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Error estimates for finite volume element methods for convection–diffusion–reaction equations

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Abstract

In this paper, we study finite volume element (FVE) method for convection–diffusion–reaction equations in a two-dimensional convex polygonal domain. These types of equations arise in the modeling of a waste scenario of a radioactive contaminant transport and reaction in flowing groundwater. Both spatially discrete scheme and discrete-in-time scheme are analyzed in this paper. For the spatially discrete scheme, optimal order error estimates in L^2 and H^1 norms are obtained for the homogeneous equation using energy method. Further, a quasi-optimal order error estimate in L^∞ norm is shown to hold in an interior subdomain away from the corners. Based on backward Euler method, a time discretization scheme is discussed and related error estimates are derived.

MSC: 65M60; 65N30; 65N15

Keywords: Convection–diffusion–reaction equation; Finite volume element method; Spatially discrete scheme; Discrete-in-time scheme; Error estimates

1. Introduction

Our mathematical formulation is based on a potential waste scenario of radioactive contaminants, which are transported and reacted with flowing groundwater in porous media (cf. [11–13]). The model is described in the formulation as an initial-boundary value problem of the form

$$u_t + \nabla \cdot \big(\mathbf{v}u - \mathcal{D}(x)\nabla u\big) + \lambda u = f(x,t) \quad \text{in } \Omega \times J \tag{1.1}$$

subject to the boundary conditions

$$u = g_1(x,t) \quad \text{on } \Gamma_1; \qquad (\mathbf{v}u - \mathcal{D}\nabla u)\cdot \mathbf{n} = g_2(x,t) \quad \text{on } \Gamma_2 \tag{1.2}$$

and initial condition

$$u(x,0) = u_0(x) \quad \text{in } \Omega. \tag{1.3}$$

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Here, $\Omega \subset \mathbb{R}^2$ is a bounded convex polygonal domain with boundary $\Gamma = \Gamma_1 \cup \Gamma_2$, $J = (0, T]$ with $T < \infty$ and $u_t = \partial u/\partial t$. Further, $\mathcal{D} = \{d_{i,j}(x)\}$ is a symmetric and uniformly positive definite dispersion–diffusion matrix of size 2×2 in Ω. The parameter $\mathbf{v}$ is the divergence free groundwater velocity and λ is the constant reaction parameter. The nonhomogeneous term f and the coefficients $d_{ij}(x)$ are assumed to be smooth for our purpose.

With the substitution $\tilde{u} = u - g_1$ on Γ_1, we rewrite Eqs. (1.1)–(1.3) as

$$\tilde{u}_t + \nabla \cdot \big(\mathbf{v}\tilde{u} - \mathcal{D}(x)\nabla\tilde{u}\big) + \lambda\tilde{u} = f(x,t) \quad \text{in } \Omega \times J \tag{1.4}$$

subject to the boundary conditions

$$\tilde{u} = 0 \quad \text{on } \Gamma_1; \qquad (\mathbf{v}\tilde{u} - \mathcal{D}\nabla\tilde{u}) \cdot \mathbf{n} = g_2 \quad \text{on } \Gamma_2 \tag{1.5}$$

and initial condition

$$\tilde{u}(x,0) = u_0(x) \quad \text{in } \Omega. \tag{1.6}$$

Thus, study of problem (1.1)–(1.3) now reduces to the study of equivalent problem (1.4)–(1.6).

In the recent years, the use of finite volume element methods has become popular due to its certain conservation feature that are desirable in many applications (cf. [8–10]). The FVE method considered in these paper are based on Petrov–Galerkin formulation in which solution space consisting of continuous piecewise polynomial functions and the test space consisting of piecewise constant functions. The test space essentially conserve the local conservation property of the method. In [8,9], the authors have studied this type of problem with self-adjoint elliptic operator and proved optimal L^2 and H^1 error estimates which requires higher regularity requirement on the solution when compared to that of finite element method (cf. [17,20]). Recently, the authors of [6] have studied FVE for self-adjoint parabolic problem with homogeneous Dirichlet boundary condition and derived optimal error estimate in L^2 and H^1 norms, and suboptimal order of error estimate in L^∞ norm. They have used semigroup theory in a crucial way in their analysis.

In this present paper, we study the convergence of FVE methods for a non-selfadjoint parabolic problem. Both spatially discrete scheme and discrete-in-time scheme are discussed and optimal error estimates in L^2 and H^1 norms are proved using only energy method. In addition, a quasi-optimal order in L^∞ norm is obtained in an interior subdomain away from the corners. Our analysis avoid the use of semigroup theory and the regularity requirement on the solution is same as that of finite element method. Further, based on backward Euler method the fully discrete scheme is analyzed and related optimal error estimates are established. To the best our knowledge error estimates for the problem (1.1)–(1.3) using FVE method have not been established earlier.

The literature on the theoretical framework and the basic tools for the analysis of the finite volume element methods for elliptic and parabolic problems are described in [3–5,7,10,15,16,18,19] and references therein.

A brief outline of this paper is as follows. In Section 2, we introduce some notations and present some preliminary materials to be used in our subsequent sections. The Petrov–Ritz projection is introduced and related estimates are carried out in Section 3. Section 4 is devoted to the error estimates for the FVE method. Finally, the backward Euler time discretization scheme is discussed in Section 5.

Throughout this paper, C denotes a generic positive constant which does not depend on the spatial and time discretization parameters h and k, respectively.

2. Notations and preliminaries

Let $V = \{\phi \in H^1(\Omega) \mid \phi = 0 \text{ on } \Gamma_1\}$. For the purpose of finite volume element approximation of (1.4)–(1.6), the weak formulation of the problem may be stated as follows: Find $\tilde{u} : \bar{J} \to V$ such that

$$(\tilde{u}_t, \phi) + A(\tilde{u}, \phi) = \langle g_2, \phi\rangle + (f, \phi), \quad \forall \phi \in V \tag{2.1}$$

with $\tilde{u}(0) = u_0$, where the bilinear form $A(\cdot,\cdot)$ is given by

$$A(\tilde{u}, \phi) = \int_\Omega \big(\mathcal{D}(x)\nabla\tilde{u} \cdot \nabla\phi - \mathbf{v}\tilde{u}\nabla\phi + \lambda\tilde{u}\phi\big)\,\mathrm{d}x. \tag{2.2}$$

Here and below, we denote $(\cdot,\cdot)$ and $\|\cdot\|$ by L^2 inner product and the induced norm on $L^2(\Omega)$. The notation $\langle\cdot,\cdot\rangle$ is used to denote boundary integral over Γ_2. Further, we shall use the standard notation for Sobolev spaces $W^{m,p}(\Omega)$ with $1 \leqslant p \leqslant \infty$. The norm on $W^{m,p}(\Omega)$ is defined by

$$\|u\|_{m,p,\Omega} = \|u\|_{m,p} = \left(\int_\Omega \sum_{|\alpha|\leqslant m} \left|D^\alpha u\right|^p \mathrm{d}x\right)^{1/p}, \quad 1 \leqslant p < \infty$$

with the standard modification for $p=\infty$. When $p=2$, we write $W^{m,2}(\Omega)$ by $H^m(\Omega)$ and denote the norm by $\|\cdot\|_m$. For a fractional number s, Sobolev space H^s is defined in [1].

