

SFB 649 Discussion Paper 2005-051

# Optimal investments for risk- and ambiguity- averse preferences: A duality approach

Alexander Schied\*



\* Technische Universität Berlin, Germany

This research was supported by the Deutsche  
Forschungsgemeinschaft through the SFB 649 "Economic Risk".

<http://sfb649.wiwi.hu-berlin.de>  
ISSN 1860-5664

SFB 649, Humboldt-Universität zu Berlin  
Spandauer Straße 1, D-10178 Berlin



SFB 649 ECONOMIC RISK BERLIN

# Optimal investments for risk- and ambiguity-averse preferences: a duality approach

Alexander Schied\*

TU Berlin, Institut für Mathematik, MA 7-4

Strasse des 17. Juni 136

10623 Berlin, Germany

e-mail: `schied@math.tu-berlin.de`

September 26, 2005

**Abstract:** Ambiguity, also called Knightian or model uncertainty, is a key feature in financial modeling. A recent paper by Maccheroni *et al.* (2004) characterizes investor preferences under aversion against both risk and ambiguity. Their result shows that these preferences can be numerically represented in terms of convex risk measures. In this paper we study the corresponding problem of optimal investment over a given time horizon, using a duality approach and building upon the results by Kramkov and Schachermayer (1999, 2001). In many situations this seems to be the only feasible approach among the known techniques, as is illustrated by several examples.

*Key words:* Model uncertainty, ambiguity, convex risk measures, optimal investments, duality theory

## 1 Introduction

In the vast majority of the literature on optimal investments in financial markets it is assumed that decisions are based on a classical expected utility criterion in the sense of John von Neumann and Oscar Morgenstern. Underlying this concept is the assumption that expected utility is computed in terms of a probability measure that accurately models future stock price evolutions. In reality, however, the choice of this probability measure is itself subject to *model uncertainty*, often also called *ambiguity* or *Knightian uncertainty*. Economists have long been aware of this fact, and in the late 1980's Gilboa and Schmeidler [33, 18] formulated axioms on investor preferences that should account for aversion against both risk and ambiguity. They showed that these preferences can be numerically represented by a robust utility functional of the form

$$X \longmapsto \inf_{Q \in \mathcal{Q}} E_Q[U(X)], \quad (1)$$

---

\*Supported by Deutsche Forschungsgemeinschaft through the SFB 649 "Economic Risk".

where  $U$  is a utility function and  $\mathcal{Q}$  is a class of probability measures; see also [16, Section 2.5] for a survey. The elements of  $\mathcal{Q}$  can be interpreted as prior models, which possibly describe the probabilities of future scenarios. Taking the infimum of all expected utilities for these prior models thus corresponds to a worst-case approach. Systematic analyses of optimal investment decisions under this type of preferences were first given independently by Quenez [28] and the author [31, 30]. Subsequent studies were conducted by Gundel [19], Burgert and Rüschendorf [4], Wu and the author [32], Müller [27], and Föllmer and Gundel [13].

One might object that robust utility functionals of the form (1) leave no room for discriminating models in  $\mathcal{Q}$  according to their plausibility. If, for instance, the class of prior models arises as a confidence set in statistical estimation, then the original estimate might have a higher plausibility, and thus should receive a higher weight, than a model at the boundary of the confidence set. Or one might wish to include the results of certain stress test models when their outcomes differ significantly from the ones of plausible priors; see, e.g., Carr et al. [5] and [16, Section 4.8]. These objections to robust utility functionals of the form (1) correspond to objections that can be raised on an axiomatic level against the axiom of ‘certainty independence’ introduced in [18]. By weakening this axiom, Maccheroni *et al.* [24] recently obtained a numerical representation of the form

$$X \longmapsto \inf_Q (E_Q[U(X)] + \gamma(Q)), \quad (2)$$

where the function  $\gamma$  assigns a penalization weight  $\gamma(Q)$  to each possible probabilistic model  $Q$ . This class of robust utility functionals clearly extends the class (1) and leaves room for a discrimination among possible prior models. The move from (1) to (2) is similar to the generalization of coherent by convex risk measures [14, 15, 16, 17].

Our goal in this paper is to study the problem of constructing dynamic investment strategies whose terminal wealth maximizes a functional (2) for a given initial investment. For the maximization of classical von Neumann-Morgenstern utility, basically two methods are available: the duality method (sometimes called the ‘martingale method’) and stochastic control techniques based on backward stochastic differential equations or Hamilton-Jacobi-Bellmann PDEs. The latter technique was applied by Hansen and Sargent [20] in a case study on optimal investment and consumption under entropic penalties as in Example 3.3 below. The stochastic control approach requires, however, the *time consistency* of the underlying convex risk measure

$$\rho(Y) := \sup_Q (E_Q[-Y] - \gamma(Q))$$

in the sense described, e.g., by Epstein and Schneider [12] for coherent risk measures and by Cheridito *et al.* [6] for the general case. This property of time consistency rules out many examples. Moreover, we believe that it is too restrictive as a normative postulate: It would require that the investor does not change the prior models and the penalty function for the entire investment period. But financial models are typically not accurate, and each piece of freshly revealed information might require to adjust models and hence penalty functions. In reality, this fact is usually taken into account by a periodic

model recalibration. It results in ever changing model parameters and clearly spoils time consistency.

If we are interested in the maximization of general robust utility functionals, the discussion in the preceding paragraph suggests the use of the duality method, and this is the approach we will take in this note. In doing so, we will build upon the results of Kramkov and Schachermayer [22, 23] and extend the results from Wu and the author [32] to general robust utility functionals (2).

In Section 2 we formulate our hypotheses and state our main results. As in standard expected utility maximization, we observe that the duality for the value functions of the robust problem holds under rather mild conditions, while a stronger condition is necessary to guarantee the existence of optimal strategies. In Section 3 we present possible choices for penalty functions  $\gamma(\cdot)$  and argue that for many of these examples the stochastic control method does not seem to be available, so that duality appears to be the method of choice. We also give examples showing that the value function of the robust problem may not be continuously differentiable. Equivalently, the dual value function may not be strictly convex. We also illustrate that the maximal solution of the dual problem may fail to have full support. Proofs are given in Section 4.

## 2 Statement of main results

As Kramkov and Schachermayer [22, 23], we assume that the utility function of the investor is a strictly increasing and strictly concave function  $U : (0, \infty) \rightarrow \mathbb{R}$ , which also is continuously differentiable and satisfies the Inada conditions

$$U'(0+) = +\infty \quad \text{and} \quad U'(\infty-) = 0.$$

Payoffs are modeled as random variables  $X$  on a given probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . Their utility shall be assessed in terms of a robust utility functional of the form

$$X \mapsto \inf_Q (E_Q[U(X)] + \gamma(Q)). \quad (3)$$

Here we assume that  $\gamma$  is bounded from below and equal to the minimal penalty function of the convex risk measure

$$\rho(Y) := \sup_{Q \ll \mathbb{P}} (E_Q[-Y] - \gamma(Q)), \quad Y \in L^\infty(\mathbb{P}),$$

that is,

$$\gamma(Q) = \sup_{Y \in L^\infty(\mathbb{P})} (E_Q[-Y] - \rho(Y)); \quad (4)$$

see [14, 16]. We may assume without loss of generality that  $\rho(0) = -\inf_Q \gamma(Q) = 0$ . We also assume the following conditions:

**Assumption 2.1** *The risk measure  $\rho$  is continuous from below: If  $Y_n \in L^\infty$  increases a.s. to  $Y \in L^\infty$ , then  $\rho(Y_n) \rightarrow \rho(Y)$ . It is also sensitive<sup>1</sup> in the sense that every nonzero  $Y \in L^\infty_-$  satisfies  $\rho(Y) > 0$ .*

---

<sup>1</sup>Sensitivity is also called *relevance*.

In Section 3 we have collected a number of particular examples for economically and statistically meaningful choices for  $\gamma$ . If  $\gamma$  only takes the values 0 and  $+\infty$ , then (3) reduces to the representation of a robust utility functional in the sense of Gilboa and Schmeidler [18]:

$$X \longmapsto \inf_{Q \in \mathcal{Q}} E_Q[U(X)] \quad (5)$$

for a convex set  $\mathcal{Q}$  of probability measures. In this case, Assumption 2.1 is equivalent to [32, Assumption 2.1], as can be seen by combining the general representation theory of convex and coherent risk measures [16] with [32, Lemma 3.2] and Lemma 4.1 below. Even when restricted to this special case, our results will be stronger than those obtained in [32]. Particular examples for optimal investment problems with robust utility functionals of type (5) were analyzed by Quenez [28] and the author [30].

**Remark 2.2** If the utility function  $U$  is not bounded from below, we must be careful in defining the expression  $\inf_Q (E_Q[U(X)] + \gamma(Q))$ . First, it is clear that probabilistic models with an infinite penalty  $\gamma(Q)$  should not contribute to the value of the robust utility functional. We therefore restrict the infimum to models  $Q$  in the domain

$$\mathcal{Q} := \{Q \ll \mathbb{P} \mid \gamma(Q) < \infty\}$$

of  $\gamma$ . That is, we precise (3) by writing

$$X \longmapsto \inf_{Q \in \mathcal{Q}} (E_Q[U(X)] + \gamma(Q)).$$

Second, we have to address the problem that the  $Q$ -expectation of  $U(X)$  might not be well-defined in the sense that  $E_Q[U^+(X)]$  and  $E_Q[U^-(X)]$  are both infinite. This problem will be resolved by extending the expectation operator  $E_Q[\cdot]$  to the entire set  $L^0$ :

$$E_Q[F] := \sup_n E_Q[F \wedge n] = \lim_{n \uparrow \infty} E_Q[F \wedge n] \quad \text{for arbitrary } F \in L^0. \quad (6)$$

It is easy to see that in doing so we retain the concavity of the functional  $X \mapsto E_Q[U(X)]$  and hence of the robust utility functional.  $\diamond$

For the financial market model, we use the same setup as Kramkov and Schachermayer [22, 23]. The discounted price process of  $d$  assets is modeled by a stochastic process  $S = (S_t)_{0 \leq t \leq T}$ . We assume that  $S$  is a  $d$ -dimensional semimartingale on  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ . A self-financing trading strategy can be regarded as a pair  $(x, \xi)$ , where  $x \in \mathbb{R}$  is the initial investment and  $\xi = (\xi_t)_{0 \leq t \leq T}$  is a  $d$ -dimensional predictable and  $S$ -integrable process. The value process  $X$  associated with  $(x, \xi)$  is given by  $X_0 = x$  and

$$X_t = X_0 + \int_0^t \xi_r dS_r, \quad 0 \leq t \leq T.$$

For  $x > 0$  given, we denote by  $\mathcal{X}(x)$  the set of all value processes  $X$  that satisfy  $X_0 \leq x$  and are admissible in the sense that  $X_t \geq 0$  for  $0 \leq t \leq T$ . We assume that our model is

arbitrage-free in the sense that  $\mathcal{M} \neq \emptyset$ , where  $\mathcal{M}$  denotes the set of measures equivalent to  $\mathbb{P}$  under which each  $X \in \mathcal{X}(1)$  is a local martingale; see [22]. Thus, our main problem can be stated as follows:

$$\text{Maximize } \inf_{Q \in \mathcal{Q}} (E_Q[U(X_T)] + \gamma(Q)) \text{ among all } X \in \mathcal{X}(x).$$

