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O$(p+1)\times$ O$(q+1)$-invariant $(r-1)$-minimal hypersurfaces in Euclidean space $\mathbb{R}^{p+q+2}$

Paulo Sousa*

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Abstract. The aim of the paper is to present a classification of nonextendable immersed O$(p+1)\times$ O$(q+1)$-invariant $(r-1)$-minimal hypersurfaces in the Euclidean space $\mathbb{R}^{p+q+2}$, with $p,q>1$ and $2\le r\le \min\{p,q\}$, by analyzing embeddedness as well as $(r-1)$-stability. The case $r=1$ and $r=2$ were treated in [1] and [17], respectively. Generalizing the seminal work of Bombieri et al. we also present a $(r-1)$-stable complete embedded hypersurface of $\mathbb{R}^{p+q+2}$ with $H_r=0$ and O$(p+1)\times$ O$(q+1)$-invariant, where $p+q\ge r+5$, that is not homeomorphic to $\mathbb{R}^{p+q+1}$.

Key words. Stability, invariant hypersurfaces, r^{th} mean curvature.

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1 Introduction

Equivariant geometry methods have been applied for many mathematicians to get and to classify explicit examples of hypersurfaces with one given condition on H_r and invariant for the action of a group of isometry. One of the first known papers is made by Delaunay [5], in which rotational surfaces of $\mathbb{R}^3$ with constant mean curvature are classified. After the classification of the groups of isometry of low cohomogeneity due to Hsiang and Lawson in [10], much work approaching equivariant geometry has been done. Studying hypersurfaces of $\mathbb{R}^{2m}$ invariant for O$(m)\times$ O(m), Hsiang, Teng and Yu in [11] show the existence of immersions of $\mathbb{S}^{2m-1}$ on $\mathbb{R}^{2m}$ with constant mean curvature that are not round spheres. These immersions jointly with the work of Wente (see [21]) show that the so called Hopf conjecture is false for all dimension.

In [6], do Carmo and Dajczer extended the classic notion of surface of rotation of $\mathbb{R}^3$ for rotational hypersurface of a space form $\overline{M}^{n+1}(c)$ where they also classified rotational hypersurfaces with constant mean curvature.

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A classification of rotational hypersurfaces with zero scalar curvature and $\mathrm{O}(n)$-invariant, as defined in [6], of a space form was made by Leite (see [12]) and later generalized by Palmas for H_r constant, see [13].

A classification of complete minimal hypersurfaces of $\mathbb{R}^{2m}$ invariant under the action of $\mathrm{SO}(m) \times \mathrm{SO}(m)$ was done by Alencar [1] where he used a seminal idea contained in the work of Bombieri et al. [4]. Years later Alencar et al. [2] presented a study of minimal hypersurfaces of $\mathbb{R}^{p+q+2}$ invariant by the action of $\mathrm{O}(p+1) \times \mathrm{O}(q+1)$ with $p, q > 1$. The study of $\mathrm{O}(p+1) \times \mathrm{O}(q+1)$-invariant hypersurfaces in $\mathbb{R}^{p+q+2}$ with zero scalar curvature began with the work due to Palmas [14] when $p = q = 1$ whereas the case $p = q > 1$ was generalized by Sato [16]. Finally, Sato and Souza Neto [17] closed the case of zero scalar curvature for $p \neq q$.

Our aim here is to present a classification of $\mathrm{O}(p+1) \times \mathrm{O}(q+1)$-invariant hypersurfaces of $\mathbb{R}^{p+q+2}$ with $H_r = 0, p, q > 1$ and $2 \leq r \leq \min\{p, q\}$. Moreover, we will analyze embeddedness and $(r-1)$-stability of such hypersurfaces.

2 Statement of results

First of all let us consider $\mathbb{R}^{p+q+2} = \mathbb{R}^{p+1} \times \mathbb{R}^{q+1}$ and $G_{pq} = \mathrm{O}(p+1) \times \mathrm{O}(q+1)$, the group of isometries. We also consider the standard action $G_{pq} \times \mathbb{R}^{p+q+2} \to \mathbb{R}^{p+q+2}$ given by $(A, B, z, w) \to (Az, Bw)$. We notice that the orbit space of this action can be identified with

$$\Omega = \pi(\mathbb{R}^{p+q+2}) = \big\{(x, y) \in \mathbb{R}^2; x \geq 0, y \geq 0\big\},$$

where $\pi : \mathbb{R}^{p+1} \times \mathbb{R}^{q+1} \to \mathbb{R}^2$ is defined by $\pi(z, w) = (|z|, |w|)$. In this way, every hypersurface $M^{p+q+1} \subset \mathbb{R}^{p+q+2}$ invariant under the action of G_{pq} is generated by a profile curve $\gamma(t) = (x(t), y(t))$, i.e., $M = \pi^{-1}(\gamma)$.

Now let us introduce the polynomial

$$Q_2(t) = \sum_{i=0}^{r} (-1)^i \binom{p}{r-i} \binom{q}{i} t^{r-i}.$$

We will show that Q_2 has r distinct positive real roots. Let us assume that the roots of Q_2 are distributed in the following form $0 < \beta_1 < \cdots < \beta_r$. Hence the profile curve of the invariant hypersurfaces $M = \pi^{-1}(\gamma)$, with $H_r = 0$, is one of the following types:

(A) $\gamma(t)$ is one of the following rays $l_j(t) = (\cos(\rho_j)t, \sin(\rho_j)t)$, where $t \geq 0$ and $\rho_j = \arctan(\sqrt{\beta_j})$, $j = 1, \ldots, r$;

(B) $\gamma(t)$ is regular, intersects orthogonally one of the half-axis $x \geq 0$ or $y \geq 0$ and asymptotes one of the rays l_j when $t \to +\infty$ or $t \to -\infty$;

(C) $\gamma(t)$ is a union of two curves $\gamma_1 : (-\infty, 0] \to \Omega$ and $\gamma_2 : [0, +\infty) \to \Omega$, $\gamma_1(0) = \gamma_2(0)$ being a singularity. Moreover, the curve γ does not intersect the boundary of the orbit space, and asymptotes two rays l_j and l_{j+1} given in the Case (A) when $t \to \pm\infty$;

(D) $\gamma(t)$ is regular and does not intersect the boundary of the orbit space and asymptotes both of the rays l_1 and l_r.

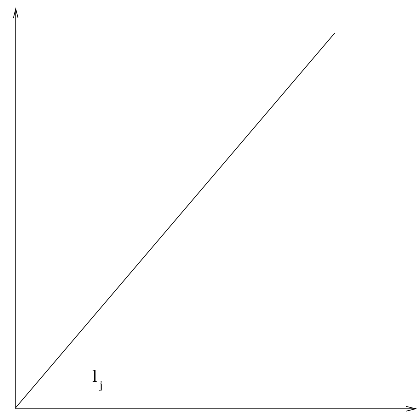

Figure 1. Curve Type A

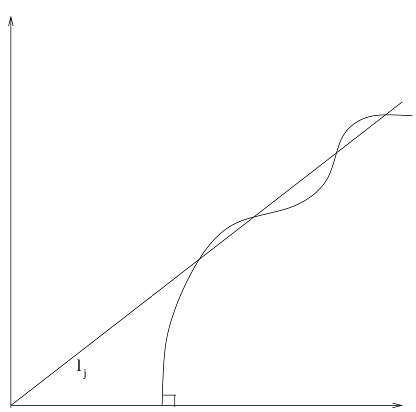

Figure 2. Curve Type B $(p+q \leq r+4)$

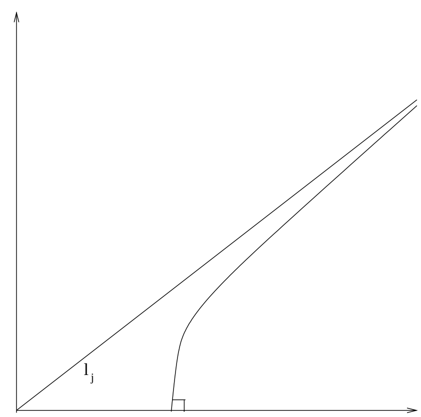

Figure 3. Curve Type B $(p+q \geq r+5)$

We will also denote the cones generated by the half-straight lines of Type A by C_j, where $j = 1, \ldots, r$. With this initial considerations we will state the main results of the paper according to the next theorems.

