Creating Synthetic Option Strategies for Asset Allocation with Transaction Costs Using Multi-Period Stochastic Programming *

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Abstract

We discuss a new approach to asset allocation with transaction costs. A multi-period stochastic linear programming model is developed where the risk is based on the worst case payoff which is endogenously determined by the model. Utilizing portfolio protection and dynamic hedging, an investment strategy similar to that of a "multiple asset option" on the initial investment portfolio is characterized. The relative changes in the expected terminal wealth, planning target and risk aversion are studied theoretically and illustrated by a numerical example. This model dominates a static mean-variance model when the optimal portfolio is measured by the Sharpe ratio.

1 Introduction

Dynamic asset allocation concerns the selection of asset categories and the accompanying proportion of wealth placed in them over time. A problem is the potential decline of the investment portfolio below some critical limit. Therefore, instead of focusing only on expected return, investors may prefer to control the downside risk. This can be done using option strategies in a multiperiod stochastic linear programming model that considers the distributions of the random returns and transaction costs. Synthetic option strategies provide an approach in managing an investment portfolio with risk control; see e.g. Arnott(1998), Boyle and Vorst (1992), Leland (1985), Leland and Rubinstein(1995), and Tilley and Latainer(1985). This strategy

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characterizes a payoff structure similar to a European call option on the initial portfolio. The investor's portfolio is W_0 at time 0. After T periods, it is desired that the portfolio W_T is worth at least K. What can be done to achieve this goal? A simple answer is to buy or create a put option on the investment portfolio with a strike price equal to the target. A shortcoming of the synthetic option strategy is that the investors' choice of target (the strike price) is exogeneous to asset movements and readjusting strategies. This K has to be "reasonably" chosen, since for a specific investor, different economic situations may influence the choice of the worst case payoff K of his portfolio. For example, Value at Risk (VaR) is a measure of potential change in value of a portfolio of financial instruments with a given probability p over a pre-set horizon. Using VaR, investors maximize the expected return given their VaR at the risk level p. It is preferable to determine K endogenously by the model. The investor may maximize K as an overall potential payoff for all scenarios at the end of planing horizon, i.e. what is the amount that should be insured for the whole planning horizon. In standard synthetic option strategies, the objective of maximizing the expected return is triggered purely by the *ex ante* choice of the target, no matter whether this choice is consistent with asset movements. Increasing K reduces the expected return, hence, the investor's objective should include K as a model variable. While a diversified portfolio is easy to construct, an option market may not be available to the options on this portfolio. Also, investors may buy index futures for hedging purposes and index options for downside risk control. Many traders adopt this strategy and try to manage the downward risk against a benchmark using optimal portfolio replication; see Dembo (1991).

In the finance literature, intertemporal decision making is modeled as dynamic stochastic control problems over discrete or continuous time. The approach lies in finding the implementable optimal policies for each period as a function of the current or past states (observation) in order to meet the investment objective. Multiperiod stochastic programing can also provide a methodology for planning under uncertainty with respect to given constraints. Bradley and Crane (1972) and Kusy

and Ziemba (1986) describe stochastic linear programs for bank asset/liability management. Cariño, Ziemba et al. (1994, 1998ab) discuss the Russell Yausuda Kasai asset/liability management model. Mulvey and Vladimirou (1992) discuss a multiperiod stochastic network model for asset allocation, and Zenios (1993) describes stochastic programming models for fixed-income asset/liability management. Ziemba and Mulvey (1998) survey this field. One of the advantages of using a stochastic programming approach for financial planning is that it can handle "irregular" objective functions and complex constraints. The complexity decreases the solvability when only stochastic control is used. Now that more versatile computer packages for solving mathematical programming modles are available, a stochastic programming model can be easily solved. The model developed here considers transaction costs and does not allow short sales in executing its trading strategy. The objective function depends on all possible states at the horizon given the strategies used in each period with the planning target determined endogenously by the model. This modeling technique facilitates a new approach of allocating asset among cash, bonds and stocks with downside risk control.

2 Dynamic Replication with Transaction Cost

Buying put options has many drawbacks. In an orderly market, this is an executable strategy. However, if the market is not perfect, holding a put option and the stock portfolio has to bear the default risk as well as the risk from biased prices. Exchange-traded options may not be suitable for investors because their strike prices may be biased from their objective. Also, market options may not be written on the securities chosen, and purchasing a put option to protect a stock portfolio requires paying the market price of the option up-front, which may not reflect the true value of the options because a biased pricing formula might have been used.

Assume that the number of states of the world is finite, and time evolves discretely taking the values $\{0, 1, \cdots, T\}$ and there is a filtration of information $(\mathcal{F}_t)_{0 \leq t \leq T}$. Since

the investment in options does not require further injection of funds we assume a self-financing strategy, so the opening value of the portfolio at time $i+1$ is the closing value at time i less transaction costs. The investor can move funds from asset to asset at each period incurring transaction costs. The presence of financial market frictions qualitatively changes the nature of the optimization faced by an investor. It requires one to either act or do nothing, an issue which does not arise in frictionless situations. Transaction costs changes decisions made under "perfect" conditions. See Boyle and Vorst(1992) and Edirisinghe et al.(1993) for studies of replication with transaction costs.

