Arbitrage Pricing Simplified¹

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Abstract

This paper derives fundamental arbitrage pricing results in finite dimensions in a simple unified framework using Tucker's theorem of the alternative. Frictionless results plus those with dividends, periodic interest payments, transaction costs, different interest rates for lending and borrowing, shorting costs and constrained short selling are presented. While the results are mostly known and appear in various places, our contribution is to present them in a coherent and comprehensive fashion with very simple proofs. The analysis yields a simple procedure to prove new results and some are presented for cases with frictions.

Keywords: Asset Pricing, Arbitrage Theory, Derivatives Pricing, Options Pricing, Incomplete Markets, Imperfect Markets

JEL Classification Numbers: G11, G12, D52, C63

1. Introduction

This paper derives known and some new arbitrage pricing results in frictionless and friction cases in a unified and simple finite dimensional setting. The main tool of analysis is Tucker's theorem of the alternative and some special cases of it.

Arbitrage theory of asset pricing was introduced by Ross [39, 41]. The underlying argument is that no-arbitrage is a necessary condition for an equilibrium. The roots of such theory appear in a general equilibrium model by Arrow (see e.g. Debreu [14]). Replication arguments involving assumption of no-arbitrage in continuous time were introduced by Black and Scholes [7] and Merton [33] for the valuation of contingent claims; for a discrete time approach, see also Cox, Ross and Rubinstein [10]. Harrison and Kreps [21] generalized the arbitrage theory of dynamic security pricing. Subsequently an extensive literature appeared; see e.g., Merton [34], Duffie [17], Karatzas [25], Pliska [36], Luenberger [29], and Björk [6].

Most of the early literature deals with frictionless markets, complete and incomplete. Later market frictions were introduced into arbitrage pricing models. For a survey of continuous time models with market frictions, see Cvitanić [12]. Several authors study the impact of transaction costs; e.g., Magill and Constantinides [31], Constantinides [8], Garman and Ohlson [19], Leland [30], Zhao and Ziemba [45], Davis and Norman [13], Jouini and Kallal [23], Bensaid, Lesne, Pagés and Scheinkman [2], Boyle and Vorst [5], Dermody and Prisman [15], Cvitanić and Karatzas [11]. Other studies involving different borrowing and lending rates, short sales considerations and trade restrictions appear in Gilster and Lee [20], Edirisinghe, Naik and Uppal [18], Bergman [3], Jouini and Kallal [24] and Karatzas and Kou [26]. Prisman [37] introduces taxation into single period arbitrage theory. Ross [42] includes taxation along with different borrowing and lending rates. King [27] derives valuation results for contingent claims under several simultaneous market frictions. His perspective is managing the claim as a liability of the writer. The hedging problem is modeled as a stochastic program and analyzed using conjugate duality.

We use Tucker's theorem, which allows a simple framework to derive and easily understand the fundamental pricing results (with and without market frictions) and to derive more results; some of the results on imperfect market are new. Section 2 describes the problem and Tucker's theorem of the alternative. Section 3 describes the frictionless case of dynamic asset pricing. Section 4 discusses imperfect markets including transaction costs, short selling constraints, differing borrowing and lending rates, and dividends. Section 5 concludes.

2. Preliminaries

We employ a discrete time approach, where the time horizon is subdivided into a finite number of periods. Uncertainty concerns the total returns of financial instruments. Realizations of uncertainties over time are depicted by a scenario tree with a finite number of nodes and branches. Let $k = 0, 1, \dots$ be nodes of the scenario tree. Assume that each node has a strictly positive probability. Node $k = 0$ is the root node appearing at the beginning of the time horizon. The set of terminal nodes is denoted by T . Each node $k \in T$ appears at the end of the time horizon, and each node $k \notin T$ appears at the beginning of some period. Let k_- denote the predecessor of node k , for $k > 0$, and S_k the set of immediate successor nodes of k . Hence, for all $s \in S_k$, $s_- = k$. For terminal nodes $k \in T$, S_k is empty. We consider financial instruments $i = 1, \dots, n$. An n vector P_k denotes the prices at node k . Such price processes may involve jumps and stochastic volatilities. An example of a scenario tree is in Figure 1.

Figure 1. A scenario tree with two periods, nodes $k = 0, \dots, 6$, and price vectors P_k .

Prices, shadow prices and returns are given by row vectors while quantities are given by column vectors. A sum vector is $e = (1, \dots, 1)$ and an identity matrix is I , both of appropriate dimension.

Throughout the paper we rely on the following result due to Tucker [44]; see also Mangasarian [32, p. 29].

Theorem 1. (Tuckers's theorem of the alternative) *Let A_1 , A_2 and A_3 be given matrices, with A_1 nonvacuous. Then*

$$A_1 z \geq 0, A_1 z \neq 0, A_2 z \geq 0 \text{ and } A_3 z = 0 \text{ has a solution } z$$

or

$$\pi A_1 + \nu A_2 + \mu A_3 = 0, \pi > 0 \text{ and } \nu \geq 0 \text{ has a solution } \pi, \nu, \mu$$

but never both.

3. Frictionless market

This section presents basic results in arbitrage pricing theory with frictionless markets.

Hence, transaction costs and other market imperfections are excluded from consider-

ation. First, we introduce state price valuation, risk neutral valuation and martingale valuation; an analysis which is similar to the simplified treatment by Dybvig and Ross [16]. Thereafter, valuation of contingent claims and general options are discussed. Finally, dividends and periodic interest payments are considered.

Under frictionless market, the economics of arbitrage pricing theory is well understood; for elaborate treatments, see Ross [40] and Dybvig and Ross [16].

3.1 Basic arbitrage theory

Let an n vector x_k denote the amounts invested in financial assets at node $k \notin T$. The investment expenditure at node k is $P_k x_k$. Let $x = (x_k)$ denote an investment strategy containing a subvector x_k for all $k \notin T$.

For each node k , let c_k denote the net cash flow generated by strategy x at node k . For the root, $-c_0$ is the initial investment, for terminal nodes $k \in T$, c_k is the wealth at the end of the horizon, and for all other nodes k , c_k is the difference of returns from investments made at node k_- and investments made at k

$$-P_0 x_0 = c_0 \quad \text{for } k = 0 \tag{1}$$

$$P_k x_{k_-} = c_k \quad \forall \quad k \in T \tag{2}$$

$$P_k x_{k_-} - P_k x_k = c_k \quad \text{otherwise.} \tag{3}$$

Let $c = (c_k)$ denote the vector of cash flows with one component for each node k . Using (1)-(3), define the matrix C via

$$Cx = c. \tag{4}$$

$$C = \begin{bmatrix} -P_0 & & & & & & \\ & P_1 & -P_1 & & & & \\ & & & & & & \\ & P_2 & & -P_2 & & & \\ & & & & & & \\ & & & P_3 & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & P_5 \\ & & & & & & \\ & & & & & & P_6 \end{bmatrix}$$

Figure 2. The matrix C in (4) for the scenario tree of Figure 1.

