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Positivity in power series rings

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Abstract. We extend and generalize results of Scheiderer (2006) on the representation of polynomials nonnegative on two-dimensional basic closed semialgebraic sets. Our extension covers some situations where the defining polynomials do not satisfy the transversality condition. Such situations arise naturally when one considers semialgebraic sets invariant under finite group actions.

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1 Introduction

Let $\mathbb{R}[\mathbf{x}] := \mathbb{R}[x_1, \dots, x_n]$ be the ring of polynomials in n variables with real coefficients. A *preordering* of a general ring A (commutative with 1) is a subsemiring of A which contains the squares. In other words, a preordering of A is a subset of A which contains all f^2, $f \in A$, and is closed under addition and multiplication. For a finite subset $S = \{g_1, \dots, g_s\}$ of $\mathbb{R}[\mathbf{x}]$, we write T_S for the preordering of $\mathbb{R}[\mathbf{x}]$ generated by S, and K_S for the set of all $x \in \mathbb{R}^n$ satisfying $g_1(x) \geq 0, \dots, g_s(x) \geq 0$ (the basic closed semialgebraic set defined by S). Note that K_S is uniquely determined by T_S, but typically T_S is not uniquely determined by K_S. For a subset K of $\mathbb{R}^n$, we write $\operatorname{Psd}(K)$ for the set of all elements of $\mathbb{R}[\mathbf{x}]$ that are nonnegative on K. We always have that $T_S \subseteq \operatorname{Psd}(K_S)$. The preordering T_S is said to be *saturated* if $T_S = \operatorname{Psd}(K_S)$.

In this paper we investigate what geometric properties of S imply that T_S is saturated. This line of investigation has been pursued by Scheiderer in a series of papers. In [9], Scheiderer showed that T_S is never saturated if $\dim(K_S) \geq 3$. The case $\dim(K_S) \leq 1$ is fairly well understood; see [5], [6], [8], [10]. We focus here on the 2-dimensional case, more precisely, on the affine 2-dimensional case, i.e., $n = \dim(K_S) = 2$.

We consider only the compact case. In the non-compact case little is known; see [5, Open Problem 6] and [11, Remark 3.16]. By [9, Remark 6.7], T_S is not saturated if K_S contains a two-dimensional cone. In the compact case, we have the following result of Scheiderer [11, Corollary 3.3]:

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Theorem 1. *Let $S = \{g_1, \dots, g_s\}$ be irreducible polynomials in $\mathbb{R}[x,y]$, let C_i be the plane affine curve $g_i = 0$ ($i = 1, \dots, s$). Assume:*

(1) *K_S is compact,*
(2) *C_i has no real singular points ($i = 1, \dots, s$),*
(3) *the real points of intersection of any two of the C_i are transversal, and no three of the C_i intersect in a real point.*

Then T_S is saturated.

The main goal of this paper is to show that saturation holds in certain other compact cases as well, e.g., if $S = \{x, 1-x, y, x^2-y\}$ or $S = \{1+x, 1-x, y, x^2-y\}$. In these examples, the boundary curves $y = 0$ and $y = x^2$ share a common tangent at the origin, so Theorem 1 does not apply. The fact that saturation holds in these examples is a consequence of our main result, Corollary 6, which is an extension of Theorem 1.

Our original motivation comes from examples which arise naturally while studying semialgebraic sets $K_{S'}$ described by a set S' of polynomials invariant under an action of a finite group G. The corresponding preordering $T_{S'}$ will typically not be saturated but it can still be "saturated for invariant polynomials" (we refer to this as "G-saturation"). The orbit map π (see [3]) relates the G-saturation of $T_{S'}$ to the saturation of a certain preordering $T_{\tilde{S}'}$ corresponding to $\pi(K_{S'}) = K_{\tilde{S}'}$. In many cases, the latter follows from our Corollary 6. An example is given in Section 3.

At the same time, Corollary 6 does not cover all interesting cases; in the concluding remarks, we consider some of the remaining cases.