Suppose the investment opportunities consist of n risky assets and a riskless asset. Consider the following notation.

W_0 : investor's initial wealth,

T : planning horizon,

ρ : continuously compound riskless rate of return equal in all periods,

r_t^i : continuously compound rate of return for risky asset i at time t, $i = 1, \cdots, n$ and $t = 0, 1, \cdots, T-1$,

α_t : amount allocated in the riskless asset at time t,

x_t^i : amount allocated to asset i at time t,

A_t^i : additional amount bought of asset i at time t,

D_t^i : additional amount sold of asset i at time t, and

θ_t^i : proportional transaction costs for purchases and sales of asset i at time t.

The initial portfolio satisfies

$$\alpha_0 + x_0^1 + \cdots + x_0^n = W_0. \tag{1}$$

At time $t+1$, investment in the riskless asset is α_{t+1}, where

$$\alpha_t e^{\rho} - \alpha_{t+1} - \sum_{i=1}^{n}(1+\theta_{t+1}^i)A_{t+1}^i + \sum_{i=1}^{n}(1-\theta_{t+1}^i)D_{t+1}^i = 0. \tag{2}$$

Buying and selling the same risky asset at the same time is not optimal with transaction costs. Hence, the following equations must hold for each risky asset at any time $t \in (0, T)$

$$x_t^i e^{r_{t+1}} - x_{t+1}^i + A_{t+1}^i - D_{t+1}^i = 0. \tag{3}$$

Divide the terminal portfolio payoff into two parts, a target K and the surplus over the target. Then the terminal value of the replicating portfolio is characterized by

$$e^{\rho}\alpha_{T-1} + \sum_{i=1}^{n}(1-\theta_T)r_T^i x_{T-1}^i - z - K = 0. \tag{4}$$

Here K is deterministic and z is an $\mathcal{F}_T$ measurable variable. Let ξ_t be the n-dimensional factor $(\xi_t^1, \xi_t^2, \cdots, \xi_t^n)'$, then the portfolio $\mathbf{x_{t+1}}$ not only depends on $\mathbf{x_t}$ but also $\mathbf{A_{t+1}}$ and $\mathbf{D_{t+1}}$.

3 A Stochastic Linear Programming Model

In decision making, one must define the objective. For the option strategy, this is a minimum payoff in all scenarios via the put option. In asset/liability management, a penalty may be subtracted for targets not met as in the Russell-Yasuda model (Cariño and Ziemba et al (1994, 1998ab). Interdependence of the asset movement and the prescribed target is not considered in these models. In asset allocation models, maximizing expected asset value is a primary objective, but the dispersion among scenarios may yield large portfolio losses.

How can this risk be controlled? In mean-variance models, one adjusts the expected value by the variance measure of dispersion. The VaR approach addresses this issue. However it is typically based on a normal distribution assumption which is inconsistent with the evidence of fat tail in real asset markets. Hence, investors should choose their objective function so that the downside risk is considered. We utilize a new approach to measure risk, namely, the reward for the worst case payoff.

Definition 1 *Let $Y > 0$ be a random variable as the terminal portfolio value. Then the worst case payoff of Y is*

$$Y^L = \sup\{K \in \Re; \Pr\{Y \geq K\} = 1\}.$$

The investors' payoff is characterized by the pair (z, W^L). The portfolio (α_t, x_t) is called self financing with transaction cost if Equations (1)-(4) are satisfied.

Assume that the investors' preference between the worst case payoff W^L and the expected surplus over W^L is given by

$$E(z) + \mu W^L \tag{5}$$

where the coefficient μ reflects risk aversion between a target and the expected surplus over this target. Investors are more likely to put wealth in the riskless asset to guarantee a minimum payoff with a large μ. The choice of target and expected surplus are discussed in the next section.

Formulation as a Recourse problem A stochastic programming with recourse problem is a decision problem that maximizes the expected utility gained from the immediate decision at the current stage plus the expected utility that will be realized with constraints satisfied in the second stage. For our problem, the utility received at the first stage is μK and the second stage utility is measured by the expected surplus over the target K. The constraints that need to be satisfied are the allocation of wealth with the front load transaction costs at the first stage and the liability constraints at the second stage. Let $\mathbf{r}$ be the return vector of the risky assets, $\mathbf{x}$ the vector of the wealth portions invested in the risky assets, and θ the transaction cost vector of risky assets. The dynamic recourse problem can be formulated as

$$\begin{aligned} \underset{K,\alpha,x_0}{\textbf{Maximize}} \quad & \mu K + Q_0(K, \alpha_0, \mathbf{x_0}) \\ \textbf{Subject to} \quad & \alpha_0 + \mathbf{x_0}'(1+\theta) = W_0 \\ & \alpha_0, \mathbf{x_0} \geq 0 \end{aligned} \tag{6}$$

where

$$Q_t(K, \alpha_t, \mathbf{x_t}) = \max_{\mathbf{x_{t+1}}} \ \{E\left(Q_{t+1}(K, \alpha_{t+1}, \mathbf{x_{t+1}}, \omega)|\mathcal{F}_t\right); s.t. \ (2) \text{and} \ (3)\}, \quad i = 1, \cdots, T-2,$$

and

$$\begin{aligned} Q_{T-1}(k, \alpha_{T-1}, \mathbf{x_{T-1}}, \omega) = \ & \max \ \ E\left(z(\omega)|\mathcal{F}_{T-1}\right) \\ & \textbf{Subject to} \ \ z(\omega) = \alpha_{T-1}e^{\rho} + (1-\theta)\mathbf{x_{T-1}}'e^{\mathbf{r_T}(\omega)} - K \\ & \qquad\qquad z(\omega) \geq 0, \forall \omega \in \Omega. \end{aligned} \tag{7}$$

It can be proved that the optimal solution K is equal to the worst case payoff W^L of the terminal portfolio value. Since the risk aversion μ in this problem is a constant, the two-period recourse problem has the following simple solution.

