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Overlapping Schwarz wave form relaxation for the solution of coupled and decoupled system of convection diffusion reaction equation

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Abstract

In this article we study the convergence and the error bound for the solution of the convection diffusion reaction equation using overlapping Schwarz wave form relaxation method combined with the first order fractional splitting method (Strang's splitting) as basic solver. We extended the study to solve decoupled and coupled system of equations of same class in order to demonstrate the effect of the coupling in the system, through the reaction term, on the convergence and error decay. The accuracy and the efficiency of the methods are investigated through the solution of different model problems of scalar, coupled and decoupled systems of convection diffusion reaction equations.

Keywords: Numerical simulation; Operator-splitting methods; Domain-decomposition-methods; Parabolic partial differential equations; Convergence analysis; Error analysis

1. Introduction

The first known method for solving partial differential equations over overlapped domains is the Schwarz method, which was first presented in [16] in 1869. The method has regained its popularity after the development of the computational numerical algorithms and the computer architecture, especially the parallel processing computations.

Further techniques have been developed for the general cases of overlapped and non-overlapped domains. For each class of methods there are some interesting features and both share same concepts, which are on how to define the interface boundary conditions over the overlapped or along the non-overlapped subdomains. The general solution methods over the whole subdomains together with the interface boundary conditions estimations are either iterative or non-iterative methods.

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For the overlapping subdomains the determination of the interface boundary conditions is defined using a predictor corrector type method. The predictor will provide an estimation of the boundary condition while the correction is performed on the updated solution over the subdomains. This class of algorithms is of an iterative type with the advantage that the stability of the solution by any unconditional difference approximation will not be affected by the predicted interface values. The well-known and intensively studied algorithm in this field is overlapping Schwarz wave form relaxation method.

Overlapping Schwarz waveform relaxation is the name for a combination of two standard algorithms, Schwarz alternating method and wave form relaxation algorithm, to solve evolution problems in parallel. The method is defined by partitioning the spatial domain into overlapping subdomains, as in the classical Schwarz method. However, on subdomains, time dependent problems are solved inside the iteration and thus the algorithm is also of waveform relaxation type. Furthermore, the problem is solved using the operator splitting of first order over each subdomain. The overlapping Schwarz waveform relaxation is introduced in [8] and independently in [6] for a solver method of evolution problems in a parallel environment with slow communication links. The idea is to solve over several time steps before communicating information to the neighboring subdomains and updating the calculated interface boundary conditions for the overlapped domains.

These algorithms contrast with the classical approach in domain decomposition for evolution problems, where time is first discretized uniformly using an implicit discretization and then at each time step a problem in space only is solved using domain decomposition, see for example [2,3,13].

In this work we will consider the overlapping type of domain decomposition method for solving the studied models of constant coefficients. Decoupled and coupled systems are solved using the first order operator splitting algorithm with backward Euler difference scheme. The most recent method in this field is the overlapping Schwarz waveform relaxation scheme, as appearing in [4].

Furthermore, in this work the operator splitting method will be considered using an implicit Euler-method for the time-discretization [17]. The main advantage in considering the overlapping Schwarz wave form relaxation method is the flexibility that we can solve over each sub-domain with different time steps and different spatial steps in the whole time-interval. For systems of convection diffusion reaction equations we study the decoupled case, i.e. m scalar equations and the coupled case, i.e. m equations coupled by the reaction-terms.

The outline of the paper is as follows. For our mathematical model we describe the convection diffusion reaction equation in Section 2. The Fractional Splitting is described in Section 2.2. For the overlapping Schwarz waveform-relaxation method we derive the error-analysis for the scalar equation and for systems (coupled or decoupled systems) and demonstrate the results in Section 3. In Section 4 we present the numerical results from the solution of selected model problems. We end the article in section 6 with conclusion and comments.

2. Mathematical model and methods

2.1. Model-problem

The motivation for the study presented below comes from a computational simulation of heat-transfer [9] and convection diffusion reaction-equations, cf. [10,11].

In our paper we concentrate on an one dimensional convection diffusion reaction equation, as in our model problem, which is given by

$$u_t - Du_{xx} + vu_x = -\lambda u, \quad \text{in } \Omega \times (0, T), \tag{1}$$

$$u(x, 0) = u_0, \quad \text{(Initial-Condition)}, \tag{2}$$

$$u(x, t) = u_1, \quad \text{on } \partial\Omega \times (0, T), \text{(Dirichlet-Boundary-Condition)}. \tag{3}$$

The unknown $u = u(x, t)$ is considered in $\Omega \times (0, T) \subset \mathbb{R} \times \mathbb{R}$, where $\Omega = [0, L]$. The parameters $u_0, u_1 \in \mathbb{R}^+$ are constants and are used as initial- and boundary-parameter respectively. The parameter λ is a constant factor, for example the decay-rate of a chemical reaction. D is a constant factor, for example the diffusion factor of a transport-process and v is a constant factor, for example the velocity-rate of a transport-process.

The aim of this paper is to present a new method based on a mixed discretization method combining Fractional-Splitting with Domain decomposition methods, which provides an effective solver-methods of strong coupled parabolic differential equations.

In the next subsection we discuss the decoupling of the time-scale with a first order fractional splitting-method.

2.2. Fractional splitting methods of first order for linear equations

First we describe the simplest operator-splitting , which is called *sequential operator splitting*for the following linear system of ordinary differential equations:

$$\partial_t u(t) = Au(t) + Bu(t), \tag{4}$$

where the initial-conditions are $u^n = u(t^n)$. The operators A and B are spatially discretized operators, i.e. they correspond to the discretized in space convection and diffusion operators (matrices). Hence, they can be considered as bounded operators.

The sequential operator-splitting method is introduced as a method that solves two sub-problems sequentially, where the different sub-problems are connected via the initial conditions. This means that we replace the original problem (4) with the sub-problems

$$\begin{aligned} \frac{\partial u^*(t)}{\partial t} &= Au^*(t), \quad \text{with } u^*(t^n) = u^n, \\ \frac{\partial u^{**}(t)}{\partial t} &= Bu^{**}(t), \quad \text{with } u^{**}(t^n) = u^*(t^{n+1}), \end{aligned} \tag{5}$$

where the splitting time-step is defined as $\tau_n = t^{n+1} - t^n$. The approximated split solution is defined as $u^{n+1} = u^{**}(t^{n+1})$.

Clearly, the replacement of the original problem with the sub-problems usually results some error, called *splitting error*. The splitting error of the sequential operator splitting method can be derived as follows (cf. e.g. [12,15,17])

$$\rho_n = \frac{1}{\tau_n}(\exp(\tau_n(A+B)) - \exp(\tau_n B)\exp(\tau_n A))u(t^n) = \begin{cases} 0, & \text{for } [A,B] = 0, \\ \mathrm{O}(\tau_n), & \text{for } [A,B] \neq 0, \end{cases} \tag{6}$$

where $[A,B] := AB - BA$ is the commutator of A and B. Consequently, the splitting error is $\mathrm{O}(\tau_n)$ when the operators A and B do not commute, otherwise the method is exact. Hence, by definition, the sequential operator splitting is called *first order splitting method*.

We apply the first order fractional splitting method to our model-equation (1) and divide the equation to the following operators $A = D\frac{\partial^2}{\partial x^2} - v\frac{\partial}{\partial x}$ and $B = -\lambda$. Our splitting-scheme is given as

$$\begin{aligned} \frac{\partial u^*(x,t)}{\partial t} &= Du^*_{xx} - vu^*_x, \quad \text{with } u^*(x,t^n) = u_0, \\ \frac{\partial u^{**}(x,t)}{\partial t} &= -\lambda u^{**}(t), \quad \text{with } u^{**}(x,t^n) = u^*(x,t^{n+1}), \end{aligned} \tag{7}$$

where $u^*(x,t) = u^{**}(x,t) = u_1$, on $\partial\Omega \times (0,T)$, are the Dirichlet-Boundary-Conditions for the equations. The solution is given as $u(x,t^{n+1}) = u^{**}(x,t^{n+1})$. We obtain an exact method because of commuting operators.

For the discretization of Eq. (7) we apply the finite-difference method for the spatial discretization and the implicit Euler method for the time discretization. The discretization is given as

$$\begin{aligned} \frac{1}{t^{n+1}-t^n}(u^*(x_i,t^{n+1}) - u^*(x_i,t^n)) = {} & D\frac{1}{h_i^2}(-u^*(x_{i+1},t^{n+1}) + 2u^*(x_i,t^{n+1}) - u^*(x_{i-1},t^{n+1})) \\ & - v\frac{1}{h_i}(u^*(x_i,t^{n+1}) - u^*(x_{i-1},t^{n+1})), \\ & \text{with } u^*(x_1,t^n) = u^*(x_2,t^n) = u_0 \text{ and } u^*(x_0,t^n) = u^*(x_m,t^n) = 0, \end{aligned} \tag{8}$$

$$u^{**}(x,t) = \exp(-\lambda(t-t^n)u^*(x,t^{n+1})), \tag{9}$$

where $h_i = x_{i+1} - x_i$ and we assume a partition with m-nodes.

Now we introduce the domain-decomposition methods as an idea for splitting methods to decompose complex domains and solve the resulting problems effectively using an adaptive method.

3. Overlapping Schwarz wave form relaxation for the solution of convection diffusion reaction equation

In this section we present the necessary conditions for the convergence of the overlapping Schwarz wave form relaxation method for the solution of the convection-reaction diffusion equation with constant coefficients. We will utilize the convergence analysis for the solution of the decoupled and coupled system of convection diffusion reaction equation in order to elaborate the impact of the coupling on the convergence of the overlapping Schwarz wave form relaxation.

