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Nearly flag-transitive affine planes

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Dedicated to Tim Penttila on the occasion of his 50th birthday

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Abstract. Spreads of orthogonal vector spaces are used to construct many translation planes of even order q^m, for odd $m > 1$, having a collineation with a $(q^m - 1)$-cycle on the line at infinity and on each of two affine lines.

Key words. Affine plane, translation plane, spread, flag, symplectic geometry, orthogonal geometry.

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1 Introduction

In [13, 14] we used the relationship between symplectic and orthogonal geometries in characteristic 2 in order to construct new affine planes: flag-transitive planes in [13], and semifield planes in [14]. In this paper we continue those papers by proving the following theorem (where $\rho(m)$ denotes the number of prime factors of m, counting multiplicities, and logarithms are always to the base 2):

Theorem 1.1. *Let $q \geq 4$ be a power of 2, and let $m > 1$ be an odd integer. Then there are more than $q^{3^{\rho(m)-2}}$ pairwise nonisomorphic translation planes of order q^m, with kernel of order q, for which there is a collineation of order $q^m - 1$ having a $(q^m - 1)$-cycle on the line at infinity and on each of two affine lines.*

For better estimates on the number of planes in the theorem, see Theorem 9.2 and Corollary 9.3. These planes are constructed using explicitly defined *prequasifields* (cf. (4.2)). They are *nearly flag-transitive affine planes*: their collineation groups have 2 or 3 flag-orbits. (In [6, p. 794] these are called "triangle-transitive planes", which suggests even more transitivity than we obtain.)

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The proof combines methods in [8] and [18]: these planes arise using symplectic and orthogonal spreads together with changing from fields to proper subfields, keeping track of these field changes using kernels of the associated planes. We settle the isomorphism problem for these planes using an elementary Sylow argument (Proposition 5.2 and Theorem 8.5). A similar argument is used in [13, Proposition 5.1] (and in [9, III.C]) for flag-transitive planes; but no such argument is possible for the semifield planes in [14]. We also relate the present planes to those in [13]:

Theorem 1.2. *There is a natural bijection between the set of isomorphism classes of flag-transitive planes in* [13] *and a subset of the isomorphism classes of nearly flag-transitive planes in Theorem* 1.1*; each plane in* [13] *is a Baer subplane of the corresponding plane in Theorem* 1.1.

Our prequasifields all have the form $(F, +, *)$ with F a finite field and $x * y = yL(xy)$ for an additive map $L\colon F \to F$ such that $x \to xL(x)$ is a permutation of F. Some of these maps L are essentially in [8, II p. 312], and also appear in [1]. In view of Theorem 1.1, one of our goals is to produce large numbers of such maps L.

The planes studied in [7, 8, 10–14] and here are symplectic translation planes. Remarkable results of Maschietti [16] use line ovals to distinguish such planes among all translation planes of characteristic 2.

2 Up and down: from symplectic to orthogonal spreads and back

Throughout this paper, all fields are finite of characteristic 2. We briefly review some of the background required from [3] and [8, 14]. The best background source is probably [11], with the coding-theoretic aspects discarded.

Spreads. Consider a $2m$-dimensional vector space W over $K = \mathrm{GF}(q)$. A *spread* in W is a family $\mathcal{S}$ of $q^m + 1$ subspaces of dimension m that partition the nonzero vectors. The corresponding *affine translation plane* $\mathfrak{A} = \mathfrak{A}(\mathcal{S})$ has as points the vectors of V and as lines all cosets of members of $\mathcal{S}$ [3, pp. 131–133]. The collineations fixing 0 and stabilizing every member of $\mathcal{S}$ generate a field $\mathfrak{K}(\mathfrak{A})$, called the *kernel* of the plane. This is the largest field over which the members of $\mathcal{S}$ can be viewed as subspaces.

The full collineation group of $\mathfrak{A}$ is

$$\mathrm{Aut}\,\mathfrak{A} = W \rtimes (\mathrm{Aut}\,\mathfrak{A})_0 = W \rtimes \Gamma\mathrm{L}(W)_{\mathcal{S}},$$

where $\Gamma\mathrm{L}(W)_{\mathcal{S}} = (\mathrm{Aut}\,\mathfrak{A})_0$ denotes the setwise stabilizer of $\mathcal{S}$, and is a group of semilinear transformations over $\mathfrak{K}(\mathfrak{A})$. More generally, any isomorphism between translation planes sends the kernel of one to the kernel of the other.

One standard way translation planes are constructed is through the use of coordinatizing *quasifields* or *prequasifields* [3, p. 129]. We will see these starting in Section 3.

Symplectic spreads. If there is also a nondegenerate alternating bilinear form (,) on W such that each $X \in \mathcal{S}$ is totally isotropic (i.e., $(X, X) = 0$), then $\mathcal{S}$ is called a *symplectic spread* and $\mathfrak{A}(\mathcal{S})$ is a *symplectic translation plane*.

Orthogonal spreads. Consider a $2m+2$-dimensional vector space V over $K = \mathrm{GF}(q)$, equipped with a nondegenerate quadratic form Q of maximal Witt index. This means that there are $m+1$-spaces X that are *totally singular* (i.e., $Q(X) = 0$). Moreover, V has $(q^{m+1}-1)(q^m+1)$ nonzero singular vectors.

An *orthogonal spread* of V is a family Σ of q^m+1 totally singular $m+1$-spaces that partition the set of nonzero singular vectors. Note that an orthogonal spread is not a spread in the sense of the earlier definition. We recall that there are two types of totally singular $m+1$-spaces such that totally singular $m+1$-spaces X and Y have the same type if and only if $\dim X \cap Y \equiv m+1$ (mod 2) [17, pp. 170–172]. Thus, *if V has an orthogonal spread then m must be odd.*

Down: from orthogonal spreads to symplectic spreads. There is also a symplectic structure on the orthogonal vector space V, determined by the nondegenerate alternating bilinear form $(u,v) := Q(u+v) - Q(u) - Q(v)$. If z is *any* nonsingular point of V, then $z^\perp/z$ inherits the nondegenerate alternating bilinear form $(u+z, v+z) := (u,v)$ for $u, v \in z^\perp$. If $X \in \Sigma$ then $\langle X \cap z^\perp, z\rangle/z$ is an m-dimensional totally isotropic subspace. Moreover,

$$\Sigma/z := \{\langle X \cap z^\perp, z\rangle/z \mid X \in \Sigma\} \tag{2.1}$$

is a symplectic spread of $z^\perp/z$.

Up: from symplectic spreads to orthogonal spreads. This process can be reversed: any symplectic spread $\mathcal{S}$ of $z^\perp/z$ can be lifted to an essentially unique orthogonal spread $\Sigma_{\mathcal{S}}$ of V such that $\Sigma_{\mathcal{S}}/z = \mathcal{S}$ (see [4,5] and [8, I]). We will exhibit such a lifting explicitly in (3.7).

The simplest example of this lifting process was studied at length in [8, I]. It arises from the orthogonal spread Σ that determines the desarguesian plane $\mathfrak{A}(\mathcal{S})$, and hence is called the (orthogonal) *desarguesian spread*.

Down: from symplectic spreads to symplectic spreads. Given a vector space V over a field F, with associated nondegenerate alternating bilinear form $(\ ,\ )$, if K is a subfield of F and $T\colon F \to K$ is the trace map, then $T(\ ,\ )$ defines a nondegenerate alternating K-bilinear form on the K-space V. If $\mathcal{S}$ is a symplectic spread of the F-space V then $\mathcal{S}$ is also a symplectic spread of the K-space V.

Scions. Let $\mathcal{S}$ be a symplectic spread. Suppose that $\mathcal{S}'$ is another symplectic spread arising via a (repeated) up and down process of passing between symplectic and orthogonal geometries, or passing to subfields, as above. Then we call $\mathcal{S}'$ a *scion* of $\mathcal{S}$ [18]. If $\mathcal{S}'$ is a scion of $\mathcal{S}$ then $\mathfrak{A}(\mathcal{S}')$ will be called a *scion* of $\mathfrak{A}(\mathcal{S})$.

Instances of this notion are crucial for Theorem 1.1.

Groups. For a symplectic vector space W, let $\Gamma\mathrm{Sp}(W)$ (or $\Gamma\mathrm{Sp}(W_K)$ if we need to specify the underlying field K) denote the subgroup of $\Gamma\mathrm{L}(W)$ consisting of those $g \in \Gamma\mathrm{L}(W)$ that preserve the symplectic structure, so that $(u^g, v^g) = k(u,v)^\alpha$ for some $k \in K^*$, some $\alpha \in \mathrm{Aut}(K)$ and all $u, v \in W$. Similarly, for an orthogonal vector space V, let $\Gamma\mathrm{O}^+(V)$ (or $\Gamma\mathrm{O}^+(V_K)$) denote the group of all $g \in \Gamma\mathrm{L}(V)$ that preserve the orthogonal structure determined by the quadratic form Q, so that $Q(v^g) = kQ(v)^\alpha$ for some $k \in K^*$, some $\alpha \in \mathrm{Aut}(K)$ and all $v \in V$.

Equivalences. The *automorphism group* of an orthogonal spread Σ is the set-stabilizer $\Gamma\mathrm{O}^+(V)_\Sigma$ of Σ in $\Gamma\mathrm{O}^+(V)$. Two orthogonal spreads are *equivalent* if some element of $\Gamma\mathrm{O}^+(V)$ sends one to the other. According to [8, Theorem 3.5 and Corollary 3.6] (cf. [12]), equivalences among orthogonal spreads are closely related to isomorphisms among the affine planes they spawn:

Theorem 2.2. *Let $\mathcal{S}_i$ be a symplectic spread in the symplectic K-space W_i, $i = 1, 2$, such that $\mathfrak{A}(\mathcal{S}_1)$ and $\mathfrak{A}(\mathcal{S}_2)$ are isomorphic.*

(i) *There is a K-semilinear map $g\colon W_1 \to W_2$ sending $\mathcal{S}_1$ to $\mathcal{S}_2$ and preserving the symplectic structure.*
(ii) *Suppose that there are orthogonal spreads Σ_i in an orthogonal vector space V, and nonsingular points z_i of V, such that $\mathcal{S}_i = \Sigma_i/z_i$ for $i = 1, 2$. Then some element of $\Gamma\mathrm{O}^+(V)$ sends Σ_1 to Σ_2 and z_1 to z_2.*

Moreover, $\mathrm{Aut}\,\mathfrak{A}(\mathcal{S}_1) = W \rtimes \big(\mathfrak{K}(\mathfrak{A}(\mathcal{S}_1))^* \,\Gamma\mathrm{Sp}(W)_{\mathcal{S}_1}\big)$ in (i) [8, Theorem 3.5 and Corollary 3.6].