Theorem 1. *Let $M^{p+q+1} \subset \mathbb{R}^{p+q+2}$ be a G_{pq}-invariant hypersurface with $p, q > 1$, $H_r = 0$ and $2 \leq r \leq \min\{p, q\}$, whose profile curve makes a constant angle with the x-axis. Then M is one of the cones $C_j, 1 \leq j \leq r$.*

Theorem 2 (Classification Theorem). *Let $M^{p+q+1} \subset \mathbb{R}^{p+q+2}$ be a G_{pq}-invariant hypersurface with $H_r = 0$, $p, q > 1$ and $2 \leq r \leq \min\{p, q\}$. Then M^{p+q+1} belongs to one of the following classes:*

(1) *Cones with singularity at the origin of $\mathbb{R}^{p+q+2}$ (Type A).*
(2) *Regular hypersurfaces asymptoting one of the cones C_j (Type B).*

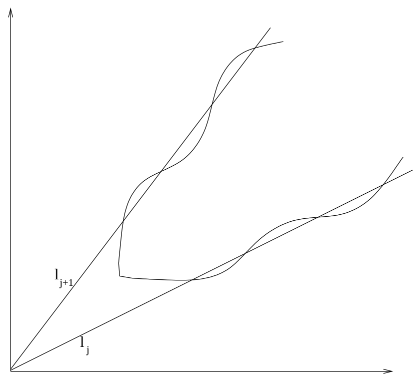

Figure 4. Curve Type C $(p+q \leq r+4)$

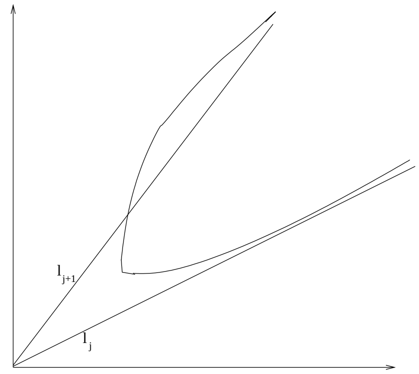

Figure 5. Curve Type C $(p+q \geq r+5)$

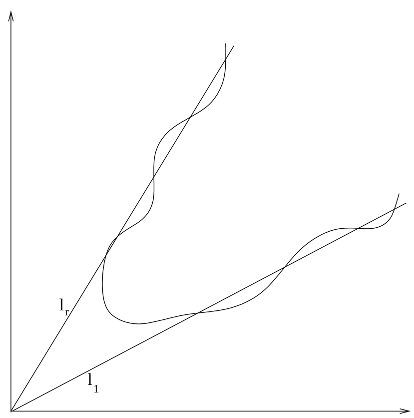

Figure 6. Curve Type D $(p+q \leq r+4)$

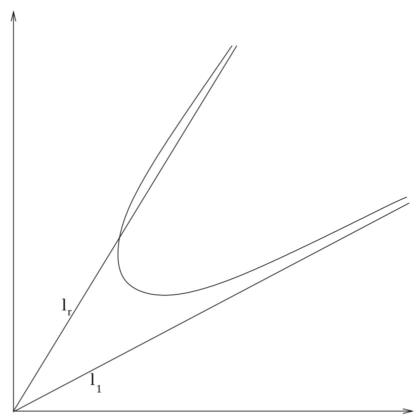

Figure 7. Curve Type D $(p+q \geq r+5)$

(3) *Hypersurfaces having one orbit of singularities that asymptote two cones C_j and C_{j+1} (Type C).*
(4) *Regular hypersurfaces asymptoting both of the cones C_1 and C_r (Type D).*

Theorem 3. *Let $M^{p+q+1} \subset \mathbb{R}^{p+q+2}$ be a complete G_{pq}-invariant hypersurface with $H_r = 0$, $p, q > 1$ and $2 \leq r \leq \min\{p, q\}$. Then M is generated by a curve of type B or D. Moreover,*

(1) *If M is generated by a curve of type B, then it is embedded and asymptotes one of the cones C_j;*
(2) *If M is generated by a curve of type D, then it is embedded and asymptotes both of the cones C_1 and C_r.*

In the last part of this paper we will discuss the stability of these hypersurfaces, obtaining the following result.

Theorem 4. *Let $M^{p+q+1} \subset \mathbb{R}^{p+q+2}$ be a complete G_{pq}-invariant hypersurface with $H_r = 0$, $p, q > 1$ and $2 \leq r \leq \min\{p, q\}$ which is generated by a curve of type B or D;*
(1) *If $p + q \leq r + 4$, then $\mathrm{Ind}_{J_{r-1}}(M)$ is infinity.*
(2) *If $p + q \geq r + 5$, then the hypersurface generated by a curve of type B is globally $(r-1)$-stable ($\mathrm{Ind}_{J_{r-1}}(M) = 0$).*

In particular, the Bernstein theorem does not hold for $H_r = 0$, when $2 \leq r \leq \min\{p, q\}$, according to the following result.

Theorem 5. *There exists an embedded, complete hypersurface $M^{p+q+1} \subset \mathbb{R}^{p+q+2}$, homeomorphic to $\mathbb{S}^p \times \mathbb{R}^{q+1}$ or $\mathbb{R}^{p+1} \times \mathbb{S}^q$ with $H_r = 0$, $2 \leq r \leq \min\{p, q\}$, that is globally $(r-1)$-stable.*

3 O$(p+1)\times$O$(q+1)$-invariant hypersurfaces with $H_r = 0$

Under the above considerations we point out that the orbital distance is the standard metric of $\mathbb{R}^2$ and to each interior point of Ω there corresponds a principal orbit given as the product of sphere $\mathbb{S}^p(x) \times \mathbb{S}^q(y)$ (see [10]). As the invariant hypersurfaces are generated by curves $\gamma(t) = (x(t), y(t))$ in the orbit space, an explicit parametrization of the invariant hypersurfaces $M = \pi^{-1}(\gamma)$ is given by

$$\varphi(t, a, b) = (x(t)\Phi(a), y(t)\Psi(b)),$$

where Φ and Ψ are parametrizations (in polar coordinates) of the unit spheres $\mathbb{S}^p(1) \subset \mathbb{R}^{p+1}$ and $\mathbb{S}^q(1) \subset \mathbb{R}^{q+1}$, respectively. From now on, we suppose that the curve $\gamma(t)$ is parametrized by arc length t. Using the parametrization above and the normal vector

$$N(t, a, b) = (-y'(t)\Phi(a), x'(t)\Psi(b))$$

it can shown that the principal curvatures associated to M are

$$\begin{aligned} k_0 &= x'y'' - x''y'; \\ k_i &= \frac{y'}{x}, \qquad i = 1, 2, \ldots, p; \\ k_j &= -\frac{x'}{y}, \qquad j = p+1, p+2, \ldots, p+q. \end{aligned}$$

The r-mean curvature of the hypersurface is defined by $\binom{n}{r} H_r = S_r$, where S_r is the r^{th} symmetric function of the principal curvatures which is given by

$$S_r = \sum_{i_1 < \cdots < i_r} k_{i_1} \ldots k_{i_r}, \quad 1 \leq r \leq p + q + 1.$$

Let $M = \pi^{-1}(\gamma)$ be an invariant hypersurface with $H_r = 0$. Since the profile curve $\gamma(t) = (x(t), y(t))$ is parametrized by arc length t, $H_r = 0$ yields the following equation:

$$\sum_{i=0}^{r-1} (-1)^i c_{1+i} d_i (x'y'' - x''y') \left(\frac{y'}{x}\right)^{r-1-i} \left(\frac{x'}{y}\right)^i + \sum_{i=0}^{r} (-1)^i c_i d_i \left(\frac{y'}{x}\right)^{r-i} \left(\frac{x'}{y}\right)^i = 0,$$

where $c_j = \binom{p}{r-j}$ and $d_j = \binom{q}{j}$.

We point out that regular curves $\gamma(t) = (x(t), y(t))$ satisfying the equation of the $(r-1)$-minimal hypersurfaces (equation above) are invariant for homotheties. Therefore, for each invariant solution $\gamma(t)$ we have a family M_λ of hypersurfaces with $H_r = 0$, generated by the curves $\gamma_\lambda(t) = (\lambda x(t), \lambda y(t))$.

Using that $(x')^2 + (y')^2 = 1$, we get $x'y'' - x''y' = \frac{y''}{x'}$ and $x'y'' - x''y' = -\frac{x''}{y'}$. From where we obtain

$$y'' = -x' \frac{\sum_{i=0}^{r}(-1)^i c_i d_i \left(\frac{y'}{x}\right)^{r-i}\left(\frac{x'}{y}\right)^i}{\sum_{i=0}^{r-1}(-1)^i c_{1+i} d_i \left(\frac{y'}{x}\right)^{r-1-i}\left(\frac{x'}{y}\right)^i},$$

$$x'' = y' \frac{\sum_{i=0}^{r}(-1)^i c_i d_i \left(\frac{y'}{x}\right)^{r-i}\left(\frac{x'}{y}\right)^i}{\sum_{i=0}^{r-1}(-1)^i c_{1+i} d_i \left(\frac{y'}{x}\right)^{r-1-i}\left(\frac{x'}{y}\right)^i}.$$

Locally the curve is the graph of a function either on the x-axis or on the y-axis. Assuming that the curve is a graph over the x-axis, i.e., $y = y(x)$ we have that $y' = \frac{dy}{dx}x'$ and $y'' = \frac{d^2y}{dx^2}(x')^2 + \frac{dy}{dx}x''$. Therefore, $(x')^2\frac{d^2y}{dx^2} = y'' - \frac{dy}{dx}x''$. Thus we may write the first equation as follows:

$$\frac{d^2y}{dx^2} = -\frac{\left(1 + \left(\frac{dy}{dx}\right)^2\right)}{xy} \frac{\sum_{i=0}^{r}(-1)^i c_i d_i \left(y\frac{dy}{dx}\right)^{r-i} x^i}{\sum_{i=0}^{r-1}(-1)^i c_{1+i} d_i \left(y\frac{dy}{dx}\right)^{r-1-i} x^i}.$$

In a similar way if $x = x(y)$, we may write the second equation in the following form:

$$\frac{d^2x}{dy^2} = \frac{\left(1 + \left(\frac{dx}{dy}\right)^2\right)}{xy} \frac{\sum_{i=0}^{r}(-1)^i c_i d_i \left(x\frac{dx}{dy}\right)^{i} y^{r-i}}{\sum_{i=0}^{r-1}(-1)^i c_{1+i} d_i \left(x\frac{dx}{dy}\right)^{i} y^{r-1-i}}.$$

These equations show that the profile curves have singularities at the zeros of the equations below:

$$\sum_{i=0}^{r-1}(-1)^i c_{1+i} d_i \left(y\frac{dy}{dx}\right)^{r-1-i} x^i = 0, \tag{1}$$

$$\sum_{i=0}^{r-1}(-1)^i c_{1+i} d_i \left(x\frac{dx}{dy}\right)^{i} y^{r-1-i} = 0. \tag{2}$$

4 Analysis of the associated vector field

Proceeding as in [1] and [4], we introduce the Bombieri–De Giorgi–Giusti coordinate transformation $(x, y) \to (u, v)$ given by

$$u = \arctan\left(\frac{y}{x}\right) \quad \text{and} \quad v = \arctan\left(\frac{y'}{x'}\right).$$

Since the orbit space is the region $\Omega = \{(x,y)\in\mathbb{R}^2; x\geq 0, y\geq 0\}$, the parameters u and v are on the closure of $(0,\frac{\pi}{2})\times(-\pi,\pi)$.