Proposition 1 *If there are only two assets, one riskless and one risky, and one investment period, then there exists a μ_0 such that all investors with a $\mu < \mu_0$ will invest fully in the risky asset, and investors with $\mu > \mu_0$ will invest fully in the risk free asset. Investors with $\mu = \mu_0$ will be indifferent between the riskless and risky assets and any combination of them.*

If r follows a normal distribution, then $\mu_0 = e^{\overline{r}+\frac{1}{2}\sigma^2-\rho}$, where $\overline{r} + \frac{1}{2}\sigma^2 - \rho$ is the risk compensated rate of return. For a more general discussion of this concept see Zhao and Ziemba (1999). However, if the number of assets is greater than 2 and the number of periods is greater than 1, then the result is not trivial.

The multi-period stochastic linear programming model. A multi-period stochastic programming model considers the interdependency of uncertainty across the periods of the planning horizon in making decisions. A single period model with roll over cannot replace multi-period models, because the uncertainties across periods are correlated and the future transaction costs may affect the initial decisions on portfolio construction. The model objective is characterized by the immediate rewards after actions have been taking at each period. The first period reward is given by the selected target K and the risk aversion coefficient μ, the intermediate

period rewards are zeros in this model, and the last period reward z is the excess return over the target K. The investor can control the portfolio by readjusting the weights subject to proportional transaction costs, assuming no injection and withdraw of funds. The downside risk control problem can be formulated as the multi-period stochastic programming model

$$
\begin{array}{llllllll}
\underset{\mathbf{x_0}\geq 0, K}{\textbf{Maximize}} & E(\mu K+ & \max\limits_{x_1\geq 0} E(0+ & \max\limits_{x_2\geq 0} E(0+ & \cdots & & \max\limits_{x_{T-1},z\geq 0} E(\mathbf{z}))) & \\
\textbf{Subject to} & \alpha_0 + \mathbf{x_0}'\mathbf{1} & & & & & & = W_0 \\
 & \ddots & & \ddots & & & & \\
 & & & \alpha_t e^{\rho} - \Lambda_{\mathbf{t+1}} & & & & = 0 \\
 & & & \mathbf{x_t e^{r_{t+1}}} - \mathbf{Q_{t+1}} & & & & = 0 \\
 & & & \ddots & & \ddots & & \\
 & & & & \alpha_{T-2} e^{\rho} - & & \Lambda_{\mathbf{T-1}} & = 0. \\
 & & & & \mathbf{x_{T-2} e^{r_{T-1}}} - & & \mathbf{Q_{T-1}} & = 0 \\
 & K- & & & \alpha_{T-1} e^{\rho} + & e^{\mathbf{r}_{T'}} \mathbf{x_{T-1}} + & \mathbf{z} & = 0.
\end{array}
$$

where

$$
\begin{aligned}
\Lambda_{\mathbf{t+1}} &= \alpha_{t+1} - (1+\theta_{\mathbf{t+1}})'\mathbf{A_{t+1}} + (\mathbf{1}-\theta_{\mathbf{t+1}})')\mathbf{D_{t+1}}, \text{and} \\
Q_{t+1} &= \mathbf{x_{t+1}} + \mathbf{A_{t+1}} - \mathbf{D_{t+1}}.
\end{aligned} \tag{8}
$$

Λ_t is equal to the portion of wealth invested in the riskless asset at the start of period t after transaction costs, and Q_t is the portion of the wealth invested in risky assets. The first stage decision variables are the portfolio weights, $\mathbf{x_0} = (x_0^0, \cdots, x_0^n)'$, and the target K. Nonanticipativity is satisfied if the $\mathbf{x_t}$ is $\mathcal{F}_t$-measurable. The optimal K depends upon the choice of μ, the risk aversion parameter. The second and third sets of constraints formulate the self-financing strategy. The last constraint represents that the terminal wealth is no less than the target K which is determined at stage 1.

4 Model Implications

We now discuss how the initial wealth W_0 and investor's reward coefficient μ affect the optimal solution. Denote the optimal solution financed by $(\alpha_t(W_0, \mu), x_t(W_0, \mu))$

by $(z(W_0,\mu), K(W_0,\mu))$ and the optimal objective by $J(W_0,\mu)$

Proposition 2 *For given $\mu \geq 1$,*

(a) $K(W_0,\mu)$ is nonnegative and bounded above by $\rho^T W_0$ if there are no arbitrage opportunities.

(b) $K(W_0,\mu)$ is increasing in μ, and $E(z(W_0,\mu))$ is decreasing in μ.