Methodological remarks. The common thread in [2, 7, 8, 10, 13, 14] and the present paper is the use of scions of desarguesian planes. In those references and here, the specific up and down process employed is designed to preserve subgroups of $\mathrm{SL}(2, q^m)$ that fix nonsingular points in the $2m + 2$-dimensional orthogonal space obtained by lifting the desarguesian spread. A start in this direction can be seen in [8], using subgroups of order $q^m + 1$, q^m or $q^m - 1$ in order to obtain flag-transitive, semifield and nearly flag-transitive planes, respectively. These groups also appear in [13], in [14] and here, but using arbitrarily long chains of subfields and hence of up and down moves. Moreover, up to conjugacy (the normalizers of) these groups are precisely the stabilizers in $\mathrm{SL}(2, q^m)$ of nonsingular points in the space underlying the (orthogonal) desarguesian spread; no further planes can be obtained in this up and down manner that are preserved by other subgroups of $\mathrm{SL}(2, q^m)$.

Slight variations on these constructions are used in [7, 14] to obtain affine planes whose full collineation groups are unusually small.

At the moment, *every* known symplectic plane in characteristic 2 having odd dimension over its kernel is a scion of a desarguesian plane. There must be many others, but we have no idea where to look for them.

3 Prequasifields

Consider finite fields $F \supseteq K$ of characteristic 2, with corresponding trace map $T\colon F \to K$, where $[F\colon K]$ is odd.

Hypothesis 3.1. $\mathfrak{P}_* = (F, *, +)$ *satisfies the following for all* $x, y, z \in F$*, all* $k \in K$*, and some* $l \in F^*$*.*

(i) $(x + y) * z = x * z + y * z$,
(ii) $xy = 0 \Rightarrow x * y = 0$,
(iii) $x * y = x * z \Rightarrow x(y + z) = 0$,
(iv) $k(x * y) = (kx) * y$,
(v) $T(x(x * y)) = T(lxy)^2$, *and*
(vi) $z(x * y) = (z^{-1}x) * (zy)$ *if* $z \neq 0$.

Here (i)–(iii) are precisely the definition of a *prequasifield*. (This is not a quasifield since it does not necessarily have an identity element.) The associated spread

$$\mathcal{S}_* := \{\mathcal{S}_*[s] \mid s \in F \cup \infty\}$$

of the K-vector space F^2 consists of the following K-subspaces (cf. (iv)):

$$\mathcal{S}_*[s] := \{(x, x * s) \mid x \in F\}, s \in F, \quad \text{and} \quad \mathcal{S}_*[\infty] := \{(0, y) \mid y \in F\}. \tag{3.2}$$

The kernel $\mathfrak{K}(\mathfrak{A}(\mathcal{S}))$ contains K since $\mathcal{S}$ consists of K-subspaces.

Conditions (v) and (vi) are of special interest. Namely, (v) implies that $\mathcal{S}_*$ is a symplectic spread with respect to the alternating K-bilinear form

$$\big((x, y), (v, w)\big) := T(xw + yv), \tag{3.3}$$

while (vi) produces collineations of the affine plane $\mathfrak{A}(\mathcal{S}_*)$, and automorphisms of the orthogonal spread Σ_*, discussed in the next proposition.

Remark 3.4. A slight modification of the multiplication $*$ would allow us to assume that $l = 1$ (namely, use $x *' y := x * (ly)$). Instead we will choose l so that the formula for the binary operation $*$ is as nice as possible.

Equip the K-vector space $V := F \oplus K \oplus F \oplus K$ with the nondegenerate quadratic form $Q(x, a, y, b) := T(xy) + ab$. We can identify the symplectic K-spaces $\langle 0, 1, 0, 1\rangle^\perp / \langle 0, 1, 0, 1\rangle$ and F^2. Define $\phi_\zeta \colon F^2 \to F^2$ and $\varphi_\zeta \colon V \to V$, $\zeta \in F^*$, by

$$(x, y)^{\phi_\zeta} = (\zeta^{-1}x, \zeta y) \quad \text{and} \quad (x, a, y, b)^{\varphi_\zeta} = (\zeta^{-1}x, a, \zeta y, b), \quad \text{respectively.} \tag{3.5}$$

Proposition 3.6. *Suppose that* $\mathfrak{P}_*(F, *, +)$ *satisfies Hypothesis* 3.1.

(i) *The symplectic spread* $\mathcal{S}_*$ *is invariant under the cyclic group* $G := \{\phi_\zeta \mid \zeta \in F^*\}$ *of symplectic isometries of* F^2*. Moreover,* G *induces a group of collineations of the affine plane* $\mathfrak{A}(\mathcal{S}_*)$ *that regularly permutes* $q^m - 1$ *points at infinity, fixes the remaining two points, and regularly permutes* $q^m - 1$ *points of each of the two lines joining either of these two points to* 0*.*
(ii) $\mathcal{S}_*$ *lifts to an orthogonal spread* $\Sigma_* = \{\Sigma_*[s] \mid s \in F \cup \{\infty\}\}$ *of* V*, where*

$$\begin{aligned} \Sigma_*[\infty] &= 0 \oplus 0 \oplus F \oplus K \\ \Sigma_*[s] &= \big\{\big(x, a, x * s + ls(a + T(lxs)), T(lxs)\big) \mid x \in F, a \in K\big\}, s \in F. \end{aligned} \tag{3.7}$$

Here $\Sigma_*/\langle 0,1,0,1\rangle$ *is* $\mathcal{S}_*$. *The group* $\hat{G} := \{\varphi_\zeta \mid \zeta \in F^*\}$ *consists of isometries of* V*, and acts on* Σ_* *by stabilizing* $\Sigma_*[0]$ *and* $\Sigma_*[\infty]$ *and regularly permuting the remaining members of* Σ_*.

(iii) $C_V(\hat{G}) = \{(0,a,0,b) \mid a,b \in K\}$*, and the nonsingular points fixed by* $\hat{G}$ *are* $\langle 0,\lambda,0,1\rangle$, $\lambda \in K^*$.

(iv) *For each* $\lambda \in K^*$*, the symplectic spread* $\Sigma_*/\langle 0,\lambda^2,0,1\rangle$ *is coordinatized by a prequasifield* $\mathfrak{P}_\circ = (F,\circ,+)$*, defined by*

$$x \circ y := x * y + lyT(lxy) + l\lambda yT(l\lambda xy) = x * y + ly(1+\lambda^2)T(lxy),$$

that satisfies Hypothesis 3.1 *with* $l\lambda$ *in place of* l.

Proof. These are straightforward calculations, most of which are given in [14, Theorem 2.18]. Hypothesis 3.1(v) is proved as follows:

$$\begin{aligned} z(x \circ y) &= z(x * y) + (1+\lambda^2)lzyT(lxy) \\ &= (z^{-1}x) * (zy) + (1+\lambda^2)l(zy)T\big(l(z^{-1}x)(zy)\big) = (z^{-1}x) \circ (zy). \end{aligned}$$

For (iv), note that

$$\begin{aligned} &\big(x, \lambda^2 T(lxs), x * s + ls(\lambda^2 T(lxs) + T(lxs)), T(lxs)\big) \\ &\qquad = \big(x, 0, x * s + ls(1+\lambda^2)T(lxs)), 0)\big) + T(lxs)(0,\lambda^2,0,1\big). \qquad \square \end{aligned}$$

We will need some elementary properties of trace functions.

Lemma 3.8. *Suppose that* $F \supseteq F' \supseteq K$ *are fields with* $[F\colon K]$ *odd and corresponding trace maps* $T\colon F \to K$ *and* $T'\colon F \to F'$. *If* $x \in F$ *and* $u \in F'$*, then*

(i) $TT'(x) = T(x)$,
(ii) $T(ux) = T(uT'(x))$,
(iii) $T\big(uxT'(x)\big) = T(ux^2)$*, and*
(iv) $T'(u) = u$ *and* $T(1) = 1$.

Proof. See [14, Lemma 2.14] for assertions (i), (ii), (iv). For (iii), use (i): $T\big(uxT'(x)\big) = TT'\big(uxT'(x)\big) = T\big(uT'\big(xT'(x)\big)\big) = T\big(uT'(x)T'(x)\big) = TT'(ux^2) = T(ux^2)$. $\square$

4 Nearly flag-transitive planes

Nearly flag-transitive planes were defined in Section 1. We now give examples that are scions of desarguesian planes (cf. Section 2).

Let $F = F_0 \supset \cdots \supset F_n$ be a chain of fields, with $[F\colon F_n]$ odd and corresponding trace maps $T_i\colon F \to F_i$. If $1 \le i \le n$ let V_i be the F_i-vector space $F \oplus F_i \oplus F \oplus F_i$, equipped with the quadratic form $Q_i(x,a,y,b) := T_i(xy) + ab$.

Proposition 4.1. *For* $1 \le i \le n$ *let* $\lambda_i \in F_i^*$*; set* $\lambda_0 = 1$ *and* $c_i := \prod_{j=0}^{i} \lambda_j$ *whenever* $0 \le i \le n$*. If*

$$x * y := xy^2 + \sum_{i=1}^{n} \big[c_{i-1} y T_i(c_{i-1}xy) + c_i y T_i(c_i xy)\big], \tag{4.2}$$

then $\mathfrak{P}_*(F, *, +)$ *satisfies Hypothesis* 3.1 *with* $l = c_n$*. Furthermore,* $\mathfrak{A}(\mathcal{S}_*)$ *is a symplectic nearly flag-transitive scion of a desarguesian plane.*

Thus, c_i is any element of F_1^* such that $c_i/c_{i-1} = \lambda_i \in F_i$ when $i \ge 1$; but (4.2) makes it clear that we will want to require that $c_{i-1} \ne c_i$. We begin with some observations concerning the sum in (4.2):

Lemma 4.3. (i) $x * y = y\big(xy + f_1(xy)\big)$*, where* $f_1(u) := \sum_{i=1}^{n} (1 + \lambda_i^2) c_{i-1} T_i(c_{i-1}u)$ *is in* F_1.

(ii) $T_j\Big(x^2 + \sum_{i=1}^{j} \big[c_{i-1} x T_i(c_{i-1}x) + c_i x T_i(c_i x)\big]\Big) = T_j(c_j x)^2$ *whenever* $x \in F$ *and* $1 \le j \le n$.

(iii) *If* $j < i \le n$ *and* $x, y \in F$*, then* $T_j(c_{i-1}x)T_i(c_{i-1}y) + T_j(c_i x)T_i(c_i y) = [c_{i-1}/c_j](1 + \lambda_i^2) T_j(c_j x) T_i(c_{i-1}y)$.

Proof. (i) $x * y = xy^2 + y \sum_{i=1}^{n} \big(c_{i-1} T_i(c_{i-1}xy) + \lambda_i c_{i-1} T_i(\lambda_i c_{i-1} xy)\big)$, $\lambda_i \in F_i^*$.