Note that the bilinear form $A(\cdot,\cdot)$ given by (2.2) may not be coercive but it can be made coercive by adding a sufficiently large constant $\kappa \in \mathbb{R}$ times the L^2-inner product. That is, it satisfies Gärding's type inequality (cf. [2])

$$A(\phi,\phi) + \kappa\|\phi\|^2 \geqslant \frac{\alpha}{2}\|\phi\|_1^2, \quad \forall \phi \in V.$$

Introducing the transformation $\bar{u} = e^{-\kappa t}\tilde{u}$ as a new dependent variable, we rewrite (1.4) as

$$\bar{u}_t + A_\kappa \bar{u} = \bar{f} = e^{-\kappa t} f, \quad t \in J \tag{2.3}$$

with $\bar{u}(0) = u_0$, where

$$A_k\bar{u} = \nabla\cdot\big(\mathbf{v}\bar{u} - \mathcal{D}(x)\nabla\bar{u}\big) + (\lambda+\kappa)\bar{u}.$$

The weak form corresponding to (2.3) is defined to be the function $\bar{u} : \bar{J} \to V$ such that

$$(\bar{u}_t, \phi) + A_\kappa(\bar{u}, \phi) = \langle \bar{g}_2, \phi\rangle + (\bar{f}, \phi), \quad \forall \phi \in V \tag{2.4}$$

with $\bar{g}_2 = e^{-\kappa t} g_2$ and $\bar{u}(0) = u_0$. The bilinear form $A_\kappa(\cdot,\cdot)$ is given by

$$A_\kappa(\bar{u},\phi) = \int_\Omega \mathcal{D}(x)\nabla\bar{u}\cdot\nabla\phi\,\mathrm{d}x - \int_\Omega \mathbf{v}\bar{u}\nabla\phi\,\mathrm{d}x + \int_\Omega (\lambda+\kappa)\bar{u}\phi\,\mathrm{d}x. \tag{2.5}$$

2.1. A priori estimates

Following the lines of proof in [17], it is easy to derive a priori bounds for the solution $\bar{u}$ satisfying (2.3) under appropriate regularity assumption on the initial function u_0. The details are thus omitted.

Lemma 2.1. *Let $u_0 \in L^2(\Omega)$ and $g_2 \in H^{1/2}(\Gamma_2)$. Then, for $f=0$, we have*

$$\left\|\bar{u}(t)\right\|^2 + \int_0^t \left\|\bar{u}(s)\right\|_1^2 \mathrm{d}s \leqslant C\left(\|u_0\|^2 + \int_0^t \|g_2\|^2_{H^{1/2}(\Gamma_2)}\,\mathrm{d}s\right).$$

Moreover, when $u_0 \in V$, we have

$$\left\|\bar{u}(t)\right\|_1^2 + \int_0^t \left\{\left\|\bar{u}_s(s)\right\|^2 + \left\|\bar{u}(s)\right\|_2^2\right\}\mathrm{d}s \leqslant C\left(\|u_0\|_1^2 + \int_0^t \|g_2\|^2_{H^{1/2}(\Gamma_2)}\,\mathrm{d}s\right).$$

Lemma 2.2. *Assume that $u_0 \in H^2(\Omega)\cap V$, $\frac{\partial^j g_2}{\partial t^j} \in H^{1/2}(\Gamma_2)$ $(j=0,1,2)$ and $f=0$. Then, we have*

$$\left\|\bar{u}_t(t)\right\|^2 + \int_0^t \left\|\bar{u}_s(s)\right\|_1^2 \mathrm{d}s \leqslant C\left(\|u_0\|_2^2 + \sum_{j=0}^1 \int_0^t \left\|\frac{\partial^j g_2}{\partial t^j}\right\|^2_{H^{1/2}(\Gamma_2)} \mathrm{d}s\right),$$

$$t\left\|\bar{u}_t(t)\right\|_1^2 + \int_0^t s\|\bar{u}_{ss}\|^2 \mathrm{d}s \leqslant C\left(\|u_0\|_2^2 + \sum_{j=0}^1 \int_0^t \left\|\frac{\partial^j g_2}{\partial t^j}\right\|^2_{H^{1/2}(\Gamma_2)} \mathrm{d}s\right),$$

$$t^2\|\bar{u}_{tt}(t)\|^2+\int_0^t s^2\|\bar{u}_{ss}(s)\|_1^2\,\mathrm{d}s\leqslant C\left(\|u_0\|_2^2+\sum_{j=0}^{2}\int_0^t\left\|\frac{\partial^j g_2}{\partial t^j}\right\|^2_{H^{1/2}(\Gamma_2)}\mathrm{d}s\right),$$

$$t^i\left\|\frac{\partial^i\bar{u}}{\partial t^i}\right\|_2\leqslant C\left\{\|u_0\|_2+\left(\sum_{j=0}^{i}\int_0^t\left\|\frac{\partial^j g_2}{\partial t^j}\right\|^2_{H^{1/2}(\Gamma_2)}\mathrm{d}s\right)^{1/2}\right\},\quad i=0,1,\ t\in J.$$

2.2. Finite volume element approximation

Let T_h be a quasi-uniform triangulation of Ω such that $\overline{\Omega}=\bigcup_{K\in T_h}K$, where K is a closed triangle element. Let N_h be the set of all nodes or vertices of T_h, i.e.,

$$N_h=\{p:\ p \text{ is a vertex of element } K\in T_h \text{ and } p\in\overline{\Omega}\}.$$

Further, we denote $N_h^0=N_h\cap\Omega$. For a vertex $x_i\in N_h$, let $\Pi(i)$ be the index set of those vertices that, along with x_i, are in some element of T_h.

For the triangulation T_h, we now introduce a dual mesh T_h^* as follows: In each element $K\in T_h$ consisting of vertices x_i, x_j and x_k, select a point $q\in K$, and select a point x_{ij} by straight lines $\gamma_{ij,K}$. Then, for a vertex x_i, we let V_i be the polygon whose edges are $\gamma_{ij,K}$ in which x_i is a vertex of the element K. We call this V_i a *control volume* centered at x_i. Further, we note that $\bigcup_{x_i\in N_h}V_i=\overline{\Omega}$. Thus, the dual mesh T_h^* is then defined as the collection of these *control volumes*. A *control volume* centered at a vertex x_i is given in Fig. 1.

We call the control volume mesh T_h^* regular or quasi-uniform if there exists a positive constant $C>0$ such that

$$C^{-1}h^2\leqslant \operatorname{meas}(V_i)\leqslant Ch^2\quad\text{for all } V_i\in T_h^*,$$

where h is the maximum diameter of all elements $K\in T_h$.