Arbitrage A. An arbitrage opportunity exists if $c \geq 0$ and $c \neq 0$ in (4), for some vector x .

The following result by Stiemke [43] is Tucker's Theorem 1 with void matrices A_2 and A_3 ; see also Mangasarian [32, p. 32].

Lemma 1. (Stiemke's theorem of the alternative) *For each given matrix A_1 , either*

$$A_1 z \geq 0 \text{ and } A_1 z \neq 0 \text{ has a solution } z$$

or

$$\pi A_1 = 0, \pi > 0 \text{ has a solution } \pi$$

but never both.

Hence, if an arbitrage does not exist, then by Stiemke's theorem, there is a vector

$\pi = (\pi_k) > 0$ such that $\pi C = 0$. Without loss of generality, assume $\pi_0 = 1$, for the root node. Hence, referring to Figure 2 depicting the structure of the matrix C , we obtain

Theorem 2. (State price valuation) *An arbitrage A does not exist if and only if there exists a node price $\pi_k > 0$, for all k , independent of the node probabilities, with $\pi_0 = 1$, such that*

$$\pi_k P_k = \sum_{s \in S_k} \pi_s P_s \quad \forall k \notin T. \quad (5)$$

This result was first developed in Ross [40] and [41]. The result can be proved based on Farkas' lemma of the alternative (Ross [40]), employing a separating hyperplane theorem (Dybvig and Ross [16]) or by applying duality theory (Prisman [37] and King [27]). We rely on Tucker's Theorem of the alternative from which the result is immediate. It also allows straightforward extensions under imperfect markets.

If a single period risk free asset exists at each node $k \notin T$ with risk free return R_k^0 , then by (5), $\sum_{s \in S_k} \pi_s = \pi_k / R_k^0$. For each $k \notin T$ and $s \in S_k$, define conditional risk neutral probabilities (Arrow [1])

$$\gamma_s = \pi_s / \sum_{j \in S_k} \pi_j > 0.$$

Hence, $\sum_{s \in S_k} \gamma_s = 1$, $\pi_s / \pi_k = \gamma_s / R_k^0$, and Theorem 2 implies the well known result

Theorem 3. (Risk neutral valuation) *If a single period risk free asset exists at each node $k \notin T$, then an arbitrage A does not exist if and only if there exist conditional risk neutral probabilities γ_k , for all $k > 0$, such that*

$$P_k = \frac{1}{R_k^0} \sum_{s \in S_k} \gamma_s P_s \quad \forall k \notin T. \quad (6)$$

If an n vector R_k denotes the total returns of all assets from node k_- to k , then equivalently with (6),

$$R_k^0 e = \sum_{s \in S_k} \gamma_s R_s \quad \forall k \notin T. \quad (7)$$

The risk neutral representation was first employed by Cox and Ross [9].

If for each node $k \notin T$, there is an asset (a numeraire) with a strictly positive price $P_s^0 > 0$, for all $s \in S_k$, then $P_k^0 > 0$ in (5). Hence, we may define normalized prices $\bar{P}_k = P_k/P_k^0$ and $\bar{P}_s = P_s/P_s^0$, and parameters $\sigma_s = (\pi_s P_s^0)/(\pi_k P_k^0)$ such that (5) becomes $\bar{P}_k = \sum_{s \in S_k} \sigma_s \bar{P}_s$. As $\sum_{s \in S_k} \sigma_s = 1$, and $\sigma_s > 0$, for all $s \in S_k$, we interpret parameters σ_s as conditional probabilities. Theorem 2 implies the following result stating that the normalized price process under no arbitrage is a martingale.

Theorem 4. (Martingale valuation) *If at each stage an asset exists with a strictly positive return, then an arbitrage A does not exist if and only if there exist conditional probabilities σ_k , for all $k > 0$, under which the normalized price process is a martingale; i.e.,*

$$\bar{P}_k = \sum_{s \in S_k} \sigma_s \bar{P}_s \quad \forall k \notin T. \quad (8)$$

This result is in a much more general framework, including vector diffusion processes for the price vector, developed in Harrison and Kreps [21].

3.2 Valuation of a contingent claim

Consider a contingent claim with cash flow $v = (v_k)$. For $k > 0$, v_k is the cash flow provided by the asset at node k , and for the root node, $V = -v_0$ is the price of the asset. We determine V based on a no arbitrage assumption. Let ξ denote the amount invested in the contingent claim. Then the net cash flow vector is $c = Cx + v\xi$ and an

arbitrage exists, if there is x and ξ such that $c \geq 0$ and $c \neq 0$. By Stiemke's theorem, there is no arbitrage if and only if the set

$$\Pi = \{\pi \mid \pi C = 0, \pi > 0, \pi_0 = 1\} \quad (9)$$

of state price vectors π is nonempty and v_0 is chosen such that $\pi v = 0$, for some $\pi \in \Pi$.

Hence, under no arbitrage assumption, for $\pi \in \Pi$,

$$V = \sum_{k>0} \pi_k v_k. \quad (10)$$

If $v = Cx$ for some x and v_0 , then v is replicated by portfolio strategy x . If C_k denotes row k in C , then $v_k = C_k x$ and (10) with $\pi C = 0$ yields $V = \sum_{k>0} \pi_k C_k x = -\pi_0 C_0 x = -C_0 x$, for all $\pi \in \Pi$. Hence, if replication is possible, the value $V = -C_0 x$ in (10) is the initial investment needed for replication. Under no arbitrage, this value is the same for all replicating portfolio strategies x .

The market is complete, if any contingent claim v can be replicated. The following observation follows immediately from elementary linear algebra applied to $\pi C = 0$; i.e.

$$\sum_{k>0} \pi_k C_k = -\pi_0 C_0$$

Lemma 2. *Assume the set Π in (9) is nonempty. Then*

Π consists of a single state price vector π

$\Leftrightarrow$

vectors C_k , for $k > 0$, are linearly independent

$\Leftrightarrow$

the market is complete.

As a special case, if a nonsingular matrix results from C when the row vector C_0 of the root node is omitted, then the state price vector $\pi \in \Pi$ is unique, and the market

is complete. However, it is possible, that a contingent claim v can be replicated even if the market is incomplete as the following result shows.