Proof (a) Assume for some $\mu \geq 1$, $K(W_0,\mu) < 0$. Then, $(z(W_0,\mu) + K(W_0,\mu), 0)$ is also a feasible solution and can be financed by the same optimal portfolio, but the value $E(z(W_0,\mu) + K(W_0,\mu)) > E(z(W_0,\mu)) + \mu K(W_0,\mu)$ since $\mu \geq 1$. Hence, $(z(W_0,\mu), K(W_0,\mu))$ cannot be an optimal pair with $K(W_0,\mu) < 0$, and $K(W_0,\mu) \geq 0$. Given the presence of a riskless asset, $K(W_0,\mu) \leq \rho^T W_0$ if there is no arbitrage opportunity,

(b). Let (z_1, K_1) and (z_2, K_2) be the optimal solutions for μ_1 and μ_2, respectively. By optimality

$$E(z_1) + \mu_2 K_1 \leq E(z_2) + \mu_2 K_2 \tag{9}$$

$$E(z_2) + \mu_1 K_2 \leq E(z_1) + \mu_1 K_1. \tag{10}$$

Combining (9) and (10), yields

$$(\mu_2 - \mu_1)(K_2 - K_1) \geq 0$$

which implies that $K(W_0,\mu)$ is increasing in μ. Similarly, it can be proved that $E(z(W_0,\mu))$ is decreasing with μ.

Proposition 2 shows that, for a risk averse investor, the target K is always positive and is bounded above by the total return if all the fortune is invested in the cash instrument in an arbitrage free market. As the reward coefficient μ increases, the investor is more risk averse and increases the target, therefore, decreases the expected surplus over the target.

Proposition 3 *The optimal value function $J(W_0, \mu)$ is linearly increasing in W_0 and convex increasing in μ*

Proof For $\forall \lambda > 0$, if (α_t, x_t) finances the optimal solution pair $(z(W_0, \mu), K(W_0, \mu))$ then $(\lambda\alpha_t, \lambda x_t)$ finances $(\lambda z(W_0, \mu), \lambda K(W_0, \mu))$, therefore, $J(\lambda W_0, \mu) \geq \lambda J(W_0, \mu)$. Since this inequality is also true for any (W_0, λ), it follows that $J(\lambda W_0, \mu) = \lambda J(W_0, \mu)$.

From (9) and (10), if $\mu_1 \leq \mu_2$, then

$$J(W_0, \mu_1) \leq E(z_1) + \mu_2 K_1 \leq J(W_0, \mu_2) \tag{11}$$

which proves that $J(W_0, \mu)$ is increasing in μ.
Given $\lambda_1, \lambda_2 \geq 0, \lambda_1 + \lambda_2 = 1$, we want to prove that

$$J(W_0, \lambda_1\mu_1 + \lambda_2\mu_2) \leq \lambda_1 J(W_0, \mu_1) + \lambda_2 J(W_0, \mu_2). \tag{12}$$

Let (z^*, K^*) be the optimal solution for $\lambda_1 + \lambda_2$, then

$$E(z_1) + \mu_1 K_1 \geq E(z^*) + \mu_1 K^*, \tag{13}$$

and

$$E(z_2) + \mu_2 K_2 \geq E(z^*) + \mu_2 K^*. \tag{14}$$

Combining (13) and (14) proves (12).

Thus, it is implied in the proof of Proposition 2 that the optimal strategy is linear homogeneous so that the proportion of asset allocation to different asset category is independent of the amount of wealth in each period.

5 Asset Liability and Synthetic Put Option

Institutional investors are often obligated to the future promised cash outflow as a liability. For our model, we use the constant μ to represent investors' risk attitude between a target and its expected surplus. We can extend this to an arbitrary concave function which represents an investor's risk aversion, i.e. the preference regarding downside risk control. The liability constraint is not included in this model. What if

the liability constraint is included? For problems for which the liability constraint is binding, there is an intuitive way to interpret the optimal solution. Investors place part of the wealth in the risky and/or risk free asset and use the rest of the wealth to buy insurance on the initial portfolio that pays the liability. This can be done by a synthetic approach through option replication or by buying put options. The latter method can be implemented only if these options are market traded. The former method is more plausible if market factors are carefully manipulated. We now discuss the relation between the liability constrained and unconstrained model solutions.

How does an investor with risk coefficient μ, endowment M, and a horizon liability design his investment plan? The investor can find the optimal solution by solving the stochastic programming formulated in (8) with an additional liability constraint $K \geq L$. Derivatives can be used for downside control (cf. Cariño and Turner(1998)), but how much should be spent on the derivative asset to optimize the portfolio value with the liability being met? The following theorem illustrates the relation between these two solutions. We call the model without the liability constraint the *unconstrained problem* ;and with the liability constraint the *constrained problem.*

Theorem 1 *Let $(z(W_0, \mu), K(W_0, \mu))$ be the optimal solution to the unconstrained problem with a starting wealth W_0 and the optimal self-financing portfolio is $(\alpha_t, \mathbf{x_t})$. If*

$$\frac{L}{K_{W_0}} = \frac{M}{W_0} := \beta \qquad (\geq 1) \tag{15}$$

then $(\beta z(W_0, \mu), L)$ is the optimal solution to the constrained problem with starting wealth M and the optimal self-financing portfolio is $(\beta\alpha_t, \beta\mathbf{x_t})$.

