
(*i*) $\#(\,\mathbb{X}(k,p)\,) = \sum\limits_{i_1=0}^{k}\sum\limits_{i_2=0}^{i_1}\cdots\sum\limits_{i_p=0}^{i_{p-1}} 1.$

(*ii*) $\#(\,\mathbb{X}(k,p)\,) = \int\limits_0^k \int\limits_0^{\lceil t_1\rceil} \cdots \int\limits_0^{\lceil t_{p-1}\rceil} 1 dt_p \ldots dt_2 dt_1.$

(*iii*) $\#(\,\mathbb{X}(k,p)\,) = f(k,p)$ *where* $f(i,j) = \begin{cases} i & j=1, \\ 0 & i=0, \\ f(i-1,j)+f(i,j-1) & \textit{otherwise.} \end{cases}$

(*iv*) $\frac{(k+1)^p}{p!} \le \#(\,\mathbb{X}(k,p)\,) \le \frac{(k+2)^p}{p!}.$

Proof: Deferred to section 4. □

The following lemma shows that there is no reordering of $\mathbb{X}(k,p)$ such that the corresponding reordering of the (k,p) amplification matrix is a block-diagonal matrix which would allow one to compute the stability polynomial $|A(\alpha, \prec_{k,p}) - \lambda I|$ as a product of determinants of submatrices with a lower and hence computationally more favourable dimension.

LEMMA 3.1.2
There is no triple of matrices $(P, B_1, B_2) \in (\mathbb{O}(f(k,p)) \cap \{1,-1,0\}^{f(k,p)\times f(k,p)}, \mathbb{M}_{k_1}, \mathbb{M}_{k_2})$ *where* $k_1, k_2 \in \mathbb{N}_{f(k,p)} \setminus \{0\}$, $k_1 + k_2 = f(k,p)$, P *permutation matrix, such that*

$$A(\alpha, \prec_{k,p}) = P^{-1} \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} P.$$

Proof: Deferred to section 4. □

3.2 Some properties of (k,p) amplification matrices

A closer look at the matrices $A(\alpha, \prec_{3,2})$ and $A(\alpha, \prec_{2,3})$ in the examples of section 2 suggests, that (k,p) amplification matrices might have a sparse structure with a large number of rows containing exactly one 1-element. In this subsection we want to make this observation rigorous and investigate some properties of (k,p)-amplification matrices. At the beginning some technicalities are necessary.
First it proves to be helpful to define some special sequences of (k,p)-indices for indexing purposes.

DEFINITION 3.2.1

$$\begin{aligned} I_i^{\prec}(k,p) &:= \{\, x \,\}_{x\in\mathbb{X}(k,p),\, x_1=i}\,, \qquad i \in \mathbb{N}_k, \\ \bar{I}^{\prec}(k,p) &:= \{\, x \,\}_{x\in\mathbb{X}(k,p)}\,, \\ \bar{I}_0^{\prec}(k,p) &:= \bigcup_{i=1}^{k} I_i^{\prec}(k,p), \\ \bar{I}_1^{\prec}(k,p) &:= \bigcup_{i=2}^{k} I_i^{\prec}(k,p). \end{aligned}$$

So far, the indexing of (k,p)-amplification matrices and recurrence vectors has been done in terms of (k,p) indices from $\mathbb{X}(k,p)$ provided with the complete ordering $\prec_{k,p}$. Now we introduce a function F, which allows to work with an integer indexing of amplification matrices and recurrence vectors, that is equivalent to the previous one. Whereas the indexing by $\mathbb{X}(k,p)$ elements is helpful for proving structural facts, the integer indexing is in some cases more handy for matrix calculations and the implementation of algorithms.

DEFINITION 3.2.2
Let $k \in \mathbb{N}$, $p \in \mathbb{N} \setminus \{0\}$.
Define, for all $x \in \mathbb{X}(k,p)$: $F(k,p,x) := \#\{\, y \in \mathbb{X}(k,p) \mid y \preccurlyeq_{k,p} x \,\}$.

REMARK 3.2.3
Let $k \in \mathbb{N}$, $p \in \mathbb{N} \setminus \{0\}$. Then

(1) $\forall x \in \mathbb{X}(k,p) : F(k,p,x) = 1 + \#\{\, y \in \mathbb{X}(k,p) \mid y \prec_{k,p} x \,\}$

(2) $\forall x, y \in \mathbb{X}(k,p)$:

$$\begin{array}{llll} (i) & F(k,p,x) < F(k,p,y) & \Longleftrightarrow & x \prec_{k,p} y, \\ (ii) & F(k,p,x) = F(k,p,y) & \Longleftrightarrow & x = y, \\ (iii) & F(k,p,x) > F(k,p,y) & \Longleftrightarrow & y \prec_{k,p} x. \end{array}$$

In the following we need a property of (k,p) indices from $(\mathbb{X}(k,p), \prec_{k,p})$.

LEMMA 3.2.4
Let

$k \in \mathbb{N}$, $l \in \mathbb{N}_k$, $p \in \mathbb{N} \setminus \{0\}$

Then

$$\begin{array}{llll} (i) & \Phi: \ \mathbb{X}_l(k,p) & \longrightarrow \ \mathbb{X}(k-l,p) & \textit{is a bijection.} \\ & \qquad x & \longrightarrow \ \Phi(x) := (x_1 - l, \ldots, x_p - l) & \end{array}$$

(ii) $\#(\mathbb{X}_l(k,p)) = f(k-l,p)$.

Proof:
Deferred to section 4. □

With the preceding lemma we are in the position to state a recursive algorithm for the computation of F.

Lemma 3.2.5

Let $k \in \mathbb{N},\ p \in \mathbb{N} \setminus \{0\}$

$x \in \mathbb{X}(k,p)$

f *the function defined in lemma 3.1.1.*

Then

$$F(k,p,x) = \begin{cases} 1 & k=0,\ p \geq 1, \\ p+2-\min\{\, p+1 \,\} \cup \{\, i \in \mathbb{N}_p \setminus \{0\} \mid x_i = 1 \,\} & k=1,\ p \geq 1, \\ 1+x_1 & k>1,\ p=1, \\ F(k-x_1, p-1, (x_2-x_1, \ldots, x_p-x_1)) + (1-\delta_0(x_1)) \sum_{i=0}^{x_1-1} f(k-i,p-1) & k>1,\ p>1. \end{cases}$$

Proof:

Deferred to section 4. □

Now we are ready to investigate the structure of $A(\alpha, \prec_{k,p})_{\bar{I}_0^{\prec}(k,p), I^{\prec}(k,p)}$.

Lemma 3.2.6

Let

$k \in \mathbb{N},\ p \in \mathbb{N} \setminus \{0\},$

$x, y \in \mathbb{X}_1(k,p).$

Then

(i) $\forall z \in \mathbb{X}_1(k,p): \quad A(\alpha, \prec_{k,p})_{x,z} = \delta_{-(x)}(z),$

(ii) $x \prec_{k,p} y \iff -(x) \prec_{k,p} -(y),$

(iii) $F(y) = F(x)+1 \implies F(-(y)) - F(-(x)) = 1 + max\{\, i \in \mathbb{N}_{p-1} \mid x_{p-i} = k \,\},$

(iv) $i, j (\neq i+1) \in \mathbb{N}_k \setminus \{0\}, \implies A(\alpha, \prec_{k,p})_{I_i^{\prec}(k,p), I_j^{\prec}(k,p)} = 0_{f(k-i,p), f(k-j,p)}.$

Proof:

Deferred to section 4. □

Corollary 3.2.7

(i) $A(\alpha, \prec_{k,p})_{\bar{I}_0^{\prec}(k,p), I(k,p)}$ *has only rows with exactly on 1-element.*
The 1-elements are $A(\alpha, \prec_{k,p})_{x, F(-(x))},\ x \in \mathbb{X}_1(k,p)$.

(ii) *Let* $x \in \mathbb{X}_{1,k}(k,p)$ *and* y *its successor in* $X(k,p)$. *Then the columns* $F(-(x))$ *and* $F(-(y))$ *are separated by* $max\{\, i \in \mathbb{N}_{p-1} \mid x_{p-i} = k \,\}$ *zero-columns.*

(iii) *The submatrices* $A(\alpha, \prec_{k,p})_{I_{i+1}^{\prec}(k,p), I_i^{\prec}(k,p)},\ i \in \mathbb{N}_{k-1}$ *are the only nonzero block matrices among* $A(\alpha, \prec_{k,p})_{I_i^{\prec}(k,p), I_j^{\prec}(k,p)},\ i, j, \in \mathbb{N}_k \setminus \{0\}$.

Finally, the structure of $A(\alpha, \prec_{k,p})_{\bar{I}_0^{\prec}(k,p), \bar{I}_0^{\prec}(k,p)}$ is completely described with the next lemma. This requires beforehand the definition of matrix operations and some special matrices.

Definition 3.2.8

(i) *Let* $A \in \mathbb{M}_{m,n},\ m, n \in \mathbb{N} \setminus \{0\}$.
Define $cat_0(A) := \begin{pmatrix} A & 0 \end{pmatrix} \in \mathbb{M}_{m,n+1}(\mathbb{R})$.

(ii) *Let* $A \in \mathbb{M}_{m,n},\ B \in \mathbb{M}_{p,q}(\mathbb{R}),\ m, n, p, q \in \mathbb{N} \setminus \{0\}$.
Define $diag(A) := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathbb{M}_{m+p,n+q}(\mathbb{R})$.

(iii) *Let* $\{m_i\}_{i \in \mathbb{N} \setminus \{0\}}, \{n_i\}_{i \in \mathbb{N} \setminus \{0\}} \subset \mathbb{N} \setminus \{0\},$
$A_i \in \mathbb{M}_{m_i,n_i}(\mathbb{R}),\ i \in \mathbb{N} \setminus \{0\}.$
Define $diag^2(A_1, A_2) := diag(A_1, A_2),$
$diag^{l+1}(A_1, A_2, \ldots, A_l, A_{l+1}) := diag(diag^l(A_1, A_2, \ldots, A_l), A_{l+1}), \quad l \geq 2,$

DEFINITION 3.2.9

(i) $\quad E_1^-(l) := cat_0(I_l), \qquad l \geq 1.$

(ii) $\quad E_{q+1}^-(l) := cat_0(diag^l(E_q^-(l), \ldots, E_q^-(1))), \quad l \geq 1,\ q \geq 1.$

LEMMA 3.2.10

Let $k \in \mathbb{N} \setminus \{0,1\}$, $p \in \mathbb{N} \setminus \{0\}$.

$$(i) \quad A(\alpha, \prec_{k,p})_{I_l^{\prec}(k,p), I_{l-1}^{\prec}(k,p)} = E_{p-1}^-(l), \quad l \in \mathbb{N}_k \setminus \{0\}.$$

$$(ii) \quad A(\alpha, \prec_{k,p})_{\bar{I}_0^{\prec}(k,p), I^{\prec}(k,p)} = E_p^-(k).$$

Proof:

Deferred to section 4. □

3.3 A representation of a stability polynomial as determinant of a matrix with polynomial coefficients

In this subsection we want to exploit the sparse structure of $A(\alpha, \prec_{k,p})_{\bar{I}_0^{\prec}(k,p), I^{\prec}(k,p)}$, in order to show that the determinant $|A(\alpha, \prec_{k,p}) - \lambda I_{f(k,p)}|$ of dimension $f(k,p)$ can actually be computed as the determinant $|B(\alpha, \prec_{k,p}, \lambda)|$ of dimension $f(k, p-1)$, where $B(\alpha, \prec_{k,p}, \lambda) \in M_{f(k,p-1)}(P_{k+1}(\lambda))$ is a matrix, whose coefficients are polynomials with a maximum order higher than 1.

The basic idea for the reduction is provided with the following well known lemma.

LEMMA 3.3.1

Let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathbb{M}_n(\mathbb{C}),\ n \in \mathbb{N} \setminus \{0,1\},$$

$A_{22} \in \mathbb{M}_l(\mathbb{C})$ *regular*, $l \in \mathbb{N}_n \setminus \{0, n\}$.

Then

$$|A| = |A_{11} - A_{12}A_{22}^{-1}A_{21}|\,|A_{22}|.$$

Proof:

Deferred to section 4. □

For our purposes we choose

$$\begin{aligned} A &= A(k,p,\lambda) &&:= A(\alpha, \prec_{k,p}) - \lambda I_{f(k,p)}, \\ A_{11} &= A_{11}(k,p,\lambda) &&:= A(\alpha, \prec_{k,p})_{I_0^{\prec}(k,p), I_0^{\prec}(k,p)} - \lambda I_{f(k,p-1)}, \\ A_{12} &= A_{12}(k,p) &&:= A(\alpha, \prec_{k,p})_{I_0^{\prec}(k,p), \bar{I}_0^{\prec}(k,p)}, \\ A_{21} &= A_{21}(k,p) &&:= A(\alpha, \prec_{k,p})_{\bar{I}_0^{\prec}(k,p), I_0^{\prec}(k,p)}, \\ A_{22} &= A_{22}(k,p,\lambda) &&:= A(\alpha, \prec_{k,p})_{\bar{I}_0^{\prec}(k,p), \bar{I}_0^{\prec}(k,p)} - \lambda I_{f(k-1,p)}. \end{aligned} \tag{3.1}$$

In the following we investigate the matrix $A_{12}(k,p)A_{22}^{-1}(k,p,\lambda)A_{21}(k,p)$. Define first:

DEFINITION 3.3.2

Let $\lambda \in \mathbb{C} \setminus \{0\}$, $r \in \mathbb{N} \setminus \{0\}$.

Then

$$\begin{aligned} L(r,\lambda) &:= (\, l_r(i,j)\,)_{i,j \in \mathbb{N}_r \setminus \{0\}}, \text{ where } l_r(i,j) = -\lambda\delta_j(i) + \delta_{j-1}(i), \\ \bar{L}(r,\lambda) &:= (\, \bar{l}_r(i,j)\,)_{i,j \in \mathbb{N}_r \setminus \{0\}}, \text{ where } \bar{l}_r(i,j) = -\sum_{l=1}^{i} \lambda^{l-1-i}\delta_l(j), \\ \tilde{L}(r,\lambda) &:= (\, \tilde{l}_r(i,j)\,)_{i,j \in \mathbb{N}_r \setminus \{0\}}, \text{ where } \tilde{l}_r(i,j) = \sum_{l=1}^{i} \lambda^{r-i}\delta_l(j). \end{aligned}$$

With the above notation we have the following result.

LEMMA 3.3.3

Let $\lambda \in \mathbb{C} \setminus \{0\}$, $r \in \mathbb{N} \setminus \{0\}$

Then $L(r,\lambda)^{-1} = \bar{L}(r,\lambda)$.

Proof: Deferred to section 4. □

Next it is necessary to define further index sequences.

DEFINITION 3.3.4

Let $k \in \mathbb{N} \setminus \{0\}$, $p \in \mathbb{N} \setminus \{0\}$.

Then define: $\forall x \in \mathbb{X}_1(k,p)$

$$\begin{aligned} J^{\prec}(k,p,x) &:= \{ (x_1+i,\ldots,x_p+i) \}_{i\in\mathbb{N}_{k-x_p}}, \\ FJ^{\prec}(k,p,x) &:= \{ F(x_1+i,\ldots,x_p+i) - f(k,p-1)) \}_{i\in\mathbb{N}_{k-x_p}}, \\ n(k,p,x) &:= \#(\, J^{\prec}(k,p,x)\,). \end{aligned}$$

Then we can show the following properties:

LEMMA 3.3.5

Let $k \in \mathbb{N} \setminus \{0\}$, $p \in \mathbb{N} \setminus \{0\}$, $\lambda \in \mathbb{C} \setminus \{0\}$.

Then

(*i*) $\forall x \in \mathbb{X}_1(k,p)$: $A(k,p,\lambda)_{J^{\prec}(k,p,x),J^{\prec}(k,p,x)} = L(k+1-x_p,\lambda)$.

(*ii*) $\forall x \in \mathbb{X}_1(k,p)$: $A(k,p,\lambda)_{J^{\prec}(k,p,x),\bar{I}_0^{\prec}(k,p)\setminus J^{\prec}(k,p,x)} = 0_{i,j}$ *with* $i = k+1-x_p$, $j = f(k,p-1)-i$.

(*iii*) $\{ J^{\prec}(k,p,x) \mid x \in \mathbb{X}_{1,2}(k,p) \}$ *is a partition of* $\mathbb{X}_1(k,p)$.

(*iv*) $A_{22}(k,p,\lambda)^{-1}$ *exists.*

(*v*) $\forall x \in \mathbb{X}_{1,2}(k,p)(k,p):\ (A_{22}(k,p,\lambda)^{-1})_{J_x,J_x} = \bar{L}(k+1-x_p)$, *where* $J_x = FJ^{\prec}(k,p,x)$.

Proof:

Deferred to section 4. □

The last lemma, with its characterization of $A_{22}^{-1}(k,p,\lambda)$, has been the first step on the way to characterize explicitly the product $A_{22}(k,p,\lambda)^{-1}A_{21}(k,p)$. The second step is based on the following two key observations about $A_{21}(k,p)$. First, due to lemma 3.3.5 $A_{21}(k,p)$ consists of unit vectors of $\mathbb{R}^{f(k-1,p)}$ and of zero columns. A second key observation is that the submatrix $A(\alpha,\prec_{k,p})_{I_1^{\prec}(k,p),I_0^{\prec}(k,p)}$ of $A_{22}(k,p,\lambda)$ consists of the columns of the identity matrix $I_{f(k-1,p-1)}$ augmented by $f(k,p-1)-f(k-1,p-1)$ zero columns. This means that $A_{22}(k,p,\lambda)^{-1}A_{21}(k,p)$ consists of the first $f(k-1,p-1)$ columns of $A_{22}(k,p,\lambda)^{-1}$ and additional $f(k,p-1)-f(k-1,p-1)$ zero columns. The next lemma states this exactly. But before this, we introduce special matrices and vectors for an appropriate description.

