MARTINGALE PRICING MEASURES IN INCOMPLETE MARKETS VIA STOCHASTIC PROGRAMMING DUALITY IN THE DUAL OF L^∞

Alan King
kingaj@us.ibm.com

Mathematical Sciences Department
IBM T.J. Watson Research Center
Yorktown Heights, NY

Lisa Korf
korf@math.washington.edu

Department of Mathematics
University of Washington
Seattle, WA

Abstract. We propose a new framework for analyzing pricing theory for incomplete markets and contingent claims, using conjugate duality and optimization theory. Various statements in the literature of the *fundamental theorem of asset pricing* give conditions under which an essentially arbitrage-free market is equivalent to the existence of an equivalent martingale measure, and a formula for the *fair* price of a contingent claim as an expectation with respect to such a measure. In the setting of incomplete markets, the fair price is not attainable as such a particular expectation, but rather as a supremum over an infinite set of equivalent martingale measures. Here, we consider the problem as a stochastic program and derive pricing results for quite general discrete time processes. It is shown that in its most general form, the martingale pricing measure is attainable if it is permitted to be *finitely additive*. This setup also gives rise to a natural way of analyzing models with risk preferences, spreads and margin constraints, and other problem variants. We consider a discrete time, multi-stage, infinite probability space setting and derive the basic results of arbitrage pricing in this framework.

Keywords: arbitrage pricing, conjugate duality, contingent claims, martingales, stochastic program, relatively complete recourse, singular multipliers, finitely additive measures

AMS Classification: 90C15, 90A60, 49N15, 49J99

Date: September 9, 2001

1. Introduction

Since the Nobel prize winning work of Black, Scholes and Merton [2, 12] in the 1970's that developed an arbitrage pricing formula for options, much attention has been devoted to understanding, generalizing, and applying this pricing model and its variants. Their techniques, which are still in use today, involved stochastic differential equations, in particular relying on the assumption that the market price process (in continuous time) behaves like geometric Brownian motion. In that setting, a unique fair option price could be obtained in the form of a linear pricing rule. Since then, however, different perspectives have emerged because of the need to model market price processes that do not necessarily conform to diffusion processes amenable to the stochastic differential equations framework, other types of options and related financial instruments, and situations in which instead of a unique price, a range of fair prices exist.

Dominant among the new mathematical perspectives, in the 1980's Harrison, Pliska and Kreps [8, 7] studied the Black-Scholes formula and rederived results in a functional analytic setting involving the representation of martingales. Harrison and Kreps [7] saw how these results could be generalized in this new setting to the relatively general assumption that the market prices are square integrable. But one should take care to note that the original Black-Scholes type models, which derive the price explicitly from a partial differential equation, were not exactly being challenged by the development of this more general framework because the new framework did not offer a means of obtaining the option price explicitly (or otherwise).

What did come out of this new approach is what is now known as the fundamental theorem of asset pricing, which depends on the concept of no arbitrage (i.e. no guaranteed profit without risk; this condition was automatically satisfied in the geometric Brownian motion setting). The theorem gives conditions under which a market is (essentially) arbitrage-free if and only if there is an equivalent probability measure for which the price process is a martingale. If the measure is unique, the "fair" price can then be determined by taking an expectation with respect to it. It was shown in Harrison and Pliska [8] that in the Black-Scholes model, the martingale measure exists and is unique, hence leading to a unique price, but this is not true in general as will be seen.

Also in contrast to the continuous time stochastic differential equation framework, from the new functional analytic approach did come many variations and generalizations in the late 1980's and 1990's, to problems with incomplete markets, transaction costs, spreads and margins and other important variations as in Jouini and Kallal [9, 10]. These kinds of issues were unmanageable and left relatively untouched in the stochastic differential equations setting.

In [5], Delbaen and Schachermayer extended the fundamental theorem of asset pricing in the functional analytic setting to include incomplete markets in a setting of semimartingale price processes, ultimately providing a supremum formula for pricing via martingale measures. This indicates that the pricing measure may not be attainable in general incomplete markets.

Convexity and duality are at the heart of the functional analytic pricing results, cf. Cvitanic and Karatzas [3]. Because the pricing problems and closely related portfolio optimization problems are so naturally cast in an optimization setting, it is our proposal to analyze these from the perspective of conjugate duality and optimization, following [14]. This leads to the modeling of the pricing problems as well as the related portfolio optimization problems as *stochastic programs*, thus offering a very rich and natural framework for real problem descriptions, inclusion of additional variables (e.g. to include the additional possibility of power production for contracts in the deregulated energy market), constraints, and other problem variations welcome in the stochastic programming setting that appear unnatural in the purely functional analytic framework, and would not appear at all in a stochastic differential equations framework. Additionally and perhaps most significantly, a stochastic programming framework provides an ideal avenue for eventual computation, that is, a way for us to get our hands on fair prices in a general setting.

Duality in stochastic programming on infinite dimensional spaces first appeared in the work of Eisner and Olsen [6] and Wets [21]. Eisner and Olsen proposed an L_p, L_q duality framework for stochastic linear programs with a very specific structure, obtaining theorems of the type $\min P = \sup D$ (where min/max indicates the solution is attainable). At around the same time, Wets made a similar contribution in an L_∞, L_1 duality setting for stochastic linear programs with more general structure but no random recourse matrix. Then in the late 1970's, Rockafellar and Wets joined forces to produce a series of seminal papers on stochastic programming duality for general convex problems in an L_∞, L_1 setting cf. [16, 17, 15, 19]. The reason for this choice of spaces is that one has to deal with tricky constraint qualifications in general L_p spaces unless the problems have a special structure. Out of their work came the significance of the notion of *relatively complete recourse* and the related notion of *induced constraints* (implicit constraints induced on past decision variables by the future) in obtaining strong duality results of the form $\inf P = \max D$ in a very general setting with possibly unbounded constraint sets. Additionally, they considered other duality settings such as pairing continuous functions and measures, which could be useful in a variety of applications, and strong duality theorems ($\inf \mathcal{P} = \max \mathcal{D}$) obtained in an L^∞, $(L^\infty)^*$ setting where singular linear functionals play a role [18]. This latter concept is what we use to arrive at the general results here. In 1987, Back and Pliska [1] proposed a different duality framework for continuous-time models, pairing spaces of functions of bounded variation with a class of measures, and applying this to pricing problems. This last paper is the only attempt the authors are aware of to consider arbitrage pricing in an infinite-dimensional stochastic programming duality setting.

In [11], King takes a first step toward the goal of casting pricing problems for contingent claims in an optimization setting by analyzing what arbitrage and arbitrage pricing mean in discrete time, on a finite probability space, where the incomplete market pricing model may be cast as a linear program. The associated dual problem is analyzed through linear programming duality, leading to the derivation of the generalized pricing results comparable to those in Harrison and Kreps [7] and Harrison and Pliska [8] (but now in a finite optimization setting). In addition, he observes that boundedness of the portfolio optimization problem associated with a contingent claim is equivalent to the no arbitrage condition. It is then demonstrated how naturally one can add risk preferences (utilities), spreads, margins and other variations to the problem and derive/compute their associated (modified) pricing results simply by analyzing/solving the new dual problems obtained. A significant aspect of this work is that the measure yielding the fair price in the dual is always attainable in the discrete time/probability space setting.

The goal of this paper is to use stochastic programming duality of $L^\infty/(L^\infty)^*$ type to set down a natural framework for arbitrage and arbitrage pricing in incomplete markets, that includes attainment of the pricing measures. The approach taken is to generalize the discrete time optimal portfolio and pricing models in King [11] to apply to more general (not finite) probability spaces, to derive arbitrage pricing results for incomplete markets in a multi-stage (discrete-time) stochastic programming duality setting.

Section 2 introduces the requisite terminology from mathematical finance. The two main problems are introduced: First, we introduce a portfolio optimization problem for the seller (writer) of a contingent claim (e.g. an option). Then we pass to the writer's fair price model, which is the feasibility problem associated with the portfolio optimization problem. Section 3 reviews conjugate duality and optimization.

In Section 4, stochastic programming duality of $L^\infty/(L^\infty)^*$ type is applied to the two main problems, to obtain relationships between the optimal values of the original problems and the optimal values of their duals. This in turn leads to equivalent expressions of boundedness of the optimization problems in terms of feasibility of the duals.

In Section 5, it is shown that the unboundedness of the writer's portfolio optimization problem is equivalent to a condition we call *free lunch in the limit* closely related to but slightly weaker than (i.e. implied by) the *free lunch* characterization of arbitrage. It is closely related to the concept of a free lunch with vanishing risk, described in Delbaen [4], but slightly more intuitive from an investor's perspective. Section 6 introduces the necessary concepts from probability theory, culminating in the definition of an equivalent finitely additive martingale measure for a stochastic process.

Through the results in the previous sections, and an analysis of the dual portfolio optimization problems obtained in Section 4, it is shown in Section 7 that there are no free lunches in the limit if and only if there exists an equivalent finitely additive martingale measure for the market price process, thus establishing the fundamental theorem of asset pricing in this setting. It is then shown through examination of solutions to the dual pricing optimization problem that such solutions exist, and that the fair writer's price is given as the expectation with respect to such a solution, when considered as an absolutely continuous finitely additive martingale measure for the market price process.

In each case, the dual variables determine the associated martingale measures, and in this sense the measures can be broken into parts corresponding to the constraints in the primal problem that are being dualized. Induced constraints play a role here which warrants future investigation. A subsequent paper will develop continuous time stochastic programming models, in which trades occur at a finite number of unspecified trading dates, where simple predictable processes are the key to the generalization.

2. The Writer's Problems

We begin with a mathematical overview of the necessary financial terminology. Underlying all of our considerations is the *market*, by which we mean a collection of $J+1$ traded assets indexed by $j = 0, \ldots, J$. Each asset has an initial *market price* at time $t = 0$, and future market prices at times $t = 1, \ldots, T$. The prices are described by a nonnegative vector $S_0 := (S_0^0, \ldots, S_0^J)^* \in \mathbb{R}_+^{J+1}$, of initial known market prices and nonnegative-valued random vectors $S_t : \Xi \to \mathbb{R}_+^{J+1}$ of future market prices, where $(\Xi, \mathcal{F}, P)$ is an underlying probability space with P-complete sigma-algebra $\mathcal{F}$ generated by a filtration $\mathcal{F}_t$ with $\mathcal{F}_T = \mathcal{F}$. It is assumed that the first asset in the price vectors is risk-free in the sense that it is always strictly positive ($S_t^0 > 0$, $t = 0, \ldots, T$). We call this riskless asset the *numeraire*, and proceed immediately to normalize the values of all other assets based on the numeraire's value and obtain the new *discounted price* vectors, $Z_t := S_t / S_t^0$. The numeraire's value is identically one for $t = 0, \ldots, T$, i.e. it is the price by which all other prices are measured. From here on it is harmlessly assumed that all prices and cash flows have been similarly adjusted to reflect this normalization. Prices in the price vector Z_t are assumed $\mathcal{F}_t$-measurable and essentially bounded, i.e. $\operatorname{ess\,sup} |Z_t^j| < \infty$ for $i = 1, \ldots, J$. Henceforth assume all variables to be defined up to measure zero, so that in particular $Z_t \in L^\infty(\Xi, \mathcal{F}_t, P; \mathbb{R}_+^{J+1})$. This is a technical consideration that should be generalizable to square integrable price vectors, in keeping with the results in Harrison and Kreps [7] as well as the case of diffusion processes, however we won't concern ourselves with that generalization in the present paper.