The overlapping Schwarz wave form relaxation method is considered for solving the scalar convection diffusion reaction equation, decoupled and coupled systems of convection diffusion reaction equations with the first order operator splitting method as basic solution method. To visualize the efficiency of the method we compared the accuracy with the accuracy from the solution by the first order operator splitting method.

3.1. Overlapping Schwarz wave form relaxation for the scalar convection diffusion reaction equation

We consider the convection diffusion reaction equation, given by

$$u_t = Du_{xx} - vu_x - \lambda u, \tag{10}$$

defined on the domain $\Omega \times T$, where $\Omega = [0, L]$ and $T = [T_0, T_f]$, with the following boundary and initial conditions

$$u(0,t) = f_1(t), \quad u(L,t) = f_2(t), \quad u(x,T_0) = u_0.$$

To solve the model problem using overlapping Schwarz wave form relaxation method, we subdivide the domain Ω in two overlapping sub-domains $\Omega_1 = [0, L_2]$ and $\Omega_2 = [L_1, L]$, where $L_1 < L_2$ and $\Omega_1 \bigcap \Omega_2 = [L_1, L_2]$ is the overlapping region for Ω_1 and Ω_2.

To start the wave form relaxation algorithm we consider first the solution of the model problem (10) over Ω_1 and Ω_2 as follows:

$$\begin{aligned} &v_t = Dv_{xx} - vv_x - \lambda v \text{ over } \Omega_1, \quad t \in [T_0, T_f],\\ &v(0,t) = f_1(t), \quad t \in [T_0, T_f],\\ &v(L_2,t) = w(L_2,t), \quad t \in [T_0, T_f],\\ &v(x,T_0) = u_0, \quad x \in \Omega_1, \end{aligned} \tag{11}$$

$$\begin{aligned} &w_t = Dw_{xx} - vw_x - \lambda w \text{ over } \Omega_2, \quad t \in [T_0, T_f],\\ &w(L_1,t) = v(L_1,t), \quad t \in [T_0, T_f],\\ &w(L,t) = f_2(t), \quad t \in [T_0, T_f],\\ &w(x,T_0) = u_0, \quad x \in \Omega_2, \end{aligned} \tag{12}$$

where $v(x,t) = u(x,t)|_{\Omega_1}$ and $w(x,t) = u(x,t)|_{\Omega_2}$.

Then the Schwarz wave form relaxation is given by

$$\begin{aligned} &v_t^{k+1} = Dv_{xx}^{k+1} - vv_x^{k+1} - \lambda v^{k+1} \text{ over } \Omega_1, \quad t \in [T_0, T_f],\\ &v^{k+1}(0,t) = f_1(t), \quad t \in [T_0, T_f],\\ &v^{k+1}(L_2,t) = w^k(L_2,t), \quad t \in [T_0, T_f],\\ &v^{k+1}(x,T_0) = u_0, \quad x \in \Omega_1, \end{aligned} \tag{13}$$

$$\begin{aligned} &w_t^{k+1} = Dw_{xx}^{k+1} - vw_x^{k+1} - \lambda w^{k+1} \text{ over } \Omega_2, \quad t \in [T_0, T_f],\\ &w^{k+1}(L_1,t) = v^k(L_1,t), \quad t \in [T_0, T_f],\\ &w^{k+1}(L,t) = f_2(t), \quad t \in [T_0, T_f],\\ &w^{k+1}(x,T_0) = u_0, \quad x \in \Omega_2. \end{aligned} \tag{14}$$

We are interested in estimating the decay of the error of the solution over the overlapping subdomains obtained with the overlapping Schwarz wave form relaxation method over long time interval.

Let us assume that $e^{k+1}(x,t) = u(x,t) - v^{k+1}(x,t)$ and $d^{k+1}(x,t) = u(x,t) - w^{k+1}(x,t)$ are the errors of (13) and (14) over Ω_1 and Ω_2 respectively. The corresponding differential equations satisfied by $e^{k+1}(x,t)$ and $d^{k+1}(x,t)$ are

$$\begin{aligned}
&e_t^{k+1} = De_{xx}^{k+1} - ve_x^{k+1} - \lambda e^{k+1} \text{ over } \Omega_1, \quad t \in [T_0, T_f],\\
&e^{k+1}(0,t) = 0, \quad t \in [T_0, T_f],\\
&e^{k+1}(L_2,t) = d^k(L_2,t), \quad t \in [T_0, T_f],\\
&e^{k+1}(x,T_0) = 0, \quad x \in \Omega_1,
\end{aligned} \tag{15}$$

$$\begin{aligned}
&d_t^{k+1} = Dd_{xx}^{k+1} - vd_x^{k+1} - \lambda d^{k+1} \text{ over } \Omega_2, \quad t \in [T_0, T_f],\\
&d^{k+1}(L_1,t) = e^k(L_1,t), \quad t \in [T_0, T_f],\\
&d^{k+1}(L,t) = 0, \quad t \in [T_0, T_f],\\
&d^{k+1}(x,T_0) = 0, \quad x \in \Omega_2.
\end{aligned} \tag{16}$$

We define for bounded functions $h(x,t) : \Omega \times [T_0, T_f] \to \mathbf{R}$ the norm

$$\|h(.,.)\|_\infty := \sup_{x\in\Omega, t\in[T_0,T_f]} |h(x,t)|.$$

The theory behind our error-estimates is based on the positivity lemma by Pao (or the maximum principle theorem), see [14], that is introduced as

Lemma 1. *Let $u \in C(\overline{\Omega}_T) \cap C^{1,2}(\Omega_T)$, where $\Omega_T = \Omega \times (0,T]$ and $\partial\Omega_T = \partial\Omega \times (0,T]$, be such that*

$$u_t - Du_{xx} + vu_x + cu \geqslant 0, \quad \text{in } \Omega_T, \tag{17}$$

$$\alpha_0 \partial u \partial v + \beta_0 u \geqslant 0, \quad \text{on } \partial\Omega_T, \tag{18}$$

$$u(x,0) \geqslant 0, \quad \text{in } \Omega, \tag{19}$$

where $\alpha_0 \geqslant 0$, $\beta_0 \geqslant 0$, $\alpha_0 + \beta_0 > 0$ on $\partial\Omega_T$, and $c \equiv c(x,t)$ is a bounded function in Ω_T, Then $u(x,t) \geqslant 0$ in Ω_T.

The convergence and error-estimates of e^{k+1} and d^{k+1} given by (15) and (16) respectively, are presented in the following theorem:

Theorem 1. *Let e^{k+1} and d^{k+1} be the error from the solution of the subproblems (11) and (12) by Schwarz wave form relaxation over Ω_1 and Ω_2, respectively, then*

$$\|e^{k+2}(L_1,t)\|_\infty \leqslant \gamma \|e^k(L_1,t)\|_\infty, \quad \text{and}$$

$$\|d^{k+2}(L_2,t)\|_\infty \leqslant \gamma \|d^k(L_1,t)\|_\infty,$$

where

$$\gamma = \frac{\sinh(\beta L_1)}{\sinh(\beta L_2)} \frac{\sinh(\beta(L_2 - L))}{\sinh(\beta(L_1 - L))} < 1,$$

with $\beta = \frac{\sqrt{v^2+4D\lambda}}{2D}$.

Proof. In order to estimate the errors e^{k+1} and d^{k+1}, we consider the following differential equations containing $\hat{e}^{k+1}$ and $\hat{d}^{k+1}$:

$$\begin{aligned}
&\hat{e}_t^{k+1} = D\hat{e}_{xx}^{k+1} - v\hat{e}_x^{k+1} - \lambda \hat{e}^{k+1} \text{ over } \Omega_1, \quad t \in [T_0, T_f],\\
&\hat{e}^{k+1}(0,t) = 0, \quad t \in [T_0, T_f],\\
&\hat{e}^{k+1}(L_2,t) = \|d^k(L_2,t)\|_\infty, \quad t \in [T_0, T_f],\\
&\hat{e}^{k+1}(x,T_0) = e^{(x-L_2)\alpha} \frac{\sinh(\beta x)}{\sinh(\beta L_2)} \|d^k(L_2,t)\|_\infty, \quad x \in \Omega_1
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
&\hat{d}_t^{k+1} = D\hat{d}_{xx}^{k+1} - v\hat{d}_x^{k+1} - \lambda\hat{d}^{k+1} \text{ over } \Omega_2, \quad t \in [T_0, T_f],\\
&\hat{d}^{k+1}(L_1,t) = \|e^k(L_1,t)\|_\infty, \quad t \in [T_0, T_f],\\
&\hat{d}^{k+1}(L,t) = 0, \quad t \in [T_0, T_f],\\
&\hat{d}^{k+1}(x,T_0) = e^{(x-L_1)\alpha}\frac{\sinh\beta(x-L)}{\sinh\beta(L_1-L)}\|e^k(L_1,t)\|_\infty, \quad x \in \Omega_2,
\end{aligned} \tag{21}$$

where $\alpha = \frac{v}{2D}$ and $\beta = \frac{\sqrt{v^2+4D\lambda}}{2D}$. □

The solution of (20) and (21) is the steady state solution given by

$$\hat{e}^{k+1}(x) = e^{(x-L_2)\alpha}\frac{\sinh(\beta x)}{\sinh(\beta L_2)}\|d^k(L_2,t)\|_\infty$$

and

$$\hat{d}^{k+1}(x) = e^{(x-L_1)\alpha}\frac{\sinh\beta(x-L)}{\sinh\beta(L_1-L)}\|e^k(L_1,t)\|_\infty,$$

respectively.