Proof If $(z(W_0, \mu), K(W_0, \mu), \alpha_t(W_0, \mu), x_t(W_0, \mu))$ is an optimal solution to model (8) with initial wealth W_0, the unconstraint problem, then

$$\beta \cdot (z(M, \mu), K(M, \mu), \alpha_t(M, \mu), x_t(M, \mu))$$

is a feasible solution to the constraint problem with the liability L and the starting wealth M. By Proposition 3, it is actually an optimal solution. The converse of the above argument is also true. This proves Theorem 1.

Investors can make their required insurance plan by solving the unconstrained problem to obtain the optimal target $K_{W_0} = K(W_0, \mu)$ and by solving (15) for W_0 which is then invested in the risk free and/or risky assets. Thus, the rest of the fund $M - W_0$ is used to buy insurance (put option) on his investment portfolio. In this way investors' liabilities at the horizon are guaranteed to be met. This also provides an approach for practitioners to implement derivative strategies.

During any performance measurement period, pension plan or balanced fund managers would like to allocate funds to the risky asset classes (stocks or bonds) that will perform best over the period. Unfortunately, managers cannot know in advance which asset will perform better. The synthetic call option will allow for a change from worse performed assets to better ones. The optimal investment strategy is to invest in the riskless asset that portion of total funds sufficient to achieve a desired minimum return, and use the remaining funds to purchase the "multiple" risky asset call options. This synthetic option is "multiple", because the option will pay off if any one of asset performs well. The execution of this option is done by dynamic replication. Investors will find themselves in the best performing assets eventually at the horizon as they had expected.

6 Asset Returns and Scenario Generation

Generally, there are two ways of modeling future asset returns. The *adaptive expectations* approach depends only on past observations of the explanatory variables. Alternatively, a *rational expectations* model can be used using forecasts produced by conceptually macroeconomic models where expectations are used. The former approach is easy to deal with standard assumptions and past data. The latter becomes a benchmark for the estimation of unobserved expectations, but it assumes that investors have "common" knowledge of the structure of the future events (e.g. coupons

and yields for bonds, dividends and earnings for equities). This paper does not focus on the evaluation of these strategies. We adopt the *adaptive expectations* approach, because modeling future events is as hard as choosing a Data Generating Process that fits historical observations. To model the price interactions between assets, we use the **Vector Auto Regression** model for future asset returns:

$$\mathbf{r_t} = \mathbf{C} + \mathbf{D_1 r_{t-1}} + \mathbf{D_2 r_{t-2}} + \cdots + \mathbf{D_p r_{t-l}} + \epsilon_\mathbf{t}. \tag{16}$$

Then,

$$E(\mathbf{r_t}|\mathcal{F}_{t-1}) = \mathbf{C} + \mathbf{D_1 r_{t-1}} + \mathbf{D_2 r_{t-2}} + \cdots + \mathbf{D_p r_{t-p}}$$

where $\mathbf{r_t}$ is the vector of logrithmic rates of return of the risky asset. ϵ_t is a vector of random disturbances with mean zero which are assumed identically and independently distributed across time periods, and p is the number of lags used in the regression.

The proper number of lags to use is not known *ex ante*. There are three criteria for determining how many to use. The first is to have enough so that $\mathbf{r_{t-p-1}}$ is insignificant in the regression. The second is to have enough so that the assumption that $\epsilon_\mathbf{t}$ is independently and identically distributed is satisfied. The third is not to include unnecessary lags that would reduce the precision of the estimates.

We assume a second order stationary process for asset returns, where the first two moments of the process are independent of time t

$$\begin{aligned} E(\mathbf{r_t}) &= E(\mathbf{r_s}) \\ E(\mathbf{r_{t+h}}'\mathbf{r_{s+h}}) &= E(\mathbf{r_t}'\mathbf{r_s}) \end{aligned} \tag{17}$$

for $\forall s, t, 0 \leq s, t \leq T$.

Proposition 4 *Equation (16) is second-order stationary if* $\forall \lambda$*, such that*

$$|\mathbf{I_n}\lambda^l - \mathbf{D_1}\lambda^{l-1} - \cdots - \mathbf{D_p}| = 0$$

implies $||\lambda|| < 1$.

Proof: see Hamilton (1994).

The stationary process is based on the assumption that the returns and the volatility should be stationary in time. That is the effect that seasonal anomaly returns are negligible; for evidence on this see Fama (1998) and Keim and Ziemba (1999).

Scenario Generation. Generating good scenarios is an important aspect of any stochastic programming application. Mulvey and Thorlacius (1996) describe a global scenario system, developed by Towers Perrin, based on a set of differential equations consistent with the underlying economic factors, such as, price and wage information, interest rates, growth rates, stock dividend yields, etc., for pension plans and insurance companies throughout the world. In the example below we utilize a vector auto-regression model where the current return structure is forecast by past returns. The residuals of the past data will be used to model the disturbances of return for each period.