DEFINITION 3.3.6

Let $k \in \mathbb{N} \setminus \{0\}$, $p \in \mathbb{N} \setminus \{0\}$, $\lambda \in \mathbb{C} \setminus \{0\}$.

Define

$$\begin{aligned} J_1(k,p) &= \{ y \}_{\exists x\in\mathbb{X}_{1,2}(k,p):y=-(x)}, \\ FJ_1(k,p) &= \{ i \}_{\exists x\in\mathbb{X}_{1,2}(k,p):i=F(k,p,-(x))}, \\ \bar{J}_1(k,p) &= \{ y \}_{\forall x\in\mathbb{X}_{1,2}(k,p):y\neq-(x)}, \\ \overline{FJ}_1(k,p) &= \{ i \}_{\forall x\in\mathbb{X}_{1,2}(k,p):i\neq F(k,p,-(x))}. \end{aligned}$$

Define

$\bar{B}(k,p,\lambda) = (v_1,\ldots,v_{f(k,p-1)}) \in \mathbb{M}_{f(k-1,p),f(k,p-1)}(\mathbb{C})$ *with*

(*i*) $\forall i \in \overline{FJ}_1(k,p) : v_i := 0$.

(*ii*) $\forall i \in FJ_1(k,p)\ \exists x \in \mathbb{X}_{1,2} :\ i = F(-(x)) - f(k,p-1)$. *Then*

$$v_{i,FJ^{\prec}(k,p,x))} := (\bar{L}(\,n(k,p,x),\lambda\,))_{\_,1} \text{ and } \forall j \notin FJ^{\prec}(k,p,x) : v_{i,j} := 0.$$

$\tilde{B}(k,p,\lambda) := (v_1,\ldots,v_{f(k,p-1)}) \in \mathbb{M}_{f(k-1,p),f(k,p-1)}(\mathbb{C})$ *with*

(*i*) $\forall i \in \overline{FJ}_1(k,p) : v_i := 0$.

(*ii*) $\forall i \in FJ_1(k,p)\ \exists x \in \mathbb{X}_{1,2} :\ i = F(-(x)) - f(k,p-1)$. *Then*

$$v_{i,FJ^{\prec}(k,p,x))} := (\tilde{L}(\,n(k,p,x),\lambda\,))_{\_,1} \text{ and } \forall j \notin FJ^{\prec}(k,p,x) : v_{i,j} := 0.$$

Definition 3.3.7
Let $k \in \mathbb{N} \setminus \{0\}$, $p \in \mathbb{N} \setminus \{0\}$.
Define $E(k,p,\lambda) = diag(\lambda^{c_1}, \ldots, \lambda^{c_{f(k,p-1)}})$, *where*

$$c_i = \begin{cases} n(k,p,x) \text{ with } x \in \mathbb{X}_{1,2}(k,p) : i = F(x) - f(k,p-1) & i \in FJ_1(k,p), \\ 0 & i \in \overline{FJ}_1(k,p). \end{cases}$$

Lemma 3.3.8
Let $k \in \mathbb{N} \setminus \{0\}$, $p \in \mathbb{N} \setminus \{0\}$, $\lambda \in \mathbb{C} \setminus \{0\}$.
Then

$$\begin{array}{llcl} (i) & (A(\alpha, \prec_{k,p}) - \lambda I_{f(k,p)})_{I_1^{\prec}(k,p),J_1(k,p)} & = & I_{i,i}, \quad i = f(k-1,p-1), \\ (ii) & (A(\alpha, \prec_{k,p}) - \lambda I_{f(k,p)})_{I_1^{\prec}(k,p),\bar{J}_1(k,p)} & = & 0_{i,j}, \quad i = f(k-1,p-1),\ j = f(k,p-1) - i, \\ (iii) & A_{22}(k,p,\lambda)^{-1} A_{21}(k,p) & = & \bar{B}(k,p,\lambda), \\ (iv) & A_{22}(k,p,\lambda)^{-1} A_{21}(k,p) E(k,p,\lambda) & = & -\tilde{B}(k,p,\lambda). \end{array}$$

Proof:

Deferred to section 4. □

Finally, the representation of the determinant $|A(\alpha, \prec_{k,p}) - \lambda I_{f(k,p)}|$ as a determinant of a matrix with polynomial coefficients and order lower $f(k,p-1)$ requires the property stated in the next lemma.

Lemma 3.3.9
Let $k \in \mathbb{N} \setminus \{0\}$, $p \in \mathbb{N} \setminus \{0\}$, $\lambda \in \mathbb{C}$.
Then

$$|A_{22}(k,p,\lambda)| = (-\lambda)^{f(k-1,p)} = (-1)^{f(k-1,p)}\, |E(k,p,\lambda)|.$$

Proof:

Deferred to section 4. □

Now, defining

$$B(\alpha, \prec_{k,p}, \lambda) := A_{11}(k,p,\lambda) E(k,p,\lambda) \; + \; A_{12}(k,p) \tilde{B}(k,p,\lambda)$$

we can conclude

Lemma 3.3.10
Let $k \in \mathbb{N} \setminus \{0\}$, $p \in \mathbb{N} \setminus \{0\}$.

Then: $\quad | A(\alpha, \prec_{k,p}) - \lambda I_{f(k,p)} | = (-1)^{f(k-1,p)} \, | B(\alpha, \prec_{k,p}, \lambda) |$.

Proof:

Deferred to section 4. □

Remark 3.3.11
(*i*) The elements of $B(\alpha, \prec_{k,p}, \lambda)$ are polynomials of maximum order $k+1$.
(*ii*) The dimension of $B(\alpha, \prec_{k,p}, \lambda)$ is $f(k,p-1) = \mathcal{O}(k^{p-1})$.

From the representation lemma an algorithm for computing the matrix $B(\alpha, \prec_{k,p}, \lambda)$ can be derived:

```
Procedure B := Generate_reduced_matrix (k, p, A,λ)
{
    B:=0_{f(k,p-1),f(k,p-1)};
    col:=0;
    for  x:=(1,...,1)  to  (1,k,...,k)
    {
        Λ:=λ^{n(k,p,x)};
        col:=col + 1 + n(k,x);
        B(_,col):=(A_11(k,p,λ))_{_,col} * Λ;
        for  y  in  J^≺(k,p,x)
        {
            Λ:=Λ/λ;
            col_2:=F(y);
```

```
            for  z  in  { z ∈ X_{0,1}(k,p) | x ∈ T_m({z}) }
            {
                row:=F(z);
                B(row,col):=B(row,col) + (A_{12}(k,p))_{row,col_2} * Λ;
            };
        };
    };
}
```

3.4 The semi-simplicity of eigenvalues of $A(\alpha, \prec_{k,p})$

As in the introductory section 1 was indicated, a linear recurrence scheme with the recurrence matrix A generates bounded sequences, if $\sigma(A) = \sigma^{s,1}_{\leq 1}(A)$. The criterion $\sigma(A) = \sigma^{s,1}_{\leq 1}(A)$ includes to check the semi-simplicity of eigenvalues. As usually for a specific eigenvalue its geometric multiplicity can be determined without computing eigenvectors, it would be helpful to have a criterion to decide, whether a particular eigenvalue is semi-implicit or not. A test like this would not avoid the determination of a particular eigenvalue and its multiplicity, but would at least avoid to compute its eigenspace.

In the following we re-state an algorithm to generate generalized eigenvectors of a matrix and an eigenvalue which can be used to generate the Jordan block structure and properties of generalized eigenspaces. This shows the relation between geometric and algebraic multiplicity of an eigenvalue $\lambda \in \sigma(A(\alpha, \prec_{k,p}))$, the length of the Jordan blocks corresponding to λ and the rank $A(\alpha, \prec_{k,p}) - \lambda I_{f(k,p)}$. With this result we can derive a criterion for checking the semi-simplicity of an eigenvalue λ in terms of the reduced matrix $B(\alpha, \prec_{k,p}, \lambda)$.

The algorithm to generate generalized eigenvectors of a matrix A belonging to a given eigenvalue λ consists of two procedures and is

```
Procedure x := Solve( A , λ , y )
{
    If ( ∃ a solution z of (A − λI)z = y)
        return z;
    else
        return fail;
}

Procedure (list,dimensions) := Generalized_eigenvectors( A , λ )
{
    γ_λ := dim( Ker(A − λI) );
    Determinate linearly independent vectors e^1_1, ..., e^1_{γ_λ} solving (A − λI)x = 0;
    foreach i := 1 to γ_λ
    {
        k_i := 0;
        while ( fail ≠ e^i_j)
        {
            k_i := k_i + 1;
            list := append(list,e^i_{k_i});
            e^i_{k_i+1} := Solve(A,λ,e^i_{k_i});
        }
        dimensions := append(dimensions,k_i);
    }
    return list,dimensions;
}
```

For the next definition and four lemma let $n \in \mathbb{N} \setminus \{0\}$, $A \in \mathbb{M}_n(\mathbb{R})$ and $\lambda \in \mathbb{C}$.
Some obvious properties of the null-spaces of the operators $\{(A - \lambda I)^n\}_{n \in \mathbb{N}}$ are presented:

LEMMA 3.4.1
Let $\lambda \in \mathbb{C}$. Then $\forall n \in \mathbb{N}$:

(i) $\quad Ker\,((A-\lambda I)^n) \subset Ker\,((A-\lambda I)^{n+1})$
(ii) $\quad Ker\,((A-\lambda I)^n) = Ker\,((A-\lambda I)^{n+1}) \longrightarrow \forall m \in \mathbb{N} : Ker\,((A-\lambda I)^{n+m}) = Ker\,((A-\lambda I)^{n+m+1})$

Proof:
Deferred to section 4. □

Furthermore we need the following definitions:

DEFINITION 3.4.2
Define:
$$n_\lambda := min\{\, r \in \mathbb{N} \mid Ker\,((A-\lambda I)^{r+1}) = Ker\,((A-\lambda I)^r)\,\}$$
Let
$\{e_1^i\}_{i \in \mathbb{N}_{\gamma(\lambda)} \setminus \{0\}}$ *the basis in* $Ker\,(A-\lambda I)$, *generated by procedure* Generalized_eigenvectors.
Define then:
$$k_\lambda^i = \max\{\, m \in \mathbb{N} \setminus \{0,1\} \mid \exists\, x \text{ with } (A-\lambda I)^m x = e_1^i\,\},\ i \in \mathbb{N}_{\gamma(\lambda)} \setminus \{0\}$$

The following lemma are used to establish a rank criterion for semi-simplicity of eigenvalues.

LEMMA 3.4.3
Let $\ i \in \mathbb{N}_{\gamma(\lambda)} \setminus \{0\},\ m \in \mathbb{N}_{k_\lambda^i} \setminus \{0\}$
Then $\ e_m^i \in Ker\,((A-\lambda I)^m) \setminus Ker\,((A-\lambda I)^{m-1})$

Proof:
Deferred to section 4. □

LEMMA 3.4.4
$\bigcup_{i=1}^{k} \{\, e_m^i \mid m \in \mathbb{N}_{k_\lambda^i} \setminus \{0\}\,\}$ *is a set of linear independent vectors.*

Proof: Deferred to section 4. □

LEMMA 3.4.5
$Ker\,((A-\lambda I)^{n_\lambda}) = span\{\bigcup_{i=1}^{k} \{\, e_m^i \mid m \in \mathbb{N}_{k_\lambda^i} \setminus \{0\}\,\}\}.$

Proof: Deferred to section 4. □

COROLLARY 3.4.6
Let $\ A \in \mathbb{M}_n(\mathbb{R})$,
$\lambda \in \sigma(A)$.
Then:
λ *has only Jordan blocks of length* 1 *if and only if* $\mu(\lambda) = \gamma(\lambda)$

Proof: Deferred to section 4. □

The above result will now be applied to (k,p)-amplification matrices.

DEFINITION 3.4.7
Let $k \in \mathbb{N},\ p \in \mathbb{N} \setminus \{0\}$.
Then define: $\forall x \in \mathbb{X}_{1,2}(k,p)$
$$\begin{aligned} J_+^{\prec}(k,p,x) &:= \{\,-(x)\,\} \cup J^{\prec}(k,p,x),\\ n_+(k,p,x) &:= n(k,p,x)+1,\\ y_+^{\prec}(k,p,x) &:= y \in J_+^{\prec}(k,p,x) \text{ with } \forall z \in J_+^{\prec}(k,p,x)\, z \preccurlyeq_{k,p} y. \end{aligned}$$

LEMMA 3.4.8
(i) *Let* $\ \lambda \in \mathbb{C} \setminus \{0\},\ a \in Ker\,(A(\alpha, \prec_{k,p}) - \lambda I_{f(k,p)})$.
For all $y \in \mathbb{X}(k,p)$ *denote with* y_p *the p-th component of* y.
Then $\forall\, x \in \mathbb{X}_{1,2}(k,p)\ \ \exists\, c(x) \in \mathbb{R}\ \ \forall y \in J_+^{\prec}(k,p,x):\ \ a_{F(y)} = -c(x)\, \lambda^{y_p - y_m^{\prec}(x)_p}$.
(ii) $dim(\,Ker\,(A(\alpha, \prec_{k,p}) - \lambda I_{f(k,p)})\,) = dim(\,Ker\,(B(\alpha, \prec_{k,p}, \lambda))\,)$.
(iii) *Then* $A(\alpha, \prec_{k,p}) - \lambda I_{f(k,p)}$ *has at most Jordan blocks of lengths 1 (is semi-simple) if and only if* $\mu(\lambda) = dim(\,Ker\,(B(\alpha, \prec_{k,p}, \lambda))\,) = f(k,p-1) - rank(\,B(\alpha, \prec_{k,p}, \lambda)\,)$.

Proof: Deferred to section 4. □

COROLLARY 3.4.9
Let $\lambda \in \sigma(A(\alpha, \prec_{k,p}))$,
B=*Generate_reduced_matrix* (k, p, A, λ).
Then λ *has at most Jordan blocks of length 1 (is semi-simple) if and only if* $\mu(\lambda) = f(k, p-1) - rank(B)$.

Proof: Deferred to section 4. □

3.5 Some special cases

3.5.1 The case p=2, k=m

A quite common case from a practical point of view is the case $p = 2$.
For any $k \in \mathbb{N} \setminus \{0\}$ assume now $m = k$ and that we are given the coefficients $\alpha = (\alpha_0, \ldots, \alpha_k)$. We extend α to a countable infinite vector

$$\bar{\alpha} := (\ldots, \alpha_{-2}, \alpha_{-1}, \alpha_0, \ldots, \alpha_k, \alpha_{k+1}, \ldots) \text{ where } \alpha_{-1} := -1,\ \alpha_i := 0,\ i \in \mathbb{Z} \setminus (\{-1\} \cup \mathbb{N}_k). \tag{3.2}$$

For this case we easily can derive that

$$\forall \pi \in \mathbb{X}_d(k,2) : \binom{2}{\pi} = 1, \quad \forall \pi \in \mathbb{X}(k,2) \setminus \mathbb{X}_d(k,2) : \binom{2}{\pi} = 2, \quad \forall \pi \in \mathbb{X}(k,1) : \binom{2}{\pi} = 1. \tag{3.3}$$

We can also explicitly compute the functions related to integer indexing.

LEMMA 3.5.1
Let $k \in \mathbb{N} \setminus \{0\}$. *Then*
(*i*) $f(k,2) = \frac{(k+1)(k+2)}{2}$.
(*ii*) $\forall x = (x_1, x_2) \in \mathbb{X}(k,2) : \quad F(k,2,x) = 1 + x_2 - x_1 + \frac{x_1(2k+3-x_1)}{2} = 1 + x_2 + \frac{x_1(2k+1-x_1)}{2}$.
$\forall i \in \mathbb{N}_k :\ l_1(i) := F(k,2,(i,i)) = 1 + \frac{i(2k+3-i)}{2},\ \ l_2(i) := F(k,2,(i,k)) = 1 + k + \frac{i(2k+1-i)}{2}$.
(*iii*) $\forall x = (x_1, x_2) \in \mathbb{X}_1(k,2)$ *with* $x_2 \neq k, y = (x_1, x_2+1) \in \mathbb{X}_1(k,2) :\ F(k,2,-(y)) = F(k,2,-(x)) + 1$.
$\forall x = (x_1, k) \in \mathbb{X}_1(k,2)$ *with* $x_1 \neq k, y = (x_1+1, x_1+1) \in \mathbb{X}_1(k,2) :\ F(k,2,-(y)) = F(k,2,-(x)) + 2$.

Proof: Deferred to section 4. □
Furthermore the submatrix $A(\alpha, \prec_{k,2})_{\bar{I}_0^{\prec}(k,2), I^{\prec}(k,2)}$ of $A(\alpha, \prec_{k,2})$ can be characterized as

LEMMA 3.5.2
Let $\forall i \in \mathbb{N}_k$: $l_1(i) := F(k,2,(i,i))$, $l_2(i) := F(k,2,(i,k))$, $l(i) := l_2(i) + 1 - l_1(i)$, $I(i) := \mathbb{N}_{l_2(i)} \setminus \mathbb{N}_{l_1(i)-1}$
Then
(*i*) $\forall i \in \mathbb{N}_k \setminus \{k\} : (A(\alpha, \prec_{k,2}))_{I_{i+1}, I_i} = cat_0(I_{k-i})$.
(*ii*) $\forall i \in \mathbb{N}_k \setminus \{0\}, i \in \mathbb{N}_k \setminus \{0, i+1\} : A(\alpha, \prec_{k,2})_{I_i, I_j} = 0_{l(i), l(j)}$.
(*iii*) $A(\alpha, \prec_{k,2})_{\bar{I}, I} = cat_0(\,diag(\,cat_0(I_k), \ldots, cat_0(I_1)\,)\,)$.