Theorem 5. *If no arbitrage opportunities exist, then the value V of a contingent claim v satisfies $V^- \leq V \leq V^+$, where V^- and V^+ are obtained from the linear programming problems*

$$V^+ = \max_{\pi \in \bar{\Pi}} \sum_{k>0} \pi_k v_k, \quad (11)$$

$$V^- = \min_{\pi \in \bar{\Pi}} \sum_{k>0} \pi_k v_k, \quad (12)$$

where $\bar{\Pi}$ is the closure of Π . If v can be replicated, then $V^- = V^+ = V$.

The replication principle for valuation was first employed by Black and Scholes [7] for option pricing.

The duals of (11) and (12) are

$$V^+ = \min_x \{-c_0 \mid c = Cx, \ c_k \geq v_k \ \forall k > 0\}, \quad (13)$$

$$V^- = \max_x \{-c_0 \mid c = Cx, \ c_k \leq v_k \ \forall k > 0\}. \quad (14)$$

Hence, the bound V^+ is equivalently given by the smallest initial investment $-c_0$ of a portfolio strategy x for which the net cash flow c superreplicates the contingent claim; i.e. $c_k \geq v_k$, for all $k > 0$. Similarly, a subreplicating portfolio strategy with $c_k \leq v_k$ defines V^- in (14). Recently, Bertsimas, Kogan and Lo [4] proposed an ϵ -arbitrage approach for approximate replication, which can provide option prices free of arbitrage in an incomplete market.

3.3 General option valuation

Consider an option allowing choices a in a compact feasible set A . If a strategy $a \in A$ is chosen for executing the option, then a contingent claim $v(a) = (v_k(a))$ is obtained. For

$k > 0$, $v_k(a)$ is the cash flow at node k , and for the root node, $V_0 = -v_0$, independent of a , is the price of the option. Hence, if there are no arbitrage opportunities and if $v(a)$ can be replicated, for all $a \in A$, then by Theorem 5, the value of the option is uniquely determined with any $\pi \in \Pi$ as

$$V_0 = \max_{a \in A} \sum_{k > 0} \pi_k v_k(a). \quad (15)$$

If the market is complete, then by Lemma 2, the state price vector is uniquely determined, and (15) applies; see Luenberger [29, p. 345].

If $v(a)$ cannot be replicated, for all $a \in A$, then upper and lower bounds V_0^+ and V_0^- , respectively, are obtained for V_0 as follows

$$V_0^+ = \max_{\pi \in \Pi} \max_{a \in A} \sum_{k > 0} \pi_k v_k(a), \quad (16)$$

$$V_0^- = \min_{\pi \in \Pi} \max_{a \in A} \sum_{k > 0} \pi_k v_k(a). \quad (17)$$

The upper limit in (16) is found by optimization, while the lower limit in (17) is obtained as a saddle value, provided one exists (Rockafellar [38, p. 3]).

3.4 Impact of periodic dividends and interest payments

For each node $k \notin T$ and for all $s \in S_k$, let D_s denote a vector of dividends per share and interest payments. Then the net cash flow vector $c = (c_k)$ is defined by

$$-P_0 x_0 = c_0 \quad \text{for } k = 0$$

$$(P_k + D_k) x_{k-} = c_k \quad \forall \quad k \in T$$

$$(P_k + D_k) x_{k-} - P_k x_k = c_k \quad \text{otherwise}$$

or by (4) with matrix C including dividends and interest payments; see Figure 3. Theorem 2, with definition of arbitrage A, remains valid with equation (5) replaced by

$$\pi_k P_k = \sum_{s \in S_k} \pi_s (D_s + P_s) \quad \forall k \notin T. \quad (18)$$

$$C = \begin{bmatrix} & & & & & & \\ & -P_0 & & & & & \\ & D_1 + P_1 & & -P_1 & & & \\ & D_2 + P_2 & & & & -P_2 & \\ & & D_3 + P_3 & & & & \\ & & D_4 + P_4 & & & & \\ & & & & D_5 + P_5 & & \\ & & & & & D_6 + P_6 & \end{bmatrix}$$

Figure 3. The matrix C in (4) with dividends and interest payments for the scenario tree of Figure 1.

4. Imperfect markets

This section introduces various imperfections in the model. The first subsection discusses proportional transaction costs for selling and buying. Other imperfections might be studied similarly one by one. However, to shorten the discussion, we directly study the simultaneous impacts of several market frictions. Taxes are not discussed. Our simple activity analysis does not apply to account for taxes in a serious way. There

are several reasons for this: rules of tax laws may involve nonlinearities, discontinuities etc.; taxation is country (region) dependent; and taxation rules change occasionally over time.

4.1 Impact of transaction costs

In this subsection, we assume proportional transaction costs of buying and selling. For simplicity, transaction costs of closing terminal positions (converting terminal wealth into cash) are omitted. Such omission may be justified by utility functions accounting for terminal wealth. The impact of relaxing this assumptions is discussed in Section 4.2. Furthermore, transaction costs of short selling any given quantity are assumed the same as transaction costs of selling the same quantity to reduce long position. Similarly, transaction costs of reducing short positions by any given quantity are assumed the same as transaction costs of buying the same quantity to increase long positions.

The vector of prices at node k is P_k . Let the vector $y_k^+ \geq 0$ be the quantities bought and the vector $y_k^- \geq 0$ the quantities sold at node k , for all $k \notin T$. The quantity y_k^+ can be interpreted as an increase in long position or as a reduction in short position. Similarly, the quantity y_k^- can be interpreted as a reduction in long position or as an increase in short position. It will become clear shortly, that the outcome of our analysis is independent of such alternative interpretations. For all nodes k and instruments i , assume that the fractional shares δ_{ik} and λ_{ik} , with $0 \leq \delta_{ik}, \lambda_{ik} < 1$, are incurred in transaction costs while buying and selling, respectively. Defining diagonal scaling matrices

$$\Delta_k = \text{diag}(1 + \delta_{ik}), \quad \Lambda_k = \text{diag}(1 - \lambda_{ik}),$$

yields the price vectors at node k , including transaction costs, for buying and selling, respectively, as

$$P_k^+ = P_k \Delta_k, \quad P_k^- = P_k \Lambda_k. \quad (19)$$

The net cash flow at node k is

$$c_k = -P_k^+ y_k^+ + P_k^- y_k^- \quad \forall k \notin T, \quad (20)$$

and the wealth at a terminal node k is

$$c_k = P_k \sum_{k \in B_k} (y_k^+ - y_k^-) \quad \forall k \in T, \quad (21)$$

where B_k is the set of nodes preceding $k \in T$ in the path from terminal node k to the root node 0. Observe, that net cash flows c_k both in (20) and in (21) are the same irrespective of quantities y_k^- being interpreted as short sales or not. In matrix notation, the net cash flow vector c having one component for each node is

$$Cy = c \quad (22)$$

where $y = (y_k^+, y_k^-) \geq 0$ is a vector with a pair y_k^+, y_k^- of subvectors for each $k \notin T$, and C is a matrix defined by (20) and (21); see Figure 4.