7 An Integrated Application

Developing financial planning models under uncertainty consists of modeling the investment asset returns, and developing a good strategy within the framework of risk/return tradeoff. Suppose the investments are Cash, Bond and Stock. Assume a one year horizon with quarterly portfolio reviews subject to transactions costs. The Cash returns $\rho = 0.0095$ quarterly. The Solomon Bond index and S&P 500 are used as the bond and stock benchmarks, respectively. Using Quarterly returns from January 1985 to December 1998, see Figure 1, the expected logarithmic rates of return are $ms = 0.04$ for S&P 500 and $mb = 0.019$ for Solomon Bond which we take as the estimates of the unconditional expected rates of return.

An appropriate vector auto regression model of order 2 is estimated as

$$\begin{aligned} s_t &= 0.037 - 0.193 s_{t-1} + 0.418 b_{t-1} - 0.172 s_{t-2} + 0.517 b_{t-2} + \epsilon_t \\ b_t &= 0.007 + 0.140 s_{t-1} + 0.175 b_{t-1} + 0.023 s_{t-2} + 0.122 b_{t-2} + \eta_t. \end{aligned} \tag{18}$$

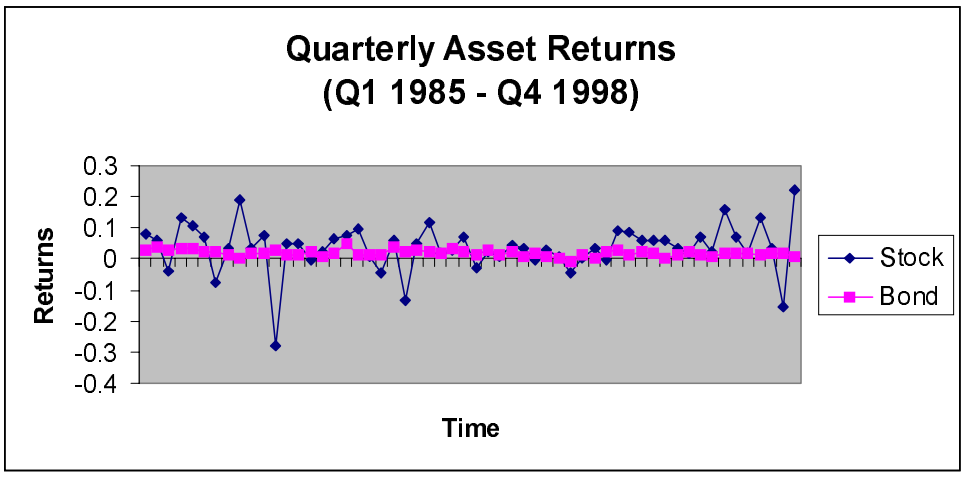

Figure 1: Historical Stock Returns

The first lag coefficients of s_{t-1} and b_{t-1} are statistically significant at the 10% level. The model assumption of an identical disturbances are checked by testing the auto correlation of residuals. The conditions of Proposition 4 is satisfied, therefore, model (18) is second-order stationary as required. Uncertainty is characterized by the pair (ϵ_t, η_t). A random sampling approach is used to estimate the joint distribution of (ϵ_t, η_t), which eventually forms the total number of scenarios for solving the model. Since (ϵ_t, η_t) are identically and independently distributed, we randomly select 20 pairs of (ϵ_t, η_t) to estimate the empirical distribution of one period uncertainty. Then a database of $20 \times 20 \times 20 \times 20 = 160,000$ uncertainty ponits across the horizon is generated. Model (18) is applied for generating the scenarios of the next period rate of return using the last two observations and the period disturbances re-sampled from the big sample. In the process, we may need to adjust the sample mean to zero if the sample size is small. With the random sampling approach, it is not possible to specify *ex ante* the tree structure of the scenarios for solving the model. Instead, we build a random tree for the model from the uncertainty set.

Transaction cost is imposed to prevent investors from buying and selling whenever the gain from transaction is less than the cost. The proportional transaction cost of

$\theta_s = 1\%$ for stock and $\theta_b = 0.5\%$ for bonds are the same for all investment points. The investment environment assumes no shortselling. The results, using the IBM OSL Stochastic Programming package, with varying risk aversion μ, are shown in Table 1.

Table 1 : Risk Aversion, Portfolio Weights, Expected Payoff, Standard Deviation and Sharpe Ratio

Risk Aversion (μ)	Cash (α)	Stock (x1)	Bond (x2)	Target (K)	Expected Payoff	Standard Deviation	Sharpe Ratio
1.0	0.000	100.000	0.000	80.745	127.248	17.289	1.353
1.5	0.000	100.000	0.000	88.609	126.567	18.136	1.252
2.0	24.715	75.285	0.000	91.880	124.132	16.230	1.249
2.5	14.271	35.865	49.865	97.295	118.023	12.205	1.161
3.0	31.763	15.780	52.457	99.963	113.907	9.426	1.067
3.5	47.253	14.760	37.988	100.041	113.731	9.361	1.055
4.0	53.066	14.289	32.645	100.067	113.653	9.336	1.050
4.5	73.237	12.656	14.107	100.156	113.370	9.258	1.028
5.0	73.237	12.656	14.107	100.156	113.370	9.258	1.028
5.5	73.237	12.656	14.107	100.156	113.370	9.258	1.028
6.0	73.237	12.656	14.107	100.156	113.371	9.258	1.028
6.5	73.237	12.656	14.107	100.156	113.371	9.258	1.028
7.0	73.807	12.610	13.583	100.159	113.353	9.262	1.025
7.5	73.807	12.610	13.583	100.159	113.353	9.262	1.025
8.0	73.807	12.610	13.583	100.159	113.352	9.262	1.025
8.5	75.583	12.466	11.951	100.166	113.300	9.254	1.021
9.0	75.583	12.466	11.951	100.166	113.300	9.254	1.021
9.5	79.642	12.138	8.220	100.185	113.142	9.261	1.003
10.0	79.642	12.138	8.220	100.185	113.142	9.261	1.003

