

Polarities of shift planes

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Abstract. We construct polarities for arbitrary shift planes and develop criteria for conjugacy under the normalizer of the shift group. Under suitable assumptions (in particular, for finite or compact planes) we construct all shift groups on a given plane, and our constructions yield all conjugacy classes of polarities. We show that a translation plane admits an orthogonal polarity if, and only if, it is a shift plane. The corresponding planes are exactly those that can be coordinatized by commutative semifields. The orthogonal polarities form a single conjugacy class. Finally, we construct examples of compact connected shift planes with more conjugacy classes of polarities than the corresponding classical planes.

Introduction

Shift planes (see 1.1 for the definition) form important classes of projective planes, in particular, among finite planes [11], and also among compact connected planes, cf. 1.11. It is known that finite and locally compact connected shift planes contain lots of ovals (see [8], [33]). Every shift plane possesses polarities (see 3.1). In the present paper, we determine all polarities in the normalizer of the shift group (5.3), solve the problem of conjugacy (6.4), determine the centralizers (7.1) and obtain partial information about the corresponding sets of absolute points.

In many cases, the shift group is uniquely determined (and thus normal in the whole group of automorphisms of the plane). Planes admitting more than one shift group tend to be translation planes (of Lenz type V), see 10.1, 10.2 and 10.4. For such planes, one has complete information about all possible shift groups: the set of shift groups is parameterized by the middle nucleus of a coordinatizing commutative semifield, see 9.4.

We also prove that the standard polarities of shift planes of Lenz type V are orthogonal ones (9.11). Conversely, every orthogonal polarity of a translation plane gives rise to a shift group (8.8), and is a standard polarity for that shift group. Thus shift planes of Lenz type V and translation planes with orthogonal polarities are the same thing (9.12). As a corollary, we obtain that every translation plane admitting an orthogonal polarity can be

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coordinatized by a commutative semifield. If the left and middle nucleus coincide or if every element in the middle nucleus is a square, then any two shift groups (and any two orthogonal polarities) of such a plane are conjugate (9.13).

Finally, we exhibit examples of shift planes that have more conjugacy classes of polarities than the corresponding classical planes. This should be seen in the context that compact projective planes of (topological) dimension 8 or 16 with large automorphism groups possess at most three conjugacy classes of polarities, where the upper bound is only reached by the classical planes (over Hamilton's quaternions \mathbb{H} or Cayley's octonions \mathbb{O}), cf. [38] and [39].

1 Shift planes

1.1 Definition. A projective plane $\mathcal{P} = (P, \mathcal{L})$ is called a *shift plane* if there exists a flag (∞, L_∞) and a commutative group Δ of collineations fixing (∞, L_∞) and acting regularly both on $P \setminus L_\infty$ and on $\mathcal{L} \setminus L_\infty$.

The group Δ will be written additively. Adopting an affine point of view, we identify Δ with the affine point set $P \setminus L_\infty$, and each line in $\mathcal{L} \setminus \{L_\infty\}$ with its affine point row. We pick a representative $L(0) \subseteq \Delta$ for the orbit $\mathcal{L} \setminus L_\infty$ such that $0 \in L(0)$. For $x \in \Delta$ we write $L(x) := L(0) + x = \{s + x \mid s \in L(0)\}$.

1.2 Remark. Throughout the first two sections we do not need that $0 \in L(0)$. However, this assumption plays a role when we consider conjugacy of standard polarities and their absolute points.

1.3 Lemma. *The group $\Delta_{[\infty, L_\infty]}$ of all elations in Δ with axis L_∞ and center ∞ is linearly transitive. In particular, the subgroup $V := \Delta_{[\infty, L_\infty]}$ is an (affine) line through 0, with ∞ as its point at infinity.*

Proof. This is a known result for the finite case (see [9] Satz 3), we give a simple argument for the general case: Let $a, b \in \Delta$ be two affine points such that the joining line L contains ∞ . Then $b - a \in \Delta$ maps a to b and fixes L . Since Δ is commutative, the collineation $b - a$ fixes each element of the orbit $\mathcal{L}_\infty \setminus \{L_\infty\}$. Thus $b - a$ belongs to $\Delta_{[\infty, L_\infty]}$. \square

Our result on linear transitivity now allows to describe the parallelism:

1.4 Corollary. *Lines are parallel in the affine plane $(P \setminus L_\infty, \mathcal{L} \setminus \{L_\infty\})$ if, and only if, one of the following cases occurs:*

- (1) *We have lines $L(x)$ and $L(y)$, and the difference $x - y$ lies in $\Delta_{[\infty, L_\infty]}$.*
- (2) *Both lines belong to \mathcal{L}_∞ . Then they are of the form $V + x$ and $V + y$, with $x, y \in \Delta$.*

1.5 Lemma. *No line $L(x)$ contains a nontrivial subgroup. In particular, there are no involutions in the set $\Delta \setminus V$.*

Proof. Assume that $L(x)$ contains 0 , a and $-a \neq 0$. Then $0 \vee a = L(x) = -a \vee 0 = L(x - a)$. Now $-a$ fixes the line, contradicting our assumption that the action of Δ on $\mathcal{L} \setminus \mathcal{L}_\infty$ is regular.

Every element $a \in \Delta \setminus V$ describes a point lying on a line $L(x)$ through 0 . If a were an involution then $\{0, a\}$ would be a subgroup contained in $L(x)$. \square

We indicate several classes of shift planes in the sequel. In order to describe the set of lines and the action of Δ on it, we may pick a set $R \subset \Delta$ of representatives for the cosets in Δ/V ; our condition $0 \in L(0)$ means $0 \in R$. Each $s \in \Delta$ can then uniquely be written as $r_s + \tau_s$ with $(r_s, \tau_s) \in R \times V$. We thus identify Δ with $R \times V$. Then $L(0)$ is the graph of a function $f: R \rightarrow V$. For $(r_s, \tau_s) \in \Delta$ we obtain $L((r_s, \tau_s)) = L(0) + (r_s, \tau_s) = \{(r + r_s, f(r) + \tau_s) \mid r \in R\}$.

1.6 Example. The smallest example of a shift plane is obtained from the cyclic group $\Delta := \mathbb{Z}/4\mathbb{Z}$: we put $L(0) := \{4\mathbb{Z}, 4\mathbb{Z} + 1\}$. With $V = \{4\mathbb{Z}, 4\mathbb{Z} + 2\} = 2\Delta$ we obtain a model $(\Delta, \Delta/2\Delta \cup \{L(x) \mid x \in \Delta\})$ of the affine plane of order 2.

1.7 Example. Let F be a commutative semifield with $\text{char } F \neq 2$, and let s be a non-zero element of the middle nucleus of F . For $\Delta := F^2$, put $L_s^2(0) := \{(x, xsx) \mid x \in F\}$. With $V := \{0\} \times F$ we obtain an affine plane

$$\mathcal{A}_s(F) := (F^2, \{ \{c\} \times F \mid c \in F \} \cup \{ L_s^2(v) \mid v \in F^2 \}).$$

This affine plane is desarguesian if (and only if) the semifield F is a field; an isomorphism from $\mathcal{A}_s(F)$ onto the affine plane over F is then given by $(x, y) \mapsto (x, y - xsx)$.

In the case of a general commutative semifield F (with $\text{char } F \neq 2$), this map describes an isomorphism from $\mathcal{A}_s(F)$ onto the translation plane (of Lenz type V) over F .

In standard coordinates for the affine plane over F the shift group consists of all maps of the form $(x, y) \mapsto (x + a, y + b + 2asx)$ with $a, b \in F$. We will see in 9.7 below that this class of examples contains all shift groups on translation planes of characteristic different from 2.

1.8 Example. Let F be a commutative semifield (we also allow $\text{char } F = 2$ here), and pick $s \neq 0$ in the middle nucleus. Then $\gamma_{a,b,as}: F^2 \rightarrow F^2: (x, y) \mapsto (x + a, y + b + asx)$ is a collineation of the affine plane over F , and the group $\Delta_s := \{ \gamma_{a,b,as} \mid (a, b) \in F^2 \}$ acts as a shift group on the projective plane $\mathcal{P}_2 F$ over F . We shall see in 9.4 below that every shift group on a plane over a commutative semifield is a conjugate of Δ_s for some suitable s .

If $\text{char } F \neq 2$ then the mapping $\gamma_{a,b,as}$ to $(a, b - \frac{1}{2}bsb)$ is an isomorphism from Δ_s onto F^2 . We obtain the planes of 1.7.

If $\text{char } F = 2$ then $\gamma_{a,b,as}^2 = \gamma_{0,asa,0}$. Thus the group $2\Delta_s := \{ \delta^2 \mid \delta \in \Delta_s \}$ is contained in the group $\{ \delta \in \Delta_s \mid \delta^2 = \text{id} \} = \{ \gamma_{0,b,0} \mid b \in F \} = V$, and Δ_s is a module over the ring $\mathbb{Z}/4\mathbb{Z}$ (we shall see in 5.8 below that this is a very general phenomenon in characteristic 2). The groups $2\Delta_s$ and V coincide if, and only if, every element of F is of the form asa with $a \in F$ (for a field F this means that the field is perfect). The shift group Δ_s is a free module over $\mathbb{Z}/4\mathbb{Z}$ precisely if $2\Delta_s = V$, cf. 5.8.

1.9 Examples. For $\Delta := \mathbb{R}^2$, we put $L^4(0) := \{(x, x^4) \mid x \in \mathbb{R}\}$ and obtain an affine plane $Q(\mathbb{R}) := (\mathbb{R}^2, \{\{c\} \times \mathbb{R} \mid c \in \mathbb{R}\} \cup \{L^4(v) \mid v \in \mathbb{R}^2\})$. This affine plane is not desarguesian; it is not even a translation plane (cf. [35] 36.3).

1.10 Examples. Finite shift planes are described in [6]; in fact, that paper contains all presently known finite shift planes that are not translation planes. The full automorphism group and the polarities of these planes are determined in [26]. We discuss shift planes that are also translation planes in Section 9 below, see 9.12.

1.11 Examples. Many examples of shift planes may be found in [23], [25], [3], [30], [35] Sections 36 and 74. In these examples, the set R of representatives is a subgroup, and $L(r_s, \tau_s) = \{(r, f(r - r_s) + \tau_s) \mid r \in R\}$. The function $f: R \rightarrow V$ is a so-called *planar function*, i.e., for each $d \in R \setminus \{0\}$ the map $f_d: R \rightarrow V: x \mapsto f(x + d) - f(x)$ is a bijection. For instance, a continuously differentiable function $f: \mathbb{R} \rightarrow \mathbb{R}$ is planar if and only if $f': \mathbb{R} \rightarrow \mathbb{R}$ is bijective ([34], cf. [35] 31.25, see [14] for an even more general result). The resulting plane is desarguesian exactly if f is a quadratic polynomial (as in $\mathcal{A}_s(\mathbb{R})$, cf. 1.7), see [35] 74.12.

1.12 Remark. In order to include examples in characteristic 2, an approach different from 1.11 is needed. One may also use the affine line $L(0)$ for R . Then $f \equiv 0$, and R is not a subgroup (cf. 1.5). The geometry is encoded in a cocycle for the group extension $0 \rightarrow V \rightarrow \Delta \rightarrow \Delta/V \rightarrow 0$, see [16].

