STRUCTURAL STABILITY OF VECTOR OPTIMIZATION PROBLEMS

by

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Abstract. — We study global stability properties for vector optimization problems of the type:

$$\mathcal{VOP}(f, H, G): \quad \min \{f(x) = (f_1(x), \dots, f_l(x)) \mid x \in M[H, G]\},$$

where

$$M[H, G] := \{x \in \mathbb{R}^n \mid h_i(x) = 0, \quad g_j(x) \leq 0, \quad i \in I, j \in J\}$$

with

$$I := \{1, \dots, m\}, \quad J := \{1, \dots, s\}, \quad L := \{1, \dots, l\}.$$

We extend Guddat/Jongen's [5] concept of structural stability of scalar nonlinear optimization problems to vector optimization problems. Under the assumption that $M[H, G]$ is compact we prove the necessary condition for the structural stability of a vector optimization problem, i.e. the scalar problem

$$\mathcal{P}^{\max}(f, H, G): \quad \min \left\{ \max_{l \in L} f_l(x) \mid x \in M[H, G] \right\}$$

has to be structurally stable.

1. Introduction

We consider the vector optimization problem

$$\mathcal{VOP}(f, H, G): \quad \min \{f(x) = (f_1(x), \dots, f_l(x)) \mid x \in M[H, G]\},$$

where

$$M[H, G] := \{x \in \mathbb{R}^n \mid h_i(x) = 0, \quad g_j(x) \leq 0, \quad i \in I, j \in J\}$$

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with

$$I := \{1, \dots, m\}, \quad J := \{1, \dots, s\}, \quad L := \{1, \dots, l\}.$$

Unless otherwise stated, we will assume that the functions f_k, g_j , $k \in L, j \in J$ belong to $C^2(\mathbb{R}^n, \mathbb{R})$. The lower level set corresponding to $u \in \mathbb{R}^{|L|}$ will be denoted as follows:

$$(1) \quad \mathcal{L}^u(f, H, G) := \{x \in M[H, G] \mid f(x) \leq_D u\}, \quad u \in \mathbb{R}^{|L|}.$$

The partial ordering $\leq_D$ is induced by a convex cone D with

$$(2) \quad x \leq_D y \iff y - x \in D.$$

Definition 1. — The vector optimization problems $\mathcal{VOP}(f, H, G)$ and $\mathcal{VOP}(\tilde{f}, \tilde{H}, \tilde{G})$ are called equivalent if there exist continuous mappings $\phi : \mathbb{R}^{|L|} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\psi : \mathbb{R}^{|L|} \rightarrow \mathbb{R}^{|L|}$ with the properties P1 – P3:

P1 For every $u \in \mathbb{R}^{|L|}$ the mapping $\phi_u : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a homeomorphism from $\mathbb{R}^n$ onto itself, where $\phi_u := \phi(u, x)$.

P2 The mapping ψ is a homeomorphism from $\mathbb{R}^{|L|}$ onto itself and ψ is D -monotonically increasing; i.e.

$$(3) \quad q_1 \leq_D q_2 \text{ implies } \psi(q_1) \leq_D \psi(q_2) \text{ for } q_1, q_2 \in \mathbb{R}^{|L|}.$$

P3 $\phi_u[\mathcal{L}^u(f, H, G)] = \mathcal{L}^{\psi(u)}(\tilde{f}, \tilde{H}, \tilde{G})$ for all $u \in \mathbb{R}^{|L|}$.

Definition 2. — The vector optimization problems $\mathcal{VOP}(f, H, G)$ is called structurally stable if there exists a C_S^2 -neighborhood $\mathcal{O}$ of (f, H, G) with the property that $\mathcal{VOP}(f, H, G)$ and $\mathcal{VOP}(\tilde{f}, \tilde{H}, \tilde{G})$ are equivalent for all $(\tilde{f}, \tilde{H}, \tilde{G}) \in \mathcal{O}$.