Consequently, the *value function of the robust problem* is defined as

$$u(x) := \sup_{X \in \mathcal{X}(x)} \inf_{Q \in \mathcal{Q}} (E_Q[U(X_T)] + \gamma(Q)).$$

One of our first results will be the minimax identity

$$u(x) = \inf_{Q \in \mathcal{Q}} (u_Q(x) + \gamma(Q)), \quad \text{where} \quad u_Q(x) := \sup_{X \in \mathcal{X}(x)} E_Q[U(X_T)].$$

The function  $u_Q$  is the value function of the optimal investment problem for an investor with subjective measure  $Q \in \mathcal{Q}$ . Next, we define as usual the convex conjugate function  $V$  of  $U$  by

$$V(y) := \sup_{x > 0} (U(x) - xy), \quad y > 0.$$

With this notation, it was stated in Theorem 3.1 of [22] that, for  $Q \sim \mathbb{P}$  with finite value function  $u_Q$ ,

$$u_Q(x) = \inf_{y > 0} (v_Q(y) + xy) \quad \text{and} \quad v_Q(y) = \sup_{x > 0} (u_Q(x) - xy), \quad (7)$$

where the dual value function  $v_Q$  is given by

$$v_Q(y) = \inf_{Y \in \mathcal{Y}_Q(y)} E_Q[V(Y_T)], \quad Q \in \mathcal{Q},$$

and the space  $\mathcal{Y}_Q(y)$  is defined as the set of all positive  $Q$ -supermartingales such that  $Y_0 = y$  and  $XY$  is a  $Q$ -supermartingale for all  $X \in \mathcal{X}(1)$ . We thus define the *dual value function of the robust problem* by

$$v(y) := \inf_{Q \in \mathcal{Q}} (v_Q(x) + \gamma(Q)) = \inf_{Q \in \mathcal{Q}} \inf_{Y \in \mathcal{Y}_Q(y)} (E_Q[V(Y_T)] + \gamma(Q)).$$

**Definition 2.3** Let  $y > 0$  be such that  $v(y) < \infty$ . A pair  $(Q, Y)$  is a *solution of the dual problem* if  $Q \in \mathcal{Q}$ ,  $Y \in \mathcal{Y}_Q(y)$ , and  $v(y) = E_Q[V(Y_T)] + \gamma(Q)$ .

Let us finally introduce the set  $\mathcal{Q}_e$  of measures in  $\mathcal{Q}$  that are equivalent to  $\mathbb{P}$ :

$$\mathcal{Q}_e := \{Q \in \mathcal{Q} \mid Q \sim \mathbb{P}\}.$$

Our assumptions on  $\gamma$  guarantee that  $\mathcal{Q}_e$  is always nonempty; see Lemma 4.1.

**Theorem 2.4** *In addition to the above assumptions, let us assume that*

$$u_{Q_0}(x) < \infty \text{ for some } x > 0 \text{ and some } Q_0 \in \mathcal{Q}_e \quad (8)$$

and that

$$v(y) < \infty \text{ implies } v_{Q_1}(y) < \infty \text{ for some } Q_1 \in \mathcal{Q}_e. \quad (9)$$

Then the robust value function  $u$  is concave, takes only finite values, and satisfies

$$u(x) = \sup_{X \in \mathcal{X}(x)} \inf_{Q \in \mathcal{Q}} (E_Q[U(X_T)] + \gamma(Q)) = \inf_{Q \in \mathcal{Q}} \sup_{X \in \mathcal{X}(x)} (E_Q[U(X_T)] + \gamma(Q)).$$

Moreover, the two robust value functions  $u$  and  $v$  are conjugate to another:

$$u(x) = \inf_{y > 0} (v(y) + xy) \quad \text{and} \quad v(y) = \sup_{x > 0} (u(x) - xy). \quad (10)$$

In particular,  $v$  is convex. The derivatives of  $u$  and  $v$  satisfy

$$u'(0+) = \infty \quad \text{and} \quad v'(\infty-) = 0. \quad (11)$$

If furthermore  $v(y) < \infty$ , then the dual problem admits a solution  $(\widehat{Q}, \widehat{Y})$  that is maximal in the sense that any other solution  $(Q, Y)$  satisfies  $Q \ll \widehat{Q}$  and  $Y_T = \widehat{Y}_T$   $Q$ -a.s.

It is possible that the maximal  $\widehat{Q}$  is *not* equivalent to  $\mathbb{P}$ ; see Example 3.2 below. If this happens, then  $\widehat{Q}$  considered as a financial market model on its own may admit arbitrage opportunities. In this light, one also has to understand the conditions (8) and (9): They exclude the possibility that the value functions  $u_Q$  and  $v_Q$  are only finite for some degenerate model  $Q \in \mathcal{Q}$ , for which the duality relations (7) need not hold.

The situation simplifies considerably if we assume that *all* measures in  $\mathcal{Q}$  are equivalent to  $\mathbb{P}$ . In this case, condition (9) is always satisfied and (8) can be replaced by the assumption that  $u(x) < \infty$  for some  $x > 0$ . Moreover, the optimal  $\widehat{Y}$  is then  $\mathbb{P}$ -almost surely unique. Despite this fact, however, and in contrast to the situation in [22, 23], it can happen that the dual value function  $v$  is not strictly convex—even if all measures in  $\mathcal{Q}$  are equivalent to  $\mathbb{P}$ . Equivalently, the value function  $u$  may fail to be continuously differentiable. This fact will be illustrated in Example 3.1 below. A sufficient condition for the strict convexity of  $v$  and the continuous differentiability of  $u$  is given in the next result. It applies in particular to entropic penalties (Example 3.3) and to penalty functions defined in terms of many other statistical distance functions (Example 3.6).

**Proposition 2.5** *Under the assumptions of Theorem 2.4,  $u$  is continuously differentiable and  $v$  is strictly convex on its domain if  $\gamma$  is strictly convex on  $\mathcal{Q}$ .*

Our next aim is to get existence results for optimal strategies. In the classical case  $\mathcal{Q} = \{\mathbb{P}\}$ , it was shown in [23] that a necessary and sufficient condition for the existence of optimal strategies at each initial capital is the finiteness of the dual value function  $v_{\mathbb{P}}$ . This condition translates as follows to our robust setting:

$$v_Q(y) < \infty \quad \text{for all } y > 0 \text{ and each } Q \in \mathcal{Q}_e. \quad (12)$$

Recall from [23, Note 2] that (12) holds as soon as  $u_Q$  is finite for all  $Q \in \mathcal{Q}_e$  and the asymptotic elasticity of the utility function  $U$  is strictly less than one:

$$AE(U) = \limsup_{x \uparrow \infty} \frac{xU'(x)}{U(x)} < 1.$$

**Theorem 2.6** *In addition to the assumptions of Theorem 2.4, let us assume (12). Then both value functions  $u$  and  $v$  take only finite values and satisfy*

$$u'(\infty-) = 0 \quad \text{and} \quad v'(0+) = -\infty. \quad (13)$$

*The robust value function  $u$  is strictly concave, and the dual value function  $v$  is continuously differentiable. Moreover, for any  $x > 0$  there exist an optimal strategy  $\hat{X} \in \mathcal{X}(x)$  for the robust problem. If  $y > 0$  is such that  $v'(y) = -x$  and  $(\hat{Q}, \hat{Y})$  is a solution of the dual problem, then*

$$\hat{X}_T = I(\hat{Y}_T) \quad \hat{Q}\text{-a.s.} \quad (14)$$

*for  $I := -V'$  and  $(\hat{Q}, \hat{X})$  is a saddlepoint for the robust problem:*

$$u(x) = \inf_{Q \in \mathcal{Q}} (E_Q[U(\hat{X}_T)] + \gamma(Q)) = E_{\hat{Q}}[U(\hat{X}_T)] + \gamma(\hat{Q}) = u_{\hat{Q}}(x) + \gamma(\hat{Q}).$$

*Furthermore,  $\hat{X}\hat{Y}\hat{Z}$  is a martingale under  $\mathbb{P}$ , where  $(\hat{Z}_t)_{0 \leq t \leq T}$  is the density process of  $\hat{Q}$  with respect to  $\mathbb{P}$ .*

In the preceding theorem, let us take  $(\hat{Q}, \hat{Y})$  as a maximal solution of the dual problem as constructed in Theorem 2.4. Then the solution  $\hat{X}_T$  will be  $\mathbb{P}$ -a.s. unique as soon as  $\hat{Q} \sim \mathbb{P}$ . This equivalence holds trivially if all measures in  $\mathcal{Q}$  are equivalent to  $\mathbb{P}$ . In the general case, however, Example 3.2 will show that  $\hat{Q}$  need not be equivalent to  $\mathbb{P}$ , so that (14) cannot guarantee the  $\mathbb{P}$ -a.s. uniqueness of  $\hat{X}_T$ . Nevertheless, we can construct an optimal strategy from a given solution of the dual problem by superhedging an appropriate contingent claim  $H \geq 0$ :

**Corollary 2.7** *Suppose the assumptions of Theorem 2.6 hold. Let  $(\hat{Q}, \hat{Y})$  be a solution of the dual problem at level  $y > 0$  and consider the contingent claim*

$$H := I(\hat{Y}_T) \mathbf{1}_{\{\hat{Z} > 0\}},$$

*where  $d\hat{Q} = \hat{Z} d\mathbb{P}$ . Then  $x = -v'(y)$  is the minimal initial investment  $x' > 0$  for which there exists some  $X \in \mathcal{X}(x')$  such that  $X_T \geq H$   $\mathbb{P}$ -a.s. If furthermore  $\hat{X} \in \mathcal{X}(x)$  is such a strategy, then it is a solution for the robust utility maximization problem at initial capital  $x$ .*

**Remark 2.8** Instead of working with the terminal values of processes in the space  $\mathcal{Y}_Q(y)$ , it is sometimes more convenient to work with the densities of in the set  $\mathcal{M}$  of equivalent local martingale measures. In fact, the dual value function satisfies

$$v(y) = \inf_{P^* \in \mathcal{M}} \inf_{Q \in \mathcal{Q}_e} \left( E_Q \left[ V \left( y \frac{dP^*}{dQ} \right) \right] + \gamma(Q) \right). \quad (15)$$

This identity follows from Lemma 4.4 below and the corresponding identity in [22, 23]. Since the infimum in (15) need not be attained, it is often not possible to represent the optimal solution  $\widehat{X}_T$  in terms of the density of an equivalent martingale measure. Nevertheless, Föllmer and Gundel [13] recently observed that the elements of  $\mathcal{Y}_Q(1)$  can be interpreted as density processes of ‘extended martingale measures’.  $\diamond$

### 3 Examples and Counterexamples

The first example in this section illustrates that the value function  $u$  need not be continuously differentiable and its dual  $v$  need not be strictly convex, even if all measures in  $\mathcal{Q}$  are mutually equivalent. The second example illustrates that the maximal solution of dual problem, as constructed in Theorem 2.4, may not have full support. The subsequent examples provide explicit choices for penalty functions  $\gamma$ , which are natural from an economical or statistical point of view. They will also illustrate that the stochastic control method is often not feasible for robust optimization problems.