For the error between the steady state and time-dependent solution, defined by $E(x,t) = \hat{e}^{k+1} - e^{k+1}$, it holds

$$\begin{aligned}
&E_t - DE_{xx} + vE_x + \lambda E \geqslant 0, \text{ over } \Omega_1, \quad t \in [T_0, T_f],\\
&E(0,t) = 0, \quad t \in [T_0, T_f],\\
&E(L_2,t) \geqslant 0, \quad t \in [T_0, T_f],\\
&E(x,T_0) \geqslant 0, \quad x \in \Omega_1.
\end{aligned} \tag{22}$$

$E(x,t)$ satisfies Lemma 1, therefore

$$E(x,t) \geqslant 0,$$

i.e.

$$|e^{k+1}| \leqslant \hat{e}^{k+1},$$

for all (x,t) and similarly we conclude that

$$|d^{k+1}| \leqslant \hat{d}^{k+1},$$

for all (x,t).

Then

$$|e^{k+1}(x,t)| \leqslant e^{(x-L_2)\alpha}\frac{\sinh(\beta x)}{\sinh(\beta L_2)}\|d^k(L_2,t)\|_\infty \tag{23}$$

and

$$|d^{k+1}(x,t)| \leqslant e^{(x-L_1)\alpha}\frac{\sinh\beta(x-L_1)}{\sinh\beta(L_1-L)}\|e^k(L_1,t)\|_\infty. \tag{24}$$

Evaluating (24) at $x = L_2$

$$|d^k(L_2,t)| \leqslant \frac{\sinh\beta(L_2-L)}{\sinh\beta(L_1-L)}\|e^{k-1}(L_1,t)\|_\infty \tag{25}$$

and substituting in (23) gives

$$|e^{k+1}(x,t)| \leqslant e^{(x-L_2)\alpha}\frac{\sinh(\beta x)}{\sinh(\beta L_2)}e^{(L_2-L_1)\alpha}\frac{\sinh\beta(L_2-L)}{\sinh\beta(L_1-L)}\|e^{k-1}(L_1,t)\|_\infty,$$

therefore

$$|e^{k+1}(L_1,t)| \leqslant e^{(L_1-L_2)\alpha}\frac{\sinh(\beta L_1)}{\sinh(\beta L_2)}e^{(L_2-L_1)\alpha}\frac{\sinh\beta(L_2-L)}{\sinh\beta(L_1-L)}\|e^{k-1}(L_1,t)\|_\infty,$$

i.e.

$$|e^{k+2}(L_1,t)| \leqslant \frac{\sinh(\beta L_1)}{\sinh(\beta L_2)}\frac{\sinh\beta(L_2-L)}{\sinh\beta(L_1-L)}\|e^{k}(L_1,t)\|_\infty.$$

Similarly for $d^{k+1}(x,t)$ we conclude that

$$|d^{k+2}(L_2,t)| \leqslant \frac{\sinh(\beta L_1)}{\sinh(\beta L_2)}\frac{\sinh\beta(L_2-L)}{\sinh\beta(L_1-L)}\|d^{k}(L_1,t)\|_\infty.$$

Theorem 1 shows that the convergence of the overlapping Schwarz method depend on $\gamma = \frac{\sinh(\beta L_1)}{\sinh(\beta L_2)}\frac{\sinh\beta(L_2-L)}{\sinh\beta(L_1-L)}$. Due to the characteristic of the sinh function we will have sharp decay of the error for any $L_1 < L_2$, and also for large size of overlapping.

3.2. Overlapping Schwarz wave for relaxation for decoupled system of convection diffusion reaction equation

In the following part of this section we are going to present the overlapping Schwarz wave form relaxation method defined for a system of convection diffusion reaction equation. Such a system defined by $u_i(x,t)$ for $i = 1,\ldots,I$ is given by

$$\begin{aligned}
&R_i u_{i,t} = D_i u_{i,xx} - v_i u_{i,x} - \lambda_i u_i \text{ over } \Omega, \quad t \in [T_0, T_f],\\
&u_i(0,t) = f_{i,1}(t), \quad t \in [T_0, T_f],\\
&u_i(L,t) = f_{i,2}(t), \quad t \in [T_0, T_f],\\
&u_i(x,T_0) = u_0, \quad x \in \Omega.
\end{aligned} \tag{26}$$

The considered system (26) is defined over the spatial domain $\Omega = \{0 < x < L\}$ and the overlapping Schwarz over relaxation method is constructed over the overlapping sub-domains $\Omega_1 = \{0 < x < L_2\}$ and $\Omega_2 = \{L_1 < x < L\}$ $L_1 < L_2$ with an overlapping size $(L_2 - L_1)$. In this work we are going to consider two types of systems of convection diffusion reaction equations, the decoupled and coupled systems.

To construct the wave form relaxation algorithm for (26) we will treat the case I=2. We consider first the solution of (26) over Ω_1 and Ω_2 as follows:

$$\begin{aligned}
&R_i v_{i,t} = D_i v_{i,xx} - v_i v_{i,x} - \lambda_i v_i \text{ over } \Omega_i, \quad t \in [T_0, T_f],\\
&v_i(0,t) = f_{i,1}(t), \quad t \in [T_0, T_f],\\
&v_i(L_2,t) = w_i(L_2,t), \quad t \in [T_0, T_f],\\
&v_i(x,T_0) = u_0, \quad x \in \Omega_i,
\end{aligned} \tag{27}$$

$$\begin{aligned}
&R_i w_{i,t} = D_i w_{i,xx} - v_i w_{i,x} - \lambda_i w_i \text{ over } \Omega_i, \quad t \in [T_0, T_f],\\
&w_i(L_1,t) = v_i(L_1,t), \quad t \in [T_0, T_f],\\
&w_i(L,t) = f_{i,2}(t), \quad t \in [T_0, T_f],\\
&w_i(x,T_0) = u_0, \quad x \in \Omega_i,
\end{aligned} \tag{28}$$

where $v_i(x,t) = u_i(x,t)|\Omega_1$ and $w_i(x,t) = u_i(x,t)|\Omega_2$.

Then the overlapping Schwarz wave form relaxation method for the decoupled system over the two overlapped sub-domains, Ω_1 and Ω_2, is given by

$$\begin{aligned}
&R_1 v_{1,t}^{k+1} = D_1 v_{1,xx}^{k+1} - v_1 v_{1,x}^{k+1} - \lambda_1 v_1^{k+1} \text{ over } \Omega_1, \quad t \in [T_0, T_f],\\
&v_1^{k+1}(0,t) = f_{1,1}(t), \quad t \in [T_0, T_f],\\
&v_1^{k+1}(L_2,t) = w_1^{k}(L_2,t), \quad t \in [T_0, T_f]\\
&v_1^{k+1}(x,T_0) = u_0, \quad x \in \Omega_1,
\end{aligned} \tag{29}$$

$$\begin{aligned}
&R_1 w_{1,t}^{k+1} = D_1 w_{1,xx}^{k+1} - v_1 w_{1,x}^{k+1} - \lambda_1 w_1^{k+1} \text{ over } \Omega_2, \quad t \in [T_0, T_f],\\
&w_1^{k+1}(L_1,t) = v_1^k(L_1,t), \quad t \in [T_0, T_f],\\
&w_1^{k+1}(L,t) = f_{1,2}(t), \quad t \in [T_0, T_f],\\
&w_1^{k+1}(x,T_0) = u_0, \quad x \in \Omega_2
\end{aligned} \tag{30}$$

for the system defined by u_1, for $i = 1$, and for the system defined by u_2, $i = 2$, it is given by

$$\begin{aligned}
&R_2 v_{2,t}^{k+1} = D_2 v_{2,xx}^{k+1} - v_2 v_{2,x}^{k+1} - \lambda_2 v_2^{k+1} \text{ over } \Omega_1, \quad t \in [T_0, T_f],\\
&v_2^{k+1}(0,t) = f_{2,1}(t), \quad t \in [T_0, T_f],\\
&v_2^{k+1}(L_2,t) = w_2^k(L_2,t), \quad t \in [T_0, T_f],\\
&v_2^{k+1}(x,T_0) = u_0, \quad x \in \Omega_1,
\end{aligned} \tag{31}$$

$$\begin{aligned}
&R_2 w_{2,t}^{k+1} = D_2 w_{2,xx}^{k+1} - v_2 w_{2,x}^{k+1} - \lambda_2 w_2^{k+1} \text{ over } \Omega_2, \quad t \in [T_0, T_f],\\
&w_2^{k+1}(L_1,t) = v_2^k(L_1,t), \quad t \in [T_0, T_f],\\
&w_2^{k+1}(L,t) = f_{2,2}(t), \quad t \in [T_0, T_f],\\
&w_2^{k+1}(x,T_0) = u_0, \quad x \in [T_0, T_f],
\end{aligned} \tag{32}$$

where k represents the iteration index.

Define $e_i^{k+1} = u - v_i^{k+1}$ and $d_i^{k+1} = u - w_i^{k+1}$, $i = 1,2$, to be the errors from the solution given by (29), (30) and (31), (32) over Ω_1 and Ω_2, respectively.