(ii) Use $x^2 = c_0 x T_0(c_0 x)$ and Lemma 3.8(iii):

$$\begin{aligned}
T_j\Big(x^2 &+ \sum_{i=1}^{j} \big[c_{i-1} x T_i(c_{i-1}x) + c_i x T_i(c_i x)\big]\Big) \\
&= \sum_{i=0}^{j-1} \big[T_j(c_i x T_{i+1}(c_i x)) + T_j(c_i x T_i(c_i x))\big] + T_j\big(c_j x T_j(c_j x)\big) \\
&= \sum_{i=0}^{j-1} \big[T_j\big((c_i x)^2\big) + T_j\big((c_i x)^2\big)\big] + T_j\big((c_j x)^2\big) = T_j(c_j x)^2.
\end{aligned}$$

(iii) Since $c_i = \lambda_i c_{i-1}$ and $c_{i-1}/c_j = \prod_{l=j+1}^{i-1} \lambda_l \in F_{j+1} \subset F_j$,

$$\begin{aligned}
T_j(c_{i-1}x)T_i(c_{i-1}y) + T_j(c_i x)T_i(c_i y) &= (1 + \lambda_i^2) T_j(c_{i-1}x) T_i(c_{i-1}y) \\
&= [c_{i-1}/c_j](1 + \lambda_i^2) T_j(c_j x) T_i(c_{i-1}y). \qquad \square
\end{aligned}$$

We now give two proofs that $\mathfrak{P}_*$ satisfies Hypothesis 3.1.

First proof of Proposition 4.1 (*geometric*). We apply the up and down process along the chain $(F_i)_0^n$ of fields, beginning with the desarguesian spread. Define $\mathfrak{P}_{*_j} = (F, *_j, +)$ by

$$x *_j y = xy^2 + \sum_{i=1}^{j} \big[c_{i-1} y T_i(c_{i-1}xy) + c_i y T_i(c_i xy)\big].$$

Clearly, $x *_0 y = xy^2$ coordinatizes the desarguesian plane. Since $x(x *_0 y) = (xy)^2$, $\mathfrak{P}_{*_0}$ satisfies Hypothesis 3.1, with $F = K$, $T = 1$ and $l = c_0 = 1$.

Suppose that $0 \le j-1 \le n-1$ and $\mathfrak{P}_{*_{j-1}} = (F, *_{j-1}, +)$ satisfies Hypothesis 3.1 with $K = F_{j-1}$, $T = T_{j-1}$ and $l = c_{j-1}$. Then $\mathfrak{P}_{*_{j-1}}$ also satisfies Hypothesis 3.1 with $K = F_j$, $T = T_j$ and $l = c_{j-1}$, since $T_j T_{j-1} = T_j$ by Lemma 3.8(i). In the preceding section we saw that this implies that $\mathfrak{P}_{*_{j-1}}$ defines a symplectic spread $S_{*_{j-1}}$ in F^2, and hence also an orthogonal spread $\Sigma_{*_{j-1}}$ in the orthogonal space $V_{j-1} = F \oplus F_j \oplus F \oplus F_j$ admitting the group $\hat{G}$ in Proposition 3.6(ii). By Proposition 3.6(iv), $\Sigma_{*_j}/\langle 0, \lambda_j^2, 0, 1\rangle$ is (equivalent to) the symplectic spread of the F_j-vector space F^2 coordinatized by $\mathfrak{P}_{*_j}$, and $\mathfrak{P}_{*_j}$ satisfies Hypothesis 3.1 with $K = F_j$, $T = T_j$ and $l = \lambda_j c_{j-1} = c_j$.

Hence, the desired result holds by induction. □

Second proof of Proposition 4.1 (*algebraic*). For completeness, as in [14, p. 908] we give a direct proof that $\mathfrak{P}_*$ satisfies Hypothesis 3.1. Parts (i), (ii), (iv) and (vi) are straightforward calculations, and (v) holds by Lemma 4.3(ii) with $l = c_j$, so we focus on part (iii): we assume that $x * y_1 = x * y_2$, and deduce that $x(y_1 + y_2) = 0$. Let $z = xy_1$ and $w = xy_2$. By (4.2), $x(x * y_1) = x(x * y_2)$ becomes

$$z^2 + \sum_{i=1}^{n} \big[c_{i-1} z T_i(c_{i-1} z) + c_i z T_i(c_i z)\big] = w^2 + \sum_{i=1}^{n} \big[c_{i-1} w T_i(c_{i-1} w) + c_i w T_i(c_i w)\big]. \quad (4.4)$$

We use backwards induction to show that

$$T_j(c_j z) = T_j(c_j w) \quad \text{whenever } 0 \le j \le n. \quad (4.5)$$

Applying T_n to (4.4), by Lemma 4.3(ii) we obtain $T_n(c_n z)^2 = T_n(c_n w)^2$, so that (4.5) holds when $j = n$.

Now suppose that, for some j such that $0 \le j < n$, whenever $j < l \le n$ we have $T_l(c_l z) = T_l(c_l w)$ (and then also $T_l(c_{l-1} z) = T_l(c_{l-1} w)$ since $c_l = \lambda_l c_{l-1}$ with $\lambda_l \in F_l^*$). *We must show that* $T_j(c_j z) = T_j(c_j w)$. By our induction hypothesis, (4.4) states that

$$\Big\{z^2 + \sum_{i=1}^{j}\big[c_{i-1} z T_i(c_{i-1} z) + c_i z T_i(c_i z)\big]\Big\} + \sum_{i=j+1}^{n}\big[c_{i-1} z T_i(c_{i-1} z) + c_i z T_i(c_i z)\big]$$
$$= \Big\{w^2 + \sum_{i=1}^{j}\big[c_{i-1} w T_i(c_{i-1} w) + c_i w T_i(c_i w)\big]\Big\} + \sum_{i=j+1}^{n}\big[c_{i-1} w T_i(c_{i-1} w) + c_i w T_i(c_i w)\big].$$

Apply T_j, using Lemma 4.3(ii) and the fact that $T_i(c_{i-1}z), T_i(c_iz) \in F_j$ for $i \geq j+1$:

$$T_j(c_jz)^2 + \sum_{i=j+1}^{n} \big[T_j(c_{i-1}z)T_i(c_{i-1}z) + T_j(c_iz)T_i(c_iz)\big]$$
$$= T_j\big(c_jw\big)^2 + \sum_{i=j+1}^{n} \big[T_j(c_{i-1}w)T_i(c_{i-1}w) + T_j(c_iw)T_i(c_iw)\big].$$

By Lemma 4.3(iii),

$$T_j(c_jz)^2 + \sum_{i=j+1}^{n} [c_{i-1}/c_j](1+\lambda_i^2)T_j(c_jz)T_i(c_{i-1}z)$$
$$= T_j(c_jw)^2 + \sum_{i=j+1}^{n} [c_{i-1}/c_j](1+\lambda_i^2)T_j(c_jw)T_i(c_{i-1}w).$$

By induction, $T_i(\lambda_i c_{i-1}z) = T_i(\lambda_i c_{i-1}w)$ if $i \geq j+1$, so that

$$\big[T_j(c_jz) + T_j(c_jw)\big]^2 = \big[T_j(c_jz) + T_j(c_jw)\big] \sum_{i=j+1}^{n} [c_{i-1}/c_j](1+\lambda_i^2)T_i(c_{i-1}z).$$

If $T_j(c_jz) \neq T_j(c_jw)$, then

$$T_j(c_jz) + T_j(c_jw) = \sum_{i=j+1}^{n} [c_{i-1}/c_j](1+\lambda_i^2)T_i(c_{i-1}z) \in F_{j+1}$$

since $c_{i-1}/c_j, \lambda_i \in F_{j+1}$ for $i \geq j+1$. Since $T_{j+1}(\lambda_{j+1}c_jz) = T_{j+1}(\lambda_{j+1}c_jw)$ by our inductive hypothesis, from Lemma 3.8(i),(iv) we obtain

$$T_j(c_jz) + T_j(c_jw) = T_{j+1}\big(T_j(c_jz) + T_j(c_jw)\big) = T_{j+1}(c_jz) + T_{j+1}(c_jw) = 0,$$

which contradicts the fact that $T_j(c_jz) \neq T_j(c_jw)$.

By induction, this proves (4.5). In particular, $xy_1 = z = T_0(c_0z) = T_0(c_0w) = w = xy_2$, as required. □

Definition 4.6. Let $(F_i)_0^n$ be a chain of distinct fields with $[F_0\colon F_n]$ odd, and let $(\lambda_i)_0^n$ be a sequence of elements with $\lambda_0 = 1$ and $\lambda_i \in F_i^*$, $1 \leq i \leq n$. Then we call $\big((F_i)_0^n, (\lambda_i)_0^n\big)$ a *defining pair* for various objects obtained *along the chain* $(F_i)_0^n$:

- the prequasifield $\mathfrak{P}_*\big((F_i)_0^n, (\lambda_i)_0^n\big)$ in (4.2),
- the symplectic spread $\mathcal{S}_*\big((F_i)_0^n, (\lambda_i)_0^n\big)$ in (3.2), and
- the orthogonal spread $\Sigma_*\big((F_i)_0^n, (\lambda_i)_0^n\big)$ in (3.7).

It is a *reduced* defining pair if $\lambda_i \neq 1$ for all $i \geq 1$.

Each defining pair $\big((F_i)_0^n, (\lambda_i)_0^n\big)$ determines a reduced defining pair $\big((F_i')_0^{n'}, (\lambda_i')_0^{n'}\big)$ obtained by deleting all entries F_j and λ_j with $j \geq 1$ and $\lambda_j = 1$. Then $\big((F_i)_0^n, (\lambda_i)_0^n\big)$

and $\big((F_i')_0^{n'}, (\lambda_i')_0^{n'}\big)$ determine the *same* prequasifield by (4.2) or Lemma 4.3(i). Hence, we will only consider reduced defining pairs. Since $\lambda_n \neq 1$ for a reduced pair, we have $|F_n| > 2$: *each of our nearly flag-transitive scions of the desarguesian plane has kernel properly containing* $\mathrm{GF}(2)$.

When $n = 1$, a reduced pair is $(F, 1)$, and produces the desarguesian plane by (4.2).

In Lemma 4.3(i) we introduced the function $f_1(x)$. Later we will need a slightly more general function:

Lemma 4.7. *Let $\big((F_i)_0^n, (\lambda_i)_0^n\big)$ be a reduced defining pair. For $1 \le j \le n$ and for all $x \in F$ set*

$$f_j(x) := (1+\lambda_j^2)T_j(c_{j-1}x) + \lambda_j \sum_{i=j+1}^{n} [c_{i-1}/c_j](1+\lambda_i^2)T_i\big(c_{i-1}x\big).$$

Then F_j is the subfield of F generated by $f_j(F)$.