**Example 3.1 (Non-differentiability of the value function)** We consider a one-period trinomial model where the risky asset starts off at  $S_0 = 1$ . At time  $t = 1$ , it can take the values 0, 1, and 2. Consequently, we let  $\Omega := \{\omega_-, \omega_0, \omega_+\}$  and define  $S_1(\omega_\pm) := 1 \pm 1$  and  $S_1(\omega_0) := 1$ . A probability measure  $Q$  on  $\Omega$  is determined by  $p := Q[\{\omega_+\}]$  and  $q := Q[\{\omega_-\}]$ . This model fits into the semimartingale framework by taking  $S_t := 1$  and  $\mathcal{F}_t := \{\emptyset, \Omega\}$  as long as  $t < 1$  and  $\mathcal{F}_1 := \sigma(S_1)$ . It is arbitrage-free and satisfies the assumption  $\mathcal{M} \neq \emptyset$  iff  $p$  and  $q$  are both strictly positive. An investment  $\xi$  in the risky asset made for an initial wealth  $x$  results in a terminal payoff  $X_1 = x + \xi(S_1 - S_0)$ . Hence,  $\xi$  is admissible iff  $|\xi| \leq x$ . Let us take  $U(x) = \sqrt{x}$ . Then the  $Q$ -expected utility of an admissible investment  $\xi$  is given by

$$E_Q[U(x + \xi(S_1 - S_0))] = p\sqrt{x + \xi} + (1 - p - q)\sqrt{x} + q\sqrt{x - \xi}.$$

Optimizing over  $\xi$  yields that

$$\xi = x \cdot \frac{p^2 - q^2}{p^2 + q^2}$$

is the unique optimal strategy for  $Q$ . Now we take  $0 < a < b < 2/3$  and define  $\mathcal{Q}$  as the set of all measures  $Q_p$  for which  $q = p/2$  and  $a \leq p \leq b$ . This set  $\mathcal{Q}$  is parameterized by  $p$  and consists of mutually equivalent measures. For  $Q_p \in \mathcal{Q}$ , the value function is given by

$$u_{Q_p}(x) = \sqrt{x}(1 + \beta p),$$

where  $\beta = \sqrt{8/5} - 3/2 + 1/\sqrt{10} > 0$ . The penalty function

$$\gamma(Q) := \begin{cases} \beta(b - p) & \text{if } Q = Q_p \in \mathcal{Q}, \\ +\infty & \text{otherwise,} \end{cases}$$

is convex and lower semicontinuous and thus satisfies our assumptions. By Theorem 2.4, the robust value function is given by

$$u(x) = \inf_{Q \in \mathcal{Q}} (u_Q(x) + \gamma(Q)) = \sqrt{x} + \beta b + \beta \inf_{a \leq p \leq b} (p\sqrt{x} - p).$$

The infimum on the right equals  $b\sqrt{x} - b$  for  $x < 1$  and  $a\sqrt{x} - a$  for  $x > 1$ . Hence,  $u$  is not continuously differentiable at  $x = 1$ , and  $v$  cannot be strictly convex; see, e.g., [29, Theorem V.26.3].  $\diamond$

**Example 3.2 (The maximal  $\widehat{\mathcal{Q}}$  may fail to have full support)** The fact that the measure  $\widehat{\mathcal{Q}}$  associated with the maximal solution for the dual problem may not be equivalent to  $\mathbb{P}$  can be deduced from [32, Example 2.5 and Theorem 2.6]. Here we give a more direct argument within the setting of [32, Example 2.5]. We consider a one-period model in discrete time ( $t = 0, 1$ ) with two risky assets  $S^1, S^2$  satisfying  $S_0^1 = S_0^2 = 1$ . Under the measure  $\mathbb{P} := Q_1$ , the first asset  $S_1^1$  has the distribution

$$Q_1[S_1^1 = 2] =: q = 1 - Q_1[S_1^1 = 0],$$

where  $1/2 < q < 1$ . The second asset  $S_1^2$  has support  $\{0, 1, \dots\}$ , and finite expected value  $E_{Q_1}[S_1^2] > S_0^2 = 1$ . We take  $\mathbb{P} := Q_1$  as our reference measure. We introduce another measure  $Q_0 \ll \mathbb{P}$  by requiring that

$$Q_0[S_1^1 = 2] = Q_0[S_1^1 = 0] = 1/2 \quad \text{and} \quad Q_0[S_1^2 = 0] = 1.$$

We define  $\gamma(Q) = 0$  if  $Q = Q_\alpha := \alpha Q_1 + (1 - \alpha)Q_0$  for some  $0 \leq \alpha \leq 1$  and  $\gamma(Q) = \infty$  otherwise. Note that a trading strategy can only be admissible for  $\mathbb{P} = Q_1$  if it does not contain short positions in the second asset, because  $S_1^2$  is unbounded. Hence, under  $Q_0$  any strategy  $X \in \mathcal{X}(1)$  is a supermartingale, and it follows that  $v_{Q_0}(y) = V(y)$ . Under  $Q_\alpha$  with  $\alpha > 0$ , any long position in the first asset will be a submartingale, and so we must have  $v_{Q_\alpha}(y) > V(y)$ . This shows that  $\widehat{\mathcal{Q}} = Q_0$  and  $\widehat{Y}_1 = yI_{\{S_1^2=0\}}$  is the unique solution of the dual problem. Moreover, one can easily show that  $\widehat{X}_1 \equiv x$  is the unique solution of the primal problem; see [32, Example 2.5]. The constant  $y = I(x)$ , however, does not belong to any of the spaces  $\mathcal{Y}_{Q_\alpha}(y)$  for  $\alpha > 0$ . This illustrates that it is possible that the duality relation

$$\widehat{X}_T = I(\widehat{Y}_T)$$

cannot be extended to a  $\mathbb{P}$ -a.s. identity. Finally, note that  $Q_0$  considered as a market model on its own has not the same admissible strategies than  $\mathbb{P}$ , since short selling the second asset is admissible in the model  $Q_0$ . In fact, such short sales even create arbitrage opportunities under  $Q_0$ .  $\diamond$

**Example 3.3 (Entropic penalties)** A popular choice for  $\gamma$  is taking (a multiple of) the relative entropy with respect to  $\mathbb{P}$ , which is defined as

$$H(Q|\mathbb{P}) = \int \frac{dQ}{d\mathbb{P}} \log \frac{dQ}{d\mathbb{P}} d\mathbb{P} = \sup_{Y \in L^\infty} (E_Q[Y] - \log \mathbb{E}[e^Y]), \quad Q \ll \mathbb{P};$$

see, e.g., [16, Sections 3.2 and 4.9]. Due to the classical duality formula

$$\log \mathbb{E}[e^X] = \sup_{Q \in \mathcal{Q}} (E_Q[X] - H(Q|\mathbb{P})), \quad (16)$$

the choice  $\gamma(Q) = \frac{1}{\theta}H(Q|\mathbb{P})$  corresponds to the utility functional

$$\inf_{Q \in \mathcal{Q}} (E_Q[U(X_T)] + \gamma(Q)) = -\frac{1}{\theta} \log \mathbb{E}[e^{-\theta U(X_T)}]$$

of the terminal wealth, which obviously satisfies Assumption 2.1. Its maximization is equivalent to the maximization of the ordinary expected utility  $\mathbb{E}[\tilde{U}(X_T)]$ , where  $\tilde{U}(x) = -e^{-\theta U(x)}$  is strictly concave, increasing, and satisfies the Inada conditions. Thus, robustness effects are only felt in *intertemporal* optimization problems; see Hansen and Sargent [20] or Barrieu and El Karoui [2]. On the other hand, the use of entropic penalties in intertemporal optimization problems is facilitated by the *dynamic consistency* of the corresponding conditional risk measure  $\rho_t(X) := \frac{1}{\theta} \log \mathbb{E}[e^{-\theta X} | \mathcal{F}_t]$ , namely,

$$\rho_0(-\rho_t(Y)) = \rho_0(Y) \quad \text{for all } Y \in L^\infty. \quad (17)$$

This property of dynamic consistency corresponds to the Bellman principle in dynamic programming and is the essential ingredient for the application of the stochastic control method; see Hansen and Sargent [20].  $\diamond$

Recently, the dynamic consistency (17) of risk measures has been the subject of intense study; see, e.g., [6], [11], and the references therein. As explained above, it is the crucial property for an application of the stochastic control method. As a normative postulate, however, it seems to be debatable as it would require that the investor does not change the penalty function for the entire investment period  $[0, T]$ . But financial models are typically not accurate, and each piece of freshly revealed information might require to adjust models and hence penalty functions. In reality, this fact is usually taken into account by a periodic model recalibration, resulting in ever changing model parameters. In addition, the following examples will illustrate that some natural risk measures do not satisfy the property (17). These examples all belong to the class of law-invariant convex risk measures. The failure of dynamic consistency for law-invariant *coherent* risk measures has already been pointed out by Delbaen [9].

**Example 3.4 (Shortfall risk)** Let  $\ell : \mathbb{R} \rightarrow \mathbb{R}$  be convex, increasing, and nonconstant and take  $x$  in the interior of  $\ell(\mathbb{R})$ . The associated *shortfall risk measure*

$$\rho(Y) := \inf \{ m \in \mathbb{R} \mid \mathbb{E}[\ell(-Y - m)] \leq x \}, \quad Y \in L^\infty, \quad (18)$$

was introduced by Föllmer and the author in [14]. Assumption 2.1 is satisfied due to [16, Proposition 4.104]. Using (16), one sees that the choice  $\ell(y) = e^{\theta y}$  corresponds to the entropic penalty  $\gamma(Q) = \frac{1}{\theta}H(Q|\mathbb{P})$ . For general  $\ell$ , the penalty function is given by

$$\gamma(Q) = \inf_{\lambda > 0} \frac{1}{\lambda} \left( x + \mathbb{E} \left[ \ell^* \left( \lambda \frac{dQ}{d\mathbb{P}} \right) \right] \right) = \inf_{\lambda > 0} \left( \lambda x + \lambda \mathbb{E} \left[ \ell^* \left( \lambda^{-1} \frac{dQ}{d\mathbb{P}} \right) \right] \right), \quad Q \ll \mathbb{P}, \quad (19)$$

where  $\ell^*$  is the Fenchel-Legendre transform of  $\ell$ ; see [14, Theorem 10] or [16, Theorem 4.106]. The risk measure  $\rho$  satisfies  $\rho(0) = 0$  if we take  $x = \ell(0)$ . It induces a dynamic risk measure  $\rho_t$ ,  $0 \leq t \leq T$ , in a canonical way by replacing the expectation operator in (18) with a conditional expectation. It is easy to see that this dynamic risk measure is *weakly dynamically consistent* in the sense that

$$\rho_t(Y) \leq 0 \text{ } \mathbb{P}\text{-a.s.} \Rightarrow \rho_0(Y) \leq 0 \quad \text{and} \quad \rho_t(Y) \geq 0 \text{ } \mathbb{P}\text{-a.s.} \Rightarrow \rho_0(Y) \geq 0; \quad (20)$$

see Weber [34]. This weak property, however, does not guarantee the validity of (17), as is illustrated by the following simple example.  $\diamond$

**Example 3.5 (Shortfall risk may not be dynamically consistent)** As a loss function we take  $\ell(y) = (y + \varepsilon)^+$ , where  $0 < \varepsilon < 1/4$ . Let  $Y_1, Y_2$  be two Bernoulli random variables such that  $\mathbb{P}[Y_i = 0] = \mathbb{P}[Y_i = 1] = 1/2$ ,  $i = 1, 2$ . Suppose that  $Y_1$  is  $\mathcal{F}_1$ -measurable and  $Y_2$  is independent of  $\mathcal{F}_1$ , while  $\mathcal{F}_0$  is trivial. We let  $Y := -Y_1 Y_2$  and compute its risk under the dynamic shortfall risk measure arising from (18), which is normalized if we choose  $x = \varepsilon$ . A straightforward computation then shows that  $\rho_0(Y) = 1 - 3\varepsilon$ , while  $\rho_1(Y) = (1 - \varepsilon)Y_1$  and  $\rho_0(-\rho_1(Y)) = 1 - 2\varepsilon$ . We believe that this failure of dynamic consistency for shortfall risk is the rule rather than the exception.  $\diamond$

Note the the condition of weak dynamic consistency (20) is necessary for (17). Yet, Weber [34] showed that, under certain technical regularity conditions, shortfall risk is the only law-invariant risk measure such that the associated canonical dynamic risk measure is weakly dynamically consistent. Here is another natural choice for a law-invariant risk measure, which may not even satisfy (20).