1.13 Remark. Every affine line $L(s)$ is a difference set of the group Δ relative to the subgroup V , i.e. $L(s)$ intersects each coset of V exactly once and every element of $\Delta \setminus V$ can be uniquely represented as a difference $x - y$ with $x, y \in L(s)$. The converse is also true. The study of shift planes via relative difference sets seems to be particularly useful in the finite case, see e.g. [18], [32], [8], or [4].

2 The normalizer of the shift group

2.1 Definition. In the group Γ of all collineations of \mathcal{P} , let Σ denote the normalizer of Δ .

As Δ fixes no flag except (∞, L_∞) and acts regularly on affine points, we have:

- 2.2 Lemma.** (1) *The group Σ fixes ∞ and L_∞ .*
 (2) *The centralizer of Δ in Γ is Δ .*
 (3) *The stabilizer Σ_0 acts by automorphisms on $(\Delta, +)$.*
 (4) *Every element $\sigma \in \Sigma$ is of the form $x \mapsto Ax + t$, where $A \in \Sigma_0$ and $t \in \Delta$. The pair $(A, t) \in \Sigma_0 \times \Delta$ is determined uniquely by σ ; we write $(A_\sigma, t_\sigma) := (A, t)$ and $\sigma_{A,t} := \sigma$.*

2.3 Lemma. For $\sigma_{A,t} \in \Sigma$ we define $c_A \in \Delta$ by $A(L(0)) = L(c_A)$. Then the line map corresponding to $\sigma_{A,t}$ is given by $L(s) \mapsto L(c_A + As + t)$. In particular, c_A does not depend on t .

For $A, B \in \Sigma_0$, we have $c_{AB} = c_A + Ac_B$. This implies $c_{B^{-1}} = -B^{-1}c_B$ and $c_{BAB^{-1}} = Bc_A + c_B - BAB^{-1}c_B$.

Proof. We compute $\sigma_{A,t}(L(s))$ to obtain the first assertion, the second one then follows from $AB(L(0)) = A(L(c_B)) = L(c_A + Ac_B)$:

$$\begin{aligned} \sigma_{A,t}(L(s)) &= \{\sigma_{A,t}(x+s) \mid x \in L(0)\} = \{A(x+s) + t \mid x \in L(0)\} \\ &= \{Ax + As + t \mid x \in L(0)\} = \{Ax \mid x \in L(0)\} + As + t \\ &= A(L(0)) + As + t = L(c_A) + As + t = L(c_A + As + t). \quad \square \end{aligned}$$

2.4 Examples. In 1.6, let $Ax := -x$. Then A belongs to Σ_0 , in fact, we have $c_A = 4\mathbb{Z} - 1$, and A describes a shear in $\Sigma_{[0,V]}$. Moreover, the group $\Sigma_0 = \text{Aut}(\mathbb{Z}/4\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ is generated by A .

In 1.7, the map A defined by $A(x, y) := (-x, y)$ fixes $L_s^2(0)$, and belongs to Σ_0 (with $c_A = 0$). This map describes a reflection with axis V .

Analogously, we obtain a reflection in 1.9.

3 Standard polarities

3.1 Lemma. A polarity J of \mathcal{P} is given via extension of $J(x) := L(-x)$.

Proof. We have prescribed the action of J on affine points. Since J shall be an involution, we have $J(L(x)) = -x$. The only remaining affine lines are those parallel to the “vertical” $V := \Delta_{[\infty, L_\infty]}$; we map $V + x$ to the parallel class $[L(-x)]_{\parallel}$ of $L(-x)$.

Using the equivalences $x \in L(y) \iff x - y \in L(0) \iff -y \in L(-x)$ and $x \in V + y \iff x - y \in V \iff L(0) \parallel L(0) - y + x \iff L(-x) \parallel L(-y)$ we see that this defines a collineation of affine planes from $(P \setminus L_\infty, \mathcal{L} \setminus \{L_\infty\})$ onto $(\mathcal{L} \setminus L_\infty, P \setminus \{\infty\})$. This collineation has a unique extension to a collineation from the projective closure \mathcal{P} onto its dual.

For the sake of completeness, we note that $J([L(x)]_{\parallel}) = V - x$ and $J(\infty) = L_\infty$. Obviously, the square of J is the identity. \square

3.2 Definition. We call J a *standard polarity* of \mathcal{P} . Note that J depends on the choice of $L(0) \in \mathcal{L}_0$. In order to obtain a notion that is independent of this choice, we call a polarity ψ of \mathcal{P} a *standard polarity with respect to Δ* if conjugation by ψ induces inversion on Δ and there is an affine *absolute point* (i.e., an affine point lying on its image under the polarity).

We discuss conjugacy of (standard) polarities in 6.5 and 6.7. In particular, it turns out that the standard polarities form a single conjugacy class if Δ is 2-divisible.

The existence of affine absolute points is a problem only if Δ is not 2-divisible (see 3.4). The following assertions give a partial understanding of the set of absolute

points. More information can be obtained under additional hypotheses, see Section 9 below.

3.3 Lemma. *The point ∞ is the only absolute point on the line L_∞ . An affine point $x \in \Delta$ is absolute if, and only if, it satisfies $2x \in L(0)$.*

Proof. The polar $L_\infty = J(\infty)$ of the absolute point ∞ does not contain any more absolute points. An affine point $x \in \Delta$ is absolute if $x \in J(x) = L(-x) = L(0) - x$. This means $2x = x + x \in L(0)$. \square

If $2V = \{0\}$ then the set of absolute points contains the line $V \cup \{\infty\}$ (and then coincides with it). For instance, this happens if there exists a shear of order 2 (see 5.7). In particular, this occurs in planes over commutative semifields of characteristic 2, cf. 1.8.

3.4 Lemma. *Every standard polarity belongs to $J \circ \Delta$. Conversely, a polarity $J \circ x$ with $x \in \Delta$ is a standard polarity precisely if $x \in 2\Delta - L(0)$.*

Proof. Since Δ is its own centralizer in Γ , every polarity that induces inversion on Δ belongs to the coset $J \circ \Delta$.

Let $J \circ x$ be a standard polarity, and choose an affine absolute point a of $J \circ x$. This means $a \in L(-a-x)$, which is equivalent to $-2a-x \in L(0)$ and thus to $x \in 2\Delta - L(0)$. Conversely, for $b \in \Delta$ and $y \in L(0)$ the polarity $J \circ (2b-y)$ has the absolute point b . \square

3.5 Remark. Using a non-perfect commutative field of characteristic 2 one may construct a shift group Δ where $2\Delta - L(0) \neq \Delta$, cf. 1.8. If we choose $x \in \Delta \setminus (2\Delta - L(0))$ we obtain a polarity $J \circ x$ with exactly one absolute point: this is not a standard polarity!

The situation looks quite different if we stay away from characteristic 2 phenomena:

3.6 Lemma. *If V is uniquely 2-divisible, then each line $L \in \mathcal{L}_\infty \setminus \{L_\infty\}$ contains exactly two absolute points.*

Proof. The affine points on L form a set $V + x$. Since $V + 2x$ and $L(0)$ are not parallel, there is a unique element $v \in V$ such that $v + 2x \in L(0)$, and the unique element $w \in V$ with $2w = v$ yields the unique affine absolute point $w + x$ on L . \square

4 Other polarities

4.1 Theorem. *For $\sigma \in \Sigma$, the composition $J \circ \sigma$ is a polarity if, and only if, we have $A_\sigma^2 = \text{id}$ and $(A_\sigma - \text{id})t_\sigma = c_{A_\sigma}$.*

Proof. Evaluating the condition $x = (J \circ \sigma)^2(x) = (J \circ \sigma)(L(-A_\sigma x - t_\sigma)) = J(L(c_{A_\sigma} - A_\sigma^2 x - A_\sigma t_\sigma + t_\sigma)) = -c_{A_\sigma} + A_\sigma^2 x + A_\sigma t_\sigma - t_\sigma$ at $x = 0$ we obtain $(A_\sigma - \text{id})t_\sigma = c_{A_\sigma}$. Then the general condition becomes $A_\sigma^2 x = x$. \square

4.2 Definition. We write $\Pi := \{\alpha \in \Sigma \mid (J \circ \alpha)^2 = \text{id}\}$. Note that Δ is contained in Π .

4.3 Lemma. *If $A \in \Sigma_0$ satisfies $A^2 = \text{id}$ then $Ac_A = -c_A$.*

Proof. Evaluate $L(0) = A^2(L(0)) = A(L(c_A)) = L(c_A + Ac_A)$. \square

4.4 Theorem. *If Δ is uniquely 2-divisible then for each involution $A \in \Sigma_0$ there exists at least one $t \in \Delta$ such that $\sigma_{A,t}$ belongs to Π .*

Proof. We have to solve the equation $(A - \text{id})t = c_A$. Take $t \in \Delta$ such that $2t = -c_A$, then $2(A - \text{id})t = -Ac_A + c_A = 2c_A$, and t is a solution, as required. \square

5 Involutions in Σ

As 4.4 requires that Δ is uniquely 2-divisible (thus excluding, in particular, the case of finite shift planes of even order), we supplement 4.4 with more geometrical considerations, using fixed elements of involutions in Σ .

5.1 Lemma. *Let $A \in \Sigma_0$ with $A^2 = \text{id}$, and let $t \in \Delta$. Then the map $\sigma_{A,t}$ belongs to Π if, and only if, the map A fixes the line $L(-t)$.*

Proof. We know from 4.1 that the map $x \mapsto Ax+t$ belongs to Π exactly if $(A-\text{id})t = c_A$. This means that the image $L(c_A + A(-t))$ of $L(-t)$ under A is $L(-t)$, again. \square

5.2 Lemma. *No collineation in Σ has an axis in $\mathcal{L} \setminus \mathcal{L}_\infty$, or a center in $P \setminus L_\infty$.*

Proof. If $\sigma \in \Sigma$ has axis $L(x) \in \mathcal{L} \setminus \mathcal{L}_\infty$ then $L(x)$ is the set of affine fixed points of σ . Since σ induces an automorphism of the group Δ , we reach a contradiction to 1.5. The dual assertion follows by an application of a standard polarity. \square

5.3 Theorem. *If $A \in \Sigma_0$ is an involution, then one of the following occurs:*

- (1) *A is a Baer involution. In this case, the involution fixes some lines in $\mathcal{L} \setminus \mathcal{L}_\infty$, and there exist elements $t \in \Delta$ such that $\sigma_{A,t}$ belongs to Π , see 5.1.*
- (2) *$A \in \Sigma_{[p,V]}$ for some $p \in L_\infty$. We have two subcases:*
 - (a) *If $p \neq \infty$, let $L(z)$ be the line joining 0 and p . Then $\sigma_{A,t}$ belongs to Π exactly if $t \in V - z$.*
 - (b) *If $p = \infty$ then there is no $t \in \Delta$ with $\sigma_{A,t} \in \Pi$.*

Proof. If A is not a Baer involution then A has an axis W and a center p , and every fixed line $L \neq W$ passes through p . From 5.2 we know $p \in L_\infty$ and $W \in \mathcal{L}_\infty$. This yields $W = o \vee \infty = V$ because A fixes o .