2 Saturation in dimension two

We focus on the case of a compact basic closed semialgebraic set. In [10, Corollary 3.17], Scheiderer proves a useful 'local-global' criterion, extending [12, Corollary 3], for deciding when a polynomial non-negative on a compact basic closed semialgebraic set lies in the associated preordering of the polynomial ring:

Theorem 2. *Suppose $f, g_1, \dots, g_s \in \mathbb{R}[\mathbf{x}]$, the subset K of $\mathbb{R}^n$ defined by the inequalities $g_i \geq 0$, $i = 1, \dots, s$, is compact, $f \geq 0$ on K, and f has just finitely many zeros in K. Then the following are equivalent:*

(1) *f lies in the preordering of $\mathbb{R}[\mathbf{x}]$ generated by $g_1, \dots, g_s$.*
(2) *For each zero p of f in K, f lies in the preordering of the completion of $\mathbb{R}[\mathbf{x}]$ at p generated by $g_1, \dots, g_s$.*

In the two-dimensional case this allows one to show that certain finitely generated preorderings are saturated; see [11]. For example, Theorem 1 can be obtained by combining Theorem 2 with the following result for power series rings, using the Transfer Principle:

Theorem 3. *Suppose $f \in \mathbb{R}[[x,y]]$.*

(1) *If $f \geq 0$ at each ordering of $\mathbb{R}((x,y))$ then f is a sum of squares in $\mathbb{R}[[x,y]]$.*

(2) *If $f \geq 0$ at each ordering of $\mathbb{R}((x,y))$ satisfying $x > 0$ then f lies in the preordering of $\mathbb{R}[[x,y]]$ generated by x.*
(3) *If $f \geq 0$ at each ordering of $\mathbb{R}((x,y))$ satisfying $x > 0$ and $y > 0$ then f lies in the preordering of $\mathbb{R}[[x,y]]$ generated by x and y.*

Proof. (1) is well known. It can be proved using a modification of the analytic argument given in [2, Lemma 7a]. The proof shows, in fact, that f is a sum of two squares. See [7, Theorem 1.6.3] for more details. (2) (respectively, (3)) follows immediately from (1) by going to the extension ring $\mathbb{R}[[\sqrt{x},y]]$ (respectively, to the extension ring $\mathbb{R}[[\sqrt{x},\sqrt{y}]]$). E.g., to prove (2), apply (1) to $\mathbb{R}[[\sqrt{x},y]]$ to deduce $f = \sum f_i^2$, $f_i \in \mathbb{R}[[\sqrt{x},y]]$. Decomposing $f_i = f_{i1} + f_{i2}\sqrt{x}$, $f_{ij} \in \mathbb{R}[[x,y]]$, and expanding, yields $f = \sum f_{i1}^2 + \sum f_{i2}^2 x$.

We will prove the following extension of Theorem 3.

Theorem 4. *Suppose $f \in \mathbb{R}[[x,y]]$ and n is a positive integer.*
(1) *If $f \geq 0$ at each ordering of $\mathbb{R}((x,y))$ satisfying $y > 0$ and $x^{2n} - y > 0$ then f lies in the preordering of $\mathbb{R}[[x,y]]$ generated by y and $x^{2n} - y$.*
(2) *If $f \geq 0$ at each ordering of $\mathbb{R}((x,y))$ satisfying $x > 0$, $y > 0$ and $x^n - y > 0$ then f lies in the preordering of $\mathbb{R}[[x,y]]$ generated by x, y and $x^n - y$.*

Remark 5. Suppose n is odd, $n \geq 3$. Then:
(i) For every ordering of $\mathbb{R}[[x,y]]$, $y \geq 0$ and $x^n - y \geq 0 \Rightarrow x \geq 0$, but x is not in the preordering of $\mathbb{R}[[x,y]]$ generated by y and $x^n - y$. This shows that an obvious attempt to strengthen Theorem 4 fails.
(ii) Similarly, for every ordering of $\mathbb{R}[[x,y]]$, $x^n - y^2 \geq 0 \Rightarrow x \geq 0$, but x is not in the preordering of $\mathbb{R}[[x,y]]$ generated by $x^n - y^2$.

Note: Going to the extension ring $\mathbb{R}[[x,\sqrt{y}]]$, we see that assertions (i) and (ii) are essentially equivalent.