**Example 3.6 (Statistical distance functions)** Let  $g : [0, \infty) \rightarrow \mathbb{R} \cup \{+\infty\}$  be a lower semicontinuous convex function satisfying  $g(1) = 0$  and the superlinear growth condition  $g(x)/x \rightarrow +\infty$  as  $x \uparrow \infty$ . Associated to it is the *g-divergence*

$$I_g(Q|\mathbb{P}) := \mathbb{E} \left[ g \left( \frac{dQ}{d\mathbb{P}} \right) \right], \quad Q \ll \mathbb{P},$$

as introduced by Csiszar [7, 8]; see Liese and Vajda [26] for a systematic study. The *g-divergence*  $I_g(Q|\mathbb{P})$  can be interpreted as a statistical distance between the hypothetical model  $Q$  and the reference measure  $\mathbb{P}$ , so that taking  $\gamma(Q) := I_g(Q|\mathbb{P})$  is a natural choice for a penalty function. The particular choice  $g(x) = x \log x$  corresponds to the relative entropy  $I_g(Q|\mathbb{P}) = H(Q|\mathbb{P})$ . Taking  $g(x) = 0$  for  $x \leq \lambda^{-1}$  and  $g(x) = \infty$  otherwise corresponds to the coherent risk measure Average Value at Risk,

$$\text{AVaR}_\lambda(Y) = \sup \{ E_Q[-Y] \mid dQ/d\mathbb{P} \leq \lambda^{-1} \},$$

which is also called Expected Shortfall or Conditional Value at Risk. One easily sees that  $\text{AVaR}_\lambda$  does *not* satisfy the condition of weak dynamic consistency (20); see also [34].

In particular it does not satisfy (17). See [30, 31] for an analysis of optimal investment problems for  $AVaR_\lambda$  in complete market models.

For general  $g$  the penalty function  $\gamma(Q) = I_g(Q|\mathbb{P})$  corresponds to the convex risk measure

$$\rho(Y) = \sup_{Q \ll \mathbb{P}} (E_Q[-Y] - \gamma(Q)), \quad Y \in L^\infty,$$

which satisfies Assumption 2.1. Indeed, the level sets  $\{dQ/d\mathbb{P} \mid I_g(Q|\mathbb{P}) \leq c\}$  are weakly compact in  $L^1(\mathbb{P})$  due to the superlinear growth condition, and so continuity from below follows from [25, Lemma 2] together with [16, Corollary 4.35]; see also [21, Theorem 2.4]. The convex risk measure  $\rho$  satisfies the variational identity

$$\rho(Y) = \sup_{Q \ll \mathbb{P}} (E_Q[-Y] - \gamma(Q)) = \inf_{z \in \mathbb{R}} (\mathbb{E}[g^*(z - Y)] - z), \quad Y \in L^\infty, \quad (21)$$

where  $g^*(y) = \sup_{x > 0} (xy - g(x))$ . This identity was obtained by Ben-Tal and Teboulle [3] for finite  $g$ . In the case of  $AVaR_\lambda$ , we have  $g^*(y) = 0 \vee y/\lambda$  and hence recover [16, Lemma 4.46] as a special case of (21). Below we will give a proof, which works in the general case and is based on the results from Föllmer and the author [14, 16] quoted in Example 3.4.

◇

**Proof of (21):** For  $\lambda > 0$  let  $g_\lambda(x) := \lambda g(x/\lambda)$ . Then  $(\lambda, x) \mapsto g_\lambda(x)$  is convex due to (25) below. Let  $\gamma_\lambda(Q) = I_{g_\lambda}(Q|\mathbb{P})$  be the corresponding  $g_\lambda$ -divergence. Then  $(\lambda, Q) \mapsto \gamma_\lambda(Q)$  is a convex functional, and it follows easily that

$$h(\lambda) := \inf_{Q \ll \mathbb{P}} (E_Q[Y] + \gamma_\lambda(Q))$$

is a convex function in  $\lambda$  if  $Y \in L^\infty$  is fixed. Our aim is to compute  $h(1)$ . The idea is to use the fact that (19) is the penalty function of the risk measure in (18) in order to identify the Fenchel-Legendre transform  $h^*$  of  $h$ . We only have to observe that  $\ell := g^*$  satisfies the assumptions of Example 3.4 and that  $\ell^* = g^{**} = g$  so as to apply (19):

$$\begin{aligned} f(x) &:= \inf \{ m \in \mathbb{R} \mid \mathbb{E}[g^*(-m - Y)] \leq x \} \\ &= \sup_{Q \ll \mathbb{P}} \left( E_Q[-Y] - \inf_{\lambda > 0} \left( \lambda x + \mathbb{E} \left[ g_\lambda \left( \frac{dQ}{d\mathbb{P}} \right) \right] \right) \right) \\ &= - \inf_{\lambda > 0} \inf_{Q \ll \mathbb{P}} \left( E_Q[Y] + \lambda x + \gamma_\lambda(Q) \right) \\ &= - \inf_{\lambda > 0} (\lambda x + h(\lambda)) = h^*(-x), \end{aligned}$$

for all  $x$  in the interior of  $g^*(\mathbb{R})$ , which is equal to  $(-\inf g, \infty)$  and thus coincides with the interior of  $\text{dom } f$ . Convexity hence yields  $h(1) = h^{**}(1) = \sup_x (x - f(-x))$ . The definition of  $f$  yields that  $x = -\mathbb{E}[g^*(-f(-x) - Y)]$ . Hence,

$$h(1) = \sup_{x \in \mathbb{R}} \left( -\mathbb{E}[g^*(-f(x) - Y)] - f(-x) \right)$$

and the assertion follows by noting that the range of  $f$  is equal to  $(-\infty, \text{ess sup}(-Y))$ . □

## 4 Proofs

For  $c \geq 0$ , let us introduce the sets

$$\mathcal{Q}(c) := \{Q \in \mathcal{Q} \mid \gamma(Q) \leq c\} \quad \text{and} \quad \mathcal{Q}_e(c) := \{Q \in \mathcal{Q}(c) \mid Q \sim \mathbb{P}\}.$$

With  $\mathcal{Z}(c)$ ,  $\mathcal{Z}$ ,  $\mathcal{Z}_e(c)$ , and  $\mathcal{Z}_e$ , we will denote the corresponding sets of densities, e.g.,

$$\mathcal{Z} := \left\{ \frac{dQ}{d\mathbb{P}} \mid Q \in \mathcal{Q} \right\}, \quad \mathcal{Z}_e(c) := \left\{ \frac{dQ}{d\mathbb{P}} \mid Q \in \mathcal{Q}_e(c) \right\}.$$

In the sequel, we will identify measures  $Q \in \mathcal{Q}$  with their densities  $Z = dQ/d\mathbb{P}$ , and we will also write  $\gamma(Z)$ ,  $u_Z$ ,  $v_Z$  for  $\gamma(Q)$ ,  $u_Q$ , and  $v_Q$ , respectively. Due to (4),  $Z \mapsto \gamma(Z)$  is a convex and weakly lower semicontinuous functional on  $L^1(\mathbb{P})$ .

**Lemma 4.1** *For every  $c > 0$ , the level set  $\mathcal{Z}(c)$  is weakly compact, and  $\mathcal{Z}_e(c)$  is nonempty. Moreover,  $Z \mapsto \gamma(Z)$  is lower semicontinuous with respect to  $\mathbb{P}$ -a.s. convergence on  $\mathcal{Z}(c)$ .*

**Proof:** The set  $\mathcal{Z}(c)$  is weakly closed by the weak lower semicontinuity of  $\gamma$  and uniformly integrable due to [16, Lemma 4.22]. Hence,  $\mathcal{Z}(c)$  is weakly compact according to the Dunford-Pettis theorem. Next, for all  $c > 0$  we have that  $\mathbb{P}[A] > 0$  implies  $Q[A] > 0$  for some  $Q \in \mathcal{Q}(c)$ . Indeed, the sensitivity of  $\rho$  gives

$$0 < \rho(-c\mathbb{I}_A) = \sup_{Q \in \mathcal{Q}(c)} (cQ[A] - \gamma(Q)).$$

Hence, the assertion  $\mathcal{Z}_e(c) \neq \emptyset$  follows from the Halmos-Savage theorem. Finally, if  $Z_n \rightarrow Z$   $\mathbb{P}$ -a.s. and all  $Z_n$  belong to some level set  $\mathcal{Z}(c)$ , then convergence also holds in  $L^1(\mathbb{P})$ , and the lower semicontinuity of  $\gamma$  follows from (4).  $\square$

We note next that the space  $\mathcal{Y}_Q(y)$  can easily be related to  $\mathcal{Y}(y) := \mathcal{Y}_{\mathbb{P}}(y)$ :

**Lemma 4.2** *Let  $(Z_t)_{0 \leq t \leq T}$  be the density process of  $Q \ll \mathbb{P}$  with respect to  $\mathbb{P}$ . Then a process  $Y^Q$  belongs to  $\mathcal{Y}_Q(y)$  if and only if  $Y^Q Z \in \mathcal{Y}(y)$ . In particular, we have*

$$v(y) = \inf_{Z \in \mathcal{Z}} \inf_{Y \in \mathcal{Y}(y)} \left( \mathbb{E} \left[ ZV \left( \frac{Y_T}{Z} \right) \right] + \gamma(Z) \right). \quad (22)$$

**Proof:** Take  $0 \leq s < t \leq T$ . If  $Y^Q \in \mathcal{Y}_Q(y)$  and  $X \in \mathcal{X}(1)$ , then

$$X_s Y_s^Q \geq E_Q[X_t Y_t^Q \mid \mathcal{F}_s] = \frac{1}{Z_s} \mathbb{E}[X_t Y_t^Q Z_t \mid \mathcal{F}_s] \quad \mathbb{P}\text{-a.s. on } \{Z_s > 0\}.$$

On  $\{Z_s = 0\}$  we have  $\mathbb{P}$ -a.s.  $Z_t = 0$  and hence  $\mathbb{E}[X_t Y_t^Q Z_t \mid \mathcal{F}_s] = 0 = X_s Y_s^Q Z_s$ . Combining these two facts shows that  $XY^Q Z$  is a  $\mathbb{P}$ -supermartingale and hence that  $Y^Q Z \in \mathcal{Y}(y)$ . Conversely, suppose that  $Y := Y^Q Z \in \mathcal{Y}(y)$ . Then we have  $Q$ -a.s. for each  $X \in \mathcal{X}(1)$

$$E_Q[X_t Y_t^Q \mid \mathcal{F}_s] = \frac{1}{Z_s} \mathbb{E}[X_t Y_t \mid \mathcal{F}_s] \leq \frac{X_s Y_s}{Z_s} = X_s Y_s^Q.$$