The corresponding differential equations satisfied by e_i^{k+1} and d_i^{k+1} over Ω_1 and Ω_2 for $i = 1,2$, are given by

$$\begin{aligned}
&R_1 e_{1,t}^{k+1} = D_1 e_{1,xx}^{k+1} - v_1 e_{1,x}^{k+1} - \lambda_1 e_1^{k+1} \text{ over } \Omega_1, \quad t \in [T_0, T_f],\\
&e_1^{k+1}(0,t) = 0, \quad t \in [T_0, T_f],\\
&e_1^{k+1}(L_2,t) = d_1^k(L_2,t), \quad t \in [T_0, T_f],\\
&e_1^{k+1}(x,T_0) = 0, \quad x \in \Omega_1,
\end{aligned} \tag{33}$$

$$\begin{aligned}
&R_1 d_{1,t}^{k+1} = D_1 d_{1,xx}^{k+1} - v_1 d_{1,x}^{k+1} - \lambda_1 d_1^{k+1} \text{ over } \Omega_2, \quad t \in [T_0, T_f],\\
&d_1^{k+1}(L_1,t) = e_1^k(L_1,t), \quad t \in [T_0, T_f],\\
&d_1^{k+1}(L,t) = 0, \quad t \in [T_0, T_f],\\
&d_1^{k+1}(x,T_0) = 0, \quad x \in [T_0, T_f]
\end{aligned} \tag{34}$$

for $i = 1$, and for $i = 2$ are given by

$$\begin{aligned}
&R_2 e_{2,t}^{k+1} = D_2 e_{2,xx}^{k+1} - v_2 e_{2,x}^{k+1} - \lambda_2 e_2^{k+1} \text{ over } \Omega_1, \quad t \in [T_0, T_f],\\
&e_2^{k+1}(0,t) = 0, \quad t \in [T_0, T_f],\\
&e_2^{k+1}(L_2,t) = d_2^k(L_2,t), \quad t \in [T_0, T_f],\\
&e_2^{k+1}(x,T_0) = u_0, \quad x \in \Omega_1,
\end{aligned} \tag{35}$$

$$\begin{aligned}
&R_2 d_{2,t}^{k+1} = D_2 d_{2,xx}^{k+1} - v_2 d_{2,x}^{k+1} - \lambda_2 d_2^{k+1} \text{ over } \Omega_2, \quad t \in [T_0, T_f],\\
&d_2^{k+1}(L_1,t) = e_2^k(L_1,t), \quad t \in [T_0, T_f],\\
&d_2^{k+1}(L,t) = 0, \quad t \in [T_0, T_f],\\
&d_2^{k+1}(x,T_0) = 0, \quad x \in [T_0, T_f].
\end{aligned} \tag{36}$$

The convergence and the error bound for the solution of (29), (30) and (31), (32) are given by the following theorem.

Theorem 2. *Let* e_i^{k+1} *and* d_i^{k+1} $(i = 1,2)$ *be the errors of the subproblems defined by the differential equations* (29), (30) *and* (31), (32) *over* Ω_1 *and* Ω_1, *respectively, then*

$$\|e_i^{k+2}(L_1,t)\|_\infty \leqslant \gamma_i \|e_i^k(L_1,t)\|_\infty$$

and

$$\|d_i^{k+2}(L_2,t)\|_\infty \leqslant \gamma_i \|d_i^k(L_1,t)\|_\infty,$$

where

$$\gamma_i = \frac{\sinh(\beta_i L_1)}{\sinh(\beta_i L_2)} \frac{\sinh(\beta_i (L_2 - L))}{\sinh(\beta_i (L_1 - L))} < 1$$

for $i = 1, 2$.

Proof. The proof will be carried out with utilization of the proof given for Theorem 1.

Let e_i and d_i be the errors of the approximated solutions u_i, i=1,2, over Ω_1 and Ω_2, respectively.

Similarly to the proof presented for Theorem 1 for each of the error differential equations (33), (34) and (35), (36), we will conclude the following relation:

$$\|e_i^{k+2}(L_1,t)\|_\infty \leqslant \gamma_i \|e_i^k(L_1,t)\|_\infty$$

and

$$\|d_i^{k+2}(L_2,t)\|_\infty \leqslant \gamma_i \|d_i^k(L_1,t)\|_\infty,$$

where

$$\gamma_i = \frac{\sinh(\beta_i L_1)}{\sinh(\beta_i L_2)} \frac{\sinh(\beta_i (L_2 - L))}{\sinh(\beta_i (L_1 - L))} < 1$$

and $\beta_i = \frac{\sqrt{v_i^2 + 4D_i\lambda_i}}{2D_i}$ for $i = 1, 2$. $\square$

3.3. Overlapping Schwarz wave form relaxation for coupled system of convection diffusion reaction equation

In the following part we are going to present the convergence and the error bound for the solution of the coupled system of convection diffusion reaction equations, defined by two functions u_1 and u_2, with the overlapping Schwarz wave form relaxation. The coupling criteria in this case are imposed within the source term of the second solution component. The considered coupled system defined by u_1 and u_2 is given by

$$\begin{aligned}
&R_1 u_{1,t} = D_1 u_{1,xx} - v_1 u_{1,x} - \lambda_1 u_1 \text{ over } \Omega = \{0 < x < L\}, \quad t \in [T_0, T_f],\\
&u_1(0,t) = f_{1,1}(t), \quad t \in [T_0, T_f],\\
&u_1(L,t) = f_{1,2}(t), \quad t \in [T_0, T_f],\\
&u_1(x, T_0) = u_0,
\end{aligned} \tag{37}$$

for u_1, and for u_2 is given by

$$\begin{aligned}
&R_2 u_{2,t} = D_2 u_{2,xx} - v_2 u_{2,x} - \lambda_2 u_2 + \lambda_1 u_1 \text{ over } \Omega, \quad t \in [T_0, T_f],\\
&u_2(0,t) = f_{2,1}(t), \quad t \in [T_0, T_f],\\
&u_2(L,t) = f_{2,2}(t), \quad t \in [T_0, T_f],\\
&u_2^{k+1}(x, T_0) = u_0.
\end{aligned} \tag{38}$$

In (38) the coupling appeared in the source term and is defined by the parameter λ_1 with the first component u_1. The strength or the *bound* of the coupling and the contribution is related to the value of the scalar λ_1. The coupled case (38) is reduced to the decoupled case (26), by assuming $\lambda_1 = 0$ for $i = 2$.

The overlapping Schwarz wave form relaxation for (37) over Ω_1 and Ω_2 is given by

$$\begin{aligned}
&R_1 v_{1,t}^{k+1} = D_1 v_{1,xx}^{k+1} - v_1 v_{1,x}^{k+1} - \lambda_1 v_1^{k+1} \text{ over } \Omega_1, \quad t \in [T_0, T_f],\\
&v_1^{k+1}(0,t) = f_{1,1}(t), \quad t \in [T_0, T_f],\\
&v_1^{k+1}(L_2,t) = w_1^k(L_2,t), \quad t \in [T_0, T_f],\\
&v_1^{k+1}(x, T_0) = u_0, \quad x \in \Omega_1
\end{aligned} \tag{39}$$

$$\begin{aligned}
&R_1 w_{1,t}^{k+1} = D_1 w_{1,xx}^{k+1} - v_1 w_{1,x}^{k+1} - \lambda_1 w_1^{k+1} \text{ over } \Omega_2, \quad t \in [T_0, T_f],\\
&w_1^{k+1}(L_1, t) = v_1^k(L_1, t), \quad t \in [T_0, T_f],\\
&w_1^{k+1}(L, t) = f_{1,2}(t), \quad t \in [T_0, T_f],\\
&w_1^{k+1}(x, T_0) = u_0, \quad x \in \Omega_2
\end{aligned} \tag{40}$$

and for the system defined by (38) the Schwarz wave form relaxation is given as

$$\begin{aligned}
&R_2 v_{2,t}^{k+1} = D_2 v_{2,xx}^{k+1} - v_2 v_{2,x}^{k+1} - \lambda_2 v_2^{k+1} + \lambda_1 v_1^{k+1} \text{ over } \Omega_1, \quad t \in [T_0, T_f],\\
&v_2^{k+1}(0, t) = f_{2,1}(t), \quad t \in [T_0, T_f],\\
&v_2^{k+1}(L_2, t) = w_2^k(L_2, t), \quad t \in [T_0, T_f],\\
&v_2^{k+1}(x, T_0) = u_0, \quad x \in \Omega_1,
\end{aligned} \tag{41}$$

$$\begin{aligned}
&R_2 w_{2,t}^{k+1} = D_2 w_{2,xx}^{k+1} - v_2 w_{2,x}^{k+1} - \lambda_2 w_2^{k+1} + \lambda_1 w_1^{k+1} \text{ over } \Omega_2, \quad t \in [T_0, T_f],\\
&w_2^{k+1}(L_1, t) = v_2^k(L_1, t), \quad t \in [T_0, T_f],\\
&w_1^{k+1}(L, t) = f_{2,2}(t), \quad t \in [T_0, T_f],\\
&w_1^{k+1}(x, T_0) = u_0, \quad x \in \Omega_2.
\end{aligned} \tag{42}$$

Define $e_i^{k+1} = u - v_i^{k+1}$ and $d_i^{k+1} = u - w_i^{k+1}$, $i = 1, 2$, to be the errors from the solution given by (39), (40) and (41), (42) over Ω_1 and Ω_2, respectively.

The convergence and the error bound for $e_i^{k+1} = u - v_i^{k+1}$ and $d_i^{k+1} = u - w_i^{k+1}$ are given by the following theorem.