Using that $x'\cdot\tan(v) = y'$, we get $(x')^2(1+\tan^2(v)) = 1$. Therefore, $x' = \cos(v)$ and $y' = \sin(v)$. Now using $x\cdot\tan(u) = y$, we have

$$x^2+y^2 = y^2\left(1+\frac{x^2}{y^2}\right) = y^2\left(1+\frac{\cos^2(u)}{\sin^2(u)}\right).$$

Then $y^2 = \sin^2(u)(x^2+y^2)$. In an analogous way $x^2 = \cos^2(u)(x^2+y^2)$. Let us introduce the following notation:

$$F = u'\frac{x^r y^r}{[x^2+y^2]^{\frac{r}{2}}}\sum_{i=0}^{r}(-1)^i c_i d_i\left(\frac{y'}{x}\right)^{r-i}\left(\frac{x'}{y}\right)^{i},$$
$$E = u'\frac{x^r y^r}{[x^2+y^2]^{\frac{r}{2}}}\sum_{i=0}^{r-1}(-1)^i c_{1+i} d_i (x'y''-x''y')\left(\frac{y'}{x}\right)^{r-1-i}\left(\frac{x'}{y}\right)^{i}.$$

We notice that

$$\begin{aligned}
F &= u'\frac{x^r y^r}{[x^2+y^2]^{\frac{r}{2}}}\sum_{i=0}^{r}(-1)^i c_i d_i\left(\frac{y'}{x}\right)^{r-i}\left(\frac{x'}{y}\right)^{i}\\
&= u'\sum_{i=0}^{r}(-1)^i c_i d_i (y')^{r-i}(x')^i\frac{y^{r-i}}{[x^2+y^2]^{\frac{r-i}{2}}}\frac{x^i}{[x^2+y^2]^{\frac{i}{2}}}\\
&= u'\sum_{i=0}^{r}(-1)^i c_i d_i \sin^{r-i}(v)\cos^i(v)\sin^{r-i}(u)\cos^i(u)\\
&= u'\sum_{i=0}^{r}(-1)^i c_i d_i (\sin(u)\sin(v))^{r-i}(\cos(u)\cos(v))^i.
\end{aligned}$$

On the other hand $\tan(u) = \frac{y}{x}$ implies $u'\sec^2(u) = \frac{xy'-x'y}{x^2}$, i.e., $u' = \frac{xy'-x'y}{x^2+y^2}$. From where we may write

$$u' = -\frac{1}{\sqrt{x^2+y^2}}\left(\frac{y}{\sqrt{x^2+y^2}}x' - \frac{x}{\sqrt{x^2+y^2}}y'\right) = -\sin(u-v)\frac{1}{\sqrt{x^2+y^2}}.$$

Then,

$$u'\frac{x^r y^r}{[x^2+y^2]^{\frac{r}{2}}} = -\sin(u-v)\frac{x^r y^r}{[x^2+y^2]^{\frac{r+1}{2}}}.$$

Using the expression $v' = x'y''-x''y'$ we get that $\overline{E} = -\frac{E}{v'\sin(u-v)}$ is given by

$$\overline{E} = \sum_{i=0}^{r-1}(-1)^i c_{1+i} d_i (y')^{r-1-i}(x')^i\frac{x^{i+1}y^{r-i}}{[x^2+y^2]^{\frac{r+1}{2}}}$$

$$
\begin{aligned}
&= \sum_{i=0}^{r-1} (-1)^i c_{1+i} d_i \sin^{r-1-i}(v) \cos^i(v) \frac{x^{i+1}}{\left[x^2+y^2\right]^{\frac{i+1}{2}}} \frac{y^{r-i}}{\left[x^2+y^2\right]^{\frac{r-i}{2}}} \\
&= \sum_{i=0}^{r-1} (-1)^i c_{1+i} d_i \sin^{r-1-i}(v) \cos^i(v) \sin^{r-i}(u) \cos^{i+1}(u) \\
&= \sin(u)\cos(u) g(u,v),
\end{aligned}
$$

where $g(u,v) = \sum_{i=0}^{r-1} (-1)^i c_{1+i} d_i (\sin(u)\sin(v))^{r-1-i} (\cos(u)\cos(v))^i$. From where we have that $E = -v' \sin(u)\cos(u)\sin(u-v) g(u,v)$.

Multiplying the equation of the $(r-1)$-minimal by $u' \frac{x^r y^r}{[x^2+y^2]^{\frac{r}{2}}}$ we get $E + F = 0$. Then,

$$
u' \sum_{i=0}^{r} (-1)^i c_i d_i (\sin(u)\sin(v))^{r-i} (\cos(u)\cos(v))^i = v' \sin(u)\cos(u)\sin(u-v) g(u,v).
$$

This equation provides a system of ordinary differential equations for u and v, to which is associated the vector field $X(u,v) = (X_1(u,v), X_2(u,v)) = (u', v')$ in the plane $\mathbb{R}^2$ given by

$$
\begin{aligned}
X_1(u,v) &= \sin(u)\cos(u)\sin(u-v) g(u,v), \\
X_2(u,v) &= \sum_{i=0}^{r} (-1)^i c_i d_i (\sin(u)\sin(v))^{r-i} (\cos(u)\cos(v))^i.
\end{aligned}
$$

Observe that $X(u, v-\pi) = (-1)^r X(u,v)$. Then it is enough to analyze the field in $[0, \frac{\pi}{2}] \times [0, \pi]$. In order to do that we will determine its singularities, which are described in the next four cases.

Case 1. Observe that $X_1(0,v) = 0$. Hence $X_2(0,v) = 0$ is necessary that $\cos(v) = 0$, i.e., $v = \frac{\pi}{2}$. This says that $(0, \frac{\pi}{2})$ is a singularity of X.

Case 2. It is easy to see that $X_1(\frac{\pi}{2}, v) = 0$. So $X_2(\frac{\pi}{2}, v) = 0$ yields $\sin(v) = 0$. Therefore, $v = 0$ or $v = \pi$. Then, $(\frac{\pi}{2}, 0)$ and $(\frac{\pi}{2}, \pi)$ are singularities of X.

Case 3. As $X_1(u,u) = 0$, we get that the points of the set $K := X_2^{-1}(0) \cap \{(u,u); u \in [0, \frac{\pi}{2}]\}$ are also singularities of X.

Case 4. Let us assume $g(u,v) = X_2(u,v) = 0$. If $\sin(u)\sin(v) = 0$, we get that $\cos(u)\cos(v) = 0$. In the same way $\cos(u)\cos(v) = 0$ implies $\sin(u)\sin(v) = 0$. With this we have that the points $(0, \frac{\pi}{2})$; $(\frac{\pi}{2}, 0)$ and $(\frac{\pi}{2}, \pi)$ are singularities of X. On the other hand, $\sin(u)\sin(v) \neq 0$ and $\cos(u)\cos(v) \neq 0$ yield

$$
\frac{g(u,v)}{(\cos(u)\cos(v))^{r-1}} = \sum_{i=0}^{r-1} (-1)^i c_{1+i} d_i [\tan(u)\tan(v)]^{r-1-i}
$$

and

$$\frac{X_2(u,v)}{(\cos(u)\cos(v))^r}=\sum_{i=0}^{r}(-1)^i c_i d_i[\tan(u)\tan(v)]^{r-i}.$$

Let us consider the following polynomials that play an important role in our analysis:

$$Q_1(t)=\sum_{i=0}^{r-1}(-1)^i\binom{p}{r-1-i}\binom{q}{i}t^{r-1-i},$$
$$Q_2(t)=\sum_{i=0}^{r}(-1)^i\binom{p}{r-i}\binom{q}{i}t^{r-i}.$$

We will show that these polynomials do not have roots in common. In particular, there does not exist (u,v) such that $Q_1(\tan(u)\tan(v))=Q_2(\tan(u)\tan(v))=0$.

By using the previous cases we derive the following lemma.

Lemma 1. *The singularities of the field* $X=(X_1,X_2)$ *in* $[0,\frac{\pi}{2}]\times[0,\pi]$ *occur at* $P_1=(0,\frac{\pi}{2})$, $P_2=(\frac{\pi}{2},0)$, $P_3=(\frac{\pi}{2},\pi)$ *and at the points of* K.

Before announcing the main tool of the polynomials Q_1 and Q_2 we will introduce some notation.

Notation 1. Let $P:\mathbb{R}\to\mathbb{R}$ be a general polynomial.

(1) The degree of P is indicated by ∂P;
(2) R_P^+ stands for the set of the positive real roots of the polynomial P;
(3) $|P|$ stands for the number of elements of R_P^+;
(4) $R_P^+=\{t_1,\dots,t_k\}$ indicates that P has k distinct positive real roots satisfying $t_1<\cdots<t_k$;
(5) $m_P(a)$ stands for the multiplicity of $a\in R_P^+$ as a root of P;
(6) P' indicates the derivative of P.