$\square$

The formula (22) is more convenient than our original definition of  $v$ , as the infimum is now taken over two sets that are no longer related to another. As in [22, 23], we obtain “abstract versions” of our theorems if we replace the spaces  $\mathcal{X}(x)$  and  $\mathcal{Y}_Q(y)$  by the respective spaces

$$\mathcal{C}(x) = \{ g \in L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}) \mid 0 \leq g \leq X_T \text{ for some } X \in \mathcal{X}(x) \}.$$

and

$$\mathcal{D}_Q(y) = \{ h \in L_+^0(\Omega, \mathcal{F}_T, Q) \mid 0 \leq h \leq Y_T \text{ for some } Y \in \mathcal{Y}_Q(y) \}.$$

Obviously, this substitution does not affect the values of our value functions, i.e., using our convention (6) we have  $u_Q(x) = \sup_{g \in \mathcal{C}(x)} E_Q[U(g)]$ ,  $v_Q(y) = \inf_{h \in \mathcal{D}_Q(y)} E_Q[V(h)]$ ,

$$u(x) = \sup_{g \in \mathcal{C}(x)} \inf_{Q \in \mathcal{Q}} (E_Q[U(g)] + \gamma(Q)),$$

and

$$v(y) = \inf_{Q \in \mathcal{Q}} \inf_{h \in \mathcal{D}_Q(y)} (E_Q[V(h)] + \gamma(Q)) = \inf_{Z \in \mathcal{Z}} \inf_{h \in \mathcal{D}(y)} \left( \mathbb{E} \left[ ZV \left( \frac{h}{Z} \right) \right] + \gamma(Z) \right).$$

Moreover, any optimal  $g$  or  $h$ , if they exist, can be taken as the terminal value of some process  $X \in \mathcal{X}(x)$  or  $Y \in \mathcal{Y}_Q(y)$ . Next, recall from [22] that for  $Q \sim \mathbb{P}$

$$\begin{aligned} g \in \mathcal{C}(x) &\iff g \geq 0 \text{ and } \sup_{h \in \mathcal{D}_Q(y)} E_Q[hg] \leq xy \\ h \in \mathcal{D}_Q(y) &\iff h \geq 0 \text{ and } \sup_{g \in \mathcal{C}(x)} E_Q[hg] \leq xy. \end{aligned} \tag{23}$$

We point out that validity of this relation is not clear for  $Q \not\sim \mathbb{P}$ , and this will create a few technical difficulties in the sequel.

Under the convention (6),  $g \mapsto E_Q[U(g)]$  is a concave functional on  $\mathcal{C}(x)$  for each  $Q \in \mathcal{Q}$  and all  $x > 0$ . Using the fact that

$$\{ \alpha g + (1 - \alpha)g' \mid g \in \mathcal{C}(x), g' \in \mathcal{C}(x') \} \subset \mathcal{C}(\alpha x + (1 - \alpha)x')$$

then yields the concavity of the value functions  $u_Q$  and  $u$ . The concavity of  $u_Q$  implies in turn that

$$u_Q \equiv +\infty \text{ as soon as } E_Q[U^+(g)] = +\infty \text{ for some } g \in \bigcup_{x>0} \mathcal{C}(x); \tag{24}$$

see [32, Lemma 3.1].

A key observation for our future analysis is the convexity of the function  $(z, y) \mapsto zV(y/z)$ . In fact, one has

$$(\alpha z_0 + (1 - \alpha)z_1)V\left(\frac{\alpha y_0 + (1 - \alpha)y_1}{\alpha z_0 + (1 - \alpha)z_1}\right) < \alpha z_0 V\left(\frac{y_0}{z_0}\right) + (1 - \alpha)z_1 V\left(\frac{y_1}{z_1}\right) \tag{25}$$

as soon as  $y_0/z_0 \neq y_1/z_1$  and  $0 < \alpha < 1$ ; see Equation (21) in [32].

**Lemma 4.3** *If  $v(y) < \infty$ , then there exist  $\hat{h} \in \mathcal{D}(y)$  and  $\hat{Z} \in \mathcal{Z}$  such that*

$$v(y) = \mathbb{E}[\hat{Z}V(\hat{h}/\hat{Z})] + \gamma(\hat{Z}).$$

Moreover,  $\hat{Z} =: d\hat{Q}/d\mathbb{P}$  and  $\hat{h}$  can be chosen in such a way that  $\hat{h}/\hat{Z}$  coincides  $\hat{Q}$ -a.s. with the terminal value of some  $\hat{Y} \in \mathcal{Y}_{\hat{Q}}(y)$  and such that  $(\hat{Q}, \hat{Y})$  is a solution of the dual problem, which is maximal in the sense of Theorem 2.4.

**Proof:** Let  $(Z_n, h_n) \in \mathcal{Z} \times \mathcal{D}(y)$  be a sequence such that  $\mathbb{E}[Z_n V(h_n/Z_n)] + \gamma(Z_n) \rightarrow v(y)$ . Jensen's inequality implies that

$$\mathbb{E}[ZV(h/Z)] \geq V(\mathbb{E}[h\mathbf{I}_{\{Z>0\}}]) \geq V(y) \quad \text{for all } Z \text{ and } h \in \mathcal{D}(y). \quad (26)$$

Hence we must have  $c := 1 + \limsup_n \gamma(Z_n) < \infty$ , and so we can assume without loss of generality that  $Z_n \in \mathcal{Z}(c)$  for all  $n$ .

Applying twice the standard Komlos-type argument of Lemma A1.1 in [10], we obtain a sequence

$$(\tilde{Z}_n, \tilde{h}_n) \in \text{conv}\{(Z_n, h_n), (Z_{n+1}, h_{n+1}), \dots\} \subset \mathcal{Z}(c) \times \mathcal{D}(y)$$

that converges  $\mathbb{P}$ -a.s. to some  $(\hat{Z}_0, \hat{h}_0)$ . From (23) we get  $\hat{h}_0 \in \mathcal{D}(y)$ . Lemma 4.1 implies  $\hat{Z}_0 \in \mathcal{Z}(c)$ . It was shown in the proof of [32, Lemma 3.6] that the function

$$\mathcal{Z}(c) \times \mathcal{D}(y) \ni (Z, h) \mapsto \mathbb{E}[ZV(h/Z)]$$

is lower semicontinuous with respect to  $\mathbb{P}$ -a.s. convergence. By the convexity of  $(x, z) \mapsto zV(x/z)$  and Lemma 4.1 we thus get

$$\mathbb{E}[\hat{Z}_0 V(\hat{h}_0/\hat{Z}_0)] + \gamma(\hat{Z}_0) \leq \liminf_{n \uparrow \infty} \left( \mathbb{E}[Z_n V(h_n/Z_n)] + \gamma(Z_n) \right) = v(y).$$

In this sense, the pair  $(\hat{h}_0, \hat{Z}_0)$  is optimal.

Suppose  $(\hat{h}_1, \hat{Z}_1)$  is another optimal pair, and let  $\hat{h}_t := t\hat{h}_1 + (1-t)\hat{h}_0$  and  $\hat{Z}_t := t\hat{Z}_1 + (1-t)\hat{Z}_0$  for  $0 \leq t \leq 1$ . The convexity of  $(h, Z) \mapsto \mathbb{E}[ZV(h/Z)] + \gamma(Z)$  implies that each pair  $(\hat{h}_t, \hat{Z}_t)$  is also optimal. If  $0 < t < 1$ , then  $\{\hat{Z}_t > 0\} = \{\hat{Z}_0 > 0\} \cup \{\hat{Z}_1 > 0\}$ . Moreover, (25) shows that the ratio  $h_t/Z_t$  does not depend on  $t \in (0, 1)$ . Hence, there exists a random variable  $\hat{Y}_T \geq 0$  and a sequence  $\hat{Z}_1, \hat{Z}_2, \dots$  such that the following hold:

- (a)  $\mathbb{P}[\hat{Z}_n > 0]$  tends to the maximum  $\mathbb{P}$ -probability for the support of any optimal  $\hat{Z}$ ;
- (b)  $\{\hat{Z}_1 > 0\} \subset \{\hat{Z}_2 > 0\} \subset \dots$ ;
- (c) for each  $n$ , we have  $\hat{h}_n := \hat{Y}_T \hat{Z}_n \in \mathcal{D}(y)$ , and the pair  $(\hat{h}_n, \hat{Z}_n)$  is optimal.

By using a Komlos-type argument, we may assume that the  $\hat{Z}_n$  converge  $\mathbb{P}$ -a.s. to some  $\hat{Z} \in \mathcal{Z}$ . Then  $\hat{Y}_T \hat{Z} \in \mathcal{D}(y)$  by (23) and in turn  $\hat{Y}_T \in \mathcal{D}_{\hat{Q}}(y)$  due to Lemma 4.2. Hence, we may assume that  $\hat{Y}_T$  is the terminal value of some  $\hat{Y} \in \mathcal{Y}_{\hat{Q}}(y)$ . As above, we then conclude  $E_{\hat{Q}}[V(\hat{Y}_T)] + \gamma(\hat{Q}) \leq v(y)$ , that is,  $(\hat{Q}, \hat{Y})$  is a solution of the dual problem. Clearly,  $(\hat{Q}, \hat{Y})$  is maximal.  $\square$

Let  $\mathcal{Q}^f$  denote the set of  $Q \in \mathcal{Q}$  such that  $u_Q(x) < \infty$  for some and hence all  $x > 0$ . Similarly we define  $\mathcal{Q}_e^f$ ,  $\mathcal{Z}^f$ , and  $\mathcal{Z}_e^f$ . We will show next that in (22) the set  $\mathcal{Q}$  can be replaced by the smaller sets  $\mathcal{Q}_e$  and  $\mathcal{Q}_e^f$ .