Theorem 3. *Let e_i^{k+1} and d_i^{k+1} $(i = 1, 2)$ be the errors from the solution of the subproblems (39), (40) and (41), (42) occurring by Schwarz wave form relaxation over Ω_1 and Ω_2, respectively. Then the error bounds for (39), (40) (coupled equations) defined by e_1 and d_1 over Ω_1 and Ω_2 are given by*

$$\|e_1^{k+2}(L_1, t)\|_\infty \leqslant \gamma_1 \|e_1^k(L_1, t)\|_\infty \tag{43}$$

and

$$\|d_1^{k+2}(L_2, t)\|_\infty \leqslant \gamma_1 \|d_1^k(L_1, t)\|_\infty, \tag{44}$$

respectively.

Furthermore, the error bound for (41), (42) (decoupled equations) defined by e_2 and d_2 over Ω_1 and Ω_2 are given by

$$\begin{aligned}
\|e_2^{k+2}(L_1, t)\|_\infty \leqslant{}& \|e_2^k(L_1, t)\|_\infty \gamma_2 + \gamma_2 \frac{\lambda_1}{\lambda_2} \Psi \left[1 + e^{\alpha_2(L_1 - L)} e^{\beta_2(L - L_1)}\right]\\
&+ \frac{\lambda_1}{\lambda_2} \Psi \left[e^{\alpha_2(L_1 - L_2)} \frac{\sinh \beta_2 L_1}{\sinh \beta_2 L_2} - e^{\alpha_2(L_1 - L)} e^{\beta_2(L - L_2)} \frac{\sinh \beta_2 L_1}{\sinh \beta_2 L_2}\right]\\
&+ \frac{\lambda_1}{\lambda_2} \Psi \left[e^{\alpha_2 L_1} \frac{\sinh \beta_2 (L_1 - L_2)}{\sinh \beta_2 L_2} - e^{\alpha_2(L_1 - L_2)} \frac{\sinh \beta_2 L_1}{\sinh \beta_2 L_2} + 1\right]
\end{aligned} \tag{45}$$

and

$$\begin{aligned}
\|d_2^{k+2}(L_2, t)\|_\infty \leqslant{}& \|d_2^k(L_2, t)\|_\infty \gamma_2 + \gamma_2 \frac{\lambda_1}{\lambda_2} \Psi \left[1 + e^{\alpha_2(L_1 - L)} e^{\beta_2(L - L_1)}\right]\\
&+ \frac{\lambda_1}{\lambda_2} \Psi \left[e^{\alpha_2(L_1 - L_2)} \frac{\sinh \beta_2 L_1}{\sinh \beta_2 L_2} - e^{\alpha_2(L_1 - L)} e^{\beta_2(L - L_2)} \frac{\sinh \beta_2 L_1}{\sinh \beta_2 L_2}\right]\\
&+ \frac{\lambda_1}{\lambda_2} \Psi \left[e^{\alpha L_1} \frac{\sinh \beta_2 (L_1 - L_2)}{\sinh \beta_2 L_2} - e^{\alpha_2(L_1 - L_2)} \frac{\sinh \beta_2 L_1}{\sinh \beta_2 L_2} + 1\right],
\end{aligned} \tag{46}$$

respectively, *where*

$$\gamma_i = \frac{\sinh \beta_i L_1}{\sinh \beta_i L_2} \frac{\sinh \beta_i (L_2 - L)}{\sinh \beta_i (L_1 - L)} < 1,$$

with $\alpha_i = \frac{v_i}{2D_i}$, $\beta_i = \frac{\sqrt{v_i^2 + 4D_i\lambda_i}}{2D_i}$, *for* $i = 1, 2$, *and* $\Psi = \max_\Omega\{e_1, e_2\}$.

Proof. For the proof of (43) and (44) we work in the same manner as in the proof of Theorem 2 for the decoupled system. The remaining part of the proof is dealing with the convergence and the error bound for the coupled equations of the system.

Let $e_2^{k+1}(x,t) := u_2(x,t) - v_2^{k+1}(x,t)$ and $d_2^{k+1}(x,t) := u_2(x,t) - w_2^{k+1}(x,t)$ be the errors of (41) and (42) over Ω_1 and Ω_2 respectively. Then the corresponding differential equations defined by $e_2(x,t)$ and $d_2(x,t)$ are given by:

$$\begin{aligned}
&R_2 e_{2,t}^{k+1} = D_2 e_{2,xx}^{k+1} - v_2 e_{2,x}^{k+1} - \lambda_2 e_2^{k+1} + \lambda_1 e_1^{k+1} \text{ over } \Omega_1, \quad t \in [T_0, T_f],\\
&e_2^{k+1}(0,t) = 0, \quad t \in [T_0, T_f],\\
&e_2^{k+1}(L_2,t) = d_2^k(L_2,t), \quad t \in [T_0, T_f],\\
&e_2^{k+1}(x,T_0) = 0, \quad x \in \Omega_2,
\end{aligned} \tag{47}$$

$$\begin{aligned}
&R_2 d_{2,t}^{k+1} = D_2 d_{2,xx}^{k+1} - v_2 d_{2,x}^{k+1} - \lambda_2 d_2^{k+1} + \lambda_1 d_1^{k+1} \text{ over } \Omega_2, \quad t \in [T_0, T_f],\\
&d_2^{k+1}(L_1,t) = e_2^k(L_1,t), \quad t \in [T_0, T_f],\\
&d_1^{k+1}(L,t) = 0, \quad t \in [T_0, T_f],\\
&d_1^{k+1}(x,T_0) = 0, \quad x \in \Omega_2. \qquad \square
\end{aligned} \tag{48}$$

Furthermore we consider the following differential equations defined by $\hat{e}^{k+1}$ and $\hat{d}^{k+1}$ given by

$$\begin{aligned}
&R_2 \hat{e}_{2,t}^{k+1} = D_2 \hat{e}_{2,xx}^{k+1} - v_2 \hat{e}_{2,x}^{k+1} - \lambda_2 \hat{e}_2^{k+1} + \lambda_1 \Psi \text{ over } \Omega_1, \quad t \in [T_0, T_f],\\
&\hat{e}_2^{k+1}(0,t) = 0, \quad t \in [T_0, T_f],\\
&\hat{e}_2^{k+1}(L_2,t) = \|d_2^k(L_2,t)\|_\infty, \quad t \in [T_0, T_f],\\
&\hat{e}_2^{k+1}(x,T_0) = \mathscr{A}(x), \quad x \in \Omega_1,
\end{aligned} \tag{49}$$

where

$$\mathscr{A}(x) = \|d_2^k(L_2,t)\|_\infty e^{\alpha_2(x-L_2)} \frac{\sinh(\beta_2 x)}{\sinh(\beta_2 L)} + \frac{\lambda_1}{\lambda_2}\Psi\left[e^{\alpha_2 x}\frac{\sinh(\beta_2(x-L_2))}{\sinh(\beta_2 L_2)} - e^{\alpha_2(x-L_2)}\frac{\sinh \beta_2 x}{\sinh \beta_2 L_2} + 1\right]$$

and

$$\begin{aligned}
&R_2 \hat{d}_{2,t}^{k+1} = D_2 \hat{d}_{2,xx}^{k+1} - v_2 \hat{d}_{2,x}^{k+1} - \lambda_2 \hat{d}_2^{k+1} + \lambda_1 \Psi \text{ over } \Omega_2, \quad t \in [T_0, T_f],\\
&\hat{d}_2^{k+1}(L_1,t) = \|e_2^k(L_1,t)\|_\infty, \quad t \in [T_0, T_f],\\
&\hat{d}_2^{k+1}(L,t) = 0, \quad t \in [T_0, T_f],\\
&\hat{d}_2^{k+1}(x,T_0) = \mathscr{B}(x), \quad x \in \Omega_2,
\end{aligned} \tag{50}$$

where

$$\begin{aligned}
\mathscr{B}(x) = {} & \|e^k(L_1,t)\|_\infty e^{\alpha_2(x-L_1)} \frac{\sinh(\beta_2(x-L))}{\sinh(\beta_2(L_1-L))} + \frac{\lambda_1}{\lambda_2}\Psi \frac{\sinh(\beta_2(L-x))}{\sinh(\beta_2(L_1-L))}\left[e^{\alpha_2(x-L_1)} - e^{\alpha_2(x-L)}e^{\beta_2(L-L_1)}\right]\\
& - \frac{\lambda_1}{\lambda_2}\Psi\left[1 - e^{\alpha_2(x-L)}e^{\beta_2(L-x)}\right].
\end{aligned}$$

Then the solution of (49) and (50) is the steady state solution given by

$$\hat{e}_2^{k+1}(x) = \|d_2^k(L_2,t)\|_\infty e^{\alpha_2(x-L_2)} \frac{\sinh(\beta_2 x)}{\sinh(\beta_2 L)} + \frac{\lambda_1}{\lambda_2}\Psi\left[e^{\alpha_2 x}\frac{\sinh(\beta_2(x-L_2))}{\sinh(\beta_2 L_2)} - e^{\alpha_2(x-L_2)}\frac{\sinh \beta_2 x}{\sinh \beta_2 L_2} + 1\right]$$

and

$$\hat{d}_2^{k+1}(x) = \|e^k(L_1,t)\|_\infty e^{\alpha_2(x-L_1)}\frac{\sinh(\beta_2(x-L))}{\sinh(\beta_2(L_1-L))} + \frac{\lambda_1}{\lambda_2}\Psi\frac{\sinh(\beta_2(L-x))}{\sinh(\beta_2(L_1-L))}\left[e^{\alpha_2(x-L_1)} - e^{\alpha_2(x-L)}e^{\beta_2(L-L_1)}\right] - \frac{\lambda_1}{\lambda_2}\Psi\left[1 - e^{\alpha_2(x-L)}e^{\beta_2(L-x)}\right],$$

respectively.