We postpone the proof of Theorem 4 to Section 4. For now we only explain how Theorems 2, 3 and 4 can be combined to yield the promised extension of Theorem 1:

Corollary 6. *Let $S = \{g_1,\dots,g_s\}$ be irreducible polynomials in $\mathbb{R}[x,y]$. Suppose that $K = K_S \subseteq \mathbb{R}^2$ is compact, and, for each boundary point p of K, either*
(1) *there exists i such that p is a non-singular zero of g_i, and K is defined locally at p by the single inequality $g_i \geq 0$; or*
(2) *there exists i,j such that p is a non-singular zero of g_i and g_j, g_i and g_j meet transversally at p, and K is defined locally at p by $g_i \geq 0$, $g_j \geq 0$; or*
(3) *there exists i,j such that p is a non-singular zero of g_i and g_j, g_i and g_j share a common tangent at p but do not cross each other at p, and K is described locally at p as the region between $g_i = 0$ and $g_j = 0$; or*
(4) *there exists i,j,k such that p is a non-singular zero of g_i, g_j and g_k, g_i and g_j share a common tangent at p, g_i and g_k meet transversally at p, and K is described locally at p as the part of the region between $g_i = 0$ and $g_j = 0$ defined by $g_k \geq 0$.*

Then the preordering of $\mathbb{R}[x,y]$ generated by $g_1,\dots,g_s$ is saturated.

Proof. Let T denote the preordering of $\mathbb{R}[x,y]$ generated by $g_1,\dots,g_s$. We wish to show that $f \in \mathbb{R}[x,y]$, $f \geq 0$ on $K \Rightarrow f \in T$. We may assume $K \neq \emptyset$, $f \neq 0$. The hypothesis implies, in particular, that K is the closure of its interior. This allows us to reduce further to the case where f is square-free and $g_i \nmid f$ for each i. In this situation, f has only finitely many zeros in K, so Theorem 2 applies, i.e., to show $f \in T$, it suffices to show that, for each zero p of f in K, f lies in the preordering of the completion of $\mathbb{R}[x,y]$ at p generated by $g_1,\dots,g_s$. If p is an interior point of K this follows from Theorem 3(1). If p is a boundary point of K satisfying (1) (respectively, (2), respectively, (3), respectively, (4)) then it follows from Theorem 3(2) (respectively, Theorem 3(3), respectively, Theorem 4(1), respectively, Theorem 4(2)). We use the Transfer Principle and apply Theorems 3 and 4 with $x = \overline{x}$, $y = \overline{y}$, where $\overline{x}, \overline{y}$ are suitably chosen local parameters at p. If p is an interior point of K we choose $\overline{x} = x - a$, $\overline{y} = y - b$ where $p = (a,b)$. In Case (1), we choose local parameters $\overline{x}$, $\overline{y}$ with $\overline{x} = g_i$. In Case (2), we choose local parameters $\overline{x}$, $\overline{y}$ with $\overline{x} = g_i$, $\overline{y} = g_j$. In Case (3), choose local parameters $\overline{x}$, g_i. By the Preparation Theorem [13, Corollary 1, p. 145], $hg_j = g_i + \overline{x}^n k$ for some unit h, some $n \geq 1$ and some unit $k \in \mathbb{R}[[\overline{x}]]$. Then $sg_i + tg_j = \overline{x}^n$ where $s = -\frac{1}{k}$ and $t = \frac{h}{k}$. By the geometry of the situation, the units s,t are positive units and n is even. Take $\overline{y} = sg_i$, so $\overline{x}^n - \overline{y} = tg_j$, and apply Theorem 4(1). In Case (4) choose local parameters $\overline{x}$, g_i with $\overline{x} = g_k$. As before, this yields $sg_i + tg_j = \overline{x}^n$ for some units s,t and some $n \geq 1$. By the geometry of the situation, s,t are positive units. Take $\overline{y} = sg_i$, so $\overline{x}^n - \overline{y} = tg_j$, and apply Theorem 4(2).

3 Application to equivariant saturated preorderings

If $S = \{1-x, 1+x, 1-y, 1+y\}$ and $S' = \{2 - x^2 - y^2, (1-x^2)(1-y^2)\}$ then $K_S = K_{S'}$ is the unit square. Note that T_S is saturated, by Theorem 1. On the other hand, it can be easily verified that $1 - x \notin T_{S'}$, hence $T_{S'}$ is not saturated.