Proof. Since $\lambda_j \neq 1$, $x \to (1+\lambda_j^2)T_j(c_{j-1}x)$ maps onto F_j. This proves the lemma if $j = n$. If $j < n$ then $\sum_{i=j+1}^{n}[c_{i-1}/c_j](1+\lambda_i^2)T_i\big(c_{i-1}x\big) \in F_{j+1}$ since c_{i-1}/c_j, $\lambda_i \in F_{j+1}$. Thus, f_j maps onto $F_j\big/(\lambda_j F_{j+1})$. Since $[F_j : F_{j+1}] \ge 3$, this implies that the subfield of F_j generated by $f_j(F)$ has size $> |F_j|^{1/2}$ and hence is F_j. □

5 Isomorphisms between nearly flag-transitive planes

We digress for a general though elementary observation concerning symplectic nearly flag-transitive planes that permits us to decrease the amount of calculation used in the proof of Theorem 1.1 (cf. [9, III.C]). Let F be a finite field of characteristic 2, let K, T, F^2 and G be as in Section 3 (but here we can allow m to be even), and consider the G-invariant nondegenerate alternating K-bilinear form (3.3) on the K-space F^2. Then G leaves invariant the totally isotropic subspaces $X := F \oplus 0$ and $Y := 0 \oplus F$.

Assume that we do not have both $|F| = 64$ and $|K| = 2$. By Zsigmondy's Theorem [19], there is a Sylow p-subgroup P of G that acts irreducibly on both X and Y. By checking orders we find that P is a Sylow subgroup of $\Gamma\mathrm{Sp}(F^2, K)$.

Lemma 5.1. (i) $C_{\Gamma\mathrm{L}(F^2,K)}(P) = \{(x,y) \to (ax, by) \mid a, b \in F^*\}$, *and*
(ii) $N_{\Gamma\mathrm{Sp}(F^2,K)}(P) = \{(x,y) \to (kax^\alpha, ka^{-1}y^\alpha) \mid k \in K^*, a \in F^*, \alpha \in \mathrm{Aut}(F)\}\langle\theta\rangle$ *where $\theta\colon (x,y) \to (y,x)$.*

Proof. By Schur's Lemma, $C_{\Gamma\mathrm{L}(F^2,K)}(P|_X)$ is the multiplicative group of a field of size $|X|$, and hence is isomorphic to F^*. The same holds for $C_{\Gamma\mathrm{L}(F^2,K)}(P|_Y)$. No semilinear transformation can centralize P and interchange X and Y (note that θ inverts P). This proves the first assertion.

For the second one, note that all scalars are in $C_{\Gamma\mathrm{Sp}(F^2,K)}(P)$, so that we can view $C_{\Gamma\mathrm{Sp}(F^2,K)}(P)$ as $G \times K^*$. All field automorphisms are present in $N_{\Gamma\mathrm{Sp}(F^2,K)}(P)$. Any $g \in N_{\Gamma\mathrm{Sp}(F^2,K)}(P)$ acts on the pair $\{X, Y\}$ and hence has the form $(x,y) \to (ax^\alpha, by^\alpha)$

or $(x, y) \to (ay^\alpha, bx^\alpha)$ for some a, b, α. By (3.3), $T(abz^\alpha) = kT(z)^\beta$ for some $\beta \in \mathrm{Aut}(F)$, $k \in K^*$, and all $z \in F$. Thus, $ab = k$, as asserted. □

The following consequence is analogous to [13, Proposition 5.1]:

Proposition 5.2. *Let $\mathcal{S}$ and $\mathcal{S}'$ be G-invariant symplectic spreads in the K-space F^2, both containing the G-invariant subspaces $F \oplus 0$ and $0 \oplus F$, such that G is transitive on the remaining members of both $\mathcal{S}$ and $\mathcal{S}'$. If $\mathfrak{A}(\mathcal{S})$ and $\mathfrak{A}(\mathcal{S}')$ are isomorphic, then $\mathcal{S}' = \mathcal{S}^\alpha$ or $\mathcal{S}^{\theta\alpha}$ for some $\alpha \in \mathrm{Aut}(F)$, where θ is as above.*

Proof. By Theorem 2.2(i), we may assume that the given isomorphism is induced by a symplectic transformation f of the K-space F^2. Let P be the Sylow subgroup of G used above. Then P and P^f are Sylow subgroups of $(\mathrm{Aut}\,\mathfrak{A}(\mathcal{S}')) \cap \mathrm{Sp}(F^2, K)$, so that $P^{fh} = P$ for some $h \in (\mathrm{Aut}\,\mathfrak{A}(\mathcal{S}')) \cap \mathrm{Sp}(F^2, K)$. Now $fh \in \Gamma\mathrm{Sp}(F^2, K)$ is an isomorphism $\mathfrak{A}(\mathcal{S}) \to \mathfrak{A}(\mathcal{S}')$ that normalizes P. Since K^* and G leave both spreads invariant, the second part of the preceding lemma concludes the proof. □

6 Kernels

Our first use of the preceding section is to calculate the kernels of our nearly flag-transitive scions of desarguesian planes. See [13, Section 6] for an argument based on the same idea.

Theorem 6.1. *If $\big((F_i)_0^n, (\lambda_i)_0^n\big)$ is a reduced defining pair, then F_n is the kernel of the associated plane $\mathfrak{A}\big((F_i)_0^n, (\lambda_i)_0^n\big)$.*

Proof. By (4.2), the kernel $\mathfrak{K}$ of $\mathfrak{A}\big((F_i)_0^n, (\lambda_i)_0^n\big)$ contains F_n. Since $|F_n| > 2$, we can apply Section 5 (with $K = F_n$). Let P be as in that section. Then P normalizes $\mathfrak{K}$, hence induces semilinear transformations on the $\mathfrak{K}$-space F^2, and hence centralizes $\mathfrak{K}^*$ in view of $|P|$. By Lemma 5.1(i), each element of $\mathfrak{K}^*$ has the form $h\colon (x, y) \to (ax, by)$ for some $a, b \in F^*$. We must show that $a = b \in F_n$.

In the notation of (3.2), $\mathcal{S}_*[s]^g = \mathcal{S}_*[s]$ for each s. By Lemma 4.3(i),

$$(ax)s^2 + sf_1(axs) = (ax) * s = b(x * s) = b\big(xs^2 + sf_1(xs)\big)$$

for all $x \in F, s \in F^*$. Then $(a - b)xs = -f_1(axs) + bf_1(xs)$. Vary x in order to see that the left side produces either 0 or F. However, the right side lies in the 2-dimensional F_1-subspace $F_1 + bF_1$ of F, which is not all of F since $[F\colon F_1] \geq 3$. Hence, $(a - b)xs$ must be 0, so that $a = b$.

Thus, $f_1(ax) = af_1(x)$ for all $x \in F$.

Suppose that $a \notin F_n$, and choose $j \leq n$ such that $a \in F_{j-1} - F_j$. By Lemma 4.3(i),

$$\sum_{i=1}^{n} (1 + \lambda_i^2)c_{i-1}T_i(c_{i-1}ax) = a \sum_{i=1}^{n} (1 + \lambda_i^2)c_{i-1}T_i(c_{i-1}x)$$

for all $x \in F$, and hence $f_j(ax) = af_j(x)$ in the notation of Lemma 4.7. By that lemma, $f_j(F)$ generates F_j, so that $f_j(x) \neq 0$ and hence $a = f_j(ax)/f_j(x) \in F_j$ for some x.

This contradiction proves that $a = b \in F_n$, as required. □

7 Interchanging X and Y

In the next section we will determine all isomorphisms among the nearly flag-transitive planes $\mathfrak{A}\big((F_i)_0^n,(\lambda_i)_0^n\big)$. We first need to see what happens when X and Y are interchanged using the symplectic map $\theta\colon (x,y)\to(y,x)$.

Theorem 7.1. *θ is an isomorphism $\mathfrak{A}\big((F_i)_0^n,(\lambda_i)_0^n\big)\to\mathfrak{A}\big((F_i)_0^n,(\lambda_i^{-1})_0^n\big)$ for any reduced defining pair $\big((F_i)_0^n,(\lambda_i)_0^n\big)$.*

We will prove this by induction on n, using the following inductive step:

Lemma 7.2. *In Proposition* 3.6*, assume that θ sends $\mathcal{S}_*$ to $\mathcal{S}_{*'}$, where $(F,+,*')$ satisfies Hypothesis* 3.1(i)–(iv),(vi) *and $T(x(x*'y))=T(l^{-1}xy)$ for all $x,y\in F$. Define $x\circ y$ as in Proposition* 3.6(iv)*, so that $\mathcal{S}_\circ=\Sigma_*/\langle 0,\lambda^2,0,1\rangle$; and similarly define*

$$x\circ' y:=x*'y+(1+\lambda^{-2})l^{-1}yT(l^{-1}xy),$$

so that $\mathcal{S}_{\circ'}=\Sigma_{*'}/\langle 0,\lambda^{-2},0,1\rangle$. Then $\mathcal{S}_\circ^\theta=\mathcal{S}_{\circ'}$.

Moreover, if θ sends the line $y=x*s$ to $y=x*s'$ for a permutation $s\to s'$ of F^*, then it also sends $y=x\circ s$ to $y=x\circ' s'$.

Proof. By Proposition 3.6(ii), $\mathcal{S}_*$ and $\mathcal{S}_{*'}$ lift to unique orthogonal spreads Σ_* and $\Sigma_{*'}$ (respectively) of $V=F\oplus K\oplus F\oplus K$ containing $\widehat{X}:=F\oplus K\oplus 0\oplus 0$ and $\widehat{Y}:=0\oplus 0\oplus F\oplus K$. For the present proof it is convenient to write $\widehat{\mathcal{S}_*}:=\Sigma_*$ and $\widehat{\mathcal{S}_{*'}}:=\Sigma_{*'}$.

Note that θ lifts to a unique orthogonal map $\hat\theta\colon (x,a,y,b)\to(y,b,x,a)$ sending $\widehat{X}\leftrightarrow\widehat{Y}$ and fixing $\langle 0,1,0,1\rangle$. For, θ lifts to a unique isometry of $\langle 0,1,0,1\rangle^\perp$ fixing $\langle 0,1,0,1\rangle$, and hence to a unique isometry of V sending $\widehat{X}$ to the subspace $\widehat{Y}$ of the same type.

Each $Z\in\mathcal{S}_*$ lifts to the subspace $Z\hat{}$ of singular points of the hyperplane of $\langle 0,1,0,1\rangle^\perp$ that contains $\langle 0,1,0,1\rangle$ and projects onto Z in $\langle 0,1,0,1\rangle^\perp/\langle 0,1,0,1\rangle$; and then lifts further to the unique totally singular subspace $\widehat{Z}$ of V that contains $Z\hat{}$ and has the same type as $\widehat{X}$ and $\widehat{Y}$.