There are various ways to introduce a regular dual mesh T_h^* depending on the choices of the point q in an element $K\in T_h$ and the points x_{ij} on its edges. In this paper, we choose q to be the barycenter of an element $K\in T_h$, and the points x_{ij} are chosen to be the midpoints of the edges of K. In addition, if T_h is locally regular, i.e., there is a constant C such that

$$Ch_K^2\leqslant \operatorname{meas}(K)\leqslant h_K^2,$$

where $h_K=\operatorname{diam}(K)$ for all elements $K\in T_h$. Then the dual mesh T_h^* is also locally regular. For the purpose of finite volume element approximation let S_h be the linear finite element space defined on the triangulation T_h,

$$S_h=\{v\in C(\Omega):\ v|_K \text{ is linear for all } K\in T_h \text{ and } v|_{\Gamma_1}=0\},$$

and its dual volume element space S_h^*,

$$S_h^*=\{v\in L^2(\Omega):\ v|_V \text{ is constant for all } V\in T_h^* \text{ and } v|_{\Gamma_1}=0\}.$$

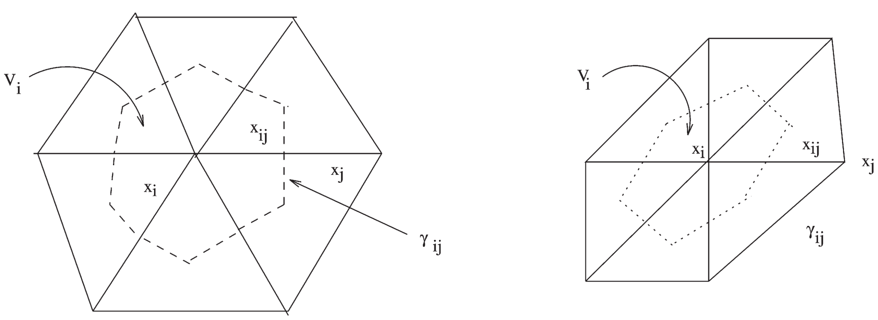

Fig. 1. Control volumes with barycenter as internal point and interface γ_{ij} of V_i and V_j.

Obviously, $S_h = \operatorname{span}\{\phi_i(x)\colon\ x_i \in N_h^0\}$ and $S_h^* = \operatorname{span}\{\chi_i(x)\colon\ x_i \in N_h^0\}$, where ϕ_i are the standard nodal basis functions associated with the node x_i, and χ_i are the characteristic functions of the volume V_i. Let $I_h : C(\Omega) \to S_h$ and $I_h^* : C(\Omega) \to S_h^*$ be the usual interpolation operators, i.e.,

$$I_h\bar{u} = \sum_{x_i \in N_h} \bar{u}_i \phi_i(x) \quad \text{and} \quad I_h^*\bar{u} = \sum_{x_i \in N_h} \bar{u}_i \chi_i(x),$$

where $\bar{u}_i = \bar{u}(x_i)$.

The FVE approximation corresponding to (2.4) is defined to be the function $\bar{u}_h(t) : \bar{J} \to S_h$ such that

$$(\bar{u}_{h,t}, I_h^*\chi) + A_\kappa(\bar{u}_h, I_h^*\chi) = \langle \bar{g}_2, I_h^*\chi\rangle + (\bar{f}, I_h^*\chi) \tag{2.6}$$

for all $\chi \in S_h$ with $\bar{u}_h(0) = u_{0,h}$, where $u_{0,h}$ is a suitable projection of u_0 onto S_h to be defined later.

The bilinear forms $A_\kappa(\cdot,\cdot)$ in (2.6) is defined by

$$A_\kappa(\bar{u}, w) = \sum_{x_i \in N_h} \left\{ -w_i \int_{\partial V_i} \big(D(x)\nabla\bar{u} - \mathbf{v}\bar{u}\big)\cdot \mathbf{n}\,\mathrm{d}S_x + w_i \int_{V_i} (\lambda + \kappa)\bar{u}\,\mathrm{d}x \right\}$$

for $(\bar{u}, w) \in ((V \cap H^2) \cup S_h) \times S_h^*$, where $\mathbf{n}$ is the outer-normal vector of the involved integration domain. Note that when $(\bar{u}, w) \in V \times V$ the bilinear form $A_\kappa(\cdot,\cdot)$ is given by (2.5). Similarly, the FVE approximation to (2.1) is easily obtained by taking $\kappa = 0$ in (2.6).

In order to describe features of the bilinear forms defined in (2.4) and (2.6) we define some discrete norms on S_h and S_h^*,

$$|u_h|_{0,h}^2 = (u_h, u_h)_{0,h}, \qquad |u_h|_{1,h}^2 = \sum_{x_i \in N_h} \sum_{x_j \in \Pi(i)} \operatorname{meas}(V_i)\big((u_{hi} - u_{hj})/d_{ij}\big)^2,$$

$$\|u_h\|_{1,h}^2 = |u_h|_{0,h}^2 + |u_h|_{1,h}^2, \qquad |\!|\!|u_h|\!|\!| = (u_h, I_h^* u_h),$$

where $(u_h, v_h)_{0,h} = \sum_{x_i \in N_h} \operatorname{meas}(V_i) u_{hi} v_{hi} = (I_h^* u_h, I_h^* v_h)$ and $d_{ij} = d(x_i, x_j)$ is the distance between x_i and x_j. These norms are well defined for $u_h \in S_h^*$ as well and $\|u_h\|_{0,h} = |\!|\!|u_h|\!|\!|$.

Below, we state the equivalence of the discrete norms $|\cdot|_{0,h}$ and $\|\cdot\|_{1,h}$ with usual norms $\|\cdot\|$ and $\|\cdot\|_1$, respectively on S_h. Further, some properties of the bilinear forms are stated without proof. For a proof, we refer to [9,10].

Lemma 2.3. *There exist two positive constants C_1 and C_2 such that for all $v_h \in S_h$, we have*

$$C_1|v_h|_{0,h} \leqslant \|v_h\| \leqslant C_2|v_h|_{0,h},$$
$$C_1|\!|\!|v_h|\!|\!| \leqslant \|v_h\| \leqslant C_2|\!|\!|v_h|\!|\!|,$$
$$C_1\|v_h\|_{1,h} \leqslant \|v_h\|_1 \leqslant C_2\|v_h\|_{1,h}.$$

Lemma 2.4. *There exist positive constants C and c such that, for all $\phi_h, \psi_h \in S_h$, the boundedness property*

$$\big|A_\kappa(\phi_h, I_h^*\psi_h)\big| \leqslant C\|\phi_h\|_1\|\psi_h\|_1$$

and the coercive property

$$A_\kappa(\phi_h, I_h^*\phi_h) \geqslant c\|\phi_h\|_1^2$$

hold true.

The following lemma gives the key feature of the bilinear forms in the finite volume element method. For a proof, see [10].

Lemma 2.5. *Let $\phi \in (V \cap H^2) \cup S_h$. Then we have*

$$A_\kappa(\phi,\chi) - A_\kappa(\phi, I_h^*\chi) = \sum_{K\in T_h}\int_K \{-\nabla\cdot(\mathcal{D}\nabla\phi - \mathbf{v}\phi) + (\lambda+\kappa)\phi\}(\chi - I_h^*\chi)\,\mathrm{d}x + \sum_{K\in T_h}\int_{\partial K}\{(\mathcal{D}\nabla\phi - \mathbf{v}\phi)\cdot\mathbf{n}\}(\chi - I_h^*\chi)\,\mathrm{d}S, \quad \forall\chi\in S_h.$$

3. Petrov–Ritz projection and related estimates

Following [8,9], define the Petrov–Ritz projection $R_h : V\cap H^2(\Omega)\to S_h$ by

$$A_\kappa(\bar{u} - R_h\bar{u}, I_h^*\chi) = 0, \quad \forall\chi\in S_h. \tag{3.1}$$

The following lemma prove to be convenient for obtaining H^1 and L^2 error estimates for the Petrov–Ritz projection.