In case 2a, we have $c_A = z - Az$. Then 4.1 yields $\sigma_{A,t} \in \Pi \iff t \in V - z$. In case 2b, the involution A fixes none of the lines $L(-t)$, and 5.1 applies. \square

5.4 Example. Let E/F be a separable quadratic field extension, and let $s \mapsto \bar{s}$ denote the generator of the Galois group $\text{Gal}(E/F)$. Then $(u, v) \mapsto (\bar{u}, \bar{v})$ yields a Baer involution of $\mathcal{A}_s(E)$, cf. 1.7.

5.5 Example. Let F be a commutative semifield with $\text{char } F \neq 2$, and consider the shift plane $\mathcal{A}_s(F)$. The map $R: (u, v) \mapsto (-u, v)$ yields an involution $\sigma_{R,0} \in \Sigma_{[p,V]}$ with $p \in L_\infty \setminus \{\infty\}$. Actually, this involution fixes $L(x)$ exactly if $x \in V$; the center p is the point at infinity of $L(0)$. We have $\sigma_{R,t} \in \Pi \iff t \in V$.

5.6 Example. For the shift plane with shift group $\mathbb{Z}/4\mathbb{Z}$ defined in 1.6, the map $x \mapsto 3x$ defines an involution in $\Sigma_{[\infty,V]}$.

We remark that the existence of involutions with incident center and axis is a ‘‘characteristic 2 phenomenon’’, occurring in finite planes only in the case of even order. We add a result that corroborates this impression also in the infinite case:

5.7 Proposition. *If $A \in \Sigma_{[\infty,V]}$ is an involution, then there is a surjective group homomorphism $\zeta: \Delta \rightarrow V$ such that $A = \zeta + \text{id}$. Moreover, we have $2V = \{0\}$ and $-c_A \in L(0)$.*

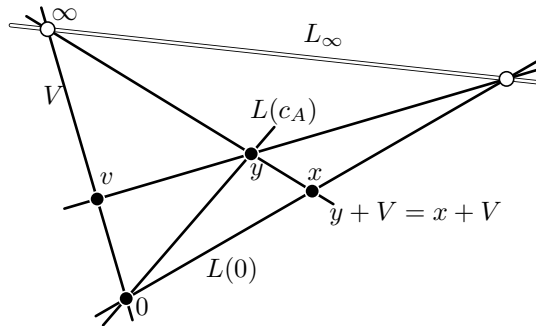


Figure 1. Surjectivity of ζ .

Proof. Our assumption yields $Ax + V = x + V$ for each $x \in \Delta$. Thus $\zeta(x) := Ax - x$ lies in V . Since Δ is a commutative group, the map $\zeta = A - \text{id}$ is a homomorphism.

In order to show that ζ is surjective, we construct (see Fig. 1) for $v \in V \setminus \{0\}$ the intersection points $y := L(v) \wedge L(c_A)$ and $x := (y + V) \wedge L(0)$. Then $Ax = (x + V) \wedge A(L(0)) = (y + V) \wedge L(c_A) = y$ yields $\zeta(x) = Ax - x = y - x$. The translation $v \in V = \Delta_{[\infty,L_\infty]}$ maps x to $(x + V) \wedge (L(0) + v) = (y + V) \wedge L(v) = y$, and we have $v = y - x = \zeta(x)$.

Since A fixes 0 , the line $A(L(0)) = L(c_A) = L(0) + c_A$ contains the point 0 , and $-c_A \in L(0)$ follows. It remains to show $2V = \{0\}$: as $V = \zeta(\Delta)$, this follows from $x = A^2x = A(x + \zeta(x)) = Ax + A(\zeta(x)) = x + \zeta(x) + \zeta(x) = x + 2\zeta(x)$. \square

Recall from 1.5 that the set $\Delta \setminus V$ does not contain any involutions. We can say more:

5.8 Theorem. *If Σ contains an involutory shear with axis V , then Δ is a module over the ring $\mathbb{Z}/4\mathbb{Z}$, and $V = \{\delta \in \Delta \mid 2\delta = 0\}$. The module is free over $\mathbb{Z}/4\mathbb{Z}$ if, and only if, we have $V = 2\Delta$.*

Proof. Since our plane admits nontrivial elations with different axes through ∞ , all elations have the same order (cf. [17] Theorem 4.14). Thus $2V = \{0\}$. From 5.7 we know that there is a surjective group homomorphism $\zeta: \Delta \rightarrow V$ such that the shear is described by $\zeta + \text{id}$. We infer $\zeta^2 = -2\zeta$ because the shear is an involution. For $v \in V$, we pick $a \in \Delta$ such that $v = \zeta(a)$, and infer $\zeta(v) = \zeta^2(a) = -2\zeta(a) = 0$. Conversely, every element in the kernel of ζ is fixed by the shear, and belongs to its axis V . Thus $\ker \zeta = V \subseteq \{\delta \in \Delta \mid 2\delta = 0\} \subseteq V$. For each $a \in \Delta$ we compute $\zeta(2a) = 2\zeta(a) = 0$. This means $2\Delta \subseteq \ker \zeta = V$, and $4\Delta = \{0\}$ follows.

We have proved that the commutative group Δ is a module over $\mathbb{Z}/4\mathbb{Z}$. In the case $V = 2\Delta$, pick any subset B of Δ such that $2B$ forms a basis for the vector space V over $\mathbb{Z}/2\mathbb{Z}$. Then B is a basis for the module Δ over $\mathbb{Z}/4\mathbb{Z}$, and that module is free. \square

5.9 Remarks. In the infinite case the module Δ is not always free, cf. 1.8. If Δ is finite then V and Δ/V have the same size (namely, the order of the plane), and Δ is a free module. Thus 5.8 generalizes a result due to Ganley [13], cf. [18].

5.10 Example. In order to construct a shift plane with shift group $\Delta = (\mathbb{Z}/4\mathbb{Z})^2$, we have to take $V := 2\Delta$, and to choose a set $L(0) = \{0, a, b, c\}$ of representatives for the cosets modulo V . It is easy to see that a, b form a basis of Δ over $\mathbb{Z}/4\mathbb{Z}$, and that $c \in V + a + b$. Because the group $\text{Aut}(\Delta)$ acts transitively on the set of bases, it suffices to consider the cases where $a = (1, 0)$, $b = (0, 1)$, and $c = (1 + x, 1 + y)$ with $x, y \in V$.

For $(x, y) \in \{(0, 0), (2, 0), (0, 2)\}$ we have $c \in \{a + b, -a + b, a - b\}$, and the candidate $L(0)$ intersects one of the translates $L(a), L(b)$ in more than one point. There remains the possibility $L(0) = \{0, a, b, -a - b\}$. Since there exists a shift plane with $|\Delta| = 2^4$ (namely, the plane over the field with 4 elements, see 1.8) and since the shift group in that case is $(\mathbb{Z}/4\mathbb{Z})^2$ by 5.8, the last remaining choice indeed yields a shift plane.

5.11 Proposition. *If $A \in \Sigma_0$ is a Baer involution, then the subplane consisting of the fixed elements of A is also a shift plane: the group $\Phi := \{x \in \Delta \mid Ax = x\}$ centralizes the Baer involution and induces a shift group for that subplane.*

6 Conjugacy of polarities

6.1 Lemma. *For $\beta \in \Sigma$ we have $A_{J \circ \beta \circ J} = A_\beta$ and $t_{J \circ \beta \circ J} = -t_\beta - c_{A_\beta}$.*

Proof. We compute $(J \circ \beta \circ J)(x) = (J \circ \beta)(L(-x)) = J(L(c_{A_\beta} - A_\beta x + t_\beta)) = -c_{A_\beta} + A_\beta x - t_\beta$. \square

6.2 Lemma. *Let $\alpha, \gamma \in \Pi$.*

- (1) *The polarities $J \circ \alpha$ and $J \circ \gamma$ are conjugate under an element of Σ if, and only if, there exist $B \in \Sigma_0$ and $u \in \Delta$ such that $A_\gamma = BA_\alpha B^{-1}$ and $(A_\gamma + \text{id})u = Bt_\alpha - c_B - t_\gamma$.*
- (2) *If $B \in \Sigma_0$ satisfies $A_\gamma = BA_\alpha B^{-1}$ then $Bt_\alpha - c_B - t_\gamma$ is fixed by A_γ .*

Proof. Let $\beta: x \mapsto Bx + u$ be an element of Σ . Calculating $\beta \circ (J \circ \alpha) \circ \beta^{-1} = J \circ (J \circ \beta \circ J \circ \alpha \circ \beta^{-1})$ we infer that $J \circ \gamma = \beta \circ (J \circ \alpha) \circ \beta^{-1}$ holds exactly if $\gamma = J \circ \beta \circ J \circ \alpha \circ \beta^{-1}$. Using 6.1 we see that the latter condition means $A_\gamma = BA_\alpha B^{-1}$ and $(A_\gamma + \text{id})u = Bt_\alpha - c_B - t_\gamma$.

It remains to show that $x := Bt_\alpha - c_B - t_\gamma$ is fixed by $A_\gamma = BA_\alpha B^{-1}$. Using 2.3, we compute

$$\begin{aligned} BA_\alpha B^{-1}x &= BA_\alpha B^{-1}(Bt_\alpha - c_B - t_\gamma) \\ &= BA_\alpha t_\alpha - BA_\alpha B^{-1}c_B - BA_\alpha B^{-1}t_\gamma \\ &= BA_\alpha t_\alpha + c_{BA_\alpha B^{-1}} - Bc_{A_\alpha} - c_B - BA_\alpha B^{-1}t_\gamma \\ &= B(A_\alpha t_\alpha - c_{A_\alpha}) - c_B + c_{BA_\alpha B^{-1}} - BA_\alpha B^{-1}t_\gamma. \end{aligned}$$

Applying 4.1 to $\alpha, \gamma \in \Pi$ we find $A_\alpha t_\alpha - c_{A_\alpha} = t_\alpha$ and $c_{BA_\alpha B^{-1}} - BA_\alpha B^{-1}t_\gamma = c_{A_\gamma} - A_\gamma t_\gamma = -t_\gamma$. Thus $A_\gamma x = x$ is proved. \square

6.3 Theorem. *For $\alpha, \beta \in \Pi$, the polarities $J \circ \alpha$ and $J \circ \beta$ are conjugate under the group $\langle J \rangle \circ \Sigma$ if, and only if, they are conjugate under Σ .*

Proof. From 6.1 we know $A_{J \circ \alpha \circ J} = A_\alpha$. Putting $B := A_\alpha$ and $u := t_\alpha$ in 6.2.1 we find that $J \circ \alpha$ and $J \circ (J \circ \alpha) \circ J$ are conjugate under Σ . Now the assertion follows easily. \square

6.4 Theorem. *Assume that the group Δ is uniquely 2-divisible. Then $\alpha, \gamma \in \Pi$ yield polarities $J \circ \alpha$ and $J \circ \gamma$ that are conjugate under Σ if, and only if, the elements A_α and A_γ are conjugate in Σ_0 . In particular, all the elements in $J \circ \Delta$ are conjugate.*

Proof. If $J \circ \alpha$ and $J \circ \gamma$ are conjugate under Σ then 6.2.1 yields that A_α and A_γ are conjugate in Σ_0 .