The C_S^2 -topology above for the product $\prod_{i=1}^r C^2(\mathbb{R}^n, \mathbb{R})$ will be the product topology generated by the strong (or Whitney-) C^2 -topology C_S^2 on each factor $C^2(\mathbb{R}^n, \mathbb{R})$ (cf. [6],[9]). A typical base-neighbourhood of a function $\rho \in C^2(\mathbb{R}^n, \mathbb{R})$ is the set $\rho + \mathcal{W}_\epsilon$, where $\mathcal{W}_\epsilon$ is defined as follows with the aid of a continuous positive $\epsilon : \mathbb{R}^n \rightarrow \mathbb{R}$,

$$(4) \quad \mathcal{W}_\epsilon = \left\{ \eta \in C^2(\mathbb{R}^n, \mathbb{R}) \mid |\eta(x)| + \sum_i \left| \frac{\partial \eta}{\partial x_i}(x) \right| + \sum_{i,j} \left| \frac{\partial^2 \eta}{\partial x_i \partial x_j}(x) \right| < \epsilon(x), \forall x \in \mathbb{R}^n \right\}.$$

We have the chance of a more local control for data corresponding to the scalar problem $\mathcal{P}(r, H, G)$, $r \in C^2(\mathbb{R}^n, \mathbb{R})$ by putting for any subset U of $\mathbb{R}^n$:

$$(5) \quad \text{norm}[(r, H, G), U] =$$

$$\sup_{x \in U} \max_{\eta \in \{r, h_i, i \in I, g_j, j \in J\}} \left\{ |\eta(x)| + \sum_i \left| \frac{\partial \eta}{\partial x_i}(x) \right| + \sum_{i,j} \left| \frac{\partial^2 \eta}{\partial x_i \partial x_j}(x) \right| \right\}.$$

Remark 1. — If $|L| = 1$ and $D = \mathbb{R}_+^{|L|}$, then we have the structural stability for the scalar nonlinear problem introduced in [5].

We consider the following scalar optimization problem

$$\mathcal{P}^{\max}(f, H, G): \quad \min \left\{ \max_{l \in L} f_l(x) \mid x \in M[H, G] \right\}.$$

For the corresponding lower level set we shall use the notation

$$\mathcal{L}_{\max}^t(f, H, G) := \left\{ x \in M[H, G] \mid \max_{l \in L} f_l(x) \leq t \right\}, \quad t \in \mathbb{R}.$$

We use the notations given in Weber [14] and denote the canonical projection of the space $\mathbb{R}^{n+1}$ on the first n components or on the last component by $pr_{(1, \dots, n)}$ and pr_{n+1} , respectively; i.e.

$$\tilde{x} = (pr_{(1, \dots, n)}(\tilde{x}), pr_{n+1}(\tilde{x})) = (x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}.$$

Let us now look at the following equivalent differentiable optimization problem which in particular reveals an objective function of class C^2 .

$$\mathcal{P}^{\vee}(\check{f}, \check{H}, \check{G}): \quad \min \{ \check{f}(\tilde{x}) := pr_{n+1}(\tilde{x}) \mid x \in M[\check{H}, \check{G}] \},$$

where

$$M[\check{H}, \check{G}] := \{ (x, x_{n+1}) \in \mathbb{R}^{n+1} \mid x \in M[H, G], f_l(x) - x_{n+1} \leq 0 \quad (l \in L) \},$$

namely with $\check{h}_i(\tilde{x}) := h_i(x)$ ($i \in I$) and

$$\check{g}_j(\tilde{x}) := \begin{cases} g_j(x), & \text{for } j \in J \\ f_l(x) - x_{n+1}, & \text{for } j = s + l, \quad l \in L. \end{cases}$$

$\mathcal{P}^{\vee}(\check{f}, \check{H}, \check{G})$ is strongly related to scalarization methods for vector optimization problems. As pointed out in Weidner [16], this scalarization method includes the known procedures of Brosowski and Conci [3], Helbig [8], Wierzbicki [17] and other papers cited in [7]. Roughly speaking, each minimal point of $\mathcal{P}^{\vee}(\check{f}, \check{H}, \check{G})$ is weakly efficient and vice versa each weakly efficient and hence each efficient point solves the scalar problem $\mathcal{P}^{\vee}(\check{f}, \check{H}, \check{G})$.

We note that the lower level sets $\mathcal{L}_{\max}^t(f, H, G)$ of $\mathcal{P}^{\max}(f, H, G)$ are just the projection in the sense of $pr_{(1, \dots, n)}$ of the corresponding level sets $\mathcal{L}^{\vee t}(\check{f}, \check{H}, \check{G})$ due to $\mathcal{P}^{\vee}(\check{f}, \check{H}, \check{G})$. Both sets are homotopy-equivalent (cf. Jongen et al. [9], Lemma 4.2.1) and $M[\check{H}, \check{G}]$ is just the part of the epigraph $E(\max_{l \in L} f_l)$ over the feasible set $M[H, G]$.