Lemma 3.1. *Assume that $\phi\in S_h$ and $\mathcal{D}\in W^{2,\infty}(\Omega)$. Then we have*

$$\left|A_\kappa(\phi,\chi) - A_\kappa(\phi, I_h^*\chi)\right| \leqslant Ch\|\phi\|_1\|\chi\|_1, \quad \forall\chi\in S_h.$$

Further, for $\phi\in V\cap H^2(\Omega)$, we have

$$\left|A_\kappa(\phi,\chi) - A_\kappa(\phi, I_h^*\chi)\right| \leqslant Ch\|\phi\|_2\|\chi\|_1, \quad \forall\chi\in S_h.$$

Proof. Since the dual mesh is formed by the barycenters, we have for $\chi\in S_h$

$$\int_K (\chi - I_h^*\chi)\,\mathrm{d}x = 0 \quad \text{for all } K\in T_h.$$

Thus, in view of Lemma 2.5, we have for $\phi,\chi\in S_h$

$$A_\kappa(\phi,\chi) - A_\kappa(\phi, I_h^*\chi) = \sum_{K\in T_h}\int_K \{-\nabla\cdot(\mathcal{D}\nabla\phi - \mathbf{v}\phi) + (\lambda+\kappa)\phi\}(\chi - I_h^*\chi)\,\mathrm{d}x + \sum_{K\in T_h}\int_{\partial K}\{(\mathcal{D} - \overline{\mathcal{D}}_K)(\nabla\phi - \mathbf{v}\phi)\cdot\mathbf{n}\}(\chi - I_h^*\chi)\,\mathrm{d}S := I_1 + I_2. \tag{3.2}$$

Here, $\overline{\mathcal{D}}_K$ is a function designed in a piecewise manner such that for any edge E of a triangle $K\in T_h$ and $x\in E$, $\overline{D}_K(x) = D(x_c)$, where x_c is the mid point of E. Noting that, for $\phi\in S_h$, $\nabla\phi$ is a constant on K, we have $\nabla\cdot(\mathcal{D}\nabla\phi) = (\nabla\cdot\mathcal{D})\nabla\phi$. Now, applying Cauchy–Schwarz's inequality and using the fact that $\|\chi - I_h^*\chi\| \leqslant Ch\|\chi\|_1$, we obtain

$$|I_1| \leqslant Ch\|\phi\|_1\|\chi\|_1. \tag{3.3}$$

Since $|\mathcal{D}(x) - \overline{\mathcal{D}}_K| \leqslant h\|\mathcal{D}\|_{1,\infty}$ and $\|\chi - I_h^*\chi\|_{L^2(\partial K)} \leqslant Ch^{1/2}\|\chi\|_{1,K}$ (cf. [10]), the term I_2 is bounded by

$$|I_2| \leqslant Ch\sum_{K\in T_h} h^{1/2}\|\nabla\phi\|_{L^2(\partial K)}\|\chi\|_{1,K} \leqslant Ch\sum_{K\in T_h}\|\phi\|_{1,K}\|\chi\|_{1,K} \leqslant Ch\|\phi\|_1\|\chi\|_1, \tag{3.4}$$

where in the second inequality we have used the fact that $\nabla\phi$ is constant on K. Combine (3.2) and (3.4) to prove the first inequality.

Next, for $\phi\in V\cap H^2(\Omega)$, we have

$$|I_1| \leqslant Ch\|\phi\|_2\|\chi\|_1. \tag{3.5}$$

For I_2, using the trace theorem [2], we obtain

$$|I_2| \leqslant Ch\sum_{K\in T_h} h^{1/2}\|\nabla\phi\|_{L^2(\partial K)}\|\chi - I_h^*\chi\|_{L^2(\partial K)} \leqslant Ch\|\phi\|_2\|\chi\|_1. \tag{3.6}$$

Combine (3.2), (3.5) and (3.6) to obtain the second inequality and this completes the proof. □

Set $\rho = \bar{u} - R_h\bar{u}$. We now establish H^1-error estimate for ρ and its temporal derivative.

Lemma 3.2. *Let ρ satisfy* (3.1). *Then we have*

$$\|\rho\|_1 \leqslant Ch\|\bar{u}\|_2, \qquad \|\rho_t\|_1 \leqslant Ch\|\bar{u}_t\|_2.$$

Proof. With $\phi_h = I_h\bar{u} - R_h\bar{u}$, we obtain using (3.1)

$$\begin{aligned} c\|\rho\|_1^2 &\leqslant A_\kappa(\rho, \rho) \\ &= A_\kappa(\rho, \bar{u} - I_h\bar{u}) + A_\kappa(\rho, I_h\bar{u} - R_h\bar{u}) \\ &= A_\kappa(\rho, \bar{u} - I_h\bar{u}) + A_\kappa(\rho, \phi_h) - A_\kappa(\rho, I_h^*\phi_h). \end{aligned}$$

An application of Lemma 3.1 yields

$$\begin{aligned} A_\kappa(\rho, \phi_h) - A_\kappa(\rho, I_h^*\phi_h) &= \{A_\kappa(\bar{u}, \phi_h) - A_\kappa(\bar{u}, I_h^*\phi_h)\} - \{A_\kappa(R_h\bar{u}, \phi_h) - A_\kappa(R_h\bar{u}, I_h^*\phi_h)\} \\ &\leqslant Ch(\|\bar{u}\|_2 + \|\bar{u}\|_1)\|\phi_h\|_1 \\ &\leqslant Ch\|\bar{u}\|_2(\|\rho\|_1 + h\|\bar{u}\|_2), \end{aligned}$$

where in the last inequality we have used $\|\phi_h\|_1 \leqslant C(h\|\bar{u}\|_2 + \|\rho\|_1)$. Thus, we obtain

$$c\|\rho\|_1^2 \leqslant Ch\|\bar{u}\|_2\|\rho\|_1 + Ch^2\|\bar{u}\|_2^2.$$

Kickback the term $\|\rho\|_1$ to obtain the first inequality. For the second inequality, differentiate (3.1) with respect to time t to have

$$A_\kappa(\rho_t, I_h^*\chi) = 0. \tag{3.7}$$

Then the rest of the proof follows in a similar fashion. □

We shall prove the L^2 estimates of ρ and its temporal derivatives in the following theorem.

Lemma 3.3. *Let ρ satisfy* (3.1). *Then we have*

$$\|\rho(t)\| \leqslant Ch^2\|\bar{u}\|_2, \qquad \|\rho_t(t)\| \leqslant Ch^2\|\bar{u}_t\|_2.$$

Proof. The proof will proceed by duality argument. Let $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of

$$A_\kappa^*\psi = \rho \quad \text{in } \Omega, \qquad \psi = 0 \quad \text{on } \partial\Omega, \tag{3.8}$$

where A_κ^* is the formal adjoint of A_κ. The solution ψ satisfies the following regularity estimate

$$\|\psi\|_2 \leqslant C\|\rho\|. \tag{3.9}$$

Multiplying (3.8) by ρ and then taking L^2 inner-product over Ω, we obtain

$$\|\rho\|^2 = A_\kappa(\rho, \psi - I_h\psi) + A_\kappa(\rho, I_h\psi) = I_1 + I_2. \tag{3.10}$$

Using Lemma 3.2, I_1 is bounded as

$$|I_1| \leqslant Ch^2\|\bar{u}\|_2\|\psi\|_2. \tag{3.11}$$

Following the line of arguments of [10, Theorem 3.5], the term I_2 is bounded as

$$|I_2| \leqslant Ch^2\|u\|_2\|\psi\|_2 \tag{3.12}$$

which combine with (3.10), (3.11) and (3.9) completes the proof. □

4. Error estimates for the spatially discrete scheme

In this section, the error analysis for the spatially discrete FVE approximation will be carried out. For homogeneous problem, optimal order error estimates are established in L^2 and H^1 norms when $u_0 \in H^2 \cap V$. In addition, a quasi-optimal order error estimate in L^∞ norm is proved in an interior sub-domain away from the corners.