Let $G = \langle a, b \mid a^4 = b^2 = (ab)^2 = 1 \rangle$ be the fourth dihedral group acting on $\mathbb{R}^2$ and $\mathbb{R}[x,y]$ in a "standard way". For every G-invariant subset M of $\mathbb{R}[x,y]$ write $M^G = \{m \in M \mid \forall g \in G\colon m^g = m\}$. We would like to show that $T_{S'}$ is *G-saturated*, i.e. $\mathrm{Psd}(K_S)^G \subseteq T_{S'}$ or equivalently, $\mathrm{Psd}(K_S)^G = (T_{S'})^G$.

Clearly, $\mathbb{R}[x,y]^G$ is an $\mathbb{R}$-algebra containing

$$u(x,y) = x^2 + y^2 \text{ and } v(x,y) = x^2y^2.$$

It can be shown that $u(x,y)$ and $v(x,y)$ are algebraically independent and that they generate $\mathbb{R}[x,y]^G$. Hence, the mapping

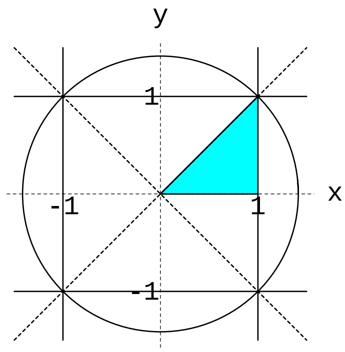
$$\tilde{\pi}\colon \mathbb{R}[u,v] \to \mathbb{R}[x,y]^G, \quad \tilde{\pi}(f)(u,v) = f(u(x,y), v(x,y))$$

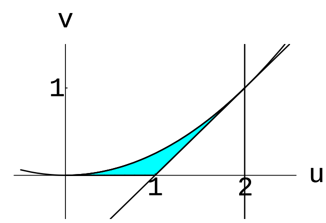
is an isomorphism. On the other hand, the mapping

$$\pi\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad \pi(x,y) = (u(x,y), v(x,y))$$

is not onto. It is easy to see that $\pi(\mathbb{R}^2) = K_{\{u,v,u^2-4v\}}$. The mapping π is not one-to-one either. It can be shown that two points have the same image if and only if they lie in the same G-orbit. (We call π the *orbit map* and $\pi(\mathbb{R}^2)$ the *orbit space*.)

The set $\Delta = \{(x,y) \mid 0 \le y \le x \le 1\}$ (picture on the left) contains exactly one point from each orbit of K_S.





Now we can compute $\pi(K_S) = \pi(\Delta)$ (picture on the right) by either parametrizing the boundary of Δ or the following way:

$$\begin{aligned}\pi(K_S) &= \pi(K_{S'}) = K_{\tilde{\pi}^{-1}(S')} \cap \pi(\mathbb{R}^2) \\ &= K_{\{2-u,1-u+v\}} \cap K_{\{u,v,u^2-4v\}} = K_{\{2-u,1-u+v,u,v,u^2-4v\}}.\end{aligned}$$

By Corollary 6, the preordering $T_{\{2-u,1-u+v,u,v,u^2-4v\}}$ is saturated. Hence

$$\mathrm{Psd}(K_S)^G = \tilde{\pi}(\mathrm{Psd}(\pi(K_S))) \subseteq \tilde{\pi}(T_{\{2-u,1-u+v,u,v,u^2-4v\}}) \subseteq T_{S'}.$$

4 Proof of Theorem 4

Assertion (2) follows from assertion (1), by going to the extension ring $\mathbb{R}[[\sqrt{x}, y]]$, so it suffices to prove (1). We can assume $f \neq 0$. We know $\mathbb{R}[[x,y]]$ is a UFD [13, Theorem 6, p. 148]. Factor f into irreducibles in $\mathbb{R}[[x,y]]$. Using the Preparation Theorem, we can assume the factorization has the form

$$f = ux^m g = ux^m \prod_{i=1}^{\ell} p_i^{m_i}$$

where u is a unit and each $p_i = p_i(y)$ is a monic polynomial in y with coefficients in $\mathbb{R}[[x]]$, with all coefficients except the leading coefficient in the maximal ideal of $\mathbb{R}[[x]]$. We can reduce to the case where $m = 0$ or 1 and g has no repeated irreducible factors. Since $\pm u$ is a square in $\mathbb{R}[[x,y]]$, we can assume further that $u = \pm 1$.