Similarly, $Z^\theta\in\mathcal{S}_*^\theta$ lifts to $(Z^\theta)\hat{}$ and then to $\widehat{Z^\theta}$. Since θ sends Z to Z^θ, $\hat\theta$ sends lifts to lifts: $(\widehat{Z})^{\hat\theta}=\widehat{Z^\theta}$. Consequently, by hypothesis, $\widehat{\mathcal{S}_*}^{\hat\theta}=\widehat{\mathcal{S}_*^\theta}=\widehat{\mathcal{S}_{*'}}$.

Since $\lambda\in K$, $\hat\theta$ sends $\langle 0,\lambda^2,0,1\rangle$ to $\langle 0,1,0,\lambda^2\rangle=\langle 0,\lambda^{-2},0,1\rangle$ and $\langle 0,\lambda^2,0,1\rangle^\perp$ to $\langle 0,\lambda^{-2},0,1\rangle^\perp$, and hence also $\widehat{\mathcal{S}_*}/\langle 0,1,0,\lambda^2\rangle=\mathcal{S}_\circ$ to $\widehat{\mathcal{S}_{*'}}/\langle 0,\lambda^{-2},0,1\rangle=\mathcal{S}_{\circ'}$, using the multiplications $\circ$ and $\circ'$ of the prequasifields in Proposition 3.6(iv) and the present lemma. This proves the first assertion of the lemma.

If $s\in F^*$ and $Z=\mathcal{S}_*[s]$ as in (3.2), then $Z^\theta=\mathcal{S}_{*'}[s']$ by the definition of s', so that $\hat{Z}^{\hat\theta}=\widehat{Z^\theta}=\Sigma_{*'}[s']$. Then $\big(\hat{Z}^{\hat\theta}\cap\langle 0,\lambda^{-2},0,1\rangle^\perp\big)/\langle 0,\lambda^{-2},0,1\rangle=\Sigma_{\circ'}[s']$, which is also $\big[\big(\hat{Z}\cap\langle 0,\lambda^2,0,1\rangle^\perp\big)/\langle 0,\lambda^2,0,1\rangle\big]^{\hat\theta}=\Sigma_\circ[s]^{\hat\theta}$.

If $x\in F$ and $b\in K$, then

$$\begin{aligned}(x,\lambda^2 b,y,b)&=(x,0,y,0)+b(0,\lambda^2,0,1)\\(y,b,x,\lambda^2 b)&=(y,0,x,0)+\lambda^2 b(0,\lambda^{-2},0,1).\end{aligned}$$

Thus, the map $\langle 0,\lambda^2,0,1\rangle^\perp/\langle 0,\lambda^2,0,1\rangle \to \langle 0,\lambda^{-2},0,1\rangle^\perp/\langle 0,\lambda^{-2},0,1\rangle$ induced by $\hat\theta$ behaves like θ, and sends $\Sigma_\circ[s]$ to $\Sigma_{\circ'}[s']$, as required. □

Proof of Theorem 7.1. We use induction. When $n=0$, $x*y=xy^2=x*'y$, $\lambda=1=\lambda^{-1}$, and θ is a collineation of the corresponding desarguesian plane interchanging the lines $y=xs$ and $y=xs^{-1}$ for $s\neq 0$.

Suppose that the theorem holds for some $n-1\geq 0$. By the lemma (with $l=c_{n-1}$ and $\lambda=\lambda_n$ as in the first proof of Proposition 4.1), since $\mathcal{S}_*^\theta=\mathcal{S}_{*'}$ we also have $\mathcal{S}_\circ^\theta=\mathcal{S}_{\circ'}$. The element "$l$" for $*'$ is c_{n-1}^{-1} by our inductive hypothesis. The definition of $\circ'$ shows that the element "λ" for $\circ'$ in Proposition 3.6(iv) is λ_n^{-1}. Hence, that proposition implies that "l" for $\circ'$ is $\lambda_n^{-1}c_{n-1}^{-1}=c_n^{-1}$, so that $\mathcal{S}_{\circ'}$ is the spread for $\mathfrak{A}\big((F_i)_0^n,(\lambda_i^{-1})_0^n\big)$. □

Remark 7.3. We conclude with computational remarks concerning the preceding results that are not needed for Theorem 1.1.

(1) As in Hypothesis 3.1(vi), we have $z(x*'y)=(z^{-1}x)*'(zy)$ if $z\neq 0$. For, $(x,y)\to(z^{-1}x,zy)$ is a collineation of the plane determined by $*$ and hence also of the plane determined by $*'$.

(2) Since $y=x*s$ implies that $x=y*'s'$, for all $x\in F$ we have

$$x=(x*s)*'s'.$$

(3) There is an element $c\in F^*$ such that $s'=cs^{-1}$. For, by (1), (2) and Hypothesis 3.1(v), if $t\neq 0$ then

$$\begin{aligned}\{(t^{-1}x)*(ts)\}*'(ts)'&=t^{-1}x\\&=t^{-1}\{(x*s)*'s'\}\\&=\{t(x*s)\}*'(t^{-1}s')\\&=\{(t^{-1}x)*(ts)\}*'(t^{-1}s').\end{aligned}$$

Thus, $(ts)'=t^{-1}s'$. Use $s=1$ in order to obtain $t'=t^{-1}c$ with $c=1'$, as required.

(4) For the planes in Theorem 7.1, $s'=s^{-1}$. Namely, this holds for desarguesian planes (when $n=0$), and hence it holds by induction using the final part of the lemma.

(5) By the lemma together with Proposition 3.6(iv) and (2),

$$\begin{aligned}x\circ y&=x*y+\mu lyT(lxy)\\x\circ' y&=x*'y+\mu'l'yT(l'xy)\\x&=(x*y)*'y'\end{aligned}$$

where $\mu=1+\lambda^2$, $\mu'=1+\lambda^{-2}$ and $l'=l^{-1}$. Hence, again by (2),

$$\begin{aligned}x&=(x\circ y)\circ' y'\\&=\{x*y+\mu lyT(lxy)\}\circ' y'\\&=\{x*y+\mu lyT(lxy)\}*'y'+\mu'l'y'T(l'y'\{x*y+\mu lyT(lxy)\})\\&=x+\mu T(lxy)\cdot(ly)*'y'+\mu'l'y'T(l'y'(x*y))+\mu'l'y'\mu T(lxy)T(l'y'ly).\end{aligned}$$

By (3), $yy' = c$ for some constant c. By (1) and Hypothesis 3.1(vi),

$$(ly) *' y' = y^{-1}(l *' c) = l^{-1}y^{-1}(1 *' (cl))$$
$$y'(x * y) = (xy'^{-1}) * (y'y) = (xyc^{-1}) * c = c((xy) * 1).$$

Also, $v \to T(l^{-1}c(v * 1))$ is a linear functional $F \to K$, so for a unique $a \in F^*$ we have $T(l^{-1}c(v * 1)) = T(av)$ for all $v \in F$. It follows that

$$0 =$$
$$\mu T(lxy)l^{-1}y^{-1}(1 *' (cl)) + l^{-1}\mu' cy^{-1}T(l^{-1}c((xy) * 1)) + \mu'\mu l^{-1}cy^{-1}T(lxy)T(c) =$$
$$\mu T(lxy)l^{-1}y^{-1}(1 *' (cl)) + c\mu' l^{-1}y^{-1}T(axy) + \mu'\mu cl^{-1}y^{-1}T(lxy)T(c).$$

Hence, for all $v \in F$,

$$0 = \mu(1 *' (cl))T(lv) + c\mu' T(av) + c\mu'\mu T(lv)T(c)$$
$$T(av) = \{\mu(1 *' (cl)) + c\mu'\mu T(c)\}(c\mu')^{-1}T(lv).$$

Since the left side is in K, it follows that

$$a = \{\mu(1 *' (cl)) + c\mu'\mu T(c)\}(c\mu')^{-1}l.$$

Thus, we have proved that the parameters c and a are related as follows:

$$a(c\mu') = \mu l(1 *' (cl)) + cl\mu'\mu T(c)$$
$$T(av) = T(cl^{-1}(v * 1)). \tag{7.4}$$

These parameters evidently depend somehow on $*$ and λ.

(6) These parameters can, however, be determined in the situation of Theorem 7.1. We assume that we are passing from the case $n-1$ to n, just as in the proof of the theorem. We have $c = 1$ by (4), and we use $\mu = \mu_n$, $T = T_n$ and $l = c_{n-1}$. By Lemma 4.3(i) and (7.4),

$$\begin{aligned} T_n(av) &= T_n\big(c_{n-1}^{-1}\{v + \sum_{i=1}^{n-1}(1+\lambda_i^2)c_{i-1}T_i(c_{i-1}v)\}\big) \\ &= T_n(c_{n-1}^{-1}v) + \sum_{i=1}^{n-1} T_n\big((1+\lambda_i^2)c_{n-1}^{-1}c_{i-1}T_i(c_{i-1}v)\big) \\ &= T_n(c_{n-1}^{-1}v) + \sum_{i=1}^{n-1} T_nT_i\big((1+\lambda_i^2)c_{n-1}^{-1}c_{i-1}c_{i-1}v\big) \\ &= T_n(c_{n-1}^{-1}v) + \sum_{i=1}^{n} T_n(c_{n-1}^{-1}c_{i-1}^2v) + \sum_{1}^{n} T_n(c_{n-1}^{-1}c_i^2v) \\ &= T_n(c_{n-1}^{-1}v) + T_n(c_{n-1}^{-1}v) + T_n(c_{n-1}^{-1}c_{n-1}^2v), \end{aligned}$$

so that $a = c_{n-1}$. Also, since $c'_{i-1}c_{n-1} = c_{i-1}^{-1}c_{n-1} \in F_i$ and $cl = c_n$,

$$\begin{aligned}
\mu_n l(1 *' (cl)) &= \mu_n c_{n-1}\big\{c_{n-1}^2 + \sum_{i=1}^{n-1}(1+\lambda_i'^2)c'_{i-1}c_{n-1}T_i(c'_{i-1}c_{n-1})\big\} \\
&= \mu_n c_{n-1}\big\{c_{n-1}^2 + \sum_{i=1}^{n-1}(1+\lambda_i'^2)c'_{i-1}c_{n-1}c'_{i-1}c_{n-1}\big\} \\
&= \mu_n c_{n-1}\big\{c_{n-1}^2 + \sum_{i=1}^{n-1} c_{i-1}'^2 c_{n-1}^2 + \sum_{i=1}^{n-1} c_i'^2 c_{n-1}^2\big\} \\
&= \mu_n c_{n-1}\{c_{n-1}^2 + c_{n-1}^2 + c_{n-1}^{-2}c_{n-1}^2\} \\
&= \mu_n c_{n-1},
\end{aligned}$$

while

$$\begin{aligned}
a(c\mu') + cl\mu'_n\mu_n T(c) &= c_{n-1}\mu'_n + c_{n-1}\mu'_n\mu_n \\
&= c_{n-1}\mu'_n(1+\mu_n) \\
&= c_{n-1}\mu_n,
\end{aligned}$$

as required in (7.4).