As usual we split the error $e = \bar{u} - \bar{u}_h$ as

$$e = (\bar{u} - R_h\bar{u}) + (R_h\bar{u} - \bar{u}_h) = \rho + \theta.$$

Since the estimates of ρ are already known, it is enough to have estimates for θ.

Using (2.6), an equation of the form (2.6) with u_h replaced by u and (3.1), it is easy to verify that θ satisfies an error equation

$$(\theta_t, I_h^*\chi) + A_\kappa(\theta, I_h^*\chi) = -(\rho_t, I_h^*\chi), \quad \forall \chi \in S_h. \tag{4.1}$$

Define $\hat{\theta}(t) = \int_0^t \theta(s)\,\mathrm{d}s$. Then, clearly $\hat{\theta}(0) = 0$ and $\hat{\theta}_t = \theta$. We shall prove a sequence of lemmas which lead to the desired result.

Lemma 4.1. *Assume that $\bar{u}_h(0) = R_h u_0$. There is a positive constant C independent of h such that*

$$\int_0^t \|\theta(s)\|^2\,\mathrm{d}s + \|\hat{\theta}(t)\|_1^2 \leqslant C\left(t\|\rho(0)\|^2 + \int_0^t \|\rho(s)\|^2\,\mathrm{d}s\right).$$

Proof. Integrate (4.1) from 0 to t and use the fact $\theta(0) = 0$ to have

$$(\theta, I_h^*\chi) + A_\kappa(\hat{\theta}, I_h^*\chi) = -\big(\rho(t), I_h^*\theta\big) + \big(\rho(0), I_h^*\theta\big). \tag{4.2}$$

Choose $\chi = \theta$ in (4.2) to obtain

$$\begin{aligned}|\!|\!|\theta|\!|\!|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\{A_\kappa(\hat{\theta}, \hat{\theta})\big\} &= -(\rho, I_h^*\theta) + \big(\rho(0), I_h^*\theta\big) + \big\{A_\kappa(\hat{\theta}, \theta) - A_\kappa(\hat{\theta}, I_h^*\theta)\big\}\\ &\leqslant \big(\|\rho\| + \|\rho(0)\|\big)\|\theta\| + C\|\hat{\theta}\|_1\|\theta\|,\end{aligned} \tag{4.3}$$

where in the last step, we have used the fact that (cf. [6, Lemma 4.1])

$$\big|A_\kappa(\hat{\theta}, \theta) - A_\kappa(\hat{\theta}, I_h^*\theta)\big| \leqslant C\|\hat{\theta}\|_1\|\theta\|.$$

Integrating (4.3) from 0 to t and using Lemma 2.3, we obtain

$$\int_0^t \|\theta(s)\|^2\,\mathrm{d}s + \|\hat{\theta}(t)\|_1^2 \leqslant C\int_0^t \big(\|\rho\|^2 + \|\rho(0)\|^2\big)\,\mathrm{d}s + \frac{1}{2}\int_0^t \|\theta\|^2\,\mathrm{d}s + \int_0^t \|\hat{\theta}\|_1^2\,\mathrm{d}s.$$

Kickback the term $\frac{1}{2}\int_0^t \|\theta\|^2\,\mathrm{d}s$ and then apply Gronwall's lemma to complete the rest of the proof. □

Lemma 4.2. *Let θ satisfy* (4.1) *with $\bar{u}_h(0) = R_h u_0$. Then there is a positive constant C independent of h such that*

$$t\|\theta(t)\|^2 + \int_0^t s\|\theta(s)\|_1^2\,\mathrm{d}s \leqslant C\left(t\|\rho(0)\|^2 + \int_0^t \big\{\|\rho(s)\|^2 + s^2\|\rho_s(s)\|^2\big\}\,\mathrm{d}s\right).$$

Proof. Set $\chi = t\theta$ in (4.1). Then using the symmetry of $(\psi, I_h^*\chi)$, $\psi,\ \chi \in S_h$ on S_h, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\{t|\!|\!|\theta|\!|\!|^2\big\} + tA_\kappa(\theta, I_h^*\theta) \leqslant |\!|\!|\theta|\!|\!|^2 + t\|\rho_t\|\|I_h^*\theta\|.$$

Integrating from 0 to t and using the weak coercivity in Lemma 2.4, it now leads to

$$\frac{1}{2}t\left|\!\left|\!\left|\theta(t)\right|\!\right|\!\right|^2 + \int_0^t s\left\|\theta(s)\right\|_1^2 \mathrm{d}s \leqslant C\int_0^t \|\theta\|^2 \mathrm{d}s + \int_0^t s\|\rho_t\|\|\theta\| \mathrm{d}s.$$

Apply Young's inequality to have

$$t\left\|\theta(t)\right\|^2 + \int_0^t s\left\|\theta(s)\right\|_1^2 \mathrm{d}s \leqslant C\left(\int_0^t \left\{\|\theta\|^2 + s^2\|\rho_s\|^2\right\} \mathrm{d}s\right).$$

Finally, use Lemma 4.1 to complete the rest of the proof. □

Lemma 4.3. *Let the hypotheses in Lemma* 4.2 *hold true. Then there is a positive constant C independent of h such that*

$$\int_0^t s^2\left\|\theta(s)\right\|^2 \mathrm{d}s + t^2\left\|\theta(t)\right\|_1^2 \leqslant C\left(t\left\|\rho(0)\right\|^2 + \int_0^t \left\{\left\|\rho(s)\right\|^2 + s^2\left\|\rho_s(s)\right\|^2\right\} \mathrm{d}s\right).$$

Proof. Choose $\chi = t^2\theta_t$ in (4.1) to have

$$t^2\left|\!\left|\!\left|\theta_t(t)\right|\!\right|\!\right|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{t^2 A_\kappa(\theta,\theta)\right\} = -t^2(\rho_t, I_h^*\theta_t) + tA_\kappa(\theta,\theta) + t^2\left\{A_\kappa(\theta,\theta_t) - A_\kappa(\theta, I_h^*\theta_t)\right\}. \tag{4.4}$$

It follows from [6, Lemma 4.1] that

$$\left|A_\kappa(\theta,\theta_t) - A_\kappa(\theta, I_h^*\theta_t)\right| \leqslant C\|\theta\|_1\|\theta_t\|.$$

Now integrate (4.4) from 0 to t. Then apply Lemmas 2.3 and 2.4 and standard kickback argument to obtain

$$\int_0^t s^2\left\|\theta_s(s)\right\|^2 \mathrm{d}s + t^2\left\|\theta(t)\right\|_1^2 \leqslant C\int_0^t s\left\|\theta(s)\right\|_1^2 \mathrm{d}s + C\int_0^t s^2\|\rho_s\|^2 \mathrm{d}s + C\int_0^t s^2\|\theta\|_1^2 \mathrm{d}s.$$

Finally, apply Lemma 4.2 and Gronwall's lemma to complete the proof. □

The main results of this section is given in the following theorems.