$$C = \begin{bmatrix} -P_0^+ & P_0^- & & & & & \\ & & -P_1^+ & P_1^- & & & \\ & & & & -P_2^+ & P_2^- & \\ P_3 & -P_3 & P_3 & -P_3 & & & \\ P_4 & -P_4 & P_4 & -P_4 & & & \\ P_5 & -P_5 & & & P_5 & -P_5 & \\ P_6 & -P_6 & & & P_6 & -P_6 & \end{bmatrix}$$

Figure 4. The matrix C in (22) with transaction costs for the scenario tree of Figure 1.

Arbitrage B. An arbitrage opportunity exists if $c \geq 0$ and $c \neq 0$ in (22), for some vector $y \geq 0$.

If $\Delta_k = \Lambda_k = I$, then arbitrage B is equivalent with arbitrage A above. To study the case with transaction costs we employ the following special case of Tucker's Theorem 1 with $A_2 = I$ and A_3 void.

Lemma 3. *For each given matrix A_1 , either*

$$A_1 z \geq 0, A_1 z \neq 0 \text{ and } z \geq 0 \text{ has a solution}$$

or

$$\pi A_1 \leq 0 \text{ and } \pi > 0 \text{ has a solution}$$

but never both.

If an arbitrage B does not exist, then Lemma 3. implies existence of a state price vector $\pi = (\pi_k) > 0$, with $\pi_0 = 1$, such that $\pi C \leq 0$. Taking into account the structure of C in (22), we have

$$\pi_k P_k^- \leq \sum_{t \in T_k} \pi_t P_t \leq \pi_k P_k^+ \quad \forall k \notin T, \quad (23)$$

where $T_k \subset T$ is the set of terminal nodes succeeding node k . Hence, we have

Theorem 6. *An arbitrage B does not exist if and only if there exists a node price $\pi_k > 0$, for all k , with $\pi_0 = 1$, satisfying (23).*

If $\delta_{ik} = \lambda_{ik} = 0$, for all i ; i.e. $P_k^- = P_k^+ = P_k$, then (23) is equivalent with recursive application of (5). For transaction cost parameters $\delta_{ik}, \lambda_{ik} > 0$, Theorem 6 involves a set of positive state price vectors π that can be used for valuation of contingent claims employing standard software; see e.g., Murtagh and Saunders [35]. This result appears to be new. Valuation formulas (11)-(12) and (16)-(17) for contingent claims and general options in Sections 3.2 and 3.3 remain valid if the set of state price vectors Π in (9) is replaced by the set of vectors π satisfying (23) with $\pi > 0$, and $\pi_0 = 1$. Similarly as in (13)-(14), duality can be applied to obtain bounds V^- and V^+ on the price V of a contingent claim with a cash flow vector $v = (v_k)$. In this case, V^+ is the smallest initial investment needed in an investment strategy defined by $y \geq 0$ satisfying (22) such that c superreplicates v ; i.e. $c_k \geq v_k$, for all $k > 0$. The lower bound V^- is the largest initial net cash flow c_0 created by such an investment strategy y such that $-c$ subreplicates v ; i.e. $-c_k \leq v_k$, for all $k > 0$.

We interpret the result of Theorem 6 in terms of martingales. For a state price vector $\pi > 0$ with $\pi_0 = 1$ satisfying (23) define a price process $\tilde{P}_k$ as follows. For

each terminal node $k \in T$, $\tilde{P}_k = P_k$. For all other nodes k , define $\tilde{P}_k$ such that $\pi_k \tilde{P}_k = \sum_{s \in S_k} \pi_s \tilde{P}_s$. Recursively we obtain $\pi_k \tilde{P}_k = \sum_{t \in T_k} \pi_t P_t$. Hence, by (23) $P_k^- \leq \tilde{P}_k \leq P_k^+$. Repeating the arguments for Theorem 4, if there is an asset (a numeraire) with a strictly positive price $\tilde{P}_k^0 > 0$, for all k , define normalized prices $\bar{P}_k = \tilde{P}_k / \tilde{P}_k^0$ and parameters $\sigma_s = (\pi_s \tilde{P}_s^0) / (\pi_k \tilde{P}_k^0)$. Then $\bar{P}_k = \sum_{s \in S_k} \sigma_s \bar{P}_s$, $\sum_{s \in S_k} \sigma_s = 1$, and $\sigma_s > 0$. Hence, the normalized price process $\bar{P}_k$ is a martingale. Consequently, if an arbitrage B does not exist, then there is a price process $\tilde{P}_k$ with $P_k^- \leq \tilde{P}_k \leq P_k^+$, such that the normalized price process $\bar{P}_k$ is a martingale. Conversely, suppose that a price process $\tilde{P}_k$ and the probabilities $\sigma_s > 0$ exist such that $P_k^- \leq \tilde{P}_k \leq P_k^+$, $\tilde{P}_k^0 > 0$ for the numeraire process, and the normalized price process $\bar{P}_k$ is a martingale. Then forward recursion $\pi_s = \sigma_s \pi_k \tilde{P}_k^0 / \tilde{P}_s^0$, for $s \in S_k$, with $\pi_0 = 1$ yields state prices $\pi_k > 0$ satisfying (23). In conclusion, for arbitrage B, we obtain the result by Jouini and Kallal [23] stating that the absence of arbitrage is equivalent to the existence of an equivalent probability measure that transforms some process $\tilde{P}_k$ between the bid and ask price processes P_k^- and P_k^+ of traded securities (after normalization) into a martingale. While Jouini and Kallal characterize valuation of contingent claims using martingale measures, evaluation is not explicitly considered.

4.2 Combined impact of imperfections

Next, consider simultaneously several market imperfections: (i) transaction costs as above, (ii) an interest rate spread between borrowing and lending, (iii) charges for short positions, and (iv) restricted short selling. As in Section 4.1, we assume proportional transaction costs of buying and selling, transaction costs of short selling are assumed the same as transaction costs of reducing long positions, and transaction costs of reducing

short positions are assumed the same as transaction costs of buying. However, we now apply an additional holding cost for short positions and restrict short positions relative to long positions. Unlike in arbitrage B, for the definition of an arbitrage, all wealth at terminal nodes may be converted to cash taking into account transaction costs. We also include dividends and periodic interest payments for lending and borrowing as in Section 3.4. Independence of imperfections is not assumed.