Definition 3. — The Linear Independence Constraint Qualification (shortly LICQ) is said to hold at $x \in M[H, G]$ if the vectors $Dh_i(x), i \in I, Dg_j(x), j \in J_o(x)$, are linearly independent. The Mangasarian-Fromovitz Constraint Qualification (shortly MFCQ) is said to hold at $x \in M[H, G]$ if the following conditions MF1 and MF2 are satisfied:

MF1 The vectors $Dh_i(x), i \in I$ are linearly independent.

MF2 There exists a vector $\xi \in \mathbb{R}^n$ satisfying:

$$(6) \quad Dh_i(x)\xi = 0, \quad i \in I$$

$$(7) \quad Dg_j(x)\xi \leq 0, \quad j \in J_o(x).$$

A vector ξ satisfying (6) and (7) will be called an *MF-vector*.

For $x \in \mathbb{R}^n$ and $\rho > 0$ let $B(x, \rho)$ denote the Euclidean ball in $\mathbb{R}^n$, centered at x with radius ρ . Following M. Kojima ([11]) we define:

Definition 4. — Let $x^u \in M[H, G]$ be a Kuhn-Tucker point for $\mathcal{P}(r, H, G)$, $r \in C^2(\mathbb{R}^n, \mathbb{R})$. Then x^u is called *strongly stable* if for some $\bar{\delta} > 0$ and each $\delta \in (0, \bar{\delta}]$ there exists an $\alpha > 0$ such that whenever $(\tilde{r}, \tilde{H}, \tilde{G})$ satisfies $\text{norm}[(r - \tilde{r}, H - \tilde{H}, G - \tilde{G}), B(x^u, \delta)] \leq \alpha$, the ball $B(x^u, \delta)$ contains a Kuhn-Tucker point for $\mathcal{P}(\tilde{r}, \tilde{H}, \tilde{G})$ which is unique in $B(x^u, \delta)$.

Lemma 1 (Kojima [11]). — Let $\bar{x}^u \in M[H, G]$ be a Kuhn-Tucker point for $\mathcal{P}(r, H, G)$, $r \in C^2(\mathbb{R}^n, \mathbb{R})$.

1. If LICQ is satisfied at $\bar{x}^u$, then $\bar{x}^u$ is a strongly stable Kuhn-Tucker point if and only if the matrix $D^2L_{[\lambda, \mu]}(\bar{x}^u)$ has nonvanishing determinants with a common sign on the subspace $W(\bar{x}^u, \bar{J})$ for all $\bar{J}$ with $J_+(\bar{x}^u) \subset \bar{J} \subset J_o(\bar{x}^u)$, where

$$J_+(\bar{x}^u) = \{j \in J_o(\bar{x}^u) \mid \mu_j > 0\}$$

$$W(\bar{x}^u, \bar{J}) = \{\xi \in \mathbb{R}^n \mid Dh_i(\bar{x}^u)\xi = 0, i \in I, Dg_j(\bar{x}^u)\xi = 0, j \in \bar{J}\}$$

$$L_{[\lambda, \mu]}(x) := r(x) + \sum_{i \in I} \lambda_i h_i(x) + \sum_{j \in J_o(\bar{x}^u)} \mu_j g_j(x).$$

2. Let MFCQ be satisfied at $\bar{x}^u$, but LICQ not. Then, $\bar{x}^u$ is a strongly stable Kuhn-Tucker point if and only if for every (λ, μ) the matrix $D^2L_{[\lambda, \mu]}(\bar{x}^u)$ is positive definite on the subspace $W(\bar{x}^u, J_+(\bar{x}^u))$.

Firstly we note that the structural stability of the problem $\mathcal{P}^{\max}(f, H, G)$ was characterized in Weber [14].

Theorem 1 (Main result). — Let $D = \mathbb{R}_+^{|L|}$ and the vector optimization problem $\mathcal{VOP}(f, H, G)$ with compact feasible set $M[H, G]$ be structurally stable. Then the problem $\mathcal{P}^{\max}(f, H, G)$ is *structurally stable*.