A market is meaningless without the possibility of trading (buying and selling), which we take up next. An *investor* may hold a *portfolio* of shares of assets $j = 0, \ldots, J$, described by a vector $\theta_t := (\theta_t^0, \ldots, \theta_t^J)^*$, $t = 0, \ldots, T$. The investor generally has some initial wealth to invest, and may change his or her portfolio at each time $t = 0, \ldots, T$. The decision of what assets to hold in the portfolio will depend on what the market does. A *trading strategy* describes all investment decisions based on all possible outcomes of the market. Therefore, $\theta := (\theta_0, \theta_1, \ldots, \theta_T)$ describes a trading strategy, where at time $t = 0$, the market prices are known and θ_0 is described by a vector in $I\!R^{J+1}$, and at time t, the market prices are $\mathcal{F}_t$-measurable functions on Ξ, so that $\theta_t : \Xi \to I\!R^{J+1}$ is also $\mathcal{F}_t$-measurable, and describes the portfolio at time t. Note that θ_t is allowed to take on negative values, which corresponds to borrowing. The class of all possible strategies is limited to those which are essentially bounded, i.e. $\operatorname{ess\,sup} |\theta_t^i| < \infty$, $i = 0, \ldots, J$. A *self-financing trading strategy* is one in which no new money is either required or generated to create it. This is expressed by $Z_t \cdot \theta_t = Z_t \cdot \theta_{t-1}$ P-a.s. for all $t = 1, \ldots, T$. It is convenient to adopt the notation

$$\Delta\theta_t := \theta_t - \theta_{t-1}$$

to indicate trading; obviously, $\Delta\theta_t$ is $\mathcal{F}_t$-measurable.

Next we turn to the definition of a *contingent claim*. A contingent claim is a type of contract that is contingent on the underlying market. Precisely, in our setting, it is a promise to pay $F_t : \Xi \to I\!R$ at each time t, where F_t is $\mathcal{F}_t$-measurable. An example of a contingent claim is a European option that offers its buyer the option to buy a certain asset *iota* at fixed times in the future at a certain *strike price* K. If the asset's market price at time t is above the strike price ($Z_t^\iota > K$), then the buyer would exercise the option at price K, immediately sell the asset at its then current price Z_t^ι for a net gain of $Z_t^\iota - K$ dollars. If the asset's market price at time t is less than the strike price, then the buyer would not exercise the option, and gain nothing. Therefore the option is a contingent claim described formally by the payouts,

$$F_t(\xi) := (Z_t^\iota(\xi) - K)^+ = \begin{cases} Z_t^\iota(\xi) - K & \text{if } Z_t^\iota(\xi) > K \\ 0 & \text{if } Z_t^\iota(\xi) \le K \end{cases}, \quad t = 1, \ldots, T.$$

In the model we consider, T is fixed but one could also consider the case in which the payout date is unknown, as is the case with American options. We leave this for subsequent investigation. We assume again that F_t is $\mathcal{F}_t$-measurable and essentially bounded. It could take negative values, as would be the case for futures contracts in which one is obligated to buy or sell an asset at a specified date in the future at a specified price.

The *writer*, or seller of a contingent claim will price the claim at a *fair* price in consideration of the fact that he will be able to invest his earnings from the sale in the market. Assuming for now that this price has been fixed at F_0, one version of the *writer's portfolio optimization problem* is given by

$$\begin{array}{lll} \text{Maximize}_\theta & E\{Z_T \cdot \theta_T\} & \\ & & \\ \text{subject to} & Z_0 \cdot \theta_0 \le F_0 & (\mathcal{P}_w) \\ & Z_t \cdot \Delta\theta_t \le -F_t \quad P\text{-a.s.}, \quad t = 1, \ldots, T & \\ & Z_T \cdot \theta_T \ge 0 \quad P\text{-a.s.}, & \end{array}$$

where $E\{\cdot\} := \int_\Xi \cdot dP(\xi)$ denotes expectation. In other words, the writer wants to maximize the expected terminal wealth by investing the initial endowment (F_0) subject to the conditions that he cover the requisite payouts F_t through profits from trades $Z_t \cdot \Delta\theta_t$ and that the terminal wealth is (almost surely) nonnegative.

This statement of the problem is a particular version of a more general statement

$$\begin{array}{ll} \text{Maximize}_\theta & E\{u(Z_T \cdot \theta_T)\} \\ \\ \text{subject to} & Z_0 \cdot \theta_0 \leq F_0 \\ & Z_t \cdot \Delta\theta_t \leq -F_t \quad P\text{-a.s.}, \quad t = 1, \dots, T, \end{array} \qquad (\mathcal{P}_u)$$

where in the particular instance $\mathcal{P}_w$, the utility function takes the form

$$u_w(v) = \begin{cases} v & \text{if } v \geq 0 \\ -\infty & \text{if } v < 0 \end{cases}$$

The requirement that the writer not lose money in the hedge is modeled by the *effective domain* (denoted $\operatorname{dom} u$) of the utility function, which in that case is the set $[0, +\infty)$. Bringing constraints up into the objective function in this way is standard practice (and a powerful analytical tool) in modern convex and variational analysis, cf. [20].

The generic assumptions on the utility function $u(\cdot)$ will be that $u(\cdot) : I\!R \to \overline{I\!R}$ is concave, strictly increasing, and upper semi-continuous, with $u(v) \to \infty$ as $v \to \infty$. In particular this means that $u(\cdot)$ is a continuous function on the interior of its domain $\operatorname{dom} u$, and is continuous from the right at the boundary of $\operatorname{dom} u$. In addition, the domain $\operatorname{dom} u$ is either all of $I\!R$ or a semi-infinite interval containing $+\infty$, which may be either closed or open depending on the behavior of $u(v)$ as v approaches the boundary of $\operatorname{dom} u$ from the right.

Two canonical utility functions to which our analyses will apply are the function $u_w(\cdot)$ as described above, and the logarithm function $u_l(v) = \log v$. Each of these fits the assumptions and are easily handled in the framework of convex analysis. Their domains are respectively, $\operatorname{dom} u_w = [0, +\infty)$ and $\operatorname{dom} u_l = (0, \infty)$. The boundary of both domains is the origin, 0. The logarithm's value at 0 is *defined* to be $-\infty$, making that function's *hypograph* closed.

Associated with the problem $\mathcal{P}_w$ is a problem which determines *the writer's fair price* as the minimum price F_0 such that $\mathcal{P}_w$ is feasible. This corresponds to the *writer's pricing problem* given by

$$\begin{array}{ll} \text{Minimize} & V \\ \\ \text{subject to} & Z_0 \cdot \theta_0 - V \leq 0 \\ & Z_t \cdot \Delta\theta_t \leq -F_t \quad P\text{-a.s.}, \quad t = 1, \dots, T, \\ & Z_T \cdot \theta_T \geq 0 \quad P\text{-a.s.} \end{array} \qquad (\mathcal{P}_{wp})$$

The pricing problem for the more general statement $\mathcal{P}_u$ is nearly the same; the modification reflects the domain of the writer's utility in the *feasibility* conditions:

$$\begin{array}{ll} \text{Minimize}_{V,\theta} & V \\ \\ \text{subject to} & Z_0 \cdot \theta_0 - V \leq 0 \\ & Z_t \cdot \Delta\theta_t \leq -F_t \quad P\text{-a.s.}, \quad t = 1, \ldots, T, \\ & Z_T \cdot \theta_T \in \operatorname{cl}\operatorname{dom} u \quad P\text{-a.s.} \end{array} \qquad (\mathcal{P}_{up})$$

The analysis to follow will be concerned with establishing conditions under which the problems $\mathcal{P}_u$ and $\mathcal{P}_{up}$ are well-formulated, i.e. conditions that imply that the problems are bounded and feasible, using the tools and techniques of conjugate duality and convex analysis.

The economic meaning of well-formedness of $\mathcal{P}_u$ is straightforward. Boundedness of problem $\mathcal{P}_u$ corresponds to existence of feasible solutions in a certain dual problem $\mathcal{D}_u$ that will be formulated in Section 4. The famous theorem connecting no-arbitrage to the existence of a certain martingale measure will arise from the examination of the feasibility region of this dual. Feasibility of problem $\mathcal{P}_u$ corresponds to an initial payment F_0 that is at least as large as the optimal value of the pricing problem $\mathcal{P}_{up}$. By duality, this value is also the optimal value of the problem dual to $\mathcal{P}_{up}$ which will be shown to have a natural relationship to the dual problem $\mathcal{D}_u$ and thence, to pricing by martingale measures. The attainability of dual solutions corresponds to whether feasible martingale measures exist that actually equal the predicted value.

3. Duality and Optimization

We turn now to review some of the underlying theory of duality and optimization. This is key to all of the results in the sequel. For further details and proofs of the results in this section, consult Rockafellar's concise book, *Conjugate Duality and Optimization* [14].

For a real linear topological space X, let C be a closed convex subset of X, $f_0 : X \to \overline{I\!R}$ be convex and lsc (lower semi-continuous), $f_i : X \to U_i$, $i = 1, \ldots, m$, be continuous convex functions on X with values in linear spaces U_i on which a partial ordering with respect to a closed convex cone E_i has been selected, which for the sake of concreteness and applicability we identify with a nonpositive orthant, so that $u_i \in E_i$ is equivalent to $u_i \leq 0$. Consider the primal optimization problem:

$$\text{Minimize } f_0(x) \text{ subject to } x \in C, \;\; f_i(x) \leq 0, \; i = 1, \ldots, m. \qquad (\mathcal{P})$$

By convention we stick to the setting of minimization, but it is easy to convert a maximization problem into a minimization problem simply by changing the sign in the objective. One may then apply duality in the minimization setting, and convert back to maximization at the end.