By defining the function $E(x,t) = \hat{e}^{k+1} - e^{k+1}$, as in the proof of Theorem 1, and by the maximum principle theorem we conclude that

$$|e_2^{k+1}| \leqslant \hat{e}_2^{k+1}$$

for all (x,t) and similarly

$$|d_2^{k+1}| \leqslant \hat{d}_2^{k+1}.$$

Then

$$|e_2^{k+1}(x,t)| \leqslant \|d_2^k(L_2,t)\|_\infty e^{\alpha_2(x-L_2)}\frac{\sinh(\beta_2 x)}{\sinh(\beta_2 L)} + \frac{\lambda_1}{\lambda_2}\Psi\left[e^{\alpha_2 x}\frac{\sinh(\beta_2(x-L_2))}{\sinh(\beta_2 L_2)} - e^{\alpha_2(x-L_2)}\frac{\sinh\beta_2 x}{\sinh\beta_2 L_2} + 1\right] \tag{51}$$

and

$$|d_2^{k+1}(x,t)| \leqslant \|e^k(L_1,t)\|_\infty e^{\alpha_2(x-L_1)}\frac{\sinh(\beta_2(x-L))}{\sinh(\beta_2(L_1-L))} + \frac{\lambda_1}{\lambda_2}\Psi\frac{\sinh(\beta_2(L-x))}{\sinh(\beta_2(L_1-L))}\left[e^{\alpha_2(x-L_1)} - e^{\alpha_2(x-L)}e^{\beta_2(L-L_1)}\right] - \frac{\lambda_1}{\lambda_2}\Psi\left[1 - e^{\alpha_2(x-L)}e^{\beta_2(L-x)}\right]. \tag{52}$$

By evaluating (52) at $x = L_2$, substituting the results in (51) and afterwards evaluating the resulting relation at $x = L_1$ we observe that (45) holds in general.

Similarly (46) will follow from the evaluation of (51) at $x = L_1$, substituting in (52) and then by evaluating the resulting relation at $x = L_2$.

For the decoupled case of the convection diffusion reaction-equation, the convergence of the overlapping Schwarz wave form relaxation method given by Theorem 2 depends on the factor $\gamma_i = \frac{\sinh(\beta_i L_1)}{\sinh(\beta_i L_2)}\frac{\sinh\beta_i(L_2-L)}{\sinh\beta_i(L_1-L)}$ which is obtained earlier for single scalar convection diffusion reaction equation in Theorem 1.

For the coupled system we demonstrated Theorem 3 and assume that the error depends on two main factors, the convergence parameter γ_i and the coupling parameter λ_1 defining the coupled system (37), (38). Its obvious that for the coupling parameter $\lambda_1 = 0$ we retain the decoupled system and faster convergence rate is achieved if we have a small ratio $\frac{\lambda_1}{\lambda_2}$.

4. Numerical results

In this section we will illustrate the numerical results from the solution of several model problems using the proposed methods. The problems are discretized using second order approximation with respect to the spatial variable using regular mesh spacing $h(= L/N)$ and backward approximation with respect to the time using Δt time stepping. The first order operator splitting method (FOP) is considered to be the basic solution algorithm for the overlapping Schwarz waveform relaxation method (FOPSWR).

4.1. First example: Convection diffusion reaction equation

We consider the one-dimensional convection diffusion reaction equation given by

$$\partial_t u + v\partial_x u - \partial_x D\partial_x u = -\lambda u, \tag{53}$$

defined over $\Omega \times [T_0, T_f] = [0,150] \times [100, 10^5]$, with an exact solution given by

$$u(x,t) = \frac{u_0}{2\sqrt{D\pi t}}\exp\left(-\frac{(x-vt)^2}{4Dt}\right)\exp(-\lambda t). \tag{54}$$

The initial condition and the Dirichlet boundary condition are defined using the exact solution (54) at starting time $T_0 = 100$ and with $u_0 = 1.0$. We have $\lambda = 10^{-5}$, $v = 0.001$ and $D = 0.0001$.

We considered the backward Euler discretization for both of the splitted operators, i.e. the convection and the diffusion operator, to simulate the solution over the time interval $[100, 10^5]$.

The model problem (53) is solved using first order operator splitting (FOP), and also the operator splitting as basic solver for the overlapping Schwarz wave form relaxation method (FOPSWR).

We compare the accuracy of the solution over the entire spatial domain with different h values, and different time steps δt, using FOP-method, and FOPSWR-method over two subdomains with different size of overlapping. The errors of the solution are given in Tables 1 and 2, respectively. The FOPSWR-method is considered over two overlapping subdomains of different overlapping size $L_2 - L_1$, in order to discuss on the accuracy of the algorithm with the operator splitting. The considered subdomains were $\Omega_1 = [0, 60]$, and $\Omega_2 = [30, 150]$ with size of overlapping 30, and $\Omega_1 = [0, 100]$, and $\Omega_2 = [30, 150]$ with size of overlapping 70.

The graphical output for the FOP-method is presented in Fig. 1.

In our numerical computations we refined the time and space partitions systematically in order to visualize the accuracy and error reduction through the simulation over the time interval for refined time and space steps. From Table 1 we observed that by the FOP-method the error reduced as first and second order with respect to the time and space.

Table 1
The L_∞-error in time and space for the convection diffusion reaction-equation using FOP-method

Time	err_{u_1}	err_{u_1}	err_{u_1}
$\frac{\Delta t}{4} = 5$	0.001108	2.15813e−4	6.55262e−5
$\frac{\Delta t}{2} = 10$	0.00113	2.3942e−4	8.6641e−5
$\Delta t = 20$	0.001195	2.86514e−4	1.2868e−4
Mesh size	$h_0 = 1$	$h = h_{0/2}$	$h = h_{0/4}$

Table 2
The L_∞-error in time and space for the scalar convection diffusion reaction-equation using FOPSWR-method for two different sizes of overlapping 30 and 70

Time	err_{u_1}	err_{u_1}	err_{u_1}	err_{u_1}	err_{u_1}	err_{u_1}
$\Delta\frac{t}{4} = 5$	1.108e−3	1.08e−3	2.159e−4	2.158e−4	6.782e−5	6.552e−5
$\Delta\frac{t}{2} = 10$	1.138e−3	1.137e−3	2.397e−4	2.394e−4	8.681e−5	8.681e−5
$\Delta t = 20$	1.196e−3	1.195e−3	2.871e−4	2.865e−4	1.290e−4	1.286e−4
Overlap size	30	70	30	70	30	70
Mesh size	$h_0 = 1$		$h = h_{0/2}$		$h = h_{0/4}$	

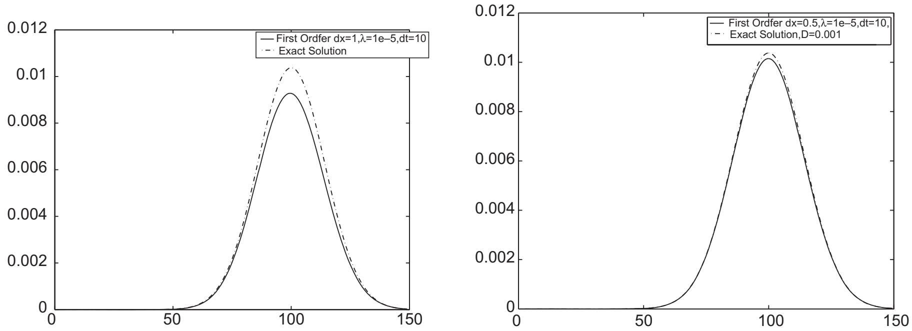

Fig. 1. The solution by the FOP-method.

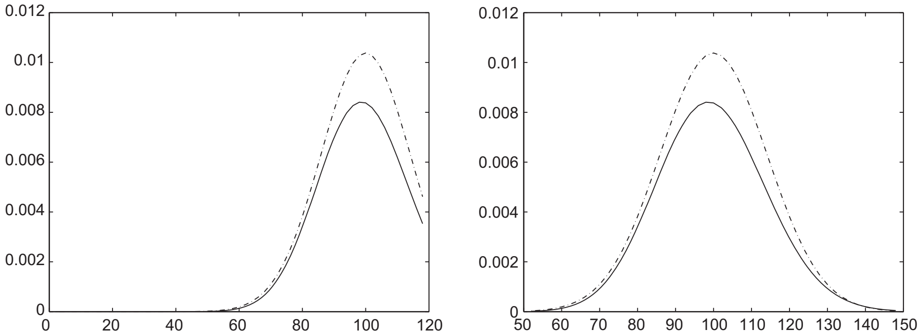

Fig. 2. The solution by the FOPSWR method using two subdomains with size of overlapping = 30.

For the solution by the FOPSWR-method, using the FOP-method as basic solver, the accuracy of the solution, over subdomains Ω_1 and Ω_2 with different size of overlapping, is compared with the accuracy from the solution by the FOP-method.

From Tables 1 and 2 we observe that the accuracy of the solution improved when we increased the size of overlapping of the subdomains Ω_1 and Ω_2 from 30 to 70. Furthermore, the accuracy of the solution with overlapping 70 possesses same accuracy as the solution by the FOP method only.

The numerical solution by the FOPSWR method is presented in Fig. 2.