As μ increases, i.e. risk aversion increases, investors move funds from stocks to bonds and/ or to cash. This results in a large K and achieves the purpose of downside risk control. The changing weights are strongly dependent on the correlation of the risky assets. Even an extremely risk averse investor should have been willing to invest certain amount in the risky assets, stocks and/or bonds; see Figure 2.

How does this compare to a static mean-variance model? The mean-variance efficient frontier is a line through the market portfolio and the riskless asset. The results derived from this model are used to analyze the performance using the Sharpe ratio, and the comparison is made to the mean variance model. Using the data, the expected annual rate of return is estimated as 0.156 for S&P 500 and 0.067 for Solomon Bond, and the corresponding standard deviations are 0.143 and 0.029. So, The Sharpe ratios for S&P 500 and Solomon Bond for yearly investment are 0.821 and

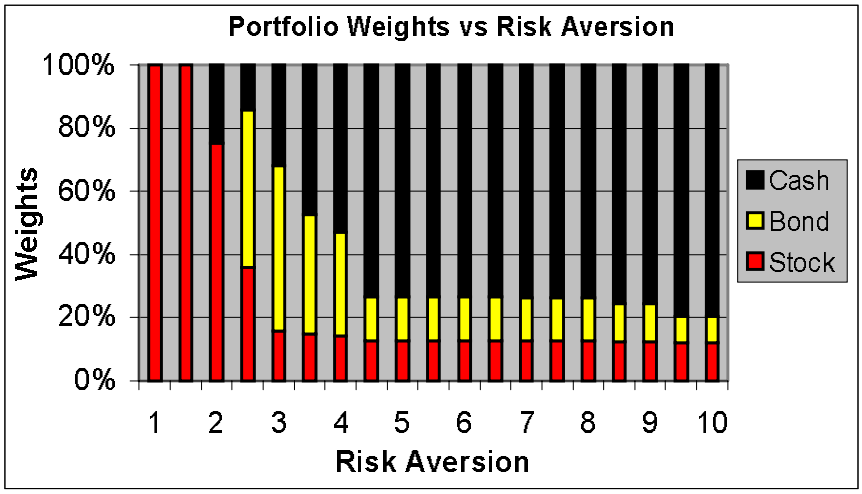

Figure 2: Optimal Weights vs Risk Aversion

0.92, respectively. To locate the mean-variance efficient frontier, we find the portfolio of risky assets that maximizes the Sharpe ratio. The optimal portfolio weights are 0.13 and 0.87 with Sharpe Ratio of 1.18. The low stock weight of 13% occured because the bond was performing extremely well. As shown in Figure 2, that as the portfolio tends to be more risky, *i.e.*, lower risk aversion index, the downside risk control model becomes dominant. This is because of the functioning of dynamic control utilizing the dynamic forecasts which are correlated across investment periods; see Figure 3.

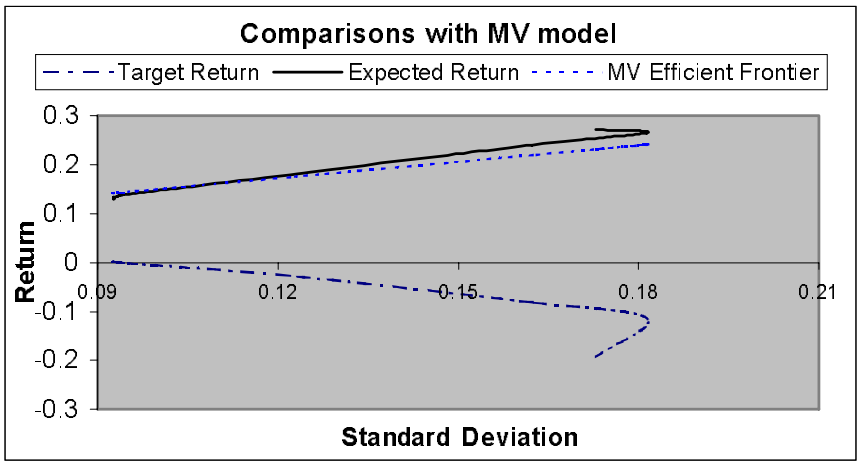

Figure 3: The Mean-Variance Efficient Frontier

The effect of this downside risk control model on the Sharpe ratio is investigated. Not only is this result preferable but also the "downside control" makes the model more practical. The target K is the minimum terminal return. How is that convincing? The answer is given in Figure 3. Even for the most risky portfolio ($\mu = 1$), the floor of the terminal portfolio is above 80%, with an expected rate of return of 27%. The chance of losing more than 20% is zero and the chance of losing more than 10% is less than 7%. Using a mean-variance model, the portfolio would have lost more than 14.3% with propability 10% and would have lost 43% with probability 1%. Figure 4 gives a typical distribution with $\mu = 2.5$. The distribution is highly skewed to the right because of the effect of dynamic downside risk control.