Now assume, conversely, that there is $B \in \Sigma_0$ such that $A_\gamma = BA_\alpha B^{-1}$. In order to show conjugacy of the polarities, we have to find a solution $u \in \Delta$ for the equation $(A_\gamma + \text{id})u = Bt_\alpha - c_B - t_\gamma$. By our divisibility assumption, there exists $u \in \Delta$ with $u + u = Bt_\alpha - c_B - t_\gamma$. According to 6.2.2, the right hand side of this equation is fixed by A_γ . Now u is fixed by A_γ because of its uniqueness, and u is a solution for our equation. \square

6.5 Theorem. *Let $x, y \in \Delta$. Then the polarities $J \circ x$ and $J \circ y$ are conjugate under Σ if, and only if, we have $y \in 2\Delta - \{c \in \Delta \mid L(c) \in \Sigma_0(L(-x))\}$.*

Proof. According to 6.2, conjugacy of $J \circ x$ and $J \circ y$ is equivalent to the existence of $B \in \Sigma_0$ and $u \in \Delta$ such that $2u = Bx - c_B - y$. The observation $B(L(-x)) = L(-Bx + c_b)$ yields the claim. \square

Of course, our result 6.5 is interesting only if Δ is not 2-divisible.

6.6 Examples. For the shift plane of order 2 introduced in 1.6, the shift group Δ is not 2-divisible, but nonetheless $J \circ \Delta$ forms a single conjugacy class: this is due to the facts that $2\Delta = V$ and that Σ_0 acts transitively on $\mathcal{L}_0 \setminus \{V\}$ in this case. Analogously, the set $J \circ \Delta$ forms a single conjugacy class in the plane of order 4 characterized in 5.10.

The crucial point (apart from $2\Delta = V$) in these two small examples is that Σ_0 contains a group of shears that acts transitively on $\mathcal{L}_0 \setminus \{V\}$. We formulate a general result (which applies to all known finite shift planes except those described in [6]):

6.7 Theorem. *Assume that \mathcal{P} is a translation plane and also a shift plane, and $V \subseteq 2\Delta$. Then $J \circ \Delta$ forms a single conjugacy class in $\langle J \rangle \circ \Sigma$.*

7 Centralizers of polarities

7.1 Theorem. *Let $\sigma_{A,t} \in \Pi$. An element $\sigma_{B,u} \in \Sigma$ centralizes the polarity $J \circ \sigma_{A,t}$ if, and only if, the following two conditions are satisfied:*

- (1) $AB = BA$,
- (2) $(A + \text{id})u = (B - \text{id})t - c_B$.

If Δ is 2-divisible there exist solutions u for condition 2 for each B in the centralizer of A with $(B - \text{id})c_A = (A - \text{id})c_B$.

In the general case, this is a necessary (but usually not sufficient) condition.

Proof. The centralizer condition $(J \circ \sigma_{A,t}) \circ \sigma_{B,u} = \sigma_{B,u} \circ (J \circ \sigma_{A,t})$ is equivalent to $\sigma_{A,t} \circ \sigma_{B,u} = J \circ \sigma_{B,u} \circ J \circ \sigma_{A,t}$. Using 6.1, we translate this into $\sigma_{A,t} \circ \sigma_{B,u} = \sigma_{B,-u-c_B} \circ \sigma_{A,t}$. Evaluating the products at $x = 0$, we find condition 2. Then the general case yields $AB = BA$.

If Δ is 2-divisible then the image of $A + \text{id}$ coincides with the kernel of $A - \text{id}$, that is, with the set of fixed points of A . Using $AB = BA$ and the condition $(A - \text{id})t = c_A$ that we know from 4.1, we translate the fixed point condition $A((B - \text{id})t - c_B) = (B - \text{id})t - c_B$ into $(B - \text{id})c_A = (A - \text{id})c_B$. In the general case, we only know that $(B - \text{id})t - c_B$ is in the image $(A + \text{id})\Delta \subseteq \ker(A - \text{id})$ and thus fixed by A . \square

As a corollary, we obtain:

7.2 Theorem. *The centralizer Ψ_J of the standard polarity J in Σ consists of all elements $\sigma_{B,u}$ that satisfy $2u = -c_B$.*

If Δ is uniquely 2-divisible then $\Psi_J = \{\sigma_{B,u} \mid B \in \Sigma_0, u = -\frac{1}{2}c_B\} \cong \Sigma_0$.

7.3 Example. For the shift plane of order 2 (see 1.6), we have $\Sigma_0 = \text{Aut}(\mathbb{Z}/4\mathbb{Z}) = \{\text{id}, -\text{id}\}$ and $c_{-\text{id}} = 3 + 4\mathbb{Z}$. The equation $2u = -3 + 4\mathbb{Z}$ has no solution in $\mathbb{Z}/4\mathbb{Z}$, and the centralizer of J in Σ_0 is trivial. Note also that $\Pi = \Delta$ holds in this case, and that $J \circ \Pi$ forms a single conjugacy class.

We will give an alternative description of Ψ_J in 9.9 for the case where the shift plane is also a translation plane.

8 Orthogonal polarities of translation planes

In this section, our aim is to show that translation planes admitting orthogonal polarities (in the sense of 8.3 below) are shift planes (and thus of Lenz type V). We consider a more general situation first.

8.1 Lemma. *Let ω be a polarity of some projective plane, and assume that there are (at least) two absolute points ∞ and a . Let Z be the group of all collineations with center ∞ and axis $\omega(\infty)$. Then the set of absolute points in the orbit $Z(a)$ is the orbit of a under the centralizer $C_Z(\omega) := \{\zeta \in Z \mid \omega \circ \zeta \circ \omega = \zeta\}$.*

Proof. For $\zeta \in Z$ we note that $\zeta^\omega := \omega \circ \zeta \circ \omega$ belongs to Z again. Let b denote the intersection point $\omega(a) \wedge \omega(\infty)$, then $\omega(b) = a \vee \infty$. We obtain $\zeta^\omega(a) = \omega(\zeta(\omega(a))) = \omega(\zeta(a \vee b)) = \omega(\zeta(a) \vee b) = \omega(\zeta(a)) \wedge \omega(b) = \omega(\zeta(a)) \wedge (a \vee \infty)$. This shows that $\zeta^\omega(a) = \zeta(a)$ holds exactly if $\zeta(a)$ is an absolute point, and our assertion follows from the fact that $\zeta \in Z$ is determined by $\zeta(a)$. \square

8.2 Lemma. *Let ω be a polarity of some projective plane, and assume that there is an absolute point ∞ such that the group Z of collineations with center ∞ and axis $\omega(\infty)$ is linearly transitive.*

- (1) *If the set of absolute points is an oval and Z is commutative then $\omega \circ \zeta \circ \omega = \zeta^{-1}$ holds for each $\zeta \in Z$. In this case, the group Z does not contain any involution.*
- (2) *If the set of absolute points is a point row, then $\omega \circ \zeta \circ \omega = \zeta$ holds for each $\zeta \in Z$.*

Proof. In each of the cases, our assumptions entail the existence of at least 3 absolute points. Since the polar of an absolute point does not contain more than one absolute point, there exists an absolute point $a \notin \omega(\infty)$. From 8.1 we know that the set of absolute points on the line $a \vee \infty$ consists of ∞ and the orbit of a under $C_Z(\omega)$.

Let $\tilde{\omega}$ denote the automorphism of Z that is induced by conjugation with ω . If the set of all absolute points is an oval then $\tilde{\omega}$ fixes no element of $Z \setminus \{\text{id}\}$. Thus $\tilde{\omega}$ maps each element to its inverse (because $\tilde{\omega}$ fixes $\zeta^\omega \circ \zeta = \zeta \circ \zeta^\omega$), and there are no involutions in Z because these would be fixed by $\tilde{\omega}$.

If the set of absolute points of ω is the point row of a line, this line is $a \vee \infty$. In this case, the automorphism $\tilde{\omega}$ is the identity. \square

8.3 Definition. Following [2], a polarity ω of a translation plane is called an *orthogonal polarity* if one of the following holds:

- (1) The characteristic of the translation group is different from 2, and the set of absolute points forms an oval.
- (2) The characteristic of the translation group is 2, and the set of absolute points forms the point row of a line.

The definition is motivated by the examples of polarities of pappian planes that are given by non-degenerate symmetric bilinear forms: there the absolute points form a conic which is an oval or empty, except in characteristic 2 where one indeed obtains a point row or a single point.

For a finite plane of order n , the orthogonal polarities are just those with at most (and then exactly) $n + 1$ absolute points, cf. [2] or [17] Theorems 12.5, 12.6.

Now let $\mathcal{P} = (P, \mathcal{L})$ be a projective translation plane with translation group T . We adopt an affine point of view, using the translation axis L_∞ as the line at infinity.

We assume that \mathcal{P} admits a polarity ω and write $\tau^\omega := \omega \circ \tau \circ \omega$ for $\tau \in T$. If \mathcal{P} is not a Moufang plane (where coordinate methods are available to determine the polarities) then \mathcal{P} is a plane of Lenz type V, and there is a unique point ∞ on L_∞ such that \mathcal{P} allows all shears with (affine) axes through ∞ . Then ∞ is the unique translation axis for the dual plane, and it follows that ω interchanges ∞ with L_∞ . In other words, the flag (∞, L_∞) is absolute. We will assume that there is at least one more absolute point 0; necessarily, this is an affine point. The existence of such a point is clear if we assume that the polarity is an orthogonal one. For arbitrary polarities, the existence of a second absolute point is secured in the finite or compact connected case, cf. [2] and [39] 1.1. See 3.5 for an example of a polarity with exactly one absolute point.

8.4 Notation. We pick an affine absolute point 0 and write $V := 0 \vee \infty$; then $\omega(V) = \omega(0) \wedge L_\infty$. Let Z denote the group of all collineations with center ∞ and axis L_∞ , and let Λ be the group of all collineations with center $\omega(V)$ and axis L_∞ . We define $\Delta_\omega := \{ \zeta \circ \lambda \circ \omega \circ \lambda^{-1} \circ \omega \mid \zeta \in Z, \lambda \in \Lambda \}$.

8.5 Remark. The conjugate $\Lambda^* := \omega \circ \Lambda \circ \omega$ consists of all collineations with axis $\omega(\omega(V)) = V$ and center $\omega(L_\infty) = \infty$, and each of the groups Z, Λ, Λ^* is linearly transitive; in fact, the groups $Z\Lambda$ and $Z\Lambda^*$ are the groups T and T^* , respectively.

Our aim is to show that Δ_ω is a shift group if ω is an orthogonal polarity.

8.6 Lemma. *The set Δ_ω is a subgroup of $\Lambda Z \Lambda^*$ acting regularly both on the set $P \setminus L_\infty$ of affine points and on the set $\mathcal{L} \setminus \mathcal{L}_\infty$ of non-vertical lines.*

Proof. We show $\Delta_\omega \Delta_\omega \subseteq \Delta_\omega$ first. Since Z is contained in the center of TT^* (because the translation groups T and T^* are commutative), it suffices to show that the product $(\lambda \circ \omega \circ \lambda^{-1} \circ \omega) \circ (\mu \circ \omega \circ \mu^{-1} \circ \omega)$ belongs to Δ_ω for all $\lambda, \mu \in \Lambda$.