Theorem 4.1. *Let $\tilde{u}$ satisfy* (1.4) *with $f = 0$, and let $\tilde{u}_h$ be its FVE approximation. Then, for $u_0 \in H^2 \cap V$, $\frac{\partial^j g_2}{\partial t^j} \in H^{1/2}(\Gamma_2)$ $(j = 0, 1, 2)$ and $\bar{u}_h(0) = R_h u_0$, we have*

$$\left\|\tilde{u}(t) - \tilde{u}_h(t)\right\|_1 \leqslant Cht^{-1/2}\left\{\|u_0\|_2 + \left(\sum_{j=0}^{2}\int_0^t \left\|\frac{\partial^j g_2}{\partial t^j}\right\|_{H^{1/2}(\Gamma_2)}^2 \mathrm{d}s\right)^{1/2}\right\}$$

and

$$\left\|\tilde{u}(t) - \tilde{u}_h(t)\right\| \leqslant Ch^2\left\{\|u_0\|_2 + \left(\sum_{j=0}^{2}\int_0^t \left\|\frac{\partial^j g_2}{\partial t^j}\right\|_{H^{1/2}(\Gamma_2)}^2 \mathrm{d}s\right)^{1/2}\right\}$$

hold true for $t \in J$.

Proof. By triangle inequality, we have

$$\left\|\bar{u}(t) - \bar{u}_h(t)\right\|_1 \leqslant \left\|\rho(t)\right\|_1 + \left\|\theta(t)\right\|_1.$$

From Lemma 4.3, we obtain

$$t\|\theta\|_1 \leqslant C\left(t\|R_h u_0 - u_0\|^2 + \int_0^t \left\{\left\|\rho(s)\right\|^2 + s^2\left\|\rho_s(s)\right\|^2\right\} \mathrm{d}s\right)^{1/2}$$
$$\leqslant Ch\left(t\|u_0\|_2^2 + \int_0^t \left\{\|\bar{u}\|_2^2 + s^2\|\bar{u}_s\|_2^2\right\} \mathrm{d}s\right)^{1/2}.$$

In view of Lemma 2.2, it now follows that

$$t\|\theta\|_1 \leqslant Cht^{1/2}\left\{\|u_0\|_2 + \left(\sum_{j=0}^{2}\int_0^t \left\|\frac{\partial^j g_2}{\partial t^j}\right\|^2_{H^{1/2}(\Gamma_2)} \mathrm{d}s\right)^{1/2}\right\}, \tag{4.5}$$

and this together with Lemmas 3.2, 2.2 and the identity

$$\bar{u} - \bar{u}_h = e^{-\kappa t}(\tilde{u} - \tilde{u}_h) \tag{4.6}$$

yield the first inequality. Similarly, for the second inequality, we use Lemmas 3.3, 4.2, a priori estimates in Lemma 2.2 and the identity (4.6). This completes the rest of the proof. □

We shall close this section by showing a quasi-optimal order error estimate in maximum norm in an interior domain $\Omega_0 \subset \Omega$ with $\overline{\Omega}_0$ not containing any vertex of Ω.

Theorem 4.2. *Let $\Omega_0 \subset \Omega$ be such that $\overline{\Omega}_0$ does not contain any vertex of Ω. Further, let $\tilde{u}$ satisfy* (1.4) *with $f = 0$, and let $\tilde{u}_h$ be its FVE approximation. Assume that $u_0 \in H^2 \cap V$, $\frac{\partial^j g_2}{\partial t^j} \in H^{1/2}(\Gamma_2)$ $(j = 0, 1, 2)$ and $\bar{u}_h(0) = R_h u_0$. Then there is a positive constant C such that*

$$\left\|\tilde{u}(t) - \tilde{u}_h(t)\right\|_{L^\infty(\Omega_0)} \leqslant Ct^{-1}h^2 \log\frac{1}{h}\left\{\|u_0\|_2 + \left(\sum_{j=0}^{2}\int_0^t \left\|\frac{\partial^j g_2}{\partial t^j}\right\|^2_{H^{1/2}(\Gamma_2)} \mathrm{d}s\right)^{1/2}\right\}, \quad t \in J.$$

Proof. By triangle inequality, we have

$$\left\|\bar{u}(t) - \bar{u}_h(t)\right\|_{L^\infty(\Omega_0)} \leqslant \left\|\theta(t)\right\|_{L^\infty(\Omega_0)} + \left\|\rho(t)\right\|_{L^\infty(\Omega_0)}. \tag{4.7}$$

Recall that S_h is the linear finite element space and triangulation is quasi-uniform, we thus have (cf. [20, Chapter 5])

$$\left\|\theta(t)\right\|_{L^\infty} \leqslant C\left(\log\frac{1}{h}\right)^{1/2}\|\theta(t)\|_1,$$

and hence, using (4.5), it now follows that

$$\left\|\theta(t)\right\|_{L^\infty(\Omega_0)} \leqslant Ch^2 t^{-1/2}\log\frac{1}{h}\left[\|u_0\|_2 + \left(\sum_{j=0}^{2}\int_0^t \left\|\frac{\partial^j g_2}{\partial t^j}\right\|^2_{H^{1/2}(\Gamma_2)} \mathrm{d}s\right)^{1/2}\right]. \tag{4.8}$$

Thus, the first term in (4.7) is bounded as desired. It now remains to bound $\|\rho\|_{L^\infty(\Omega_0)}$. Let Ω_2 and Ω_3 be domains with $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \Omega$ and smooth boundaries. Further, let Ω_3 does not contain any corner of Ω and the distances between $\partial\Omega_3 \cap \Omega$, $\partial\Omega_2 \cap \Omega$, and $\partial\Omega_1 \cap \Omega$ are positive. Let ω be a smooth function such that $\omega|_{\Omega_2} = 1$ and $\omega|_{\partial\Omega_3\cap\Omega} = 0$. It is well known that (cf. [6])

$$\left\|\rho(t)\right\|_{L^\infty(\Omega_0)} \leqslant Ch^2 \log\frac{1}{h}\left\|\bar{u}(t)\right\|_{W^{2,\infty}(\Omega_2)} + C\left\|\rho(t)\right\|. \tag{4.9}$$

