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Linear and Quasi-Linear Iterative Splitting Methods: Theory and Applications

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Abstract

In this paper we consider time-decomposition methods and present interesting model problems as benchmark problems in order to study the numerical analysis of the proposed methods. For the time-decomposition methods we discuss the iterative operator-splitting methods with respect to the stability and consistency. The main idea for deriving the error estimates is the Taylor expansion in time of the linearized operators. The stability analysis is based on the A-stability of ordinary differential equations, and the importance of including weighted parameters for relaxing the iterative operator-splitting methods can be seen. The exactness and the efficiency of the methods are investigated through solutions of nonlinear model problems of parabolic differential equations, for example systems of convection-reaction-diffusion equations. Finally we discuss the future works and the usefulness of this study in real-life applications.

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1 Introduction

In this paper we consider the numerical solutions of linear and nonlinear time-dependent partial differential equations (PDEs) of reaction-transport problems. These equations are numerically studied and convergence results are presented, e.g. in [20, 15, 13]. We concentrate on the time-decomposition methods and decouple the multi-operator equations in simpler equations, see [18]. The idea behind is to decouple into different time scales and therefore have more efficient computations. These methods are well-known in applications for large equation systems with slow and fast time scales, for example in environmental models, such as air pollution models, see [1, 4, 8, 21]. Our contribution is the analysis of the consistency and stability of the linear and quasi-linear iterative splitting methods, see [9, 12]. Under certain assumptions to the regularity and the boundedness we can extend our linear theory. Based on this results the stability of the methods is also discussed. The main advantage of the method lies in the higher-order results if the initial conditions are sufficiently exact. Numerical results of non-stiff, linear and nonlinear models can support our contributions, see [8, 15, 17].

The paper is outlined as follows. We introduce our mathematical model of parabolic differential equations 2. In section 3 we describe the iterative operator-splitting method. The consistency and stability analysis is presented for the linear and nonlinear case. We discuss the variational splitting and the *a posteriori* error estimates. The parallelization is presented in section 4. Our numerical results with linear and nonlinear examples are discussed in section 5. Finally we discuss our future works in section 6 with respect to our research area.

2 Mathematical Model

We deal with systems of parabolic differential equations containing a first-order temporal derivation and second-order spatial derivations. The equations are used for modeling transport-reaction processes in environmental problems, see [10, 15, 21]. Such systems of n parabolic differential equations are of the form

$$\begin{aligned}
&\frac{\partial u}{\partial t} = F_1(u)u + F_2(u)u \ , \ \text{in } \Omega \times (0,T) \ , \qquad (1)\\
&u(x,t) = g(x,t) \ , \ \text{on } \partial\Omega \times (0,T) \ , \ \text{(boundary condition)} \ , \\
&u(x,0) = u_0(x) \ , \ \text{in } \Omega \ , \ \text{(initial condition)} \ ,
\end{aligned}$$

where the solution is given as $u = (u_1, \ldots, u_n)$, $\Omega \subset I\!R^d$, with $d = 2,3$ is the spatial dimension, and

$$F_1(u) = \begin{pmatrix} -v_{11} \cdot \nabla u_1 & \cdots & -v_{n1} \cdot \nabla u_n \\ \ldots & \cdots & \ldots \\ -v_{1n} \cdot \nabla u_1 & \cdots & -v_{nn} \cdot \nabla u_n \end{pmatrix},$$

$$F_2(u) = \begin{pmatrix} \nabla D_{11} \cdot \nabla u_1 + f_1(u) & \ldots & \nabla D_{n1} \cdot \nabla u_1 + f_n(u) \\ \ldots & \ldots & \ldots \\ \nabla D_{1n} \cdot \nabla u_1 + f_1(u) & \ldots & \nabla D_{nn} \cdot \nabla u_n + f_n(u) \end{pmatrix}.$$

with $F_1(u)$ being the nonlinear convection and $F_2(u)$ the nonlinear diffusion and nonlinear reaction operator . We assume sufficient smoothness for the solution vector $u = (u_1, \ldots, u_n)^t$ with $u_i \in C^{2,1}(\Omega, [0,T])$, for $i = 1, \ldots, n$, where n is the number of equations. The solution of the model corresponds to the concentration of the pollution. The velocity parameters are given as $v_{i,j} \in I\!R^{d,+}$, with $i, j = 1, \ldots, n$. The diffusion parameters are given as $D_{i,j} \in I\!R^{d,+} \times I\!R^{d,+}$, with $i, j = 1, \ldots, n$. The source term or reaction term is a nonlinear function given as $f_i : (C^{2,1}(\Omega, [0,T]))^n \rightarrow I\!R^+$, with $i = 1, \ldots, n$, see [10].

In the following analysis we assume the spatial discretization of our convection and diffusion operators, e.g. Finite Difference or Finite Element methods. Therefore we obtain an ordinary differential equation, which is a Cauchy problem of the following form:

$$\frac{dc(t)}{dt} = A(c(t))c(t) + B(c(t))c(t) \quad t \in (0,T), \quad c(0) = c_0, \tag{2}$$

where the initial function c_0 is given, and the operators $A(u),\ B(u) : \mathbf{X} \rightarrow \mathbf{X}$ are linear and densely defined in the real Banach-space $\mathbf{X}$, see [3].

Thus they correspond with the operators given in equation (1), whereby $A(u)$ represents the convection operator, $B(u)$ the diffusion and reaction operator.

In the next section we introduce the iterative splitting method.

3 Iterative Splitting Method

We introduce the iterative splitting method and concentrate on two operators. The method is studied as a global approximation method on the whole time interval $[0,T]$ in [16]. As a numerical method it was introduced in [6]. In this paper, we discuss the linear and nonlinear case of the method.

3.1 Linear iterative splitting method

The linear iterative operator-splitting method is described in [6] and has its benefits in being a higher-order method and a physical splitting of the problem, while the operators are still in the sub-problems, see [6]. The resulting new operator equations are dominated by each separated physical effect, see [8] and [10]. Due to this in each operator equation we can specialize the discretization and solver methods to the dominating physical effect. We present an algorithm which is based on the iteration for the fixed discretization with the step size τ_n. On the time interval $[t^n, t^{n+1}]$ we solve the following sub-problems consecutively:

$$\frac{dc_i(t)}{dt} = Ac_i(t) \ + \ Bc_{i-1}(t), \text{ with } \ c_i(t^n) = c_{\rm sp}^n, \tag{3}$$

$$\frac{dc_{i+1}(t)}{dt} = Ac_i(t) \ + \ Bc_{i+1}(t), \text{ with } \ c_{i+1}(t^n) = c_{\rm sp}^n, \tag{4}$$

where $c_0(t)$ is any fixed function for each iteration and $i = 1, 3, 5, \dots 2m + 1$. $c_{\rm sp}^n$ denotes the known split approximation at the time level $t = t^n$. The split approximation at the time level $t = t^{n+1}$ is defined as $c_{\rm sp}^{n+1} = c_{2m+1}(t^{n+1})$. We assume that the starting function $c_0(t^{n+1})$ satisfies $c_0(t^n) = c_{\rm sp}^n$. Therefore the iterative splitting method is consistent, see [6].