For a linear space U, a *perturbation function* $F : X \times U \to \overline{I\!R}$ for $\mathcal{P}$ is a convex function satisfying

$$F(x,0) =: f(x) = \begin{cases} f_0(x) & \text{if } x \in C,\ f_i(x) \le 0,\ i = 1,\dots,m, \\ +\infty & \text{otherwise.} \end{cases} \tag{1}$$

Thus F defines a family of convexly parameterized problems, and the original optimization problem $\mathcal{P}$ may be given by the *full objective function* f. A common choice of F for $\mathcal{P}$ is defined on $X \times U$, where $U = \times_{i=1}^m U_i$, and given by

$$F(x,u) := \begin{cases} f_0(x) & \text{if } x \in C,\ f_i(x) \le u_i,\ i = 1,\dots,m, \\ +\infty & \text{otherwise.} \end{cases}$$

This choice of F is clearly convex and satisfies (1), hence it is a valid perturbation function.

To the linear space U is associated a *dual linear space* Y along with a bilinear form $\langle \cdot,\cdot \rangle : U \times Y \to I\!R$. A topology on U is compatible with this pairing if it is a locally convex topology such that for each $y \in Y$, the linear functionals $u \mapsto \langle u, y\rangle$ are all continuous and every continuous linear functional on U can be represented in this form for some $y \in Y$. Similarly, a topology on Y is compatible with the pairing if it is a locally convex topology such that for each $u \in U$, the linear functionals $y \mapsto \langle u, y\rangle$ are all continuous and every continuous linear functional on Y can be represented in this form for some $u \in U$. It is assumed that U and Y have been equipped with compatible topologies with respect to the given bilinear form.

It is often useful to work with the *Legendre-Fenchel transform*. For an lsc function $f : U \to \overline{I\!R}$, the Legendre-Fenchel *conjugate* of f (in the convex sense) is the function $f^* : Y \to \overline{I\!R}$ given by

$$f^*(y) = \sup_{u \in U} \{\langle y, u\rangle - f(u)\}.$$

The conjugate in the concave sense just replaces "sup" with "inf."

Next we define the Lagrangian function $L : X \times Y \to \overline{I\!R}$,

$$L(x,y) := \inf\{F(x,u) + \langle u, y\rangle \mid u \in U\}.$$

Note that $L(x,y)$ is the negative of the conjugate of $F(x,\cdot)$ evaluated at $-y$.

For the given perturbation function F above, we have

$$L(x,y) = \begin{cases} f_0(x) + \sum_i \langle f_i(x), y_i\rangle & \text{if } x \in C,\ y \ge 0, \\ +\infty & \text{if } x \notin C, \\ -\infty & \text{if } x \in C,\ y \not\ge 0, \end{cases} \tag{2}$$

where $y \ge 0$ refers to the componentwise dual partial ordering induced by the closed convex cones E_i° that are polar to E_i, $i = 1,\dots,m$. The Lagrangian function is closed concave in $y \in Y$ for each $x \in X$, closed convex in $x \in X$ for each $y \in Y$, and satisfies

$$\sup_{y \in Y} L(x,y) = \begin{cases} f_0(x) & \text{if } x \in C,\ f_i(x) \le 0,\ i = 1,\dots,m \\ +\infty & \text{otherwise,} \end{cases}$$

i.e. the supremum over all $y \in Y$ of the Lagrangian function yields the original problem $\mathcal{P}$. This leads to the definition of the problem *dual* to $\mathcal{P}$, defined on Y by

$$\text{maximize } g(y) \text{ so that } y \geq 0, \tag{$\mathcal{D}$}$$

where

$$g(y) := \inf_{x \in X} L(x, y).$$

The concave function g is closed, cf. [14]. Properties relating $\mathcal{P}$ and $\mathcal{D}$ are intimately tied to the convex optimal value function $\varphi : U \to \overline{I\!R}$ defined by

$$\varphi(u) := \inf_{x \in X} F(x, u).$$

This is due to the fact that $\inf \mathcal{P}$, the optimal value of $\mathcal{P}$ is given by $\varphi(0)$, whereas $\sup \mathcal{D}$, the optimal value of $\mathcal{D}$ is equal to $\varphi^{**}(u) = \liminf_{u \to 0} \varphi(u)$, assuming $\mathcal{P}$ is feasible, where $u \to 0$ is in the designated topology. Thus duality results of the form $\inf \mathcal{P} = \sup \mathcal{D}$ reduce to whether $\liminf_{u \to 0} \varphi(u) = \varphi(0)$. In particular, we have the following theorem.

Theorem 3.1 [14, Theorem 15]. *Suppose $\mathcal{P}$ is feasible and* $\liminf_{u \to 0} \varphi(u) \geq \varphi(0)$. *Then* $\inf \mathcal{P} = \sup \mathcal{D}$.

Attainment of dual solutions is equivalent to the subgradient of φ at 0 being nonempty. A condition that ensures this is the continuity of φ at 0. The next theorem provides conditions that imply the continuity of φ at 0, hence the existence of dual solutions.

Theorem 3.2. *Suppose there exists $x \in C$ such that $f_i(x) \in \operatorname{int} E_i$, $i = 1, \ldots, m$. Then* $\inf \mathcal{P} = \max \mathcal{D}$ *(i.e. there exists at least one $\bar{y}$ solving $\mathcal{D}$).*

The condition, "there exists $x \in C$ such that $f_i(x) \in \operatorname{int} E_i$, $i = 1, \ldots, m$," is a strict feasibility condition. In the setting of L^∞, it corresponds to the existence of an $\varepsilon > 0$ such that $f_i(x) \leq -\varepsilon$ almost surely, $i = 1, \ldots, m$. Stochastic programming duality involves a particular choice of X that allows for the description of the evolving probabilistic information present in the problem, as laid out in the next section.

4. $L^\infty/(L^\infty)^*$ Stochastic Programming Duality Applied to Writer's Problems

We are now prepared to derive duality theorems for these problems which will, in Section 7, yield the existence and attainment of a finitely additive martingale measure for the market price process, as well as a formula for the fair price of a contingent claim in terms of this measure. The $L^\infty/(L^\infty)^*$ stochastic programming duality scheme considered here was inspired by [18].

Let's return to the writer's problem $\mathcal{P}_u$ first. In keeping with the duality discussion in §3, with the solution space X now denoted by Θ for convenience, let

$$\Theta := \{\theta = (\theta_0, \ldots, \theta_T) \mid \theta_0 \in I\!R^{J+1},\ \theta_t \in L^\infty(\Xi, \mathcal{F}_t, P; I\!R^{J+1}),\ t = 1, \ldots, T\},$$

equipped with the strong product topology. Let the perturbation space U be defined by

$$U := \{u = (u_0, \ldots, u_T) \mid u_0 \in I\!R,\ u_t \in L^\infty(\Xi, \mathcal{F}_t, P; I\!R),\ t = 1, \ldots, T\}.$$

The dual linear space Y is then

$$Y := \{y = (y_0, \bar{y}_1, \ldots, \bar{y}_T) \mid y_0 \in I\!R,\ \bar{y}_t = (y_t, y_t^0) \in (L^\infty)^*(\Xi, \mathcal{F}_t, P; I\!R),\ t = 1, \ldots, T\},$$

with the compatible topologies the strong product topology on U and the weak* product topology on Y.

It is useful to understand how elements of $(L^\infty)^*$ behave. Each such element $\bar{y}$ may be uniquely decomposed into an L^1 component $\hat{y}$ and a *singular* component y^0. An element $\bar{y}$ of $(L^\infty)^*$ is singular if there exists sets E^n with $P(E^n) \searrow 0$ such that if $z\mathbf{1}_{E^n} = 0$ almost surely for some n, then $\langle \bar{y}, z\rangle = 0$.

The problem which will turn out to be dual to $\mathcal{P}_u$ is

$$\begin{array}{rl}
\text{Minimize}_{y\in Y} & F_0 y_0 - \sum_{t=1}^T E\{F_t y_t\} - \sum_{t=1}^T \langle y_t^0, F_t\rangle - (Eu)^*(y_T, y_T^0) \\[2ex]
\text{subject to} & E\{Z_t y_t \cdot \theta_{t-1}\} + \langle y_t^0, Z_t \cdot \theta_{t-1}\rangle = E\{Z_{t-1} y_{t-1} \cdot \theta_{t-1}\} + \langle y_{t-1}^0, Z_{t-1} \cdot \theta_{t-1}\rangle \\
\text{for all} & \theta_{t-1} \in L^\infty(\Xi, \mathcal{F}_{t-1}, P; I\!R^{J+1}),\ t = 1, \ldots, T, \\
& y \geq 0,
\end{array} \qquad (\mathcal{D}_u)$$

where $Eu : L^\infty(\Xi, \mathcal{F}, P; I\!R) \to I\!R$ is the functional defined by

$$Eu(w) := E\{u(w)\},$$

and $(Eu)^*$ is the conjugate of Eu in the concave sense, defined on $(L^\infty)^*(\Xi, \mathcal{F}, P; I\!R)$, cf. §3. Here, $y \geq 0$ in Y means that $y_0 \geq 0$, $y_t \geq 0$ P-almost surely, and $\langle y_t^0, z\rangle \geq 0$ for all $z \in L_+^\infty(\Xi, \mathcal{F}_t, P; I\!R)$, $t = 1, \ldots, T$.

Theorem 4.1. *$\mathcal{P}_u$ and $\mathcal{D}_u$ are the primal and dual problems associated with the Lagrangian $L : \Theta \times Y \to \overline{I\!R}$ given by*

$$L(\theta, y) := \begin{cases} -E\{u(Z_T \cdot \theta_T)\} + Z_0 \cdot \theta_0 y_0 - F_0 y_0 + \sum_{t=1}^T E\{Z_t \cdot \Delta\theta_t y_t\} \\ + \sum_{t=1}^T \langle y_t^0, Z_t \cdot \Delta\theta_t\rangle + \sum_{t=1}^T E\{F_t y_t\} + \sum_{t=1}^T \langle y_t^0, F_t\rangle & \text{if } y \geq 0 \\ \\ -\infty & \text{otherwise.} \end{cases}$$

Proof. It's straightforward to see that one can obtain the primal problem $\mathcal{P}_u$ from L. We are thinking of $\mathcal{P}_u$ now as a minimization problem, mentally changing the sign in the objective.

$$\sup_{y \in Y} L(\theta, y) = \begin{cases} -E\{u(Z_T \cdot \theta_T)\} & \text{if } Z_0 \cdot \theta_0 \leq F_0 \\ & \quad Z_t \cdot \Delta\theta_t \leq -F_t \ P\text{-a.s.}, \ t = 1, \dots, T \\ +\infty & \text{otherwise} \end{cases}$$

yields the problem to be minimized. Converting the sign of the objective to make it into a maximization problem yields $\mathcal{P}_u$.