8 Isomorphisms among nearly flag-transitive scions

Our goal in this section is Theorem 8.5, which completely solves the isomorphism problem for the planes $\mathfrak{A}\big((F_i)_0^n, (\lambda_i)_0^n\big)$. As in [13, Lemma 5.3] and [14, Proposition 3.38], we first need to know when two of our spreads coincide. While the required result offers no surprises, proving it appears to be harder than one might expect, in fact slightly harder than in the preceding two references:

Proposition 8.1. *Let* $\big((F_i)_0^n, (\lambda_i)_0^n\big)$ *and* $\big((F'_i)_0^{n'}, (\lambda'_i)_0^{n'}\big)$ *be reduced defining pairs with* $F_0 = F = F'_0$. *They determine the same prequasifield* (*i.e., the exact same multiplication*) *if and only if* $n = n'$, $F_i = F'_i$ *and* $\lambda_i = \lambda'_i$ *whenever* $1 \le i \le n$.

Proof. We may assume without loss of generality that $n' \ge n$. We will prove that

$$F_j = F'_j \quad \text{and} \quad \lambda_{j-1} = \lambda'_{j-1} \quad \text{whenever } 1 \le j \le n.$$

For this purpose we use induction to prove that, for each j with $1 \le j \le n$,

$$F_l = F'_l \quad \text{whenever } 0 \le l \le j, \quad \text{and} \quad \lambda_l = \lambda'_l \quad \text{whenever } 0 \le l < j. \tag{8.2}$$

When $j = 1$, we have $\lambda_0 = 1 = \lambda'_0$ by definition. We must show that $F_1 = F'_1$. By hypothesis, $x * y = x \circ y$ for all $x, y \in F$. Then Lemma 4.3(i) implies that $yf_1(z) = yf'_1(z)$ for all $z \in F$ (where $z = xy$, and f_1 and f'_1 are as in that lemma). Thus, $F_1 = F'_1$ by Lemma 4.7, and (8.2) holds when $j = 1$.

Now assume that (8.2) holds for some j with $1 \le j < n$. Then $c'_k = \prod_{i=0}^{k} \lambda'_i = \prod_{i=0}^{k} \lambda_i = c_k$ whenever $0 \le k < j$.

We first show that $\lambda_j = \lambda'_j$. By Lemma 4.3(i), if $x, y \in F^*$ then

$$\begin{aligned} &xy^2 + \sum_{i=1}^{j-1}(1+\lambda_i^2)c_{i-1}yT_i(c_{i-1}xy) \\ &\quad + c_{j-1}\Big[(1+\lambda_j^2)yT_j(c_{j-1}xy) + \sum_{i=j+1}^{n}(1+\lambda_i^2)[c_{i-1}/c_{j-1}]yT_i(c_{i-1}xy)\Big] \\ &= xy^2 + \sum_{i=1}^{j-1}(1+\lambda_i^2)c_{i-1}yT_i(c_{i-1}xy) \\ &\quad + c_{j-1}\Big[(1+{\lambda'_j}^2)yT_j(c_{j-1}xy) + \sum_{i=j+1}^{n'}(1+{\lambda'_i}^2)[c'_{i-1}/c_{j-1}]yT'_i(c'_{i-1}xy)\Big]. \end{aligned}$$

By our inductive hypothesis,

$$\begin{aligned} &(1+\lambda_j^2)T_j(c_{j-1}xy) + \sum_{i=j+1}^{n}(1+\lambda_i^2)[c_{i-1}/c_{j-1}]T_i(c_{i-1}xy) \\ &= (1+{\lambda'_j}^2)T_j(c_{j-1}xy) + \sum_{i=j+1}^{n'}(1+{\lambda'_i}^2)[c'_{i-1}/c_{j-1}]T'_i(c'_{i-1}xy). \end{aligned} \tag{8.3}$$

Let $x = c_{j-1}^{-1}$. By Lemma 3.8(iv), if $y \in F_j$ then

$$\begin{aligned} &(1+\lambda_j^2)y + \lambda_j \sum_{i=j+1}^{n}(1+\lambda_i^2)[c_{i-1}/c_j]T_i([c_{i-1}/c_{j-1}]y) \\ &= (1+{\lambda'_j}^2)y + \lambda'_j \sum_{i=j+1}^{n'}(1+{\lambda'_i}^2)[c'_{i-1}/c_j]T'_i([c'_{i-1}/c_{j-1}]y), \end{aligned}$$

where $1+\lambda_i^2, c_{i-1}/c_j = \prod_{j+1}^{i-1}\lambda_j \in F_{j+1}$ and $1+\lambda_i'^2, c'_{i-1}/c_j \in F'_{j+1}$ for $i \ge j+1$. Then

$$(\lambda_j + \lambda'_j)^2 y = \lambda_j g(y) + \lambda'_j g'(y) \quad \text{for all } y \in F_j,$$

for additive maps $g\colon F_j \to F_{j+1}$ and $g'\colon F_j \to F'_{j+1}$. Since $[F_j : F_{j+1}] \ge 3$ and $[F_j : F'_{j+1}] \ge 3$, we have $|\ker g| \ge (2/3)|F_j|$ and $|\ker g'| \ge (2/3)|F_j|$, so that there is some $y \ne 0$ in $\ker g \cap \ker g'$. Then $(\lambda_j + \lambda'_j)^2 y = 0$, so that $\lambda_j = \lambda'_j$, as claimed.

Next we show that $F_{j+1} = F'_{j+1}$. By induction and the preceding paragraph, we have $F_i = F'_i$ and $\lambda_i = \lambda'_i$ whenever $i \le j$. Now (8.3) states that $c_j f_{j+1}(xy) = c_j f'_{j+1}(xy)$ for all $x, y \in F^*$, in the notation of Lemma 4.7. Then $F_{j+1} = F'_{j+1}$ by that lemma.

By induction, this proves (8.2).

It remains to prove that $n = n'$ and $\lambda_n = \lambda'_n$. By (8.3) with $j = n$,

$$(1+\lambda_n^2)T_n(c_{n-1}xy) = (1+\lambda_n'^2)T_n(c_{n-1}xy) + \sum_{i=n+1}^{n'} (1+\lambda_i'^2)[c'_{i-1}/c_{n-1}]T'_i(c'_{i-1}xy)$$

for all $x, y \in F$. If $n' = n$, then the latter sum is empty and hence $\lambda_n = \lambda'_n$, as claimed.

On the other hand, if $n' > n$, then

$$(\lambda_n^2 + \lambda_n'^2)T_n(c_{n-1}x) = \lambda_n'^{-1} \sum_{i=n+1}^{n'} (1+\lambda_i'^2)[c'_{i-1}/c'_n]T'_i(c'_{i-1}x) \tag{8.4}$$

for all $x \in F$, where $1+\lambda_i'^2, c'_{i-1}/c'_n \in F'_{n+1}$ for $i \geq n+1$. If $\lambda_n \neq \lambda'_n$, vary x and obtain F_n on the left side of (8.4) and a subset of $\lambda_n'^{-1}F'_{n+1}$ on the right side, which is impossible since $F_n = F'_n \supset F'_{n+1}$. Thus, $\lambda_n = \lambda'_n$ and (8.4) yield $(1+\lambda_{n+1}'^2)T'_{n+1}(c'_n x) = \sum_{i=n+2}^{n'} (1+\lambda_i'^2)[c'_{i-1}/c'_n]T'_i(c'_{i-1}x) \in F'_{n+2}$ for all x. As above, vary x in order to obtain a contradiction. □

Theorem 8.5. *Suppose that* $\big((F_i)_0^n, (\lambda_i)_0^n\big)$ *and* $\big((F'_i)_0^{n'}, (\lambda'_i)_0^{n'}\big)$ *are reduced defining pairs with* $F_0 = F = F'_0$. *Then* $\mathfrak{A}\big((F_i)_0^n, (\lambda_i)_0^n\big)$ *and* $\mathfrak{A}\big((F'_i)_0^{n'}, (\lambda'_i)_0^{n'}\big)$ *are isomorphic if and only if* $n' = n$, $F'_i = F_i$ *for all* i, *and there is a sign* $\varepsilon = \pm 1$ *and a field automorphism* $\alpha \in \mathrm{Aut}(F_1)$ *such that* $\lambda'_i = (\lambda_i^\varepsilon)^\alpha$ *whenever* $1 \leq i \leq n$.

Proof. If $n' = n$, $F'_i = F_i$ and $\lambda'_i = \lambda_i^\alpha$ whenever $1 \leq i \leq n$, for some $\alpha \in \mathrm{Aut}(F_1)$, then extend α to $\beta \in \mathrm{Aut}(F)$ and observe that (4.2) implies that $(x * s)^\beta = x^\beta \circ s^\beta$ for all $x, s \in F$; the map $(x, y) \to (x^\beta, y^\beta)$ induces an isomorphism $\mathfrak{A}(\mathcal{S}_*) \to \mathfrak{A}(\mathcal{S}_\circ)$. Similarly, Theorem 7.1 takes care of the case $\lambda'_i = (\lambda_i^{-1})^\alpha$ for all i.

Assume that the two planes are isomorphic. Then, by Theorem 6.1, both have kernel $K := F_n = F'_{n'}$, which has order greater than 2 (see Definition 4.6). By Theorem 2.2, we may assume that an isomorphism between the two planes is induced by an element $g \in \Gamma\mathrm{Sp}(F^2, K)$. By Proposition 5.2, we may assume that g has the form $(x, y) \to (x^\alpha, y^\alpha)$ or $(x, y) \to (y^\alpha, x^\alpha)$ for all $x, y \in F$ and some $\alpha \in \mathrm{Aut}(F)$.

Case 1. $g\colon (x, y) \to (x^\alpha, y^\alpha)$ with $\alpha \in \mathrm{Aut}(F)$. In the notation of (3.2), $\mathcal{S}_*[\infty]^g = \mathcal{S}_\circ[\infty]$ and, for each $s \in F$, there is some $s' \in F$ such that $\mathcal{S}_*[s]^g = \mathcal{S}_\circ[s']$. Then $(x * s)^\alpha = x^\alpha \circ s'$ for all $x \in F$.