We can derive the following error of the linear iterative splitting method. We can obtain a higher-order method, if our starting conditions are equal to our initial conditions and if the approximating error is sufficient small, e.g. $O(\tau^2)$. Then we have the following theorem for the splitting error.

Theorem 3.1 *Let $A, B \in \mathcal{L}(X)$ be given linear bounded operators. We consider the abstract Cauchy problem:*

$$\begin{aligned} \partial_t c(t) = Ac(t) + Bc(t), \quad 0 < t \leq T, \\ c(0) = c_0. \end{aligned} \tag{5}$$

Then the problem (5) has a unique solution.

The error for the splitting methods (3)–(4), for $i = 1, 3, \dots, 2m+1$, is given as:

$$||e_i|| = K||B||\tau_n||e_{i-1}|| + O(\tau_n^2) \tag{6}$$

and hence

$$||e_{2m+1}|| = K_m||e_0||||B||^{2m}\tau_n^{2m} + O(\tau_n^{2m+1}), \tag{7}$$

where τ_n is the time step, e_0 the initial error $e_0(t) = c(t) - c_0(t)$ and m the number of iteration steps. $K \in I\!R^+$ and $K_m < C \in R^+$ for $m \to \infty$ are constants, thus we can bound the operators. Furthermore $||B||$ is the maximum norm of operator B. We also assume that A and B are bounded and monotone operators.

For the proof of the linear case we refer to the ideas of the Taylor expansion and the estimation of exp-functions, as done in the work [6].

Proof 3.2 *Since $A + B \in \mathcal{L}(X)$, therefore it is a generator of a uniformly continuous semi-group, hence the problem (5) has a unique solution $c(t) = \exp((A + B)t)c_0$.*

Let us consider the iteration (3)–(4) on the subinterval $[t^n, t^{n+1}]$. For the local error function $e_i(t) = c(t) - c_i(t)$ we have the following relations:

$$\begin{aligned} \partial_t e_i(t) &= Ae_i(t) + Be_{i-1}(t), \quad t \in (t^n, t^{n+1}], \\ e_i(t^n) &= 0, \end{aligned} \tag{8}$$

and

$$\begin{aligned} \partial_t e_{i+1}(t) &= Ae_i(t) + Be_{i+1}(t), \quad t \in (t^n, t^{n+1}], \\ e_{i+1}(t^n) &= 0, \end{aligned} \tag{9}$$

for $i = 1, 3, 5, \ldots$, with $e_0(0) = 0$ and $e_0(t) = c(t)$. We use the notations $\mathbf{X}^2$ for the product space $\mathbf{X} \times \mathbf{X}$ supplied with the norm $\|(u, v)\| = \max\{\|u\|, \|v\|\}$ $(u, v \in \mathbf{X})$. The elements $\mathcal{E}_i(t)$, $\mathcal{F}_i(t) \in \mathbf{X}^2$ and the linear operator $\mathcal{A} : \mathbf{X}^2 \to \mathbf{X}^2$ are defined as follows:

$$\mathcal{E}_i(t) = \begin{bmatrix} e_i(t) \\ e_{i+1}(t) \end{bmatrix}, \quad \mathcal{F}_i(t) = \begin{bmatrix} Be_{i-1}(t) \\ 0 \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} A & 0 \\ A & B \end{bmatrix}. \tag{10}$$

Then, using the notations (10), the relations (8)–(9) can be written in the form

$$\begin{aligned} \partial_t \mathcal{E}_i(t) &= \mathcal{A}\mathcal{E}_i(t) + \mathcal{F}_i(t), \quad t \in (t^n, t^{n+1}], \\ \mathcal{E}_i(t^n) &= 0. \end{aligned} \tag{11}$$

Due to our assumptions, $\mathcal{A}$ is a generator of the one-parameter C_0 semi-group $(\exp \mathcal{A}t)_{t \geq 0}$, hence using the variations of constants formula, the solution of the abstract Cauchy problem (11) with homogeneous initial conditions can be written as:

$$\mathcal{E}_i(t) = \int_{t^n}^{t} \exp(\mathcal{A}(t - s))\mathcal{F}_i(s) ds, \quad t \in [t^n, t^{n+1}]. \tag{12}$$

Hence, using the denotation

$$\|\mathcal{E}_i\|_\infty = \sup_{t \in [t^n, t^{n+1}]} \|\mathcal{E}_i(t)\|, \tag{13}$$

we have

$$\begin{aligned} \|\mathcal{E}_i(t)\| &\leq \|\mathcal{F}_i\|_\infty \int_{t^n}^{t} \|\exp(\mathcal{A}(t - s))\| ds = \\ &= \|B\| \|e_{i-1}\| \int_{t^n}^{t} \|\exp(\mathcal{A}(t - s))\| ds, \quad t \in [t^n, t^{n+1}]. \end{aligned} \tag{14}$$

Since $(\mathcal{A}(t))_{t\geq 0}$ *is a semi-group, therefore the so-called growth estimation,*

$$\|\exp(\mathcal{A}t)\| \leq K\exp(\omega t); \quad t \geq 0, \tag{15}$$

holds with some numbers $K \geq 0$ *and* $\omega \in I\!R$.

- *Assume that* $(\mathcal{A}(t))_{t\geq 0}$ *is a bounded or exponentially stable semi-group, i.e. (15) holds with some* $\omega \leq 0$. *Then obviously the estimate*

$$\|\exp(\mathcal{A}t)\| \leq K, \quad t \geq 0, \tag{16}$$

holds, and hence, according to (14), we have the relation

$$\|\mathcal{E}_i\|(t) \leq K\|B\|\tau_n\|e_{i-1}\|, \quad t \in [t^n, t^{n+1}]. \tag{17}$$

- *Assume that* $(\exp\mathcal{A}t)_{t\geq 0}$ *has an exponential growth with some* $\omega > 0$. *Using (15) we have*

$$\int_{t^n}^{t} \|\exp(\mathcal{A}(t-s))\| ds \leq K_\omega(t), \quad t \in [t^n, t^{n+1}], \tag{18}$$

where

$$K_\omega(t) = \frac{K}{\omega}\left(\exp(\omega(t-t^n)) - 1\right), \quad t \in [t^n, t^{n+1}]. \tag{19}$$

Hence

$$K_\omega(t) \leq \frac{K}{\omega}\left(\exp(\omega\tau_n) - 1\right) = K\tau_n + \mathcal{O}(\tau_n^2). \tag{20}$$

The estimations (17) and (20) result in

$$\|\mathcal{E}_i\|_\infty = K\|B\|\tau_n\|e_{i-1}\| + \mathcal{O}(\tau_n^2). \tag{21}$$

Taking into account the definition of $\mathcal{E}_i$ *and the norm* $\|\cdot\|_\infty$, *we obtain*

$$\|e_i\| = K\|B\|\tau_n\|e_{i-1}\| + \mathcal{O}(\tau_n^2), \tag{22}$$

and hence

$$\|e_{i+1}\| = K\|B\|||e_i|| \int_{t^n}^{t} \|\exp(\mathcal{A}(t-s))\| ds, \tag{23}$$

$$\|e_{i+1}\| = K\|B\|\tau_n(K\|B\|\tau_n\|e_{i-1}\| + \mathcal{O}(\tau_n^2)), \tag{24}$$

$$\|e_{i+1}\| = K_1\tau_n^2\|e_{i-1}\| + \mathcal{O}(\tau_n^3), \tag{25}$$

we apply the recursive argument which proves our statement .