Since the term $\|\rho\|$ is bounded as desired by Lemma 3.3, it now remains to bound the first term $\|\bar{u}(t)\|_{W^{2,\infty}(\Omega_2)}$. Using Sobolev inequality and elliptic regularity estimate in Ω_3 (recall that $\partial\Omega_3$ is smooth), we obtain, with $\bar{\bar{u}} = \omega\bar{u}$,

$$\|\bar{u}\|_{W^{2,\infty}(\Omega_2)} \leqslant C\|\bar{u}\|_{W^{3,p}(\Omega_2)} \leqslant C\|\bar{\bar{u}}\|_{W^{3,p}(\Omega_3)} \leqslant C\|A_\kappa\bar{\bar{u}}\|_{W^{1,p}(\Omega_3)}$$
$$\leqslant C\left(\|A_\kappa\bar{u}\|_{W^{1,p}} + \|\bar{u}\|_{W^{2,p}}\right) \leqslant C\|A_\kappa\bar{u}\|_{W^{1,p}}, \tag{4.10}$$

where $2 < p < 2/(2-\beta)$ with $1 < \beta$. In the last inequality, we have used the following regularity estimate (cf. [14, Theorem 5.2.7])

$$\|\bar{u}\|_{W^{2,p}} \leqslant C\|A_\kappa \bar{u}\|_{L^p}.$$

Using (2.3) with $f = 0$, Sobolev inequality and Lemma 2.2, it now follows that

$$\begin{aligned}\|A_\kappa \bar{u}\|_{W^{1,p}} &\leqslant C\|\bar{u}_t\|_{W^{1,p}} \leqslant C\|\bar{u}_t\|_{H^2}\\ &\leqslant C\|\bar{u}_{tt}\| \leqslant Ct^{-1}\left[\|u_0\|_2 + \left(\sum_{j=0}^{2}\int_0^t \left\|\frac{\partial^j g_2}{\partial t^j}\right\|^2_{H^{1/2}(\Gamma_2)} \mathrm{d}s\right)^{1/2}\right]\end{aligned} \tag{4.11}$$

for $t \in J$. Combine (4.7)–(4.11) with (4.6) to complete the rest of the proof. □

5. Discrete-in-time scheme

In this section, based on backward Euler method we shall discuss fully discrete approximations to (2.6). While optimal order error estimates are obtained in L^2 and H^1 norms, a quasi-optimal order error estimate in L^∞ norm is established in any sub-domain away from the corners.

Let $k > 0$ be the time step and $t_n = nk$ with $T = Nk$. For any continuous function $\psi(t)$, set $\psi^n = \psi(t_n)$ and $\bar{\partial}_t \psi^n = k^{-1}(\psi^n - \psi^{n-1})$. For $\phi \in S_h$, define $\|\phi\|_{-j,h}$ as

$$\|\phi\|_{-j,h} = \sup_{g \in S_h} \frac{(\phi, I_h^* g)}{\|g\|_j}, \quad j = 0, 1.$$

The discrete in time Euler scheme is to seek a function U^n, $n = 1, 2, \ldots, N$ satisfying

$$\left(\bar{\partial}_t U^n, I_h^* \chi\right) + A_\kappa\left(U^n, I_h^* \chi\right) = \langle \bar{g}_2, I_h^* \chi\rangle + \left(\bar{f}^n, I_h^* \chi\right) \quad \forall \chi \in S_h, \tag{5.1}$$

with given $U^0 = R_h u_0$.

Set $U^n = e^{-\kappa t_n}\widetilde{U}^n$, where $\widetilde{U}^n$ is the backward Euler approximation to (1.4) which may be obtained by putting $\kappa = 0$ in (5.1). Note that if U^n's are known then we can easily compute $\widetilde{U}^n$'s.

Denote $\eta^n = U^n - \bar{u}_h^n$. Then, from (2.6) and (5.1), η^n satisfies

$$\left(\bar{\partial}_t \eta^n, I_h^* \chi\right) + A_\kappa\left(\eta^n, I_h^* \chi\right) = \left(\tau^n, I_h^* \chi\right), \quad \chi \in S_h \tag{5.2}$$

with $\eta^0 = 0$, where $\tau^n = \bar{u}_{ht}^n - \bar{\partial}_t \bar{u}_h^n$.

Lemma 5.1. *Let η^n satisfy* (5.2) *and $\bar{u}_h(0) = R_h u_0$. Then there exists a constant C independent of k such that*

$$\|\eta^n\|^2 + k\sum_{j=1}^{n}\|\eta^j\|_1^2 \leqslant Ck^2\left(\|u_0\|_2^2 + \sum_{j=0}^{1}\int_0^t \left\|\frac{\partial^j g_2}{\partial t^j}\right\|^2_{H^{1/2}(\Gamma_2)} \mathrm{d}s\right).$$

Proof. Taking $\chi = \eta^n$ in (5.2) and using the symmetry of $(\chi, I_h^* \psi)$, $\chi, \psi \in S_h$ on S_h, and the identity $(\bar{\partial}_t \eta^n, I_h^* \eta^n) = \frac{1}{2}\bar{\partial}_t\{|||\eta^n|||^2\} + \frac{k}{2}|||\bar{\partial}_t \eta^n|||^2$ leads to

$$\frac{1}{2}\bar{\partial}_t\left\{|||\eta^n|||^2\right\} + A_\kappa\left(\eta^n, I_h^* \eta^n\right) + \frac{k}{2}|||\bar{\partial}_t \eta^n|||^2 = \left(\tau^n, I_h^* \eta^n\right) \leqslant \|\tau^n\|_{-1,h}\|\eta^n\|_1.$$

Apply Young's inequality and kickback the term $\|\eta^n\|_1^2$ to obtain

$$\frac{1}{2}\bar{\partial}_t\left\{|||\eta^n|||^2\right\} + \|\eta^n\|_1^2 \leqslant C\|\tau^n\|_{-1,h}^2.$$

Summing over n from 1 to m and using Lemma 2.3, it now leads to

$$\|\eta^m\|^2 + k\sum_{n=1}^{m}\|\eta^n\|_1^2 \leqslant C\left(\|\eta^0\|^2 + k\sum_{n=1}^{m}\|\tau^n\|_{-1,h}^2\right).$$

Since $\eta^0 = 0$, it now remains to estimate the second term on the right. We write τ^j as

$$\tau^j = \frac{1}{k}\int_{t_{j-1}}^{t_j} (s - t_{j-1})\bar{u}_{hss}(s)\,\mathrm{d}s \tag{5.3}$$

and hence,

$$\|\tau^j\|_{-1,h}^2 \leqslant k \int_{t_{j-1}}^{t_j} \|\bar{u}_{hss}(s)\|_{-1,h}^2\,\mathrm{d}s.$$

Differentiating (2.6) with respect to time t, we obtain for $f = 0$

$$\big(\bar{u}_{htt}(t), I_h^*\chi\big) = -A_\kappa\big(\bar{u}_{ht}(t), I_h^*\chi\big) + \left(\frac{\partial \bar{g}_2}{\partial t}, I_h^*\chi\right),$$

and this implies

$$\|\bar{u}_{htt}(t)\|_{-1,h} \leqslant C\left(\|\bar{u}_{ht}(t)\|_1 + \sum_{j=0}^{1}\left\|\frac{\partial^j g_2}{\partial t^j}\right\|_{H^{1/2}(\Gamma_2)}\right).$$