Again, for each node k , P_k is the vector of prices at node k . Price vectors P_k^+ and P_k^- , including transaction costs, are defined by (19) such that $P_k^+ \geq P_k^-$. The vector $y_k^+ \geq 0$ denotes the quantities bought and the vector $y_k^- \geq 0$ the quantities sold at node k , for all $k \notin T$. Again, the vector y_k^+ is interpreted as an increase in long positions or as a reduction in short positions. Similarly, the quantity y_k^- is interpreted as a reduction in long positions or as an increase in short positions. It will become clear shortly, that for our analysis it is unnecessary to keep track of these alternative interpretations.

Let $x_k = x_k^+ - x_k^-$ denote the vector of quantities held in each instrument at node k , with $x_k^+, x_k^- \geq 0$ referring to long and short positions, respectively. For each $k \notin T$, and for each $s \in S_k$, define vectors D_s^+ and D_s^- as follows. For long positions x_k^+ , $D_s^+ \geq 0$ is the the vector of cash value per share provided by all instruments. This vector consists of dividends and interest, and if s is a terminal node, then also the selling price P_s^- is included in D_s^+ . For short positions x_k^- , D_s^- is composed of dividends, interest, shorting costs, and for terminal nodes, also the buying price P_s^+ is included. The borrowing and lending components in D_s^+ and D_s^- are interpreted as interest payments. Interest rates for short positions is assumed at least as large as for long positions. The shorting

cost can be infinite, if the short position is prohibited. Hence, for all $s \neq 0$, we assume

$$D_s^- \geq D_s^+. \quad (24)$$

The net cash flow at the root node is

$$c_0 = -P_0^+ y_0^+ + P_0^- y_0^-. \quad (25)$$

For nodes $k > 0$ and $k \notin T$

$$c_k = -P_k^+ y_k^+ + P_k^- y_k^- + D_k^+ x_{k-}^+ - D_k^- x_{k-}^-. \quad (26)$$

Finally, for terminal nodes $k \in T$

$$c_k = D_k^+ x_{k-}^+ - D_k^- x_{k-}^-. \quad (27)$$

Hence, the net cash flow vector c having one component for each node is

$$Ax + By = c \quad (28)$$

where $x = (x_k^+, x_k^-) \geq 0$ is a vector with a pair x_k^+, x_k^- of subvectors for each $k \notin T$, $y = (y_k^+, y_k^-) \geq 0$ is a vector with a pair y_k^+, y_k^- of subvectors for each $k \notin T$, and A and B are matrices defined by (25) - (27).

For nodes $k \notin T$, a margin requirement limits short positions $P_k x_k^-$ to a share $\rho \leq 1$ of long positions $P_k x_k^+$; i.e.

$$\rho P_k x_k^+ - P_k x_k^- \geq 0 \quad \forall k \notin T \quad (29)$$

which in matrix notation is

$$Ex \geq 0. \quad (30)$$

The quantities held at the root node 0 satisfy

$$-x_0^+ + x_0^- + y_0^+ - y_0^- = 0, \quad (31)$$

and for all other nodes $k \notin T$

$$-x_k^+ + x_k^- + x_{k-}^+ - x_{k-}^- + y_k^+ - y_k^- = 0. \quad (32)$$

In matrix notation,

$$Cx + Dy = 0, \quad (33)$$

with C and D defined by (31) and (32).

$$\begin{array}{c|c}
 \begin{array}{cc} x & y \end{array} \\
 \hline
 \begin{array}{cc} A & B \\ E & 0 \\ C & D \end{array}
 \end{array} = \begin{array}{c|c}
 \begin{array}{cccccc} x_0^+ & x_0^- & x_1^+ & x_1^- & x_2^+ & x_2^- \end{array} & \begin{array}{cccccc} y_0^+ & y_0^- & y_1^+ & y_1^- & y_2^+ & y_2^- \end{array} \\
 \hline
 \begin{array}{cc} D_1^+ & -D_1^- \\ D_2^+ & -D_2^- \\ & D_3^+ & -D_3^- \\ & D_4^+ & -D_4^- \\ & & D_5^+ & -D_5^- \\ & & D_6^+ & -D_6^- \end{array} & \begin{array}{cc} -P_0^+ & P_0^- \\ & -P_1^+ & P_1^- \\ & & -P_2^+ & P_2^- \end{array} \\
 \hline
 \begin{array}{cc} \rho P_0 & -P_0 \\ & \rho P_1 & -P_1 \\ & & \rho P_2 & -P_2 \end{array} & \\
 \hline
 \begin{array}{cc} -I & I \\ I & -I & -I & I \\ I & -I & & -I & I \end{array} & \begin{array}{cc} I & -I \\ & I & -I \\ & & I & -I \end{array}
 \end{array}$$

Figure 5. The structure of the model of (28), (30) and (33) for the scenario tree of

Figure 1.

Arbitrage C. An arbitrage opportunity exists if $c \geq 0$ and $c \neq 0$ in (28), for vectors $x \geq 0$ and $y \geq 0$ satisfying (30) and (33).²

If $\Delta_k = \Lambda_k = I$, if the margin requirement is omitted, and if $D_k^+ = D_k^- = P_k$, for $k \in T$, and $D_k^+ = D_k^- = 0$ otherwise, then arbitrage C is equivalent with arbitrage A. To analyze the case with mixed imperfections, we employ another special case of Tucker's Theorem 1.

Lemma 4. *Let A_1 , A_2 and A_3 be given matrices, with A_1 nonvacuous. Then*

$$A_1 z \geq 0, A_1 z \neq 0, A_2 z \geq 0, z \geq 0, \text{ and } A_3 z = 0 \text{ has a solution } z$$