2. Lemmas and preliminary results

Lemma 2. — Let $t \in \mathbb{R}$, $D = \mathbb{R}_+^{|L|}$ and $u := (t, \dots, t) = te$, $e = (1, \dots, 1) \in \mathbb{R}^{|L|}$. Then the following relations between lower level sets are fulfilled:

$$(8) \quad \mathcal{L}_{\max}^t(f, H, G) = \text{pr}_{(1, \dots, n)} \mathcal{L}^{\vee t}(\check{f}, \check{H}, \check{G}) = \mathcal{L}^u(f, H, G).$$

Proof. — The first equality is obvious. We show the second equality.

$\mathcal{L}^u(f, H, G) \subseteq \text{pr}_{(1, \dots, n)} \mathcal{L}^{\vee t}(\check{f}, \check{H}, \check{G})$: Let $x \in \mathcal{L}^u(f, H, G)$. Then $x \in M[H, G]$, $f_l(x) \leq t$, $l \in \{1, \dots, L\}$. This implies that $(x, t) \in M(\check{H}, \check{G})$ and so $x \in \text{pr}_{(1, \dots, n)} \mathcal{L}^{\vee t}(\check{f}, \check{H}, \check{G})$.

$\mathcal{L}^u(f, H, G) \supseteq pr_{(1, \dots, n)} \mathcal{L}^{\vee t}(\check{f}, \check{H}, \check{G})$: Let $x \in pr_{(1, \dots, n)} \mathcal{L}^{\vee t}(\check{f}, \check{H}, \check{G})$. Then, there exists a $z \in \mathbb{R}$ with $(x, z) \in \mathcal{L}^{\vee t}(\check{f}, \check{H}, \check{G})$. This implies that $x \in M[H, G]$, $f_l(x) \leq z$, $l \in L$ and $z \leq t$ and so $f_l(x) \leq t$, $l \in L$. That means $x \in \mathcal{L}^u(f, H, G)$. $\square$

Lemma 3. — *Let D be a pointed, closed and convex cone. Then there exists a $\bar{u} \in \mathbb{R}^{|L|}$ with:*

$$(9) \quad u_1 \leq_{\bar{D}} \bar{u}, \quad u_2 \leq_{\bar{D}} \bar{u}$$

for all $u_1, u_2 \in \mathbb{R}^{|L|}$.

Proof. — We follow the idea by Tanino in [12] (Proof of the Theorem 3.3.1). Pick

$$d \in \text{int}D$$

and define

$$y(\alpha) = \alpha d \quad \alpha \in \mathbb{R}.$$

We claim that, for any $y \in \mathbb{R}^{|L|}$, there exists $\alpha > 0$ such that

$$y \in y(\alpha) - D.$$

If it is not true, then applying the standard separation theorem to the ray $\{y - \alpha d : \alpha > 0\}$ and the convex cone $-D$, we deduce the existence of $\nu \in \mathbb{R}^p \setminus \{0\}$ such that

$$\langle \nu, y - \alpha d \rangle \geq 0 \quad \text{for all } \alpha > 0$$

and

$$\langle \nu, d' \rangle \geq 0 \quad \text{for all } d' \in D.$$

Since $d \in \text{int}D$, $\langle \nu, d \rangle > 0$ from the latter inequality and so

$$\langle \nu, y - \alpha d \rangle < 0$$

for sufficiently large α , which is a contradiction.

We can therefore choose $\alpha_1, \alpha_2 > 0$ with

$$u_1 \in \alpha_1 d - D \quad \text{and } u_2 \in \alpha_2 d - D.$$

Take $\alpha_{\max} := \max\{\alpha_1, \alpha_2\}$. Since $\alpha_i d \leq_{\bar{D}} \alpha_{\max} d, i = 1, 2$, we can observe that the inequalities (9) are satisfied by $\bar{u} := \alpha_{\max} d$. $\square$

3. Proof of the Necessity Part

Let us now assume that the problem $\mathcal{VOP}(f, H, G)$ is *structurally stable* according to the Definition 2. Then from the results given by Weber in [10] (see also [14] and [5]) we have to verify the following conditions:

- (C1) The Mangasarian-Fromovitz constraint qualification is satisfied at every point of $M[H, G]$.
- (C2) Every Kuhn-Tucker point of $\mathcal{P}^{\max}(f, H, G)$ is strongly stable.
- (C3) Different Kuhn-Tucker points have different $(f-)$ values.