Proof: Deferred to section 4. □
The reduced matrix can then be computed by using the following properties

LEMMA 3.5.3
(*i*) $A_{21}(k,2) = (e_1, \ldots, e_k, 0)$ *with* $\{e_i\}_{i \in \mathbb{N}_k \setminus \{0\}}$ *unit vectors and* 0 *zero vector in* $\mathbb{R}^{\frac{1}{2}k(k+1)}$.
(*ii*) $A_{22}(k,2,\lambda)^{-1} A_{21}(k,2) = (v_1, \ldots, v_k, 0)$ *where for all* $i \in \mathbb{N}_k \setminus \{0\}$, $j \in \mathbb{N}_{k-i}$

$$v_{l_{i,j}, i} = \begin{cases} -\lambda^{-1-j} & l_{i,j} = i + \frac{j(2k+1-j)}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Deferred to section 4. □

With this the reduced matrix can explicitly computed.

LEMMA 3.5.4

$B(\alpha, \prec_{k,2}, \lambda) = (b(\alpha, \prec_{k,2}, \lambda)_{i,j})_{i,j \in \mathbb{N}_{\frac{1}{2}k(k+1)} \setminus \{0\}}$ *with* $b(\alpha, \prec_{k,2}, \lambda)_{i,j} =$

$$= \begin{cases} \sum\limits_{l=-1}^{k} \mathbb{E}[\, \alpha_l^2 \,] \, \lambda^{k-l} & i = 1,\ j = 1, \\ \sum\limits_{l=0}^{k-(j-1)} \mathbb{E}[\, 2\alpha_l \alpha_{l+(j-1)} \,] \, \lambda^{k-l-(j-1)} & i = 1,\ j = 2, \dots, k+1, \\ \mathbb{E}[\, \alpha_{i-2} \,] \, \lambda^{k-(i-2)} & i = 2, \dots, k+1,\ j = 1, \\ \mathbb{E}[\, \alpha_{i-2-(j-1)} \,] \, \lambda^{k-(i-2)} + \mathbb{E}[\, \alpha_{(i-2)+(j-1)} \,] \, \lambda^{k-(i-2)-(j-1)} & i = 2, \dots, k+1,\ j = 2, \dots, k+1. \end{cases}$$

Proof: Deferred to section 4. □

EXAMPLE 3.5.5

For $p = 2$, $k = m = 4$ lemma 3.5.4 states that $B(\alpha, \prec_{4,2}, \lambda) =$

$\sum\limits_{l=-1}^{4} \mathbb{E}[\alpha_l^2] \, \lambda^{4-l}$	$\sum\limits_{l=0}^{3} \mathbb{E}[2\alpha_l \alpha_{l+1}] \, \lambda^{3-l}$	$\sum\limits_{l=0}^{2} \mathbb{E}[2\alpha_l \alpha_{l+2}] \, \lambda^{2-l}$	$\sum\limits_{l=0}^{1} \mathbb{E}[2\alpha_l \alpha_{l+3}] \, \lambda^{1-l}$	$\sum\limits_{l=0}^{0} \mathbb{E}[2\alpha_l \alpha_{l+4}] \, \lambda^{0-l}$
$\mathbb{E}[\alpha_0] \, \lambda^4$	$\mathbb{E}[\alpha_{-1}] \, \lambda^4$ $+\mathbb{E}[\alpha_1] \, \lambda^3$	$\mathbb{E}[\alpha_2] \, \lambda^2$	$\mathbb{E}[\alpha_3] \, \lambda^1$	$\mathbb{E}[\alpha_4] \, \lambda^0$
$\mathbb{E}[\alpha_1] \, \lambda^3$	$\mathbb{E}[\alpha_0] \, \lambda^3$ $+\mathbb{E}[\alpha_2] \, \lambda^2$	$\mathbb{E}[\alpha_{-1}] \, \lambda^3$ $+\mathbb{E}[\alpha_3] \, \lambda^1$	$\mathbb{E}[\alpha_4] \, \lambda^0$	0
$\mathbb{E}[\alpha_2] \, \lambda^2$	$\mathbb{E}[\alpha_1] \, \lambda^2$ $+\mathbb{E}[\alpha_3] \, \lambda^1$	$\mathbb{E}[\alpha_0] \, \lambda^2$ $+\mathbb{E}[\alpha_4] \, \lambda^0$	$\mathbb{E}[\alpha_{-1}] \, \lambda^2$	0
$\mathbb{E}[\alpha_3] \, \lambda^1$	$\mathbb{E}[\alpha_2] \, \lambda^1$ $+\mathbb{E}[\alpha_4]\lambda^0$	$\mathbb{E}[\alpha_1] \, \lambda^1$	$\mathbb{E}[\alpha_0] \, \lambda^1$	$\mathbb{E}[\alpha_{-1}] \, \lambda^1$

Considering the rows 2, 3, 4 and 5 in the previous example one can notice that

– $\mathbb{E}[\alpha_{-1}]$ contributes with the summand $\mathbb{E}[\alpha_{-1}]\lambda^{k-(j-2)}$ to the matrix elements $B(\alpha, \prec_{4,2}, \lambda)_{j,j}$, $j = 2, \dots, 5$;

– $\mathbb{E}[\alpha_0]$ contributes with the summand $\mathbb{E}[\alpha_0]\lambda^{k-(j-2)}$ to the matrix elements $B(\alpha, \prec_{4,2}, \lambda)_{j,j-1}$, $j = 2, \dots, 5$;

– $\mathbb{E}[\alpha_i]$ contributes with the summand $\mathbb{E}[\alpha_i]\lambda^{k-(j-2)-i}$ to the matrix elements $B(\alpha, \prec_{4,2}, \lambda)_{j,j-2-i}$, $j = 2+i, \dots, 5$, $i = 1, 2, 3$;

– $\mathbb{E}[\alpha_i]$ contributes with the summand $\mathbb{E}[\alpha_i]\lambda^{k-i}$ to the matrix elements $B(\alpha, \prec_{4,2}, \lambda)_{j,3+i-j}$, $j = 2, \dots, 2+i$, $i = 1, 2, 3$;

– $\mathbb{E}[\alpha_4]$ contributes with the summand $\mathbb{E}[\alpha_4]\lambda^{k-4}$ to the matrix elements $B(\alpha, \prec_{4,2}, \lambda)_{j,k+1-j}$.

This observation suggests to characterize $B(\alpha, \prec_{k,p}, \lambda)$ by the regularity in terms of λ-potentials or factors $\{\, \mathbb{E}[\alpha_i] \,\}_{i \in \mathbb{N}_k}$. For this we fix, what is understood by „contribute to an matrix element“.

DEFINITION 3.5.6

Let $k \in \mathbb{N} \setminus \{0\}$, $p = 2$,
$i, j \in \mathbb{N}_{k+1} \setminus \mathbb{N}_1$,
$l \in \mathbb{N}_k$.

If $B(\alpha, \prec_{k,2})_{i,j} = \mathbb{E}[\alpha_{k-l}] \, \lambda^l$ $\quad j = 1$,
or $B(\alpha, \prec_{k,2})_{i,j} = \mathbb{E}[\alpha_{k-l-(j-1)}] \, \lambda^l + \mathbb{E}[\alpha_{k-l+(j-1)}] \, \lambda^{l-(j-1)}$ $\quad j = 2, \dots, k$,
or $B(\alpha, \prec_{k,2})_{i,j} = \mathbb{E}[\alpha_{k-l-2(j-1)}] \, \lambda^{l+(j-1)} + \mathbb{E}[\alpha_{k-l}] \, \lambda^l$ $\quad j = 2, \dots, k$,

then we say that λ^l **contributes to** $B(\alpha, \prec_{k,2})_{i,j}$ *(with summand* $\mathbb{E}[\alpha_{k-l}] \, \lambda^l$, $\mathbb{E}[\alpha_{k-l-(j-1)}] \, \lambda^l$ *or* $\mathbb{E}[\alpha_{k-l}] \, \lambda^l$*, respectively).*

If $B(\alpha, \prec_{k,2})_{i,j} = \mathbb{E}[\alpha_l] \, \lambda^{k-l}$ $\quad j = 1$,
or $B(\alpha, \prec_{k,2})_{i,j} = \mathbb{E}[\alpha_l] \, \lambda^{k-l-(j-1)} + \mathbb{E}[\alpha_{l+2(j-1)}] \, \lambda^{k-l-2(j-1)}$ $\quad j = 2, \dots, k$,
or $B(\alpha, \prec_{k,2})_{i,j} = \mathbb{E}[\alpha_{l-2(j-1)}] \, \lambda^{k-l+2(j-1)} + \mathbb{E}[\alpha_l] \, \lambda^{k-l+(j-1)}$ $\quad j = 2, \dots, k$,

then we say that $\mathbb{E}[\alpha_l]$ **contributes to** $B(\alpha, \prec_{k,2})_{i,j}$ *(with summand* $\mathbb{E}[\alpha_l] \, \lambda^{k-l}$, $\mathbb{E}[\alpha_l] \, \lambda^{k-l-(j-1)}$ *or* $\mathbb{E}[\alpha_l] \, \lambda^{k-l+(j-1)}$*, respectively).*

Now we can characterize $B(\alpha, \prec_{k,2})$ by powers of λ and presence of $\mathbb{E}[\alpha_l]$, $l \in \{-1\} \cup \mathbb{N}_k$, with

LEMMA 3.5.7
Let $k \in \mathbb{N} \setminus \{0\}, p = 2$. *Then*

(*i*) $\forall i \in \mathbb{N}_{k+1} \setminus \mathbb{N}_1$, $\forall j \in \mathbb{N}_i \setminus \{0\}$
$\lambda^{k-(i-2)}$ *contributes to* $B(\alpha, \prec_{k,p}, \lambda)_{i,j}$ *with summand* $\mathbb{E}[\alpha_{(i-2)-(j-1)}]\, \lambda^{k-(i-2)}$.

(*ii*) $\forall i \in \mathbb{N}_{k+1} \setminus \mathbb{N}_1$, $\forall j \in \mathbb{N}_{k+1-(i-2)} \setminus \{0\}$
$\lambda^{k-(i-2)-(j-1)}$ *contributes to* $B(\alpha, \prec_{k,p}, \lambda)_{i,j}$ *with summand* $\mathbb{E}[\alpha_{(i-2)+(j-1)}]\, \lambda^{k-(i-2)-(j-1)}$.

(*iii*) $\forall l \in \{-1\} \cup \mathbb{N}_{k-1}$, $\forall i \in \mathbb{N}_{k+1} \setminus \mathbb{N}_{(l+1)\vee 1}$:
$\mathbb{E}[\alpha_l]$ *contributes to* $B(\alpha, \prec_{k,p}, \lambda)_{i,i-1-l}$ *with summand* $\mathbb{E}[\alpha_l]\, \lambda^{k-(i-2)}$.

(*iv*) $\forall l \in \mathbb{N}_k$, $\forall i \in \mathbb{N}_{(l+2)\wedge(k+1)} \setminus \mathbb{N}_1$:
$\mathbb{E}[\alpha_l]$ *contributes to* $B(\alpha, \prec_{k,p}, \lambda)_{i,l+3-i}$ *with summand* $\mathbb{E}[\alpha_l]\, \lambda^{k-l}$.

(*v*) *Let* $l \in \mathbb{N}_k$.
Then $\forall i \in \mathbb{N}_{k+1} \setminus \mathbb{N}_1$: $\mathbb{E}[\alpha_l]$ *contributes to at most one* $B(\alpha, \prec_{k,p}, \lambda)_{i,j}$, $j \in \mathbb{N}_{k+1} \setminus \mathbb{N}_0$.

Proof: Deferred to section 4. □

3.5.2 The case p=2, k>m and discretized SDDE

Consider the linear SDDE

$$dX(t) = (a_0 X(t) + a_1 X(t-1) + a_2 X(t-2))\, dt + X(t) dW(t) \tag{3.4}$$

under the conditions described in section 1. Consider as a numerical method the Euler-Maruyama-Scheme

$$\begin{aligned} X_{n+1} &= X_n + a_0 h\, X_n + a_1 h\, X(n-K) + a_2 h\, X(n-2K) + X_n \sqrt{h}\epsilon_{n+1} \\ &= (1 + a_0 h + \sqrt{h}\epsilon_{n+1})\, X_n + a_1 h\, X(n-K) + a_2 h\, X(n-2K) \end{aligned} \tag{3.5}$$

where $K = 1/h$ and step width h is chosen such that $1/h \in \mathbb{N}$. Define now $k(h) := 2/h$, $\alpha(h) = (\alpha_0(h), \ldots, \alpha_k(h))$ with $\alpha_0(h) = 1 + a_0 h + \sqrt{h}\epsilon$, $\epsilon \sim N(0,1)$, $\alpha_{k(h)/2} = a_1 h$, $\alpha_{k(h)} = a_2 h$, $\alpha_i = 0$, $i \in \mathbb{N}_{k(h)} \setminus \{0, k(h)/2, k(h)\}$. In order to investigate now the influence of the step with h on the asymptotic behaviour of the numerical scheme, one can to consider a family of coefficients $\alpha(h)$, whose dimension increases for $h \downarrow 0$ and which have three non-zero elements at positions $k_0(h) = 0$, $k_1(h) = 1/h$ and $k_2(h) = 2/h$. A characteristic property of the non-zero elements of the family $\alpha(h)$ is that $\frac{k_0(h)}{k(h)} = c_0$, $\frac{k_1(h)}{k(h)} = c_1$ and $\frac{k_2(h)}{k(h)} = c_2$ with h-independent constants $c_0 = 0$, $c_1 = \frac{1}{2}$, $c_2 = 1$.

Consider the affine SDDE

$$dX(t) = (b + a_0 X(t) + a_1 X(t-1) + a_2 X(t-2))\, dt + X(t) dW(t) \tag{3.6}$$

under the conditions described in section 1. As a numerical method choose as before the Euler-Maruyama-Scheme

$$\begin{aligned} X_{n+1} &= X_n + bh + a_0 h X_n + a_1 h X(n-K) + a_2 h X(n-2K) + X(n)\sqrt{h}\epsilon_{n+1} \\ &= bh + (1 + a_0 h + \sqrt{h}\epsilon_{n+1}) X_n + a_1 h X(n-K) + a_2 h X(n-2K) \end{aligned} \tag{3.7}$$

where $K = 1/h$ and step width h is chosen such that $1/h \in \mathbb{N}$. Now the as indicated in section 1, it is of interest to investigate the error propagation of the initial error. But this question is related to the homogeneous scheme

$$\delta_{n+1} = (1 + a_0 h + \sqrt{h}\epsilon_{n+1})\, \delta_n + a_1 h\, \delta_{n-K} + a_2 h\, \delta_{n-2K} \tag{3.8}$$

Define again $k(h) := 2/h$, $\alpha(h) = (\alpha_0(h), \ldots, \alpha_k(h))$ with $\alpha_0(h) = 1 + a_0 h + \sqrt{h}\epsilon$, $\epsilon \sim N(0,1)$, $\alpha_{k(h)/2} = a_1 h$, $\alpha_{k(h)} = a_2 h$, $\alpha_i = 0$, $i \in \mathbb{N}_{k(h)} \setminus \{0, k(h)/2, k(h)\}$. In order to investigate now the influence of the step with h on the error propagation of the initial error δ_0, one has to consider a family of coefficients $\alpha(h)$, whose dimension increases for $h \downarrow 0$ and which have three non-zero elements at positions

$k_0(h) = 0$, $k_1(h) = 1/h$ and $k_2(h) = 2/h$. A characteristic property of the non-zero elements of the family $\alpha(h)$ is, that $\frac{k_0(h)}{k(h)} = c_0$, $\frac{k_1(h)}{k(h)} = c_1$ and $\frac{k_2(h)}{k(h)} = c_2$ with h-independent constants $c_0 = 0$, $c_1 = \frac{1}{2}$, $c_2 = 1$.

The two previous examples motivate the investigation of the case $p = 2$ and $k > m$. For technical reasons we deal with this case by restating it in terms of the previous case. Define $K := \{ k_j \mid j \in \mathbb{N}_m \}$ and $\Phi \in (\{-1\} \cup \mathbb{N}_m, \{-1\} \cup K)$ with: $\Phi(-1) = -1$, $\forall l \in \mathbb{N}_m\ \Phi(l) = k_l$. Consider instead of 2.1

$$\tilde{X}_{n+1} \quad = \quad \sum_{j=0}^{k} \tilde{\alpha}_{j,n} \tilde{X}_{n-j} \tag{3.9}$$

with stochastic coefficients $\tilde{\alpha}_n := (\tilde{\alpha}_{0,n}, \ldots, \tilde{\alpha}_{m,n})$, $n \in \mathbb{N}$, which satisfy

$\forall n \in \mathbb{N} : \tilde{\alpha}_n$ is independent of $\{\tilde{X}_l\}_{l \in \mathbb{N}_{n-1}}$

$\forall n \in \mathbb{N} : \tilde{\alpha}_n \sim \tilde{\alpha} := (\tilde{\alpha}_0, \ldots, \tilde{\alpha}_m)$.

$\forall n \in \mathbb{N} : \tilde{\alpha}_{n,l} = \begin{cases} \alpha_{n,l} & l \in K, \\ 0 & \text{otherwise}, \end{cases} \quad \tilde{\alpha}_l = \begin{cases} \alpha_l & k_l \in K, \\ 0 & \text{otherwise}. \end{cases}$

REMARK 3.5.8
By definition 2.2.8 of (*k*,*p*)-amplification matrices one observes $A(\alpha, \prec_{k,2}) = A(\tilde{\alpha}, \prec_{k,2})$. Then lemma 3.5.3 provides an exact characterization of $B(\alpha, \prec_{k,2}, \lambda)$ as from $A(\alpha, \prec_{k,2}) = A(\tilde{\alpha}, \prec_{k,2})$ follows also $B(\alpha, \prec_{k,2}, \lambda) = B(\tilde{\alpha}, \prec_{k,2}, \lambda)$.