²At each node k , (33) allows long and short positions to be incremented by the same quantity, but we must ensure that both x_k^+ and x_k^- remain non-negative. Reasonable interpretations exist. For example, one might simultaneously reduce both long and short position without transaction cost. On the other hand, we might consider a revised model (28), (33), (30), defining the increments in long positions by $y_k^+ \geq 0$ and $y_k^- \geq 0$, and the increments in short positions by separate vectors $s_k^+ \geq 0$ and $s_k^- \geq 0$. For the revised model, the original requirement (30) remains unchanged, while the original equations (28) and (33) are reformulated as follows. Equation (28) is replaced by net cash flow equations $c_0 = -P_0^+ y_0^+ + P_0^- s_0^+$, for the root node, $c_k = D_k^+ x_{k-}^+ - D_k^- x_{k-}^- - P_k^+ y_k^+ - P_k^+ s_k^- + P_k^- y_k^- + P_k^- s_k^+$, for nodes $k \neq 0$ and $k \notin T$, and $c_k = D_k^+ x_{k-}^+ - D_k^- x_{k-}^-$, for terminal nodes $k \in T$. Equation (33) is replaced by separate equations for long and short positions: $x_0^+ = y_0^+$ and $x_0^- = s_0^+$, for the root node, and $x_k^+ = x_{k-}^+ + y_k^+ - y_k^-$ and $x_k^- = x_{k-}^- + s_k^+ - s_k^-$, for other nodes $k \notin T$. If an arbitrage opportunity exists in the revised model, then such an arbitrage strategy immediately defines an arbitrage opportunity in the original model. On the other hand, suppose that an arbitrage opportunity exists in the original model. Then, because $D_k^- \geq D_k^+$, for all k , and $\rho \leq 1$, it is straightforward to check that reducing at each node k both positions x_k^+ and x_k^- by $\max\{x_k^+, x_k^-\}$ (the maximum component-wise), results in another arbitrage strategy. In the latter strategy, no asset is held simultaneously long and short, and hence this arbitrage provides an arbitrage in the revised model. In conclusion, the revised model, which is shown in the Appendix as well, exhibits an arbitrage if and only if the original model does.

or

$$\pi A_1 + \nu A_2 + \mu A_3 \leq 0, \pi > 0, \nu \geq 0 \text{ has a solution } \pi, \nu, \mu$$

but never both.

If an arbitrage C does not exist, then Lemma 4. with $A_1 = [A, B]$, $A_2 = [E, 0]$ and $A_3 = [C, D]$ with submatrices A , B , C , D and E in (28), (30) and (33), implies existence of a state price vector $\pi = (\pi_k) > 0$, with $\pi_0 = 1$, a vector $\nu = (\nu_k) \geq 0$, and a vector $\mu = (\mu_k)$, with components ν_k and subvectors μ_k , for all $k \notin T$, such that

$$\pi A + \nu E + \mu C \leq 0 \quad (34)$$

$$\pi B + \mu D \leq 0. \quad (35)$$

Taking into account the structure of matrices A , B , C , D and E , (34) becomes

$$\rho \nu_k P_k + \sum_{s \in S_k} (\pi_s D_s^+ + \mu_s) \leq \mu_k \leq \nu_k P_k + \sum_{s \in S_k} (\pi_s D_s^- + \mu_s) \quad \forall k \notin T, \quad (36)$$

with $\mu_s \equiv 0$, for $s \in T$, and (35) becomes

$$\pi_k P_k^- \leq \mu_k \leq \pi_k P_k^+ \quad \forall k \notin T. \quad (37)$$

Hence, we conclude with the following result, which also is new

Theorem 7. *An arbitrage C does not exist if and only if there exists a node price vector $\pi > 0$, $\nu \geq 0$ and μ satisfying (36) and (37), with $\pi_0 = 1$.*

We discuss valuation procedures shortly. However, in order to help understand and interpret Theorem 7, we first discuss a number of special cases, starting with the most simple ones. In these cases the margin requirement is omitted, so that we think of P_k in (29) being replaced by a zero vector and thereby $\nu_k P_k$ is set to zero in (36).

(a) *The basic model (Section 3.1).* We now have $P_k^- = P_k^+ = P_k$, for all k , $D_k^- = D_k^+ = 0$, for $k \notin T$ and $D_k^- = D_k^+ = P_k$, for $k \in T$. Condition (37) implies $\mu_k = \pi_k P_k$ so that (36) yields the basic state price valuation result (5).

(b) *The model with dividends and interest payments (Section 3.4).* Again $P_k^- = P_k^+ = P_k$, for all k . For the cash payments, $D_k^- = D_k^+ = D_k$, for $k \notin T$, and $D_k^- = D_k^+ = P_k + D_k$, for $k \in T$. Then, condition (37) implies $\mu_k = \pi_k P_k$. Hence, (36) yields the result (18).

(c) *The model with transaction costs (Section 4.1).* Let $P_k^- \leq P_k \leq P_k^+$ and $D_k^- = D_k^+ = 0$, for $k \notin T$, and $P_k^- = P_k^+ = P_k = D_k^- = D_k^+$, for terminal nodes $k \in T$. Then, for k such that $S_k \subset T$, condition (36) implies $\mu_k = \sum_{s \in S_k} \pi_s P_s$. For all other nodes $k \notin T$, (36) implies $\mu_k = \sum_{s \in S_k} \mu_s$ so that with backward recursion we obtain $\mu_k = \sum_{t \in T_k} \pi_t P_t$. This with (37) implies (23). Related models have been studied by Jouini and Kallal [23] (for more details, see Section 4.1 above), Leland [30], Boyle and Vorst [5], and Gilster and Lee [20], for instance.

(d) *Restricted short and long positions with interest rate spread.* This model is a discretized version of the model of Jouini and Kallal [24]. Consider a finite set of marketed instruments i with prices P_k^i . There are no transaction costs so that $P_k^+ = P_k^- = P_k$, for all k . Instrument $i = 0$ refers to lending and borrowing. We set $P_k^0 = 1$, for all k . For $k \notin T$, the interests of lending and borrowing are paid in cash and given by $D_k^{0+} = r_k^+$ and $D_k^{0-} = r_k^-$, respectively. For $k \in T$, the face value is included as well, so that $D_k^{0+} = 1 + r_k^+$ and $D_k^{0-} = 1 + r_k^-$. To avoid arbitrage, $r_k^+ \leq r_k^-$. For all other instruments i , either short positions are prohibited with $D_k^{i-} = \infty$ or long positions are prohibited with $D_k^{i+} = -\infty$. In the former case we set $D_k^{i+} = 0$, for

$k \notin T$, and $D_k^{i+} = P_k^i$, for $k \in T$, and in the latter case $D_k^{i-} = 0$, for $k \notin T$, and $D_k^{i-} = P_k^i$, for $k \in T$. If both short and long positions are possible for some securities, such instruments are duplicated to fit in the model.