We write $\lambda^* := \omega \circ \lambda^{-1} \circ \omega$ and note that for all $\alpha, \beta \in TT^*$ the commutator $[\alpha, \beta] := \alpha \circ \beta \circ \alpha^{-1} \circ \beta^{-1}$ belongs to Z and thus commutes with each element of TT^* . Now we compute the product in question as

$$\begin{aligned} (\lambda \circ \lambda^*) \circ (\mu \circ \mu^*) &= \lambda \circ \mu \circ [\mu^{-1}, \lambda^*] \circ \lambda^* \circ \mu^* \\ &= \lambda \circ \mu \circ [\mu^{-1}, \lambda^*] \circ [\lambda^*, \mu^*] \circ (\lambda \circ \mu)^* \\ &= [\mu^{-1}, \lambda^*] \circ [\lambda^*, \mu^*] \circ (\lambda \circ \mu) \circ (\lambda \circ \mu)^* \in \Delta_\omega. \end{aligned}$$

We compute $(\zeta \circ \lambda \circ \lambda^*)^{-1} = \lambda^\omega \circ \lambda^{-1} \circ \zeta^{-1} = [\lambda^\omega, \lambda^{-1}] \circ \zeta^{-1} \circ \lambda^{-1} \circ \lambda^\omega \in Z \circ \lambda^{-1} \circ \lambda^\omega \subseteq \Delta_\omega$, and see that Δ_ω is closed under inversion.

Thus Δ_ω is a group. From $(\zeta \circ \lambda \circ \lambda^*)(0) = (\zeta \circ \lambda)(0)$ we infer that Δ_ω acts transitively on $P \setminus L_\infty$. Moreover $\zeta \circ \lambda \circ \lambda^*$ fixes 0 exactly if the translation $\zeta \circ \lambda$ is trivial: this means $\lambda = \text{id} = \zeta$, and $\zeta \circ \lambda \circ \lambda^* = \text{id}$ follows. Conjugation with ω translates this to the action on $\mathcal{L} \setminus \mathcal{L}_\infty$. □

8.7 Lemma. *If ω is an orthogonal polarity then ω induces inversion on Δ_ω .*

Proof. First of all, we compute

$$\omega \circ (\lambda \circ \omega \circ \lambda^{-1} \circ \omega) \circ \omega = \omega \circ \lambda \circ \omega \circ \lambda^{-1} = (\lambda \circ \omega \circ \lambda^{-1} \circ \omega)^{-1}.$$

Thus conjugation by ω induces inversion on the set $\{\lambda \circ \omega \circ \lambda^{-1} \circ \omega \mid \lambda \in \Lambda\}$. Since $Z = T \cap T^*$ is contained in the center of TT^* , it remains to understand the action on the factor Z . We distinguish two cases:

If the characteristic is different from 2 then the set of absolute points is an oval, and 8.2 yields $\omega \circ \zeta \circ \omega = \zeta^{-1}$ for each $\zeta \in Z$.

If the characteristic is 2, we infer from 8.2 that ω induces the identity on Z . However, this does not matter because every element of Z is its own inverse. \square

8.8 Theorem. *If ω is an orthogonal polarity then Δ_ω is commutative, and thus a shift group. The polarity ω is a standard polarity of the corresponding shift plane.*

Proof. From 8.6 we know that Δ_ω is a group. Conjugation by ω is an automorphism of this group. According to 8.7, this automorphism is inversion: this means that Δ_ω is commutative.

We use 8.7 to see that ω is a standard polarity: choose $L(0) := \omega(0)$ and note that for $\delta \in \Delta_\omega$ we have $\omega(\delta(0)) = (\omega \circ \delta \circ \omega)(\omega(0)) = \delta^{-1}(L(0))$. \square

8.9 Remark. Conversely, any translation plane that is also a shift plane admits a polarity (by 3.1). Section 9 contains a proof of the fact that each standard polarity is an orthogonal one.

8.10 Remarks. For each translation plane that is also a shift plane a commutative semifield coordinatizing that plane will be constructed in 9.3.

Note that a non-desarguesian plane of Lenz type V may be coordinatized by some commutative semifield and also by some non-commutative semifield (which is isotopic to the commutative one).

Finite projective planes over commutative semifields have been characterized in [15] by the existence of sufficiently many orthogonal polarities. In that paper, the set of orthogonal polarities with two given absolute points is parameterized by the middle nucleus of the semifield, the corresponding shift groups are parameterized in the same way.

Shift groups for translation planes are also characterized in [29]: if a commutative subgroup of the group generated by the translation group and a group of shears acts regularly on the set of affine points, it is the translation group or a shift group. That paper also includes a discussion of non-commutative groups that fulfill the regularity conditions for shift groups.

8.11 Remark. We conjecture that every shift plane can be coordinatized by a ternary field with commutative multiplication. This is known for finite shift planes (see [11] Theorem 6) and for compact connected shift planes (see [22], [41] Satz 5.2). See also [40].

9 Shift groups on translation planes

Let $\mathcal{P} = (P, \mathcal{L})$ be a projective plane of Lenz type V, with $\Gamma := \text{Aut}(\mathcal{P})$. More precisely, assume that there is a triangle (∞, o, u) such that each of the elation groups $\Lambda := \Gamma_{[u, u \vee \infty]}$, $Z := \Gamma_{[\infty, u \vee \infty]}$, and $\Lambda^* := \Gamma_{[\infty, o \vee \infty]}$ is linearly transitive. Our aim is to determine all candidates for shift groups fixing ∞ and $L_\infty := u \vee \infty$. We adopt an affine point view, with L_∞ as the line at infinity. With respect to any quadrangle (o, u, ∞, e) , the plane \mathcal{P} is coordinatized by some semifield $(S, +, \cdot)$.

We remark that the group $\Lambda Z \Lambda^*$ with the pair $(\Lambda Z, Z \Lambda^*)$ of subgroups forms a T -group in the sense of [7]. Our arguments in the present section may be interpreted as a discussion of the P -system $\langle \Lambda Z, Z \Lambda^* \rangle$ (in the sense of [7]) and its polarities.

From now on, assume that a shift group exists. Surely this means some restriction on the semifield S . We shall prove in this section (see 9.3 below) that S has to be isotopic to a commutative semifield. In particular, non-commutative alternative fields are excluded because all ternary fields for a given plane over an alternative field (i.e., a Moufang plane) are isomorphic. Commutative alternative fields are fields ([36], [5], cf. [28] 6.3, 6.4). Thus the existence of a shift group on a translation plane implies that the plane is pappian, or coordinatized by a proper semifield (which may be chosen as a commutative one).

From 1.3 we know that Z is contained in each shift group that fixes (∞, L_∞) . Since shift groups are commutative, we have to search inside the centralizer Ξ of Z . As the group $\Lambda Z \Lambda^*$ acts transitively on the set of affine points and the stabilizer Λ^* of o acts transitively on $L_\infty \setminus \{\infty\}$, the stabilizer of the flag (∞, L_∞) is the product of $\Lambda Z \Lambda^*$ and the stabilizer ∇ of the triangle (∞, o, u) . If $\delta \in \nabla$ centralizes Z we find that δ fixes each point on $V := o \vee \infty$ and each line through u . Thus δ belongs to $\Gamma_{[u, V]}$, and we obtain (cf. [17] Theorem 8.2):

9.1 Lemma. *The centralizer Ξ of Z in $\Gamma_{\infty, L_\infty}$ consists of maps of the form $(x, y) \mapsto (sx + a, y + cx + b)$, with $a, b, c \in S$ and $s \in N_m \setminus \{0\}$ where N_m denotes the middle nucleus of S .*

9.2 Lemma. *For every shift group inside $\Gamma_{\infty, L_\infty}$ there is a function $C : S \rightarrow N_m$ such that the shift group is $\Delta_C := \{((x, y) \mapsto (x + a, y + C(a)x + b)) \mid (a, b) \in S^2\}$.*

Proof. Let $(a, b) \in S^2$ be arbitrary. Since the shift group acts regularly on the set of affine points, it contains a unique element $\delta_{a,b}$ such that $\delta_{a,b}(0, 0) = (a, b)$. For each $b \in S$, the element $\delta_{0,b}$ belongs to Z , and we find $\delta_{0,b}(x, y) = (x, y + b)$.

In the general case 9.1 yields that there are s and $c = c_{a,b} \in S$ (depending on a, b) such that $\delta_{a,b}(x, y) = (sx + a, y + cx + b)$. We claim $s = 1$: otherwise, we could find $w \in S$ such that $(1 - s)w = a$, and $\delta_{0, -cw - b} \circ \delta_{a,b}$ would fix the point $(w, 0)$. Regularity of the action then yields $\delta_{a,b} \in Z$, contradicting $s \neq 1$.

It remains to show that c depends only on a . We evaluate both $\delta_{a,b}$ and $\delta_{a,0} \circ \delta_{0,b}$ first at $(0, 0)$ to see that they coincide, and then at $(x, 0)$, finding $(x + a, c_{a,b}x + b) = \delta_{a,b}(x, 0) = \delta_{a,0}(\delta_{0,b}(x, 0)) = (x + a, b + c_{a,0}x)$. Putting $C(a) := c_{a,0}$ we obtain the assertion. \square

Evaluating at $(0, 0)$ we find the composition rule $\delta_{u,v} \circ \delta_{a,b} = \delta_{a+u, b+v+C(u)a}$. It remains to determine the function C . Using commutativity of the shift group, we observe $\delta_{1+a, C(1)a} = \delta_{1,0} \circ \delta_{a,0} = \delta_{a,0} \circ \delta_{1,0} = \delta_{a+1, C(a)1}$ and obtain that $C(a) = C(1)a$ holds for each $a \in S$. Put $s := C(1)$. From $\delta_{a+u, b+v+C(u)a} = \delta_{u,v} \circ \delta_{a,b} = \delta_{a,b} \circ \delta_{u,v} = \delta_{u+a, v+b+C(a)u}$ and $C(x) = sx$ we infer that

$$(su)a = (sa)u \quad \text{for } a, u \in S.$$

9.3 Proposition. *We obtain an isotopism $(1, C, C)$ from $(S, +, \cdot)$ onto $(S, +, *)$, where $x*y := C^{-1}(C(x)\cdot y) (= C^{-1}((s\cdot x)\cdot y))$. Clearly, the semifield $(S, +, *)$ is commutative.*

For finite translation planes [12] contains a forerunner to this result – with a different definition of “orthogonal polarity” which in the finite case is equivalent to our definition by [2].

The isotopism corresponds to changing the unit point e for the quadrangle (o, u, ∞, e) used for the introduction of coordinates.

From now on, assume that we have chosen the quadrangle in such a way that S is a commutative semifield. Then s belongs to the middle nucleus of S . Conversely, every element s of the middle nucleus gives rise to a shift group Δ_C , where $C(x) := sx$. We have thus proved:

9.4 Theorem. *If a translation plane \mathcal{P} admits a shift group, then the plane may be coordinatized by a commutative semifield $(S, +, \cdot)$.*

If \mathcal{P} is not a pappian plane, then every shift group of \mathcal{P} is contained in $\Lambda Z\Lambda^$, and all these shift groups are of the form*

$$\Delta_s := \{((x, y) \mapsto (x + a, y + b + (as)x)) \mid (a, b) \in S^2\},$$

where s belongs to the middle nucleus of the (commutative) semifield S . If the middle nucleus coincides with the left (right) nucleus, then all these shift groups are conjugate under the triangle stabilizer ∇ . In the pappian case, the shift groups form a single conjugacy class, as well.

9.5 Remarks. The connection between shift groups and planes over commutative semifields has already been observed in [37], in the context of incidence groups. See also [29] and [9] Zusatz 4.1.