However, due to the possibility of a larger or large number of zero coefficients in $\tilde{\alpha}$, the characterization of $B(\alpha, \prec_{k,2}, \lambda)$ by means of the non-zero coefficients is useful.

LEMMA 3.5.9
Let $k \in \mathbb{N} \setminus \{0\}$.
Then

(*i*) $\quad B(\alpha, \prec_{k,2}, \lambda)_{1,j} = \sum_{l=0}^{m} \mathbb{E}[\, \alpha_l^2 \,] \lambda^{k-k_l}, \qquad j = 1.$

(*ii*) $\quad B(\alpha, \prec_{k,2}, \lambda)_{1,j} = \sum_{l \in \mathbb{N}_m : k_l + (j-1) \in K} \mathbb{E}[\, \alpha_l \alpha_{\Phi^{-1}(k_l + (j-1))} \,] \lambda^{k-k_l-(j-1)}, \qquad j = 2, \ldots, k+1.$

(*iii*) $\quad \forall l \in \{-1\} \cup \mathbb{N}_m$, $\forall i \in \mathbb{N}_{k+1} \setminus \mathbb{N}_{(\Phi(l)+1) \vee 1}$:
$\mathbb{E}[\, \alpha_l \,]$ *contributes to* $B(\alpha, \prec_{k,p}, \lambda)_{i,i-1-k_l}$ *with summand* $\mathbb{E}[\, \alpha_l \,] \lambda^{k-(i-2)}$.

(*iv*) $\quad \forall l \in \mathbb{N}_m$, $\forall i \in \mathbb{N}_{(k_l+2) \wedge (k+1)} \setminus \mathbb{N}_1$:
$\mathbb{E}[\, \alpha_l \,]$ *contributes to* $B(\alpha, \prec_{k,p}, \lambda)_{i,k_l+3-i}$ *with summand* $\mathbb{E}[\, \alpha_l \,] \lambda^{k-k_l}$.

Proof: Deferred to section 4. □

3.5.3 The case p≥2, k≥m=1

In contrast to the two special cases considered before, the number m of memory coefficients and not the power p is limited. This case has been the starting point for proving the main lemma of section 2.2. The advantage of this case is that for an (k,p)-index $x = (0^q, x_{>0})$ with $q \in \mathbb{N}_p \setminus \{0\}$ and $x_{>0} \in \mathbb{X}_1(k, p-q)$ T_1 maps $\{x\}$ on $\{ (0^i, x, k^j) \mid i, j \in \mathbb{N}_p, i + j = p - q \}$. The elements of $T_m(\{x\})$ have a properties that allow us to characterize the corresponding matrix entries in $A(\alpha, \prec_{k,p})$ and hence $B(\alpha, \prec_{k,p}, \lambda)$ easily.

LEMMA 3.5.10
Let $k \in \mathbb{N} \setminus \{0\}$, $p \in \mathbb{N} \setminus \mathbb{N}_2$,
$x \in \mathbb{X}_{0,1}(k, p)$.
Then $\forall j \in \mathbb{N}_{n(0,x)-1}$: $r(j, n(0,x), -(x)) \notin J_1(k,p)$.

Proof: Deferred to section 4. □

Now we are ready to characterize $B(\alpha, \prec_{k,p}, \lambda)$.

LEMMA 3.5.11

Let $k \in \mathbb{N} \setminus \{0\}$, $p \in \mathbb{N} \setminus \mathbb{N}_2$,

$x, y \in \mathbb{X}_{0,1}(k,p)$.

Then $B(\alpha, \prec_{k,p}, \lambda) = B_1(\alpha, k, p, \lambda) - B_2(k, p, \lambda)$, *where*

$$B_1(\alpha, k, p, \lambda) = \begin{cases} c(n(0,x), n(0,x), \alpha)\, \lambda^{k+1-x_p} + \delta_y(z - (k+1-x_p))\, c(0,p,\alpha)) & y = r(n(0,x), n(0,x), -(x)) \text{ and } \\ & z = r(0, n(0,x), -(x)), \\ c(j, n(0,x), \alpha) & y = r(j, n(0,x), -(x)) \text{ and } \\ & j \in \mathbb{N}_{n(0,x)-1} \setminus \{0\}, \\ c(0,p,\alpha) & y = z - z_1 \neq \bar{z} \text{ where } \\ & z = r(n(0,x), n(0,x), -(x)), \\ & \bar{z} = r(0, n(0,x), -(x)), \\ 0 & \text{otherwise,} \end{cases}$$

$B_2(k,p,\lambda) = diag^{f(k,p-1)}(\lambda^{d_1}, \ldots, \lambda^{d_{f(k,p-1)}})$ *where* $\forall x \in \mathbb{X}_{0,1}(k,p) : d_{F(x)} = k + 1 - x_p$,

Proof: Deferred to section 4. □

4 Appendix

4.1 Proofs of section 2

Proof of lemma 2.2.2:

$0 < n(0,x)$:

$$\begin{aligned} X^x_{n+1} &= X^{0_{n(0,x)}}_{n+1} X^{s(n(0,x),p,x)}_{n+1} \\ &\underset{(2.4)}{=} \sum_{j=0}^{n(0,x)} \beta(j, n(0,x), \alpha) X^j_n X^{n(0,x)-j}_{n-k} X^{s(n(0,x),p,x)}_{n+1} \\ &= \sum_{j=0}^{n(0,x)} \beta(j, n(0,x), \alpha) X^j_n X^{n(0,x)-j}_{n-k} X_n^{-(s(n(0,x),p,x))} \\ &= \sum_{j=0}^{n(0,x)} \beta(j, n(0,x), \alpha) X^j_n X^{n(0,x)-j}_{n-k} X_n^{(s(n(0,x),p,-(x)))} \\ &= \sum_{j=0}^{n(0,x)} \beta(j, n(0,x), \alpha) X_n^{r(j,n(0,x),-(x))}. \\ Y^x_{n+1} &= \sum_{j=0}^{n(0,x)} \gamma(j, n(0,x), \alpha) Y_n^{r(j,n(0,x),-(x))}. \end{aligned}$$

$0 = n(0,x)$:

$$\begin{aligned} Y^x_{n+1} &= \mathbb{E}[\prod_{j=1}^{p} X_{n+1-x_j}] \\ &= \mathbb{E}[\prod_{j=1}^{p} X_{n-(x_j-1)}]. \\ &= Y_n^{-(x)} \end{aligned}$$

□

Proof of lemma 2.2.4:

1) $\mathcal{X}(k,p) \subseteq \mathbb{X}(k,p)$ holds by definition.

2) Define for $p \in \mathbb{N} \setminus \{0\}$, $l \in \mathbb{N}_{p-1}$, $r \in \mathbb{N}_k$

$$M(l,p)^{-r} := \begin{cases} \{ x^{-r} \in \mathbb{X}(k,p) \mid x_p = k \} & l = 0, \\ \{ x^{-r} \in \mathbb{X}(k,p) \mid x_1 = \ldots = x_l = 0, x_p = k \} & l > 0. \end{cases}$$

$$\overline{M}(l,p) := \begin{cases} \mathbb{X}(k,p) & l = 0, \\ \{ x \in \mathbb{X}(k,p) \mid x_1 = \ldots = x_l = 0 \} & l > 0. \end{cases}$$

Then $\bigcup_{r=0}^{p-1} M(l,p)^{-r} = \overline{M}(l,p)$.

Proof:

Let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_p) \in \overline{M}(l,p)$. Then $x = \bar{x}^{+(k-\bar{x})} \in M(l,p)^{-0}$ and $\bar{x} = x^{-(k-\bar{x})} \in M(l,p)^{-(k-\bar{x})}$.

It follows $\overline{M}(l,p) \subseteq \overset{p-1}{\underset{r=0}{\cup}} M(l,p)^{-r}$.

As $\forall r \in \mathbb{N}_k : M(l,p)^{-r} \subset \overline{M}(l,p)$, it follows $\overset{p-1}{\underset{r=0}{\cup}} M(l,p)^{-r} \subseteq \overline{M}(l,p)$. $\square$

3) $\{ (\{0\}^j, \{k\}^{p-j}) \mid j \in \mathbb{N}_p \} \in \mathcal{X}(k,p)$.

Proof:

As by (2.19) $\{0\}^p \in \mathcal{X}(k,p)$, it follows by (2.20):

$$\forall j \in \mathbb{N}_p \setminus \{0\} : r(j,p,-(x)) = (\{0\}^j, \{k\}^{p-j}) \in \mathcal{X}(k,p).$$ $\square$

4) Let $l \in \mathbb{N}_{p-1} \setminus \{0\}$.

Then: $\overset{p-1}{\underset{j=l}{\cup}} \overline{M}(j,p) \subseteq \mathcal{X}(k,p) \Longrightarrow M(l-1,p) \subseteq \mathcal{X}(k,p)$.

Proof:

For any $x \in \overset{p-1}{\underset{j=l}{\cup}} \overline{M}(j,p)$ $0 < n(0,x)$, and it holds by (2.20):

$$\forall j = l, \ldots, p-1 : \{ r(l-1, n(0,x), -(x)) \mid x \in \overline{M}(j,p) \} \subseteq \mathcal{X}(k,p).$$

As by 3) $\{ (\{0\}^{l-1}, \{k\}^{p-j}) \} \in \mathcal{X}(k,p)$ and

$$\{r(l-1,n(0,x),-(x)) \mid x \in \overline{M}(j,p)\} = \{r(l-1,j,-(x)) \mid x \in \overline{M}(j,p)\}$$
$$= \{(\{0\}^{l-1}, y_1 \ldots, y_{p-j}, \{k\}^{j+1-l}) \mid 0 \le y_1 \le \ldots y_{p-j} \le k-1\}$$

it holds

$$\mathcal{X}(k,p) \supseteq \{ (\{0\}^{l-1}, \{k\}^{p+1-l}) \} \cup \overset{p-1}{\underset{j=l}{\cup}} \{r(l-1,n(0,x),-(x)) \mid x \in \overline{M}(j,p)\}$$
$$= \{ (\{0\}^{l-1}, \{k\}^{p+1-l}) \} \cup \{(\{0\}^{l-1}, y_1 \ldots, y_{p+1-l}) \mid 0 \le y_1 \le \ldots y_{p+1-l}, y_l \le k-1\}$$
$$= M(l-1,p).$$

$\square$

5) Let $l \in \mathbb{N}_{p-1} \setminus \{0\}$.

Then: $\overset{p-1}{\underset{j=l}{\cup}} \overline{M}(j,p) \subseteq \mathcal{X}(k,p) \Longrightarrow \overset{p-1}{\underset{j=l-1}{\cup}} \overline{M}(l-1,p) \subseteq \mathcal{X}(k,p)$.

Proof:

Due to 4) it holds $\overset{p-1}{\underset{j=l}{\cup}} \overline{M}(j,p) \subseteq \mathcal{X}(k,p) \Longrightarrow M(l-1,p) \subseteq \mathcal{X}(k,p)$.

Due to (2.20) and 2) it holds $M(l-1,p) \subseteq \mathcal{X}(k,p)) \Longrightarrow \overline{M}(l-1,p) \subseteq \mathcal{X}(k,p)$.

This implies $\overset{p-1}{\underset{j=l-1}{\cup}} \overline{M}(l-1,p) \subseteq \mathcal{X}(k,p)$. $\square$

6) $\mathbb{X}(k,p) \subseteq \mathcal{X}(k,p)$.

Proof:

Due to 3) $\{ (\{0\}^j, \{k\}^{p-j}) \mid j \in \mathbb{N}_p \} \subseteq \mathcal{X}(k,p)$ implying $M(p-1,p) \subseteq \mathcal{X}(k,p)$.

Due to (2.20) and 2) then follows $\overline{M}(p-1,p) \subseteq \mathcal{X}(k,p)$.

This can be taken as initial step for a complete induction to show:

by 5) for any $l = p-1 \downarrow 1$ $\overset{p-1}{\underset{j=l}{\cup}} \overline{M}(j,p) \subset \mathcal{X}(j,p) \Longrightarrow \overset{p-1}{\underset{j=l-1}{\cup}} \overline{M}(j,p)$.

This implies $\overset{p-1}{\underset{j=0}{\cup}} \overline{M}(j,p) \subseteq \mathcal{X}(k,p)$. The result follows from $\mathbb{X}(k,p) \underset{2)}{=} \overline{M}(0,p)$. $\square$

Proof of lemma 2.3.2:

$0 < n(0,x)$:

$$\begin{aligned}
X_{n+1}^{x} &= X_{n+1}^{0_{n(0,x)}} X_{n+1}^{s(n(0,x),p,x)} \\
&\underset{(2.4)}{=} \sum_{\pi \in I(m,n(0,x))} \binom{n(0,x)}{\pi} \alpha_n^{\pi} X_n^{\pi} X_{n+1}^{s(n(0,x),p,x)} \\
&= \sum_{\pi \in I(m,n(0,x))} B(\pi,n(0,x),\alpha)\; X_n^{\pi} X_n^{s(n(0,x),p,-(x))} \\
&= \sum_{\pi \in I(m,n(0,x))} B(\pi,n(0,x),\alpha)\; X_n^{m(s(n(0,x),p,-(x)),\pi)}. \\
Y_{n+1}^{x} &= \sum_{\pi \in I(m,n(0,x))} C(\pi,n(0,x),\alpha)\; Y_n^{c(s(n(0,x),p,-(x)),\pi)}.
\end{aligned}$$

$0 = n(0,x)$: $$Y^x_{n+1} = \mathbb{E}[\prod_{j=1}^{p} X_{n+1-x_j}] = \mathbb{E}[\prod_{j=1}^{p} X_{n-(x_j-1)}] = Y_n^{-(x)}.$$ □

Proof of lemma 2.4.1:

Let $f \in \mathcal{C}(\mathbb{R},\mathbb{R})$ be a convex function. Then it is known that $\forall n \in \mathbb{N}\setminus\{0\}$, $\forall c = (c_1,\ldots,c_n) \in \mathbb{R}^n_+$ with $\sum_{i=1}^n c_i = 1$ $\forall (t_1,\ldots,t_n) \in \mathbb{R}^n$: $f(\sum_{i=1}^n c_i t_i) \le \sum_{i=1}^n c_i f(t_i)$. Choosing $f(t) = e^t$ and $t_i = ln(|\ a_i\ |)$, $i \in \mathbb{N}\setminus\{0\}$, this implies

$$\prod_{i=1}^{n} |\ a_i\ |^{c_i} \le \sum_{i=1}^{n} c_i\ |\ a_i\ |\ . \tag{4.1}$$

Let $f_i \in L^p(\Omega,\mathbb{R},\mathbb{P})\setminus\{0\}$, $i \in \mathbb{N}\setminus\{0\}$, $(p_1,\ldots,p_n) \in \mathbb{R}^n_+$ with $\sum_{i=1}^n p_i = p$. Then $\forall\, \omega \in \Omega$: set $c_i = p_i/p$, $a_i = (\ f_i(\omega)/\ |\ f_i\ |_p\ )^p$. Applying (4.1), taking the expectations and some minor modifications lead then to

$$\mathbb{E}[\prod_{i=1}^{n} |\ f_i\ |^{p_i}\ ] \le \prod_{i=1}^{n} |\ f_i\ |_p^{p_i}\ . \tag{4.2}$$

If any f_i were zero, (4.2) would hold trivially. But this enables us to conclude:

$$|\ \mathbb{E}[\prod_{i=1}^{q} X^{\pi(i)}_{n-k_i}]\ | \le \mathbb{E}[\prod_{i=1}^{q} |\ X_{n-k_i}\ |^{\pi(i)}\ ] \le \prod_{i=1}^{q} \mathbb{E}[\ |\ X_{n-k_i}\ |^q\ ]^{\frac{\pi(i)}{q}} \le \prod_{i=1}^{q} \mathbb{E}[\ |\ X_{n-k_i}\ |^p\ ]^{\frac{\pi(i)}{p}}. \tag{4.3}$$

□

Proof of lemma 2.3.4:

Observe that $\mathbb{X}(k,p) = \mathcal{X}(k,p) \subset \mathcal{X}(\vec{k},p) \subset \mathbb{X}(k,p)$. □

4.2 Proofs of section 3

Proof of lemma 3.1.1:

(i) By definition of $\mathbb{X}(k,p)$.

(ii) The integral form of (i).

(iii) Per induction using $\sum_{i_1=0}^{k+1}\sum_{i_2=0}^{i_1}\cdots\sum_{i_p=0}^{i_{p-1}} 1 = \sum_{i_1=0}^{k}\sum_{i_2=0}^{i_1}\cdots\sum_{i_p=0}^{i_{p-1}} 1 + \sum_{i_2=0}^{k+1}\cdots\sum_{i_p=0}^{i_{p-1}} 1.$

(iv) use (ii), $\int_0^t\int_0^{t_1}\cdots\int_0^{t_{p-1}} 1 dt_p \ldots dt_2 dt_1 = \frac{t^p}{p!}$ and geometrical interpretation for the approximation.

□

Proof of lemma 3.1.2:

Assume, there existed a triple $(P, B_1, B_2) \in (\mathbb{O}(f(k,p)) \cap \{1,-1,0\}^{f(k,p)\times f(k,p)}, \mathbb{M}_{k_1}, \mathbb{M}_{k_2})$ where $k_1, k_2 \in \mathbb{N}_{f(k,p)}\setminus\{0\}$, $k_1 + k_2 = f(k,p)$, P permutation matrix, such that

$$A(\alpha, \prec_{k,p}) = P^{-1}\begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} P.$$

As P is a permutation matrix describing a reordering of rows and columns, assume without loss of generality $P = I_{f(k,p)}$.