Remark 3.3 *The result shows that for large* m *we have an estimation of* $K_m = K^m||B||^m \leq \infty$, *so that means we have to restrict the number of iteration steps. In practice* $m = 2, 4, 6$ *is sufficient and we can control the estimation.*

In the following we extend the results to the quasi-linear case.

3.2 Quasi-Linear iterative splitting method

We consider the quasi-linear evolution equation

$$\frac{dc(t)}{dt} = A(c(t))c(t) \ + \ B(c(t))c(t) \ , \ \forall t \in [0, T] \ , \tag{26}$$

$$c(0) = c_0 \ , \tag{27}$$

where $T > 0$ is sufficient small and the operators $A(c), B(c) : \mathbf{X} \rightarrow \mathbf{X}$ are linear and densely defined in the real Banach-space $\mathbf{X}$, see [**?**].

In the following we modify the linear iterative operator-splitting methods to a quasi-linear operator-splitting method. The idea are to linearize the method by using the old solution for the linear operators.

The algorithm is based on the iteration with fixed splitting discretization step size τ. On the time interval $[t^n, t^{n+1}]$ we solve the following subproblems consecutively for $i = 1, 3, \dots, 2m + 1$:

$$\frac{\partial c_i(t)}{\partial t} = A(c_{i-1}(t))c_i(t) \ + \ B(c_{i-1}(t))c_{i-1}(t), \text{ with } \ c_i(t^n) = c^n \ , \tag{28}$$

$$\frac{\partial c_{i+1}(t)}{\partial t} = A(c_{i-1}(t))c_i(t) \ + \ B(c_{i-1}(t))c_{i+1}, \text{ with } \ c_{i+1}(t^n) = c^n \ , \tag{29}$$

where $c_0 \equiv 0$ and c^n is the known split approximation at the time level $t = t^n$. The split approximation at the time level $t = t^{n+1}$ is defined as $c^{n+1} = c_{2m+1}(t^{n+1})$. We assume the operators $A(c_{i-1}), B(c_{i-1}) : \mathbf{X} \rightarrow \mathbf{X}$ are linear and densely defined on the real Banach-space $\mathbf{X}$, for $i = 1, 3, \dots, 2m + 1$.

The splitting discretization step size is τ and the time interval is $[t^n, t^{n+1}]$. We solve the following subproblems consecutively for $i = 1, 3, \dots, 2m + 1$:

$$\frac{\partial c_i(t)}{\partial t} = \tilde{A} c_i(t) \ + \ \tilde{B}(c_{i-1}(t)), \text{ with } \ c_i(t^n) = c^n \ , \tag{30}$$

$$\frac{\partial c_{i+1}(t)}{\partial t} \tilde{A}(c_i(t)) \ + \ \tilde{B} c_{i+1}(t), \text{ with } \ c_{i+1}(t^n) c^n \ , \tag{31}$$

where $c_0 \equiv 0$ and c^n is the known split approximation at the time level $t = t^n$. The split approximation at the time level $t = t^{n+1}$ is defined as $c^{n+1} = c_{2m+1}(t^{n+1})$. The operators are given as:

$$\tilde{A} = A(c_{i-1}), \ \tilde{B} = B(c_{i-1}),$$

We assume bounded operators $\tilde{A}, \tilde{B} : \mathbf{X} \rightarrow \mathbf{X}$, where $\mathbf{X}$ is a general Banach space. These operators as well as their sum are generators of the C_0 semi-group. The convergence is examined in a general Banach space setting in the following theorem.

Theorem 3.4 *Let us consider the quasi-linear evolution equation*

$$\begin{aligned} &\partial_t c(t) = A(c(t))c(t) + B(c(t))c(t), \quad 0 < t \leq T \ , \\ &c(t^n) = c_n \ , \end{aligned} \tag{32}$$

*where $A(c), B(c)$ are linear and densely defined operators in a Banach-space, see [**?**].*

We apply the quasi-linear iterative operator-splitting method (28)–(29) and obtain a convergence-rate of second order.

$$||e_i|| = K\tau_n\omega_1||e_{i-1}|| + \mathcal{O}(\tau_n^2), \tag{33}$$

where K is constant. Further we assume the boundedness of the linear operators with $\max\{||A(e_{i-1}(t))||, ||B(e_{i-1}(t)||) \leq \omega_1$ *for* $t \in [0, T]$ *for* T *is sufficient small.*

We can obtain the result with Lipschitz-constants, and we prove the argument by using the semi-group theory.

Proof 3.5 *Let us consider the iteration (28)–(29) on the subinterval* $[t^n, t^{n+1}]$. *For the error function $e_i(t) = c(t) - c_i(t)$, we have the relations:*

$$\begin{aligned} &\partial_t e_i(t) = \tilde{A} e_i(t) + \tilde{B} e_{i-1}(t), \quad t \in (t^n, t^{n+1}], \\ &e_i(t^n) = 0 \ , \end{aligned} \tag{34}$$

and

$$\begin{aligned} &\partial_t e_{i+1}(t) = \tilde{A} e_i(t) + \tilde{B} e_{i+1}(t), \quad t \in (t^n, t^{n+1}], \\ &e_{i+1}(t^n) = 0 \ , \end{aligned} \tag{35}$$

for $m = 1, 3, 5, \dots$, with $e_0(0) = 0$, $e_{-1}(t) = c(t)$, $\tilde{A} = A(e_{i-1})$ and $\tilde{B} = B(e_{i-1})$.

We can rewrite the equations (34)–(35) into a system of linear first order differential equations in the following way. The elements $\mathcal{E}_i(t)$, $\mathcal{F}_i(t) \in \mathbf{X}^2$ and the linear operator $\mathcal{A} : \mathbf{X}^2 \to \mathbf{X}^2$ are defined as follows:

$$\mathcal{E}_i(t) = \left[\begin{array}{c} e_i(t) \\ e_{i+1}(t) \end{array} \right]; \quad \mathcal{A} = \left[\begin{array}{cc} \tilde{A} & 0 \\ \tilde{A} & \tilde{B} \end{array} \right] , \tag{36}$$

$$\mathcal{F}_i(t) = \left[\begin{array}{c} \tilde{B} e_{i-1}(t) \\ 0 \end{array} \right] . \tag{37}$$

Then, using the notations of Theorem 32, the relations (36)–(37) can be written in the form:

$$\begin{aligned} &\partial_t \mathcal{E}_i(t) = \mathcal{A}\mathcal{E}_i(t) + \mathcal{F}_i(t), \quad t \in (t^n, t^{n+1}], \\ &\mathcal{E}_i(t^n) = 0, \end{aligned} \tag{38}$$

due to our assumption, that $\tilde{A}$ and $\tilde{B}$ are bounded and linear operators. Furthermore we have a lipschitzian domain, and $\mathcal{A}$ is a generator of the one-parameter C_0 semi-group $(\mathcal{A}(t))_{t\geq 0}$. We also assume, that the estimation of our term $\mathcal{F}_i(t)$ with the growth conditions holds.