4.2. Second example: system of convection diffusion reaction equation (decoupled)

We consider a system of one-dimensional convection diffusion reaction equation of decoupled type

$$R_i \partial_t u_i + v \partial_x u_i - \partial_x D \partial_x u_i = -R_i \lambda_i u_i, \tag{55}$$

$$u_i(x, T_0) = u_L^i(x, t_0), \tag{56}$$

defined over $\Omega \times [T_0, T_f] = [0, 150] \times [100, 10^5]$, with the analytical solution, cf. [7]

$$u_L^i(x,t) = \frac{u_{\text{in}}^i}{2R_i\sqrt{D\pi t/R_i}} e^{-\frac{(x-vt/R_i)^2}{4Dt/R_i}} e^{(-\lambda_i t)}, \tag{57}$$

where $i = 1, 2$.

The initial and boundary conditions are defined using the exact solution (57) with the parameters $u_{\text{in}}^1 = 1.0$, $u_{\text{in}}^2 = 1.0$, the starting-time $T_0 = 100$, $\lambda_1 = 1.0 \times 10^{-5}$, $\lambda_2 = 4.0 \times 10^{-5}$, $v = 0.001$, $D = 0.0001$, $R_1 = 2.0$, and $R_2 = 1.0$.

The results for the operator-splitting splitting method for the nonoverlapping domain are given in Table 3.

In the next experiments we consider the overlapping Schwarz waveform relaxation method over two overlapped subdomains. For the overlapping we obtain the overlap size of 30 and 70, i.e. $\Omega_1 = \{0 < x < 60\}$ and $\Omega_2 = \{30 < x < 150\}$ while for the other case we have $\Omega_1 = \{0 < x < 100\}$ and $\Omega_2 = \{30 < x < 150\}$, respectively.

The errors of the solution over the subdomains Ω_1 and Ω_2 with sizes of overlapping 30 and 70 are presented in Tables 4 and 5, respectively.

Table 3
L_∞-error in time and space for the decoupled system solved by operator splitting method (FOP)

Time	err_{u_1}	err_{u_2}	err_{u_1}	err_{u_2}	err_{u_1}	err_{u_2}
$\frac{\Delta t}{4} = 5$	4.461e−4	2.045e−4	3.466e−4	1.940e−4	9.110e−5	1.792e−4
$\frac{\Delta t}{2} = 10$	4.506e−4	2.055e−04	3.515e−4	1.948e−4	9.528e−5	1.794e−4
$\Delta t = 20$	4.594e−4	2.075e−4	3.611e−4	1.963e−4	1.036e−4	1.799e−4
Mesh size	$h_0 = 1$		$h = h_{0/2}$		$h = h_{0/4}$	

Table 4
L_∞-error in time and space for the decoupled system with size of overlapping = 30 using FOPSWR method

Time	err_{u_1}	err_{u_2}	err_{u_1}	err_{u_2}	err_{u_1}	err_{u_2}
$\frac{\Delta t}{4}=5$	4.461e−4	2.051e−4	3.471e−4	1.944e−4	9.110e−5	1.792e−4
$\frac{\Delta t}{2}=10$	4.506e−4	2.058e−04	3.521e−4	1.951e−4	9.528e−5	1.795e−4
$\Delta t=20$	4.594e−4	2.080e−4	3.631e−4	1.971e−4	1.036e−4	1.800e−4
Mesh size	$h_0=1$		$h=h_{0/2}$		$h=h_{0/4}$	
Overlap size	30					

Table 5
L_∞-error in time and space for the decoupled system with size of overlapping = 70 using FOPSWR method

Time	err_{u_1}	err_{u_2}	err_{u_1}	err_{u_2}	err_{u_1}	err_{u_2}
$\frac{\Delta t}{4}=5$	4.461e−4	2.046e−4	3.466e−4	1.941e−4	9.110e−5	1.792e−4
$\frac{\Delta t}{2}=10$	4.506e−4	2.055e−04	3.515e−4	1.948e−4	9.528e−5	1.794e−4
$\Delta t=20$	4.594e−4	2.075e−4	3.611e−4	1.963e−4	1.036e−4	1.800e−4
Mesh size	$h_0=1$		$h=h_{0/2}$		$h=h_{0/4}$	
Overlap size	70					

By comparing the results in Tables 4 and 5 we could observe an improvement in the accuracy when the size of overlapping increased. Because of the decoupling, each equation could be computed separately. For the first component we derive improved results because of the smaller reaction in the equation.

However, the accuracy of the solution by the FOPSWR method remains the same as when we solved the system by the FOP method. Henceforth, from what has been presented so far, we conclude that the FOPSWR is an efficient parallel solution algorithm for the decoupled systems over different overlapped subdomains with the same accuracy as when we consider the FOP method only.

4.3. Third example: system of convection diffusion reaction equation (coupled)

Finally we deal with the following example of a coupled system of one-dimensional convection diffusion reaction equation

$$R_1\partial_t u_1 + v\partial_x u_1 - \partial_x D\partial_x u_1 = -R_1\lambda_1 u_1, \tag{58}$$

$$R_2\partial_t u_2 + v\partial_x u_2 - \partial_x D\partial_x u_2 = R_1\lambda_1 u_1 - R_2\lambda_2 u_2, \tag{59}$$

$$u_1(x,t_0) = u_L^1(x,t_0), \quad u_2(x,t_0) = u_L^2(x,t_0), \tag{60}$$

defined over $\Omega \times [T_0,T_f] = [0,150]\times[100,10^5]$, with an exact solution given by, cf. [7]

$$u_L^1(x,t) = \frac{u_{\text{in}}^1}{2R_1\sqrt{D\pi t/R_1}} e^{-\frac{(x-vt/R_1)^2}{4Dt/R_1}} e^{(-\lambda_1 t)},$$

$$u_L^2(x,t) = \frac{u_{\text{in}}^2}{2R_2\sqrt{D\pi t/R_2}} e^{-\frac{(x-vt/R_2)^2}{4Dt/R_2}} e^{(-\lambda_2 t)} + \frac{R_1\lambda_1 u_{\text{in}}^1}{2\sqrt{D\pi(R_1-R_2)}} \exp\left(\frac{xv}{2D}\right) e^{\frac{-(R_1\lambda_1-R_2\lambda_2)t}{(R_1-R_2)}} (W(v_2)-W(v_1)),$$

$$v_1 = \sqrt{R_1\lambda_1 - \frac{(R_1\lambda_1-R_2\lambda_2)}{R_1-R_2}R_1 + v^2/(4D)}, \quad v_2 = \sqrt{R_2\lambda_2 - \frac{(R_1\lambda_1-R_2\lambda_2)}{R_1-R_2}R_2 + v^2/(4D)},$$

$$W(v) = 0.5\left(\exp\left(-\frac{xvv}{2D}\right)\text{erfc}\left(\frac{x-vvt}{\sqrt{4Dt}}\right) + \exp\left(\frac{xvv}{2D}\right)\text{erfc}\left(\frac{x+vvt}{\sqrt{4Dt}}\right)\right),$$

where $\text{erfc}(\cdot)$ is the known error-function and we have the following conditions : $R_1 > R_2$ and $\lambda_2 > \lambda_1$. As shown in (59) the coupling of the system is through the source term defined by $R_1\lambda_1 u_1$.

Table 6
L_∞-error in time and space for the system of convection diffusion reaction-equation using first order splitting, with $\lambda_1 = 2e{-}5$, $\lambda_2 = 4e{-}5$

Time	err_{u_1}	err_{u_2}	err_{u_1}	err_{u_2}	err_{u_1}	err_{u_2}
$\frac{\Delta t}{4} = 5$	4.461e−4	2.403e−3	3.466–4	2.452e−3	9.110e−5	2.702e−3
$\frac{\Delta t}{2} = 10$	4.506e−4	2.39e−3	3.515e−4	2.447e−3	9.528e−5	2.697e−3
$\Delta t = 20$	4.594e−4	2.8e−3	3.611e−4	2.438e−3	1.036e−4	2.689e−3
Mesh size	$h_0 = 1$		$h = h_{0/2}$		$h = h_{0/4}$	

Table 7
L_∞-error in time and space for the system of convection diffusion reaction-equation using first order splitting and Schwarz wave form relaxation method, with $\lambda_1 = 2e{-}5$, $\lambda_2 = 4e{-}5$

Time	err_{u_1}	err_{u_2}	err_{u_1}	err_{u_2}	err_{u_1}	err_{u_2}
$\frac{\Delta t}{4} = 5$	4.461e−4	2.403e−3	3.466e−4	2.452e−3	9.110e−5	2.702e−3
$\frac{\Delta t}{2} = 10$	4.506e−4	2.398e−3	3.515e−4	2.447e−3	9.528e−5	2.697e−3
$\Delta t = 20$	4.594e−4	2.388e−3	3.611e−4	2.438e−3	1.036e−4	2.689e−3
Mesh size	$h_0 = 1$		$h = h_{0/2}$		$h = h_{0/4}$	
Overlap size	70					

The initial conditions are $u_{in}^1 = 1.0$ and $u_{in}^2 = 0.0$ and the starting-time $t_0 = 100$. As boundary conditions we use the Dirichlet-boundary-conditions defined through the exact solution. We have the following equation-parameters $\lambda_1 = 2.010^{-5}$, $\lambda_2 = 4.010^{-5}$, $v = 0.001$, $D = 0.0001$, $R_1 = 2.0$, and $R_2 = 1.0$.

In the next tables we compare the accuracy of the first order operator splitting with the accuracy of the solution by the overlapping Schwarz wave form relaxation method and we test different reaction-parameters.