Theorem 6. *Let* Q_1 *and* Q_2 *be the polynomials given previously, with* $r\geq 2$. *Then we have:*

(i) $R_{Q_2}^+\cap R_{Q_1}^+=\emptyset$; $R_{Q_2}^+\cap R_{Q_2'}^+=\emptyset$ *and* $R_{Q_1}^+\cap R_{Q_1'}^+=\emptyset$;
(ii) *Let* $\alpha_{j+1}>\alpha_j>0$ *be consecutive roots of* Q_1, *then there exists* $\beta_i\in(\alpha_j,\alpha_{j+1})$ *such that* $Q_2(\beta_i)=0$. *Reciprocally, if* $\beta_{i+1}>\beta_i>0$ *are consecutive roots of* Q_2, *then there exists* $\alpha_j\in(\beta_i,\beta_{i+1})$ *such that* $Q_1(\alpha_j)=0$;
(iii) *If* $r\leq\min\{p,q\}$, *then*
  (a) $r=|Q_2|=|Q_1|+1$;
  (b) *Writing* $R_{Q_1}^+=\{\alpha_1,\dots,\alpha_{r-1}\}$ *and* $R_{Q_2}^+=\{\beta_1,\dots,\beta_r\}$ *we have the following gap:*
$$0<\beta_1<\alpha_1<\beta_2<\cdots<\beta_{r-1}<\alpha_{r-1}<\beta_r;$$
(iv) $|Q_2|>0$ *provided* $\min\{p,q\}<r\leq 2\min\{p,q\}-1$.

Proof. Let us consider the auxiliary polynomials

$$T_1(t) := tQ_1(t) + Q_2(t) = \sum_{i=0}^{r} (-1)^i \binom{p+1}{r-i} \binom{q}{i} t^{r-i},$$
$$T_2(t) := Q_2(t) - Q_1(t) = \sum_{i=0}^{r} (-1)^i \binom{p}{r-i} \binom{q+1}{i} t^{r-i}.$$

A straightforward computation yields $(p+1)Q_1(t) = T_1'(t) = Q_1(t) + tQ_1'(t) + Q_2'(t)$. Then,

$$pQ_1(t) = tQ_1'(t) + Q_2'(t).$$

Letting $P_2(t) := t^r T_2\left(\frac{1}{t}\right)$, using the expression of T_2 we get

$$P_2'(t) = -(q+1)t^{r-1}Q_1\left(\frac{1}{t}\right).$$

As consequence we have $(q+1)Q_1(t) = -t^{r-1}P_2'\left(\frac{1}{t}\right)$. On the other hand, it is easy to derive the following equations from the definition of P_2:

$$\begin{aligned} t^r P_2\left(\frac{1}{t}\right) &= T_2(t), \\ rt^{r-1}P_2\left(\frac{1}{t}\right) - t^{r-2}P_2'\left(\frac{1}{t}\right) &= T_2'(t), \\ rt^r P_2\left(\frac{1}{t}\right) - t^{r-1}P_2'\left(\frac{1}{t}\right) &= tT_2'(t), \\ rT_2(t) + (q+1)Q_1(t) &= tT_2'(t), \\ rQ_2(t) + (q-r+1)Q_1(t) &= tQ_2'(t) - tQ_1'(t). \end{aligned}$$

With this we get the following equations:

$$(t+1)Q_2'(t) = rQ_2(t) + (p+q-r+1)Q_1(t), \tag{3}$$
$$ptQ_1(t) = rQ_2(t) + (q-r+1)Q_1(t) + t(t+1)Q_1'(t). \tag{4}$$

If $a \in R_{Q_j}$, where $j \in \{1, 2\}$, an analysis of the polynomial Q_j shows that $a > 0$. Let us suppose that exists such $a \in R_{Q_2} \cap R_{Q_1}$. On the other hand, Equations (3) and (4) imply $m_{Q_1}(a) = r - 1$ and $m_{Q_2}(a) = r$. Therefore,

$$a^r = \frac{\binom{q}{r}}{\binom{p}{r}} \quad \text{and} \quad a^{r-1} = \frac{\binom{q}{r-1}}{\binom{p}{r-1}}.$$

From where we obtain $a = \frac{q-r+1}{p-r+1}$. Since $a \cdot r = \frac{\binom{p}{r-1}\binom{q}{1}}{\binom{p}{r}} = q\frac{r}{p-r+1}$, we get $a = \frac{q}{p-r+1}$. This implies $r = 1$, which gives a contradiction, since $r \geq 2$. Therefore, $R_{Q_2} \cap R_{Q_1} = \emptyset$. Using this jointly with the Equations (3) and (4) again we conclude $R_{Q_2} \cap R_{Q_2'} = \emptyset$ and $R_{Q_1} \cap R_{Q_1'} = \emptyset$.

Let us assume $R_{Q_1} = \{\alpha_1, \dots, \alpha_n\}$ and $R_{Q_2} = \{\beta_1, \dots, \beta_m\}$. It is clear that $Q_1'(\alpha_n) > 0$ and $\operatorname{sign}[Q_1'(\alpha_1)] = (-1)^r$. Using the Equation (4) we get the item (2). Also it is easy to see that, $Q_1'(\alpha_j)Q_1'(\alpha_{j+1}) < 0$. From the Equation (4), we have $Q_2(\alpha_j)Q_2(\alpha_{j+1}) < 0$. From an analogous argument and by using the Equation (3) we get $Q_1(\beta_j)Q_1(\beta_{j+1}) < 0$. This proves that $m = n+1$ and $0 < \beta_1 < \alpha_1 < \beta_2 < \cdots < \beta_{m-1} < \alpha_{m-1} < \beta_m$.

The above arguments can be made for arbitrary s with $2 \le s \le \min\{p, q\}$; then we will use the notation Q_1^s to indicate the polynomial Q_1 of degree $s-1$ and Q_2^s to indicate the polynomial Q_2 of degree s. We point out that $Q_2^{s-1} = Q_1^s$ and $|Q_1^3| = 2$. Therefore,

$$|Q_1^3| = 2 \Rightarrow |Q_2^3| = |Q_1^4| = 3 \Rightarrow \cdots \Rightarrow |Q_2^{r-1}| = |Q_1^r| = r-1 \Rightarrow |Q_2^r| = r.$$

From where we conclude the proof of the theorem. □

We consider the functions $v_j : [0, \frac{\pi}{2}] \to \mathbb{R}$ given by $v_j(u) = \arctan\big(\alpha_j \cot(u)\big)$, where $1 \le j \le r-1$. If J_j is the graph of v_j, then $g^{-1}(0) = \bigcup_{j=1}^r J_j$. We will denote for D_{j+1} the domain delimited by J_j and J_{j+1}, for $1 \le j \le r-2$. We will call D_1 the domain delimited by J_1 and the straight lines $u = v = 0$ while D_r will be the domain delimited by J_{r-1} and the straight lines $u = v = \frac{\pi}{2}$.

Let us also consider the function $q_1 : (0, \frac{\pi}{2}) \to \mathbb{R}$ given by $q_1(u) = \frac{g(u,u)}{(\cos^2(u))^{r-1}}$, i.e., $q_1(u) = Q_1(\tan^2(u))$. If $u_0 \in (0, \frac{\pi}{2})$ is such that $g(u_0, u_0) = 0$, then $q_1(u_0) = 0$. Deriving q_1 we get $q_1'(u) = 2\tan(u)\sec^2(u)Q_1'(\tan^2(u))$. Since $R_{Q_1} \cap R_{Q_1'} = \emptyset$ we have $q_1'(u_0) \ne 0$. Therefore $g(u,u)$ changes sign at u_0. The function $g(u,u)$ also satisfies $g(0,0) = (-1)^{r-1}\binom{p}{r-1}$ and $g(\frac{\pi}{2}, \frac{\pi}{2}) = \binom{p}{r-1} > 0$. Thus $\operatorname{sign}(g|D_j) = (-1)^{r-j}$. Denoting by D_j^+ the set of the points of D_j such that $v > u$ and by D_j^- the set of the points of D_j such that $v < u$ we get $\operatorname{sign}(X_1|D_j^+) = -(-1)^{r-j}$ and $\operatorname{sign}(X_1|D_j^-) = (-1)^{r-j}$. From these remarks we obtain the next lemma.

Lemma 2. *The first coordinate X_1 of the field X satisfies:*

(1) X_1 *vanishes along of* $u = 0$, $u = \frac{\pi}{2}$, $v = u$ *and* I_j*;*
(2) $X_1(u, 0) = (-1)^{r-1}\binom{q}{r-1}\sin^2(u)\cos^r(u)$*;*
(3) $X_1(u, \frac{\pi}{2}) = -\binom{p}{r}\sin^r(u)\cos^2(u)$*;*
(4) $\operatorname{sign}(X_1|D_j^+) = -(-1)^{r-j}$ *and* $\operatorname{sign}(X_1|D_j^-) = (-1)^{r-j}$.

Now we will analyze the coordinate X_2. Let us consider the following function $q_2 : (0, \frac{\pi}{2}) \to \mathbb{R}$ given by $q_2(u) = \frac{X_2(u,u)}{(\cos^2(u))^r}$, i.e., $q_2(u) = Q_2(\tan^2(u))$.