The Dickson commutative semifields (cf. [17] IX 5) are examples of commutative semifields where the left (right) nucleus is properly contained in the middle nucleus.

9.6 Remarks. Our parameterization of the group Δ_s gives an isomorphism onto $\Delta_s^{\&} := (S^2, \&)$ where $(a, b) \& (c, d) := (a + b, c + d + (cs)a)$. The affine points on the line $o \vee u$ are just the images of o under the elements of the set $L(0) := S \times \{0\} \subseteq \Delta_s^{\&}$ of coset representatives for $\Delta_s^{\&} / (\{0\} \times S)$. Applying inversion in the group $\Delta_s^{\&}$ we obtain the set $L^-(0) = \{(-a, (as)a) \mid a \in S\}$: together with the point ∞ , the corresponding points form an oval in the shift plane (see [8]).

Using only the two distributive laws in S (but no associativity assumptions on the multiplication except the fact that $s \in N_m$), one computes the square of $(a, b) \in \Delta_s^{\&}$ as $(a, b) \& (a, b) = (2a, 2b + (as)a)$. If $\text{char } S = 2$, we find that the group $\Delta_s^{\&}$ is a module over $\mathbb{Z}/4\mathbb{Z}$ (as expected according to 5.8). This module is free if, and only if, every element of S is contained in $\{(as)a \mid a \in S\}$; cf. 5.9 and 1.8. If $\text{char } S \neq 2$ then the additive group of S is uniquely 2-divisible, and so is the group Δ_s .

9.7 Theorem. *Let Δ be a shift group on a translation plane of characteristic different from 2. Then there exists a commutative semifield S , an element s of the middle nucleus of S and an isomorphism from Δ onto $(S^2, +)$ such that $L(0)$ is mapped to $L^2(0) := \{(x, xsx) \mid x \in S\}$, see 1.7. The plane is isomorphic to the plane over the semifield S .*

9.8 Notation. The elements of $\Omega := \Lambda Z \Lambda^*$ will be denoted by

$$\gamma_{a,z,b}: S^2 \rightarrow S^2: (x, y) \mapsto (x + a, y + z + bx).$$

Note that $\gamma_{a,z,b} \circ \gamma_{c,v,d} = \gamma_{c+a,z+v+bc,b+d}$ and $\gamma_{a,z,b}^{-1} = \gamma_{-a,ba-z,-b}$. The groups Λ , Z and Λ^* are characterized by $z = 0 = b$, $a = 0 = \bar{b}$, and $a = 0 = z$, respectively. The shift groups are $\Delta_s = \{\gamma_{a,z,as} \mid a, z \in S\}$. We need to choose a standard polarity J_s for Δ_s . Since γ acts transitively on the set of flags (a, K) with $a \notin L_\infty$ and $\infty \notin K$, we may assume that J_s maps o to $o \vee u$. Together with the fact that J_s induces inversion on Δ_s , this choice determines the action of J_s on the set of affine points, and thus determines J_s .

9.9 Lemma. *The action of J_s by conjugation on $\Omega = \Lambda Z \Lambda^*$ is described by*

$$\gamma_{a,z,b}^{J_s} = \gamma_{-bs^{-1},a(bs^{-1})-z,-as}.$$

In particular, we have

- (1) $\Delta_s = \{\gamma \in \Omega \mid \gamma^{J_s} = \gamma^{-1}\}$,
- (2) $\Psi_s := C_\Omega(J_s) = \{\gamma_{a,z,-as} \mid a, z \in S, 2z = -asa\}$.

If $\text{char } S = 2$ then this simplifies to $\Psi_s = Z$.

Proof. We know that J_s induces inversion on $Z \leq \Delta_s$. Surely $(\Lambda^*)^{J_s} = \Lambda$ because J_s interchanges (∞, V) with (u, L_∞) . As conjugation by J_s is an automorphism, it thus suffices to compute $\gamma_{0,0,b}^{J_s}$ for $\gamma_{0,0,b} \in \Lambda^*$.

We find $\gamma_{0,0,b}(J_s(o)) = \gamma_{0,0,b}(o \vee u) = \gamma_{c,y,cs}(o \vee u)$ with $\gamma_{c,y,cs} \in \Delta_s$, where $c = bs^{-1}$ and $y = bc = bs^{-1}b$. This yields $(J_s \circ \gamma_{0,0,b} \circ J_s)(o) = \gamma_{c,y,cs}^{J_s}(o) = \gamma_{c,y,cs}^{-1}(o) = \gamma_{-bs^{-1},0,-b}(o) = (-bs^{-1}, 0) = \gamma_{-bs^{-1},0,0}(o)$. Now we compute $\gamma_{a,z,b}^{J_1} = (\gamma_{a,0,0} \circ \gamma_{0,z,0} \circ \gamma_{0,0,b})^{J_1} = \gamma_{a,0,0}^{J_1} \circ \gamma_{0,z,0}^{J_1} \circ \gamma_{0,0,b}^{J_1} = \gamma_{0,0,-a} \circ \gamma_{0,-z,0} \circ \gamma_{-bs^{-1},0,0} = \gamma_{-bs^{-1},a(bs^{-1})-z,-a}$, as claimed. The rest follows easily. \square

Let $\mathcal{O}_s := \{p \in P \mid p \in J_1(p)\}$ be the set of absolute points of J_s . Our characterization 3.3 of affine absolute points now reads

$$\mathcal{O}_s \setminus \{\infty\} = \{\gamma_{a,z,as}(o) \mid (2a, asa + 2z) \in o \vee u\} = \{(a, z) \mid a, z \in S, 2z = -asa\}.$$

From 9.9 we infer immediately:

9.10 Theorem. *The centralizer Ψ of J acts regularly on $\mathcal{O}_s \setminus \{\infty\}$. If $\text{char } S = 2$ this simplifies to $\mathcal{O}_s \setminus \{\infty\} = Z(o) = V$.*

9.11 Theorem. *If \mathcal{P} is a shift plane and a translation plane, then the standard polarities are orthogonal ones.*

Proof. After 9.10, it remains to discuss the case $\text{char } S \neq 2$. Moreover, any line that meets \mathcal{O}_s in an affine point may be mapped to a line through o by some element of Ψ . Thus it remains to show that each line in $\mathcal{L}_o \setminus \{o \vee u, V\}$ meets \mathcal{O}_s in exactly one point apart from o . This amounts to solving equations of the form $xsx + 2mx = 0$ in S : the only solutions are 0 and $x = -2ms^{-1}$. \square

We collect our results regarding shift planes that are translation planes:

9.12 Theorem. *Let \mathcal{P} be a translation plane. Then the following are equivalent:*

- \mathcal{P} admits a shift group.
- \mathcal{P} admits an orthogonal polarity.
- \mathcal{P} can be coordinatized by a commutative semifield.

It is well known that the orthogonal polarities of a pappian plane form a single conjugacy class. We generalize this result, giving a new proof for the pappian case, as well:

9.13 Theorem. *Let \mathcal{P} be a translation plane with an orthogonal polarity J , and assume that the middle nucleus of some coordinatizing commutative semifield coincides with its left nucleus. Then every orthogonal polarity of \mathcal{P} is a conjugate of J .*

Proof. Let ω be an orthogonal polarity. Then ω has more than one absolute flag. If \mathcal{P} is a Moufang plane, then $\Gamma := \text{Aut}(\mathcal{P})$ acts transitively on the set of flags. If \mathcal{P} is not a Moufang plane, it has Lenz type V, and every polarity fixes the special flag. Therefore, it suffices to consider polarities such that (∞, L_∞) is absolute. For some conjugate of ω the point o is absolute because $\Gamma_{\infty, L_\infty}$ acts transitively on $P \setminus P_\infty$. Finally, the transitive group Λ^* of shears with axis $o \vee \infty$ yields that ω has a conjugate $\tilde{\omega}$ with $\tilde{\omega}(o) = J(o)$.

The polarity $\tilde{\omega}$ defines a shift group $\Delta_{\tilde{\omega}}$, and induces inversion on this group. Since all shift groups fixing (∞, L_∞) are conjugate, we may also assume that $\Delta_{\tilde{\omega}} = \Delta_J$. Now it remains to note that every affine point is of the form $\delta(o)$ with $\delta \in \Delta_J$, and $\tilde{\omega}(\delta(o)) = \delta^{-1}(\tilde{\omega}(o)) = \delta^{-1}(J(o)) = J(\delta(o))$. \square

10 Uniqueness of the shift group

Our results about polarities cover only the polarities in $J \circ \Pi \subseteq J \circ \Sigma$ (and their conjugates). If the shift group Δ is not normal in the full group Γ of all collineations of (P, \mathcal{L}) then there exists a second group acting as a shift group on (P, \mathcal{L}) . The results in the present section show that this only happens in cases that are well understood, at least if we impose reasonable additional conditions like finiteness or compactness.

We conjecture that the existence of a second shift group *always* implies that the plane is a translation plane.

10.1 Theorem. *If a shift plane admits shift groups with different fixed flags, then it is a plane of Lenz type V, coordinatized by a commutative semifield. If the flags share neither the point nor the line, then the plane is pappian.*

Proof. This follows from the Lenz classification ([27], cf. [10] 3.1.20): We know from 1.3 that $\Delta_{[\infty, L_\infty]}$ is linearly transitive. If there exists a shift group fixing a flag (∞', L'_∞) different from (∞, L_∞) , then one of the following occurs:

- (1) Γ is transitive on P ,
- (2) $\infty = \infty'$ and \mathcal{L}_∞ is contained in an orbit of Γ ,
- (3) $L_\infty = L'_\infty$ and the point row L_∞ is contained in an orbit of Γ .

However, none of the Lenz types except type VII contains a flag (p, L) for each point $p \in P$. So transitivity on P implies that the plane is a Moufang plane, coordinatized (with respect to any quadrangle) by an alternative field. This alternative field has commutative multiplication by 9.3, and is a commutative field by [28] 6.1.6, p. 162.

In cases 2 and 3, the group Γ contains linearly transitive elation groups for each flag in $(\{\infty\} \times \mathcal{L}_\infty) \cup (J(\mathcal{L}_\infty) \times \{J(\infty)\}) = (\{\infty\} \times \mathcal{L}_\infty) \cup (L_\infty \times \{L_\infty\})$: this is Lenz type V. The coordinatizing semifield is commutative by 9.3. \square

10.2 Theorem. *If \mathcal{P} is a finite shift plane admitting more than one shift group, then \mathcal{P} is a plane of Lenz type V, coordinatized by a commutative semifield.*

Proof. After 10.1, it suffices to consider the case where (P, \mathcal{L}) admits two different shift groups Δ_1, Δ_2 fixing the same flag (∞, L_∞) . Then $|\Delta_1| = n^2 = |\Delta_2|$, where n is the order of the plane. We write $\Psi := \langle \Delta_1 \cup \Delta_2 \rangle$. According to 1.3, the group $\Psi_{[\infty, L_\infty]} = V$ is contained in the intersection of the abelian groups Δ_j , and thus in the center Z of Ψ .

For each affine point p , the stabilizer Ψ_p is a nontrivial group fixing every point in the orbit $Z(p)$. This orbit is a union of affine lines sharing ∞ as their point at infinity, and we infer that $Z(p)$ is contained in (and then coincides with) the affine line $V + p = \Psi_{[\infty, L_\infty]}(p)$. The stabilizer Z_p fixes each point in the orbit $\Psi(p)$ (that is, each affine point) and is therefore trivial. Thus $n = |V| \leq |\Delta_1 \cap \Delta_2| \leq |Z| \leq n$ yields $V = \Delta_1 \cap \Delta_2 = Z$.