Remark 3.6 We can estimate the linear operators $A(e_{i-1})$ and $B(e_{i-1})$ by assuming the maximal accretivity and contractivity as:

$$||A(e_{i-1})y||_{\mathbf{X}} \leq \omega_2 ||y||_{\mathbf{Y}} \, , ||B(e_{i-1})y||_{\mathbf{X}} \leq \omega_3 ||y||_{\mathbf{Y}} \, , \tag{39}$$

where we have the embedding $\mathbf{Y} \subset \mathbf{X}$ and ω_2, ω_3 are constants in $I\!R^+$.

We can estimate the right hand side $\mathcal{F}_i(t)$ in the following lemma :

Lemma 3.7 *Let us consider the linear densely operator $\tilde{B}$. Then we can estimate $\mathcal{F}_i(t)$ as follows:*

$$||\mathcal{F}_i(t)|| \leq \omega_3 ||e_{i-1}|| \, . \tag{40}$$

Proof 3.8 *We have the norm $||\mathcal{F}_i(t)|| = \max\{\mathcal{F}_{i_1}(t), \mathcal{F}_{i_2}(t)\}$ over the components of the vector.*
We have to estimate each term:

$$\begin{aligned} ||\mathcal{F}_{i_1}(t)|| &\leq ||\tilde{B}(e_{i-1}(t))|| \\ &\leq \omega_3 ||e_{i-1}(t)|| \, , && (41) \\ ||\mathcal{F}_{i_2}(t)|| &= 0 \, . && (42) \end{aligned}$$

Thus we obtain the estimation:
$||\mathcal{F}_i(t)|| \leq \omega_3 ||e_{i-1}(t)||$.

Hence, using the variations of constants formula, the solution of the abstract Cauchy problem (38) with homogeneous initial condition can be written as:

$$\mathcal{E}_i(t) = \int_{t^n}^{t} \exp(\mathcal{A}(t-s)) \mathcal{F}_i(s) ds, \quad t \in [t^n, t^{n+1}]. \tag{43}$$

(See, e.g. [3].) Hence, using the denotation

$$\|\mathcal{E}_i\|_\infty = \sup_{t \in [t^n, t^{n+1}]} \|\mathcal{E}_i(t)\| \quad , \tag{44}$$

we have:

$$\begin{aligned} \|\mathcal{E}_i\|(t) &\leq \|\mathcal{F}_i\|_\infty \int_{t^n}^{t} \|\exp(\mathcal{A}(t-s))\| ds \\ &= \omega_3 \, \|e_{i-1}\| \int_{t^n}^{t} \|\exp(\mathcal{A}(t-s))\| ds, \quad t \in [t^n, t^{n+1}]. \end{aligned} \tag{45}$$

Since $(\mathcal{A}(t))_{t\geq 0}$ is a semi-group, therefore the so-called growth estimation

$$\|\exp(\mathcal{A}t)\| \leq K\exp(\omega_1 t), \quad t \geq 0\,, \tag{46}$$

holds with some numbers $K \geq 0$ and $\omega_1 = \max\{\omega_2, \omega_3\} \in I\!R$, see Remark 3.6 and [3].

Because of $\omega_1 \geq 0$, we Assume that $(\mathcal{A}(t))_{t\geq 0}$ has an exponential growth with. Using (46) we have:

$$\int_{t^n}^{t^{n+1}} \|\exp(\mathcal{A}(t-s))\| ds \leq K_\omega(t), \quad t \in [t^n, t^{n+1}], \tag{47}$$

where

$$K_{\omega_1}(t) = \frac{K}{\omega_1}\left(\exp(\omega(t-t^n)) - 1\right), \quad t \in [t^n, t^{n+1}]\,, \tag{48}$$

and hence

$$K_{\omega_1}(t) \leq \frac{K}{\omega_1}\left(\exp(\omega_1\tau_n) - 1\right) = K\tau_n + \mathcal{O}(\tau_n^2)\,. \tag{49}$$

Thus the estimations (40) and (49) result in

$$\|\mathcal{E}_i\|_\infty = K\tau_n\|e_{i-1}\| + \mathcal{O}(\tau_n^2). \tag{50}$$

Taking into account the definition of $\mathcal{E}_i$ and the norm $\|\cdot\|_\infty$, we obtain

$$\|e_i\| = K\tau_n\|e_{i-1}\| + \mathcal{O}(\tau_n^2), \tag{51}$$

where $K = \omega_1\omega_3 \in I\!R^+$. This proves our statement.

In the next subsection we present the stability results of the linear iterative splitting methods. The results can also be generalized to the nonlinear case.

3.3 Stability of the iterative operator-splitting method

The stability of the iterative operator-splitting methods is discussed in [14], [12]. The idea is to stabilize the pure iterative method with weighted operators. So we can relax the method with weighted operators, that use the old solutions of the iterative process.

The underlying weighted iterative operator-splitting methods are given as

$$\frac{dc_i(t)}{dt} = (1-\omega_1)Ac_i(t) + \omega_1 Ac_{i-1} \ + \ \omega_2\, Bc_{i-1}(t), \tag{52}$$
$$\text{with } \ c_i(t^n) = \omega_2\, c^n + (1-\omega_2)\, c_{i-1}(t^{n+1})$$
$$\text{and } c_0(t^n) = c^n\,,\ c_{-1} = 0.0,$$

$$\frac{dc_{i+1}(t)}{dt} = \omega_3 Ac_i(t) \ + \ (1-\omega_4)\, Bc_{i+1}(t) + \omega_4 Bc_i(t), \tag{53}$$
$$\text{with } \ c_{i+1}(t^n) = \omega_3\, c^n + (1-\omega_3)\, c_i(t^{n+1})\,.$$

In the following we present the stability analysis for the continuous case with commutative operators. First we apply the recursion for the general case and then concentrate on the commutative case.

3.3.1 Recursion

For a simplification we rewrite the linear system (52) and (53) recursively. That means we studied the recursive equations, integrated over the temporal intervals. The obtained recursive linear algebraic equation system can be studied in each scalar equation.

We consider the suitable vector norm $||\cdot||$ on $I\!R^M$ together with its induced operator norm. The matrix exponential of $Z \in I\!R^{M\times M}$ is denoted by $\exp(Z)$. For the estimates we assume

$|| \exp(\tau\ A)|| \leq 1$ and $|| \exp(\tau\ B)|| \leq 1$, for all $\tau > 0$.

For the system

$$\frac{dc(t)}{dt} = A\ c(t)\ +\ B\ c(t)\ ,\quad t \in (0,T),\quad c(0) = c_0, \tag{54}$$

where A, B are bounded and linear operators, it can be shown that $\exp(\tau\ (A+B)) \leq 1$ and the system itself is stable.