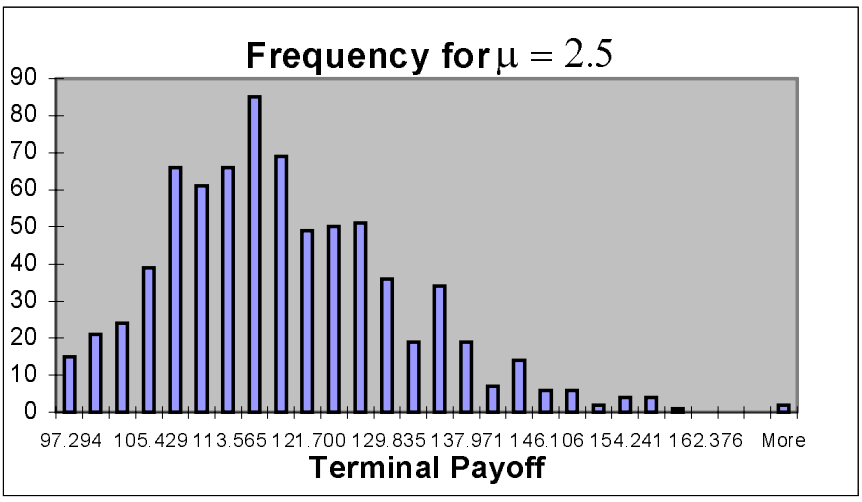

Figure 4: The Distribution of Terminal Portfolio Value

Figure 5 shows the change of weights through the horizon for a typical scenario. At the beginning of the second period, the investor needs to readjust the portfolio weights by buying \$13.118 of stock and all wealth are in the stock in period 3. In period 4, all wealth must be in the bond and finally the investor cashes out his portfolio at the end of horizon. The stream of wealth for this typical scenario is $100 \to 101.435 \to 100.958 \to 119.671 \to 118.644$.

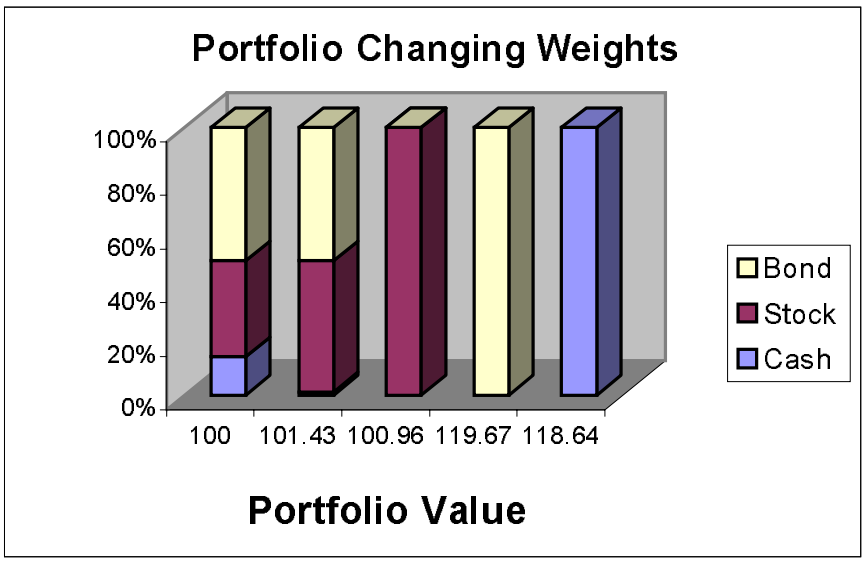

Figure 5: The Change of Weights for A Typical Scenario

8 Conclusions

In this paper, we applied a multiperiod stochastic programming model for financial investment that considers downside risk. A stochastic control approach can dynamically manage long term investment portfolios. Starting with a new feature of risk measure, the reward on the worst case payoff, and utilizing the dynamic hedging, we have shown that the downside risk control model dominates the mean-variance model as the portfolio tends to be more risky in risk aversion sense. Zhao and Ziemba (1999) prove that there exists a μ_0 such that if $\mu < \mu_0$, the dynamic downside risk control model will dominate the mean-variance model, *i.e.*, it will have a higher Sharpe ratio, when the asset prices follow a geometric Brownian motion.

A new feature of this model is that planning target is endogenously determined by the model. Unlike in the traditional insured asset allocation, the strike price or the planning target, depends on investor's risk aversion and asset movements through time. Making the downsize limit K endogenous maximizes planning target. Exogenously picking K would generally be suboptimal. We have shown how to create synthetic option strategies with transaction costs for asset allocation using stochas-

tic programming. We analyzed the changing pattern of optimal portfolios and the optimal target as risk aversion changes.

A possible suggestion for this model is to relax the risk aversion to be random i.e. dependent on past observations of economic variables, instead of being constant through time, or more generally, a concave function of the worst case payoff.

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