Since y and $x^{2n} - y$ are obviously in the preordering generated by y and $x^{2n} - y$, we can assume $y \nmid g$ and $y - x^{2n} \nmid g$. More generally, if g has an irreducible factor p which has constant sign on the set $y > 0$ in the real spectrum (see [1]) of $\mathbb{R}((x,y))$ then, by Part (2) of Theorem 3, $\pm p$ is in the preordering generated by y. Similarly, if p has constant sign on the set $x^{2n} > y$ in the real spectrum of $\mathbb{R}((x,y))$ then, by Part (2) of Theorem 3 (using the fact that $\mathbb{R}[[x,y]] = \mathbb{R}[[x, x^{2n} - y]]$), $\pm p$ is in the preordering generated by $x^{2n} - y$. Consequently, we can assume that g has no such irreducible factors.

Fix an irreducible factor p of g and consider the discrete valuation on $\mathbb{R}((x,y))$ with associated valuation ring $\mathbb{R}[[x,y]]_{(p)}$. The residue field is $L = \mathrm{qf}\frac{\mathbb{R}[[x,y]]}{(p)} = \frac{\mathbb{R}((x))[y]}{(p)}$ [13, Theorem 6, p. 148]. Set $\overline{y} = y + (p)$. Since $p \neq y$, $p \neq y - x^{2n}$, we know that $\overline{y} \neq 0$, $\overline{y} \neq x^{2n}$. L is a finite extension of the complete discrete valued field $\mathbb{R}((x))$ so it either has no orderings (if the residue field is $\mathbb{C}$) or two orderings (if the residue field is $\mathbb{R}$).

Claim 1: L has no ordering satisfying $0 < \overline{y} < x^{2n}$. Otherwise, pulling this ordering back to $\mathbb{R}((x,y))$, using Baer–Krull, yields two orderings on $\mathbb{R}((x,y))$ satisfying $0 < y < x^{2n}$, one with $p > 0$ and one with $p < 0$. Since an irreducible factor q of f different from p has the same sign at each of these two orderings, and since p has multiplicity 1 in f, one of these two orderings must make $f < 0$. This contradicts our assumption and proves the claim.

Claim 2: L has an ordering satisfying $\overline{y} > x^{2n}$ and also an ordering satisfying $\overline{y} < 0$. By assumption $p = p(y)$ is not always positive on the set $y > 0$ in the real spectrum of $\mathbb{R}((x,y))$, so there exists an ordering of $\mathbb{R}((x,y))$, with real closure R say, with $y > 0$ and $p(y) < 0$, so the polynomial $p(t)$ (obtained by replacing y by the new variable t) has a root $a > y$ in R. Then $\overline{y} \mapsto a$ defines an $\mathbb{R}((x))$-embedding of L into R, so L has an ordering satisfying $\overline{y} > 0$, i.e., $\overline{y} > x^{2n}$. We prove the second assertion when $\deg(p)$ is odd. The proof when $\deg(p)$ is even is similar. By assumption p is not always negative on the set $x^{2n} > y$ in the real spectrum of $\mathbb{R}((x,y))$, so there exists an ordering of $\mathbb{R}((x,y))$ with real closure R say, with $y < x^{2n}$ and $p(y) > 0$, so the polynomial $p(t)$ has a root $a < y$ in R. Then $\overline{y} \mapsto a$ defines an $\mathbb{R}((x))$-embedding of L into R, so L has an ordering satisfying $\overline{y} < x^{2n}$, i.e., $\overline{y} < 0$.

Denote the valuation on L by v. Since $p(\overline{y}) = 0$ we see that $v(\overline{y}) > 0$. Since L has an ordering satisfying $\overline{y} > x^{2n}$, it follows that $v(\overline{y}) \leq v(x^{2n})$. At the same time, $v(\overline{y}) = v(x^{2n})$ is not possible. (If $v(\overline{y}) = v(x^{2n})$ then $\overline{y} = ux^{2n}$, u a unit. Since $\overline{y}$ is positive at one ordering and negative at the other, the same would be true for u, which is not possible.) Thus $0 < v(\overline{y}) < v(x^{2n})$.