Using this idea, we apply an integration on the linear problem (52) and (53) and obtain the following:

$$\begin{aligned} c_i(t) &= \exp((1-\omega_1)(t-t^n)A)c^n + \int_{t^n}^t \exp((1-\omega_1)(t-s)A)\ (\omega_1\ Ac_{i-1}(s) \\ &\quad +\omega_2 Bc_{i-1}(s)\ ds\ , \qquad (55)\\ c_{i+1}(t) &= \exp((1-\omega_4)(t-t^n)B)c^n + \int_{t^n}^t \exp((1-\omega_4)(t-s)B)\ (\omega_4 B\ c_i(s) \\ &\quad +\omega_3 Ac_i(s))\ ds\ . \qquad (56) \end{aligned}$$

We eliminate c_i in the second equation with using the relation (55). Further we assume $\omega_2 = \omega_3 = \omega$ and $\omega_1 = \omega_4 = 0$ and obtain

$$\begin{aligned} c_{i+1}(t) &= \exp((t-t^n)B)c^n\ +\ \omega \int_{t^n}^t \exp((t-s)B)\ A\ \exp((t-s)A)\ ds\ c^n \qquad (57)\\ &+\ \omega^2 \int_{s=t^n}^t \int_{s'=t^n}^s \exp((t-s)B)\ A\ \exp((s-s')A)\ B\ c_{i-1}(s')\ ds'\ ds\ . \end{aligned}$$

In the next steps we estimate the resulting equation (57) with respect to commutative operators. We assume that we can evaluate the double integral $\int_{s=t^n}^t \int_{s'=t^n}^s$ as $\int_{s'=t^n}^t \int_{s=s'}^t$.

3.3.2 Commutative operators

For more transparency of the formula (56) we consider the eigenvalues λ_1 of A and λ_2 of B.

By replacing the operators A and B we obtain after some calculations

$$\begin{aligned} c_{i+1}(t) &= c^n \frac{1}{\lambda_1 - \lambda_2} (\omega \lambda_1 \exp((t - t^n)\lambda_1) + ((1-\omega)\lambda_1 - \lambda_2) \exp((t - t^n)\lambda_2)) \\ &+ c^n\, \omega^2 \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} \int_{s=t^n}^{t} (\exp((t-s)\lambda_1) - \exp((t-s)\lambda_2))\, ds\,. \end{aligned} \quad (58)$$

We point out that this relation is commutative in λ_1 and λ_2.

3.3.3 $A(\alpha)$-stability

We define $z_k = \tau \lambda_k$, $k = 1, 2$. We start with $c_0(t^n) = c^n$ and obtain

$$c_{2m}(t^{n+1}) = S_m(z_1, z_2)\, c^n\,, \quad (59)$$

where S_m is the stability function of the scheme with m iterations. We use (58) and obtain after some calculations

$$\begin{aligned} S_1(z_1, z_2)\, c^n &= \omega^2\, c^n + \frac{\omega\, z_1 + \omega^2\, z_2}{z_1 - z_2} \exp(z_1)\, c^n \\ &+ \frac{(1 - \omega - \omega^2)\, z_1 - z_2}{z_1 - z_2} \exp(z_2)\, c^n\,, \end{aligned} \quad (60)$$

$$\begin{aligned} S_2(z_1, z_2)\, c^n &= \omega^4\, c^n + \frac{\omega\, z_1 + \omega^4\, z_2}{z_1 - z_2} \exp(z_1)\, c^n \\ &+ \frac{(1 - \omega - \omega^4)\, z_1 - z_2}{z_1 - z_2} \exp(z_2)\, c^n \\ &+ \frac{\omega^2\, z_1\, z_2}{(z_1 - z_2)^2} ((\omega z_1 + \omega^2 z_2) \exp(z_1) \\ &+ (-(1 - \omega - \omega^2) z_1 + z_2) \exp(z_2))\, c^n \\ &+ \frac{\omega^2\, z_1\, z_2}{(z_1 - z_2)^3} ((-\omega z_1 - \omega^2 z_2)(\exp(z_1) - \exp(z_2)) \\ &+ ((1 - \omega - \omega^2) z_1 - z_2)(\exp(z_1) - \exp(z_2)))\, c^n\,. \end{aligned} \quad (61)$$

Let us consider the set of eigenvalues of the function $S_m(z_1, z_2)$ given as $\mathcal{W}_\alpha = \{\zeta \in \mathbb{C} : |\arg(\zeta) \leq \alpha\}$. Then we can define the $A(\alpha)$-stability as follows.

Definition 3.9 *The $A(\alpha)$-stability is defined for the function $S_m(z_1, z_2)$ if the following equations are satisfied:*
1) Boundedness of the function:

$$|S_m(z_1, z_2)| \leq 1 \ , \tag{62}$$

and
2) The eigenvalues z_1, z_2 are in the sector $\pi/2$:

$$z_1, z_2 \in \mathcal{W}_{\pi/2}. \tag{63}$$

The $A(\alpha)$-stability of the equations (60) and (61) are given in the following theorem.

Theorem 3.10 *We have the following stabilities.*

For S_1 we have the A-stability,
$\max_{z_1 \leq 0, z_2 \in W_\alpha} |S_1(z_1, z_2)| \leq 1 \ , \ \forall \ \alpha \in [0, \pi/2]$ *with* $\omega = \frac{\sqrt{2}}{2}$.

For S_2 we have the $A(\alpha)$-stability,
$\max_{z_1 \leq 0, z_2 \in W_\alpha} |S_2(z_1, z_2)| \leq 1 \ , \ \forall \ \alpha \in [0, \pi/2]$ *with* $\omega \leq \left(\frac{1}{8 \ \tan^2(\alpha) + 1} \right)^{1/8}$.

Proof 3.11 *We consider a fixed $z_1 = z$ and $z_2 \rightarrow -\infty$. Then we obtain*

$$S_1(z, -\infty) = \omega^2 (1 - e^z) \tag{64}$$

and

$$S_2(z, -\infty) = \omega^4 (1 - (1 - z) e^z) \ . \tag{65}$$

If $z = x + iy$ then the stability function of the first iteration is given as
1) For S_1 there holds:

$$|S_1(z, -\infty)|^2 = \omega^4 (1 - \exp(x) \cos(y) + \exp(2x)) \leq 1 \ . \tag{66}$$

We rewrite the inequality (66) with respect to the exp*-function and get the result*

$$\exp(2x) \leq \frac{1}{\omega^4} - 1 + 2 \exp(x) \ \cos(y) \ . \tag{67}$$

Because $x < 0$ and $y \in I\!R$, we have $-2 \leq 2 \exp(x) \ \cos(y)$ and $\exp(2x) \leq 1$.
We estimate (67) as $\omega \leq \frac{\sqrt{2}}{2}$.