We claim that $s' = s^\alpha$. For, since $[F\colon F_1] \geq 3$ and $[F\colon F'_1] \geq 3$, we have $|\ker T_1| \geq (2/3)|F|$ and $|\ker T'_1| \geq (2/3)|F|$, so that there is some $x \in F^*$ such that $T_1(xs) = 0 = T'_1(x^\alpha s')$. By Lemma 3.8(i), $T_i(c_{i-1}xs) = T_iT_1(c_{i-1}xs) = T_i\big(c_{i-1}T_1(xs)\big) = 0$ and $T'_j(c'_{j-1}x^\alpha s') = T'_jT'_1(c'_{j-1}x^\alpha s') = T'_j\big(c'_{j-1}T'_1(x^\alpha s')\big) = 0$ whenever $1 \leq i \leq n$ and $1 \leq j \leq n'$ (recall that $c_0 = 1, c_{i-1} \in F_1, c'_{j-1} \in F'_1$). By Lemma 4.3(i), $x * s = xs^2$ and $x^\alpha \circ s' = x^\alpha s'^2$. Hence $x^\alpha s^{2\alpha} = (x * s)^\alpha = x^\alpha \circ s' = x^\alpha s'^2$, so that $s^\alpha = s'$, as claimed. Thus, $(x * s)^\alpha = x^\alpha \circ s^\alpha$.

By Lemma 4.3(i) for $*$ and for $\circ$, $(x * s)^\alpha = x^\alpha \circ s^\alpha$ implies that $\circ$ arises from the reduced defining pair $\big((F_i)_0^n, (\lambda_i^\alpha)_0^n\big)$ as well as from $\big((F'_i)_0^{n'}, (\lambda'_i)_0^{n'}\big)$. By Proposition 8.1, $n' = n$, $F'_i = F_i$ and $\lambda'_i = \lambda_i^\alpha$ whenever $1 \leq i \leq n$.

Finally, since all λ_i belong to F_1 we can replace α by $\alpha|_{F_1}$ in order to obtain the desired conclusion.

Case 2. $g\colon (x,y) \to (y^\alpha, x^\alpha)$ with $\alpha \in \mathrm{Aut}(F)$. In the notation of Theorem 7.1, θg is an isomorphism $\mathfrak{A}\big((F_i)_0^n, (\lambda_i^{-1})_0^n\big) \to \mathfrak{A}\big((F_i')_0^{n'}, (\lambda_i')_0^{n'}\big)$ sending $(x,y) \to (x^\alpha, y^\alpha)$. Now apply Case 1. □

Corollary 8.6. *Suppose that* $\big((F_i)_0^n, (\lambda_i)_0^n\big)$ *is a reduced defining pair with* $n \geq 1$*. Then some collineation of* $\mathfrak{A}(\big((F_i)_0^n, (\lambda_i)_0^n\big))$ *interchanges* X *and* Y *if and only if* $\lambda_i\lambda_i^\alpha = 1$ *for all* i *and some* $\alpha \in \mathrm{Aut}(F_1)$.

Proof. By the remark following Theorem 2.2, if g is a collineation interchanging X and Y, then gk is symplectic for some k in the kernel of the plane. Then k fixes X and Y, so that gk also interchanges X and Y. Using $(gk)\theta$ in the preceding Case 2, we see that $\lambda_i = (\lambda_i^{-1})^\alpha$ for all i. □

Remark 8.7. If $\lambda_1^\alpha = \lambda_1^{-1} \neq 1$ then α has even order. Since m is odd, this means that q must be a square. We will consider the case of an involutory field automorphism in Section 10.

When $n = 1$ the planes in the corollary are among those studied in [8, II Section 6], where the interchanging collineation was not noticed.

Remark 8.8. While $\mathrm{Aut}\,\mathfrak{A}\big((F_i)_0^n, (\lambda_i)_0^n\big)$ always contains all translations and the multiplicative group of the kernel of $\mathfrak{A}\big((F_i)_0^n, (\lambda_i)_0^n\big)$, we just saw that it also contains the Galois group $\mathrm{Gal}(F/F_1)$.

9 Enumeration

Notation 9.1. Let $m = m_0$ be an odd composite integer. Let $\sigma = (m_i)_0^{l(\sigma)}$ be any sequence of $l(\sigma) + 1$ distinct integers such that $m_{l(\sigma)} \geq 1$ and m_i divides m_{i-1} for $1 \leq i \leq l(\sigma)$. Let q be a power of 2 such that $q^{m_{l(\sigma)}} \geq 4$.

Theorem 9.2. (i) *For* $\sigma = (m_i)_0^{l(\sigma)}$*, there are at least* $\prod_{i=1}^{l(\sigma)}(q^{m_i} - 2)/(2m_1 \log q)$ *pairwise nonisomorphic symplectic nearly flag-transitive scions of the desarguesian plane* $\mathrm{AG}(2, q^m)$ *obtained by applying the up and down process along the chain* $\big(\mathrm{GF}(q^{m_i})\big)_0^{l(\sigma)}$*, all having kernel* $\mathrm{GF}(q^{m_{l(\sigma)}})$*.*

(ii) *There are at least* $\sum_\sigma \big\{\prod_{i=1}^{l(\sigma)}(q^{m_i} - 2)\big\}/(2m_1 \log q)$ *pairwise nonisomorphic symplectic nearly flag-transitive scions of the desarguesian plane of order* q^m*. Here the sum runs over all sequences* σ *as above.*

Proof. (i) There are $\prod_{i=1}^{l(\sigma)}(q^{m_i} - 2)$ reduced defining pairs obtained from the stated chain $(F_i)_0^{l(\sigma)}$. By Proposition 4.1, these reduced pairs determine $\prod_{i=1}^{l(\sigma)}(q^{m_i} - 2)$ symplectic nearly flag-transitive scions of the desarguesian plane. Now apply Theorem 8.5.

(ii) Again use Proposition 4.1 and Theorem 8.5. □

An elementary calculation using the definition of $\rho(m)$ in Section 1 yields the

Corollary 9.3. *If $q \geq 4$ is a power of 2 and $m > 1$ is odd, then the number of planes in the theorem of order q^m with kernel* $\mathrm{GF}(q)$ *is greater than $q^{3^{\rho(m)-2}}$ and less than q^m.*

This completes the proof of Theorem 1.1.

10 The Baer perspective: flag-transitive subplanes

In Corollary 8.6 we saw that a field automorphism α of even order such that $\lambda_i\lambda_i^\alpha = 1$ for all i produces a collineation $(x, y) \to (y^\alpha, x^\alpha)$ of $\mathfrak{A}\big((F_i)_0^n, (\lambda_i)_0^n\big)$. This collineation is involutory if and only if α is; this case is the topic of this section. We will see that *this produces a Baer subplane that is one of the flag-transitive planes studied in* [13], *and that each of the latter planes arises in this manner from a unique one of the planes* $\mathfrak{A}\big((F_i)_0^n, (\lambda_i)_0^n\big)$ (cf. Theorem 1.2).

Write $x^\alpha = \bar{x}$ for all $x \in F$, and consider a plane $\mathfrak{A}\big((F_i)_0^n, (\lambda_i)_0^n\big)$ such that $\lambda_i\bar{\lambda}_i = 1$ for all i, so that $\tilde{\theta}\colon (x, y) \to (\bar{y}, \bar{x})$ is a Baer involution. Then $\tilde{\theta}$ sends the line $y = x * s$ to $y = x * \bar{s}^{-1}$ by Remark 7.3(4). Thus,

$$\textit{The lines through } 0 \textit{ fixed by } \tilde{\theta} \textit{ are } y = x * s \textit{ with } s\bar{s} = 1. \tag{10.1}$$

Moreover, $\tilde{\theta}$ normalizes the group G in Proposition 3.6(i), since $\phi_\zeta^{\tilde{\theta}} = \phi_{\bar{\zeta}^{-1}}$. Thus, $C_G(\tilde{\theta})$ consists of the $q^{m/2}+1$ linear transformations (3.5) for which $\zeta\bar{\zeta} = 1$. Since our subplane has order $q^{m/2}$, it follows that $C_G(\tilde{\theta})$ *is flag-transitive on this Baer subplane.*

Consider the chain $(F_i^\bullet)_0^n$ of subfields of F such that $F_i^\bullet$ is the set of fixed points in F_i of our involutory field automorphism. Then $[F_i\colon F_i^\bullet] = 2$ and $[F_0^\bullet\colon F_i^\bullet] = [F\colon F_i]$ for each i. Let $T_i^\bullet\colon F_0^\bullet \to F_i^\bullet$ be the trace map, and define $T_{n+1} = T_{n+1}^\bullet = 0$. Note that

$$T_i(x) = T_i^\bullet(x) \quad \text{if } x \in F_i^\bullet. \tag{10.2}$$

For, if $|F_i| = q^r$ then, as the integer j goes from 0 to $r-1$, if j is even then $j/2$ goes from 0 to $(r-1)/2$, and if j is odd then $(r+j)/2$ goes from $(r+1)/2$ to $r-1$. In the latter case, $x^{q^{j/2}} = x^{q^{(r+j)/2}}$. Thus, $T_i^\bullet(x) = \sum_{j=0}^{r-1} x^{q^{j/2}} = \sum_{k=0}^{r-1} x^{q^k} = T_i(x)$.

If $0 \leq i \leq n+1$ write

$$W_i := \ker T_{i+1}^\bullet|_{F_i^\bullet}; \tag{10.3}$$

for example, $W_{n+1} = F_n^\bullet$. Then the flag-transitive symplectic planes studied in [13] are produced by the spreads

$$\big\{\zeta\big(W_0\bar{c}_0 \oplus W_1\bar{c}_1 \oplus \cdots \oplus W_n\bar{c}_n\big) \mid \zeta \in F, \zeta\bar{\zeta} = 1\big\} \tag{10.4}$$

of the $F_n^\bullet$-space F corresponding to defining pairs $\big((F_i)_0^n, (\lambda_i)_0^n\big)$ such that $\lambda_i\bar{\lambda}_i = 1$ for all i. (We have slightly changed the notation in [13] so as not to conflict with the notation used earlier in the present paper.) The flag-transitive group in [13] consists of all maps $x \to \zeta x$ with $\zeta \in F$ and $\zeta\bar{\zeta} = 1$. By [13, Corollary 4.5], we can restrict to reduced defining pairs.

Theorem 10.5. *The spread* (10.4) *coincides with the spread on the flag-transitive subplane of fixed points in* $\mathfrak{A}\big((F_i)_0^n,(\lambda_i)_0^n\big)$ *of the Baer involution* $\tilde\theta\colon (x,y)\to(\bar y,\bar x)$.

Proof. By (10.1), $\tilde\theta$ fixes the line $y=x*1$ of our subplane $\{(x,\bar x)\mid x\in F\}$. As a set of points of our subplane, that line is $\{(x,\bar x)\mid x\in Z\}$, where

$$Z:=\{x\in F\mid \bar x=x*1\}=\{x\in F\mid x+\bar x=\sum_1^n(1+\lambda_i^2)c_{i-1}T_i(c_{i-1}x)\}\quad(10.6)$$

(cf. Lemma 4.3(i)). In the notation of (3.5), $\phi_{\bar\zeta}$ sends $(x,\bar x)$ to $(\zeta x,\overline{\zeta x})$ if $\zeta\bar\zeta=1$. Thus, in view of the above remark concerning the flag-transitive group in [13], it suffices to prove that

$$Z=W_0\bar c_0\oplus W_1\bar c_1\oplus\cdots\oplus W_n\bar c_n.$$

Since both sides are lines of affine planes of order $|F|^{1/2}=|F_0'|$, *it suffices to prove that*

$$W_j\bar c_j\subseteq Z\quad\text{for } 0\le j\le n.\quad(10.7)$$

Consider $x=f_jc_j^{-1}=f_j\bar c_j\in W_j\bar c_j$, so that $f_j=\bar f_j\in F_j^\bullet$ and $T_{j+1}^\bullet(f_j)=0$. Then also $T_{j+1}(f_j)=0$ by (10.2).