**Lemma 4.4** *For  $v(y) < \infty$  the dual value function of the robust problem satisfies*

$$v(y) = \inf_{Q \in \mathcal{Q}_e} (v_Q(y) + \gamma(Q)) = \inf_{Q \in \mathcal{Q}_e^f} (v_Q(y) + \gamma(Q)).$$

**Proof:** As for the proof of the first identity, suppose  $Z_0 \in \mathcal{Z} \setminus \mathcal{Z}_e$  and  $h_0 \in \mathcal{D}(y)$  are such that  $\mathbb{E}[Z_0 V(h_0/Z_0)] < \infty$ . Due to our assumption (9), we may choose  $Z_1 \in \mathcal{Z}_e$  and  $h_1 \in \mathcal{D}(y)$  such that  $\mathbb{E}[Z_1 V(h_1/Z_1)] < \infty$ . Now let  $Z_t := tZ_1 + (1-t)Z_0 \in \mathcal{Z}_e$  and  $h_t := th_1 + (1-t)h_0$  for  $0 < t \leq 1$ . Since the function  $t \mapsto \mathbb{E}[Z_t V(h_t/Z_t)] + \gamma(Z_t)$  is convex and takes finite values, it is upper semicontinuous and we get  $v_{Z_0}(y) + \gamma(Z_0) \geq \limsup_{t \downarrow 0} (v_{Z_t} + \gamma(Z_t))$ . This proves the first identity. The second identity follows from the fact that for  $Q \sim \mathbb{P}$  we have  $v_Q \equiv \infty$  as soon as  $u_Q \equiv \infty$ ; see the proof of [32, Lemma 3.5].  $\square$

**Remark 4.5** In the sequel, we will sometimes use variants of the upper semicontinuity argument in the preceding proof. For a convex set  $\mathcal{Z}' \subset \mathcal{Z}$  and  $Z_0, Z_1 \in \mathcal{Z}'$  let  $Z_t := tZ_1 + (1-t)Z_0$ . If  $f : \mathcal{Z}' \rightarrow \mathbb{R} \cup \{\infty\}$  is a convex functional and  $f(Z_t) < \infty$  for  $0 < t < 1$ , then  $t \mapsto f(Z_t)$  is upper semicontinuous on  $[0, 1]$ . If  $f$  is moreover lower semicontinuous (e.g., with respect to  $\mathbb{P}$ -a.s. convergence), then  $t \mapsto f(Z_t)$  is even continuous on  $[0, 1]$ . Due to (4), this argument applies to  $\mathcal{Z}' := \mathcal{Z}$  and  $f(Z) := \gamma(Z)$ . It also works for  $\mathcal{Z}' := \mathcal{Z}^f$  and  $f(Z) := u_Z(x)$ ; see [32, Lemma 3.3].  $\diamond$

**Lemma 4.6** *We have*

$$\begin{aligned} u(x) &= \sup_{g \in \mathcal{C}(x)} \inf_{Q \in \mathcal{Q}} (E_Q[U(g)] + \gamma(Q)) = \inf_{Q \in \mathcal{Q}} (u_Q(x) + \gamma(Q)) \\ &= \sup_{g \in \mathcal{C}(x)} \inf_{Q \in \mathcal{Q}_e} (E_Q[U(g)] + \gamma(Q)) = \inf_{Q \in \mathcal{Q}_e} (u_Q(x) + \gamma(Q)). \end{aligned}$$

**Proof:** Take  $\varepsilon \in (0, 1)$  and let  $c := 1 + u(x + 1) - U(\varepsilon) \wedge 0$  so that

$$u(x + \varepsilon) \geq \sup_{g \in \mathcal{C}(x)} \inf_{Q \in \mathcal{Q}} (E_Q[U(\varepsilon + g)] + \gamma(Q)) = \sup_{g \in \mathcal{C}(x)} \inf_{Z \in \mathcal{Z}(c)} (\mathbb{E}[ZU(\varepsilon + g)] + \gamma(Z)).$$

On the one hand, the function  $U(\cdot + \varepsilon)$  is bounded from below, and so  $Z \mapsto \mathbb{E}[ZU(\varepsilon + g)]$  is a weakly lower semicontinuous affine functional on  $\mathcal{Z}(c)$ . Furthermore,  $Z \mapsto \gamma(Z)$  is also weakly lower semicontinuous, and the set  $\mathcal{Z}(c)$  is convex and weakly compact by Lemma 4.1. On the other hand, for each  $Z \in \mathcal{Z}(c)$ ,  $g \mapsto \mathbb{E}[ZU(\varepsilon + g)]$  is a concave functional defined on the convex set  $\mathcal{C}(x)$ . Thus, the conditions of the lop sided minimax theorem [1, Chapter 6, p. 295] are satisfied, and so

$$\sup_{g \in \mathcal{C}(x)} \inf_{Z \in \mathcal{Z}(c)} (\mathbb{E}[ZU(\varepsilon + g)] + \gamma(Z)) = \inf_{Z \in \mathcal{Z}(c)} \sup_{g \in \mathcal{C}(x)} (\mathbb{E}[ZU(\varepsilon + g)] + \gamma(Z)).$$

Since this expression is bounded above by  $u(x + \varepsilon) < c + U(\varepsilon) \wedge 0$ , we may replace  $\mathcal{Z}(c)$  by  $\mathcal{Z}$ . Hence, we arrive at

$$\begin{aligned} u(x + \varepsilon) &\geq \inf_{Z \in \mathcal{Z}} \sup_{g \in \mathcal{C}(x)} (\mathbb{E}[ZU(\varepsilon + g)] + \gamma(Z)) \geq \inf_{Z \in \mathcal{Z}} \sup_{g \in \mathcal{C}(x)} (\mathbb{E}[ZU(g)] + \gamma(Z)) \\ &\geq \sup_{g \in \mathcal{C}(x)} \inf_{Z \in \mathcal{Z}} (\mathbb{E}[ZU(g)] + \gamma(Z)) = u(x). \end{aligned}$$

Sending  $\varepsilon \downarrow 0$  and using the continuity of  $u$  yields the first part of the lemma.

We still have to show that  $\mathcal{Z}$  may be replaced by  $\mathcal{Z}_e$ . To this end, let  $Z_0 \in \mathcal{Z}^f \setminus \mathcal{Z}_e$ . By assumption (8) there also exists some  $Z_1 \in \mathcal{Z}_e^f$ . Remark 4.5 then gives  $u_{Z_0}(x) = \lim_{t \downarrow 0} u_{Z_t}(x)$ , where  $Z_t := (1 - t)Z_0 + tZ_1 \in \mathcal{Z}_e$  for  $0 < t \leq 1$ . Hence, using the first part of this proof,

$$\begin{aligned} u(x) &= \inf_{Z \in \mathcal{Z}_e} (u_Z(x) + \gamma(Z)) \geq \sup_{g \in \mathcal{C}(x)} \inf_{Q \in \mathcal{Q}_e} (E_Q[U(g)] + \gamma(Q)) \\ &\geq \sup_{g \in \mathcal{C}(x)} \inf_{Q \in \mathcal{Q}} (E_Q[U(g)] + \gamma(Q)) = u(x). \end{aligned}$$

□

**Proof of Theorem 2.4:** By Lemma 4.6, (7), and Lemma 4.4,

$$\begin{aligned} u(x) &= \inf_{Q \in \mathcal{Q}_e} (u_Q(x) + \gamma(Q)) = \inf_{Q \in \mathcal{Q}_e^f} (u_Q(x) + \gamma(Q)) \\ &= \inf_{Q \in \mathcal{Q}_e^f} \inf_{y > 0} (v_Q(y) + \gamma(Q) + xy) = \inf_{y > 0} (v(y) + xy), \end{aligned}$$

which is the first identity in (10).

To prove the second one, we first observe that  $v$  is convex due to the convexity of  $(Z, h) \mapsto \mathbb{E}[ZV(h/Z)] + \gamma(Z)$ . Next we will prove that  $v$  is lower semicontinuous on  $[0, \infty)$  if we define  $v(0) := V(0) := \lim_{y \downarrow 0} V(y)$ . This will then imply that  $v$  is the conjugate function of  $u$  according to standard biduality results; see, e.g., [16, Proposition A.6 (b)]. To this end, take a sequence  $y_n > 0$  converging to  $y \geq 0$ . There is nothing to show if  $\liminf_n v(y_n) = \infty$ , so we may assume that  $\sup_n v(y_n) < \infty$ . By Lemma 4.3 there are  $\hat{h}_n \in \mathcal{D}(y_n)$  and  $\hat{Z}_n \in \mathcal{Z}$  such that  $v(y_n) = \mathbb{E}[\hat{Z}_n V(\hat{h}_n/\hat{Z}_n)] + \gamma(\hat{Z}_n)$ . By (26) we have  $v(y_n) \geq V(y_n) + \gamma(\hat{Z}_n)$ . Since  $\gamma$  is bounded from below, we must necessarily have  $V(y) < \infty$ . Moreover, all  $\hat{Z}_n$  must belong to some  $\mathcal{Z}(c)$  for some finite constant  $c$ . As above, we can pass to a sequence of convex combinations, which converges  $\mathbb{P}$ -a.s. to some  $(\hat{h}, \hat{Z})$ . Using again (23) and Lemma 4.1 yields  $(\hat{h}, \hat{Z}) \in \mathcal{D}(y) \times \mathcal{Z}(c)$ , while convexity, lower semicontinuity, and Lemma 4.1 give  $v(y) \leq \mathbb{E}[\hat{Z}V(\hat{h}/\hat{Z})] + \gamma(\hat{Z}) \leq \liminf_n v(y_n)$ .

The identities in (11) can be proved as in [22, Lemma 3.5]. □

**Proof of Proposition 2.5:** The strict convexity of  $v$  will imply the differentiability of  $u$ ; see, e.g., [29, Theorem V.26.3]. So suppose by way of contradiction that  $0 < y_0 < y_1$  are such that  $v$  is finite and affine on  $[y_0, y_1]$ . By Lemma 4.3 there are  $Z_i \in \mathcal{Z}$  and  $h_i \in \mathcal{D}(y_i)$

such that  $v(y_i) = \mathbb{E}[Z_i V(h_i/Z_i)] + \gamma(Z_i)$ . We let  $\tilde{y} := (y_1 + y_0)/2$ ,  $\tilde{h} := (h_1 + h_0)/2$  etc. Then  $\tilde{h} \in \mathcal{D}(\tilde{y})$  due to (23). Hence, the affinity of  $v$  and (25) imply that

$$\begin{aligned} v(\tilde{y}) &= \frac{v(y_1) + v(y_0)}{2} = \frac{1}{2} \left( \mathbb{E}[Z_1 V(h_1/Z_1)] + \gamma(Z_1) + \mathbb{E}[Z_0 V(h_0/Z_0)] + \gamma(Z_0) \right) \\ &\geq \mathbb{E}[\tilde{Z} V(\tilde{h}/\tilde{Z})] + \gamma(\tilde{Z}) \geq v(\tilde{y}). \end{aligned}$$

Hence, the strict convexity of  $\gamma$  implies that  $\mathbb{P}$ -a.s.  $Z_0 = Z_1$ . But then we must also  $h_1 = h_0$   $\mathbb{P}$ -a.s. on  $\{Z_i > 0\}$ , due to the strict convexity (25). Thus, we get  $v(y_0) = v(y_1)$ . However, taking a strictly positive  $h \in \mathcal{D}(1)$  (e.g. the density of some  $P^* \in \mathcal{M}$ ) we have  $\tilde{h}_1 := h_0 + (y_1 - y_0)h \in \mathcal{D}(y_1)$  and  $\tilde{h}_1 > h_0$  so that

$$v(y_1) \leq \mathbb{E}[Z_0 V(\tilde{h}_1/Z_0)] < \mathbb{E}[Z_0 V(h_0/Z_0)] = v(y_0) = v(y_1),$$

which is the desired contradiction.  $\square$

We turn now to the existence and characterization of optimal strategies.