Proof of (C1): We show that the structural stability of $\mathcal{VOP}(f, H, G)$ implies (C1). Consider a C_s^2 -neighborhood $\mathcal{O}$ of (f, H, G) with the property: for every $(\tilde{f}, \tilde{H}, \tilde{G}) \in \mathcal{O}$ the set $M[\tilde{H}, \tilde{G}]$ is compact and moreover $\mathcal{VOP}(f, H, G)$ is equivalent with $\mathcal{VOP}(\tilde{f}, \tilde{H}, \tilde{G})$. Choose an arbitrary element $(\tilde{f}, \tilde{H}, \tilde{G}) \in \mathcal{O}$. Let ϕ and ψ be mappings which establish the equivalence according Definition 1. Take $\alpha, \tilde{\alpha} \in \mathbb{R}^{|L|}$ with

$$(10) \quad \alpha := (\alpha_1, \dots, \alpha_{|L|}), \quad \alpha_i := \max_{x \in M[H, G]} f_i(x)$$

$$(11) \quad \tilde{\alpha} := (\tilde{\alpha}_1, \dots, \tilde{\alpha}_{|L|}), \quad \tilde{\alpha}_i := \max_{x \in M[\tilde{H}, \tilde{G}]} \tilde{f}_i(x)$$

and consider the vectors α and $\psi^{-1}(\tilde{\alpha})$. Lemma 9 implies the existence of a $\bar{u} \in \mathbb{R}^{|L|}$ such that $\alpha \leq_{\bar{D}} \bar{u}$ and $\psi^{-1}(\tilde{\alpha}) \leq_{\bar{D}} \bar{u}$. We note that $\mathcal{L}^{\bar{u}}(f, H, G) = M[H, G]$. Since ψ is monotonically increasing, we have $\tilde{\alpha} \leq_{\bar{D}} \psi(\bar{u})$. Thus $\mathcal{L}^{\psi(\bar{u})}(\tilde{f}, \tilde{H}, \tilde{G}) = M[\tilde{H}, \tilde{G}]$.

From $\phi_{\bar{u}}(\mathcal{L}^{\bar{u}}(f, H, G)) = \mathcal{L}^{\psi(\bar{u})}(\tilde{f}, \tilde{H}, \tilde{G})$, we have $\phi_{\bar{u}}(M[H, G]) = M[\tilde{H}, \tilde{G}]$. Hence, $\phi_{\bar{u}}$ establishes a homeomorphism between $M[H, G]$ and $M[\tilde{H}, \tilde{G}]$. So, we have shown: for every $(\tilde{H}, \tilde{G})$ in some C_s^2 -neighborhood of (H, G) the sets $M[H, G]$ and $M[\tilde{H}, \tilde{G}]$ are homeomorphic. Then, with the help of the Stability Theorem from [5] we may equivalently say that MFCQ is satisfied at every point $x \in M[H, G]$. Hence, the condition (C1) is valid.

Proof of (C2): Let us assume that there is a stationary point which is not strongly stable, i.e. condition (C2) is not valid. In order to arrive at a contradiction, the perturbation result from Weber [14] plays a central part.

Lemma 4 (Perturbation Lemma, Weber [14]). — *Let $M[H, G]$ be compact and suppose that (C1) holds. Suppose further, that $\mathcal{P}^{\max}(V, H, G)$ has a Kuhn-Tucker point $\bar{x}^u$ which is not strongly stable. Then, there exist a number $k \in \mathbb{N}$ and functions $(V^1, \tilde{H}, \tilde{G})$, $(V^2, \tilde{H}, \tilde{G})$, both arbitrarily C_s^2 -near (V, H, G) , with the following properties:*

- (i) *Problem $\mathcal{P}^{\max}(V^1, \tilde{H}, \tilde{G})$ has k Kuhn-Tucker points, exactly one of which is not strongly stable.*
- (ii) *Problem $\mathcal{P}^{\max}(V^2, \tilde{H}, \tilde{G})$ has at least $k+1$ Kuhn-Tucker points, all of them being strongly stable.*
- (iii) *Conditions C1 and C3 hold for problems $\mathcal{P}^{\max}(V^i, \tilde{H}, \tilde{G})$, $i = 1, 2$.*

For $r \in C^2(\mathbb{R}^n, \mathbb{R})$ we denote the set $\{x \in M[H, G] \mid a \leq r(x) \leq b\}$ by $\mathcal{L}_b^a(r, H, G)$ for $a \leq b$, $a, b \in \mathbb{R}$.