Let $u_0 \in (0, \frac{\pi}{2})$ be such that $X_2(u_0, u_0) = 0$, then $q_2(u_0) = 0$. Deriving q_2 we get $q_2'(u) = 2\tan(u)\sec^2(u)Q_2'(\tan^2(u))$. Since $R_{Q_2} \cap R_{Q_2'} = \emptyset$, we have $q_2'(u_0) \ne 0$. Then $X_2(u,u)$ changes sign at u_0. Now we consider the functions $w_j : [0, \frac{\pi}{2}] \to \mathbb{R}$ defined by $w_j(u) = \arctan(\beta_j \cot(u))$, $1 \le j \le r$. If I_j is the graph of w_j, then $X_2^{-1}(0) = \bigcup_{j=1}^r I_j$. We will denote by R_j the domain delimited by I_j and I_{j+1}, for $j = 1, \dots, r-1$. We will call R_0 the domain delimited by I_1 and the straight lines $u = v = 0$ while R_r will be the domain delimited by I_r and the straight lines $u = v = \frac{\pi}{2}$. The coordinate X_2 also satisfies $X_2(0,0) = (-1)^r\binom{p}{r}$ and $X_2(\frac{\pi}{2}, \frac{\pi}{2}) = \binom{p}{r} > 0$. Therefore, $\operatorname{sign}(X_2|R_j) = (-1)^{r-j}$. Summing up we derive the following lemma.

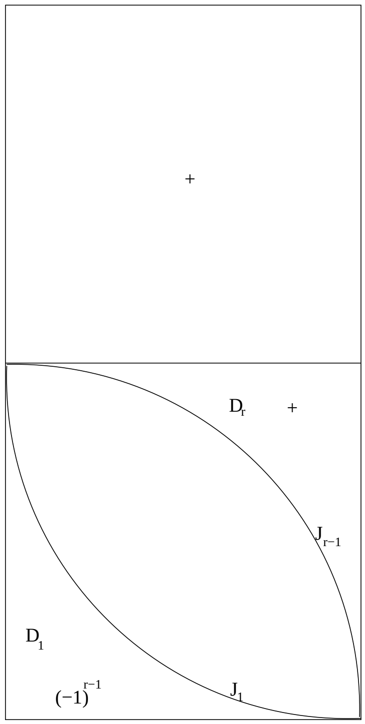

Figure 8. Sign of g

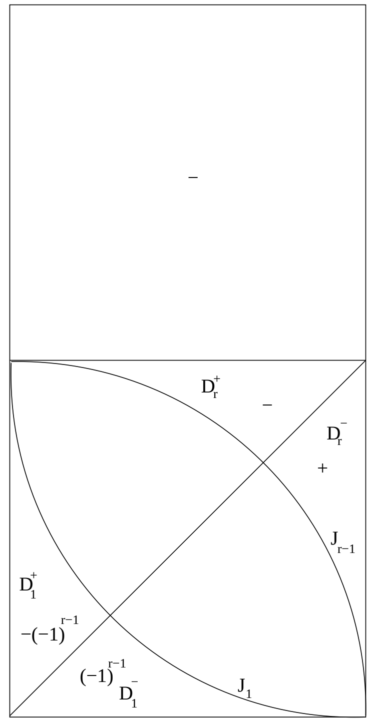

Figure 9. Sign of X_1

Lemma 3. *The second coordinate X_2 of the vector field X satisfies:*

(1) *X_2 vanishes along the graphs of the functions w_j;*
(2) $X_2(u,0) = (-1)^r \binom{q}{r} \cos^r(u)$, $X_2(u,\frac{\pi}{2}) = \binom{p}{r} \sin^r(u)$*;*
(3) $X_2(0,v) = (-1)^r \binom{q}{r} \cos^r(v)$, $X_2(\frac{\pi}{2},v) = \binom{p}{r} \sin^r(v)$*;*
(4) $\mathrm{sign}(X_2|R_j) = (-1)^{r-j}$.

Observation 1. Denoting by $K_j = (\arctan(\sqrt{\beta_j}), \arctan(\sqrt{\beta_j}))$, $1 \le j \le r$, the points of the set K we see that $K_j \in D_j$ is the only singularity of X in the domain D_j, where $1 \le j \le r$.

To classify the singularities of X we need to compute its Jacobian $DX = (A_{ij})$. In order to do that we will use the following notation:

$$L_i(u,v) = (\sin(u)\sin(v))^{r-1-i}(\cos(u)\cos(v))^i;$$
$$M_i(u,v) = (\sin(u)\sin(v))^{r-i}(\cos(u)\cos(v))^{i-1};$$
$$\begin{aligned} A_{11} &= \frac{\partial X_1}{\partial u}(u,v) \\ &= \frac{\partial}{\partial u}\left[\frac{1}{2}\sin(2u)\sin(u-v)\right] g(u,v) + \sin(u)\cos(u)\sin(u-v)\frac{\partial g}{\partial u}(u,v) \end{aligned}$$

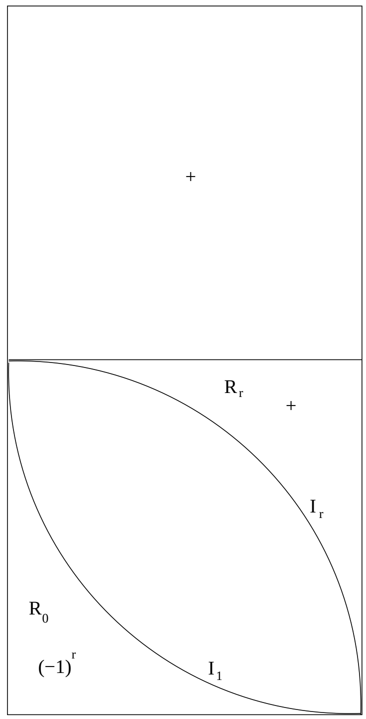

Figure 10. Sign of X_2

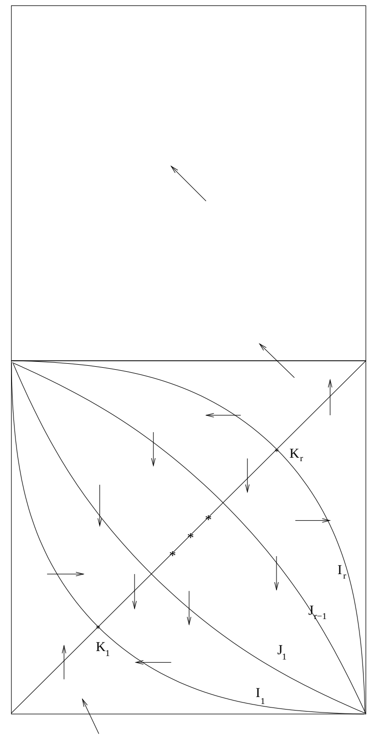

Figure 11. Transversality-r even.

$$
\begin{aligned}
&= \left[\cos(2u)\sin(u-v) + \frac{1}{2}\sin(2u)\cos(u-v)\right] g(u,v) \\
&\quad + \sin(u)\cos(u)\sin(u-v)\frac{\partial g}{\partial u}(u,v); \\
A_{22} &= \frac{\partial X_2}{\partial v}(u,v) \\
&= \sum_{i=0}^{r}(-1)^i c_i d_i (r-i)\sin(u)\cos(v)L_i(u,v) \\
&\quad - \sum_{i=0}^{r}(-1)^i c_i d_i \cdot i \cdot \cos(u)\sin(v)M_i(u,v); \\
A_{12} &= \frac{\partial X_1}{\partial v}(u,v) \\
&= \frac{\partial}{\partial v}\left[\frac{1}{2}\sin(2u)\sin(u-v)\right] g(u,v) + \sin(u)\cos(u)\sin(u-v)\frac{\partial g}{\partial v}(u,v) \\
&= -\frac{1}{2}\sin(2u)\cos(u-v)g(u,v) + \sin(u)\cos(u)\sin(u-v)\frac{\partial g}{\partial v}(u,v);
\end{aligned}
$$

$$\begin{aligned}A_{21} &= \frac{\partial X_2}{\partial u}(u,v)\\ &= \sum_{i=0}^{r}(-1)^i c_i d_i (r-i)\cos(u)\sin(v)L_i(u,v)\\ &\quad -\sum_{i=0}^{r}(-1)^i c_i d_i \cdot i \cdot \sin(u)\cos(v)M_i(u,v).\end{aligned}$$

From the above expressions we have that the singularities P_1, P_2 and P_3 are degenerated. Now, we need to classify the singularities contained in the set K. In order to do that we need to know $DX(u,u)$. First of all we have

$$\begin{aligned}A_{11}(u,u) &= \frac{1}{2}\sin(2u)\sum_{i=0}^{r-1}(-1)^i c_{1+i} d_i L_i(u,u),\\ A_{22}(u,u) &= \sin(u)\cos(u)\sum_{i=0}^{r-1}(-1)^i c_i d_i (r-i) L_i(u,u)\\ &\quad - \sin(u)\cos(u)\sum_{i=1}^{r}(-1)^i c_i d_i \cdot i \cdot M_i(u,u).\end{aligned}$$

Now it is easy to see that $A_{12}(u,u) = -A_{11}(u,u)$ and $A_{21}(u,u) = A_{22}(u,u)$. Using the equality $M_{j+1}(u,v) = L_j(u,v)$, we obtain

$$A_{22}(u,u) = \frac{1}{2}\sin(2u)\sum_{i=0}^{r-1}(-1)^i\big[c_i d_i (r-i) + c_{1+i} d_{1+i}(i+1)\big]L_i(u,u).$$

On the other hand a straightforward calculation yields

$$\frac{\binom{p}{r-i}(r-i)}{\binom{p}{r-1-i}} = p-r+1+i \quad \text{and} \quad \frac{\binom{q}{i+1}(i+1)}{\binom{q}{i}} = q-i.$$

Therefore, $A_{22}(u,u) = (p+q-r+1)A_{11}(u,u)$. Thus, we obtain the equality

$$DX(u,u) = A_{11}(u,u)A(p,q,r) = A_{11}(u,u)\begin{bmatrix} 1 & -1 \\ p+q-r+1 & p+q-r+1\end{bmatrix}.$$

Let us consider the following notation: $\sigma_0 = 0$, $\sigma_r = \frac{\pi}{2}$ and $\sigma_j = \arctan(\sqrt{\alpha_j})$, for all $1 \le j \le r-1$. Using the equality $A_{11}(u) := A_{11}(u,u) = \frac{1}{2}\sin(2u)g(u,u)$ we conclude that $\operatorname{sign}(A_{11}|(\sigma_j,\sigma_{j+1})) = (-1)^{r-1-j}$, where $0 \le j \le r-1$. We also notice that $DX(u,u-\pi) = (-1)^r DX(u,u)$.