2) For S_2 there holds:

$$|S_2(z,-\infty)|^2 = \omega^8(1-2\exp(x)((1-x)\cos(y)+y\sin(y)) + \exp(2x)((1-x)^2+y^2)) \leq 1 . \quad (68)$$

After some calculations we obtain

$$\exp(x) \leq (\frac{1}{\omega^8}-1)\frac{\exp(-x)}{(1-x)^2+y^2} - 2\frac{|1-x|+|y|}{(1-x)^2+y^2} . \quad (69)$$

Then we estimate for $x < 0$ and $y \in I\!R$, such that

$$\frac{|1-x|+|y|}{(1-x)^2+y^2} \leq 3/2 \quad (70)$$

and

$$\frac{1}{2\tan^2(\alpha)} < \frac{\exp(-x)}{(1-x)^2+y^2} \quad (71)$$

are fulfilled.
Therefore we obtain $\tan(\alpha) = y/x$ *and we get the bound for* ω

$$\omega \leq \left(\frac{1}{8\tan^2(\alpha)+1}\right)^{1/8} . \quad (72)$$

In the next section we introduce the variational splitting that respects the spatial discretization methods based on weak formulations. So we can extend the strong formulation to weak formulations.

3.4 Variational splitting method

To extend the operator splitting also to weak formulations we introduce the variational splitting. The operators are reset with the variational formulation of the spatial discretization. Due to this all proofs of the splitting methods can be extended to the weak formulations, but we obtain a weaker order of H^m (Sobolev spaces), where m is the order of the weak formulation. The error analysis of the variational splitting is considered in the H^m space. We also obtain an reduction of the error in this space for more iteration steps.

The variational formulation can be written as:
Find $u \in H^m$ such that:

$$(\frac{\partial c}{\partial t}, v) = (A_1 c, v) + (A_2 c, v) , \ \forall v \in H^m, \quad (73)$$

$$c(x,t^n) = c^n(x) , \text{ on } \Omega ,$$
$$c(x,t) = g(x,t) , \text{ on } \partial\Omega \times [0,T] ,$$
$$(74)$$

where $(Ac, v) = (A_1 c, v) + (A_2 c, v)$.

We have the following iterative splitting method:

Find $c_{i-1}, c_i \in H^m$ such that

$$(\frac{\partial c_i}{\partial t}, v) = (A_1 c_i, v) + (A_2 c_{i-1}, v) \ , \ \forall \ v \in H^m \ , \tag{75}$$

and find $c_i, c_{i+1} \in H^m$ such that

$$(\frac{\partial c_{i+1}}{\partial t}, v) = (A_1 c_i, v) + (A_2 c_{i+1}, v) \ , \ \forall \ v \in H^m \ . \tag{76}$$

Remark 3.12 *The variational splitting method is a weak formulation of the iterative operator-splitting method. We can consider Hilbert spaces and therefore apply the results for less continuous solutions.*

In the next section we derive an *a posteriori* error estimate for our splitting method.

3.5 A posteriori error estimates for the variational splitting method

We consider the *a posteriori* error estimates for the beginning time iterations. The following theorem is derived for the *a posteriori* error estimates.

Theorem 3.13 *Let us consider the iterative operator-splitting method with the operators $A_1, A_2 : \mathbf{H} \to \mathbf{H}$, where $\mathbf{H}$ is an Hilbert space. We start with the initial condition $c_0(t^n) = c^n$ and consider two iterations ($i = 2$). Then we have*

$$||c_2(x, t^{n+1}) - c_1(x, t^{n+1})||_{L^2} \leq C \ ||c^n||_{\mathbf{H}} \ \tau + O(\tau^2) \ , \ \forall x \in \Omega \subset I\!R^d \ , \tag{77}$$

where C is a constant, $\tau = t^{n+1} - t^n$, $c^n = c(x, t^n)$ and $d = 2, 3$. For more iteration steps we can increase the order of the splitting method.

Proof 3.14 *We apply the equations (75) and (76) and deal with $c_{i-1}(s) = 0$. So the first iteration c_1 is given as:*

$$(c_1(x, t), v) = (\exp(A_1(t - t_n)) \ c^n(x), v) \ . \tag{78}$$

The second iteration is given as:

$$\begin{aligned}(c_2(x, t), v) \quad = \quad & (\exp(A_2(t - t_n)) \ (\int_{t_n}^{t} \ \exp(-B(s - t_n)) \\ & A_1 \exp(A_1(s - t_n)) \ dx + c^n(x)), v) \ . \end{aligned} \tag{79}$$

The Taylor expansion for the two functions leads to

$$(c_1(x,t),v) = ((I + A_1\tau + \frac{\tau^2}{2!}A_1^2)c^n(x),v) + O(\tau^3), \tag{80}$$

and

$$(c_2(x,t),v) = ((I + A_2\tau + \frac{\tau^2}{2!}A_2^2 + A_1\tau + A_1^2\frac{\tau^2}{2!} \\ +A_2A_1\frac{\tau^2}{2!})c^n(x),v) + O(\tau^3) \; . \tag{81}$$

For the stability we insert $v = c^n$ and obtain the error estimates by the subtraction $c_2 - c_1$:

$$||c_2 - c_1||_{L_2} \leq \tau ||c^n||_{\mathbf{H}} + O(\tau^2) \; . \tag{82}$$

Remark 3.15 *For the variational splitting we can derive for the first iteration steps the same accuracy as for the iterative operator-splitting method. We can generalize the result with respect to more iteration steps.*

In the next section we introduce the parallelization of the iterative splitting methods.

4 Parallelization

The efficiency of the iterative operator-splitting method is due to the parallelization of the method. While decoupling into simpler equations, the benefit of parallel computations of each equation is important.

One of the ideas is the windowing of the time-decomposition method. So for each window we compute a more accurate starting function to the next window. Based on this we can compute the windows independently and we only communicate by the starting functions, that are the result of the end time step of each window.

To illustrate the idea we present the figure 1.

Remark 4.1 *For a parallelization on the operator level, the iterative operator-splitting method has to be reformulated as an additive splitting method, see [7]. On the equation level we can parallelize on different initial sequences in time, defined as time windows. Each sequence is computed independently and is an improved initial value for the next sequence.*

In the next section we present the numerical examples.

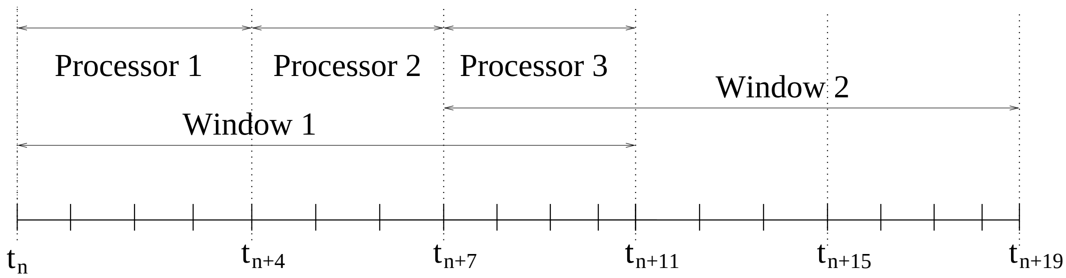

Figure 1: Parallelization of the time intervals.

5 Numerical Examples

In this section we consider linear and nonlinear examples to confirm the results of our theoretical considerations about the iterative operator-splitting methods.