**Lemma 4.7** *Under condition (12), for all  $x > 0$  there exists some  $\hat{g} \in \mathcal{C}(x)$  such that  $\inf_{Q \in \mathcal{Q}} (E_Q[U(\hat{g})] + \gamma(Q)) = u(x)$ .*

**Proof:** Due to our assumption (12) and [32, Lemma 3.5], we have  $\mathcal{Q}_e^f = \mathcal{Q}_e$ . In particular, we have  $E_Q[U^+(g)] < \infty$  for all  $Q \in \mathcal{Q}_e$  and  $g \in \mathcal{C}(x)$  by (24), and so the expectations  $E_Q[U(g)]$  are defined in the standard way. Moreover,

$$\frac{u_Q(x)}{x} \longrightarrow 0 \quad \text{as } x \uparrow \infty$$

for each  $Q \in \mathcal{Q}_e$ ; see [23, Note 1]. Hence it follows from the proof of [23, Eq. (25)] that the mapping  $\mathcal{C}(x) \ni g \mapsto E_Q[U(g)]$  is upper semicontinuous with respect to  $\mathbb{P}$ -almost-sure convergence (note that the proof of Eq. (25) in [23] does not use the assumption that  $(g^n)$  is a maximizing sequence). Hence,  $\mathcal{C}(x) \ni g \mapsto \inf_{Q \in \mathcal{Q}_e} (E_Q[U(g)] + \gamma(Q))$  is also upper semicontinuous with respect to  $\mathbb{P}$ -almost-sure convergence. Now let  $(\tilde{g}_n)$  be a maximizing sequence in  $\mathcal{C}(x)$ . By the usual Komlos-type argument there is a sequence  $g_n \in \text{conv}\{\tilde{g}_n, \tilde{g}_{n+1}, \dots\}$  converging  $\mathbb{P}$ -a.s. to some  $\hat{g} \geq 0$ . We have  $\hat{g} \in \mathcal{C}(x)$  due to (23). Moreover, the concavity of the functional  $g \mapsto \inf_{Q \in \mathcal{Q}_e} (E_Q[U(g)] + \gamma(Q))$  implies that  $(g_n)$  is again a maximizing sequence, while its upper semicontinuity yields that  $\inf_{Q \in \mathcal{Q}_e} (E_Q[U(\hat{g})] + \gamma(Q)) \geq u(x)$ .

We note next that the set  $\{Q \in \mathcal{Q} \mid E_Q[U^-(\hat{g})] = \infty\}$  must be empty, for otherwise it would have a nonvoid intersection with  $\mathcal{Q}_e$ . Hence, for  $Q \in \mathcal{Q} \setminus \mathcal{Q}_e$  and  $Q_0 \in \mathcal{Q}_e$ ,  $E_Q[U(\hat{g})]$  is the limit as  $t \uparrow 1$  of  $E_{Q_t}[U(\hat{g})]$ , where  $Q_t := tQ + (1-t)Q_0 \in \mathcal{Q}_e$ . By Remark 4.5, we also have  $\gamma(Q_t) \rightarrow \gamma(Q)$ . This shows that we have  $\inf_{Q \in \mathcal{Q}} (E_Q[U(\hat{g})] + \gamma(Q)) \geq u(x)$ .  $\square$

**Proof of Theorem 2.6:** The existence of an optimal strategy  $\widehat{X}$  follows from Lemma 4.7. The assertion that  $u'(\infty-) = 0$  follows from the fact that  $u(x)/x \rightarrow 0$  as  $x \uparrow \infty$ , which is itself a consequence of assumption (12) and [23, Note 1]. The second identity in (13) follows from the first and the duality relations between  $u$  and  $v$ .

Now let  $y > 0$  be such that  $v(y) + xy = u(x)$ . Such a  $y$  exists due to the fact that  $v'(0+) = -\infty$  and  $v'(\infty-) = 0$ . We take a solution  $(\widehat{Q}, \widehat{Y})$  to the dual problem at level  $y$  and denote by  $\widehat{Z}$  the density process of  $\widehat{Q}$  with respect to  $\mathbb{P}$ . By an abuse of notation, we will also write  $\widehat{Z}_T = \widehat{Z}$ . Our next goal is to show that  $(\widehat{Q}, \widehat{X})$  is a saddlepoint for the robust problem. To this end, take any  $Z_1 \in \mathcal{Z}_e$  and let  $Z_t := (1-t)\widehat{Z} + tZ_1 \in \mathcal{Z}_e$  for  $0 < t \leq 1$ .

We first claim that  $v_{Z_t}(y) + \gamma(Z_t) \rightarrow v(y)$  as  $t \downarrow 0$ . To prove this claim, let  $\widehat{h}, h_1 \in \mathcal{D}(y)$  be such that  $v_{\widehat{Z}}(y) = \mathbb{E}[\widehat{Z}V(\widehat{h}/\widehat{Z})]$  and  $v_{Z_1}(y) = \mathbb{E}[Z_1V(h_1/Z_1)]$ , and let  $h_t := (1-t)\widehat{h} + th_1 \in \mathcal{D}(y)$ . By the convexity of  $(y, z) \mapsto zV(y/z)$  we have

$$\begin{aligned} v(y) &\leq v_{Z_t}(y) + \gamma(Z_t) \leq \mathbb{E}\left[Z_t V\left(\frac{h_t}{Z_t}\right)\right] + \gamma(Z_t) \\ &\leq t(v_{Z_1}(y) + \gamma(Z_1)) + (1-t)(v_{\widehat{Z}}(y) + \gamma(\widehat{Z})), \end{aligned}$$

and our claim follows, since the right-hand side tends to  $v(y)$  as  $t \downarrow 0$ .

Next, due to the duality relations (7) between  $v_{Z_t}$  and  $u_{Z_t}$ , we have  $v_{Z_t}(y) + xy \geq u_{Z_t}(x)$ . Moreover, as  $t \downarrow 0$ ,  $u_{Z_t}(x) + \gamma(Z_t)$  tends to  $u_{\widehat{Z}}(x) + \gamma(\widehat{Z})$  according to Remark 4.5. Thus, we obtain

$$u(x) = v(y) + xy = \lim_{t \downarrow 0} (v_{Z_t}(y) + xy + \gamma(Z_t)) \geq \lim_{t \downarrow 0} (u_{Z_t}(x) + \gamma(Z_t)) = u_{\widehat{Z}}(x) + \gamma(\widehat{Z}).$$

Thus, Lemma 4.6 implies that  $u_{\widehat{Z}}(x) + \gamma(\widehat{Z}) = u(x)$ . Now we can conclude that

$$u(x) = u_{\widehat{Z}}(x) + \gamma(\widehat{Z}) \geq \mathbb{E}[\widehat{Z}U(\widehat{X}_T)] + \gamma(\widehat{Z}) \geq \inf_{Q \in \mathcal{Q}} (E_Q[U(\widehat{X}_T)] + \gamma(\widehat{Z})) = u(x),$$

which finishes the proof that  $(\widehat{Q}, \widehat{X})$  is a saddlepoint.

Next, we show that  $\widehat{X}_T$  coincides  $\widehat{Q}$ -a.s. with  $I(\widehat{Y}_T)$ . We have  $0 \leq V(\widehat{Y}_T) + \widehat{X}_T \widehat{Y}_T - U(\widehat{X}_T)$   $\widehat{Q}$ -a.s. and

$$E_{\widehat{Q}}[V(\widehat{Y}_T) + \widehat{X}_T \widehat{Y}_T - U(\widehat{X}_T)] = v(y) + \mathbb{E}[\widehat{X}_T \widehat{Y}_T \widehat{Z}] - u(x) \leq v(y) + xy - u(x) = 0,$$

where we have used (23) and the fact that the process  $\widehat{Y} \widehat{Z}$  belongs to  $\mathcal{Y}(y)$  due to Lemma 4.2. Thus,  $0 = V(\widehat{Y}_T) + \widehat{X}_T \widehat{Y}_T - U(\widehat{X}_T)$  and in turn  $\widehat{X}_T = I(\widehat{Y}_T)$   $\widehat{Q}$ -a.s. We also get  $\mathbb{E}[\widehat{X}_T \widehat{Y}_T \widehat{Z}] = xy$ , and this implies that the process  $\widehat{X} \widehat{Y} \widehat{Z}$  is a  $\mathbb{P}$ -martingale.

We will show next that  $u$  is strictly concave. The continuous differentiability of  $v$  will then follow by general principles (e.g., [29, Theorem V.26.3]) and from the duality relations (10). Suppose by way of contradiction that  $u$  is not strictly concave. Since  $u$  is strictly increasing with  $u'(0+) = \infty$  and  $u'(\infty-) = 0$ , there will be  $0 < x_0 < x_1$  and  $y > 0$  such that  $v(y) + x_i y = u(x_i)$  for  $i = 0, 1$ . Let  $\widehat{X}^i \in \mathcal{X}(x_i)$  be the corresponding optimal solutions, and let  $(\widehat{Q}, \widehat{Y})$  be a solution to the dual problem at level  $y$ . Then we have both  $\widehat{X}_T^0 = I(\widehat{Y}_T) = \widehat{X}_T^1$   $\widehat{Q}$ -a.s. and  $E_{\widehat{Q}}[\widehat{X}_T^0 \widehat{Y}_T] = x_0 y < x_1 y = E_{\widehat{Q}}[\widehat{X}_T^1 \widehat{Y}_T]$ , which is impossible.  $\square$

**Proof of Corollary 2.7:** The existence of a superhedging strategy for  $H$  with initial capital  $x$  follows from Theorem 2.6. That is, we have  $H \in \mathcal{C}(x)$ . Moreover, we have  $\widehat{Y}_T \widehat{Z} \in \mathcal{D}(y)$  by Lemma 4.2, and hence

$$\sup_{h \in \mathcal{D}(y)} \mathbb{E}[Hh] \geq \mathbb{E}[H\widehat{Y}_T \widehat{Z}] = xy,$$

where the equality on the right follows from Theorem 2.6. Hence, due to (23),  $H$  cannot belong to any set  $\mathcal{C}(x')$  with  $x' < x$ .  $\square$

## References

- [1] Aubin, J.-P., Ekeland, I. *Applied nonlinear analysis*. John Wiley & Sons, New York, 1984.
- [2] Barrieu, P., El Karoui, N. *Optimal Derivatives Design under Dynamic Risk Measures*. In: Mathematics of finance, 13–25, Contemp. Math., 351, Amer. Math. Soc., Providence, RI, 2004.
- [3] Ben-Tal, A., Teboulle, M. *Penalty functions and duality in stochastic programming via  $\phi$ -divergence functionals*. Math. Oper. Research **12**, 224-240 (1987).
- [4] Burgert, C., Rüschendorf, L. *Optimal consumption strategies under model uncertainty*. Preprint, 2005. To appear in Stat. Dec.
- [5] Carr, P., Geman, H., Madan, D., Pricing and hedging in incomplete markets. J. Financial Econom. 62 (2001), 131–167.
- [6] Cheridito, P., Delbaen, F., Kupper, M. *Dynamic monetary risk measures for bounded discrete-time processes*. Preprint, 2004.
- [7] Csiszar, I. *Eine informationstheoretische Ungleichung und ihre Anwendung auf den Beweis der Ergodizität von Markoffschen Ketten*. Magyar Tud. Akad. Mat. Kutató Int. Közl. 8 1963 85–108.
- [8] Csiszar, I. *On topological properties of  $f$ -divergences*. Studia. Sci. Math. Hungarica, 329-339 (1967).
- [9] Delbaen, F. *The structure of  $m$ -stable sets and in particular of the set of riskneutral measures*. Preprint, ETH Zürich, 2004.
- [10] Delbaen, F., Schachermayer, W. *A general version of the fundamental theorem of asset pricing*. Math. Ann. 300 (1994), no. 3, 463–520.
- [11] Detlefsen, K., Scandolo, G. *Conditional and dynamic convex risk measures*. To appear in Finance and Stochastics.
- [12] Epstein, L., Schneider, M. *Recursive multiple-priors*. J. Economic Theory 113 (2003), 1-31
- [13] Föllmer, H., Gundel, A. *Robust projections in the class of martingale measures*. Preprint, 2005.
- [14] Föllmer, H., Schied, A., *Convex measures of risk and trading constraints*. Finance Stoch. 6 (4) (2002).