As consequence we obtain the following proposition.

Proposition 1. *The singularities $P_1 = (0,\frac{\pi}{2})$, $P_2 = (\frac{\pi}{2},0)$ and $P_3 = (\frac{\pi}{2},\pi)$ are degenerated. If $p+q \le r+4$, the points K_j with $(r-1-j)$ odd are attractor focus and the points K_j where $(r-1-j)$ is even are repulsor focus. If $p+q \ge r+5$, the points K_j where $(r-1-j)$ is odd are attractor nodes and the points K_j with $(r-1-j)$ even are repulsor nodes.*

Proof. The first part is obvious. For the second part we need the following calculation:

$$\begin{aligned}[\operatorname{tr} A(p,q,r)]^2 - 4\det A(p,q,r) &= [p+q-(r-1)+1]^2 - 8[p+q-(r-1)] \\ &= [p+q-(r-1)]^2 - 6[p+q-(r-1)] + 1 \\ &= [p+q-r-2+2\sqrt{2}][p+q-r-2-2\sqrt{2}].\end{aligned}$$

The eigenvalues of the matrix $A(p,q,r)$ are given by

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{2}\operatorname{tr} A(p,q,r) \pm \frac{1}{2}\sqrt{[\operatorname{tr} A(p,q,r)]^2 - 4\det A(p,q,r)} \\ &= \frac{1}{2}[p+q-(r-1)+1] \pm \frac{1}{2}\sqrt{[p+q-(r-1)]^2 - 6[p+q-(r-1)]+1}.\end{aligned}$$

We also observe that $r+2-2\sqrt{2} < r+2-2 = r < 2r \leq p+q$. Therefore, $p+q-r-2+2\sqrt{2} > 0$. Assuming $p+q \leq r+4$ and using $4 < 2+2\sqrt{2}$, we have $p+q \leq r+4 \Rightarrow p+q < r+2+2\sqrt{2} \Rightarrow p+q-r-2-2\sqrt{2} < 0$. Then the eigenvalues of $A(p,q,r)$ are complex with positive real part. Now assuming $p+q \geq r+5$ and using $5 > 2+2\sqrt{2}$, we have that $p+q > r+2+2\sqrt{2}$. Soon, $p+q-r-2-2\sqrt{2} > 0$. Then the eigenvalues of $A(p,q,r)$ are positive real numbers. Analyzing the sign of the function f we get the conclusion about the points of the set K. □

Bendixson's criterion is useful to classify the orbits of X. Hence we will analyze its periodic orbits by computing the divergence of X. Again we will use the notation

$$L_i(u,v) = (\sin(u)\sin(v))^{r-1-i}(\cos(u)\cos(v))^i.$$

From previous calculations we have that

$$\begin{aligned}\operatorname{div} X(u,v) &= A_{11}(u,v) + A_{22}(u,v) \\ &= \big[\sin(u)\cos(v)(2-3\sin^2(u)) + \cos(u)\sin(v)(2-3\cos^2(u))\big]g(u,v) \\ &\quad + \cos^2(u)\sin(u-v)\sum_{i=0}^{r-1}(-1)^i\binom{p}{r-1-i}\binom{q}{i}(r-1-i)L_i(u,v) \\ &\quad - \sin^2(u)\sin(u-v)\sum_{i=0}^{r-1}(-1)^i\binom{p}{r-1-i}\binom{q}{i}iL_i(u,v) \\ &\quad + \sin(u)\cos(v)\sum_{i=0}^{r-1}(-1)^i\binom{p}{r-i}\binom{q}{i}(r-i)L_i(u,v) \\ &\quad + \cos(u)\sin(v)\sum_{i=0}^{r-1}(-1)^i\binom{p}{r-1-i}\binom{q}{i+1}(i+1)L_i(u,v).\end{aligned}$$

Therefore,

$$\begin{aligned}\operatorname{div} X &= \big[\sin(u)\cos(v)(2-3\sin^2(u)) + \cos(u)\sin(v)(2-3\cos^2(u))\big]g(u,v) \\ &\quad + (1-\sin^2(u))\sin(u)\cos(v)\sum_{i=0}^{r-1}(-1)^i\binom{p}{r-1-i}\binom{q}{i}(r-1-i)L_i(u,v)\end{aligned}$$

$$
\begin{aligned}
&- \cos^2(u)[\cos(u)\sin(v)] \sum_{i=0}^{r-1} (-1)^i \binom{p}{r-1-i} \binom{q}{i} (r-1-i) L_i(u,v) \\
&- \sin^2(u)[\sin(u)\cos(v)] \sum_{i=0}^{r-1} (-1)^i \binom{p}{r-1-i} \binom{q}{i} i L_i(u,v) \\
&+ (1-\cos^2(u))\cos(u)\sin(v) \sum_{i=0}^{r-1} (-1)^i \binom{p}{r-1-i} \binom{q}{i} i L_i(u,v) \\
&+ \sin(u)\cos(v) \sum_{i=0}^{r-1} (-1)^i \binom{p}{r-i} \binom{q}{i} (r-i) L_i(u,v) \\
&+ \cos(u)\sin(v) \sum_{i=0}^{r-1} (-1)^i \binom{p}{r-1-i} \binom{q}{i+1} (i+1) L_i(u,v).
\end{aligned}
$$

After some manipulations we arrive at

$$
\begin{aligned}
\operatorname{div} X = {} & \big[\sin(u)\cos(v)(2-3\sin^2(u)) + \cos(u)\sin(v)(2-3\cos^2(u))\big] g(u,v) \\
& + \sin(u)\cos(v) \sum_{i=0}^{r-1} (-1)^i \binom{p}{r-i} \binom{q}{i} (r-i) L_i(u,v) \\
& + \sin(u)\cos(v) \sum_{i=0}^{r-1} (-1)^i \binom{p}{r-1-i} \binom{q}{i} (r-1-i) L_i(u,v) \\
& - \sin^2(u)[\sin(u)\cos(v)] \sum_{i=0}^{r-1} (-1)^i \binom{p}{r-1-i} \binom{q}{i} (r-1) L_i(u,v) \\
& + \cos(u)\sin(v) \sum_{i=0}^{r-1} (-1)^i \binom{p}{r-1-i} \binom{q}{i+1} (i+1) L_i(u,v) \\
& + \cos(u)\sin(v) \sum_{i=0}^{r-1} (-1)^i \binom{p}{r-1-i} \binom{q}{i} i L_i(u,v) \\
& - \cos^2(u))[\cos(u)\sin(v)] \sum_{i=0}^{r-1} (-1)^i \binom{p}{r-1-i} \binom{q}{i} (r-1) L_i(u,v) \\
= {} & \big[\sin(u)\cos(v)(2-3\sin^2(u)) + \cos(u)\sin(v)(2-3\cos^2(u))\big] g(u,v) \\
& + p\sin(u)\cos(v) g(u,v) - (r-1)\sin^2(u)[\sin(u)\cos(v)] g(u,v) \\
& + q\cos(u)\sin(v) g(u,v) - (r-1)\cos^2(u)[\cos(u)\sin(v)] g(u,v).
\end{aligned}
$$

Finally we deduce

$$
\begin{aligned}
\operatorname{div} X = {} & \sin(u)\cos(v)\big[p+2-(r+2)\sin^2(u)\big] g(u,v) \\
& + \cos(u)\sin(v)\big[q+2-(r+2)\cos^2(u)\big] g(u,v).
\end{aligned}
$$

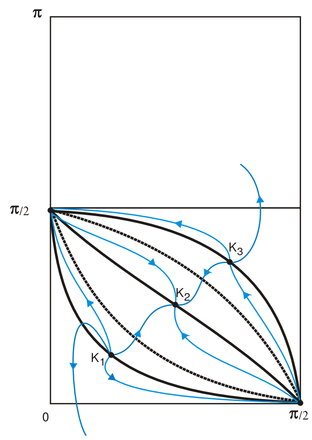

Figure 12. Orbits of X for $p+q \geq r+5$, when $r=3$.

Proposition 2. *The field X does not have periodic orbits in $D := \bigcup_{j=0}^{r} D_j$.*

Proof. As $\sin^2(u) = 1 \Leftrightarrow u = \frac{\pi}{2}$ and $\cos^2(u) = 1 \Leftrightarrow u = 0$, we obtain that $p+2-(r+2)\sin^2(u) > p+2-r-2 \geq 0$ and $q+2-(r+2)\cos^2(u) > p+2-r-2 \geq 0$. It is also easy to see that $\sin(u)\cos(v) > 0$ and $\cos(u)\sin(v) > 0$ in $(0,\frac{\pi}{2}) \times (0,\frac{\pi}{2})$. As $g(u,v) \neq 0$ for $(u,v) \in D$, the result follows from the Bendixson Theorem. □

In that follows, we will denote by $D_j^{-\pi}$, $j = 1,\ldots,r$, the translation of the domain D_j by $(0,-\pi)$. And for $K_j^{-\pi}$ the same translation of the points of the set K.