For the dual problem $\mathcal{D}_u$, observe that

$$\begin{aligned} \inf_{\theta \in \Theta} L(\theta, y) \;=\; & -F_0 y_0 + \textstyle\sum_{t=1}^T E\{F_t y_t\} + \sum_{t=1}^T \langle y_t^0, F_t \rangle \\ & + \textstyle\sum_{t=0}^{T-1} \inf_{\theta_t} \{E\{[Z_t y_t - Z_{t+1} y_{t+1}] \cdot \theta_t\} + \langle y_t^0, Z_t \cdot \theta_t \rangle - \langle y_{t+1}^0, Z_{t+1} \cdot \theta_t \rangle\} \\ & + \inf_{\theta_T} \{ -E\{u(Z_T \cdot \theta_T)\} + E\{Z_T y_T \cdot \theta_T\} + \langle y_T^0, Z_T \cdot \theta_T \rangle\}. \end{aligned}$$

For fixed $t \in \{0, \dots, T-1\}$,

$$\inf_{\theta_t} \{E\{[Z_t y_t - Z_{t+1} y_{t+1}] \cdot \theta_t\} + \langle y_t^0, Z_t \cdot \theta_t \rangle - \langle y_{t+1}^0, Z_{t+1} \cdot \theta_t \rangle\}$$

$$= \begin{cases} 0 & \text{if } E\{[Z_t y_t - Z_{t+1} y_{t+1}] \cdot \theta_t\} + \langle y_t^0, Z_t \cdot \theta_t \rangle - \langle y_{t+1}^0, Z_{t+1} \cdot \theta_t \rangle = 0 \\ & \text{for all } \theta_t \in L^\infty(\Xi, \mathcal{F}_t, P; I\!R^{J+1}) \\ -\infty & \text{otherwise.} \end{cases}$$

Also,

$$\inf_{\theta_T} \{ -E\{u(Z_T \cdot \theta_T)\} + E\{Z_T y_T \cdot \theta_T\} + \langle y_T^0, Z_T \cdot \theta_T \rangle\} = (Eu)^*(y_T, y_T^0).$$

Putting all of this together, reindexing and changing signs to reflect minimization in the dual yields the dual problem $\mathcal{D}_u$. ◻

Next we derive the duality theorem relating $\mathcal{P}_u$ and $\mathcal{D}_u$.

Theorem 4.2. *Suppose $\mathcal{P}_u$ is strictly feasible. Then $\sup \mathcal{P}_u = \min \mathcal{D}_u$.*

Proof. Strict feasibility of $\mathcal{P}_u$ implies there exists $\varepsilon > 0$, $\theta \in \Theta$, such that

$$\begin{aligned} Z_0 \cdot \theta_0 &\leq F_0 - \varepsilon, \\ Z_t \cdot \Delta\theta_t &\leq -F_t - \varepsilon \quad P\text{-a.s.}, \quad t = 1, \dots, t, \\ Z_T \cdot \theta_T - \varepsilon &\in \operatorname{cl} \operatorname{dom} u \quad P\text{-a.s.} \end{aligned}$$

Thus the assumptions of Theorem 3.2 are satisfied (with E_i the nonpositive orthant), and we immediately obtain the result $\sup \mathcal{P}_u = \min \mathcal{D}_u$ (translating to the setting of maximization). ◻

Note that $\mathcal{P}_u$ bounded, i.e. $\sup \mathcal{P}_u < +\infty$, if and only if $\min \mathcal{D}_u < +\infty$, i.e. $\mathcal{D}_u$ is feasible. This fact will be used in Section 7 to obtain the fundamental theorem of asset pricing (Theorem 7.2).

Next we turn to the writer's pricing problem $\mathcal{P}_{up}$, recalling that it is the feasibility problem for $\mathcal{P}_u$. The solution space now contains V in addition to θ, so that

$$\Theta := \{\theta = (V, \theta_0, \ldots, \theta_T) \mid V \in I\!R,\, \theta_0 \in I\!R^{J+1},\, \theta_t \in L^\infty(\Xi, \mathcal{F}_t, P : I\!R^{J+1}), t = 1, \ldots, T\},$$

equipped with the strong product topology. The perturbation space is

$$\begin{aligned} U \;:= \{u = (u_0, \ldots, u_T, s_t) \mid\, & u_0 \in I\!R,\, u_t \in L^\infty(\Xi, \mathcal{F}_t, P; I\!R),\, t = 1, \ldots, T, \\ & s_T \in L^\infty(\Xi, cF_T, P; I\!R)\}. \end{aligned}$$

The dual linear space is

$$\begin{aligned} Y \;:= \{y = (y_0, \bar{y}_1, \ldots, \bar{y}_T, \bar{x}_T) \mid\, & y_0 \in I\!R,\, \bar{y}_t = (y_t, y_t^0) \in (L^\infty)^*(\Xi, \mathcal{F}_t, P; I\!R),\, t = 1, \ldots, T, \\ & \bar{x}_T = (x_T, x_T^0) \in (L^\infty)^*(\Xi, \mathcal{F}_T, P; I\!R)\}, \end{aligned}$$

with the compatible topologies the strong product topology on U and the weak* product topology on Y.

The problem which will turn out to be dual to $\mathcal{P}_{up}$ is

$$\begin{array}{rl} \text{Maximize}_{y \in Y} & \sum_{t=1}^T E\{F_t y_t\} + \sum_{t=1}^T \langle y_t^0, F_t\rangle + E\{\alpha x_T\} + \langle x_T^0, \alpha \mathbf{1}\rangle \\ \\ \text{subject to} & E\{Z_t y_t \cdot \theta_{t-1}\} + \langle y_t^0, Z_t \cdot \theta_{t-1}\rangle = E\{Z_{t-1} y_{t-1} \cdot \theta_{t-1}\} + \langle y_{t-1}^0, Z_{t-1} \cdot \theta_{t-1}\rangle \\ \text{for all} & \theta_{t-1} \in L^\infty(\Xi, \mathcal{F}_{t-1}, P; I\!R^{J+1}), \quad t = 1, \ldots, T, \\ & x_T^0 = y_T^0,\; y_T = x_T \; P\text{-a.s.},\; y_0 = 1,\; y \geq 0 \end{array} \tag{$\mathcal{D}_{up}$}$$

where $\alpha := \inf\{\operatorname{dom} u\}$, and $\mathbf{1} := 1$ almost surely.

Theorem 4.3. *$\mathcal{P}_{up}$ and $\mathcal{D}_{up}$ are the primal and dual problems associated with the Lagrangian $L : \Theta \times Y \to \overline{I\!R}$ given by*

$$L(\theta, y) := \begin{cases} V + Z_0 \cdot \theta_0 y_0 - V y_0 + \sum_{t=1}^T E\{Z_t \cdot \Delta\theta_t y_t\} \\ + \sum_{t=1}^T \langle y_t^0, Z_t \cdot \Delta\theta_t\rangle + \sum_{t=1}^T E\{F_t y_t\} + \sum_{t=1}^T \langle y_t^0, F_t\rangle \\ -E\{Z_T \cdot \theta_T x_T\} - \langle x_T^0, Z_T \cdot \theta_T\rangle + E\{\alpha x_T\} + \langle x_T^0, \alpha \mathbf{1}\rangle & \text{if } y \geq 0 \\ \\ -\infty & \text{otherwise.} \end{cases}$$

Proof. Again, obtaining the primal problem $\mathcal{P}_{up}$ from L is straightforward.

$$\sup_{y \in Y} L(\theta, y) = \begin{cases} V & \text{if } Z_0 \cdot \theta_0 - V \leq 0 \\ & \quad Z_t \cdot \Delta\theta_t \leq -F_t \; P\text{-a.s.},\, t = 1, \ldots, T \\ & \quad Z_T \cdot \theta_T \in \operatorname{cl} \operatorname{dom} u \; P\text{-a.s.} \\ \\ +\infty & \text{otherwise} \end{cases}$$

yields $\mathcal{P}_{up}$.

For the dual problem $\mathcal{D}_{up}$, observe that

$$\begin{array}{rcl}\inf_{\theta\in\Theta} L(\theta,y) &=& \sum_{t=1}^T E\{F_t y_t\}+\sum_{t=1}^T\langle y_t^0, F_t\rangle + E\{\alpha x_T\}+\langle x_T^0,\alpha\mathbb{1}\rangle+\inf_V\{V-Vy_0\}\\ && +\sum_{t=0}^{T-1}\inf_{\theta_t}\left\{E\{[Z_t y_t - Z_{t+1}y_{t+1}]\cdot\theta_t\}+\langle y_t^0, Z_t\cdot\theta_t\rangle - \langle y_{t+1}^0, Z_{t+1}\cdot\theta_t\rangle\right\}\\ && +\inf_{\theta_T}\left\{E\{Z_T(y_T-x_T)\cdot\theta_T\}+\langle y_T^0, Z_T\cdot\theta_T\rangle - \langle x_T^0, Z_T\cdot\theta_T\rangle\right\}.\end{array}$$

We have

$$\inf_V\{V-Vy_0\}=\begin{cases}0 & \text{if } y_0=1\\ -\infty & \text{otherwise.}\end{cases}$$

For fixed $t\in\{0,\dots,T-1\}$,

$$\inf_{\theta_t}\left\{E\{[Z_t y_t - Z_{t+1}y_{t+1}]\cdot\theta_t\}+\langle y_t^0, Z_t\cdot\theta_t\rangle - \langle y_{t+1}^0, Z_{t+1}\cdot\theta_t\rangle\right\}$$

$$=\begin{cases}0 & \text{if } E\{[Z_t y_t - Z_{t+1}y_{t+1}]\cdot\theta_t\}+\langle y_t^0, Z_t\cdot\theta_t\rangle - \langle y_{t+1}^0, Z_{t+1}\cdot\theta_t\rangle = 0\\ & \text{for all } \theta_t\in L^\infty(\Xi,\mathcal{F}_t,P;\mathbb{R}^{J+1})\\ -\infty & \text{otherwise.}\end{cases}$$

Also,

$$\inf_{\theta_T}\left\{E\{Z_T(y_T-x_T)\cdot\theta_T\}+\langle y_T^0, Z_T\cdot\theta_T\rangle - \langle x_T^0, Z_T\cdot\theta_T\rangle\right\}=\begin{cases}0 & \text{if } y_T=x_T\ P\text{-a.s.}\\ & \quad y_T^0=x_T^0,\\ -\infty & \text{otherwise.}\end{cases}$$

Putting all of this together and reindexing yields the dual problem $\mathcal{D}_{up}$. □

Next we derive the duality theorem relating $\mathcal{P}_{up}$ and $\mathcal{D}_{up}$.

Theorem 4.4. *$\mathcal{P}_{up}$ is strictly feasible, and* $\inf\mathcal{P}_{up}=\max\mathcal{D}_{up}$.