If there is no arbitrage C, then condition (37) implies $\mu_k = \pi_k P_k$, for all $k \notin T$. For $i = 0$, (36) yields $\sum_{s \in S_k} \pi_s (1 + r_s^+) \leq \pi_k \leq \sum_{s \in S_k} \pi_s (1 + r_s^-)$, for all $k \notin T$. Hence, for some interest rate process $\tilde{r}_k$, such that $r_k^+ \leq \tilde{r}_k \leq r_k^-$, we have $\pi_k = \sum_{s \in S_k} \pi_s (1 + \tilde{r}_s)$, or equivalently, $1 = \sum_{s \in S_k} \sigma_s$, where $\sigma_s = (1 + \tilde{r}_s) \pi_s / \pi_k > 0$. Define recursively a numeraire process $\tilde{R}_k$ as follows. For the root node, $\tilde{R}_0 = 1$, and for other nodes, $\tilde{R}_k = (1 + \tilde{r}_k) \tilde{R}_{k_-}$, where k_- denotes the predecessor node of node k . Then the normalized price process is $\bar{P}_k = P_k / \tilde{R}_k$. For instruments $i \neq 0$, if $D_k^{i-} = \infty$ then $\pi_k P_k^i \geq \sum_{s \in S_k} \pi_s P_s^i$ by (36), or equivalently, $\bar{P}_k^i \geq \sum_{s \in S_k} \sigma_s \bar{P}_s^i$, i.e., $\bar{P}_k^i$ is a supermartingale. Similarly, if $D_k^{i+} = -\infty$, then $\pi_k P_k^i \leq \sum_{s \in S_k} \pi_s P_s^i$, and $\bar{P}_k^i \leq \sum_{s \in S_k} \sigma_s \bar{P}_s^i$ so that $\bar{P}_k^i$ is a submartingale.

It is straightforward to show the converse. Hence in conclusion, an arbitrage C does not exist if and only if there is some interest rate process $\tilde{r}_k$, with $r_k^+ \leq \tilde{r}_k \leq r_k^-$, an associated numeraire process $\tilde{R}_k$, and an equivalent probability measure such that the normalized price processes $\bar{P}_k^i$, for $i \neq 0$, are supermartingales or submartingales. This is the main result in Jouini and Kallal [24].

(e) *The model with holding costs and benefits; no transaction costs.* Assume $P_k^- = P_k^+ = P_k$ and $D_k^- \geq D_k^+$, for all k . In this case, different interest rates for lending and borrowing as well as costs for short positions may be included in the model. Such a case in continuous time has been studied by Bergman [3]. Condition (37) implies $\mu_k = \pi_k P_k$. Hence, for k such that $S_k \subset T$, (36) implies $\sum_{s \in S_k} \pi_s D_s^+ \leq \pi_k P_k \leq \sum_{s \in S_k} \pi_s D_s^-$. For

other nodes $k \notin T$, (36) implies $\sum_{s \in S_k} \pi_s(D_s^+ + P_s) \leq \pi_k P_k \leq \sum_{s \in S_k} \pi_s(D_s^- + P_s)$. Note that for terminal nodes $k \in T$, both D_k^+ and D_k^- include the price component P_k . Hence, there exists a process $\tilde{D}_k$ such that $D_k^+ \leq \tilde{D}_k \leq D_k^-$, for $k \notin T$, $D_k^+ \leq \tilde{D}_k + P_k \leq D_k^-$, for $k \in T$, and $\pi_k P_k = \sum_{s \in S_k} \pi_s(\tilde{D}_s + P_s)$, for $k \notin T$. The latter equation provides a frictionless condition (18) for the absence of arbitrage C. For further discussion, see case (f) where (e) is a special case.

(f) *The model with transaction costs, and holding costs and benefits.* Assume $P_k^- \leq P_k \leq P_k^+$ and $D_k^- \geq D_k^+$, for all k . In this case, define $\tilde{P}_k$ such that $\pi_k \tilde{P}_k = \mu_k$, for $k \notin T$, and $\tilde{P}_k = P_k$, for $k \in T$. Then condition (37) and our assumption imply $P_k^- \leq \tilde{P}_k \leq P_k^+$, for all k . Hence, for k such that $S_k \subset T$, (36) implies $\sum_{s \in S_k} \pi_s D_s^+ \leq \pi_k \tilde{P}_k \leq \sum_{s \in S_k} \pi_s D_s^-$. For other nodes $k \notin T$, (36) implies $\sum_{s \in S_k} \pi_s(D_s^+ + \tilde{P}_s) \leq \pi_k \tilde{P}_k \leq \sum_{s \in S_k} \pi_s(D_s^- + \tilde{P}_s)$. Consequently, absence of arbitrage C implies existence of a process $\tilde{D}_k$ with $D_k^+ \leq \tilde{D}_k \leq D_k^-$, for $k \notin T$, $D_k^+ \leq \tilde{D}_k + \tilde{P}_k \leq D_k^-$, for $k \in T$, such that

$$\pi_k \tilde{P}_k = \sum_{s \in S_k} \pi_s(\tilde{D}_s + \tilde{P}_s) \quad \forall k \notin T. \quad (38)$$

This equation is a frictionless condition (18) for the absence of arbitrage C. One may readily check, that the converse is true with $\mu_k = \pi_k \tilde{P}_k$, for $k \notin T$, in Theorem 7. In conclusion, an arbitrage C does not exist if and only if there is $\pi > 0$ with $\pi_0 = 1$, and a price process $\tilde{P}_k$ with $P_k^- \leq \tilde{P}_k \leq P_k^+$ satisfying (38) for some process $\tilde{D}_k$ such that $D_k^+ \leq \tilde{D}_k \leq D_k^-$, for $k \notin T$, and $D_k^+ \leq \tilde{D}_k + \tilde{P}_k \leq D_k^-$, for $k \in T$. Of course, interpretation of (38) in terms of risk neutral or martingale valuation is straightforward. Thereby our conclusion extends earlier results.

We now return to the general case of Theorem 7. Similarly as before, valuation formulas (11)-(12) and (16)-(17) for contingent claims and general options in Sections

3.2 and 3.3 remain valid if the set of state price vectors Π in (9) is replaced by the set of vectors π satisfying (36) and (37) with $\pi > 0$, $\pi_0 = 1$, and $\nu \geq 0$; for an illustration, see Example 1 below. Optimization codes, such as Minos [35], are readily available to compute bounds V^- and V^+ on the price of a contingent claim. Because Π is a convex set, every price outside the interval $[V^-, V^+]$ leads to an arbitrage C, and there is no arbitrage C for price levels in the interior of this interval; see also Karatzas and Kou [26].