Consider now the graph $G = (V,E)$, whose nodes are formed by the (k,p)-covariances of $\mathbb{X}(k,p)$ $(V = \mathbb{X}(k,p))$, and whose edges are formed by entries of the amplification matrix $A(\alpha, \prec_{k,p})$, that is for $x,y \in \mathbb{X}(k,p)$: $e = (x,y) \in E \leftrightarrow a(\alpha,\prec_{k,p})_{x,y} \ne 0$.

Lemma 2.3.2 implies that for any $x,y \in \mathbb{X}(k,p)$ there exists a path from x to y in G. On the other hand, the block diagonalization implies that G consists of at least two non-connected subgraphs. But this is a contradiction. □

Proof of lemma 3.2.4

(*i*) Let $x,y \in \mathbb{X}_l(k,p)$ with $\phi(x) = \phi(y)$. Then $\forall i \in \mathbb{N}_k\setminus\{0\} : x_i - l = y_i - l$.
But then $\forall i \in \mathbb{N}_k\setminus\{0\} : x_i = y_i$, $x = y$. So ϕ is an injection.
Let $y \in \mathbb{X}(k-l,p)$. Then define $x \in \mathbb{X}(k,p)$ by: $\forall i \in \mathbb{N}_k\setminus\{0\} : x_i = y_i + l$.
Then obviously $x \in \mathbb{X}_l(k,p)$ and $\phi(x) = y$.

$(ii)\quad \#(\,\mathbb{X}_l(k,p)\,) = \#(\,\phi(\mathbb{X}_l(k,p))\,) = \#(\,\mathbb{X}(k-l,p)\,) = f(k-l,p).$

□

Proof of lemma 3.2.5

If $k=0$, $\mathbb{X}(0,p) = \{\, 0^p \,\}$ and $F(0^p) = 1$.

If $k=1$, $\mathbb{X}(1,p) = \{\, (0^{p-i}, 1^i) \mid i \in \mathbb{N}_p \,\}$. Then $F(0^p) = 1 = p+2-min\{p+1\} \cup \emptyset$. Furthermore for $j \in \mathbb{N}_p \setminus \{0\}$: $F(\,(0^{p-j}, 1^j)\,) = j+1 = p+2-(p-j+1) = p+2-min\{p+1\} \cup \{\, i \in \mathbb{N} \mid x_i = 1 \,\}$.

If $p=1$, $\mathbb{X}(k,p) = \mathbb{N}$. But then for $j \in \mathbb{N}_k$: $F(\,(j)\,) = j+1$.

Let $k>1$, $p>1$.

Let $x_1 = 0$. Then $F(k,p,x) = 1 + \#\{\, y \in \mathbb{X}(k,p) \mid y \prec_{k,p} x \,\} = 1 + \#\{\, y \in \mathbb{X}(k,p-1) \mid y \prec_{k,p} (x_2,\dots,x_p) \,\} = F(k,p-1,(x_2,\dots,x_p))$.

Let $x_1 > 0$. Then

$$\begin{aligned}\{\, y \in \mathbb{X}(k,p) \mid y \prec_{k,p} x \,\} &= (\bigcup_{i=0}^{x_1-1} \mathbb{X}_{i,i+1}(k,p)\,) \cup \{\, y \in \mathbb{X}_{x_1}(k,p) \mid y \prec_{k,p} x \,\} \\ &= (\bigcup_{i=0}^{x_1-1} \{\, i \,\} \times \mathbb{X}_i(k,p-1)\,) \cup \{\, y \in \mathbb{X}_{x_1}(k,p) \mid y \prec_{k,p} x \,\}\end{aligned}$$

is a partition into disjoint sets. As

$$\#(\{\, i \,\} \times \mathbb{X}_i(k,p-1)) = \#\mathbb{X}_i(k,p-1) \underset{lemma\ 3.2.4}{=} f(k-i,p-1)$$

and

$$\begin{aligned}\#\{\, y \in \mathbb{X}_{x_1}(k,p) \mid y \prec_{k,p} x \,\} &= \#\{\, y \in \mathbb{X}_{x_1}(k,p-1) \mid y \prec_{k,p-1} (x_2-x_1,\dots,x_p-x_1) \,\} \\ &= F(k-x_1,p-1,(x_2-x_1,\dots,x_p-x_1)\,),\end{aligned}$$

(ii) follows.

□

Proof of lemma 3.2.6

(i) $\forall x \in \mathbb{X}_1(k,p)$: x does not contain any 0, $0 = n(0,x)$. Hence $T_m(\{x\}) = \{-(x)\}$, and by definition (2.23) (i) follows.

(ii) Let $x,y \in \mathbb{X}_1(k,p)$. Then $1 \le x_1 \le \dots \le x_p$, $1 \le y_1 \le \dots \le y_p$.
Let $x=y$. Then $\forall i \in \mathbb{N}_p \setminus \{0\}$: $x_i = y_i$, hence $\forall i \in \mathbb{N}_p \setminus \{0\}$: $x_i - 1 = y_i - 1$ and $-(x) = -(y)$.
Let $-(x) = -(y)$. Then $\forall i \in \mathbb{N}_p \setminus \{0\}$: $x_i - 1 = y_i - 1$, hence $\forall i \in \mathbb{N}_p \setminus \{0\}$: $x_i = y_i$ and $x = y$.

(iii) Let $p=1$.
Let $x \in \mathbb{X}(k,p)$ with $0 = n(k,x)$.
Then $y = (x_1+1)$, $-(x) = (x_1-1)$, $-(y) = (x_1)$, $F(-(y)) - F(-(x)) = 1$.
If $x \in \mathbb{X}_k(k,1)$, then $\nexists y \in \mathbb{X}(k,1) : F(y) = F(x)+1$.
Let $p \in \mathbb{N} \setminus \{0,1\}$.
Let $x \in \mathbb{X}(k,p)$ with $0 = n(k,x)$. Then $y = (x_1 \dots, x_{p-1}, x_p+1)$, $-(x) = (x_1-1,\dots,x_p-1)$, $-(y) = (x_1-1,\dots,x_{p-1}-1,x_p)$, $F(-(y)) - F(-(x)) = 1$.
Let $i \in \mathbb{N}_p \setminus \{0,p\}$ $x = (x_1,\dots,x_{p-i},k^i)$. Then $y = (x_1,\dots,x_{p-i}+1,(x_{p-i}+1)^i)$, $-(x) = (x_1-1,\dots,x_{p-i}-1,(k-1)^i)$, $-(y) = (x_1-1,\dots,x_{p-i},(x_{p-i})^i)$. Hence

$$\begin{aligned}&\{\, z \in \mathbb{X}(k,p) \mid z \prec_{k,p} -(y) \,\} \setminus \{\, z \in \mathbb{X}(k,p) \mid z \prec_{k,p} -(x) \,\} \\ &\qquad = \{\, z \in \mathbb{X}(k,p) \mid z = (x_1-1,\dots,x_{p-i}-1,\tilde{z}), \tilde{z} \in \mathbb{X}_{k-1}(k,i) \,\}.\end{aligned}$$

$F(-(y)) - F(-(x)) = \#\mathbb{X}_{k-1}(k,i) = \#\mathbb{X}(1,i) = 1+i$
If $x \in \mathbb{X}_k(k,p)$, then $\nexists y \in \mathbb{X}(k,p) : F(y) = F(x)+1$.

(iv) Let $i \in \mathbb{N}_k \setminus \{0\}$. Then $\forall x \in \mathbb{X}_{i,i+1}(k,p)$: $x \in I_i^{\prec}(k,p)$ and $-(x) \in I_{i-1}^{\prec}(k,p)$ by definition (3.2.1). But then (iv) follows from (i).

□

Proof of corollary 3.2.7

(i) Follows from lemma (3.2.6).(i).

(ii) Follows from lemma (3.2.6).(iii).

(iii) Follows from lemma (3.2.6).(iv).

□

Proof of lemma 3.2.10

(1) The case $p=2$ is proven in lemma 3.5.2.

(2) For $i \in \mathbb{N}_{k-1} \setminus \{0\}$: $\mathbb{X}_{i,i+1}(k,p+1) = \{\, (i,x) \mid x \in \mathbb{X}_i(k,p) \,\}$.

(3) $\forall i \in \mathbb{N}_{k-1}$ define $\phi_i\colon \begin{array}{ccc} \mathbb{X}_{i,i+1}(k,p+1) & \longrightarrow & \mathbb{X}_i(k,p) \\ (i,x) & \mapsto & x. \end{array}$
Obviously, ϕ_i is a bijection.

(4) Let $i \in \mathbb{N}_{k-1} \setminus \{0\}$, $\tilde{x} = (i, x) \in \mathbb{X}_{i,i+1}(k, p+1)$, $\tilde{y} = (i-1, y) \in \mathbb{X}_{i-1,i}(k, p+1)$.
Then $\tilde{y} = -(\tilde{x}) \longleftrightarrow y = -(x)$.

(5) $i \in \mathbb{N}_{k-1} \setminus \{0\}$.
From (4) follows $A(\alpha, \prec_{k,p+1})_{I_i^{\prec}(k,p+1), I_{i-1}^{\prec}(k,p+1)} = A(\alpha, \prec_{k,p})_{\cup_{j=i}^{k} I_j^{\prec}(k,p), \cup_{j=i-1}^{k} I_j^{\prec}(k,p)}$.

(6) For $p = 2$, (i) and (ii) are proved in lemma 3.5.2.
Assume, (i), (ii) hold for $p = n$.
From (5) we deduce that for $i \in \mathbb{N}_{k-1} \setminus \{0\}$: $A(\alpha, \prec_{k,p+1})_{I_i^{\prec}(k,p+1), I_{i-1}^{\prec}(k,p+1)} = E_n^-(i)$.
Let $y = k^{p+1}$. Then: $\forall x \in \bar{I}_0^{\prec}(k,p) : A(\alpha, \prec_{k,p})_{x,y} = 0$.
But then, applying corollary 3.2.7.(iii), it follows

$$
\begin{aligned}
& A(\alpha, \prec_{k,p+1})_{\bar{I}_0^{\prec}(k,p+1), I^{\prec}(k,p+1)} \\
&\quad = cat_0(diag^k(A(\alpha, \prec_{k,p})_{I_1^{\prec}(k,p+1), I_0^{\prec}(k,p)}, \ldots, A(\alpha, \prec_{k,p})_{I_{k-1}^{\prec}(k,p), I_k^{\prec}(k,p)})) \\
&\quad = cat_0(diag^k(E_n^-(k), \ldots, E_n^-(1))) \\
&\quad = E_{n+1}^-(k).
\end{aligned}
$$

(7) But then (i), (ii) follow per complete induction from () and (7).

□

Proof of lemma 3.3.1

Define the $\mathbb{M}_n(\mathbb{R})$-matrices:

$$
P = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, Q_l = \begin{pmatrix} I & 0 \\ A_{12}A_{22}^{-1} & I \end{pmatrix}, Q_r = \begin{pmatrix} I & A_{22}^{-1}A_{21} \\ 0 & I \end{pmatrix}, B = \begin{pmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ 0 & A_{22} \end{pmatrix}.
$$

Then $Q_l P A P Q_r = B$. As $|P| = |Q_l| = |Q_r| = 1$, the conclusion follows from $|A| = |B|$.

□

Proof of lemma 3.3.3

The case $n = 1$ is trivial. Assume therefore $n > 1$.

Consider the linear system $Lx = y$ in $\mathbb{R}^n$:

$$
\begin{array}{llcll}
\text{(i)} & -\lambda x_1 & = & y_1, & \\
\text{(ii)} & x_{i-1} - \lambda x_i & = & y_i & i \in \mathbb{N} \setminus \{0, 1\}.
\end{array}
$$

Then $x_1 = -\lambda^{-1} y_1$ and -$\sum_{l=1}^{j} \lambda^{l-1-i} y_l \underset{(ii)}{=} \lambda^{-1-i} x_1 - \sum_{l=2}^{i} \lambda^{l-1-i} x_{l-1} + \sum_{l=2}^{i} \lambda^{l-i} x_l = x_i$.

□

Proof of lemma 3.3.5

(i) Let $x \in \mathbb{X}_1$.
Due to corollary 3.2.7.(i), 3.2.7.(ii) the diagonal elements of $A(\alpha, \prec_{k,p})$ are 0, hence the diagonal elements of $A(\alpha, \prec_{k,p}) - \lambda I)_{J^{\prec}(k,p,x), J^{\prec}(k,p,x)}$ are $-\lambda$.
Let $J^{\prec}(k,p,x) = \{ y_i \mid i = 1, \ldots, n(k,p,x) \}$. W.l.o.g. let $y_{i-1} \prec_{k,p} y_i$, $i = 1, \ldots n(k,p,x)s$ hence $y_{i-1} = -(y_i)$, $i = 1, \ldots, n(k,p,x)$. From corollary 3.2.7.(i) we know that $\forall i = 2, \ldots, n(k,p,x)$: $A(\alpha, \prec_{k,p})_{y_{i-1}, y_i} = 1$. So all elements of the first lower subdiagonal of ($A(\alpha, \prec_{k,p}$) $- \lambda I$)$_{J^{\prec}(k,p,x), J^{\prec}(k,p,x)}$ are 1.
From corollary 3.2.7.(i) we know that then the other elements of $A(\alpha, \prec_{k,p}) - \lambda I)_{J^{\prec}(k,p,x), J^{\prec}(k,p,x)}$ are zero.
By definition of $J^{\prec}(k,p,x)$ we know that $\forall i = 2, \ldots, n(k,p,x)$: $y_{i-1} = -(y_i)$. Hence the last component of $y_{n(k,p,x)}$ is k. Otherwise $+(y_{n(k,p,x)}) \in \mathbb{X}(k,p)$, hence $y_{n(k,p,x)} \prec +(y_{n(k,p,x)}) \in J^{\prec}(k,p,x)$, a contradiction. But this implies $k = x_p + n(k,p,x) - 1$, $n(k,p,x) = k + 1 - x_p$.

(ii) Due to (i), $n(k,p,x) = k+1-x_p$. By definition, $\#(J^{\prec}(k,p,x)) = n(k,p,x) = k+1-x_p$. Furthermore $\#(\ \bar{I}_0^{\prec}(k,p) \setminus J^{\prec}(k,p,x)\ ) = f(k-1,p) - n(k,p,x) = f(k-1,p) - k + 1 + x_p$.
Due to corollary 3.2.7.(ii), $\forall x \in J^{\prec}(k,p,x), y \in \bar{I}_0^{\prec}(k,p) \setminus J^{\prec}(k,p,x)$: $A(k,p,\lambda)_{x,y} = 0$.
This shows (ii).

(iii) Obviously $\bigcup_{x \in \mathbb{X}_{1,2}} J^{\prec}(k,p,x) \subset \mathbb{X}_1(k,p)$.
Let $y \in \mathbb{X}_2(k,p)$. Then consider $z = \{ z_i \mid i = 1, \ldots, y_1, i < y_1 \rightarrow z_{i+1} = +(z_i) \}$. By construction of z, $z_1 \in \mathbb{X}_{1,2}(k,p)$ and $y \in J^{\prec}(k,p,z_1)$. So $\mathbb{X}_1(k,p) \subset \bigcup_{x \in \mathbb{X}_{1,2}} J^{\prec}(k,p,x)$.

Assume, $\exists x_1, x_2 \in \mathbb{X}_{1,2}(k,p)$, $y \in J^{\prec}(k,p,x_1) \cap J^{\prec}(k,p,x_2)$. But then by definition of $J^{\prec}(k,p,x_1)$, $J^{\prec}(k,p,x_2)$, $\forall \tilde{y} \in J^{\prec}(k,p,x_1)$ with $\tilde{y} \preccurlyeq_{k,p} y$: $\tilde{y} \in J^{\prec}(k,p,x_1) \cap J^{\prec}(k,p,x_2)$. But then $x_1 \in J^{\prec}(k,p,x_2)$. As by construction $J^{\prec}(k,p,x_2) \cap \mathbb{X}_{1,2}(k,p) = \{x_2\}$, as $x_1 \in \mathbb{X}_{1,2}(k,p)$, it follows $x_1 = x_2$, hence $J^{\prec}(k,p,x_1) = J^{\prec}(k,p,x_2)$.

(iv) Define a $f(k-1,p) \times f(k-1,p)$ matrix $\tilde{A}(\alpha, \prec_{k,p}, \lambda)$ such that $\forall x \in \mathbb{X}_{1,2}(k,p)$:

$$1) \quad \tilde{A}(\alpha, \prec_{k,p}, \lambda)_{J^{\prec}(k,p,x), J^{\prec}(k,p,x)} = \bar{L}(n(k,p,x), \lambda), \tag{4.4}$$

$$2) \quad \tilde{A}(\alpha, \prec_{k,p}, \lambda)_{J^{\prec}(k,p,x), I_0^{\prec}(k,p) \setminus J^{\prec}(k,p,x)} = 0_{i,j},\ i = n(k,p,x),\ j = f(k-1, p-n(k,p,x)).$$

By (i), (ii) and lemma 3.3.3 then $(A(\alpha, \prec_{k,p}) - \lambda I)\tilde{A}(\alpha, \prec_{k,p}, \lambda) = \hat{A}(\alpha, \prec_{k,p}, \lambda)$, where

1) $\hat{A}(\alpha, \prec_{k,p}, \lambda)_{J^{\prec}(k,p,x), J^{\prec}(k,p,x)} = L(n(k,p,x), \lambda)\bar{L}(n(k,p,x), \lambda) = I_{n(k,p,x)}$,

2) $\hat{A}(\alpha, \prec_{k,p}, \lambda)_{J^{\prec}(k,p,x), I_0^{\prec}(k,p) \setminus J^{\prec}(k,p,x)} = 0_{i,j}, \quad i = n(k,p,x), \quad j = f(k-1, p-n(k,p,x))$.