Proposition 3. *The orbits of $X = (X_1, X_2)$ are defined for all values of t. In the region $R = \{(u,v) \in \mathbb{R}^2 : 0 \leq u \leq \frac{\pi}{2}$ and $-\pi \leq v \leq \pi\}$ their possible behavior are one of the following:*

(1) *$\phi(t)$ is either a vertical orbit with α-limit $(\frac{\pi}{2},0)$ and ω-limit $(\frac{\pi}{2},\pi)$; or, if r is even, a vertical orbit with α-limit $(0,-\frac{\pi}{2})$ and ω-limit $(0,\frac{\pi}{2})$, or a vertical orbit with α-limit $(\frac{\pi}{2},-\pi)$ and ω-limit $(\frac{\pi}{2},0)$; or, if r is odd, a vertical orbit with α-limit $(0,\frac{\pi}{2})$ and ω-limit $(0,-\frac{\pi}{2})$, or a vertical orbit with α-limit $(\frac{\pi}{2},0)$ and ω-limit $(\frac{\pi}{2},-\pi)$;*

(2) *$\phi(t)$ is either a vertical semi-orbit with α-limit $(0,\frac{\pi}{2})$; or, if r is even, a vertical semi-orbit with ω-limit $(0,-\frac{\pi}{2})$; or, if r is odd, a vertical semi-orbit with α-limit $(0,-\frac{\pi}{2})$;*

(3) *$\phi(t)$ is either an orbit in $(0,\frac{\pi}{2})\times(0,\frac{\pi}{2})$ going through the points of J_j, $j = 1,\ldots,r-1$, such that α-limit is K_j and ω-limit is K_{j+1}, or ω-limit is K_j and α-limit is K_{j+1};*

(4) *If r is even, $\phi(t)$ is an orbit in $D_1 \bigcup\{[0,\frac{\pi}{2}]\times[-\frac{\pi}{2},0]\} \cup D_r^{-\pi}$ with ω-limit K_1 and α-limit $K_r^{-\pi}$; or, if r is odd, $\phi(t)$ is an orbit in $D_1 \bigcup\{[0,\frac{\pi}{2}]\times[-\frac{\pi}{2},0]\} \cup D_r^{-\pi}$ with α-limit K_1 and ω-limit $K_r^{-\pi}$;*

(5) *If j is even, $\phi(t)$ is a connection of saddle contained in D_j with α-limit $(0,\frac{\pi}{2})$ and ω-limit K_j; or, if j is odd, $\phi(t)$ is a connection of saddle contained in D_j with ω-limit $(0,\frac{\pi}{2})$ and α-limit K_j;*

(6) *If j is even, $\phi(t)$ is a connection of saddle contained in D_j with α-limit $(\frac{\pi}{2}, 0)$ and ω-limit K_j; or, if j is odd, $\phi(t)$ is a connection of saddle contained in D_j with ω-limit $(\frac{\pi}{2}, 0)$ and α-limit K_j;*
(7) *$\phi(t)$ is a singular orbit K_j, $j = 1, \ldots, r$;*
(8) *$\phi(t)$ is an orbit, or part of it, obtained by a translation, followed by a change of orientation if r is odd, among one of the orbits from the previous items.*

The proof of Proposition 3 is a consequence of Lemmas 2 and 3, while Proposition 2 follows from the Poincaré–Bendixson Theorem and the Tubular Flow Theorem.

5 Classification of the invariant hypersurfaces

In this section we will translate the behavior of the orbits $\phi(t) = (u(t), v(t))$ of the vector field X given by Proposition 3 into information concerning to the corresponding profile curve $\gamma(t) = (x(t), y(t))$. This geometric approach allows us to classify the $\mathrm{O}(p+1) \times \mathrm{O}(q+1)$-invariant hypersurfaces.

On the other hand, according to Proposition (3.1) and Remark (3.1) of [16], the following facts concerning $\mathrm{O}(p+1) \times \mathrm{O}(q+1)$-invariant hypersurfaces $M^{p+q+1} \subset \mathbb{R}^{p+q+2}$ will be used: M is embedded if and only if the associated profile curve is embedded. Moreover, if the orbit of X associated to the profile curve is defined for all t, then the corresponding hypersurface is complete.

Lemma 4. *Let $\phi(t) = (u(t), v(t))$ be an orbit given on item* (4) *of Proposition* 3 *and $\gamma(t) = (x(t), y(t))$ the associated profile curve. Then, $\phi(t)$ intersects the segment $L_j = \{(\sigma_j, v); \sigma_j - \pi < v < \sigma_j, j = 1, \ldots, r-1 \ : \sigma_j = \arctan(\sqrt{\alpha_j})\}$ exactly once, and so $\gamma(t)$ intersects the ray $y = \sqrt{\alpha_j}$ exactly once.*

Proof. See Lemma 4.1 of [17]. □

Proposition 4. *The profile curve $\gamma(t)$ given in Lemma* 4 *does not have self-intersection.*

Proof. The proof of this proposition is similar to that one of Proposition 4.1 of [17] and we leave it to the reader. □

Observation 2. If $0 < v < \frac{\pi}{2}$ we have $x' \neq 0$, $y' \neq 0$ thus we may see the profile curve $\gamma(t) = (x(t), y(t))$ as the graph (or an union of graphs when γ presents singularities) of a function $y = y(x)$ (or $x = x(y)$). The equations derived for $\frac{d^2y}{dx^2}$ and $\frac{d^2x}{dy^2}$ say that the singularities occur at the zeros of the Equations (1) and (2).

We also notice that the coordinates of the profile curve (x, y) associated to the coordinates (u, v) given by $v = \arctan(\alpha_j \cot(u))$, $1 \leq j \leq r-1$, satisfy

$$\frac{dy}{dx} = \frac{y'}{x'} = \tan(v) = \cot(u)\alpha_j = \frac{x}{y}\alpha_j \Rightarrow y\frac{dy}{dx} = x\alpha_j \ \textit{or}$$
$$\frac{dx}{dy} = \frac{x'}{y'} = \cot(v) = \tan(u)\frac{1}{\alpha_j} = \frac{y}{x}\frac{1}{\alpha_j} \Rightarrow x\frac{dx}{dy} = y\frac{1}{\alpha_j}.$$

As α_j is a root of Q_1, we get that the singularities of the profile curve correspond to the coordinates (u,v) above defined.

As we can see the Proposition 3 is the main tool to prove the theorems stated in the introduction.

Proof of Theorem 1. Let $M = \pi^{-1}(\gamma)$ be a hypersurface whose profile curve makes a constant angle with the x-axis. Then $u'(t) = 0$ and there exists $\rho \in \mathbb{R}$ such that $\arctan\left(\frac{y}{x}\right) = \rho$. Thus, $y(t) = \tan(\rho)x(t)$. From a direct substitution on the $(r-1)$-minimal equation we get

$$\sum_{i=0}^{r}(-1)^r\binom{p}{r-i}\binom{q}{i}\left[\tan^2(\rho)\right]^{r-i} = 0.$$

Therefore, $\tan^2(\rho) \in R_{Q_2}$ which completes the proof of the theorem. □

To prove the classification theorem we need the following lemma.

Lemma 5. *The following relationships between the coordinates (x,y) of the profile curve and the coordinates (u,v) of the field hold:*

(1) $u = 0 \Leftrightarrow y = 0$ *and* $u = \frac{\pi}{2} \Leftrightarrow x = 0$*;*
(2) $v = 0, \pm\pi \Leftrightarrow y' = 0$ *and* $v = \pm\frac{\pi}{2} \Leftrightarrow x' = 0$*;*
(3) $v = u \Leftrightarrow \frac{y}{x} = \frac{y'}{x'}$*;*
(4) $v = \frac{\pi}{2} - u \Leftrightarrow \frac{x}{y} = \frac{y'}{x'}$*.*

Proof. The proof follows immediately from the equations $x' = \cos(v)$, $y' = \sin(v)$, and $\tan(u) = \frac{y}{x}$. □

We will use the notation $l_j = \gamma_j = (\cos(\rho_j)t, \sin(\rho_j)t)$, where $t \geq 0$ and $j = 1,\dots,r$.

Proof of Theorem 2. The numbering is according to the statement of the theorem.

(1) These are the hypersurfaces given by Theorem 1 that are associated to the singular orbits of the field X.
(2) Let $\phi(t)$ be an orbit of X that intersects J_j in P, and has ω-limit K_j and α-limit K_{j+1}, or ω-limit K_{j+1} and α-limit K_j. Let $\gamma(t)$ be the associated profile curve. The point P corresponds to the singularity of the profile curve (see Observation 2). As $x^2 = \cos^2(u)(x^2+y^2)$ and $y^2 = \sin^2(u)(x^2+y^2)$ we get that $\gamma(t)$ does not intersect the boundary of the space of orbits. Moreover, $\gamma(t)$ asymptotes the profile curves l_j and l_{j+1} which are associated to the cones C_j and C_{j+1}. Therefore, the associated hypersurfaces asymptote the cones.
(3) Let $\phi(t)$ be an orbit classified in items (5) and (6) of Proposition 3 whose associated profile curve is $\gamma(t) = (x(t), y(t))$. We observe that $(u,v) \to (0, \frac{\pi}{2}) \Rightarrow y \to 0; x' \to 0$ and $(u,v) \to (\frac{\pi}{2}, 0) \Rightarrow x \to 0; y' \to 0$. Therefore, $\gamma(t)$ intersects orthogonally the x-axis or the y-axis. Moreover, $\gamma(t)$ asymptotes the profile curve l_j associated to the cone C_j. Therefore, the associated hypersurfaces asymptote the cone C_j.