5.1 First example: linear ODE

In the first example We deal with the following linear ordinary differential equation:

$$\frac{\partial u(t)}{\partial t} = \begin{pmatrix} -\lambda_1 & \lambda_2 \\ \lambda_1 & -\lambda_2 \end{pmatrix} u, \tag{83}$$

where the initial condition $u_0 = (1, 1)$ is given on the interval $[0, T]$.

The analytical solution is given by:

$$u(t) = \begin{pmatrix} c_1 - c_2 \exp\left(-(\lambda_1 + \lambda_2)t\right) \\ \frac{\lambda_1}{\lambda_2} c_1 + c_2 \exp\left(-(\lambda_1 + \lambda_2)t\right) \end{pmatrix}, \tag{84}$$

where

$$c_1 = \frac{2}{1 + \frac{\lambda_1}{\lambda_2}} \quad , \quad c_2 = \frac{1 - \frac{\lambda_1}{\lambda_2}}{1 + \frac{\lambda_1}{\lambda_2}}.$$

We split our linear operator into two operators by setting

$$\frac{\partial u(t)}{\partial t} = \begin{pmatrix} -\lambda_1 & 0 \\ \lambda_1 & 0 \end{pmatrix} u + \begin{pmatrix} 0 & \lambda_2 \\ 0 & -\lambda_2 \end{pmatrix} u. \tag{85}$$

We choose $\lambda_1 = 0.25$ and $\lambda_2 = 0.5$ on the interval [0,1].

We therefor have the operators:

$$A = \begin{pmatrix} -0.25 & 0 \\ 0.25 & 0 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 0 & 0.5 \\ 0 & -0.5 \end{pmatrix}.$$

For the integration method we use a temporal step size of $h = 10^{-3}$.

For the initialization of our iterative method we use $c_{-1} \equiv 0$.

From the examples you can see that the order increases by each iteration step.

In the following we compare the results of different discretization methods for the linear ordinary differential equation. An accuracy of at least fourth order is allowed. Our numerical results are presented in the tables 1, 2 and 3.

To compare the results we choose the same iteration steps and time partitions. The error between the analytical and numerical solution is given in the supremum norm.

Iterative steps	Number of splitting partitions	err_1	err_2
2	1	4.5321e-002	4.5321e-002
2	10	3.9664e-003	3.9664e-003
2	100	3.9204e-004	3.9204e-004
3	1	7.6766e-003	7.6766e-003
3	10	6.6383e-005	6.6383e-005
3	100	6.5139e-007	6.5139e-007
4	1	4.6126e-004	4.6126e-004
4	10	4.1883e-007	4.1883e-007
4	100	5.9520e-009	5.9521e-009
5	1	4.6828e-005	4.6828e-005
5	10	1.3954e-009	1.3953e-009
5	100	5.5352e-009	5.5351e-009
6	1	1.9096e-006	1.9096e-006
6	10	5.5527e-009	5.5528e-009
6	100	5.5355e-009	5.5356e-009

Table 1: Numerical results for the first example with the iterative splitting method and the second-order trapezoidal rule.

Iterative steps	Number of splitting partitions	err_1	err_2
2	1	4.5321e-002	4.5321e-002
2	10	3.9664e-003	3.9664e-003
2	100	3.9204e-004	3.9204e-004
3	1	7.6766e-003	7.6766e-003
3	10	6.6385e-005	6.6385e-005
3	100	6.5312e-007	6.5312e-007
4	1	4.6126e-004	4.6126e-004
4	10	4.1334e-007	4.1334e-007
4	100	1.7864e-009	1.7863e-009
5	1	4.6833e-005	4.6833e-005
5	10	4.0122e-009	4.0122e-009
5	100	1.3737e-009	1.3737e-009
6	1	1.9040e-006	1.9040e-006
6	10	1.4350e-010	1.4336e-010
6	100	1.3742e-009	1.3741e-014

Table 2: Numerical results for the first example with the iterative splitting method and third-order BDF 3 method.

The higher order in the time-discretization allows improved results with more iteration steps. Based on the theoretical results we can improve the order of the results with each iteration step. So at least with the fourth-order time-discretization we could show the highest order in our iterative method.

The convergence results of the three methods are given in figure 2.

Remark 5.1 *For the non-stiff case we obtain improved results for the iterative splitting method by increasing the number of iteration steps. Due to improved time-discretization methods, the splitting error can be reduced with higher-order Runge-Kutta methods.*

5.2 Second example: linear ODE with stiff parameters

We deal with the same equation as in the first example, now choosing $\lambda_1 = 1$ and $\lambda_2 = 10^4$ on the interval [0,1].

We therefore have the operators:

$$A = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 0 & 10^4 \\ 0 & -10^4 \end{pmatrix} .$$

Iterative steps	Number of splitting partitions	err_1	err_2
2	1	4.5321e-002	4.5321e-002
2	10	3.9664e-003	3.9664e-003
2	100	3.9204e-004	3.9204e-004
3	1	7.6766e-003	7.6766e-003
3	10	6.6385e-005	6.6385e-005
3	100	6.5369e-007	6.5369e-007
4	1	4.6126e-004	4.6126e-004
4	10	4.1321e-007	4.1321e-007
4	100	4.0839e-010	4.0839e-010
5	1	4.6833e-005	4.6833e-005
5	10	4.1382e-009	4.1382e-009
5	100	4.0878e-013	4.0856e-013
6	1	1.9040e-006	1.9040e-006
6	10	1.7200e-011	1.7200e-011
6	100	2.4425e-015	1.1102e-016

Table 3: Numerical results for the first example with the iterative splitting method and fourth-order Gauß RK method.

The discretization of the linear ordinary differential equation is done with the BDF3 method. Our numerical results are presented in table 5.2. For the stiff problem we choose more iteration steps and time partitions and show the error between the analytical and numerical solution in the supremum norm.

In table 5.2 we need more iteration steps for the same results as in the non-stiff case, so we double the number of iteration steps to obtain the same results.

Remark 5.2 *For the stiff case we obtain improved results with more than 5 iteration steps. Because of the inexact starting function, the accuracy has to be improved by more iteration steps. At least higher-order time-discretization methods, as BDF3 method and the iterative operator-splitting method, accelerate the solving process.*

5.3 Third example: linear partial differential equation

We consider the one-dimensional convection-diffusion-reaction equation given

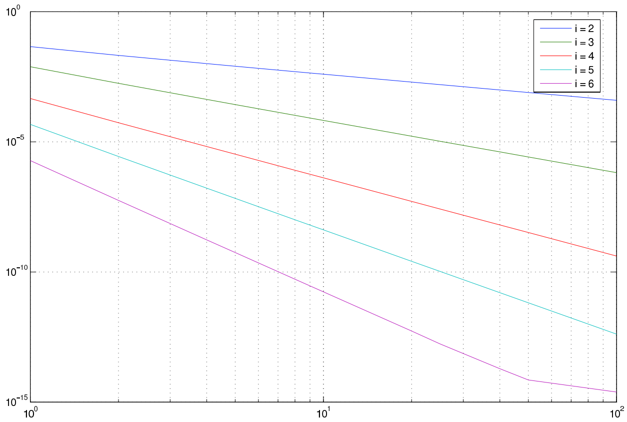

Figure 2: Convergence rates from 2 up to 6 iterations.