Proof. The strict feasibility of $\mathcal{P}_{up}$ is due to the fact that $\mathcal{P}_{up}$ is the feasibility problem for $\mathcal{P}_u$. Simply fix an $\varepsilon>0$, and let

$$\begin{array}{rcl}V &:=& \alpha+(T+2)\varepsilon+\operatorname{ess\,sup}\sum_{t=1}^T F_t\\ \theta_0 &:=& \begin{pmatrix}V-\varepsilon\\ \vec{0}\end{pmatrix}\\ \theta_t &:=& \begin{pmatrix}-\sum_{\tau=1}^t F_\tau-(t+1)\varepsilon+V\\ \vec{0}\end{pmatrix},\quad t=1,\dots,T.\end{array}$$

This is a strictly feasible point, i.e. it satisfies

$$\begin{array}{rcl}Z_0\cdot\theta_0 - V = V-\varepsilon-V &=& -\varepsilon\\ Z_t\cdot\Delta\theta_t &=& -F_t-\varepsilon\quad P\text{-a.s.},\quad t=1,\dots,T,\end{array}$$

and

$$Z_T \cdot \theta_T - \varepsilon = -\sum_{t=1}^{T} F_t - (T+1)\varepsilon + \alpha + (T+2)\varepsilon + \operatorname{ess\,sup} \sum_{t=1}^{T} F_t - \varepsilon \geq \alpha \in \operatorname{cl} \operatorname{dom} u \quad P\text{-a.s.}$$

Thus the assumptions of Theorem 3.2 are satisfied (with E_i the nonpositive orthant), and we immediately obtain the result $\inf \mathcal{P}_{up} = \max \mathcal{D}_{up}$. □

5. No Free Lunch in the Limit

The very important concept of *arbitrage* in the market, loosely the ability to generate positive wealth with no risk, is what we concern ourselves with next. The market is said to admit *no free lunches* if there are no self-financing trading strategies with zero initial wealth, nonnegative terminal wealth, and with a positive probability of strictly positive terminal wealth. Mathematically, we may write the concept of a free lunch concisely as

$$\begin{aligned} Z_0 \cdot \theta_0 &= 0 \\ Z_t \cdot \Delta\theta_t &= 0 \quad P\text{-a.s.}, \ t = 1, \dots, T \\ Z_T \cdot \theta_T &\geq 0 \quad P\text{-a.s.} \\ E\{Z_T \cdot \theta_T\} &> 0 \end{aligned}$$

where $E\{\cdot\}$ again refers to the expectation of a random variable with respect to the measure P. Slightly stronger than the no free lunch condition, and used extensively in Delbaen and Shachermayer [5] to obtain asset pricing theorems, is the concept of *no free lunch with vanishing risk* (NFLVR). This condition says there should be no sequence of final wealths $Z_T \cdot \theta_T^\nu$ such that the negative parts tend to zero uniformly and such that $Z_T \cdot \theta_T^\nu$ tends almost surely to a nonnegative-valued random variable that is strictly positive with positive probability. Put concisely, there should be no sequence of trading strategies satisfying

$$\begin{aligned} Z_0 \cdot \theta_0^\nu &= 0 \\ Z_t \cdot \Delta\theta_t^\nu &= 0 \quad P\text{-a.s.}, \quad t = 1, \dots, T \\ Z_T \cdot \theta_T^\nu &\geq -\varepsilon^\nu \quad P\text{-a.s.} \\ \lim_\nu Z_T \cdot \theta_T^\nu &= X \geq 0 \quad P\text{-a.s.} \\ E\{X\} &> 0, \end{aligned}$$

where $\varepsilon^\nu \to 0$.

We propose a third concept that is closely related to NFLVR, but even more intuitive from an investor's perspective. *No free lunch in the limit* (NFLIL) means that there is no sequence of trading strategies satisfying

$$\begin{aligned} Z_0 \cdot \theta_0^\nu &= 0 \\ Z_t \cdot \Delta\theta_t^\nu &= 0 \quad P\text{-a.s.}, \quad t = 1, \ldots, T \\ Z_T \cdot \theta_T^\nu &\geq -\varepsilon^\nu \quad P\text{-a.s.} \\ \lim_\nu E\{Z_T \cdot \theta_T^\nu\} &> 0 \end{aligned}$$

The difference between this concept and NFLVR is that the terminal wealth sequence here is not required to converge to a random variable, a somewhat artificial requirement. The next theorem demonstrates the relationship between these three concepts.

Theorem 5.1. *NFLIL implies NFLVR implies no free lunches.*

Proof. We begin with NFLIL implies NFLVR, by showing that a free lunch with vanishing risk implies the existence of a free lunch in the limit. Assuming the existence of a free lunch with vanishing risk, let $\varepsilon^\nu \searrow 0$, and let θ^ν be a sequence of trading strategies satisfying

$$\begin{aligned} Z_0 \cdot \theta_0^\nu &= 0 \\ Z_t \cdot \Delta\theta_t^\nu &= 0 \quad P\text{-a.s.}, t = 1, \ldots, T \\ Z_T \cdot \theta_T^\nu &\geq -\varepsilon^\nu \quad P\text{-a.s.} \\ \lim_\nu Z_T \cdot \theta_T^\nu &= X \geq 0 \quad P\text{-a.s.} \\ E\{X\} &> 0. \end{aligned}$$

Then, by Fatou's lemma,

$$\begin{aligned} \liminf_\nu E\{Z_T \cdot \theta_T^\nu\} &\geq E\{\liminf_\nu Z_T \cdot \theta_T^\nu\} \\ &= E\{X\} > 0, \end{aligned}$$

whereby a subsequence of the trading strategy is a free lunch in the limit.

We next show that NFLVR implies no free lunches, by showing that a free lunch implies the existence of a free lunch with vanishing risk. Suppose there exists a free lunch, that is a trading strategy which satisfies

$$\begin{aligned} Z_0 \cdot \theta_0 &= 0 \\ Z_t \cdot \Delta\theta_t &= 0 \quad P\text{-a.s.}, \quad t = 1, \ldots, T \end{aligned}$$

$$\begin{array}{rcl} Z_T \cdot \theta_T & \geq & 0 \quad P\text{-a.s.} \\ E\{Z_T \cdot \theta_T\} & > & 0. \end{array}$$

Let $\theta^\nu := \theta$ for all $\nu \in I\!N$. This creates the desired free lunch with vanishing risk, which completes the proof. ◻

The next theorem equates NFLIL with the boundedness of the particular writer's portfolio optimization problem $\mathcal{P}_w$, a significant feature that comes out of our approach. The optimization problems under consideration are *strictly feasible* if there is an $\varepsilon > 0$ such that the problems are still feasible when the inequality constraints (including the implicit constraints governed by $\operatorname{dom} u$) are modified to become stricter by a factor of ε.

Theorem 5.2. *Suppose $\mathcal{P}_w$ is strictly feasible, with $F_0 > \operatorname{ess\,inf}(\sum_{t=1}^T F_t)$ in $\mathcal{P}_w$. Then the following are equivalent.*

(a) *$\mathcal{P}_w$ is bounded,*

(b) *The market admits NFLIL.*

Proof. We will show that (a) $\iff$ (b) by contrapositive. First suppose that (a) does not hold, i.e. $\mathcal{P}_w$ is unbounded (it is feasible by assumption). Then there is a sequence of trading strategies satisfying

$$\begin{array}{rcl} Z_0 \cdot \theta_0^\nu & \leq & F_0 \\ Z_t \cdot \Delta\theta_t^\nu & \leq & -F_t \quad P\text{-a.s.}, \quad t = 1, \ldots, T \\ Z_T \cdot \theta_T^\nu & \geq & 0 \quad P\text{-a.s.} \\ E\{Z_T \cdot \theta_T^\nu\} & \nearrow & +\infty. \end{array}$$

Note that $\beta := \frac{1}{F_0 - \operatorname{ess\,inf}(\sum_{t=1}^T F_t)} > 0$ by assumption. Let $0 < \varepsilon^\nu := \frac{1}{E\{Z_T \cdot \theta_T^\nu\}} \searrow 0$. Let $\gamma^\nu = \varepsilon^\nu \beta$ and note in particular that $\gamma^\nu > 0$. Let

$$\bar{\theta}_0^\nu := \gamma^\nu \Bigg(\theta_0^\nu + \begin{pmatrix} F_0 - \operatorname{ess\,inf}(\sum_{t=1}^T F_t) - Z_0 \cdot \theta_0^\nu \\ \vec{0} \end{pmatrix} \Bigg) - \begin{pmatrix} \varepsilon^\nu \\ \vec{0} \end{pmatrix},$$

and for $t = 1, \ldots, T$,

$$\bar{\theta}_t^\nu := \gamma^\nu \Bigg(\theta_t^\nu + \begin{pmatrix} -\operatorname{ess\,inf}(\sum_{t=1}^T F_t) - \sum_{t=1}^t Z_t \cdot \Delta\theta_t^\nu \\ \vec{0} \end{pmatrix} \Bigg) - \begin{pmatrix} \varepsilon^\nu \\ \vec{0} \end{pmatrix}.$$

Then,

$$Z_0 \cdot \bar{\theta}_0^\nu \quad = \quad \gamma^\nu \Big(Z_0\theta_0^\nu + F_0 - Z_0\theta_0^\nu - \operatorname{ess\,inf}(\sum_{t=1}^T F_t) \Big) - \varepsilon^\nu$$

$$
\begin{aligned}
&= \gamma^\nu\Big(F_0 - \operatorname{ess\,inf}\,(\sum_{t=1}^T F_t)\Big) - \varepsilon^\nu \\
&= \varepsilon^\nu - \varepsilon^\nu = 0.
\end{aligned}
$$

Also, the self-financing condition holds:

$$
\begin{aligned}
Z_t \cdot \Delta\bar{\theta}_t^\nu &= \gamma^\nu\Big(Z_t \cdot \Delta\theta_t^\nu - Z_t \cdot \Delta\theta_t^\nu\Big) \\
&= 0.
\end{aligned}
$$

The negative part of the terminal wealth sequence converges uniformly to 0, as given by

$$
\begin{aligned}
Z_T \cdot \bar{\theta}_T^\nu &= \gamma^\nu\Big(Z_T \cdot \theta_T^\nu - \operatorname{ess\,inf}\,(\sum_{t=1}^T F_t) - \sum_{t=1}^T Z_t \cdot \Delta\theta_t^\nu\Big) \\
&\geq \gamma^\nu\Big(Z_T \cdot \theta_T^\nu - \operatorname{ess\,inf}\,(\sum_{t=1}^T F_t) + \sum_{t=1}^T F_t\Big) - \varepsilon^\nu \\
&\geq -\varepsilon^\nu.
\end{aligned}
$$

Finally, we have that the expected terminal wealth is positive in the limit:

$$
\begin{aligned}
E\{Z_T \cdot \bar{\theta}_T^\nu\} &\geq \gamma^\nu\Big(E\{Z_T \cdot \theta_T^\nu\} - \operatorname{ess\,inf}\,(\sum_{t=1}^T F_t) + E\{\sum_{t=1}^T F_t\}\Big) - \varepsilon^\nu \\
&= \beta\Big(1 + \varepsilon^\nu\Big(E\{\sum_{t=1}^T F_t\} - \operatorname{ess\,inf}\,(\sum_{t=1}^T F_t)\Big)\Big) - \varepsilon^\nu,
\end{aligned}
$$

so that

$$
\liminf_\nu E\{Z_T \cdot \bar{\theta}_T^\nu\} \geq \beta > 0.
$$

Thus a subsequence of $\bar{\theta}_T^\nu$ is a free lunch in the limit, establishing (b) $\implies$ (a).