(4) Finally, let $\phi(t)$ be an orbit of the field X given in the item 4 of Proposition 3 and $\gamma(t)$ the associated profile curve. Then $\gamma(t)$ does not intersect the boundary of the space of orbits and asymptotes the profile curves l_1 and l_r. This shows that the associated hypersurfaces asymptote the cones C_1 and C_r. □

Proof of Theorem 3. From the Classification Theorem, the type B hypersurfaces M are generated by the profile curves γ associated to the orbits that are connections of saddle. In this case, we have $0 < v < \frac{\pi}{2}$. Therefore, we see from Observation 2 that such a curve is a graph either of a function $y = y(x)$ or $x = x(y)$. From where we conclude that M is embedded. On the other hand, the type D hypersurfaces are generated by profile curves associated to the orbits of item (4) of Proposition 3. Hence from Proposition 4 and from Theorem 1 we get that M is embedded and asymptotes both cones C_1 and C_r. This finishes the proof of Theorem 3. □

6 Stability of $\mathbf{O}(p+1) \times \mathbf{O}(q+1)$-invariant hypersurfaces

During this section we will present a study concerning the r-stability of our hypersurfaces. Before we remember some basic facts about stability. Let $\xi : M^n \to \overline{M}^{n+1}(c)$ be an isometric immersion and $D \subset M$ be a domain with compact closure in M. A variation of D is an application $X : (-\varepsilon, \varepsilon) \times \overline{D} \to \overline{M}^{n+1}(c)$ satisfying the following proprieties:

(1) $X_t : D \to \overline{M}^{n+1}(c)$, given by $X_t(p) := X(t,p)$, is an immersion;
(2) $X_0 = \xi$ and $X_t|\partial D = \xi|\partial D$ for each $t \in (-\varepsilon, \varepsilon)$.

We define the variational field and the normal component of the variation, respectively, by

$$E(t,p) = \frac{\partial X}{\partial t}(t,p) \quad \text{and} \quad f(t,p) = \langle E(t,p), N_t(p) \rangle,$$

where N_t is a unit normal field to $X_t(D)$. When f has compact support we say that X is a variation with compact support. The volume associated to the variation is the function

$$V(t) = \int_{[0,t] \times D} X^* d\overline{M},$$

where $X^* d\overline{M}$ stands for pull-back of the element of volume $d\overline{M}$ of $\overline{M}^{n+1}(c)$. We say that X preserve volume if $V(t) = V(0)$, $\forall t \in (-\varepsilon, \varepsilon)$.

Now we consider the functional

$$A_r(t) = \int_{X_t(D)} F_r(S_1, \ldots, S_r) dM_t,$$

where the functions F_r are defined inductively by

$$F_0 = 1,$$
$$F_1 = S_1,$$

and

$$F_r = S_r + \frac{c(n-r+1)}{r-1}, \quad 2 \le r \le n-1.$$

It is well known (see [3] and [15]) that immersions $\xi : M^n \to \overline{M}^{n+1}(c)$ with constant $(r+1)$-mean curvature H_{r+1} are critical points of the variational problem of minimizing $A_r(t)$ for volume preserving variations, where their normal components have compact support in D. Then the first variation $A_r'(0)$ vanishes. Moreover, the second variation formula reads

$$A_r''(0) = -(r+1)\int_D f[L_r(f) + (S_1 S_{r+1} - (r+2)S_{r+2})f + c(n-r)S_r f)],$$

where $L_r(f) = \mathrm{tr}[P_r \operatorname{Hess}(f)] = \mathrm{div}[P_r(\operatorname{grad} f)]$. Associated to the second variation formula $A_r''(0)$ is the Jacobi operator which is a second order self-adjoint differential operator given by

$$J_r = L_r + (S_1 S_{r+1} - (r+2)S_{r+2}) + c(n-r)S_r.$$

Thus a bilinear symmetric form I_r may be defined by

$$I_r(f,g) = -\int_D g J_r(f) dM.$$

Definition 1. Let $\xi : M^n \to \overline{M}^{n+1}(c)$ be an immersion with constant $(r+1)$-mean curvature H_{r+1} and $D \subset M$ be a domain with compact closure in M. We say that D is r-stable if $I_r(f,f) \ge 0$ for all $f \in C_c^\infty(D)$. Otherwise, D is r-unstable. Moreover, we say that M is stable if every domain $D \subset M$ is r-stable.

We define the index $\mathrm{Ind}_{J_r}(D)$ of the *Jacobi operator* J_r on D as the maximal dimension of a subspace of $f \in C_c^\infty(D)$ where the quadratic form associated to I_r is negative definite. The index $\mathrm{Ind}_{J_r}(M)$ of J_r in M is then defined by

$$\mathrm{Ind}_{J_r}(M) = \sup_{D \subseteq M} \mathrm{Ind}_{J_r}(D),$$

where the supremum is taken over all domains D with compact closure in M.

In order to show our result concerning to r-stability we will need the next proposition and lemma that appear in [7], [8] and [16].

Proposition 5. *Let M be a complete noncompact Riemannian manifold and $\xi : M^n \to \overline{M}^{n+1}(c)$ be an immersion with constant $(r+1)$-mean curvature H_{r+1}. The immersion ξ is r-stable if only if there exists a positive smooth function $f : M \to \mathbb{R}$ satisfying the Jacobi equation $J_r(f) = 0$.*

Lemma 6. *Let $\xi : M^n \to \mathbb{R}^{n+1}$ be an isometric immersion of a connected, orientable Riemannian manifold M in $\mathbb{R}^{n+1}$ with constant $(r+1)$-mean curvature H_{r+1}. The support function $h(p) = \langle \xi(p), N(p)\rangle$ satisfies the Jacobi equation $J_r(h) = 0$ in M.*

Now let M be an $\mathrm{O}(p+1)\times\mathrm{O}(q+1)$-invariant hypersurface in $\mathbb{R}^{p+q+2}$ with $H_r = 0$. Since we are assuming that the profile curve is parametrized by the arc length, the rank of the second fundamental form of the immersion $\xi : M^{p+q+1} \to \mathbb{R}^{p+q+2}$ is greater than $\min\{p, q\} \geq r > r-1$. Therefore, the associated Jacobi operator J_{r-1} is an elliptic operator (see [9]).

Proof of Theorem 4. Let $h(p) = \langle \xi(p), N(p)\rangle$ be the support function of the immersion $\xi : M^{p+q+1} \to \mathbb{R}^{p+q+2}$. By Lemma 6 we have that h satisfies the Jacobi equation $J_{r-1}(h) = 0$ in every domain D with compact closure M, where J_{r-1} is an elliptic operator. Using the parametrization and the normal vector associated to this immersion, given by

$$\varphi(t,a,b) = (x(t)\Phi(a), y(t)\Psi(b)),$$
$$N(t,a,b) = (-y'(t)\Phi(a), x'(t)\Psi(b))$$

and taking into account $u' = \frac{xy'-x'y}{x^2+y^2}$, we conclude that the support function is given by

$$h(t) = -u'(t)(x^2(t) + y^2(t)).$$

Therefore h depends only on the profile curve. Moreover, it is constant along its orbits. When $\gamma(t) = (x(t), y(t))$ is a profile curve of Type B, we have that γ is associated to an orbit $\phi = (u(t), v(t))$ which has one of the points K_j, $1 \leq j \leq r$, as α-limit or ω-limit. When γ is of Type D, the associated orbit has α-limit $K_1^{-\pi}$ and ω-limit K_r (or conversely).

For $p+q \leq r+4$, the singular points K_j are hyperbolic focus (see Proposition 1). Therefore there exists a monotone sequence $(t_i)_{i\in\mathbb{N}}$ with $u'(t_i) = X_1(\phi(t_i)) = 0$, where $\phi(t)$ is associated to a profile curve of Type B or Type D. Then there exists a sequence of domains

$$D_1 \subsetneq D_2 \subsetneq \cdots \subsetneq D_i \subsetneq \cdots \subset M$$

such that $h_{|\partial D_i} = 0$, where ∂D_i is the orbit of $\gamma(t_i)$ under the action of $\mathrm{O}(p+1)\times\mathrm{O}(q+1)$. Then we may apply the Morse Index Theorem to the operator J_{r-1} (see [18], [19] and [20]) to conclude that $\mathrm{Ind}_{J_{r-1}}(M)$ is infinite.

When $p+q \geq r+5$ the singularities K_j are hyperbolic nodes (see Proposition 1). Therefore, $u'(t) \neq 0$ for every orbit associated to a Type B profile curve. In this case, either h or $-h$ is a positive function in M satisfying the Jacobi equation $J_{r-1}(\pm h) = 0$. From Proposition 5 it follows that the hypersurface generated by γ is globally $(r-1)$-stable, which completes the proof of theorem. □

Proof of Theorem 5. Finally we point out that the proof of Theorem 5 is an immediate consequence of Theorem 4. □

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P. Sousa, Universidade Federal do Piauí, Departamento de Matemática, 64049-550 Teresina, PI, Brazil
Email: pauloalexandre@ufpi.edu.br