As in (13)-(14), also duality can be applied to obtain bounds V^- and V^+ on the price V of a contingent claim with a cash flow vector $v = (v_k)$. The upper limit V^+ is the smallest initial investment needed in an investment strategy defined by $x \geq 0$ and $y \geq 0$ satisfying (28), (30) and (33) such that c superreplicates v ; i.e. $c_k \geq v_k$, for all $k > 0$. The lower limit V^- is the largest initial net cash flow c_0 created by such an investment strategy x and y such that $-c$ subreplicates v ; i.e. $-c_k \leq v_k$, for all $k > 0$. However, the replication result of valuation in frictionless markets is no longer valid; a result, which was first pointed out by Bensaid et al. [2]. Example 2 demonstrates the case.

We now show that a frictionless price is in the price interval obtained with frictions. Suppose an arbitrage C does not exist in a frictionless case (b), including (a) as a special case. Consider $\pi > 0$ with $\pi_0 = 1$ and μ_k in Theorem 7 satisfying (36) and (37). Then $\mu_k = \pi_k P_k$ by (37) and $\pi_k P_k = \sum_{s \in S_k} \pi_s (D_s + P_s)$ by (36), for $k \notin T$. Next, consider the case (f) with frictions such that $P_k^- \leq P_k \leq P_k^+$, for all k , $D_k^+ \leq D_k \leq D_k^-$, for $k \notin T$, and $D_k^+ \leq D_k + P_k \leq D_k^-$, for $k \in T$. Then $\pi_k P_k^- \leq \pi_k P_k = \mu_k \leq \pi_k P_k^+$ so that (37) holds for case (f). To check that (36) holds as well, we obtain for k such that $S_k \subset T$, $\sum_{s \in S_k} \pi_s D_s^+ \leq \sum_{s \in S_k} \pi_s (D_s + P_s) = \pi_k P_k = \mu_k \leq \sum_{s \in S_k} \pi_s D_s^-$, and for all other $k \notin T$,

$\sum_{s \in S_k} (\pi_s D_s^+ + \mu_s) \leq \sum_{s \in S_k} \pi_s (D_s + P_s) = \pi_k P_k = \mu_k \leq \sum_{s \in S_k} (\pi_s D_s^- + \mu_s)$. Hence, if π and μ_k for case (b) satisfy (36) and (37) in Theorem 7, then they apply for case (f) as well. Consequently, the values (or the value, in case it is unique) obtained for a contingent claim in the frictionless case are within the bounds V^- and V^+ obtained under transaction costs and holding costs. Such observation is demonstrated by Jouini and Kallal [23], and Karatzas and Kou [26].

Example 1. Consider an example of Hull [22, pp. 395-396]. A 5-month American put option on a non-dividend paying stock, with current stock price 50 and annual volatility 40 %, has an exercise price 50. The risk free annual interest rate is 10 %. For frictionless markets, a binomial lattice approach with five steps of equal length yields an option value 4.49. Including transaction costs of 0.1%, an annual interest rate spread of 1.0 % between borrowing and lending, and an annual holding cost of 2.0 % for short position in the stock, while omitting the margin requirement, we obtain option prices ranging from 4.01 to 5.03. This gap is relatively large. However, if additional assets are employed in arbitrage pricing, then the gap between bounds tends to narrow, because the set of state price vectors π tends to shrink; see also King, Koivu and Pennanen [28].

Example 2. Consider a single period model with two scenarios and two assets. One asset is risk free with interest rates 3 % and 4 % for lending and borrowing, respectively. The other asset is risky with long position returns 20% and 0 % in terminal nodes 1 and 2, respectively, and with a holding cost of 5 % of the initial price for short position. Consider a contingent claim v with $v_1 = 0$, $v_2 = 1$, and initial price denoted by V . Then (11) yields $V^+ = 0.971$. Let V_r^+ be the smallest initial investment needed to

replicate v . Such replication is possible and $V_r^+ = 1.068 > V^+$. In this case, there is a superreplicating strategy for which the initial investment V^+ is less than the initial investment in any replicating strategy. Consequently, any price V based on replication results in existence of arbitrage.

5. Conclusions

Tucker's theorem of the alternative provides a simple tool to prove arbitrage results in frictionless cases and when there are transaction costs and other frictions in the finite case. The approach yields known results directly and some new results and is a useful approach for further research.

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Appendix: The revised model of Section 4.2

For each $k \notin T$, $y_k^+ \geq 0$ and $y_k^- \geq 0$ denote the increase and decrease in long positions, respectively. For short positions, such increments are denoted by $s_k^+ \geq 0$ and $s_k^- \geq 0$. After such increments, $x_k^+ \geq 0$ and $x_k^- \geq 0$ are the long and short positions, respectively.

The net cash flow at the root node $k = 0$ is

$$c_0 = -P_0^+ y_0^+ + P_0^- s_0^+. \quad (39)$$

For nodes $k > 0$ and $k \notin T$

$$c_k = D_k^+ x_{k-}^+ - D_k^- x_{k-}^- - P_k^+ y_k^+ - P_k^+ s_k^- + P_k^- y_k^- + P_k^- s_k^+ \quad (40)$$

For terminal nodes $k \in T$,

$$c_k = D_k^+ x_{k-}^+ - D_k^- x_{k-}^-. \quad (41)$$

For nodes $k \notin T$,

$$\rho P_k x_k^+ - P_k x_k^- \geq 0. \quad (42)$$

The positions x_0^+ and x_0^- at the root node 0 satisfy

$$-x_0^+ + y_0^+ = 0 \quad (43)$$

$$x_0^- - s_0^+ = 0 \quad (44)$$

and for all other nodes $k \notin T$,

$$x_{k-}^+ - x_k^+ + y_k^+ - y_k^- = 0. \quad (45)$$

$$-x_{k-}^- + x_k^- + s_k^- - s_k^+ = 0. \quad (46)$$

x_0^+	x_0^-	x_1^+	x_1^-	x_2^+	x_2^-	y_0^+	s_0^+	y_1^+	s_1^-	y_1^-	s_1^+	y_2^+	s_2^-	y_2^-	s_2^+
D_1^+	$-D_1^-$					$-P_0^+$	P_0^-								
D_2^+	$-D_2^-$							$-P_1^+$	$-P_1^+$	P_1^-	P_1^-				
		D_3^+	$-D_3^-$									$-P_2^+$	$-P_2^+$	P_2^-	P_2^-
		D_4^+	$-D_4^-$												
				D_5^+	$-D_5^-$										
				D_6^+	$-D_6^-$										
ρP_0	$-P_0$														
		ρP_1	$-P_1$												
				ρP_2	$-P_2$										
$-I$						I									
	I					$-I$									
I		$-I$					I		$-I$						
	$-I$		I					I		$-I$					
I				$-I$							I		$-I$		
	$-I$				I							I		$-I$	

Figure 6. The structure of the revised model (39)-(46) for the scenario tree of Figure

1.