But due to (iii) then $\hat{A}(\alpha, \prec_{k,p}, \lambda) = I_{f(k-1,p)}$, hence $\tilde{A}(\alpha, \prec_{k,p}, \lambda) = (A(\alpha, \prec_{k,p}) - \lambda I)^{-1}$.

(v) See (iv).

□

Proof of lemma 3.3.8

(i) $\forall x \in \mathbb{X}_{1,2}(k,p) : -(x) \in J_1(k,p)$. $\forall y \in J_1(k,p) \exists x \in \mathbb{X}_{1,2}(k,p) : y = -(x)$. Hence $J_1(k,p) = \{ -(x) \mid x \in \mathbb{X}_{1,2}(k,p) \}$.

As $\forall x, y \in \mathbb{X}_{1,2}(k,p)$: $x \prec_{k,p} y \longleftrightarrow -(x) \prec_{k,p} -(y)$, $x = y \longleftrightarrow -(x) = -(y)$, it follows: if $\mathbb{X}_{1,2}(k,p) = \{ x_i \}_{i=1,\dots,f(k-1,p-1)}$ with $x_1 \prec_{k,p} \dots x_{f(k-1,p-1)}$, then $J_1(k,p) = \{ x_i \}_{i=1,\dots,f(k-1,p-1)}$ with $y_i = -(x_i)$), $i = 1, \dots, f(k-1,p-1)$.

Due to 3.2.7.(i), the statement follows.

(ii) $I_0^{\prec}(k,p) = J_1(k,p) \cup \bar{J}_1(k,p)$. Furthermore, by definition of $J_1(k,p)$, $\#(\bar{J}_1(k,p)) = \#(I_1^{\prec}(k,p))$.

But then the statement follows due to corollary 3.2.7.(i) and (i).

(iii) Due to (i) and corollary 3.2.7.(iii),

$$(A(\alpha, \prec_{k,p}) - \lambda I_{f(k,p)})_{\bar{I}_0(k,p), J_1(k,p)} = (I_{f(k-1.p-1)}, 0_{f(k-1,p-1), f(k,p-1)-f(k-1,p-1)})^{\top},$$

$$(A(\alpha, \prec_{k,p}) - \lambda I_{f(k,p)})_{\bar{I}_0(k,p), \bar{J}_1(k,p)} = 0_{f(k-1,p), f(k,p-1)-f(k-1,p-1)}.$$

But then

$$(A_{22}^{-1}(k,p,\lambda)A_{21}(k,p))_{I,J} = A_{22}^{-1}(k,p,\lambda)_{I,J}, \quad I = \mathbb{N}_{f(k-1,p)} \setminus \{0\}, J = FJ_1(k,p),$$

$$(A_{22}^{-1}(k,p,\lambda)A_{21}(k,p))_{I,J} = 0_{I,J}, \quad I = \mathbb{N}_{f(k-1,p)} \setminus \{0\}, J = \overline{FJ}_1(k,p).$$

As by definition for each $x \in \mathbb{X}_{1,2}(k,p)$ $J^{\prec}(k,p,x) \cap \mathbb{X}_{1,2}(k,p) = \{x\}$ and due to (i), $\forall j \in FJ_1(k,p) \, \exists x_j \in \mathbb{X}_{1,2}(k,p) : j = F(k,p,-(x_j))$. Hence by 4.4.$(iv)$

$A_{22}^{-1}(k,p,\lambda)_{J^{\prec}(k,p,x_j),j} = \bar{L}(n(k,p,x_j),\lambda)_{\_,1}$ and $A_{22}^{-1}(k,p,\lambda)_{\bar{I}_0(k,p) \setminus J^{\prec}(k,p,x_j),j} = 0_{f(k-1,p)-n(k,p,x_j)}$.

(iv) Let $J_1(k,p) = \{ x_j \}_{j=1,\dots,f(k-1,p-1)}$, where $\forall i = 1, \dots, f(k-1,p-1)-1 : x_j \prec_{l,p} x_{j+1}$.

By definition of $E(k,p,\lambda)$, $E(k,p,\lambda)_{J_1(k,p),J_1(k,p)} = diag(\lambda^{n(k,p,x_1)}, \dots, \lambda^{n(k,p,x_{f(k-1,p-1)})})$ and $E(k,p,\lambda)_{\bar{J}_1(k,p),\bar{J}_1(k,p)} = I_{f(k,p-1)-f(k-1,p-1)}$.

By (ii) and definition 3.3.2 then $\forall j = 1, \dots, f(k-1,p-1)$:

$A_{22}^{-1}(k,p,\lambda)_{J^{\prec}(k,p,x_j),j}\lambda^{n(k,p,x_j)} = \bar{L}(n(k,p,x_j),\lambda)_{\_,1}\lambda^{n(k,p,x_j)} = -\tilde{L}(n(k,p,x_j),\lambda)_{\_,1}\lambda^{n(k,p,x_j)}$

□

Proof of lemma 3.3.9

Due to corollary 3.2.7.(iii), $A_{22}(k,p,\lambda)$ is a lower triangular matrix with diagonal elements $-\lambda$. Hence $|A_{22}(k,p,\lambda)| = (-\lambda)^{f(k-1,p)}$.

By definition $E(k,p,\lambda)$ is a diagonal matrix. Hence $|E(k,p,\lambda)| = \lambda^{\sum_{i=1}^{f(k,p-1)} c_i}$.

As $\sum_{i=1}^{f(k,p-1)} c_i = \sum_{x \in \mathbb{X}_{1,2}(k,p)} n(k,p,x) = \#(\bigcup_{x \in \mathbb{X}_{1,2}(k,p)} J^{\prec}(k,p,x)) = \#\mathbb{X}_1(k,p) = f(k-1,p)$ it follows $|E(k,p,\lambda)| = \lambda^{f(k-1,p)}$. But then the claim is obvious.

□

Proof of lemma 3.3.10

Let $\lambda \in \mathbb{C} \setminus \{0\}$

Due to lemma 3.3.5.(iv) and lemma 3.3.1 it follows

$$\begin{aligned}
|A(\alpha, \prec_{k,p}) - \lambda I_{f(k,p)}| &= |A_{11}(k,p,\lambda) - A_{12}(k,p)A_{22}(k,p,\lambda)^{-1}A_{21}(k,p)| \; |A_{22}(k,p,\lambda| \\
&= |A_{11}(k,p,\lambda) - A_{12}(k,p)A_{22}(k,p,\lambda)^{-1}A_{21}(k,p)| \; (-1)^{f(k-1,p)}|E(k,p,\lambda)| \\
&= (-1)^{f(k-1,p)} \; |A_{11}(k,p,\lambda)E(k,p,\lambda) - A_{12}(k,p)A_{22}(k,p,\lambda)^{-1}A_{21}(k,p)E(k,p,\lambda)|
\end{aligned}$$

$= (-1)^{f(k-1,p)} \ |A_{11}(k,p,\lambda)E(k,p,\lambda) - A_{12}(k,p)\bar{B}(k,p,\lambda)E(k,p,\lambda)|$
$= (-1)^{f(k-1,p)} \ |A_{11}(k,p,\lambda)E(k,p,\lambda) + A_{12}(k,p)\tilde{B}(k,p,\lambda)|$
$= (-1)^{f(k-1,p)} \ |B(k,p,\lambda)|.$
$|A(\alpha, \prec_{k.p}) - \lambda I|$ and $|B(\alpha, \prec_{k,p}, \lambda)$ are polynomials. Hence they are complex analytic functions on $\mathbb{C}$ and are equal on $\mathbb{C} \setminus \{0\}$. Hence they are equal on $\mathbb{C}$.

□

Proof of lemma (3.4.1)
Let $x \in Ker\ ((A-\lambda I)^n)$. Then $(A-\lambda I)^{n+1}x = (A-\lambda I)(A-\lambda I)^n x = 0$.
Let $x \in Ker\ ((A-\lambda I)^{n+m+1})$. Then $(A-\lambda I)^m x \in Ker\ ((A-\lambda I)^{n+1})$. It follows $(A-\lambda I)^m x \in Ker\ ((A-\lambda I)^n)$. But then $0 = (A-\lambda I)^n (A-\lambda I)^m x = (A-\lambda I)^{n+m} x$.

□

Proof of lemma (3.4.3):
Fix an arbitrary $i \in \mathbb{N}_{\gamma(\lambda)} \setminus \{0\})$.
For $l = 1$ this lemma holds trivially due to the choice of the initial values.
Assume now that for $l = m$: $e^i_m \in (A-\lambda I)^m \wedge \exists\ x$ with $(A-\lambda I)x = e^i_m$.
Then $(A-\lambda I)^{m+1} e^i_{m+1} = (A-\lambda I)^m e^i_m = 0$, $e^i_{m+1} \in Ker\ ((A-\lambda I)^{m+1})$.
Assume that $e^i_{m+1} \in Ker\ ((A-\lambda I)^m)$.
Then $0 = (A-\lambda I)^m e^i_{m+1} = (A-\lambda I)^{m-1} e^i_m$, $e^i_m \in Ker\ ((A-\lambda I)^{m-1})$.
This contradicts the assumptions about e^i_m.

□

Proof of lemma 3.4.4:

(1) Let $\forall l \in \mathbb{N}_{n_\lambda} \setminus \{0\}$ define $E_l = \bigcup_{i=1}^{k} \{\ e^i_r \mid r \in \mathbb{N}_{l \wedge k^i_\lambda} \setminus \{0\}\ \}$.
(2) By selection of $\{\ e^i_1\ \}_{i \in \mathbb{N}_{\gamma(\lambda)} \setminus \{0\}}$ the vectors of E_1 are linear independent.
(3) Assume for $l = m < n$ all vectors of E_l are independent.
(4) Let $d = \#(E_{m+1})$
$E_{m+1} = \{\ w_i \mid i \in \mathbb{N}_d \setminus \{0\}\ \}$
$d_K = \#(E_{m+1} \cap Ker\ ((A-\lambda I)^{m+1}))$,
w.l.o.g. $K_{m+1} = \{\ w_i \in E_{m+1} \mid i \in \mathbb{N}_{d_K} \setminus \{0\}\ \} \subset Ker\ ((A-\lambda I)^{m+1})) \setminus Ker\ ((A-\lambda I)^m))$.
Assume, there exists $a = (a_1, \ldots, a_d) \in \mathbb{R}^d \setminus \{0\}$ such that $\sum_{i=1}^d a_i w_i = 0$.
(5) $\underset{(4)}{\Longrightarrow} \forall i \in \mathbb{N}_{d_K} \setminus \{0\} : w_i = u_i \oplus v_i$ with $u_i \in Ker\ ((A-\lambda I)^{m+}))\ominus Ker\ ((A-\lambda I)^m))$ and $v_i \in Ker\ ((A-\lambda I)^m))$.
(6) As $\sum_{i=1}^{d_K} a_i u_i \in Ker\ ((A-\lambda I)^{m+1})) \ominus Ker\ ((A-\lambda I)^m))$ and $\forall i \in \mathbb{N}_{d_K} \setminus \{0\} : u_i \neq 0$, $\sum_{i=1}^{d_K} a_i v_i + \sum_{i=d_K+1}^{d} a_i w_i \in Ker\ ((A-\lambda I)^m))$,
it follows $\sum_{i=1}^{d_K} a_i u_i = 0$.
(7) $\Longrightarrow 0 \underset{(5),(6)}{=} (A-\lambda I)^m \sum_{i=1}^{d_K} a_i u_i + (A-\lambda I)^m \sum_{i=1}^{d_K} a_i v_i \underset{(5)}{=} \sum_{i=1}^{d_K} a_i (A-\lambda I)^m w_i = \sum_{i=1}^{d_K} a_i e_i$.
(8) $\underset{(2),(7)}{\Longrightarrow} a_1 = \ldots = a_{d_K} = 0$.
(9) $\underset{(4),(8)}{\Longrightarrow} 0 = \sum_{i=1}^{d} a_i w_i = \sum_{i=d_K+1}^{d} a_i w_i \in Ker\ ((A-\lambda I)^m))$.
(10) $\underset{(3)}{\Longrightarrow} a_{d_K+1} = \ldots = a_d = 0$.
(11) $\underset{(8),(10)}{\Longrightarrow}$ the vectors in E_{m+1} are linear independent.
(12) The lemma follows from the complete induction over l.

□

Proof of lemma 3.4.5:

(1) By the choice of e^i_r, $i \in \mathbb{N}_{\gamma(\lambda)} \setminus \{0\}$, $r \in \mathbb{N}_{k^i_\lambda} \setminus \{0\}$: $span\{\bigcup_{i=1}^{k} \{e^i_r \mid r \in \mathbb{N}_{k^i_\lambda} \setminus \{0\}\}\} \subset Ker((A-\lambda I)^{n_\lambda})$.
(2) $Ker\ ((A-\lambda I)^1) = span\{\ \bigcup_{i=1}^{k} \{\ e^i_1\ \}\ \}$.
(3) Assume for $l = m$: $span\{\ \bigcup_{i=1}^{k} \{\ e^i_r \mid r \in \mathbb{N}_{m \wedge k^i_\lambda} \setminus \{0\}\ \}\ \} = Ker\ ((A-\lambda I)^m)$.
(4) Let $y \in Ker\ ((A-\lambda I)^{m+1}) \ominus Ker\ ((A-\lambda I)^m)$.
$\underset{Lemma\ 3.4.1}{\Longrightarrow} \forall r \in \mathbb{N}_m \setminus \{0\} : y \notin Ker\ ((A-\lambda I)^r)$.
$\Longrightarrow \exists x \in Ker\ ((A-\lambda I)^m) \ominus Ker\ ((A-\lambda I)^{m-1}) : y = (A-\lambda I)x$.

(5) As $\bigcup_{i=1}^{k}\{ e_r^i \mid r \in \mathbb{N}_{m-1\wedge k_\lambda^i} \setminus \{0\} \} \subset Ker\ ((A-\lambda I)^{m-1}),\ \bigcup_{i=1}^{k}\{ e_m^i \} \subset Ker\ ((A-\lambda I)^m)$.

$\underset{(4)}{\Longrightarrow}\quad x = \sum_{i=1}^{k} a_i e_m^i.$

$\Longrightarrow\quad y = (A-\lambda I)\sum_{i=1}^{k} a_i e_m^i = \sum_{i=1}^{k} a_i (A-\lambda I) e_m^i \in span\{ \bigcup_{i=1}^{k}\{ e_r^i \mid r \in \mathbb{N}_{m+1\wedge k_\lambda^i} \setminus \{0\} \} \}.$

(6) So for $l = m+1$: $Ker\ ((A-\lambda I)^{m+1}) = span\{ \bigcup_{i=1}^{k}\{ e_r^i \mid r \in \mathbb{N}_{m+1\wedge k_\lambda^i} \setminus \{0\} \} \}$.

The lemma follows from complete induction.

□

Proof of corollary 3.4.6:

From lemma 3.4.3, 3.4.4 and 3.4.5 follows that in a basis of $Ker\ (A-\lambda I)$ exactly one eigenvector corresponds to each Jordan block. From its definitions follows that $\gamma(\lambda) = dim(Ker\ (A-\lambda I))$ and $\mu(\lambda) = dim(Ker\ ((A-\lambda I)^{n_\lambda}))$. But then lemma 3.4.1 implies: $\mu(\lambda) = \gamma(\lambda)$ if and only if $n_\lambda = 1$, so that the result follows.

□

Proof of lemma 3.4.8:

(*i*) For each $x \in \mathbb{X}_{1,2}(k,p)$ consider $A(\alpha, \prec_{k,p})_{J^{\prec}(k,p,x),I(k,p)}$.
Then $\forall y \in J^{\prec}(k,p,x)$: $A(\alpha,\prec_{k,p})_{y,I(k,p)} a = 0 \Leftrightarrow a_{F(-(y))} - \lambda a_{F(y)} = 0 \Leftrightarrow a_{F(-(y))} = \lambda a_{F(y)}$.
Let $y_{max} \in J^{\prec}(k,p,x)$ such that $\forall y \in J^{\prec}(k,p,x) y \preccurlyeq_{k,p} y_{max}$.
Defining $c(x) = a_{F(y_{max})}$, (*i*) follows from the definition of $J^{\prec}(k,p,x)$.