The next Lemma is proved by Guddat and Jongen in [5].

Lemma 5. — *Let the feasible set $M[H, G]$ corresponding to the problem $\mathcal{P}(r, H, G)$ be compact and suppose that Condition C1 is fulfilled. Let $a, b \in \mathbb{R}$ be given with $a < b$.*

- (i) *If $\mathcal{L}_b^a(r, H, G)$ contains no Kuhn-Tucker point, then $\mathcal{L}^a(r, H, G)$ and $\mathcal{L}^b(r, H, G)$ are homeomorphic.*

- (ii) Suppose that $\mathcal{L}_b^a(r, H, G)$ contains exactly one Kuhn Tucker point $\bar{x}^u$ with $a < r(\bar{x}^u) < b$, and suppose that $\bar{x}^u$ is strongly stable. Then the sets $\mathcal{L}^a(r, H, G)$ and $\mathcal{L}^b(r, H, G)$ are not homeomorphic.

Lemma 6. — Let $M[H, G]$ be compact and suppose that Condition C1 is fulfilled. Let $V^1, V^2 \in C^2(\mathbb{R}^n, \mathbb{R})^{|L|}$ be functions satisfying Condition C3 w.r.t. $\mathcal{P}^{\max}(V^i, H, G), i = 1, 2$, with the following additional properties:

- (i) Problem $\mathcal{P}^{\max}(V^1, H, G)$ has k Kuhn-Tucker points, exactly one of which is not strongly stable.
- (ii) Problem $\mathcal{P}^{\max}(V^2, H, G)$ has at least $k+1$ Kuhn-Tucker points, all of them being strongly stable.

Then, the vector optimization problems $\mathcal{VOP}(V^i, H, G), i = 1, 2$ are not equivalent.

Proof. — We refer to the n -dimensional problems $\mathcal{P}^{\max}(V^i, H, G), i = 1, 2$ due to the corresponding $(n+1)$ -dimensional problems $\mathcal{P}^{\vee}(\tilde{f}^i, \tilde{H}, \tilde{G}), i = 1, 2$. For the two latter problems, we study the topological behaviour of the lower level sets, when the level parameter t increasingly traverses the line of real numbers. Taking account of Condition C3 formulated for both problems and the Lemma 5, it follows that the homeomorphy-type of the lower level set $\mathcal{L}^{\vee t}(\tilde{f}^2, \tilde{H}, \tilde{G})$ changes at least $k+1$, and at least $k-1$, but (only) at most k such changes for $\mathcal{L}^{\vee t}(\tilde{f}^1, \tilde{H}, \tilde{G})$. The pairwise homotopy equivalence between the lower level sets in $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$ gives rise to the analogous description in dimension n of the topological behavior of $\mathcal{L}_{\max}^t(V^i, H, G), i = 1, 2$. From Lemma 2 (8) it follows that $\mathcal{L}^{te}(V^i, H, G), i = 1, 2$ have two different numbers for the changes of the topological type in $\mathbb{R}^n$, as we move along the line $u(t) = te, e \in \mathbb{R}^{|L|}, t \in \mathbb{R}$. If the problems $\mathcal{VOP}(V^i, H, G), i = 1, 2$ would be equivalent, then these two numbers have to coincide. $\square$

We now apply the perturbational result for our indirect proof of Condition C2. If this condition does not hold, we can apply Lemma 4 and Lemma 6 in order to obtain two non-equivalent problems $\mathcal{VOP}(f^i, \tilde{H}, \tilde{G}), i = 1, 2$ with each $(f^i, \tilde{H}, \tilde{G})$ arbitrarily C_S^2 -near to (f, H, G) . This is in contradiction with structural stability.