by

$$R\partial_t u + v\partial_x u - D\partial_{xx} u = -\lambda u \ , \ \text{on } \Omega \times [t_0, t_{\text{end}}), \tag{86}$$

$$u(x, t_0) = u_{\text{exact}}(x, t_0) \ , \tag{87}$$

$$u(0, t) = u_{\text{exact}}(0, t) \ , \ u(L, t) = u_{\text{exact}}(L, t). \tag{88}$$

We choose $x \in [0, 30]$ and $t \in [10^4, 2 \cdot 10^4]$.
Furthermore we have $\lambda = 10^{-5}$, $v = 0.001$, $D = 0.0001$ and $R = 1.0$. The analytical solution is given by

$$u_{\text{exact}}(x, t) = \frac{1}{2\sqrt{D\pi t}} \exp(-\frac{(x - vt)^2}{4Dt}) \exp(-\lambda t) \ . \tag{89}$$

To be out of the singular point of the exact solution, we start from the time point $t_0 = 10^4$.

Our spitted operators are

$$A = \frac{D}{R}\partial_{xx} u \ , \ B = -\frac{1}{R}(\lambda u + v\partial_x u) \ . \tag{90}$$

For the spatial discretization we use the Finite Differences with $\Delta x = \frac{1}{10}$. The discretization of the linear ordinary differential equation is done with the BDF3 method, so we deal with a third-order method. Our numerical results are presented in table 5. We choose different iteration steps and time partitions and show the error between the analytical and numerical solution in the supremum norm.

The figure 3 shows the initial solution at $t = 10^4$ and the analytical as well as the numerical solutions at $t = 2 \ 10^4$ of the convection-diffusion-reaction equation.

Iterative steps	Number of splitting partitions	err_1	err_2
5	1	3.4434e-001	3.4434e-001
5	10	3.0907e-004	3.0907e-004
10	1	2.2600e-006	2.2600e-006
10	10	1.5397e-011	1.5397e-011
15	1	9.3025e-005	9.3025e-005
15	10	5.3002e-013	5.4205e-013
20	1	1.2262e-010	1.2260e-010
20	10	2.2204e-014	2.2768e-018

Table 4: Numerical results for the stiff example with the iterative operator-splitting method and BDF3 method with temporal step size $h = 10^{-2}$.

Iterative steps	Number of splitting partitions	error $x = 18$	error $x = 20$	error $x = 22$
1	10	9.8993e-002	1.6331e-001	9.9054e-002
2	10	9.5011e-003	1.6800e-002	8.0857e-003
3	10	9.6209e-004	1.9782e-002	2.2922e-004
4	10	8.7208e-004	1.7100e-002	1.5168e-005

Table 5: Numerical results for the second example with the iterative operator-splitting method and BDF3 method with $h = 10^{-2}$.

As one result we can see, that we can reduce the error between the analytical and the numerical solution with using more iteration steps. If we restrict us to the error of 10^{-4} we obtain an effective computation with 3 iteration steps and time-partitions 10.

Remark 5.3 *For the partial differential equations we also need to take into account the spatial discretization. We applied a fine grid-step of the spatial discretization, so that the error of the time-discretization is dominant. We obtain an optimal efficiency of the iteration steps and the time partitions, if we use* 10 *iteration steps and* 2 *time partitions.*

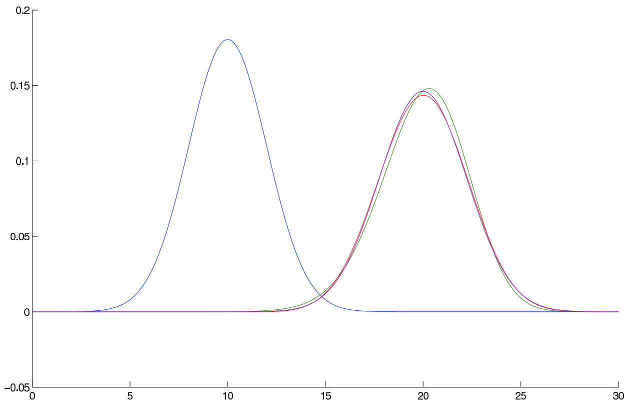

Figure 3: Initial and computed results for the second example with the iterative splitting method and BDF3 method.

5.4 Fourth example: nonlinear ordinary differential equation

As a nonlinear differential example we choose the Bernoulli equation, given as:

$$\begin{aligned} \frac{\partial u(t)}{\partial t} &= \lambda_1 u(t) + \lambda_2 u^n(t), \\ u(0) &= 1, \end{aligned}$$

with the solution

$$u(t) = \left[(1 + \frac{\lambda_2}{\lambda_1}) \exp(\lambda_1 t(1-n)) - \frac{\lambda_2}{\lambda_1}) \right]^{-\frac{1}{1-n}} .$$

We choose $n = 2$, $\lambda_1 = -1$, $\lambda_2 = -100$ and $h = 10^{-2}$.

We apply the iterative operator-splitting method with the nonlinear operators

$$A(u) = \lambda_1 u(t) \ , \ B(u) = \lambda_2 u^n(t) \ . \tag{91}$$

The discretization of the nonlinear ordinary differential equation is done with higher-order Runge-Kutta methods, precisely at least third-order methods. Our numerical results are presented in table 6. We choose different iteration steps and time partitions. The error between the analytical and numerical solution is shown with the supremum norm.

The experiments result in showing the reduced errors for more iteration steps and more time partitions. Because of the time-discretization for the ODE's, we restrict the number of iteration steps to a maximum of 5 iteration

Iterative steps	Number of splitting partitions	error
2	1	7.3724e-001
2	2	2.7910e-002
2	5	2.1306e-003
10	1	1.0578e-001
10	2	3.9777e-004
20	1	1.2081e-004
20	2	3.9782e-005

Table 6: Numerical results for the Bernoulli equation with the iterative operator-splitting method and BDF3 method.

steps. If we restrict the error bound to 10^{-4}, the most effective combination is given by 2 iteration steps and 10 time partitions.

Remark 5.4 *For the nonlinear ordinary differential equations we have the problem of the exact starting function. So the initialization process is delicate and we can decrease the splitting error at least by more iteration steps. Due to the linearization we gain at least linear convergence rates. This can be improved by a higher-order linearization, see [1, 18].*

Finally we finish with the conclusion to our paper.

6 Conclusion

In this paper we discuss the extension of iterative operator-splitting methods with respect to nonlinearity and stiffness. The analysis is based on the linearization of the nonlinear operators and on dividing into linear and linearized operators. To obtain stable methods we propose weighted operators for the algorithms. In numerical experiments the theoretical background is discussed in linear and nonlinear equations. The results reflect the application of the iterative splitting method with more iteration steps in combination of higher-order temporal and spatial discretization methods. In the future we concentrate on splitting nonlinear differential equations with nontrivial boundary conditions. We obtain equation parts that can be treated with fast solver methods based on implicit discretization methods.

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