Now the multiplication map from $\Delta_1 \times \Delta_2$ onto $\Delta_1 \Delta_2 \subseteq \Psi$ has fibers of size $|\Delta_1 \cap \Delta_2| = n$, and we infer $|\Psi| \geq \frac{|\Delta_1| \cdot |\Delta_2|}{|\Delta_1 \cap \Delta_2|} = n^3$.

In order to complete the proof, we need a result due to André (see [1] Satz 3, with the “Zusatz” on p. 32, or [17] Theorem 4.25): the set C of points outside the line $0 \vee \infty$ that occur as centers of nontrivial elements of $\Psi_{[V+p]}$ forms a single orbit under the group $\Psi_{[V+p]}$. Therefore, we have the following two cases:

If the stabilizer $\Psi_{p,q}$ is nontrivial for each $q \in L_\infty \setminus \{\infty\}$, then $\Psi_{[\infty, V+p]}$ is linearly transitive. In this case, the groups $\Psi_{[\infty, V+p]}$ and $V = \Psi_{[\infty, L_\infty]}$ generate a transitive group of translations on the affine subplane $(\mathcal{L} \setminus \mathcal{L}_\infty, P \setminus \{\infty\})$ of the dual plane (\mathcal{L}, P) . Since the shift plane is self-dual (cf. 3.1), we have a plane of Lenz type V.

If there exists $q \in L_\infty \setminus \{\infty\}$ such that $\Psi_{p,q}$ is trivial, then the orbit $\Psi_p(q)$ has $n = |\Psi_p|$ elements, and $\Psi_p \leq \Psi_{[V+p]}$ acts transitively on $L_\infty \setminus \{\infty\}$. In particular, each element of Ψ_p has center ∞ , and $\Psi_p = \Psi_{[\infty, V+p]}$ is linearly transitive, again. As before, we find that (P, \mathcal{L}) has Lenz type V. The coordinatizing semifield is commutative by 9.3. \square

10.3 Examples. Semifields yielding shift planes of Lenz type V as in 10.2 are constructed in [20], cf. [19] and [21]. Some of these semifields have a middle nucleus of order 2, and do not admit more than one shift group. In view of 9.4, this means that there is no simple converse to 10.2.

10.4 Theorem. *If \mathcal{P} is a compact connected shift plane admitting more than one shift group, then \mathcal{P} is a pappian plane (in fact, isomorphic to the projective plane over \mathbb{R} or \mathbb{C}).*

Proof. See [25] for the case of 2-dimensional planes. For planes of dimension 4, this has been proved in [24], cf. [35] 74.8(c). Compact planes of higher dimension do not admit shift groups [41], see [35] 74.6. \square

11 Examples with many classes of polarities

11.1 Example. We use the group $\Delta := \mathbb{R}^2$ and $L(0) := \{(x, f(x)) \mid x \in \mathbb{R}\}$ where $f: \mathbb{R} \rightarrow \mathbb{R}: x \mapsto x^2 + \cos(x) - 1$. One checks that the derivative f' is a homeomorphism of \mathbb{R} onto itself. Following the procedure described in 1.11 we obtain a shift plane \mathcal{P}_{\cos} . This plane is not desarguesian because f is not a quadratic polynomial, and Δ is unique by 10.4. In order to determine $\text{Aut}(\mathcal{P}_{\cos})$, it therefore suffices to determine Σ_0 .

According to [25], every automorphism of \mathcal{P}_{\cos} fixing the point $(0, 0)$ has the form $T_{a,c,d}: \mathbb{R}^2 \rightarrow \mathbb{R}^2: (x, y) \mapsto (ax, cx + dy)$. It remains to determine the admissible values for a, c, d . We apply $T_{a,c,d}$ to the (affine) line $L(0)$ and obtain the set

$$\begin{aligned} T_{a,c,d}(L(0)) &= \{(ax, cx + dx^2 + d \cos x - d) \mid x \in \mathbb{R}\} \\ &= \left\{ \left(u, \frac{c}{a}u + \frac{d}{a^2}u^2 + d \cos\left(\frac{u}{a}\right) - d \right) \mid u \in \mathbb{R} \right\}. \end{aligned}$$

This should be a (non-vertical) line through $(0, 0)$, and thus equal to $L((s, -f(-s))) = \{(u, f(u-s) - f(-s)) \mid u \in \mathbb{R}\}$ for some $s \in \mathbb{R}$. This gives

$$\begin{aligned} 0 &= \frac{c}{a}u + \frac{d}{a^2}u^2 + d \cos\left(\frac{u}{a}\right) - d - (f(u-s) - f(-s)) \\ &= \frac{c}{a}u + \frac{d}{a^2}u^2 + d \cos\left(\frac{u}{a}\right) - d - (u^2 - 2su + \cos(u-s) - \cos(s)) \\ &= \left(\frac{d}{a^2} - 1\right)u^2 + \left(\frac{c}{a} + 2s\right)u + d \cos\left(\frac{u}{a}\right) - \cos(u-s) + \cos(s) - d \end{aligned}$$

for all $u \in \mathbb{R}$. As the term $g(u) := d \cos\left(\frac{u}{a}\right) - \cos(u-s) + \cos(s) - d$ is bounded, we have $\frac{d}{a^2} - 1 = 0 = \frac{c}{a} + 2s$ and $g(u) = 0$ for all u . Now the derivative $g'(u) = -\frac{d}{a} \sin\left(\frac{u}{a}\right) + \sin(u-s)$ is vanishing identically, and $\sin\left(\frac{u}{a}\right)$ and $\sin(u-s)$ have the same set of zeros. This implies $a \in \{1, -1\}$, and $d = a^2 = 1$ follows. Then $0 = g'(u) = -\sin(u) + \sin(u-s)$ yields $s \in 2\pi\mathbb{Z}$. We obtain $c = -2as = 4k\pi$ with $k \in \mathbb{Z}$.

A simple verification shows that $T_{a,4k\pi,1}$ is indeed an automorphism of \mathcal{P}_{\cos} whenever $a \in \{1, -1\}$ and $k \in \mathbb{Z}$.

In the notation of 2.2 we have $A_{T_{a,4k\pi,1}} = \begin{pmatrix} a & 0 \\ 4k\pi & 1 \end{pmatrix}$ and $t_{T_{a,4k\pi,1}} = (0, 0)$, here automorphisms of \mathbb{R}^2 are described by matrices applied from the left. Note also that $c_{T_{a,4k\pi,1}} = (-2ak\pi, -4k^2\pi^2)$.

11.2 Theorem. *The automorphism group of \mathcal{P}_{\cos} is*

$$\Sigma := \left\{ \sigma_{A,t} \mid A = \begin{pmatrix} a & 0 \\ 4k\pi & 1 \end{pmatrix} \text{ with } a = \pm 1, k \in \mathbb{Z}, t \in \mathbb{R}^2 \right\}.$$

The involutions in Σ_0 are the elements $\sigma_{A_k,0}$, where $A_k := \begin{pmatrix} -1 & 0 \\ 4k\pi & 1 \end{pmatrix}$ and $k \in \mathbb{Z}$. Two involutions $\sigma_{A_k,0}$ and $\sigma_{A_m,0}$ are conjugate in Σ_0 if, and only if, we have $k - m \in 2\mathbb{Z}$.

The set $\Pi = \{ \alpha \in \Sigma \mid J \circ \alpha \text{ is a polarity} \}$ considered in 4.2 is obtained as

$$\Pi = \Delta \cup \{ \sigma_{A_k,(-k\pi,y)} \mid k \in \mathbb{Z}, y \in \mathbb{R} \}.$$

The conjugacy classes of polarities of \mathcal{P}_{\cos} are represented by J , $J \circ \sigma_{A_0,(0,0)}$, and $J \circ \sigma_{A_1,(-\pi,0)}$, respectively.

Proof. We know \mathcal{P}_{\cos} from the discussion in 11.1. An easy computation yields the involutions, as claimed. Surely A_k and A_m are not conjugate if $k - m \notin 2\mathbb{Z}$. Conversely, if $k - m = 2n \in 2\mathbb{Z}$, then $B := A_{k+n}$ satisfies $BA_kB^{-1} = A_m$. (This reflects the fact that Σ_0 is an infinite dihedral group generated by $\sigma_{A_0,(0,0)}$ and $\sigma_{A_1,(0,0)}$.)

In order to determine the set Π , we use $c_{A_k} = (2k\pi, -4k^2\pi^2)$ and 4.1: the solutions for $(A_k - \text{id})t = c_{A_k}$ are just the elements of $\{(-k\pi, y) \mid y \in \mathbb{R}\}$.

Finally, we apply 6.4 in order to reduce the question of conjugacy of polarities $J \circ \sigma_{A_k,(-k\pi,y)}$ and $J \circ \sigma_{A_m,(-m\pi,z)}$ to conjugacy of A_k and A_m in Σ_0 . \square

We use a known extension process to produce a shift plane with $\Delta = \mathbb{R}^4$ from one with shift group \mathbb{R}^2 . The following is contained in [25] (see [31] for a generalization of the construction):

11.3 Proposition. *Let $f: \mathbb{R} \rightarrow \mathbb{R}: x \mapsto x^2 + \cos(x) - 1$ be as in 11.1, and define*

$$f^\wedge: \mathbb{R}^2 \rightarrow \mathbb{R}^2: (x, y) \mapsto (f(x) - y^2, 2xy) = (x^2 + \cos(x) - 1 - y^2, 2xy).$$

Then f^\wedge is a planar function (in the sense of 1.11) defining a shift plane \mathcal{P}_{\cos}^ with shift group $\Delta := \mathbb{R}^4$ and*

$$L(0) = \{ (x, y, x^2 + \cos(x) - 1 - y^2, 2xy) \mid x, y \in \mathbb{R} \}.$$

Every element $\alpha = T_{a,4k\pi,1}$ of the stabilizer $\Sigma_0 := \text{Aut}(\mathcal{P}_{\cos})_0$ of 0 in $\text{Aut}(\mathcal{P}_{\cos})$ extends to an automorphism α^\wedge of \mathcal{P}_{\cos}^ via*

$$T_{a,4k\pi,1}^\wedge(x, y, u, v) = (ax, y, u + 4k\pi x, av + 4ak\pi y) \quad \text{for } a \in \{1, -1\} \text{ and } k \in \mathbb{Z}.$$

Mapping α to α^\wedge defines an embedding of the point stabilizer Σ_0 into $\Sigma_0^ := \text{Aut}(\mathcal{P}_{\cos}^*)_0$. Together with the subgroup $\Xi := \mathbb{R} \times \{0\} \times \mathbb{R} \times \{0\} \subseteq \Delta$ the image Σ_0^\wedge under this*

embedding induces the full group of automorphisms of the Baer subplane $\mathcal{P}_{\cos}^\wedge \cong \mathcal{P}_{\cos}$ with affine point set Ξ .

The map $\beta: \mathbb{R}^4 \rightarrow \mathbb{R}^4: (x, y, u, v) \mapsto (x, -y, u, -v)$ yields a Baer involution that centralizes the group Σ_0^\wedge : the fixed elements are the points and lines of $\mathcal{P}_{\cos}^\wedge$.