Of course, since the various roots a of p in the algebraic closure of $\mathbb{R}((x))$ are conjugate to $\overline{y}$ over $\mathbb{R}((x))$, they all have the same value $v(a) = v(\overline{y})$.

Write $f = \pm x^m p_1 \dots p_\ell$ where the p_i are irreducible, $p_i = \sum_{j=0}^{k_i} b_{ij} y^j$, $b_{ik_i} = 1$, $v(b_{i0}) = k_i v(a_i)$, $v(b_{ij}) \geq (k_i - j) v(a_i)$, where a_i is a fixed root of p_i. We know $0 < v(a_i) < v(x^{2n})$. Decompose f as

$$f = f(0) + \sum_{\underline{j} \neq (0,\dots,0)} \pm x^m b_{\underline{j}} y^{j_1 + \dots + j_\ell} \tag{1}$$

where $\underline{j} := (j_1, \dots, j_\ell)$, $b_{\underline{j}} := b_{1j_1} \dots b_{\ell j_\ell}$ and $f(0) := \pm x^m b_{10} \dots b_{\ell 0}$.

Claim 3: $f(0)$ is positive at both orderings of $\mathbb{R}((x))$, i.e., $f(0)$ is a square in $\mathbb{R}[[x]]$. Suppose to the contrary that $f(0)$ is negative at one of the orderings of $\mathbb{R}((x))$. Consider the discrete valuation on $\mathbb{R}((x,y))$ with valuation ring $\mathbb{R}[[x,y]]_{(y)}$ and residue field $\mathbb{R}((x))$. Pulling the culprit ordering of $\mathbb{R}((x))$ back to $\mathbb{R}((x,y))$, using Baer–Krull, yields two orderings of $\mathbb{R}((x,y))$, one of which satisfies $x^{2n} > y > 0$ and $f < 0$. This is a contradiction.

We write each term $\pm x^m b_{\underline{j}} y^{j_1+\cdots+j_\ell}$, $\underline{j} \neq (0,\dots,0)$ in (1) as

$$(c_{\underline{j}} \pm x^m b_{\underline{j}}) y^{j_1+\cdots+j_\ell} + c_{\underline{j}}(x^{2n(j_1+\cdots+j_\ell)} - y^{j_1+\cdots+j_\ell}) - c_{\underline{j}} x^{2n(j_1+\cdots+j_\ell)}.$$

Factoring in the obvious way, we see that $x^{2n(j_1+\cdots+j_\ell)} - y^{j_1+\cdots+j_\ell}$ lies in the preordering generated by $x^{2n} - y$ and y. To complete the proof, it suffices to show we can choose the elements $c_{\underline{j}} \in \mathbb{R}[[x]]$, $\underline{j} \neq (0,\dots,0)$ such that

$$c_{\underline{j}} \pm x^m b_{\underline{j}}, c_{\underline{j}} \text{ and } f(0) - \sum_{\underline{j}\neq(0,\dots,0)} c_{\underline{j}} x^{2n(j_1+\cdots+j_\ell)}$$

are squares in $\mathbb{R}[[x]]$. Since $\underline{j} \neq (0,\dots,0)$,

$$\begin{aligned} v(x^m b_{\underline{j}}) &= v(x^m) + \sum_i v(b_{ij_i}) \\ &\geq v(x^m) + \sum_i (k_i - j_i) v(a_i) \\ &= v(x^m) + \sum_i k_i v(a_i) - \sum_i j_i v(a_i) \\ &> v(x^m) + \sum_i k_i v(a_i) - \sum_i j_i v(x^{2n}) \\ &= v(x^m) + \sum_i v(b_{i0}) - \sum_i j_i v(x^{2n}) \\ &= v\Big(\frac{f(0)}{x^{2n(j_1+\cdots+j_\ell)}}\Big). \end{aligned}$$