Now let the market admit a free lunch in the limit θ^ν for a sequence $(\varepsilon^\nu)^2 \searrow 0$. By the strict feasibility of $\mathcal{P}_w$, for large enough $\bar{\nu}$, $\nu > \bar{\nu}$, the problem

$$
\begin{array}{ll}
\text{Maximize} & E\{Z_T \cdot \theta_T\} \\
\text{subject to} & Z_0 \cdot \theta_0 \leq F_0 \\
& Z_t \cdot \Delta\theta_t \leq -F_t \quad P\text{-a.s.}, \quad t = 1, \ldots, T \\
& Z_T \cdot \theta_T \geq \varepsilon^\nu \quad P\text{-a.s.}
\end{array}
$$

is feasible, so let $\bar{\theta}^\nu$ be such a feasible point for each $\nu > \bar{\nu}$. Let $\hat{\theta}^\nu := \bar{\theta}^\nu + (\varepsilon^\nu)^{-1}\theta^\nu$. Then $\hat{\theta}^\nu$ is feasible for $\mathcal{P}_w$, since

$$
Z_0 \cdot \hat{\theta}_0^\nu = Z_0 \cdot \bar{\theta}_0^\nu + Z_0 \cdot (\varepsilon^\nu)^{-1}\theta_0^\nu \leq F_0,
$$

and for $t = 1, \ldots, T$,

$$
Z_t \cdot \Delta\hat{\theta}_t^\nu = Z_t \cdot \Delta\bar{\theta}_t^\nu + (\varepsilon^\nu)^{-1} Z_t \cdot \Delta\theta_t^\nu \leq -F_t.
$$

Also,

$$\begin{aligned} Z_T \cdot \hat{\theta}_T^\nu &= Z_T \cdot \bar{\theta}_T^\nu + (\varepsilon^\nu)^{-1} Z_T \cdot \theta_T^\nu \\ &\geq Z_T \cdot \bar{\theta}_T^\nu - (\varepsilon^\nu)^{-1}(\varepsilon^\nu)^2 \\ &= \varepsilon^\nu - \varepsilon^\nu \geq 0, \end{aligned}$$

and

$$\begin{aligned} E\{Z_T \cdot \hat{\theta}_T^\nu\} &= E\{Z_T \cdot \bar{\theta}_T^\nu\} + (\varepsilon^\nu)^{-1} E\{Z_T \cdot \theta_T^\nu\} \\ &\geq \varepsilon^\nu + (\varepsilon^\nu)^{-1} E\{Z_T \cdot \theta_T^\nu\}. \end{aligned}$$

Since $\lim_\nu E\{Z_T \cdot \theta_T^\nu\} > 0$, we have

$$\lim_\nu E\{Z_T \cdot \hat{\theta}_T^\nu\} = +\infty.$$

We have shown that $\sup \mathcal{P}_w = +\infty$, i.e. $\mathcal{P}_w$ is unbounded, establishing (a) $\implies$ (b). $\square$

It is not possible to equate the boundedness for general $\mathcal{P}_u$ with the no free lunch conditions, however one may do so with some further assumptions on the utility u.

Theorem 5.3. *Suppose the utility function* $u : \mathbb{R} \to \overline{\mathbb{R}}$ *satisfies* $\lim_{x \to \alpha^+} u(x) > -\infty$ *where* $\alpha := \inf\{\operatorname{dom} u\} > -\infty$, *and* $\lim_{x\to\infty} \frac{u(x)}{x} = c > 0$. *Suppose* $\mathcal{P}_u$ *is strictly feasible, with* $F_0 > \operatorname{ess\,inf}\,(\sum_{t=1}^T F_t)$ *in* $\mathcal{P}_u$. *Then the following are equivalent.*

(a) $\mathcal{P}_u$ *is bounded,*

(b) *The market admits NFLIL.*

Proof. We proceed as in Theorem 5.2. First suppose $\mathcal{P}_u$ is unbounded, and let $\{\theta^\nu\}_{\nu=1}^\infty$ be a sequence of trading strategies satisfying

$$\begin{aligned} Z_0 \cdot \theta_0^\nu &\leq F_0 \\ Z_t \cdot [\theta_t^\nu - \theta_{t-1}^\nu] &\leq -F_t \quad P\text{-a.s.,} \quad t = 1, \ldots, T \\ Z_T \cdot \theta_T^\nu &\geq 0 \quad P\text{-a.s.} \\ E\{u(Z_T \cdot \theta_T^\nu)\} &\nearrow +\infty. \end{aligned}$$

$E\{u(Z_T \cdot \theta_T^\nu)\} \leq u(E\{Z_T \cdot \theta_T^\nu\})$ by Jensen's inequality. Thus, $u(E\{Z_T \cdot \theta_T^\nu\}) \nearrow \infty$ for a subsequence, whereby $E\{Z_T \cdot \theta_T^\nu\} \nearrow \infty$ by the assumption that $u(x) \to \infty$ as $x \to \infty$, and that u is strictly increasing. Appealing now to the proof in Theorem 5.2, the same argument yields a free lunch in the limit, establishing (b) $\implies$ (a).

Now let the market admit a free lunch in the limit θ^ν for a sequence $(\varepsilon^\nu)^2 \searrow 0$. By the strict feasibility of $\mathcal{P}_u$, for large enough $\bar{\nu}$, $\nu > \bar{\nu}$, the problem

$$\begin{array}{ll} \text{Maximize} & E\{u(Z_T \cdot \theta_T)\} \\ \text{subject to} & Z_0 \cdot \theta_0 \leq F_0 \\ & Z_t \cdot \Delta\theta_t \leq -F_t \quad P\text{-a.s.,} \quad t = 1, \ldots, T \\ & Z_T \cdot \theta_T - \varepsilon^\nu \in \operatorname{cl}\operatorname{dom} u \quad P\text{-a.s.} \end{array}$$

is feasible, so let $\bar{\theta}^\nu$ be such a feasible point for each $\nu > \bar{\nu}$. Let $\hat{\theta}^\nu := \bar{\theta}^\nu + (\varepsilon^\nu)^{-1}\theta^\nu$. Then $\hat{\theta}^\nu$ is feasible for $\mathcal{P}_u$, since

$$Z_0 \cdot \hat{\theta}_0^\nu = Z_0 \cdot \bar{\theta}_0^\nu + Z_0 \cdot (\varepsilon^\nu)^{-1}\theta_0^\nu \le F_0,$$

and for $t = 1, \ldots, T$,

$$Z_t \cdot \Delta\hat{\theta}_t^\nu = Z_t \cdot \Delta\bar{\theta}_t^\nu + (\varepsilon^\nu)^{-1} Z_t \cdot \Delta\theta_t^\nu \le -F_t.$$

Also,

$$\begin{aligned} Z_T \cdot \hat{\theta}_T^\nu &= Z_T \cdot \bar{\theta}_T^\nu + (\varepsilon^\nu)^{-1} Z_T \cdot \theta_T^\nu \\ &\ge Z_T \cdot \bar{\theta}_T^\nu - (\varepsilon^\nu)^{-1}(\varepsilon^\nu)^2 \\ &= Z_T \cdot \bar{\theta}_T^\nu - \varepsilon^\nu \in \operatorname{dom} u, \end{aligned}$$

and

$$\begin{aligned} E\{Z_T \cdot \hat{\theta}_T^\nu\} &= E\{Z_T \cdot \bar{\theta}_T^\nu\} + (\varepsilon^\nu)^{-1} E\{Z_T \cdot \theta_T^\nu\} \\ &\ge \varepsilon^\nu + (\varepsilon^\nu)^{-1} E\{Z_T \cdot \theta_T^\nu\}. \end{aligned}$$

Since $\lim_\nu E\{Z_T \cdot \theta_T^\nu\} > 0$, we have

$$\lim_\nu E\{Z_T \cdot \hat{\theta}_T^\nu\} = +\infty.$$

By the assumption $\lim_{x\to\infty} \frac{u(x)}{x} = c > 0$, for $0 < \varepsilon < c$, there exists a $K_\varepsilon > 0$ such that $x > K_\varepsilon$ implies $\frac{u(x)}{x} \ge c - \varepsilon$, or $u(x) \ge (c-\varepsilon)x$. Since u is strictly increasing,

$$E\{u(Z_T \cdot \theta_T)\} = E\Big\{u(Z_T \cdot \bar{\theta}_T^\nu + (\varepsilon^\nu)^{-1} Z_T \cdot \theta_T^\nu)\Big\} \ge E\Big\{u(\alpha + \varepsilon^\nu + (\varepsilon^\nu)^{-1} Z_T \cdot \theta_T^\nu)\Big\}.$$

Let $X^\nu := \alpha + \varepsilon^\nu + (\varepsilon^\nu)^{-1} Z_T \cdot \theta_T^\nu$. Since $\lim_{\nu\to\infty} E\{Z_T \cdot \theta_T^\nu\} > 0$, it follows that $\lim_{\nu\to\infty} E\{X^\nu\} = +\infty$, and thus also $\lim_{\nu\to\infty} E\{X^\nu \mathbf{1}_{X^\nu > K}\} = +\infty$, for $K \in I\!R$. Now observe that

$$\begin{aligned} E\{u(X^\nu)\} &= E\{u(X^\nu)\mathbf{1}_{X^\nu \le K_\varepsilon}\} + E\{u(X^\nu)\mathbf{1}_{X^\nu > K_\varepsilon}\} \\ &\ge E\{u(X^\nu)\mathbf{1}_{X^\nu \le K_\varepsilon}\} + E\{(c-\varepsilon)X^\nu \mathbf{1}_{X^\nu > K_\varepsilon}\} \\ &\ge \alpha P(X^\nu \le K_\varepsilon) + E\{(c-\varepsilon)X^\nu \mathbf{1}_{X^\nu > K_\varepsilon}\} \\ &\ge -|\alpha| + E\{(c-\varepsilon)X^\nu \mathbf{1}_{X^\nu > K_\varepsilon}\}. \end{aligned}$$

Thus,

$$\lim_{\nu\to\infty} E\{u(X^\nu)\} \ge -|\alpha| + \lim_{\nu\to\infty} E\{(c-\varepsilon)X^\nu \mathbf{1}_{X^\nu > K_\varepsilon}\} = +\infty.$$

We have thus shown that $\sup \mathcal{P}_u = +\infty$, i.e. that $\mathcal{P}_u$ is unbounded. ◻

6. Martingale Measures

This section reviews the definition of a martingale and equivalent representations for martingales. Then the latter part of the section extends the notion of martingale measures to include finitely additive measures.