(*ii*) Define $\forall x \in \mathbb{X}_{1,2}(k,p)$ $y_m(x) \in \mathbb{X}(k,p)$ as element of $J_+^{\prec}(k,p,x)$ with $\forall z \in J_+^{\prec}(k,p,x) z \preccurlyeq y_m(x)$.

$$\begin{array}{llcl} \text{Define } \phi: & \mathbb{C}^{f(k,p-1)} & \longrightarrow & \mathbb{C}^{f(k,p)} \\ & \tilde{a} = (\bar{a}_1, \ldots, \bar{a}_{f(k,p-1)}) & \mapsto & a = (a_1, \ldots, a_{f(k,p)})^\top \end{array}$$

where $\forall i \in \mathbb{N}_{f(k,p)} \setminus \{0\}$:

$$a_i = \begin{cases} \lambda^j \tilde{a}_{i_0} & \exists x \in \mathbb{X}_{1,2}(k,p) : i_0 = F(y_m(x)),\ \exists y \in J_+^{\prec}(k,p,x) : i = F(y),\ j = y_p - y_m(x)_p, \\ \tilde{a}_i & \text{otherwise.} \end{cases}$$

Then obviously ϕ is an injection.
From (3.3.5) it is known that $\exists A_{22}(k,p,\lambda)^{-1}$.
Let $a \in Ker\ (A(\alpha,\prec_{k,p}) - \lambda I)$. Then $a = (a_1,\ldots,a_{f(k,p)})^\top$, $\tilde{a} = (a_1,\ldots,a_{f(k,p-1)})^\top$, $\bar{a} = (a_{f(k,p-1)+1},\ldots,a_{f(k,p)})^\top$ and

$$\begin{array}{rrcl} & 0 & = & A_{21}(k,p)\,\tilde{a} + A_{22}(k,p,\lambda)\,\bar{a} \\ \Longrightarrow & \bar{a} & = & -A_{22}(k,p,\lambda)^{-1} A_{21}(k,p)\,\tilde{a}, \\ & 0 & = & A_{11}(k,p,\lambda)\,\tilde{a} + A_{12}(k,p)\,\bar{a} \\ \Longrightarrow & 0 & = & (\,A_{11}(k,p,\lambda) - A_{12}(k,p) A_{22}(k,p,\lambda)^{-1} A_{21}(k,p)\,)\,\tilde{a} \;=\; B(\alpha,\prec_{k,p},\lambda)\,\tilde{a}. \end{array}$$

Hence $Ker\ (A(\alpha,\prec_{k,p}) - \lambda I) \subset \phi(\ Ker\ (B(\alpha,\prec_{k,p},\lambda))\ )$.
Let $\tilde{a} \in Ker\ (B(\alpha,\prec_{k,p},\lambda))$, $a = \phi(\tilde{a}) = (a_1,\ldots,a_{f(k,p)})^\top = (\tilde{a}^\top, \bar{a}^\top)$, where $\bar{a} = (a_{f(k,p-1)+1},\ldots,a_{f(k,p)})^\top$.
By definition of ϕ and (*i*) it follows $0 = A_{21}(k,p)\tilde{a} + A_{22}(k,p,\lambda)\bar{a}$. $\bar{a} = -A_{22}(k,p,\lambda)^{-1}A_{21}(k,p)\tilde{a}$.
As $\tilde{a} \in Ker\ (B(\alpha,\prec_{k,p}))$, it follows $0 = B(\alpha,\prec_{k,p},\lambda)\,\tilde{a} = A_{11}(k,p,\lambda)\tilde{a} + A_{12}(k,p)\,\bar{a}$.
But then $(\ A(\alpha,\prec_{k,p}) - \lambda I\ )\ \phi(\tilde{a}) = 0$. Hence $\phi(\ Ker\ (B(\alpha,\prec_{k,p},\lambda))\ ) \subset Ker\ (A(\alpha,\prec_{k,p} - \lambda I))$.
Hence $Ker\ (A(\alpha,\prec_{k,p}) - \lambda I) = \phi(\ Ker\ (B(\alpha,\prec_{k,p},\lambda))\ )$, and as ϕ is an injection, (*ii*) follows.

(iii) Follows from corollary 3.4.6, (*iii*).

□

Proof of corollary 3.4.9

Follows from lemma 3.4.8 and the definition of procedure Generate _reduced_matrix.

□

Proof of lemma 3.5.1

(*i*) Due to lemma 3.1.1.(*iii*), 3.1.1.(*i*)
$f(k,2) = \sum_{i_1=0}^{k}\sum_{i_2=0}^{i_1} 1 = \sum_{i_1=0}^{k}(i_1+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$.

(*ii*) Due to lemma 3.2.5

$$\begin{aligned} F(k,2,(x_1,x_2)) &= F(k-x_1,1,(x_2-x_1)) + (1-\delta_0(x_1))\textstyle\sum_{i=0}^{x_1-1} f(k-i,1) \\ &= 1 + x_2 - x_1 + (1-\delta_0(x_1))\textstyle\sum_{i=0}^{x_1-1}(1+k-i) \\ &= 1 + x_2 - x_1 + (1-\delta_0(x_1))\tfrac{x_1(2k+3-x_1)}{2} \\ &= 1 + x_2 + \tfrac{x_1(2k+1-x_1)}{2}. \end{aligned}$$

But then for $i \in \mathbb{N}_k$: $l_1(i) = F(k,2,(i,i)) = 1 + i + \frac{i(2k+1-i)}{2} = 1 + \frac{i(2k+3-i)}{2}$,

$l_2(i) = F(k,2,(i,k)) = 1 + k + \frac{i(2k11-i)}{2}$.

(iii) Let $(x_1,x_2) \in \mathbb{X}(k,p)$ with $x_2 \neq k$. Then $(x_1,x_2+1) \in \mathbb{X}(k,p)$. Furthermore

$$\begin{aligned} F(k,p,-(x_1,x_2+1)) &= F(k,p,(x_1-1,x_2)) \\ &= 1 + x_2 + \tfrac{(x_1-1)(2k+1-(x_1-1))}{2} \\ &= 1 + F(k,p,(x_1-1,x_2-1)) \quad = \quad 1 + F(k,p,-(x_1,x_2)). \end{aligned}$$

Let $(x_1,k) \in \mathbb{X}(k,p)$ with $x_1 \neq k$. Then $(x_1+1,x_2+1) \in \mathbb{X}(k,p)$. Furthermore

$$\begin{aligned} F(k,p,-(x_1+1,x_1+1)) &= F(k,p,(x_1,x_1)) \\ &= 1 + \tfrac{x_1(2k+3-x_1)}{2} \quad = \quad 1 + \tfrac{x_1}{2} + \tfrac{(x_1-1)(2k+1-(x_1-1))}{2} + \tfrac{2(k-1)+4-x_1}{2} \\ &= 2 + F(k,p,(x_1-1,k-1)) \quad = \quad 2 + F(k,p,-(x_1,k)). \end{aligned}$$

□

Proof of lemma 3.5.2

(i) Follows from corollary 3.2.7.(i) and lemma 3.5.1.(iii).

(ii) Follows from corollary 3.2.7.(iii) and lemma 3.5.1.(iii).

(iii) Follows from (i), (ii) and $A(\alpha,\prec_{k,p})_{\bar{I}_0^{\prec}(k,p),I_k^{\prec}(k,p)} = 0_{f(k-1,p),1)}$.

□

Proof of lemma 3.5.3

(i) From lemma 3.5.1 it follows $J_1(k,2) = \{\,(1,i) \mid i = 1,\ldots,k-1\,\}$, $\bar{J}_1(k,2) = \{\,(1,k)\,\}$. But then the statement follows from lemma 3.3.8.(i), 3.3.8.(ii).

(ii) Consider $x \in J_1(k,p)$. Then $x = (1,i)$, $i \in \mathbb{N}_{k-1} \setminus \{0\}$. But then $J^{\prec}(k,p,x) = \{\,(1+j,i+j) \mid j \in \mathbb{N}_{k-i}\,\}$. Let $y = (1+j,i+j) \in J^{\prec}(k,p,x)$, $(j \in \mathbb{N}_{k-i})$. From lemma 3.5.2 we deduce that $F(y) = 1+i+j-(1+j) + \frac{(1+j)(2k+3-(1+j)}{2} = i + \frac{2k+2+-j+j(2k+2-j)}{2} = k+1+i+\frac{j(2k+1-j)}{2}$. As $F(y) = \#(\{\,z \in \mathbb{X}(k,p) \mid z \prec_{k,p} y\,\})$:
$\#(\{\,z \in \mathbb{X}_1(k,p) \mid z \prec_{k,p} y\,\}) = F(y) - F((0,k)). = k+1+i+\frac{j(2k+1-j)}{2} - (k+1) = i + \frac{j(2k+1-j)}{2}$.
The statement follows now from lemma 3.3.8.(iii).

□

Proof of lemma 3.5.4

(1) $\forall x = (1,i) \in J_1(k,p)$, $i = 1,\ldots,k-1$ $J^{\prec}(k,p,x) = \{\,(1+j,i+) \mid j \in \mathbb{N}_{k-i}\,\}$.
Hence $n(k,p,x) = k+1-i$.

(2) Let $x = (1,i) \in J_1(k,p)$, $i \in \mathbb{N}_{k-1} \setminus \{0\}$. Then $\forall y = (1+j,i+j) \in J(k,p,x)$ $(j \in \mathbb{N}_{k-i})$:
$F(y) - F((j,k)) = k+1+i+\frac{j(2k+1-j)}{2} - (1+k+\frac{j(2k+1-j)}{2}) = i$.

(3) Due to (1), $E(k,2,\lambda) = diag(\lambda^k,\ldots,\lambda^0)$.

(4) It holds $\mathbb{E}[X_{n+1}^2] = \sum\limits_{i=0}^{k}\mathbb{E}[\alpha_i^2]\mathbb{E}[X_{n-i}^2] + \sum\limits_{i=0}^{k-1}\sum\limits_{j=i+1}^{k}\mathbb{E}[2\alpha_i\alpha_j]\mathbb{E}[X_{n-i}X_{n-j}]$.

(5) With (3), (4) it follows

$$(\,(A(\alpha,\prec_{k,p}) - \lambda I_{f(k,p)})\,E(k,p,\lambda)\,)_{1,j} = \begin{cases} -\lambda^{k+1} + \mathbb{E}[\alpha_0^2]\,\lambda^k & j = 1, \\ \mathbb{E}[2\alpha_0\alpha_l]\,\lambda^{k+1-l} & j = 2,\ldots,k+1. \end{cases}$$

Let $x = (1,j) \in J_1(k,p)$. Then $j \in \mathbb{N}_{k-1} \setminus \{0\}$ and $J^{\prec}(k,p,x) = \{\,(1+l,j+l) \mid l \in \mathbb{N}_{k-j}\,\}$.

$$\text{But then } (\,A_{12}(k,p,x)\,)_{1,J^{\prec}(k,p)} = \begin{cases} (\,\mathbb{E}[\alpha_1^2],\ldots,\mathbb{E}[\alpha_k^2]\,) & j = 1, \\ (\,\mathbb{E}[2\alpha_1\alpha_j],\ldots,\mathbb{E}[2\alpha_{k+1-j}\alpha_k]\,) & j = 2,\ldots,k. \end{cases}$$

Due to lemma 3.5.3.(ii), (3) it follows that

$(\,-A_{12}(k,p)A_{22}(k,p\lambda)^{-1}A_{21}(k,p)\,E(k,p,\lambda)\,)_{1,1}$

$$= \sum_{l=1}^{k} \mathbb{E}[\alpha_l^2]\lambda^{-1-(l-1)}\lambda^k = \sum_{l=1}^{k} \mathbb{E}[\alpha_l^2]\lambda^{k-l} = \sum_{l=1}^{k} \mathbb{E}[\alpha_l^2]\lambda^{k-l},$$

$$(\,-A_{12}(k,p)A_{22}(k,p\lambda)^{-1}A_{21}(k,p)\;E(k,p,\lambda)\,)_{1,k+1} = 0$$

and $\forall j \in \mathbb{N}_k \setminus \{0,1\}$

$$(\,-A_{12}(k,p)A_{22}(k,p\lambda)^{-1}A_{21}(k,p)\;E(k,p,\lambda)\,)_{1,j}$$

$$= \sum_{l=j}^{k} \mathbb{E}[2\alpha_{l+1-j}\alpha_l]\;\lambda^{-1-(l-j)}\lambda^{k+1-j} = \sum_{l=j}^{k} \mathbb{E}[2\alpha_{l-(j-1)}\alpha_l]\;\lambda^{k-l} = \sum_{l=1}^{k-(j-1)} \mathbb{E}[2\alpha_l\alpha_{l+(j-1)}]\;\lambda^{k-l-(j-1)}.$$

But then

$$(\,(\,A(\alpha,\prec_{k,p}) - \lambda I_{f(k,p)}\,)\;E(k,p,\lambda) - A_{12}(k,p)A_{22}(k,p\lambda)^{-1}A_{21}(k,p)\;E(k,p,\lambda)\,)_{1,1}$$

$$= -\lambda^{k+1} + \mathbb{E}[\alpha_0^2]\;\lambda^k + \sum_{l=1}^{k} \mathbb{E}[\alpha_l^2]\lambda^{k-l} = \sum_{l=-1}^{k} \mathbb{E}[\alpha_l^2]\lambda^{k-l},$$

$$(\,(\,A(\alpha,\prec_{k,p}) - \lambda I_{f(k,p)}\,)\;E(k,p,\lambda) - A_{12}(k,p)A_{22}(k,p\lambda)^{-1}A_{21}(k,p)\;E(k,p,\lambda)\,)_{1,k+1}$$

$$= \mathbb{E}[2\alpha_0\alpha_k] = \sum_{l=0}^{k-(k+1-1)} \mathbb{E}[2\alpha_l\alpha_{l+(k+1-1)}]\;\lambda^{k-l-(k+1-1)},$$

$$(\,(\,A(\alpha,\prec_{k,p}) - \lambda I_{f(k,p)}\,)\;E(k,p,\lambda) - A_{12}(k,p)A_{22}(k,p\lambda)^{-1}A_{21}(k,p)\;E(k,p,\lambda)\,)_{1,j}$$

$$= \mathbb{E}[2\alpha_0\alpha_l]\;\lambda^{k+1-l} + \sum_{l=1}^{k-(j-1)} \mathbb{E}[2\alpha_l\alpha_{l+(j-1)}]\;\lambda^{k-l-(j-1)} = \sum_{l=0}^{k-(j-1)} \mathbb{E}[2\alpha_l\alpha_{l+(j-1)}]\;\lambda^{k-l-(j-1)}.$$

(6) Let $i = 2$. Then $\mathbb{E}[X_{n+1}X_{n-(i-2)}] = \sum_{j=0}^{k} \mathbb{E}[\alpha_j]\mathbb{E}[X_nX_{n-j}]$.

(7) Let $i = 2$. Due to (6) $A_{1,2}(k,p)_{2,\_} = 0$. But then follows with (3) that $\forall j = 1, \ldots, k+1$:

$$(\,A(\alpha,\prec_{k,p}) - \lambda I_{f(k,p)})\;E(k,p,\lambda) - A_{12}(k,p)A_{22}(k,p\lambda)^{-1}A_{21}(k,p)\;E(k,p,\lambda)\,)_{2,j}$$

$$= (\,A(\alpha,\prec_{k,p}) - \lambda I_{f(k,p)})\;E(k,p,\lambda)\,)_{2,j} = \mathbb{E}[\alpha_{j-1}]\lambda^{k+1-j} + \delta_2(j)\mathbb{E}[\alpha_{-1}]\lambda^k$$

But for $j = 1$: $\quad \mathbb{E}[\alpha_{j-1}]\lambda^{k+1-j} + \delta_2(j)\mathbb{E}[\alpha_{-1}]\lambda^{k+1} = \mathbb{E}[\alpha_0]\lambda^k = \mathbb{E}[\alpha_{i-2}]\lambda^{k+(i-2)}$,

for $j = 2$: $\mathbb{E}[\alpha_{j-1}]\lambda^{k+1-j} + \delta_2(j)\mathbb{E}[\alpha_{-1}]\lambda^{k+1} = \mathbb{E}[\alpha_1]\lambda^{k-1} + \mathbb{E}[\alpha_{-1}]\lambda^k$

$$= \mathbb{E}[\alpha_{(i-2)-(j-1)}]\lambda^{k-(i-2)} + \mathbb{E}[\alpha_{(i-2)+(j-1}]\lambda^{k-(i-2)-(j-1)},$$

and for $j \in \mathbb{N}_{k+1} \setminus \mathbb{N}_2$:

$$\mathbb{E}[\alpha_{j-1}]\lambda^{k+1-j} + \delta_2(j)\mathbb{E}[\alpha_{-1}]\lambda^{k+1} = \mathbb{E}[\alpha_{j-1}]\lambda^{k-(j-1)}$$

$$= \mathbb{E}[\alpha_{(i-2)-(j-1)}]\lambda^{k-(i-2)} + \mathbb{E}[\alpha_{(i-2)+(j-1}]\lambda^{k-(i-2)-(j-1)}.$$

(8) Let $i \in \mathbb{N}_k \setminus \mathbb{N}_2$. Then $\mathbb{E}[X_{n+1}X_{n-(i-2)}] = \sum_{j=0}^{i-3} \mathbb{E}[\alpha_j]\mathbb{E}[X_{n-j}X_{n-(i-2)}] + \sum_{j=i-2}^{k} \mathbb{E}[\alpha_j]\mathbb{E}[X_{n-(i-2)}X_{n-j}]$.

(9) Let $i \in \mathbb{N}_k \setminus \mathbb{N}_2$, $j = 1, \ldots, k+1$. From (8) follows

$$(\,(\,A(\alpha,\prec_{k,p}) - \lambda I_{f(k,p)}\,)\;E(k,p,\lambda)\,)_{i,j} = \begin{cases} \mathbb{E}[\alpha_0]\;\lambda^{k-(i-2)} & j = i-1, \\ \mathbb{E}[\alpha_{-1}]\;\lambda^{k-(i-2)} & j = i, \\ 0 & \text{otherwise.} \end{cases}$$

(10) Let $i \in \mathbb{N}_k \setminus \mathbb{N}_2$, $y = (1, 1+i_0) \in J_1(k,p)$, $i_0 \in \mathbb{N}_{k-1}$.

Define $M_1(k,i) = \{1, \ldots, min\{i-3, k-(i-2)\}\}$, $M_2(k,i) = \{min\{i-2, k+1-(i-2)\}, \ldots, i-3\}$, $M_3(k,i) = \{min\{i-2, k+1-(i-2)\}, \ldots, k-(i-2)\}$.

Let $i_0 = 0$, $y_0 = (i-2, i-2)$.

Then $\forall y_l = (l,l) \in \mathbb{X}_1(k,p), l \in \mathbb{N}_k \setminus \{0\}$: $A(\alpha,\prec_{k,p})_{y,y_l} \underset{(8)}{=} \delta_{y_0}(y_l)\mathbb{E}[\alpha_{i-2}]$.

Let $i_0 \in M_1(k,i)$. $y_0 = (\,(i-2) - i_0, i-2\,)$, $y_1 = (\,i-2, (i-2)+i_0\,)$. Then

$\forall y_l = (l, l+i_0) \in \mathbb{X}_1(k,p), l \in \mathbb{N}_{k-i_0} \setminus \{0\}$: $A(\alpha,\prec_{k,p})_{y,y_l} \underset{(8)}{=} \delta_{y_0}(y_l)\mathbb{E}[\alpha_{(i-2)-i_0}] + \delta_{y_1}(y_l)\mathbb{E}[\alpha_{(i-2)+i_0}]$.