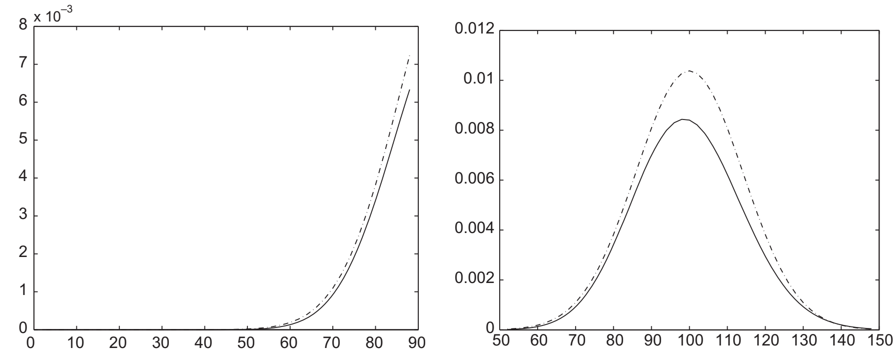

Fig. 3. The solution by the FOPSWR method using two subdomains with size of overlapping = 70.

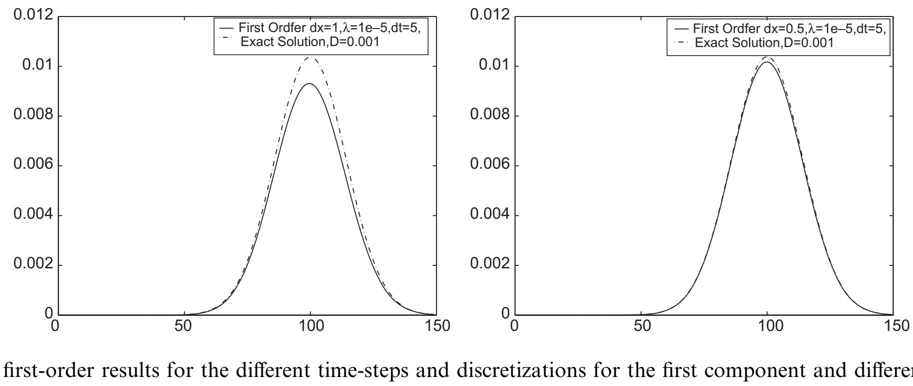

Fig. 4. The first-order results for the different time-steps and discretizations for the first component and different time-steps.

Table 8
L_∞-error in time and space for the system of convection diffusion reaction-equation using first order splitting, with $\lambda_1 = 1e{-}9$, $\lambda_2 = 4e{-}5$

Time	err_{u_1}	err_{u_2}	err_{u_1}	err_{u_2}	err_{u_1}	err_{u_2}
$\frac{\Delta t}{4} = 5$	3.297e−3	6.058e−7	2.562e−3	6.192e−7	6.753e−4	6.820e−7
$\frac{\Delta t}{2} = 10$	3.30e−3	6.044e−7	2.599e−3	6.179e−7	7.083e−4	6.808e−7
$\Delta t = 20$	3.396e−3	6.018e−7	2.673e−3	6.152e−7	7.746e−4	6.784e−7
Mesh size	$h_0 = 2$		$h = h_{0/2}$		$h = h_{0/4}$	

Table 9
L_∞-error in time and space for the system of convection diffusion reaction-equation using first order splitting and overlapping Schwarz wave form relaxation method with $\lambda_1 = 1e{-}9$, $\lambda_2 = 4e{-}5$

Time	err_{u_1}	err_{u_2}	err_{u_1}	err_{u_2}	err_{u_1}	err_{u_2}
$\frac{\Delta t}{4} = 5$	3.297e−3	6.058e−7	2.545e−3	6.192e−7	6.753e−4	6.820e−7
$\frac{\Delta t}{2} = 10$	3.314e−3	6.044e−7	2.599e−3	6.179e−7	7.083e−4	6.808e−7
$\Delta t = 20$	3.380e−3	6.018e−7	2.673e−3	6.152e−7	7.746e−4	6.784e−7
Mesh size	$h_0 = 2$		$h = h_{0/2}$		$h = h_{0/4}$	
Overlap size	70					

The results for the operator-splitting method are given in Table 6.

The results for the overlapping Schwarz wave form relaxation method are given in Table 7 when the system (58)–(60) is solved over the overlapped subdomains $\Omega_1 = \{0 < x < 100\}$ and $\Omega_2 = \{30 < x < 150\}$. From the comparison of the error given in Tables 7 and 6 we observe that the solution by the FOPSWR method for the coupled system is of the same accuracy as the solution by FOP method Fig. 3.

Fig. 4 shows the result for the system, where the solutions for different time-steps are presented.

For the second experiment we modify the reaction parameter to observe the influence between the first and the second component. First we solved the system over nonoverlapped domain Ω using the first order operator splitting and the errors of the solutions are presented in Table 8.

Then we use the overlapping Schwarz wave form relaxation method solved over the overlapped subdomains $\Omega_1 = \{0 < x < 100\}$ and $\Omega_2 = \{30 < x < 150\}$ and the results are given in Table 9.

In Tables 8 and 9 we see higher order results in space for the first component. For the second component the influence of the first component is important and we could decrease the error of the second component, while the error of the first component decreases. The results for the modified method are shown in Fig. 2.

Because of a very strong coupling of the equation the methods should be of second order because the reduction of the error increases and the computations are more effective. From Tables 9 and 8 we observe that the accuracy of the solution for the strong coupling system by the FOPSWR method, over an overlapped subdo-

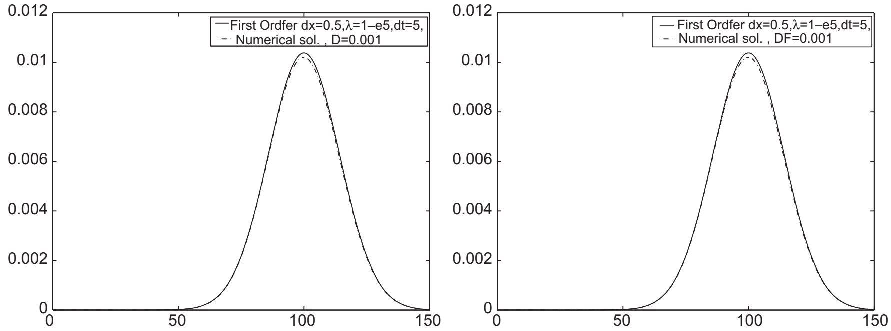

Fig. 5. The second-order results for the different time-steps and discretizations for the first component and different time-steps.

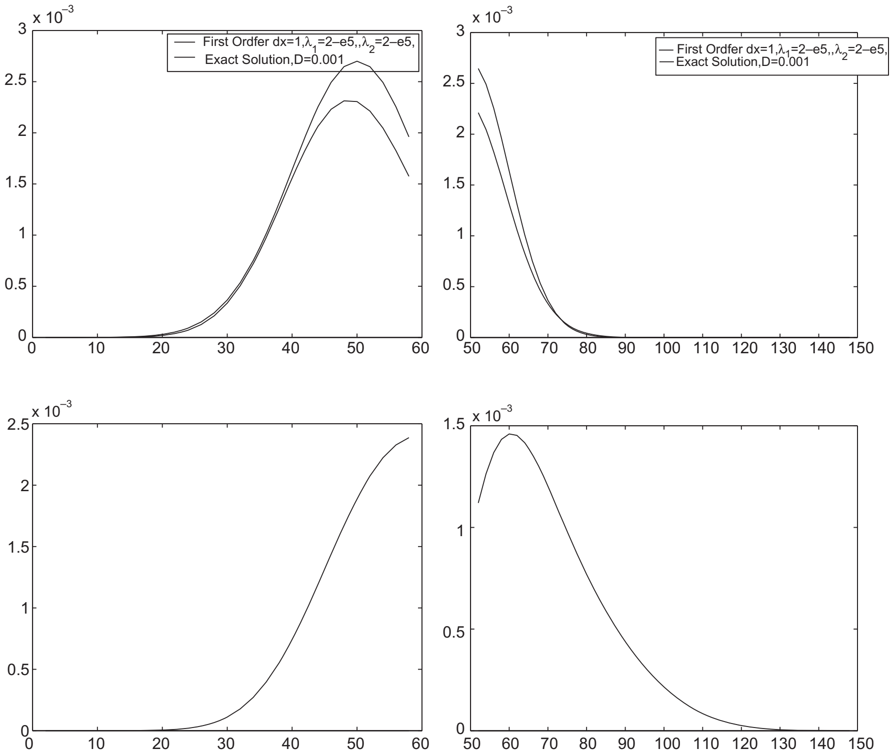

Fig. 6. The results for the Schwarz-method with 2 domains.

mains, retain same accuracy by the FOP method (Figs. 5 and 6). In the next section we present our conclusions.

5. Conclusions and discussions

We present a decomposition for a complex equation based on operator splitting and overlapping Schwarz wave form relaxation method. The error-analysis of the overlapping Schwarz wave form relaxation method, based on the analytical solution of the steady state solution, is developed for an one-dimensional system of parabolic equation. Numerical experiments are done for scalar, coupled and decoupled system of convection diffusion reaction equations. We compared the results of first order operator splitting (FOP) and first order operator splitting with overlapping Schwarz wave form relaxation method (FOPSWR). We could reach more effectivity with time- and space-decomposition. So fractional splitting methods plus overlapping Schwarz wave form relaxation methods have their benefit in decomposing complex equations with different time- and space-scales. In the future we will concentrate on more applied problems in crystal-growth, see [1] and biological models, see [5].

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