If $i-1\ge j$ then $c_{i-1}c_j^{-1}=\prod_1^{i-1}\lambda_k/\prod_1^j\lambda_k=\prod_{j+1}^{i-1}\lambda_k\in F_{j+1}$. By Lemma 3.8(i),

$$T_i(c_{i-1}c_j^{-1}f_j)=T_iT_{j+1}(c_{i-1}c_j^{-1}f_j)=T_i\big(c_{i-1}c_j^{-1}T_{j+1}(f_j)\big)=0.$$

If $1\le i\le j$ then $c_{i-1}c_j^{-1}=\prod_1^{i-1}\lambda_k/\prod_1^j\lambda_k=1/\prod_i^j\lambda_k\in F_i$. By Lemma 3.8(iv), $T_i(c_{i-1}c_j^{-1}f_j)=c_{i-1}c_j^{-1}f_j$. Since $x=f_jc_j^{-1}$, $\bar x=f_jc_j$ and $c_i=\lambda_ic_{i-1}$,

$$\begin{aligned}\sum_{i=1}^n(1+\lambda_i^2)c_{i-1}T_i(c_{i-1}x)&=\sum_{i=1}^j(1+\lambda_i^2)c_{i-1}(c_{i-1}c_j^{-1}f_j)\\&=\sum_{i=1}^j c_{i-1}^2c_j^{-1}f_j+\sum_{i=1}^j c_i^2c_j^{-1}f_j\\&=c_0^2c_j^{-1}f_j+c_j^2c_j^{-1}f_j\\&=x+\bar x\end{aligned}$$

for any $x\in W_j\bar c_j$. By (10.6), this proves (10.7) and hence the theorem. □

Proof of Theorem 1.2. Let $\big((F_i)_0^n,(\lambda_i)_0^n\big)$ and $\big((F_i')_0^{n'},(\lambda_i')_0^{n'}\big)$ be reduced defining pairs with $F_0=F=F_0'$, $1\ne\lambda_i=\bar\lambda_i^{-1}\in F_i^*$ and $1\ne\lambda_i'=\bar\lambda_i'^{-1}\in F_i^*$ for all i. By [13, Theorem 5.2], the corresponding flag-transitive planes in [13] are isomorphic if and only if $n=n'$ and there is some $\beta\in\mathrm{Aut}(F)$ such that $\lambda_i'=\lambda_i^\beta$ for all i; since all $\lambda_i\in F_1$ we can restrict β to $\mathrm{Aut}(F_1)$. By Theorem 8.5, the same is true for the corresponding nearly flag-transitive planes $\mathfrak{A}\big((F_i)_0^n,(\lambda_i)_0^n\big)$ and $\mathfrak{A}\big((F_i')_0^{n'},(\lambda_i')_0^{n'}\big)$. Thus, Theorem 10.5 implies the required result. □

11 Equivalence of orthogonal spreads

We now discuss equivalence for orthogonal spreads. By Proposition 3.6(ii), each defining pair $\big((F_i)_0^n,(\lambda_i)_0^n\big)$ with $F=F_0$ and $F_n\supset K\supseteq \mathrm{GF}(2)$ determines an orthogonal spread $\Sigma_*\big((F_i)_0^n,(\lambda_i)_0^n\big)$ of $V=F\oplus K\oplus F\oplus K$.

Theorem 11.1. *Assume that* $\big((F_i)_0^n,(\lambda_i)_0^n\big)$ *and* $\big((F_i')_0^{n'},(\lambda_i')_0^{n'}\big)$ *are reduced defining pairs, with* $F_0=F=F_0'$ *and* $F_n,F_{n'}'\supset K\supseteq \mathrm{GF}(2)$, *producing orthogonal spreads* $\Sigma_*\big((F_i)_0^n,(\lambda_i)_0^n\big)$ *and* $\Sigma_\circ\big((F_i')_0^{n'},(\lambda_i')_0^{n'}\big)$ *of the* $\mathrm{O}^+(2m+2,K)$*-space* V *as well as planes* $\mathfrak{A}(\mathcal{S}_*)$ *and* $\mathfrak{A}(\mathcal{S}_\circ)$. *Then* Σ_* *and* $\Sigma_\circ$ *are equivalent if and only if* $\mathfrak{A}(\mathcal{S}_*)$ *and* $\mathfrak{A}(\mathcal{S}_\circ)$ *are isomorphic, hence if and only if* $n'=n$, $F_i'=F_i$ *and* $\lambda_i'=(\lambda_i^\varepsilon)^\alpha$ *whenever* $1\le i\le n$, *for some* $\varepsilon=\pm1$ *and* $\alpha\in\mathrm{Aut}(F_1)$.

Proof. By Proposition 3.6(ii),

$$\mathfrak{A}(\mathcal{S}_*)\cong\mathfrak{A}(\Sigma_*/\langle 0,1,0,1\rangle)\quad\text{and}\quad\mathfrak{A}(\mathcal{S}_\circ)\cong\mathfrak{A}(\Sigma_\circ/\langle 0,1,0,1\rangle).$$

By Theorem 2.2(ii), if there is an isomorphism $\mathfrak{A}(\mathcal{S}_*)\to\mathfrak{A}(\mathcal{S}_\circ)$, then there is an equivalence $\Sigma_*\to\Sigma_\circ$ fixing $\langle 0,1,0,1\rangle$.

For the converse, suppose that $g\in\Gamma\mathrm{O}^+(2m+2,K)$ sends Σ_* to $\Sigma_\circ$. The Sylow subgroup $P\le G$ of $\mathrm{Sp}(2m,K)$ used in Section 5 induces a Sylow subgroup $\hat P$ of $\mathrm{O}^+(2m+2,K)$ lying in $\hat G$ (the group $\hat G$ was defined in Lemma 3.6(ii)). Hence, as in the proof of Proposition 5.2, we may assume that g normalizes $\hat P$. By Lemma 3.6(iii), $C_V(P)=\{(0,a,0,b)\mid a,b\in K\}$ is an invariant subspace of g. Since $g\in\Gamma\mathrm{O}^+(V)$, it permutes the set $\{\langle 0,\lambda^2,0,1\rangle\mid\lambda\in K^*\}$ of nonsingular points of this subspace. If $\langle 0,1,0,1\rangle^g=\langle 0,\lambda^2,0,1\rangle$, then g induces an isomorphism $\mathfrak{A}(\Sigma_*/\langle 0,1,0,1\rangle)\to\mathfrak{A}(\Sigma_\circ/\langle 0,\lambda^2,0,1\rangle)$. If $\lambda=1$ then $\mathfrak{A}(\mathcal{S}_*)\cong\mathfrak{A}(\mathcal{S}_\circ)$, as claimed.

Suppose that $\lambda\ne1$. By the first proof of Proposition 4.1, K is the smallest member of the chain of fields that determines $\mathfrak{A}(\Sigma_\circ/\langle 0,\lambda^2,0,1\rangle)$. Since $\lambda\in K-\{1\}$, Theorem 6.1 implies that K is the kernel of this plane. However, the same theorem also implies that F_n is the kernel of the isomorphic plane $\mathfrak{A}(\Sigma_*/\langle 0,1,0,1\rangle)$. Since $F_n\supset K$, this contradiction shows that $\lambda=1$.

The final assertion of the theorem is just Theorem 8.5. □

Theorem 11.2. *There are more than* $\sum_\sigma\big\{\prod_{i=1}^{l(\sigma)}(q^{m_i}-2)\big\}/(2m_1\log q)$ *inequivalent orthogonal spreads of the* K*-space* V, *each admitting the group* $\hat G$ *as isometries* (*cf. Proposition* 3.6(ii)). *Here the sum runs over all sequences* σ *in* (9.1) *such that* $F_n\supset K$.

Proof. This immediately follows from Theorems 9.2 and 11.1. □

12 Concluding remarks

1. Subplanes. Consider a plane $\mathfrak{A}\big((F_i)_0^n,(\lambda_i)_0^n\big)$, and let $1\le j\le n$. If all $\lambda_i\in F_j$, then $(x,y)\to(x^{|F_j|},y^{|F_j|})$ is a collineation whose set of fixed points is F_j^2. The group of all

$(x, y) \to (\zeta^{-1}x, \zeta y), \zeta \in F_j^*$, acts nearly flag-transitively on the resulting subplane $\mathfrak{A}_j$; in fact,

$$\mathfrak{A}_j \cong \mathfrak{A}\big((F_i)_j^n, (\lambda_i')_j^n\big),$$

where $\lambda_j' = 1$ and $\lambda_i' = \lambda_i'$ whenever $j < i \le n$. For, let $c_i' := c_i/c_j = \prod_{l=j}^i \lambda_l' \in F_j$ whenever $j \le i \le n$. If $x, y \in F_j$ and we write $y' = c_j y$, then

$$x * y = xy'^2 + \sum_{i=j+1}^{n} \big[c_{i-1}' y' T_i(c_{i-1}' xy') + c_i' y' T_i(c_i' xy')\big],$$

since $xy^2 + \sum_{i=1}^j \big[c_{i-1} y T_i(c_{i-1}xy) + c_i y T_i(c_i xy)\big] = xy^2 + \sum_{i=1}^j \big[xy^2 c_{i-1}^2 + xy^2 c_i^2\big] = xy^2 c_j^2 = xy'^2$ by Lemma 3.8(iv).

Thus, the scions of the desarguesian plane $\mathrm{AG}(2, |F|)$ breed scions of its subplanes $\mathrm{AG}(2, |F_j|)$.

2. Automorphism groups. Straightforward use of [15] shows that, for each nondesarguesian plane $\mathfrak{A}\big((F_i)_0^n, (\lambda_i)_0^n\big)$, the group $(\mathrm{Aut}\,\mathfrak{A})_0$ normalizes P and hence is contained in the group $N_{\Gamma\mathrm{Sp}(F^2,K)}(P)$ in Lemma 5.1. However, since [15] uses the classification of the finite simple groups, it would be far preferable to have a direct and elementary proof of this geometric result — even one that is highly computational.

Similar remarks apply to $\Gamma\mathrm{O}(V)_\Sigma$, as well as to the flag-transitive planes in [13] and Section 10.

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