Let $i_0 \in M_2(k,i)$. $y_0 = (\,(i-2) - i_0, i-2\,)$. Then

$\forall y_l = (l, l+i_0) \in \mathbb{X}_1(k,p), l \in \mathbb{N}_{k-i_0} \setminus \{0\}$: $A(\alpha,\prec_{k,p})_{y,y_l} \underset{(8)}{=} \delta_{y_0}(y_l)\mathbb{E}[\alpha_{(i-2)-i_0}]$.

Let $i_0 \in M_3(k,i)$. $y_1 = (\,i-2, (i-2)+i_0\,)$. Then

$\forall y_l = (l, l+i_0) \in \mathbb{X}_1(k,p), l \in \mathbb{N}_{k-i_0} \setminus \{0\}$: $A(\alpha,\prec_{k,p})_{y,y_l} \underset{(8)}{=} \delta_{y_1}(y_l)\mathbb{E}[\alpha_{(i-2)+i_0}]$.

In all cases $y_0, y_1 \in J^{\prec}(k,p,y)$ obviously.

By lemma 3.5.3.(ii), (3) it follows that

$$(-A_{12}(k,p)A_{22}(k,p,\lambda)^{-1}A_{21}(k,p)E(k,p,\lambda))_{i,1+i_0}$$
$$= \begin{cases} \mathbb{E}[\alpha_{i-2}]\,\lambda^{-(i-2)}\lambda^k & i_0 = 0, \\ \mathbb{E}[\alpha_{(i-2)-i_0}]\,\lambda^{-((i-2)-i_0)}\lambda^{k-i_0} + \mathbb{E}[\alpha_{(i-2)+i_0}]\,\lambda^{-(i-2))}\lambda^{k-i_0} & i_0 \in M_1(k,i), \\ \mathbb{E}[\alpha_{(i-2)-i_0}]\,\lambda^{-((i-2)-i_0)}\lambda^{k-i_0} & i_0 \in M_2(k,i), \\ \mathbb{E}[\alpha_{(i-2)+i_0}]\,\lambda^{-(i-2)}\lambda^{k-i_0} & i_0 \in M_3(k,i). \end{cases}$$

Define $\bar{M}_1(k,i) = \{2,\ldots,min\{i-2,k+1-(i-2)\}\}$, $\bar{M}_2(k,i) = \{min\{i-1,k+2-(i-2)\},\ldots,i-2\}$, $\bar{M}_3(k,i) = \{min\{i-1,k+2-(i-2)\},\ldots,k+1-(i-2)\}$.

So for $j = 1,\ldots,k$

$$(-A_{12}(k,p)A_{22}(k,p,\lambda)^{-1}A_{21}(k,p)E(k,p,\lambda))_{i,j}$$
$$= \begin{cases} \mathbb{E}[\alpha_{i-2}]\,\lambda^{k-(i-2)} & j = 1, \\ \mathbb{E}[\alpha_{(i-2)-(j-1)}]\,\lambda^{k-(i-2)} + \mathbb{E}[\alpha_{(i-2)+(j-1)}]\,\lambda^{k-(i-2)-(j-1)} & j \in \bar{M}_1(k,i), \\ \mathbb{E}[\alpha_{(i-2)-(j-1)}]\,\lambda^{k-(i-2)} & j \in \bar{M}_2(k,i), \\ \mathbb{E}[\alpha_{(i-2)+(j-1)}]\,\lambda^{k-(i-2)-(j-1)} & j \in \bar{M}_3(k,i). \end{cases}$$

and due to lemma 3.5.3.(i),

$$(-A_{12}(k,p)A_{22}(k,p,\lambda)^{-1}A_{21}(k,p)E(k,p,\lambda))_{i,k+1} = 0.$$

(11) Let $i \in \mathbb{N}_k \setminus \mathbb{N}_2$.

Define $\tilde{M}_1(k,i) = \{2,\ldots,min\{i,k+1-(i-2)\}\}$, $\tilde{M}_2(k,i) = \{min\{i+1,k+2-(i-2)\},\ldots,i-2\}$, $\tilde{M}_3(k,i) = \{min\{i+1,k+2-(i-2)\},\ldots,k+1-(i-2)\}$.

As $i > i-1 > i-2$, hence $i-1, i \notin \bar{M}_2(k,i)$, for $j = i-1 : \mathbb{E}[\alpha_0]\,\lambda^{k-(i-2)} = \mathbb{E}[\alpha_{(i-2)-(j-1)}]\,\lambda^{k-(i-2)}$, for $j = i$: $\mathbb{E}[\alpha_{-1}]\,\lambda^{k-(i-2)} = \mathbb{E}[\alpha_{(i-2)-(j-1)}]\,\lambda^{k-(i-2)}$ it follows from (9), (10): for $j = 1,\ldots,k$

$$(A_{11}(\alpha,\prec_{k,p},\lambda)\,E(k,p,\lambda) - A_{12}(k,p)A_{22}(k,p,\lambda)^{-1}A_{21}(k,p)E(k,p,\lambda))_{i,j}$$
$$= \begin{cases} \mathbb{E}[\alpha_{i-2}]\,\lambda^{k-(i-2)} & j = 1, \\ \mathbb{E}[\alpha_{(i-2)-(j-1)}]\,\lambda^{k-(i-2)} + \mathbb{E}[\alpha_{(i-2)+(j-1)}]\,\lambda^{k-(i-2)-(j-1)} & j \in \tilde{M}_1(k,i), \\ \mathbb{E}[\alpha_{(i-2)-(j-1)}]\,\lambda^{k-(i-2)} & j \in \tilde{M}_2(k,i), \\ \mathbb{E}[\alpha_{(i-2)+(j-1)}]\,\lambda^{k-(i-2)-(j-1)} & j \in \tilde{M}_3(k,i). \end{cases}$$

(12) Let $i \in \mathbb{N}_k \setminus \mathbb{N}_2$.

If $\tilde{M}_2(k,i) \neq \emptyset$, then $\forall j \in \tilde{M}_2(k,i) : (i-2)+(j-1) \geq i-2+k+2-(i-2)-1 = k+1$.
If $\tilde{M}_3(k,i) \neq \emptyset$, then $\forall j \in \tilde{M}_2(k,i) : (i-2)-(j-1) \leq i-2-(i+1-1) = -2$.
Furthermore, $\tilde{M}_2(k,i) \cap \tilde{M}_3(k,i) = \emptyset$. But then (11) can be reformulated as

$$(A_{11}(\alpha,\prec_{k,p},\lambda)\,E(k,p,\lambda) - A_{12}(k,p)A_{22}(k,p,\lambda)^{-1}A_{21}(k,p)E(k,p,\lambda))_{i,j}$$
$$= \begin{cases} \mathbb{E}[\alpha_{i-2}]\,\lambda^{k-(i-2)} & j = 1, \\ \mathbb{E}[\alpha_{(i-2)-(j-1)}]\,\lambda^{k-(i-2)} + \mathbb{E}[\alpha_{(i-2)+(j-1)}]\,\lambda^{k-(i-2)-(j-1)} & j = 2,\ldots,k+1. \end{cases}$$

The lemma follows from (5), (7) and (12).

□

Proof of lemma 3.5.7

(1) (i) and (ii) follow immediately from definition 3.5.6 and lemma 3.4.8.

(2) Let $l \in \{-1\} \cup \mathbb{N}_k$.
Let $i \in \mathbb{N}_{k+1} \setminus \mathbb{N}_1$, $j = 1$.
$\mathbb{E}[\alpha_l]$ contributes to $B(\alpha,\prec_{k,p},\lambda)_{i,j}$ iff $l = i-2$.
But then $i = l+2$ and $B(\alpha,\prec_{k,p},\lambda)_{l+2,1} = \mathbb{E}[\alpha_l]\,\lambda^{k-l}$.

(3) Let $l \in \{-1\} \cup \mathbb{N}_k$.
Let $i \in \mathbb{N}_{k+1} \setminus \mathbb{N}_1$, $j = 2,..,k+1$.
$\mathbb{E}[\alpha_l]$ contributes to $B(\alpha,\prec_{k,p},\lambda)_{i,j}$ iff either $l = (i-2)-(j-1)$ or $l = (i-2)+(j-1)$.
If $l = (i-2)-(j-1)$, then $j = i-1-l$ and $\mathbb{E}[\alpha_l]$ contributes with $\mathbb{E}[\alpha_l]\,\lambda^{k-(i-2)}$ to $B(\alpha,\prec_{k,2})_{i,i-1-l}$ and holds for $1 \leq i-1-l \leq k+1$, or $(l+2) \vee 2 \leq i \leq k+1$ due to the above choice of i, respectively.
If $l = (i-2)+(j-1)$, then $j = l+1-(i-2)$ and $\mathbb{E}[\alpha_l]$ contributes with $\mathbb{E}[\alpha_l]\,\lambda^{k-l}$ to

$B(\alpha, \prec_{k,2})_{i,l+1-(i-2)}$ and holds for $1 \leq l+1-(i-2) \leq k+1$, or $2 \leq i \leq (l+2) \wedge (k+1)$ due to the above choice of i, respectively.

(4) (2) and (3) show (iii) and (iv).

(5) Let $i \in \mathbb{N}_{k+1} \setminus \mathbb{N}_1$.
If $j = 1$, only $\mathbb{E}[\alpha_{i-2}]$ contributes to $B(\alpha, \prec_{k,2}, \lambda)_{i,1}$.
If $j \in \mathbb{N}_{k+1} \setminus \mathbb{N}_1$, then only $\mathbb{E}[\alpha_{i-2} - (j-1)]$ and $\mathbb{E}[\alpha_{i-2} + (j-1)]$ contribute to $B(\alpha, \prec_{k,2}, \lambda)_{i,j}$.
Due to the choice of j, $\mathbb{E}[\alpha_{i-2} - (j-1)] \neq \mathbb{E}[\alpha_{i-2} + (j-1)]$.
It holds: $(i-2)-(j-1) > i-2$, $(i-2)+(j-1) > (i-2)$, $(i-2)-(j-1)+(i-2)+(j-1) = 2(i-2)$ is j-independent.
But this implies that $\forall j_1, j_2\ (\neq j_1) \in \mathbb{N}_{k+1} \setminus \mathbb{N}_1$: $\mathbb{E}[\alpha_{i-2} - (j_1-1)] \neq \mathbb{E}[\alpha_{i-2} + (j_2-1)]$ and $\mathbb{E}[\alpha_{i-2} - (j_2-1)] \neq \mathbb{E}[\alpha_{i-2} + (j_1-1)]$.
This shows (v).

□

Proof of lemma 3.5.9

(1) By remark 3.5.8, $B(\tilde{\alpha}, \prec_{k,p}, \lambda) = B(\alpha, \prec_{k,p}, \lambda)$

(2) By (1), lemma 3.5.4 $B(\alpha, \prec_{k,p}, \lambda)_{1,1} = \sum_{l=0}^{k} \mathbb{E}[\tilde{\alpha}_l^2]\lambda^{k-l} = \sum_{l=0}^{m} \mathbb{E}[\alpha_l^2]\lambda^{k-k_l}$. This proves (i).

(3) Let $j \in \mathbb{N}_{k+1} \setminus \mathbb{N}_1$. By (1), lemma 3.5.4

$$\begin{aligned} B(\alpha, \prec_{k,p}, \lambda)_{1,j} &= \sum_{l=0}^{k-(j-1)} \mathbb{E}[\tilde{\alpha}_l \tilde{\alpha}_{l+(j-1)}]\lambda^{k-l-(j-1)} \\ &= \sum_{l=0}^{m} \mathbb{E}[\alpha_l \tilde{\alpha}_{k_l+(j-1)}]\lambda^{k-k_l-(j-1)} \\ &= \sum_{l\in\mathbb{N}_m:\, k_l+(j-1)\in K} \mathbb{E}[\alpha_l \alpha_{\Phi^{-1}(k_l+(j-1))}]\lambda^{k-k_l-(j-1)}. \end{aligned}$$

This proves (ii).

(4) (iii), (iv) follow from (1), lemma 3.5.7.(i) and 3.5.7.(ii), applied to $B(\tilde{\alpha}, \prec_{k,p}, \lambda)$ and taking into account the zero coefficients of $\tilde{\alpha}$ (see the definition of $\tilde{\alpha}$).

□

Proof of lemma 3.5.10

For $j \in \mathbb{N}_{n(0,x)-1}$ define $y(j) = r(j, n(0,x), -(x))$. Then $y(j)_p = k$. The equation $k = y - 1$ can't be solved in $\mathbb{N}_k$. So $y(j) \notin J_1(k,p)$.

□

Proof of lemma 3.5.11

(1) Choose $B_1(\alpha, k, p, \lambda) = (\ A(\alpha, \prec_{k,p})_{I_0^{\prec}(k,p), I_0^{\prec}(k,p)}\ E(k,p,\lambda) - A_{12}A_{22}(k,p,\lambda)^{-1}A_{21}(k,p)\ E(k,p,\lambda)\ )$,
$B_2(k,p,\lambda) = \lambda E(k,p,\lambda)$.
Then $B(\alpha, \prec_{k,p}, \lambda) = B_1(\alpha, k, p, \lambda) - B_2(k,p,\lambda)$.

(2) By definition of $E(k,p,\lambda)$ follows $B_2(k,p,\lambda) = diag(\lambda^{c_1+1}, \ldots, \lambda^{c_{f(k,p-1)}+1})$, where for $i = 1, \ldots, f(k,p-1)$ $c_i = \begin{cases} \#(J^{\prec}(k,p,+(x)) & x \in J_1(k,p), \\ 0 & \text{otherwise.} \end{cases} = \begin{cases} k+1-(x_p+1) & x \in J_1(k,p), \\ 0 & \text{otherwise.} \end{cases}$.
But then for $i = 1, \ldots, f(k,p-1)$ $d_i = c_{i+1} = \begin{cases} k+1-x_p & x \in J_1(k,p), \\ 0 & \text{otherwise.} \end{cases}$.

(3) Let $x, y \in \mathbb{X}_{0,1}(k,p)$. Then by definition of $A(\alpha, \prec_{k,p})$):
$\forall j \in \mathbb{N}_{n(0,x)}\ A(\alpha, \prec_{k,p})_{x, r(j,n(0,x),-(x))} = c(j, n(0,x), -(x))$.

(4) $\forall j \in \mathbb{N}_{n(0,x)} \setminus \{0\} : r(j, n(0,x), -(x)) \in \mathbb{X}_{0,1}(k,p)$.

(5) $\forall j \in \mathbb{N}_{n(0,x)-1} : y(j)_p = k$, where $y(j) = r(j, n(0,x), -(x)) \in \mathbb{X}_{0,1}(k,p)$.

(6) (5) and lemma 3.5.10 imply $\forall j \in \mathbb{N}_{n(0,x)-1} \setminus \{0\} : r(j, n(0,x), -(x)) \notin J_1(k,p)$.
Then by definition of $A_{12}(k,p)$, $A_{22}(k,p,\lambda)$, $A_{21}(k,p)$:
$(\ A_{12}(k,p)A_{22}(k,p,\lambda)^{-1}A_{21}(k,p)\ E(k,p,\lambda)\ )_{x,y} = 0$.
From the definition of $E(k,p,\lambda)$ it follows: $c_{F(r(j,n(0,x),-(x)))} = 0$.
With (3) this implies $B_1(\alpha, k, p, \lambda)_{x, r(j,n(0,x),-(x))} = c(j, n(0,x), -(x))$.

(7) From the definition of $E(k,p,\lambda)$ it follows: $c_{F(r(n(0,x),n(0,x),-(x)))} = k+1-x_p$.
Hence $(\ A(\alpha, \prec_{k,p})_{I_0^{\prec}(k,p), I_0^{\prec}(k,p)} E(k,p,\lambda)\ )_{x, r(n(0,x),n(0,x),-(x))} = c(n(0,x),n(0,x),-(x))\lambda^{k+1-x_p}$.
If $r(n(0,x),n(0,x),-(x)) + (k-(x_p-1) \vee 0) = r(0,n(0,x),-(x))$, then

$$r(0,n(0,x),-(x)) \in J^{\prec}(k,p,+(x)),$$
$$(\ A_{12}(k,p)A_{22}(k,p,\lambda)^{-1}A_{21}(k,p)\ E(k,p,\lambda)\ )_{x,r(n(0,x),n(0,x),-(x))} = c(0,n(0,x),-(x)).$$

But then $B_1(\alpha,k,p,\lambda)_{x,r(n(0,x),n(0,x),-(x))} = c(n(0,x),n(0,x),-(x))\lambda^{k+1-x_p}$
$+\delta_{r(0,n(0,x),-(x))}(r(n(0,x),n(0,x),-(x)) + (k-(x_p-1) \vee 0) c(0,n(0,x),-(x))$.

(8) Denote $z = r(0,n(0,x),-(x))$.
From the definition of r follows that $z \notin \mathbb{X}_{0,1}(k,p)$.
Define $\tilde{z} = z - z_1$ (componentwise subtraction of scalar z_1). Then $\tilde{z} \in \mathbb{X}_{0,1}(k,p$ and $\tilde{z}_p < k$. This implies that $\forall j \in \mathbb{N}_{n(0,x)-1} \setminus \{0\}$: $\tilde{z} \neq r(j,n(0,x),-(x))$.
Furthermore, $z \in J^{\prec}(k,p,+(\tilde{z}))$. The case $\tilde{z} = r(n(0,x),n(0,x),-(x))$ is already covered by (7). If $\tilde{z} \neq r(n(0,x),n(0,x),-(x))$, then $A(\alpha,\prec_{k,p})_{x,\tilde{z}} = 0$. But then $B_1(\alpha,k,p,\lambda)_{x,\tilde{z}} = c(0,n(0,x),-(x))$.
□

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6 References

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