Definition 6.1. *Let $(\Xi, \mathcal{F}, P)$ be a probability space and $\{\mathcal{F}_t\}$ the filtration with respect to which a vector process $\{Z_t\}_{t=0}^T$ is measurable. $\{Z_t\}_{t=0}^T$ is a martingale under P if*

$$E\{Z_t|\mathcal{F}_{t-1}\} = Z_{t-1} \quad P\text{-a.s.}, \quad t = 1, \ldots, T.$$

Equivalently,

$$\int_E Z_t dP = \int_E Z_{t-1} dP \quad \forall\, E \in \mathcal{F}_{t-1}, \quad t = 1, \ldots, T.$$

Another way of representing a martingale will be useful in the sections to follow.

Proposition 6.2. *Let $\{Z_t\}_{t=0}^T$ be a vector process defined on $(\Xi, \mathcal{F})$, and $\{\mathcal{F}_t\}$ the associated filtration. Then Z_t is a martingale with respect to a probability measure P if and only if*

$$E\{Z_t \cdot \theta_{t-1}\} = E\{Z_{t-1} \cdot \theta_{t-1}\}, \quad t = 1, \ldots, T,$$

for all $\theta_{t-1} \in L^\infty(\Xi, \mathcal{F}_{t-1}, P; I\!R^{J+1})$, $t = 1, \ldots, T$.

Proof. Suppose $\{Z_t\}_{t=0}^T$ is a martingale under P. Then for fixed $\theta = (\theta_0, \ldots, \theta_T)$ such that $\theta_t \in L^\infty(\Xi, \mathcal{F}_t, P; I\!R^{J+1})$, and fixed t,

$$\begin{aligned} E\{Z_t \cdot \theta_{t-1}|\mathcal{F}_{t-1}\} &= \theta_{t-1} \cdot E\{Z_t|\mathcal{F}_{t-1}\} \\ &= Z_{t-1} \cdot \theta_{t-1}, \end{aligned}$$

whereby $E\{Z_t \cdot \theta_{t-1}\} = E\{Z_{t-1} \cdot \theta_{t-1}\}$.

Now suppose $E\{Z_t \cdot \theta_{t-1}\} = E\{Z_{t-1} \cdot \theta_{t-1}\}$ for all $\theta_{t-1} \in L^\infty(\Xi, \mathcal{F}_{t-1}, P; I\!R^{J+1})$, $t = 1, \ldots, T$. Let $E \in \mathcal{F}_{t-1}$ and let e_i represent the $\mathcal{F}_{t-1}$-measurable coordinate vector with a $\mathbf{1}_E$ in the i'th position and 0's elsewhere, $i = 0, \ldots, J$. For each fixed i, with $\theta_{t-1} \equiv e_i$, one obtains

$$E\{Z_t^i \mathbf{1}_E\} = E\{Z_{t-1}^i \mathbf{1}_E\},$$

thus by Definition 6.1, $\{Z_t\}_{t=0}^T$ is a martingale under P. ◻

Definition 6.3. *A probability measure Q on $(\Xi, \mathcal{F})$ is said to be absolutely continuous with respect to P (denoted $Q << P$) on $\mathcal{F}$ if $P(E) = 0$ implies $Q(E) = 0$ for all $E \in \mathcal{F}$. Q is said to be equivalent to P (denoted $Q \sim P$) on $\mathcal{F}$ if P and Q have the same zero measure sets, i.e. $P(E) = 0$ if and only if $Q(E) = 0$.*

Definition 6.4. *We say that a probability measure Q is a martingale measure for the vector process $\{Z_t\}_{t=0}^T$ if $Q << P$ and $\{Z_t\}_{t=0}^T$ is a martingale under Q. It is an equivalent martingale measure if in addition $Q \sim P$.*

A probability measure that is absolutely continuous with respect to P has an equivalent representation as the Radon-Nikodym derivative of a function $y \in L_+^1(\Xi, \mathcal{F}, P; I\!R)$

such that $\int_\Xi y dP = 1$. In fact, one may translate between the space of absolutely continuous probability measures and the space of such $y's$ under the identification

$$Q(E) = \int_E y dP, \quad \forall\, E \in \mathcal{F}.$$

One may express the martingale condition in terms of its equivalent representation. Let y be the Radon-Nikodym derivative of the absolutely continuous probability measure Q, and let $y_t = E\{y|\mathcal{F}_t\}$.

Proposition 6.5. *A probability measure $Q << P$ is a martingale measure for $\{Z_t\}_{t=0}^T$ if and only if the vector process $\{Z_t y_t\}_{t=0}^T$ is a martingale under P, i.e.*

$$E\{Z_t y_t|\mathcal{F}_{t-1}\} = Z_{t-1}y_{t-1} \quad P\text{-a.s.}, \quad t = 1, \ldots, T.$$

Equivalently,

$$\int_E Z_t y_t dP = \int_E Z_{t-1}y_{t-1}dP \quad \forall E \in \mathcal{F}_{t-1}, \quad t = 1, \ldots, T.$$

Q is an equivalent martingale measure if in addition, $y > 0$ P-a.s..

Proof. This is immediate from the definition. □

All of these considerations may be extended to the space of finitely additive probability measures.

Definition 6.6. *A finitely additive probability measure Q is one that satisfies finite additivity, i.e. for E^n disjoint sets in $\mathcal{F}$, $n = 1, \ldots, N$,*

$$Q(\bigcup_{n=1}^N E^n) = \sum_{n=1}^N Q(E^n),$$

but not necessarily countable additivity (countable additivity is a property satisfied by standard probability measures).

Note that every countably additive probability measure is finitely additive. That conditional expectations with finitely additive probability measures are well-defined can be found in Regazzini [13], along with their properties. Finitely additive measures which are not countably additive are not as well behaved as standard probability measures, and thus mostly avoided. However, they should not be ignored as they do arise in natural contexts as we have been demonstrating. It was already stated that a probability measure has an equivalent representation as the Radon-Nikodym derivative of a function $y \in L^1_+(\Xi, \mathcal{F}, P; I\!R)$ such that $\int y dP = 1$. Similarly, a finitely additive probability measure Q has an equivalent representation as the set function arising from a function $y \in (L^\infty)^*_+(\Xi, \mathcal{F}, P; I\!R)$ such that $\langle y, \mathbf{1}\rangle = 1$, i.e. under the identification

$$Q(E) = \langle \mathbf{1}_E, y\rangle,$$

c.f. [22]. For a finitely additive probability measure Q, let $E_Q\{\cdot\}$ denote the expectation with respect to this measure, which is well-defined when viewed with respect to this representation. To avoid technical considerations, the definition of a finitely additive martingale we take here parallels the representation in Proposition 6.2.

Definition 6.7. *Let $\{Z_t\}_{t=0}^T$ be a vector process defined on $(\Xi, \mathcal{F})$, and $\{\mathcal{F}_t\}$ the associated filtration. Then Z_t is a martingale with respect to a finitely additive probability measure Q if*

$$E_Q\{Z_t \cdot \theta_{t-1}\} = E_Q\{Z_{t-1} \cdot \theta_{t-1}\}$$

for all $\theta_{t-1} \in L^\infty(\Xi, \mathcal{F}_{t-1}, P; I\!R^{J+1})$, $t = 1, \ldots, T$.

Definition 6.8. *A finitely additive probability measure Q on $(\Xi, \mathcal{F})$ is said to be absolutely continuous with respect to a (finitely or countably additive) probability measure P (denoted $Q << P$) on $\mathcal{F}$ if $P(E) = 0$ implies $Q(E) = 0$ for all $E \in \mathcal{F}$. Q is said to be equivalent to P (denoted $Q \sim P$) on $\mathcal{F}$ if P and Q have the same zero measure sets, i.e. $P(E) = 0$ if and only if $Q(E) = 0$.*

Definition 6.9. *Let $(\Xi, \mathcal{F}, P)$ be the underlying probability space, where P is a countably additive probability measure. We say that a finitely additive probability measure Q is a martingale measure for the vector process $\{Z_t\}_{t=0}^T$ if $Q << P$ and $\{Z_t\}_{t=0}^T$ is a martingale under Q. It is an equivalent martingale measure if in addition $Q \sim P$.*

Recall that each element of $(L^\infty)^*_+$ may be uniquely decomposed into an L^1 component and a *singular* component. Thus the associated finitely additive measure decomposes uniquely into a countably additive part and what is called a purely finitely additive part, corresponding to the singular component in $(L^\infty)^*_+$. With $(\Xi, \mathcal{F}, P)$ the underlying probability space, let (y, y^0) be the unique decomposition of the equivalent representation in $(L^\infty)^*_+$ of a finitely additive probability measure $Q << P$. Let $y_t := E\{y|\mathcal{F}_t\}$, and y_t^0 be the unique singular component which is $\mathcal{F}_t$-measurable and satisfies $\langle y_t^0, \mathbf{1}_E\rangle + E\{y\mathbf{1}_E\} = Q(E)$ for all $E \in \mathcal{F}_t$, $t = 0, \ldots, T$.

Proposition 6.10. *A finitely additive probability measure $Q << P$ is a finitely additive martingale measure for $\{Z_t\}_{t=0}^T$ if and only if*

$$E\{Z_t y_t \cdot \theta_{t-1}\} + \langle y_t^0, Z_t \cdot \theta_{t-1}\rangle = E\{Z_{t-1} y_{t-1} \cdot \theta_{t-1}\} + \langle y_{t-1}^0, Z_{t-1} \cdot \theta_{t-1}\rangle$$

for all $\theta_{t-1} \in L^\infty(\Xi, \mathcal{F}_{t-1}, P; I\!R^{J+1})$, $t = 1, \ldots, T$. Q is an equivalent finitely additive martingale measure if and only if in addition to the above, $y > 0$ P-a.s..

Proof. This is a straightforward application of the definition of a finitely additive martingale measure, observing that

$$E_Q\{Z_t \cdot \theta_{t-1}\} = E\{Z_t y_t \cdot \theta_{t-1}\} + \langle y_t^0, Z_t \cdot \theta_{t-1}\rangle,$$

and

$$E_Q\{Z_{t-1} \cdot \theta_{t-1}\} = E\{Z_{t-1}y_{t-1} \cdot \theta_{t-1}\} + \langle y^0_{t-1}, Z_{t-1} \cdot \theta_{t-1}\rangle.$$

To get the equivalence, if $Q \sim P$, then y can't be 0 on a set E of positive measure because this would mean $Q(E) = 0$ while $P(E) > 0$, thus it must be that $y > 0$ almost surely. Now, if $y > 0$, and $E \in \mathcal{F}$ is such that $P(E) = 0$, then $Q(E) = E\{\mathbb{1}_E y\} + \langle y^0, \mathbb{1}_e\rangle = 0$ since $\mathbb{1}_E = 0$ P-almost surely. If $Q(E) = 0$, this means $E\{\mathbb{1}_E y\} + \langle y^0, \mathbb{1}_E\rangle = 0$. Since both terms must be greater than or equal to 0, it follows that $E\{\mathbb{1}_E y\} = 0$. Now, using the fact that $y > 0$ P-almost surely, it must be true that $\mathbb{1}_E y = 0$ P-almost surely, and thus $\mathbb{1}_E = 0$ P-almost surely, i.e. $P(E) = 0$. □

7. The Fundamental Theorem of Asset Pricing

We now proceed to apply the results in the preceding sections to the pricing theory for contingent claims in incomplete markets.