We choose the $c_{\underline{j}}$ as follows: If $\underline{j} \neq (k_1,\dots,k_\ell)$ or $\underline{j} = (k_1,\dots,k_\ell)$ and $m = 1$, then $x^m b_{\underline{j}}$ has positive value. In this case, we choose $c_{\underline{j}}$ with small positive lowest coefficient and with

$$v(c_{\underline{j}}) = \max\Big\{v\Big(\frac{f(0)}{x^{2n(j_1+\cdots+j_\ell)}}\Big), 0\Big\}.$$

In the remaining case, where $m = 0$ and $\underline{j} = (k_1,\dots,k_\ell)$, $b_{ij_i} = 1$, $i = 1,\dots,\ell$, and we choose $c_{\underline{j}} = 1$. The point is, with this choice of $c_{\underline{j}}$, for each $\underline{j} \neq (0,\dots,0)$, either $c_{\underline{j}} x^{2n(j_1+\cdots+j_\ell)}$ has larger value than $f(0)$ or, it has the same value as $f(0)$, but its lowest coefficient is small.

5 Concluding remarks

1. Theorems 3 and 4 do not cover all interesting cases. The general question remains: When is a finitely generated preordering of $\mathbb{R}[[x,y]]$ saturated? Recall that the *saturation* of a preordering T of a general ring A (commutative with 1) is the intersection of all orderings of A containing T, and that T is said to be *saturated* if it coincides with its saturation.

2. The following preorderings of $\mathbb{R}[[x,y]]$ are saturated:

(i) The preordering of $\mathbb{R}[[x,y]]$ generated by y and $y - x^n$, n odd, $n \geq 3$.
(ii) The preordering of $\mathbb{R}[[x,y]]$ generated by $y^2 - x^n$, n odd, $n \geq 3$.

Saturation in Case (i) is a consequence of saturation in Case (ii), by going to the extension ring $\mathbb{R}[[x,\sqrt{y}]]$. In an analogous way, saturation in Case (ii) is a consequence of [4, Theorem 5.1], by going to the extension ring

$$A := \frac{\mathbb{R}[[x,y]][z]}{(z^2 - y^2 + x^n)} = \frac{\mathbb{R}[[x,y,z]]}{(z^2 - y^2 + x^n)}.$$

[4, Theorem 5.1] asserts that the ring A defined above satisfies $\mathrm{Psd} = \mathrm{sos}$, i.e., that the preordering of A consisting of sums of squares in saturated.[1] Actually, [4, Theorem 5.1] is stated in terms of analytic function germs. What we are quoting here is the formal power series version of the result. Note: Knowing saturation holds in Case (i) allows one to extend Corollary 6, adding an additional case to the list.

3. It is still not known if the following preorderings of $\mathbb{R}[[x,y]]$ are saturated:

(iii) The preordering of $\mathbb{R}[[x,y]]$ generated by $y, y - x^n$ and $x^m - y$, n odd, m even, $n > m \geq 2$.
(iv) The preordering of $\mathbb{R}[[x,y]]$ generated by $y, y - x^n, x^m - y$ and $x^m(1 + a(x)) - y$, n odd, m even, $n > m \geq 2$, $a(x) \in \mathbb{R}[[x]]$, $a(0) = 0$.

A positive answer in Cases (iii) and (iv), coupled with what we already know by Theorems 3 and 4 and Case (i) above, would complete our understanding of saturation for preorderings of $\mathbb{R}[[x,y]]$ generated by finitely many elements of order ≤ 1. The proof of this assertion will not be given here. The *order* of $f \in \mathbb{R}[[x,y]]$ is defined to be the greatest integer $k \geq 0$ such that $f \in \mathfrak{m}^k$, where $\mathfrak{m}$ denotes the maximal ideal of $\mathbb{R}[[x,y]]$.

4. The case where some of the generators have order ≥ 2 seems to be pretty much wide open. Case (ii) is of this type, as is the example given earlier, in Remark 5 (ii). If $g \in \mathbb{R}[x,y]$ and $\mathrm{Psd} = \mathrm{sos}$ holds for the ring $A = \frac{\mathbb{R}[[x,y]][z]}{(z^2 - g)}$, then the preordering of $\mathbb{R}[[x,y]]$ generated by g is saturated. Combining this with [4, Theorem 3.1] yields a variety of examples of this sort where g has order 2 or 3 and saturation holds.

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