Applying Lemma 2.2 at the discrete level, it now follows that

$$k\sum_{j=1}^{n}\|\tau^j\|_{-1,h}^2 \leqslant Ck^2\int_0^{t_n}\left\{\|\bar{u}_{hs}(s)\|_1^2 + \sum_{j=0}^{1}\left\|\frac{\partial^j g_2}{\partial t^j}\right\|_{H^{1/2}(\Gamma_2)}^2\right\}\mathrm{d}s \leqslant Ck^2\left[\|u_0\|_2^2 + \sum_{j=0}^{1}\int_0^{t_n}\left\|\frac{\partial^j g_2}{\partial t^j}\right\|_{H^{1/2}(\Gamma_2)}^2\mathrm{d}s\right],$$

and this completes the rest of the proof. □

Lemma 5.2. *Let the hypotheses in Lemma* 5.1 *hold true. Then there is a constant C independent of k such that*

$$k\sum_{j=1}^{n} t_j\|\bar{\partial}_t\eta^j\|^2 + t_n\|\eta^n\|_1^2 \leqslant Ck^2\left(\|u_0\|_2^2 + \sum_{j=0}^{1}\int_0^{t_n}\left\|\frac{\partial^j g_2}{\partial t^j}\right\|_{H^{1/2}(\Gamma_2)}^2\mathrm{d}s\right).$$

Taking $\chi = t_n\bar{\partial}_t\eta^n$ in (5.2) and using identity

$$t_nA_\kappa\big(\eta^n, \bar{\partial}_t\eta^n\big) = \frac{1}{2}\bar{\partial}_t\big\{t_nA_\kappa\big(\eta^n, \eta^n\big)\big\} + \frac{k}{2}t_nA_\kappa\big(\bar{\partial}_t\eta^n, \bar{\partial}_t\eta^n\big) - \frac{1}{2}A_\kappa\big(\eta^{n-1}, \eta^{n-1}\big),$$

we obtain

$$\begin{aligned} &t_n|||\bar{\partial}_t\eta^n|||^2 + \frac{1}{2}\bar{\partial}_t\big\{t_nA_\kappa\big(\eta^n, \eta^n\big)\big\} + \frac{k}{2}t_nA_\kappa\big(\bar{\partial}_t\eta^n, \bar{\partial}_t\eta^n\big) \\ &\quad = t_n\big(\tau^n, \bar{\partial}_t\eta^n\big) + \frac{1}{2}A_\kappa\big(\eta^{n-1}, \eta^{n-1}\big) + t_n\big\{A_\kappa\big(\eta^n, \bar{\partial}_t\eta^n\big) - A_\kappa\big(\eta^n, I_h^*\bar{\partial}_t\eta^n\big)\big\} \\ &\quad \leqslant t_n\|\tau^n\|\|\bar{\partial}_t\eta^n\| + C\|\eta^{n-1}\|_1^2 + Ct_n\|\eta^n\|_1\|\bar{\partial}_t\eta^n\|. \end{aligned}$$

Summing over n from 1 to m and using standard kickback arguments to have

$$k\sum_{n=1}^{m} t_n\|\bar{\partial}_t\eta^n\|^2 + t_m\|\eta^m\|_1^2 \leqslant Ck\sum_{n=1}^{m} t_n\|\tau^n\|^2 + Ck\sum_{n=1}^{m-1}\|\eta^n\|_1^2 + Ck\sum_{n=1}^{m} t_n\|\eta^n\|_1^2.$$

From (5.3), we note that

$$k\sum_{n=1}^{m} t_n\|\tau^n\|^2 \leqslant \sum_{n=1}^{m}\int_{t_{n-1}}^{t_n} t_n(s - t_{n-1})^2\|\bar{u}_{hss}\|^2\,\mathrm{d}s.$$

Since $(s-t_{n-1})^2 t_n \leqslant Ck^2 s$ for $s \in [t_{n-1}, t_n]$, we have

$$k\sum_{n=1}^{m} t_n \|\tau^n\|^2 \leqslant Ck^2 \int_0^{t_n} s\|\bar{u}_{hss}\|^2 \, \mathrm{d}s \leqslant Ck^2\left(\|u_0\|_2^2 + \sum_{j=0}^{1}\int_0^{t_n}\left\|\frac{\partial^j g_2}{\partial t^j}\right\|_{H^{1/2}(\Gamma_2)}^2 \mathrm{d}s\right).$$

Now using Lemma 5.1, we obtain, for sufficiently small k,

$$k\sum_{n=1}^{m} t_n \|\bar{\partial}_t \eta^n\|^2 + t_m\|\eta^m\|_1^2 \leqslant Ck^2\left(\|u_0\|_2^2 + \sum_{j=0}^{1}\int_0^{t_n}\left\|\frac{\partial^j g_2}{\partial t^j}\right\|_{H^{1/2}(\Gamma_2)}^2 \mathrm{d}s\right) + Ck\sum_{n=1}^{m-1} t_n\|\eta^n\|_1^2.$$

Finally, apply discrete version of Gronwall's lemma to completes the proof. □

Theorem 5.1. *Let $\tilde{u}$ be the exact solution of* (1.4) *with $f=0$, and $\widetilde{U}^n$ be its backward Euler approximation at $t=t_n$. Then there is a constant C independent of h and k such that, for $n=1,2,\ldots,N$, we have*

$$\|\widetilde{U}^n - \tilde{u}(t_n)\|_1 \leqslant Ct_n^{-1/2}(h+k)\left\{\|u_0\|_2 + \left(\sum_{j=0}^{2}\int_0^{t_n}\left\|\frac{\partial^j g_2}{\partial t^j}\right\|_{H^{1/2}(\Gamma_2)}^2 \mathrm{d}s\right)^{1/2}\right\}$$

and

$$\|\widetilde{U}^n - \tilde{u}(t_n)\| \leqslant C(h^2+k)\left\{\|u_0\|_2 + \left(\sum_{j=0}^{2}\int_0^{t_n}\left\|\frac{\partial^j g_2}{\partial t^j}\right\|_{H^{1/2}(\Gamma_2)}^2 \mathrm{d}s\right)^{1/2}\right\}.$$

Proof. We write $\widetilde{U}^n - \tilde{u}(t_n)$ as

$$\widetilde{U}^n - \tilde{u}(t_n) = e^{\kappa t_n}\big(U^n - \bar{u}_h(t_n)\big) - \big(\tilde{u}(t_n) - \tilde{u}_h(t_n)\big) = e^{\kappa t_n}\eta^n - \big(\tilde{u}(t_n) - \tilde{u}_h(t_n)\big). \tag{5.4}$$

Now, combine Lemmas 5.2 and 5.1 with Theorem 4.1 to obtain the desired estimates and this completes the rest of the proof. □

Theorem 5.2. *Let $\Omega_0 \subset \Omega$ be such that $\overline{\Omega}_0$ does not contain any vertex of Ω. Further, let $\tilde{u}$ be the exact solution of* (1.4) *with $f=0$, and $\widetilde{U}^n$ be its backward Euler approximation at $t=t_n$. Then there is a positive constant C such that*

$$\|\widetilde{U}^n - \tilde{u}(t_n)\|_{L_\infty(\Omega_0)} \leqslant Ct_n^{-1}\log\frac{1}{h}(h^2+k)\left\{\|u_0\|_2 + \left(\sum_{j=0}^{1}\int_0^{t_n}\left\|\frac{\partial^j g_2}{\partial t^j}\right\|_{H^{1/2}(\Gamma_2)}^2 \mathrm{d}s\right)^{1/2}\right\}$$

for $n=1,2,\ldots,N$.

Proof. By Sobolev inequality, we have

$$\|\eta^n\|_{L^\infty} \leqslant C\left(\log\frac{1}{h}\right)^{1/2}\|\eta^n\|_1.$$

The desired estimate now follows from Lemma 5.2, Theorem 4.2, (5.4) and the triangle inequality. This completes the rest of the proof. □

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