For each $r \in \mathbb{R}$ the map $S_r: \mathbb{R}^4 \rightarrow \mathbb{R}^4: (x, y, u, v) \mapsto (x, y, u - ry, v + rx)$ is an automorphism of \mathcal{P}_{\cos}^* , and $S := \{S_r \mid r \in \mathbb{R}\}$ is a subgroup of $\text{Aut}(\mathcal{P}_{\cos}^*)$ consisting of shears with axis $V = \{0\}^2 \times \mathbb{R}^2$.

For the sake of readability, we abbreviate $A_k^\wedge := T_{-1, 4k\pi, 1}^\wedge$.

11.4 Theorem. *The full group of automorphisms of \mathcal{P}_{\cos}^* is the semidirect product of the shift group $\Delta = \mathbb{R}^4$ and $\Sigma_0^* = S\langle\beta\rangle\Sigma_0^\wedge$. The conjugacy classes of involutions in Σ_0^* are represented by*

$$A_0^\wedge, \quad A_1^\wedge, \quad \beta, \quad \beta \circ A_0^\wedge, \quad \beta \circ A_1^\wedge.$$

Consequently, there are 6 conjugacy classes of polarities of \mathcal{P}_{\cos}^* , represented by

$$J, \quad J \circ \sigma_{A_0^\wedge, 0}, \quad J \circ \sigma_{A_1^\wedge, (-\pi, 0, 0, 0)}, \quad J \circ \beta, \quad J \circ \beta \circ \sigma_{A_0^\wedge, 0}, \quad J \circ \beta \circ \sigma_{A_1^\wedge, (-\pi, 0, 0, 0)}.$$

11.5 Remarks. The fixed points of the involution $\sigma_{A_0^\wedge, 0}$ form a Baer subplane, the affine points of that plane are those in $\{0\} \times \mathbb{R}^2 \times \{0\}$. The involution $\sigma_{A_1^\wedge, 0}$ is also a Baer involution, its affine point set is $\{(0, y, u, -2\pi y) \mid y, u \in \mathbb{R}\}$. In both cases, the plane of fixed elements is isomorphic to the classical plane over \mathbb{R} : this is readily seen from the fact that $L(0)$ intersects the affine set of fixed points in a parabola, cf. 1.11.

Each one of the involutions $\beta \circ \sigma_{A_k^\wedge, 0}$ is a reflection with axis $V = \{0\}^2 \times \mathbb{R}^2$, the center is the point at infinity of the parallel class $\{L(k\pi, 0, u, v) \mid u, v \in \mathbb{R}\}$.

Proof of Theorem 11.4. We already know that $S\langle\beta\rangle\Sigma_0^\wedge$ is a subgroup of Σ_0^* and that $\text{Aut}(\mathcal{P}_{\cos}^*) = \Sigma_0^* \circ \Delta$; it remains to determine Σ_0^* .

We claim that S is a normal subgroup in Σ_0^* : otherwise, we would have two full groups of shears with center ∞ , and \mathcal{P}_{\cos}^* would be a dual translation plane. As the shear plane is self-dual, this would mean that \mathcal{P}_{\cos}^* has Lenz type V (at least), and \mathcal{P}_{\cos}^* were isomorphic to the desarguesian plane over \mathbb{C} (see [35] 64.15). But this is impossible since \mathcal{P}_{\cos}^* contains the non-desarguesian Baer subplane $\mathcal{P}_{\cos}^\wedge$.

Using the action on the shift group, we describe the elements of Σ_0^* by block matrices $\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$, where $A, D \in \text{GL}_2\mathbb{R}$ and $C \in \mathbb{R}^{2 \times 2}$. The zero block is due to the fact that $V = \{0\}^2 \times \mathbb{R}^2$ is invariant under elements of Σ_0^* . We denote the neutral element of $\text{GL}_2\mathbb{R}$ by E and write $i := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $j := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. In this notation, we have $S = \left\{ \begin{pmatrix} E & 0 \\ ri & E \end{pmatrix} \mid r \in \mathbb{R} \right\}$, and the fact that S is normal in Σ_0^* implies

$$\Sigma_0^* \subseteq \left\{ \begin{pmatrix} A & 0 \\ C & -diAi \end{pmatrix} \mid A \in \text{GL}_2\mathbb{R}, C \in \mathbb{R}^{2 \times 2}, d \in \mathbb{R} \setminus \{0\} \right\}.$$

We write $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, $C = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$, and consider $M := \begin{pmatrix} A & 0 \\ C & -diAi \end{pmatrix} \in \Sigma_0^*$. Then M maps $L(0)$ to a line $L((s, t, -f^\wedge(-s, -t)) = L((s, t, -f(-s) + t^2, -2st))$ through 0 ,

with suitable $(s, t) \in \mathbb{R}^2$. This gives the conditions

$$\begin{aligned} & (a_1x + a_2y - s)^2 + \cos(a_1x + a_2y - s) - 1 - (a_3x + a_4y - t)^2 \\ & = c_1x + c_2y + da_4(x^2 + \cos(x) - 1 - y^2) + da_32xy + s^2 + \cos(-s) - 1 - t^2 \end{aligned} \quad (*)$$

and

$$\begin{aligned} & 2(a_1x + a_2y - s)(a_3x + a_4y - t) \\ & = c_3x + c_4y - da_2(x^2 + \cos(x) - 1 - y^2) - da_12xy + 2st \end{aligned} \quad (**)$$

for all $x, y \in \mathbb{R}$. Specializing $x = 0$, we obtain

$$\begin{aligned} & (a_2y - s)^2 + \cos(a_2y - s) - 1 - (a_4y - t)^2 \\ & = c_2y - da_4y^2 + s^2 + \cos(-s) - 1 - t^2 \end{aligned} \quad (I)$$

and

$$2(a_2y - s)(a_4y - t) = c_4y + da_2y^2 + 2st. \quad (II)$$

Comparing coefficients of the polynomials in (II) we find

$$\begin{aligned} 2a_2a_4 &= da_2 \\ -2a_2t - 2a_4s &= c_4, \end{aligned}$$

and (I) yields

$$a_2^2y^2 - 2a_2sy + \cos(a_2y - s) - a_4^2y^2 + 2a_4ty = c_2y - da_4y^2 + \cos(s).$$

Comparing coefficients of the bounded parts and in the remaining polynomials we find

$$\begin{aligned} \cos(a_2y - s) &= \cos(s) \\ a_2^2 - a_4^2 &= -da_4 \\ -2a_2s + 2a_4t &= c_2. \end{aligned}$$

The first of these conditions gives $a_2 = 0$. As A is invertible, this implies $a_4 \neq 0$ and then $-2a_4s = c_4$, $d = a_4$, and $c_2 = 2a_4t$.

We use these results and consider the case of general x , but specialize to $y = 0$ now. Comparing coefficients in (*) again, we obtain from

$$\begin{aligned} & (a_1x - s)^2 + \cos(a_1x - s) - 1 - (a_3x - t)^2 \\ & = c_1x + a_4^2x^2 + a_4^2 \cos(x) - a_4^2 + s^2 + \cos(s) - 1 - t^2 \end{aligned}$$

the conditions

$$\begin{aligned} a_1^2 - a_3^2 &= a_4^2 \\ -2a_1s + 2a_3t &= c_1 \\ \cos(a_1x - s) &= a_4^2 \cos(x) - a_4^2 + \cos(s). \end{aligned}$$

Derivation of the first condition yields that $\sin(a_1x - s)$ and $\sin(x)$ have the same sets of zeros: this leads to $a_1 \in \{1, -1\}$ and $s \in \pi\mathbb{Z}$. However, the choice $s \in \pi + 2\pi\mathbb{Z}$ leads to the contradiction $1 = -\cos(s) = -a_4^2$. Thus we have $s \in 2\pi\mathbb{Z}$, and $a_4^2 = 1 = a_1^2$ and $a_3 = 0$ follow. Using these relations and specializing $y = 0$ in (**) we now obtain $c_3 = -2a_1t$.

Collecting our results, we find that $\begin{pmatrix} A & 0 \\ C & -diAi \end{pmatrix} \in \Sigma_0^*$ has to be of the form

$$\begin{aligned} \begin{pmatrix} A & 0 \\ C & -diAi \end{pmatrix} &= \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_4 & 0 & 0 \\ -2a_1s & 2a_4t & 1 & 0 \\ -2a_1t & -2a_4s & 0 & a_1a_4 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -r & 1 & 0 \\ r & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix} \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 4k\pi & 0 & 1 & 0 \\ 0 & 4a_1k\pi & 0 & a_1 \end{pmatrix} \end{aligned}$$

with $a_1, a_4 \in \{1, -1\}$, $k := -a_1 \frac{s}{2\pi} \in \mathbb{Z}$, and $r := -2t \in \mathbb{R}$. These are just the elements of $S\langle\beta\rangle\Sigma_0^*$, as claimed.

It remains to determine the conjugacy classes of involutions in Σ_0^* . An easy computation shows that the involutions in Σ_0^* are of the form

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 4k\pi & -br & 1 & 0 \\ br & -4abk\pi & 0 & ab \end{pmatrix}$$

with $a, b \in \{1, -1\}$, $k \in \mathbb{Z}$, and $r \in \mathbb{R}$ as before, plus the conditions $(a + 1)k = 0$ and $(b + 1)r = 0$.

When choosing representatives for the conjugacy classes of involutions, we may assume $r = 0$ (otherwise we pass to the conjugate under $S_{-r/2}$). Since conjugation by elements of S does not change the coset modulo S (these cosets are just the connected components of Σ_0^*) it suffices to consider conjugacy classes in $\Sigma_0^*/S \cong \Sigma_0 \times \langle\beta\rangle$. This reduces the problem of conjugacy of involutions to our result 11.2.

Applying A_k^\wedge and $\beta \circ A_k^\wedge = A_k^\wedge \circ \beta$ to $L(0) = \beta(L(0))$ we find $c_{A_k^\wedge} = c_{\beta \circ A_k^\wedge} = (2k\pi, 0, -4k^2\pi^2, 0)$. Now the criterion 4.1 yields that $J \circ \sigma_{A_k^\wedge, t}$ is a polarity exactly if $t = (-k\pi, y, z, -2k\pi y)$, and $J \circ \beta \circ \sigma_{A_k^\wedge, t}$ is a polarity exactly if $t = (-k\pi, 0, z, y)$, with $y, z \in \mathbb{R}$, respectively.

The conjugacy classes of polarities are now clear from 6.4. □

11.6 Remarks. The plane \mathcal{P}_{cos} is a 2-dimensional compact projective plane. Its set of polarities splits into three conjugacy classes. The plane $\mathcal{P}_{\text{cos}}^*$ is a 4-dimensional compact projective plane with six conjugacy classes of polarities.

In either one of these cases the number of conjugacy classes of polarities is larger than in the corresponding classical compact projective plane (i.e., the plane over \mathbb{R} and \mathbb{C} , respectively, where one has 2 and 3 classes of polarities, cf. [35] 13.12, 13.18). These results

form a marked contrast to results in [39] concerning the case of 8- and 16-dimensional compact projective planes with large groups of automorphisms, where the number of conjugacy classes of polarities of the non-classical examples is always smaller than in the classical cases (i.e., the planes over \mathbb{H} and \mathbb{O} , respectively).

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