Lemma 7.1. *$\mathcal{D}_u$ is feasible if and only if there exists a finitely additive equivalent martingale measure.*

Proof. Let $y = (y_0, (y_1, y^0_1), \dots, (y_T, y^0_T)) \in Y$ be feasible for $\mathcal{D}_u$. Then y satisfies the constraints in $\mathcal{D}_u$:

$$\begin{array}{ll} & E\{Z_t y_t \cdot \theta_{t-1}\} + \langle y^0_t, Z_t \cdot \theta_{t-1}\rangle = E\{Z_{t-1}y_{t-1} \cdot \theta_{t-1}\} + \langle y^0_{t-1}, Z_{t-1} \cdot \theta_{t-1}\rangle \\ \text{for all} & \theta_{t-1} \in L^\infty(\Xi, \mathcal{F}_{t-1}, P; \mathbb{R}^{J+1}), \quad t = 1, \dots, T, \\ & (y_T, y^0_T) \in \operatorname{dom}(Eu)^*, \quad y \geq 0. \end{array}$$

We begin by showing that $y_T > 0$ almost surely. Suppose to the contrary that there is a set $E \in \mathcal{F}_T$, $P(A) > 0$, such that $y_T \mathbb{1}_E = 0$ almost surely. For y^0_T, let $E^\nu \searrow$ be the associated sets in $\mathcal{F}_T$ such that $P(E^\nu) \searrow 0$ and $\langle y^0_T, z\rangle = 0$ whenever $z\mathbb{1}_{E^\nu} = 0$ almost surely for some ν. Choose ν large enough so that $P(E \setminus E^\nu) > 0$. For $\gamma \in \operatorname{dom} u$, $\lambda > 0$, let $\bar{w}_T = \lambda \mathbb{1}_{E\setminus E^\nu} + \gamma \mathbb{1}_{E^c \cup E^\nu}$. Then,

$$\begin{array}{rcl} (Eu)^*(y_T, y^0_T) & = & \inf_{w_T \in L^\infty(\Xi, \mathcal{F}_T, P; R)} \{E\{w_T y_T\} + \langle y^0_T, w_T\rangle - Eu(w_T)\} \\ & \leq & E\{\bar{w}_T y_T\} + \langle y^0_T, \bar{w}_T\rangle - Eu(\bar{w}_T) \\ & = & \gamma E\{y_T\} + \gamma y^0_T(\mathbb{1}) - u(\lambda)P(E \setminus E^\nu) - u(\gamma)P(E^c \cup E^\nu) \\ & \searrow & -\infty \text{ as } \lambda \nearrow \infty, \end{array}$$

since u is strictly increasing. Thus $(Eu)^*(y_T, y_0) = -\infty$, which means that (y_T, y^0_T) is not in $\operatorname{dom}(Eu)^*$, contradicting our choice of y as a feasible point for $\mathcal{D}_u$. Thus $y_T > 0$ almost surely, as claimed.

Now let $\bar{v}_T := (v_T, v^0_T)$, where

$$v_T := v_T / (E\{y_T\} + \langle y^0_T, \mathbb{1}\rangle),$$

and

$$v_T^0 := y_T^0/(E\{y_T\} + \langle y_T^0, \mathbf{1}\rangle).$$

We proceed to show that the set function Q on $\mathcal{F}_T$ defined by

$$Q(E) := \langle \bar{v}_T, \mathbf{1}_E\rangle, \quad E \in \mathcal{F}_T,$$

is an equivalent finitely additive martingale measure. The finite additivity is immediate from that induced by (y_T, y_T^0). The requirement that $Q(\Xi) = 1$ follows from the normalization,

$$\begin{aligned} Q(\Xi) &= \langle \bar{v}_T, \mathbf{1}\rangle \\ &= E\{v_T\} + \langle v_T^0, \mathbf{1}\rangle \\ &= \frac{E\{y_T\}}{\left(E\{y_T\}+\langle y_T^0,\mathbf{1}\rangle\right)} + \frac{\langle y_T^0,\mathbf{1}\rangle}{\left(E\{y_T\}+\langle y_T^0,\mathbf{1}\rangle\right)} \\ &= 1. \end{aligned}$$

That Q is equivalent to P follows from $v_T > 0$. It remains to show that Q is a martingale measure for the price process. Let $v_t := E\{v_T | \mathcal{F}_t\}$, and v_t^0 be the unique singular component which is $\mathcal{F}_t$-measurable and satisfies $\langle v_t^0, \mathbf{1}_E\rangle + E\{v_T \mathbf{1}_E\} = Q(E)$ for all $E \in \mathcal{F}_t$, $t = 0, \ldots, T$. Then by the constraints in $\mathcal{D}_u$ (and the fact that $Z_t^0 \equiv 1$, $t = 0, \ldots, T$), $v_t = y_t/(E\{y_T\} + \langle y_T^0, \mathbf{1}\rangle)$ and $v_t^0 = y_t^0/(E\{y_T\} + \langle y_T^0, \mathbf{1}\rangle)$. Observe thus by the constraints in $\mathcal{D}_u$, for $t = 1, \ldots, T$, $\theta_{t-1} \in L^\infty(\Xi, \mathcal{F}_{t-1}, P; I\!R^{J+1})$,

$$\begin{aligned} E_Q\{Z_t \cdot \theta_{t-1}\} &= E\{Z_t v_T \cdot \theta_{t-1}\} + \langle v_T^0, Z_t \cdot \theta_{t-1}\rangle \\ &= E\{Z_t v_t \cdot \theta_{t-1}\} + \langle v_t^0, Z_t \cdot \theta_{t-1}\rangle \\ &= E\{Z_{t-1} v_{t-1} \cdot \theta_{t-1}\} + \langle v_{t-1}^0, Z_{t-1} \cdot \theta_{t-1}\rangle \\ &= E_Q\{Z_{t-1} \cdot \theta_{t-1}\}, \end{aligned}$$

whereby Q is an equivalent finitely additive martingale measure.

Now suppose that there exists an equivalent finitely additive martingale measure Q. Let $(y, y_0) \in (L^\infty)^*(\Xi, \mathcal{F}_T, P; I\!R)$ be the representation of Q in $(L^\infty)^*(\Xi, \mathcal{F}_T, P; I\!R)$, let $y_t := E\{y | \mathcal{F}_t\}$, y_t^0 the unique $\mathcal{F}_t$-measurable singular component such that

$$Q(E) = E\{y \mathbf{1}_E\} + \langle y_t^0, \mathbf{1}_E\rangle \ \text{for all} \ E \in \mathcal{F}_t.$$

Then $y := (y_0, (y_1, y_1^0), \ldots, (y_T, y_T^0)) \in Y$ is a feasible solution to $\mathcal{D}_u$, which completes the proof. □

Theorem 7.2. *The market admits no free lunches in the limit if and only if there exists an equivalent finitely additive martingale measure.*

Proof. Consider the writer's portfolio optimization problem $\mathcal{P}_w$, with $\mathcal{P}_w$ strictly feasible and satisfying $F_0 > \operatorname{ess\,inf} \sum_{t=1}^T F_t$. By Theorem 5.2, NFLIL is equivalent to the boundedness of $\mathcal{P}_w$. And this is equivalent to the feasibility of $\mathcal{D}_u$ through the strong duality result in Theorem 4.2, with u the utility function

$$u(v) = \begin{cases} v & \text{if } v \geq 0 \\ -\infty & \text{if } v < 0. \end{cases}$$

Lemma 7.1 gives the equivalence between the feasibility of $\mathcal{D}_u$ and the existence of an equivalent finitely additive martingale measure. ◻

Theorem 7.3. *Suppose the market admits no free lunches in the limit. Then the writer's fair price is*

$$\max\{\sum_{t=1}^{T} E_Q\{F_t\} + \alpha \mid Q \in \mathcal{Q}\},$$

where $\mathcal{Q}$ denotes the space of finitely additive martingale measures.

Proof. The writer's fair price is the optimal value in $\mathcal{P}_{up}$. By the duality result in Theorem 4.4, $\inf \mathcal{P}_{up} = \max \mathcal{D}_{up}$. The feasible region defined in $\mathcal{D}_{up}$, via the same argument as in the proof of Lemma 7.1, is the set of $y \in Y$ such that $\bar{y}_T \in (L^\infty)^*(\Xi, \mathcal{F}, P; I\!R)$ represents the absolutely continuous finitely additive martingale measures through the identification

$$\langle \bar{y}_T, \mathbf{1}_E \rangle = Q(E) \text{ for all } E \in \mathcal{F},$$

where $\bar{y}_T = (y_T, y_T^0)$. That Q is a probability measure follows from the constraint in $\mathcal{D}_{up}$ that $y_0 = 1$. ◻

We remark here that the maximum in Theorem 7.3 could be stated as a supremum with respect to the set of all *equivalent* finitely additive martingale measures, since these are dense in the space of absolutely continuous martingale measures. But in fact the *attainable* pricing measure is only guaranteed to be absolutely continuous with respect to the underlying measure P.

We have thus laid out a very natural duality framework in which these types of pricing problems and their variants lie. The usual attempts in the literature to obtain a countably additive pricing measure (i.e. with Radon-Nikodym derivative in L^1) for more than the most simple problem formulations are often ill-fated. The reason is that, as we have shown, the natural (and attainable) pricing measure includes a purely finitely additive (singular) component, and thus may be represented naturally in the dual of L^∞. Only in special cases when the singular component may be taken to be 0 will the pricing measure be countably additive (with Radon-Nikodym derivative in L^1), and then attainability is still not guaranteed.

In this paper, we have only made vague reference to the possible interpretation of the dual pricing measure, in particular the singular components. Stochastic programming duality in an L^∞/L^1 setting relying on the notion of induced constraints shows that in fact the singular multipliers are in some sense multipliers for implicit constraints at a given time period that are induced by constraints in the future. This interesting fact and its interpretation in the setting of these pricing problems warrants, and will be the topic of, further exploration. Similarly, an extension of the results here to a stochastic programming model that allows for descriptions of the price processes and trading strategies in continuous time is an obvious next step which will be taken up in a future paper.

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