Proof of (C3): We know the validity of C1 and C2. If $\mathcal{P}^{\max}(f, H, G)$ does not satisfy Condition C3, whenever the equation $\max_{l \in L} f(\bar{x}_1^u) = \max_{l \in L} f(\bar{x}_2^u)$ holds for two different Kuhn-Tucker points of $\mathcal{P}^{\max}(f, H, G)$ then an addition of one small positive and one small negative constant to all functions f_k ($k \in L$) locally at $\bar{x}_1^u$ and $\bar{x}_2^u$, respectively, leads to a problem $\mathcal{P}^{\max}(\tilde{f}, H, G)$ satisfying Condition C3. Let k the number of Kuhn-Tucker points for $\mathcal{P}^{\max}(\tilde{f}, H, G)$. Then, reasoning as in the proof of Lemma 6, the homeomorphy-type of $\mathcal{L}_{\max}^t(\tilde{f}, H, G)$ for increasing t , changes exactly k -times. The corresponding number of changes w.r.t. $\mathcal{P}^{\max}(f, H, G)$ is less than k . So, as in Lemma 6, $\mathcal{VOP}(f, H, G)$ and $\mathcal{VOP}(\tilde{f}, H, G)$ are not equivalent. This leads again to a contradiction with the condition on structural stability of $\mathcal{VOP}(f, H, G)$.

References

- [1] Bank, B.; Guddat, J.; Klatte, D.; Kummer, B. and Tammer, K.: Non-Linear Parametric Optimization. Akademie-Verlag, Berlin, 1982.
- [2] Guddat, J., Jongen, H. Th., Rueckmann, J.: On stability and stationary points in nonlinear optimization, J. Aust. Math. Soc., Ser. B 28 (1986), 36 – 56.
- [3] Brosowski, B. and Conci, A.: On vector optimization and Parametric Programming. Segundas Jornadas Latino Americanas de Matematica Aplicada, 2(1983), 483 – 495.
- [4] Guddat, J.; Guerra Vasquez, F. and Jongen, H. Th.: Parametric Optimization, Singularities, Pathfollowing and Jumps. John Wiley, Chichester 1990.
- [5] Guddat, Juergen and Jongen, H. Th.: Structural Stability in Nonlinear Optimization. In: Optimization 18 (1987)5, 617 – 631.
- [6] Hirsch, M. W.: Differential Topology, Springer-Verlag, Berlin, Heidelberg, New York, 1976.
- [7] Helbig, S.: On Quatitative and Qualitative Stability of a Scalarization Method, I. In Parametric Optimization and Related Topics III, 255 – 270, Peter Lang Verlag, Frankfurt am Main 1993.
- [8] Helbig, S.: Approximation of the Efficient Point Set by Perturbation of the Ordering Cone, Zeitschrift für Operations Research 35(1991), 197 – 220.
- [9] Jongen, H. Th.; Jonker, P., Twilt, F.: Nonlinear Optimization in $\mathbb{R}^n$, I. Morse Theory, Chebishev approximation, Peter Lang Verlag, Frankfurt, 1983.
- [10] Jongen, H. Th and Weber, G.-W. : Nonlinear Optimization: Characterization of Structural Stability. In: J. Global Optimization 1 (1991), 47 – 64.
- [11] Kojima, M.: Strongly stable stationary solutions in nonlinear programs. In: S.M. Robinson (ed.): Analysis and computation of fixed points. Academic Press, New York, 1980.
- [12] Sawaragi Y., Nakayama H., Tanino T.: Theory of Multiobjective Optimization. Academic Press, Inc., Orlando 1985.
- [13] The Luc, D.: Theory of Vector Optimization. Springer-Verlag, Berlin, 1989.
- [14] Weber, G.: Minimization of a max-type function: Characterization of structural stability. In Parametric Optimization and Related Topics III, 519 – 538, Peter Lang Verlag, Frankfurt am Main 1993.
- [15] Weber, G.: Charakterisierung struktureller Stabilität in der nichtlinearen Optimierung. Aachener Beiträge zur Mathematik, Band 5, Aachen 1992.
- [16] Weidner, P.: An approach to different scalarizations in vector optimization. Wissenschaftliche Zeitschrift der TH Ilmenau 3(1990), 103 – 110.
- [17] Wierzbicki, A. P.: On the Completeness and Constructiveness of Parametric Characterization to Vector Optimization Problem, OR Spektrum 8(1986), 73 – 87.
- [18] Wierzbicki, A. P.: Basic Properties of scalarizing functional for multiobjective optimization. Math. Oper.Stat. Optim. 8, 55-60, 1977.