- [15] Föllmer, H., Schied, A., *Robust representation of convex measures of risk*. In: *Advances in Finance and Stochastics. Essays in Honour of Dieter Sondermann*, Springer-Verlag, 2002, 39–56.
- [16] Föllmer, H., Schied, A. *Stochastic Finance: An Introduction in Discrete Time*. Berlin: de Gruyter Studies in Mathematics 27 (2002). Second edition (2004).
- [17] Frittelli, M., Rosazza Gianin, E., Putting order in risk measures. *J. Banking & Finance* 26 (2002), 1473-1486.
- [18] Gilboa, I., Schmeidler, D. *Maxmin expected utility with non-unique prior*. *J. Math. Econ.* 18, 141-153 (1989).
- [19] Gundel, A. *Robust utility maximization for complete and incomplete market models*. To appear in *Finance and Stochastics*.
- [20] Hansen, L., Sargent, T. *Robust control and model uncertainty*. *American Economic Review* 91, 60-66 (2001).
- [21] Jouini, E., Schachermayer, W., Touzi N. *Law invariant risk measures have the Fatou property*. Preprint, Université Paris Dauphine, 2005.
- [22] Kramkov, D., Schachermayer, W. *The asymptotic elasticity of utility functions and optimal investment in incomplete markets*. *Ann. Appl. Probab.* 9, no. 3, 904–950 (1999).
- [23] Kramkov, D., Schachermayer, W. *Necessary and sufficient conditions in the problem of optimal investment in incomplete markets*. *Ann. Appl. Probab.*, Vol. 13, no. 4 (2003).
- [24] Maccheroni, F., Marinacci, M., Rustichini, A. *Ambiguity aversion, malevolent nature, and the variational representation of preferences*. Preprint 2004.
- [25] Krätschmer, V. *Robust representation of convex risk measures by probability measures*. *Stoch. Finance* (to appear).
- [26] Liese, F., Vajda, I. *Convex statistical distances*. Teubner-Texte zur Mathematik 95. Teubner Verlagsgesellschaft, Leipzig, 1987.
- [27] Müller, M. *Market completion and robust utility maximization*. PhD Thesis, Humboldt-Universität zu Berlin (2005).
- [28] Quenez, M.-C. *Optimal portfolio in a multiple-priors model*. *Seminar on Stochastic Analysis, Random Fields and Applications IV*, 291–321, *Progr. Probab.*, 58, Birkhäuser, Basel, 2004.
- [29] Rockafellar, R. T. *Convex analysis*. Princeton Mathematical Series, No. 28 Princeton University Press, Princeton, N.J. 1970. Reprinted 1997.
- [30] Schied, A. *Optimal investments for robust utility functionals in complete market models*. *Math. Oper. Research.* 30, No. 3, 750-764 (2005).
- [31] Schied, A. *On the Neyman-Pearson problem for law-invariant risk measures and robust utility functionals*. *Ann. Appl. Probab.* 14, 1398-1423 (2004).
- [32] Schied, A., Wu, C.-T. *Duality theory for optimal investments under model uncertainty*. Preprint, TU Berlin (2005).

- [33] Schmeidler, D. *Subjective probability and expected utility without additivity*. *Econometrica* 57, no. 3, 571-587 (1989).
- [34] Weber, S. *Distribution-invariant risk measures, information, and dynamic consistency*. To appear in *Math. Finance*.

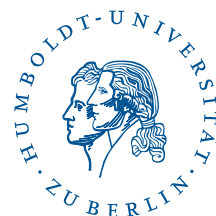
## SFB 649 Discussion Paper Series

For a complete list of Discussion Papers published by the SFB 649, please visit <http://sfb649.wiwi.hu-berlin.de>.

- 001 "Nonparametric Risk Management with Generalized Hyperbolic Distributions" by Ying Chen, Wolfgang Härdle and Seok-Oh Jeong, January 2005.
- 002 "Selecting Comparables for the Valuation of the European Firms" by Ingolf Dittmann and Christian Weiner, February 2005.
- 003 "Competitive Risk Sharing Contracts with One-sided Commitment" by Dirk Krueger and Harald Uhlig, February 2005.
- 004 "Value-at-Risk Calculations with Time Varying Copulae" by Enzo Giacomini and Wolfgang Härdle, February 2005.
- 005 "An Optimal Stopping Problem in a Diffusion-type Model with Delay" by Pavel V. Gapeev and Markus Reiß, February 2005.
- 006 "Conditional and Dynamic Convex Risk Measures" by Kai Detlefsen and Giacomo Scandolo, February 2005.
- 007 "Implied Trinomial Trees" by Pavel Čížek and Karel Komorád, February 2005.
- 008 "Stable Distributions" by Szymon Borak, Wolfgang Härdle and Rafal Weron, February 2005.
- 009 "Predicting Bankruptcy with Support Vector Machines" by Wolfgang Härdle, Rouslan A. Moro and Dorothea Schäfer, February 2005.
- 010 "Working with the XQC" by Wolfgang Härdle and Heiko Lehmann, February 2005.
- 011 "FFT Based Option Pricing" by Szymon Borak, Kai Detlefsen and Wolfgang Härdle, February 2005.
- 012 "Common Functional Implied Volatility Analysis" by Michal Benko and Wolfgang Härdle, February 2005.
- 013 "Nonparametric Productivity Analysis" by Wolfgang Härdle and Seok-Oh Jeong, March 2005.
- 014 "Are Eastern European Countries Catching Up? Time Series Evidence for Czech Republic, Hungary, and Poland" by Ralf Brüggemann and Carsten Trenkler, March 2005.
- 015 "Robust Estimation of Dimension Reduction Space" by Pavel Čížek and Wolfgang Härdle, March 2005.
- 016 "Common Functional Component Modelling" by Alois Kneip and Michal Benko, March 2005.
- 017 "A Two State Model for Noise-induced Resonance in Bistable Systems with Delay" by Markus Fischer and Peter Imkeller, March 2005.
- 018 "Yxilon – a Modular Open-source Statistical Programming Language" by Sigbert Klinke, Uwe Ziegenhagen and Yuval Guri, March 2005.
- 019 "Arbitrage-free Smoothing of the Implied Volatility Surface" by Matthias R. Fengler, March 2005.
- 020 "A Dynamic Semiparametric Factor Model for Implied Volatility String Dynamics" by Matthias R. Fengler, Wolfgang Härdle and Enno Mammen, March 2005.
- 021 "Dynamics of State Price Densities" by Wolfgang Härdle and Zdeněk Hlávka, March 2005.
- 022 "DSFM fitting of Implied Volatility Surfaces" by Szymon Borak, Matthias R. Fengler and Wolfgang Härdle, March 2005.

**SFB 649, Spandauer Straße 1, D-10178 Berlin**  
**<http://sfb649.wiwi.hu-berlin.de>**

This research was supported by the Deutsche  
Forschungsgemeinschaft through the SFB 649 "Economic Risk".



- 023 "Towards a Monthly Business Cycle Chronology for the Euro Area" by Emanuel Mönch and Harald Uhlig, April 2005.
- 024 "Modeling the FIBOR/EURIBOR Swap Term Structure: An Empirical Approach" by Oliver Blaskowitz, Helmut Herwartz and Gonzalo de Cadenas Santiago, April 2005.
- 025 "Duality Theory for Optimal Investments under Model Uncertainty" by Alexander Schied and Ching-Tang Wu, April 2005.
- 026 "Projection Pursuit For Exploratory Supervised Classification" by Eun-Kyung Lee, Dianne Cook, Sigbert Klinke and Thomas Lumley, May 2005.
- 027 "Money Demand and Macroeconomic Stability Revisited" by Andreas Schabert and Christian Stoltenberg, May 2005.
- 028 "A Market Basket Analysis Conducted with a Multivariate Logit Model" by Yasemin Boztuğ and Lutz Hildebrandt, May 2005.
- 029 "Utility Duality under Additional Information: Conditional Measures versus Filtration Enlargements" by Stefan Ankirchner, May 2005.
- 030 "The Shannon Information of Filtrations and the Additional Logarithmic Utility of Insiders" by Stefan Ankirchner, Steffen Dereich and Peter Imkeller, May 2005.
- 031 "Does Temporary Agency Work Provide a Stepping Stone to Regular Employment?" by Michael Kvasnicka, May 2005.
- 032 "Working Time as an Investment? – The Effects of Unpaid Overtime on Wages, Promotions and Layoffs" by Silke Anger, June 2005.
- 033 "Notes on an Endogenous Growth Model with two Capital Stocks II: The Stochastic Case" by Dirk Bethmann, June 2005.
- 034 "Skill Mismatch in Equilibrium Unemployment" by Ronald Bachmann, June 2005.
- 035 "Uncovered Interest Rate Parity and the Expectations Hypothesis of the Term Structure: Empirical Results for the U.S. and Europe" by Ralf Brüggemann and Helmut Lütkepohl, April 2005.
- 036 "Getting Used to Risks: Reference Dependence and Risk Inclusion" by Astrid Matthey, May 2005.
- 037 "New Evidence on the Puzzles. Results from Agnostic Identification on Monetary Policy and Exchange Rates." by Almuth Scholl and Harald Uhlig, July 2005.
- 038 "Discretisation of Stochastic Control Problems for Continuous Time Dynamics with Delay" by Markus Fischer and Markus Reiss, August 2005.
- 039 "What are the Effects of Fiscal Policy Shocks?" by Andrew Mountford and Harald Uhlig, July 2005.
- 040 "Optimal Sticky Prices under Rational Inattention" by Bartosz Maćkowiak and Mirko Wiederholt, July 2005.
- 041 "Fixed-Prize Tournaments versus First-Price Auctions in Innovation Contests" by Anja Schöttner, August 2005.
- 042 "Bank finance versus bond finance: what explains the differences between US and Europe?" by Fiorella De Fiore and Harald Uhlig, August 2005.
- 043 "On Local Times of Ranked Continuous Semimartingales; Application to Portfolio Generating Functions" by Raouf Ghomrasni, June 2005.
- 044 "A Software Framework for Data Based Analysis" by Markus Krätzig, August 2005.
- 045 "Labour Market Dynamics in Germany: Hirings, Separations, and Job-to-Job Transitions over the Business Cycle" by Ronald Bachmann, September 2005.

**SFB 649, Spandauer Straße 1, D-10178 Berlin**  
<http://sfb649.wiwi.hu-berlin.de>

This research was supported by the Deutsche  
 Forschungsgemeinschaft through the SFB 649 "Economic Risk".



- 046 "Paternal Uncertainty and the Economics of Mating, Marriage, and Parental Investment in Children" by Dirk Bethmann and Michael Kvasnicka, September 2005.
- 047 "Estimation and Testing for Varying Coefficients in Additive Models with Marginal Integration " by Lijian Yang, Byeong U. Park, Lan Xue and Wolfgang Härdle, September 2005.
- 048 "Zeitarbeit in Deutschland: Trends und Perspektiven" by Michael C. Burda and Michael Kvasnicka, September 2005.
- 049 "Courtesy and Idleness: Gender Differences in Team Work and Team Competition" by Radosveta Ivanova-Stenzel and Dorothea Kübler, September 2005.
- 050 "Do Factor Shares Reflect Technology?" by Benjamin Bental and Domonique Demougin, September 2005.
- 051 "Optimal investments for risk- and ambiguity-averse preferences: A duality approach" by Alexander Schied, September 2005.

**SFB 649, Spandauer Straße 1, D-10178 Berlin**  
**<http://sfb649.wiwi.hu-berlin.de>**

This research was supported by the Deutsche  
Forschungsgemeinschaft through the SFB 649 "Economic Risk".

